Fysikens matematiska metoder 1FA121, VT 2025

Homework 1

- Upload the solution as a PDF file to Studium **before the deadline** posted on the homework page. Late submissions will not be accepted.
- The score will count towards the final grade (scale 0-40) according to the rounding of

$$\text{bonus on final grade} = \frac{\text{your score}}{\text{maximum score}} \times 2 \tag{1}$$

  to the nearest half-integer unit (example: $1.4 \to 1.5$ whereas $1.2 \to 1$)

- The grading of each problem takes into account **how complete** the answer is, and how **well explained** the reasoning is.

Problem 1

(4 points) Solve the following problem

$$\begin{aligned}
&\partial_t u(x,t) = \kappa \partial_x^2 u(x,t) \\
&\partial_x u(0,t) = 0 = \partial_x u(L,t) = 0 \qquad (t > 0) \\
&u(x,0) = \begin{cases} 0 & 0 < x < L/2 \\ 1 & L/2 < x < L \end{cases}
\end{aligned} \tag{2}$$

Problem 2

Consider a 1-dimensional rod of length L, with one endpoint at $x = 0$ immersed in a heat bath at temperature $T = 0°C$, and the other end at $x = L$ insulated. Assume no sources, so that heat transfer is regulated by

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \tag{3}$$

1. *(3 points)* Assuming an initial ($t = 0$) temperature profile

$$\sin\left(\frac{\pi}{2L}x\right)\left(2 + \cos\left(\frac{\pi}{L}x\right)\right) + \cos\left(\frac{\pi}{2L}x\right) \cdot \sin\left(\frac{\pi}{L}x\right) \tag{4}$$

   derive the temperature at any $t > 0$.

2. *(1 point)* Use the above solution to deduce the behavior for $t \to \infty$.
3. *(1 point)* Now change the boundary conditions so that the temperature at $x = 0$ is fixed to $u(x = 0, t) = 1$ for all $t \geqslant 0$. How would the new general solution behave at late time ($t \to \infty$)?

Problem 3

1. *(3 points)* Solve the Laplace equation in the region $0 < x < \infty$ times $0 < y < H$ with boundary conditions

$$\frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,H) = 0 \qquad u(0,y) = f(y) \qquad |u(\infty,y)| < \infty \quad (5)$$

Problem 4

Consider the function $f(x) = x^2$.

1. *(2 points)* Compute its Fourier cosine series for $0 < x < L$
2. *(1 point)* Sketch, for $x \in \mathbb{R}$, the graph of the Fourier series for $0 < x < L$
3. *(1 point)* Sketch, for $x \in \mathbb{R}$, the graph of the cosine series for $0 < x < L$
4. *(1 point)* Sketch, for $x \in \mathbb{R}$, the graph of the sine series for $0 < x < L$
5. *(2 points)* Determine the value at $x = L$ of the sine and the cosine series.