Assignment 7

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CS21BTECH11021

Question

Example 5.2 [Papoulis]

Let X and Y be two continuous random variables such that $Y = y$ and $X = x$ where $x, y \in \mathbb{R}$ with x and y related by the expression $y = x^2$. Find $\mathbf{F_Y(y)}$ and $\mathbf{f_Y(y)}$

Solution

Definition/Properties

$$F_Y(y) = \Pr(Y \leq y) \tag{1}$$
$$\Pr(a \leq x \leq b) = F_X(b) - F_X(a) \tag{2}$$
$$f_Y(y) = \frac{dF_Y(y)}{dy} \tag{3}$$

If $y \geq 0$,

$$x^2 \leq y \tag{4}$$
$$-\sqrt{y} \leq x \leq \sqrt{y} \tag{5}$$
$$F_Y(y) = \Pr(-\sqrt{y} \leq x \leq \sqrt{y}) \tag{6}$$
$$F_Y(y) = F_X(\sqrt{y}) - F_Y(-\sqrt{y}) \tag{7}$$

If $y < 0$,
x has no real values satisfying the inequality $x^2 < y$

$$F_Y(y) = \Pr(\phi) \tag{8}$$

$$F_Y(y) = 0 \tag{9}$$

CDF

$$F_Y(y) = \begin{cases} F_X(\sqrt{y}) - F_X(-\sqrt{y}) & y \geq 0 \\ 0 & y < 0 \end{cases} \tag{10}$$

PDF

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}}(f_X(\sqrt{y}) + f_X(-\sqrt{y})) & y > 0 \\ 0 & y \leq 0 \end{cases} \tag{11}$$

Case: $f_X(x)$ is even

Let

$$U(y) = \begin{cases} 1 & y > 0 \\ 0 & y \leq 0 \end{cases} \tag{12}$$

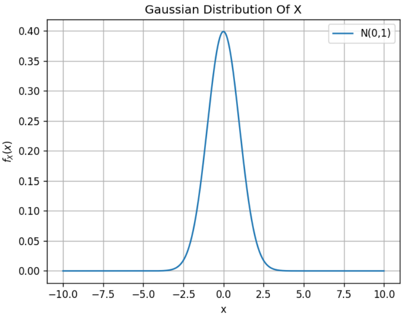
$$f_Y(y) = \frac{1}{\sqrt{y}} f_X(\sqrt{y}) U(y) \tag{13}$$

Case (a): Gaussian Distribution

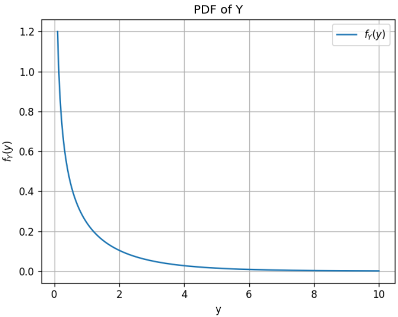
Let X be normally distributed with $\mu = 0$ and $\sigma^2 = 1$.
$X \sim N(0, 1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{14}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2} U(y) \tag{15}$$



(a) Gaussian/Normal Distribution.

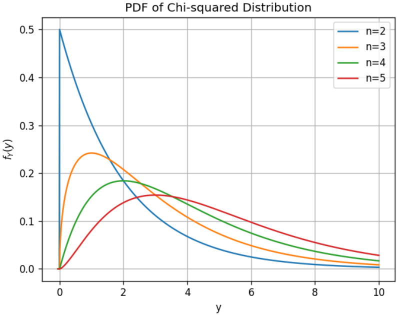


(b) PDF of Y.

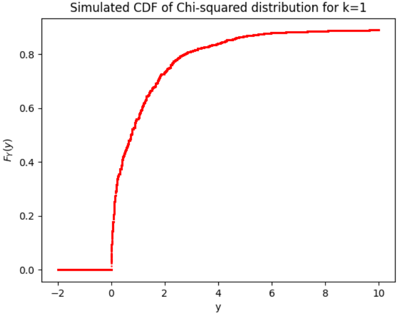
On comparing, we notice that this represents a chi-square random variable with $n = 1$

Chi-sqaured Distribution

$$f_X(x, n) = \begin{cases} \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2} & x > 0 \\ 0 & x \leq 0 \end{cases} \tag{16}$$



(c) PDF for Chi-squared distribution.



(d) Simulated CDF.

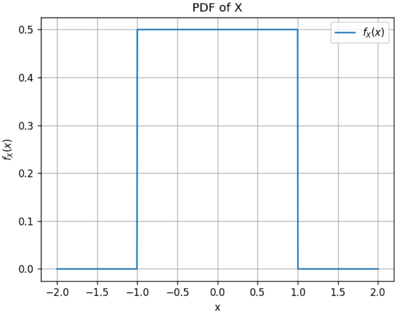
Figure: Chi-squared distribution

Case (b): Uniform Distribution

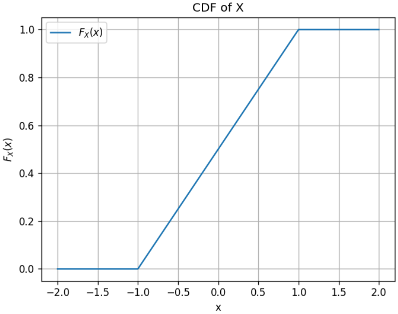
Let X be uniformly distributed in the interval $x \in (-1, 1)$

$$f_X(x) = \begin{cases} 1/2 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases} \tag{17}$$

$$f_Y(y) = \begin{cases} \dfrac{1}{2\sqrt{y}} & y \in (0, 1] \\ 0 & y \notin (0, 1] \end{cases} \tag{18}$$



(a) PDF for Uniform Distribution of X.



(b) CDF for Uniform Distribution of X.

Figure: Uniform Distribution

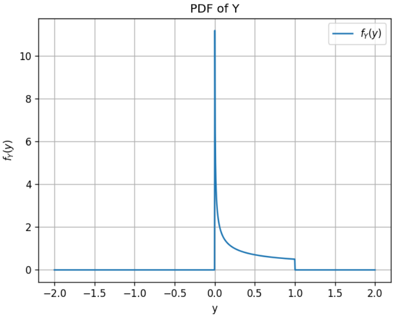


Figure: PDF of Y