Homework 1 GoA 3

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Question 1

There are several ways to compute the integral in the question but in order to make it a bit easier I applied the fundamental theorem of calculus for line integrals and tried to find a potential function.

We start with the integral

$$\int_{\gamma} e^y dx + xe^y dy + (z+1)e^z dz \tag{1}$$

Then we from this write a system of differentialequations

$$\frac{\partial U}{\partial x} = e^y \tag{2}$$
$$\frac{\partial U}{\partial y} = xe^y \tag{3}$$
$$\frac{\partial U}{\partial z} = (z+1)e^z \tag{4}$$

Solving this system is then a matter of integration. We start by solving the first equation.

$$\frac{\partial U}{\partial x} = e^y \Leftrightarrow U = \int e^y dx = xe^y + g(y,z) \tag{5}$$

This integration gives us a integration constant wich is a function of y and z in order to find out what this function is we begin by taking the y derivative of U.

$$\frac{\partial U}{\partial y} = xe^y + g_y \tag{6}$$

The right hand side of equation 3 and 6 should be equal this gives that $g_y = 0$

We do the same in z and get that $g_z = (z+1)e^z$. The problem has now been reduced to two equations

$$g_y = 0 \tag{7}$$
$$g_z = (z+1)e^z \tag{8}$$

If we then solve equation 7

$$g(y,z) = \int 0dy = 0 + h(z) \tag{9}$$

This gives a constant of integration wich is a function of z. We take the z derivative of this function and get that $h'(z) = (z+1)e^z$ we can then integrate this to find out what $h(z)$ is.

$$h(z) = \int h'(z)dz = \int (z+1)e^z dz = \int ze^z + e^z dz = \int ze^z dz + e^z \tag{10}$$

To solve $\int ze^z$ we need to use integration by parts.

$$\int ze^z dz = ze^z - \int e^z dz = e^z(z-1) \tag{11}$$

This gives that $h(z) = e^z(z-1) + e^z$ and the full potential function $U = xe^x + e^z(z-1) + e^z$

We can then evaluate this at the endpoints which are using the parametrization given $(0,0,0)$ and $(-1,-1,1)$

$$U(0,0,0) = 0e^0 + e^0(0-1) + e^0 = 0 \tag{12}$$
$$U(-1,-1,1) = -1e^-1 + e^1(1-1) + e^1 = -\frac{1}{e} + e \tag{13}$$

The integral is then $U(0,0,0) - U(-1,-1,1) = \frac{1}{e} - e$

$$\int_\gamma e^y dx + xe^y dy + (z+1)e^z dz = \frac{1}{e} - e \tag{14}$$

Question 2

In order to make this integral easier to solve we can use greens theorem to rewrite it. We have the integral

$$\int_\gamma \frac{y^3}{3}dx - \frac{x^3}{3}dy \tag{1}$$

Using greens theorem we can rewrite it amsmath

$$\iint_D -(x^2+y^2)dxdy \tag{2}$$

Where D is the volume enclosed by γ and γ is the sphere described by $x^2+y^2=3$. We can then switch to polar coordinates in order to calculate the integral.

$$\iint_D -r^2 r dr d\theta \tag{3}$$

Rewriting the domain we find that $0 \leq r \leq \sqrt{3}$ and $0 < \theta \leq 2\pi$. We can then apply Fubinis theorem to 3 and get the iterated integral

$$\int_0^{2\pi}\int_0^{\sqrt{3}} -r^3 dr d\theta = 2\pi \left[\frac{-r^4}{4}\right]_0^{\sqrt{3}} = -\frac{9\pi}{2} \tag{4}$$

So we get

$$\int_\gamma \frac{y^3}{3}dx - \frac{x^3}{3}dy = -\frac{9\pi}{2} \tag{5}$$