Conditional Probability

CONDITIONAL PROBABILITY AND INDEPENDENT EVENTS

Sometimes the probability of a given event depends on the occurrence or non occurrence of some other event.

Suppose A and B are two events in sample space S.

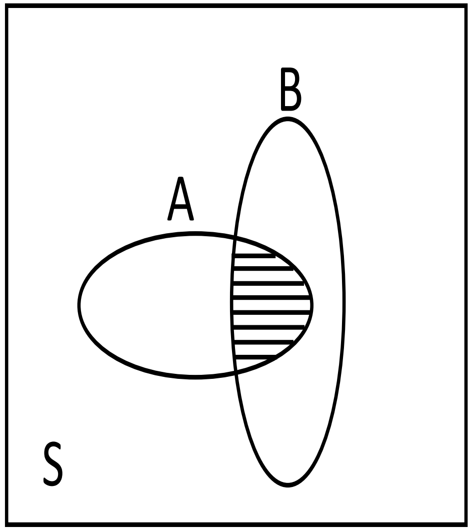
Let n = number of sample points in S

m_1 = number of sample points in A

m_2 = number of sample points in B

m_{12} = number of sample points in $A \cap B$

Then $P(A) = \frac{m_1}{n}$, $P(B) = \frac{m_2}{n}$ and $P(A \cap B) = \frac{m_{12}}{n}$



CONDITIONAL PROBABILITY AND INDEPENDENT EVENTS

- Now, suppose that the event A has taken place. On this assumption, the m_1, sample points of A constitute the sample space for other events. In particular, the event B in this sample space occurs along With A.

Hence in this sample space A, m_{12} sample points belong to B also. These are the sample points in $A \cap B$. The probability of $A \cap B$ (i.e., of B) in the sample space A is $\frac{m_{12}}{m_1}$. This is the probability of B under the assumption that A takes place. It is denoted by $P(B/A)$ and is called the **conditional**

- **probability** of B given that A takes place.

$\therefore P\left(\frac{B}{A}\right) = \frac{m_{12}}{m_1} = \frac{n(A \cap B)}{n(A)}$, provided $n(A) \neq 0$

Similarly, $P(A/B)$ is the conditional probability of A given that B has taken place and

$\therefore P\left(\frac{A}{B}\right) = \frac{m_{12}}{m_1} = \frac{n(A \cap B)}{n(B)}$, provided $n(B) \neq 0$

Two events A and B are said to be independent, if $P\left(\frac{A}{B}\right) = P(A)$ and $P\left(\frac{B}{A}\right) = P(B)$

MULTIPLICATION THEOREM

- **Theorem :** If A and B are two events, then

$$P(A \cap B) = P(A) \cdot P(B\,/\,A) = P(B) \cdot P(A\,/\,B)$$

- **Remarks:**
- **1.** If A and B are independent, then $P\left(\frac{B}{A}\right) = P(B)$ and $P\left(\frac{A}{B}\right) = P(A)$
- $\therefore P(A \cap B) = P(A) \cdot P(B)$
- **2.** In general, if $A_1, A_2, A_{3,}, \dots, A_n$ are n independent events, then probability of simultaneous
- occurrence of these n events is
- $P(A_1 \cap\ A_2 \cap\ A_3 \cap\ \dots \cap\ A_n) = P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot \cdots \dots \cdot P(A_n)$
- **3.** If A and B are mutually exclusive events, then $P(A \cap B) = 0$
- $\therefore P\left(\frac{A}{B}\right) = P\left(\frac{B}{A}\right) = 0$

Theorem: If A and B are two independent events and A', B' are their complementary events, then the events

(i) A and B' are independent

(ii) A' and B are independent

(iii) A' and B' are independent

Example - 1

- Two dice are thrown simultaneously. If at least one of the dice shows number 5, what is the probability that sum of the numbers on two dice is 9?

- **Solution:** Let A ≡ the event that one of the dice shows number 5
- and B ≡ the event that sum of numbers on two dice is 9.
- When two dice are thrown, then $n(S) = 36$
- But we are given that one of the dice shown number 5,
- therefore, the sample space reduces to the event A, where
- $A = \{(1,5), (2,5), (3,5), (4,5), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,5)\}$
- $\therefore n(A) = 11$
- Out of these, the sample points favorable to B are $\{(4,5), (5,4)\}$
- $\therefore P$ (sum of the numbers on two dice is 9, given that one of the dice shows number 5)
- $= P(B/A) = \frac{2}{11}$

Example - 2

- If $P(A') = 0.7, P(B) = 0.7, P(B/A) = 0.5$, find $P(A/B)$ and $P(A \cup B)$
- **Solution:** Since $P(A') = 0.7$,
- $P(A) = 1 - P(A') = 1 - 0.7 = 0.3$
- Now $P(B/A) = \frac{P(A \cap B)}{P(A)}$
- $\therefore 0.5 = \frac{P(A \cap B)}{0.3}$
- $\therefore P(A \cap B) = 0.15$
- Again $P(A/B) = \frac{P(A \cap B)}{P(B)}$
- $= \frac{0.15}{0.7}$
- $\therefore P(A/B) = \frac{3}{14}$
- Further, by addition theorem
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $= 0.3 + 0.7 - 0.15$
- $= 0.85$

Example - 3

Data on the readership of a certain magazine shows that the proportion of male reader under 35 is 0.40 and over 35 is 0.20. if the proportion of readers under 35 is 0.70, find the proportion of subscribers that are females over 35 years. Also calculate the probability that a randomly selected male subscriber is under 35 yrs of age.

Solution: Let us define the following events

A: Reader of the magazine is a male

B: Reader of the magazine is over 35 yrs of age

$$\therefore P(A \cap B) = 0.20, \qquad P(A \cap \bar{B}) = 0.40, \qquad P(\bar{B}) = 0.70$$

$$\therefore P(B) = 1 - P(\bar{B}) = 0.30$$

Example - 3

(i) The proportion of subscribers that are females over 35 years is :

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$
$$= 0.30 - 0.20 = 0.10$$

(ii) The probability that a randomly selected male subscriber is under 35 years is

$$P(\bar{B}/A) = \frac{P(A \cap \bar{B})}{P(A)}$$

Now $P(A) = P(A \cap B) + P(A \cap \bar{B}) = 0.20 + 0.40 = 0.60$

$$\therefore P(\bar{B}/A) = \frac{0.40}{0.60} = \frac{2}{3}$$

Example - 4

A card is drawn from a well shuffled deck of 52 cards. Consider two events A and B as
A: a club card is drawn, B: an ace card is drawn.
Determine whether the events A and B are independent or not.

Solution: One card can be drawn out of 52 cards in ${}^{52}C_1 = 52$ ways

$\therefore n(S) = 52$

Given A $\equiv$ the event that a club card is drawn and

B $\equiv$ the event that an ace card is drawn

1 club card can be drawn out of 13 club cards in ${}^{13}C_1 = 13$ ways

$\therefore n(A) = 13$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{13}{52} = \frac{1}{4}$$

1 ace card can be drawn out of 4 aces in ${}^{4}C_1 = 4$ ways

$\therefore n(B) = 4$

$$\therefore P(B) = \frac{n(B)}{n(S)} = \frac{4}{52} = \frac{1}{13}$$

Since 1 card is common between A and B,

$n(A \cap B) = 1$

$$\therefore P(A \cap B) = \frac{n(A\cap B)}{n(S)} = \frac{1}{52}$$

Now, $P(A) \times P(B) = \frac{1}{4} \times \frac{1}{13} = \frac{1}{52} = P(A \cap B)$

$\therefore$ A and B are independent events.

Example - 5

IMP In an examination, 30% of the students have failed in Mathematics, 20% of the students failed in Chemistry and 10% failed in both Mathematics and Chemistry. A student is selected at random, what is the probability that student

(i) has failed in Mathematics, if it is known that he has failed in chemistry?

(ii) has failed in at least one subject?

(iii) has failed in exactly one subject?

Solution: Let A ≡ the event that the student failed in Mathematics

and B ≡ the event that the student failed in Chemistry.

Then $P(A) = 30\% = \frac{30}{100}, P(B) = 20\% = \frac{20}{100}$ and $P(A \cap B) = 10\% = \frac{10}{100}$

(i) P (student failed in Mathematics, if he has failed in Chemistry)

$$= P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{(10/100)}{(20/100)} = \frac{10}{20} = \frac{1}{2}$$

(ii) P (student failed in at least one subject)

$$= P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{30}{100} + \frac{20}{100} - \frac{10}{100} = \frac{40}{100} = 0.40$$

(iii) P (student failed in exactly one subject)

$$= P(A \cup B) - P(A \cap B) = P(A) + P(B) - 2 \cdot$$

$P(A \cap B)$

$$= \frac{30}{100} + \frac{20}{100} - 2\left(\frac{10}{100}\right) = \frac{30}{100} = 0.30$$

Example - 6

From a city population, the probability of selecting (i) a male or a smoker is 0.70. (ii) a male smoker is 0.40 and (iii) a male, if a smoker is already selected is 2/3. Find the probability of selecting (a) a non smoker (b) a male and (c) a smoker if a male is first selected.

Solution: Define A: a male is selected

B: a smoker is selected

Given $P(A \cup B) = 0.70, \quad P(A \cap B) = 0.4 \quad , P(A/B) = \frac{2}{3}$

(a) Probability of selecting a smoker = $P(\bar{B}) = 1 - P(B) = 1 - \frac{P(A \cap B)}{P\left(\frac{A}{B}\right)}$

$$= 1 - \frac{0.4}{\frac{2}{3}} = \frac{2}{5}$$

$$\Rightarrow \quad P(B) = 1 - P(\bar{B}) = \frac{3}{5}$$

(B) Probability of selecting a male = $P(A) = P(A \cup B) + P(A \cap B) - P(B)$

$$= 0.7 + 0.4 - 0.6 = 0.5$$

(C) $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0.4}{0.5} = 0.8$