7.16. AVL Tree Performance

Before we proceed any further let's look at the result of enforcing this new balance factor requirement. Our claim is that by ensuring that a tree always has a balance factor of -1, 0, or 1 we can get better Big-O performance of key operations. Let us start by thinking about how this balance condition changes the worst-case tree. There are two possibilities to consider, a left-heavy tree and a right heavy tree. If we consider trees of heights 0, 1, 2, and 3, Figure 2 illustrates the most unbalanced left-heavy tree possible under the new rules.

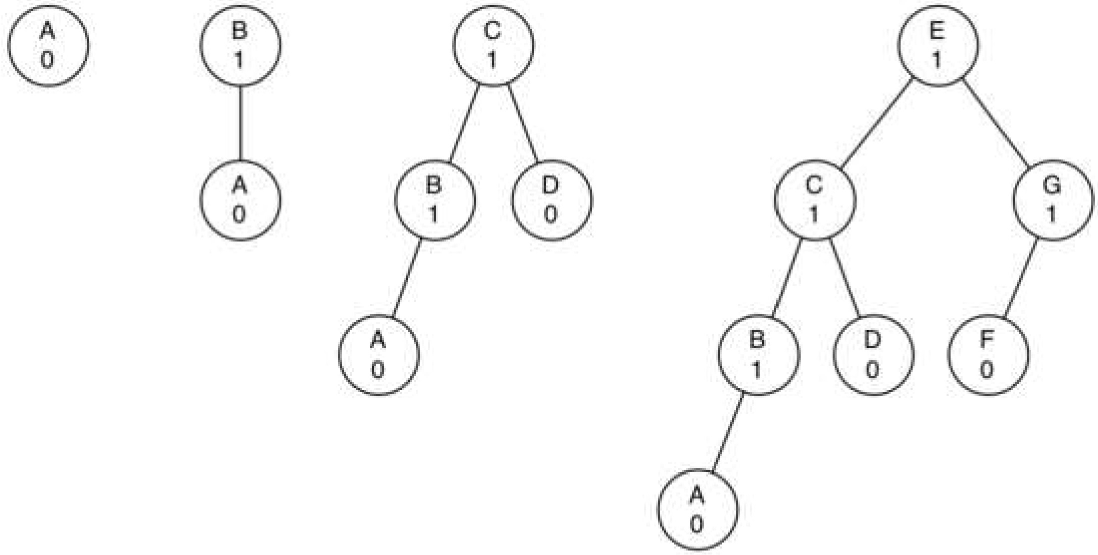

Figure 2: Worst-Case Left-Heavy AVL Trees

Looking at the total number of nodes in the tree we see that for a tree of height 0 there is 1 node, for a tree of height 1 there is $1 + 1 = 2$ nodes, for a tree of height 2 there are $1 + 1 + 2 = 4$ and for a tree of height 3 there are $1 + 2 + 4 = 7$. More generally the pattern we see for the number of nodes in a tree of height h (N_h) is:

$$N_h = 1 + N_{h-1} + N_{h-2}$$

This recurrence may look familiar to you because it is very similar to the Fibonacci sequence. We can use this fact to derive a formula for the height of an AVL tree given the number of nodes in the tree. Recall that for the Fibonacci sequence the i_{th} Fibonacci number is given by:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_i &= F_{i-1} + F_{i-2} \text{ for all } i \ge 2 \end{aligned}$$

An important mathematical result is that as the numbers of the Fibonacci sequence get larger and larger the ratio of F_i/F_{i-1} becomes closer and closer to approximating the golden ratio Φ which is defined as $\Phi = \frac{1+\sqrt{5}}{2}$. You can consult a math text if you want to see a derivation of the previous equation. We will simply use this equation to approximate F_i as $F_i = \Phi^i/\sqrt{5}$. If we make use of this approximation we can rewrite the equation for N_h as:

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$$N_h = F_{h+1} - 1, h \ge 1$$

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By replacing the Fibonacci reference with its golden ratio approximation we get:

$$N_h = \frac{\Phi^{h+2}}{\sqrt{5}} - 1$$

If we rearrange the terms, and take the base 2 log of both sides and then solve for h we get the following derivation:

$$\log N_h + 1 = (h+2)\log \Phi - \frac{1}{2}\log 5$$
$$h = \frac{\log N_h + 1 - 2\log \Phi + \frac{1}{2}\log 5}{\log \Phi}$$
$$h = 1.44 \log N_h$$

This derivation shows us that at any time the height of our AVL tree is equal to a constant(1.44) times the log of the number of nodes in the tree. This is great news for searching our AVL tree because it limits the search to $O(\log N)$.

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