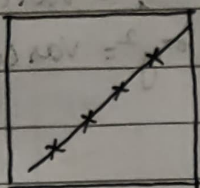
MODULE - 2

Correlation :

r = co-eff of correlation

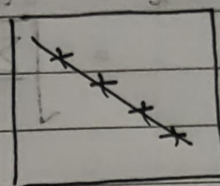
value of r : $-1 \le r \le 1$

Scatter diagram:



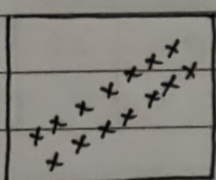
Perfect +ve Correlation

$r = 1$



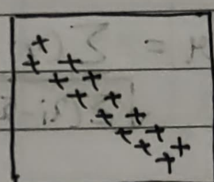
Perfect -ve correlation

$r = -1$



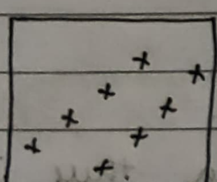
High degree +ve correlation,

$r = +ve$



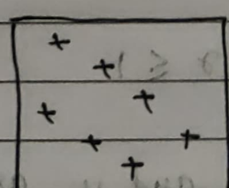
High degree -ve correlation,

$r = -ve$



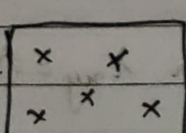
low degree +ve correl

$r = +ve$



low degree -ve correl

$r = -ve$



No correlation,

$r = 0$

→ r = Karl - Pearson's co-eff of correlation :

$$= \frac{cov(x,y)}{\sigma_x \times \sigma_y}$$ cov: co-variance.

$$= \frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{N \cdot \sigma_x \cdot \sigma_y}$$

where σ_x = Std deviation of x $\quad \sigma_x^2 = Var(x)$

$$= \sqrt{\frac{\sum(x_i-\bar{x})^2}{N}}$$ $\quad \sigma_y^2 = Var(y)$

& σ_y = std deviation of y

$$= \sqrt{\frac{\sum(y_i-\bar{y})^2}{N}}$$ N: no. of observations

OR $$r = \frac{\sum(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum(x_i-\bar{x})^2}\sqrt{\sum(y_i-\bar{y})^2}}$$

OR $$r = \frac{\sum x_i y_i - N\bar{x}\bar{y}}{\sqrt{\sum x_i^2 - N\bar{x}^2}\sqrt{\sum y_i^2 - N\bar{y}^2}}$$

Properties of r:

1. $-1 \leq r \leq 1$

2. If x and y are independent then there is no correlation between x and y.

3. Correlation co-eff is independent of change of scale and change of origin.
i.e; $u_i = \frac{x_i - a}{h}$, $v_i = \frac{y_i - b}{h}$ then $r_{xy} = r_{uv}$

Q Calculate correlation co-eff from the given data:

X	23	27	28	29	30	31	33	35	36	39
Y	18	22	23	24	25	26	28	29	30	32

Sol:

x	y	u=x-30	v=y-25	u^2	v^2	uv
23	18	-7	-7	49	49	49
27	22	-3	-3	9	9	9
28	23	-2	-2	4	4	4
29	24	-1	-1	1	1	1
30	25	0	0	0	0	0
31	26	1	1	1	1	1
33	28	3	3	9	9	9
35	29	5	4	25	16	20
36	30	6	5	36	25	30
39	32	9	7	81	49	63
		= 11	= 7	= 215	= 163	$\Sigma uv = 186$

$N = 10$,

$\bar{u} = \dfrac{\Sigma u}{N} = 1.1$

$\bar{v} = \dfrac{\Sigma v}{N} = 0.7$

$$r = \frac{\Sigma uv - N\bar{u}\bar{v}}{\sqrt{\Sigma u^2 - N\bar{u}^2}\sqrt{\Sigma v^2 - N\bar{v}^2}}$$

$$= \frac{186 - (10 \times 1.1 \times 0.7)}{\sqrt{215 - 10(1.1)^2}\sqrt{163 - 10(0.7)^2}} = \cancel{0.99} \; 0.996$$

Date

→ Spearman's Rank Correlation co. eff :

$$R = 1 - \left(\frac{6 \sum d^2}{n^3 - n}\right), \quad -1 \le R \le 1$$

where d = diff between ranks

Stud No	Rank in Eng (R_1)	Rank in Maths (R_2)	$d^2 = (R_1 - R_2)^2$
1	1	3	4
2	3	1	4
3	7	4	9
4	5	5	0
5	4	6	4
6	6	9	9
7	2	7	25
8	10	8	4
9	9	10	1
10	8	2	36

$\sum d^2 = 96, \quad N = 10$

$$\therefore R = 1 - \frac{6 \sum d^2}{N^3 - N}$$

$$= 1 - \frac{6 \times 96}{(10^3 - 10)} = 0.4181$$

→ For equal ranks:

$$R = 1 - \frac{6\left[\sum d^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \cdots\right]}{n^3 - n}$$

where m_i = no. of times any item is repeated.

BARFI Page No.
Date

Q. Find R for the following data :

X	R_1	Y	R_2	$d^2 = (R_1 - R_2)^2$
32	3	40	5	4
55	9	30	3.5	30.25
49	7.5	70	9	2.25
60	10	20	1	81
43	5.5	30	3.5	4
37	4	50	7	9
43	5.6	72	10	20.25
49	7.5	60	8	0.25
10	1	45	6	25
20	2	25	2	0

$m_1 = 2,$
$m_2 = 2,$
$m_3 = 2$

$$\Sigma d^2 = 176$$

$$R = 1 - \frac{6\left[\Sigma d^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \frac{1}{12}(m_3^3 - m_3)\right]}{n^3 - n}$$

$$= 1 - \frac{6[176 + 3/2]}{990}$$

$$= -0.075$$

Go Through

① let $r_{xy} = 0.4$
$Cov(x, y) = 1.6$
$\sigma_y^2 = 25$ find σ_x

② $R_{xy} = 0.143$
$\Sigma d^2 = 48$ find N

Date

ans ① $r_{xy} = \frac{Cov(x,y)}{\sigma_x \times \sigma_y} = \frac{1.6}{5 \times \sigma_x} = 0.4$

$\therefore \sigma_x = 0.8$

② $R_{xy} = 1 - \frac{6 \sum d^2}{n^3 - n}$

$0.143 = 1 - \frac{6 \times 48}{n^3 - n}$

$n^3 - n - 336 = 0$

$\therefore n = 7, \ -3.5 \pm 5.98i$

$\times$ not valid

$\therefore n = 7.$

Find r for : $N = 10$, $\sum x = 140$,

$\sum y = 150$

$\sum (x-10)^2 = 180$

$\sum (y-15)^2 = 215$

$\sum (x-10)(y-15) = 60$

ans: Let $u = x - 10$

$v = y - 15$

$\bar{u} = \frac{\sum u}{N} = \frac{\sum (x-10)}{N} = \frac{\sum x - \sum 10}{N}$

$= \frac{\sum x}{N} - \frac{\sum 10}{N}$

$= \bar{x} - 10$

$= 14 - 10 = 4$

$$\bar{V} = \frac{\Sigma V}{N} = \frac{\Sigma(y-15)}{N} = \frac{\Sigma y}{N} - \frac{\Sigma 15}{N}$$

$$= 15 - 15 = 0$$

$$r_{UV} = \frac{\Sigma UV - N\bar{U}\bar{V}}{\sqrt{\Sigma U^2 - N\bar{U}^2}\sqrt{\Sigma V^2 - N\bar{V}^2}}$$

$$= \frac{60 - 0}{\sqrt{180 - 10\times16}\sqrt{215 - 10\times0}}$$

$$= 0.91$$

Q. A sample of 25 pairs of values of x and y gives the following results:

$\Sigma x = 127$ $\quad \Sigma x^2 = 760$

$\Sigma y = 100$ $\quad \Sigma y^2 = 449$

$\Sigma xy = 500$

Later on it was found that 2 pairs were taken as (8, 14) and (8, 6) instead of (8, 12) and (6, 8). Find correct correlation coefficient (r)

ans: correct Σx = wrong $\Sigma x - (8+8) + (6+8) = 125$

correct Σy = wrong $\Sigma y - (14+6) + (12+8) = 100$

correct Σx^2 = wrong $\Sigma x^2 - (8^2+8^2) + (8^2+6^2) = 732$

correct Σy^2 = wrong $\Sigma y^2 - (14^2+6^2) + (12^2+8^2) = 425$

correct Σxy = wrong $\Sigma xy - [(8\times14)+(8\times6)] + [(8\times12)+(6\times8)]$

$= 484$

correct $\bar{x} = \frac{\Sigma x}{N} = \frac{125}{25} = 5$

correct $\bar{y} = \frac{\Sigma y}{N} = \frac{100}{25} = 4$

$$r = \frac{\Sigma xy - N\bar{x}\bar{y}}{\sqrt{\Sigma x^2 - N\bar{x}^2}\sqrt{\Sigma y^2 - N\bar{y}^2}}$$

$= -0.3$

→ <u>Regression lines:</u>

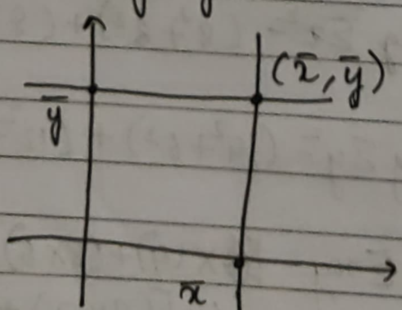
Reg line of y on x : $y - \bar{y} = \cancel{b_{yx}}\ b_{yx}(x - \bar{x})$

where $b_{yx} = \frac{r\sigma_y}{\sigma_x}$ $\qquad (\sigma_x^2 = Var(x))$

Reg line of x on y : $x - \bar{x} = b_{xy}(y - \bar{y})$
where $b_{xy} = \frac{r\sigma_x}{\sigma_y}$

b_{xy} & b_{yx} are called regression co-eff which are the slopes of the given 2 lines.

(1) If $r = 0$: $y = \bar{y}$ & $x = \bar{x}$ ⇒ these lines are parallel to y & x axis respectively & mutually perpendicular.



(ii) If $r = \pm 1$;
then the regression lines are coincidental or parallel.

(iii) $$b_{yx} = \frac{\Sigma xy - n\bar{x}\bar{y}}{\Sigma x^2 - n\bar{x}^2}$$

$$b_{xy} = \frac{\Sigma xy - n\bar{x}\bar{y}}{\Sigma y^2 - n\bar{y}^2}$$

(iv) $r = \pm\sqrt{b_{xy} b_{yx}}$

(v) If $b_{xy} > 1$ then $b_{yx} < 1$ & vice versa

(vi) $$\frac{b_{xy} + b_{yx}}{2} \geq r$$

(vii) θ : is the angle between 2 regression lines then,
$$\tan\theta = \left(\frac{1-r^2}{r}\right)\frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

→ if $r = 0$: $\tan\theta = \infty$
$\therefore \theta = \pi/2$

→ if $r = \pm 1$: $\tan\theta = 0$
$\therefore \theta = 0$

(viii) Regression coeff. are independent of change of origin but not change of scale.
i.e, if $u = \frac{x-a}{h}$, $v = \frac{y-b}{k}$

then $r_{xy} = r_{uv}$ but $\sigma_x = h\sigma_u$ &
$\sigma_y = k\sigma_v$

$\therefore\ b_{yx} = \frac{r\sigma_y}{\sigma_x} = \frac{r\,k\sigma_v}{h\sigma_u} = \frac{k}{h} b_{vu}$

$\&\ b_{xy} = \frac{h}{k} b_{uv}$

Q. A panel of 2 judges graded the following performances

Perf no.	X(A)	Y(B)	$u = x - \bar{x}$	$v = y - \bar{y}$	uv	u^2
1	36	35	3	2	6	9
2	32	33	-1	0	0	1
3	34	31	1	-2	-2	1
4	31	30	-2	-3	6	4
5	32	34	-1	1	-1	1
6	32	32	-1	-1	1	1
7	34	36	1	3	3	1
	231	231	0	0	13	18

Judge A rewarded 38 marks for the 8th Performance while Judge B was absent. How many marks Judge B would have rewarded if he was present?

Y is Judge b

ans: To find: Regression line Y on x

i.e, $y - \bar{y} = b_{yx}(x - \bar{x})$ $\qquad x \leftrightarrow u$

but, $\qquad y \leftrightarrow v$

$b_{yx} = \frac{\Sigma xy - n\bar{x}\bar{y}}{\Sigma x^2 - n\bar{x}^2}$ $\qquad b_{yx} \leftrightarrow b_{vu}$

$\bar{x} = \frac{\Sigma x}{n} = 33$ $\qquad \bar{u} = \frac{\Sigma u}{n} = 0$

$\bar{y} = \frac{\Sigma y}{n} = 33$ $\qquad \bar{v} = \frac{\Sigma v}{n} = 0$

BARFI Page No. Date

Reg n

we have Reg line of v on u:

$$v - \bar{v} = b_{vu}(u - \bar{u})$$

$$\Rightarrow v = 0.72u$$

$$\Rightarrow y - \bar{y} = 0.72(x - \bar{x})$$

$$\Rightarrow y = 33 + 0.72(x - 33)$$

when $x = 38$

$$\Rightarrow y = 36.6 \approx 37$$

ans: $\underline{\underline{37}}$.

Q Estimate marks of student in maths who scored 60 marks in english and marks of student in english who scored 70 marks in maths.

	Maths	Eng	x	y
mean:	80	50	?	60
S.D:	15	10	70	?

$r = 0.4$

ans:

Maths	Eng	x	Y
80 ($\bar{x}$)	50 ($\bar{y}$)		
15 (σ_x)	10 (σ_y)		

$$b_{xy} = \frac{r\sigma_x}{\sigma_y} = 0.4 \times \frac{15}{10} = 0.6$$

$$b_{yx} = \frac{r\sigma_y}{\sigma_x} = 0.4 \times \frac{10}{15} = 0.27$$

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$\Rightarrow x - 80 = 0.6(y - 50)$$

where $y = 60$ $\therefore$ $x = 86$

Date

$$y - \bar{y} = byx(x - \bar{x})$$

$$y - 50 = 0.27(x - 80)$$

when $x = 70$

$$y - 50 = 0.27 \times (-10)$$

$$\therefore y = 47.3$$

Q Given: $6y = 5x + 90 \quad -(1)$

$15x = 8y + 130 \quad -(2)$

$\sigma_x^2 = 16$

find : (i) $\bar{x}$ & $\bar{y}$

(ii) r

(iii) σ_y^2

ans: (1) solving eq (1) and (2) simultaneously :

$$6y = 5x + 90 \qquad x = 30 = \bar{x}$$

$$15x = 8y + 130 \Rightarrow y = 40 = \bar{y}$$

(ii) $r = \sqrt{byx \, bxy}$

① & ② $\Rightarrow y = \frac{5}{6}x + 15$

$$x = \frac{8}{15}y + \frac{130}{15}$$

$\therefore byx = 5/6$ & $bxy = 8/15$

$\therefore r = \sqrt{\frac{\cancel{5}}{\cancel{6}_3} \times \frac{\cancel{8}^4}{\cancel{15}_3}} = \frac{2}{3}$ ✓ correct as $r < 1$

BARFI Page No.
Date

(iii) $\sigma y^2 \Rightarrow b_{yx} = r \frac{\sigma y}{\sigma x}$

$\therefore \frac{5}{8} = \frac{2}{3} \times \frac{\sigma y}{4x}$

$\sigma y = 5$

Q If $\sigma x = \sigma y = 6$ and angle between the 2 lines is $\tan^{-1}(3)$. Find r

ans: $\tan\theta = 3$

$\Rightarrow 3 = \left(\frac{-r^2+1}{r}\right) \frac{\sigma x \sigma y}{\sigma x^2 + \sigma y^2}$

$\Rightarrow 3 = \left(\frac{1-r^2}{r}\right) \frac{\sigma^2}{2\sigma^2}$

$\Rightarrow 6r = 1 - r^2$

$\Rightarrow r^2 + 6r - 1 = 0$

$\Rightarrow r = -3 \pm \sqrt{10}$

$\therefore r = -3 + \sqrt{10}$ ✓

Q If $\bar{x} = 5$, $\bar{y} = 10$ and line of reg of y on x is parallel to the line: $20y = 9x + 40$. Find y of $x = 30$.

ans: Since, Reg line of y on x is parallel to the given line; the slope of Reg line of y on x = slope of : $20y = 9x + 40$

$\Rightarrow b_{yx} = \frac{9}{20}$

$\therefore y - \bar{y} = b_{yx}(x - \bar{x})$

Date

$$y - 10 = \frac{9}{20}(x-5)$$

when $x = 30$, $y = 85/4 = \underline{21.25}$

Q If the tangent of the angle made by the line of reg of y on x is 0.6 and $\sigma_y = 2\sigma_x$. Find r

ans: $\tan\theta = 0.6$

$b_{yx} = 0.6$

$b_{yx} = r\dfrac{\sigma_y}{\sigma_x}$

$0.6 = r \times \dfrac{2\cancel{\sigma_x}}{\cancel{\sigma_x}} \quad \therefore r = 0.3$

correct

$\tan\theta = 0.6 = \left(\dfrac{1-r^2}{r}\right)\dfrac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$

$0.6 = \dfrac{1-r^2}{r} \times \dfrac{2\cancel{\sigma_x^2}}{5\cancel{\sigma_x^2}}$

wrong!!

$1.5r = 1-r^2$

$r^2 + 1.5r - 1 = 0$

$r = 0.5, \cancel{-2} \qquad \underline{0.5}$

BARFI Page No.
Date

PROBABILITY DISTRIBUTION :

* Random variable :
It is a function which assigns real no to an outcome of an experiment.

Ex: x: no. of tails, total no. of outcomes $= 2^3$

S: {(HHH)(HHT)(HTH)(THH)(HTT)(TTH)(THT)(TTT)}

x: 0 : (HHH) : $P(x=0) = 1/8$
: 1 : (HHT)(HTH)(THH) : $P(x=1) = 3/8$
: 2 : (HTT)(TTH)(THT) : $P(x=2) = 3/8$
: 3 : (TTT) : $P(x=3) = 1/8$

$\Sigma P_i = 1$

* Probability distribution of a discrete random variable:

Let X be a discrete random variable taking values x_1, x_2, x_3 .... and so on with its probabilities $P(x_1), P(x_2), P(x_3)$... and so on such that $P(x_i) \geq 0 \ \forall i$ and $\Sigma P(x_i) = 1$ then $P(x_i) \ \forall i$ is called Probability mass function or PMF and $(x_i, P(x_i))$ is called Probability distribution of a random variable x.

Ex: If 2 unbaised dice are thrown then write the PD of the sum of the no's appearing on the toss of these 2 dice.

Total no. of outcomes $= 6^2 = 36$

S = { (1,1) (1,2) (1,3) (1,4) (1,5) (1,6)
(2,1) (2,2) (2,3) (2,4) (2,5) (2,6)
(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)
(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)
(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)
(6,1) (6,2) (6,3) (6,4) (6,5) (6,6) }

let X = sum of the numbers appearing on a roll of 2 dice

X:	2	3	4	5	6	7	8	9	10	11	12
$P(X=x_i)$:	$1/36$	$2/36$	$3/36$	$4/36$	$(5/36)$	$1/6$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

Q A Random variable X has following PD:

X:	0	1	2	3	4	5	6	7
$P(X=x_i)$:	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

(i) Find K

(ii) $P\left(\frac{1.5<x<4.5}{x>2}\right)$

Go through

(iii) smallest λ st $P(X\le\lambda) > 1/2$

ans:

(i) Since $\Sigma P_i = 1$.

$\Rightarrow 9K + 10K^2 = 1$

$\Rightarrow 10K^2+9K-1=0$

$\therefore K = 1/10$ or -1

✓ ✗

$\therefore K = 1/10$

(ii) $P\left(\frac{1.5<x<4.5}{x>2}\right)$

$= \frac{P[(1.5<x<4.5)\cap(x>2)]}{P(x>2)}$

$= \frac{P[2<x<4.5]}{P(x>2)}$

$= \frac{P(X=3,4)}{P(X=3,4,5,6,7)} = \frac{\frac{2}{10}+\frac{3}{10}}{\frac{2}{10}+\frac{\cancel{1}\,20}{100}+\frac{3}{10}\cancel{+}} = \frac{5}{7}$

BARFI Page No.
Date

(iii) If $\lambda = 4$ then $P(X \leq 4)$

$= P(X = 1, 2, 3, 4)$

$= 8/10 = 0.8 > 0.5$

Solve

Q A random variable X takes values : $-2, -1, 0, 1, 2$

such that $P(X > 0) = P(X = 0) = P(X < 0)$

$P(X = -2) = P(X = -1)$

$P(X = 1) = P(X = 2)$

find: (i) $P\left(\frac{-1 \leq X \leq 1}{-2 \leq X \leq 0}\right)$

(ii) $P\left(\frac{X = 1}{0 \leq X \leq 2}\right)$

ans:

$= \underbrace{P(X=-2) + P(X=-1)}_{P(X<0)} + P(X=0) + \underbrace{P(X=1) + P(X=2)}_{P(X>0)} = 1$

$= 3P(X=0) = 1$

$= P(X=0) = 1/3$

$P(X=-2) + P(X=-1) = P(X<0) = 1/3$

$P(X=-1) = 1/6 = P(X=-2)$

$P(X=1) + P(X=2) = 1/3$

$P(X=1) = 1/6 = P(X=2)$.

(i) $P\left(\frac{-1 \leq X \leq 1}{-2 \leq x \leq 0}\right)$

$= \frac{P(-1 \leq x \leq 0)}{P(-2 \leq x \leq 0)} = \frac{\frac{1}{6} + \frac{1}{3}}{\frac{1}{6} + \frac{1}{6} + \frac{1}{3}} = \frac{3}{4}$ //.

(ii) $P\left(\frac{X=1}{0\le x\le 2}\right) = \frac{1/6}{\frac{1}{6}+\frac{1}{6}+\frac{1}{3}} = \frac{1}{4}$

* The Distribution func of discrete random variable X. Consider $F(x_i) = P(X \le x_i)$

i.e, $F(x_1) = P(x_1)$

$F(x_2) = P(x_1) + P(x_2)$

$\vdots$

$F(x_n) = \sum_{i=1}^{n} P(x_i) \longrightarrow$ cdf

where, F is called cummulative distribution function abbrevated for CDF (cdf) and $(x_i, F(x_i))$ is called cummulative probability distribution.

Q. $X: 1, 2, 3, 4$

IMP

s.t: $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$

find Pmf (Probability mass function)

ans: let $P(X=3) = K$

$\Rightarrow$

X :	1	2	3	4
$P(X=x_i)$:	$\frac{K}{2}$	$\frac{K}{3}$	K	$\frac{K}{5}$

$\because \sum P(x_i) = 1 \Rightarrow K =$

$\frac{K}{2} + \frac{K}{3} + K + \frac{K}{5} = 1$

$30K + 20K + 60K + 12K = 60$

$K = \frac{60}{122} = \frac{30}{61}$

110
116
6
61
×2
122

$\therefore$

X :	1	2	3	4
$P(X=x_i)$:	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

$\longrightarrow$ Pmf

Eg:

$P(x = x_i)$: $\frac{15}{61}$ $\frac{10}{61}$ $\frac{30}{61}$ $\frac{6}{61}$ → Pmf

$F(x_i)$: $\frac{15}{61}$ $\frac{25}{61}$ $\frac{55}{61}$ $\frac{61}{61}$ → Cdf

* **PDF of continious random variable X:**

Consider a continious random function $y = f(x)$

s.t: (i) f is integrable

(ii) $f(x) \geq 0$

(iii) $\int_a^b f(x)\,dx = 1$, $x \in [a,b]$

(iv) $P(\alpha \leq X \leq \beta) = \int_\alpha^\beta f(x)\,dx$ $a < \alpha < \beta < b$

then f is called probability density function (Pdf) of a random variable X.

Q Continuous R.V X has the following pdf $f(x) = kx^2$. where $0 \leq x \leq 2$.

Find (i) K,

(ii) $P(0.2 \leq x \leq 0.5)$

(iii) $P[(x \geq 3/4) / (x \geq 1/2)]$

ans: (i) Since, $f(x)$ is a pdf,

$$\int_0^2 f(x)\,dx = 1$$

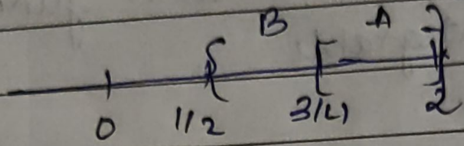
$$\Rightarrow K\left[\frac{x^3}{3}\right]_0^2 = 1 \quad \Rightarrow K = \frac{3}{8}.$$

(ii) $P(0.2 \le x \le 0.5) = \frac{3}{8}\int_{0.2}^{0.5} x^2 dx = 0.015$

(iii) let $A : x \geq 3/4$

$B : x \geq 1/2$

$A = [3/4, 2]$, $B = [1/2, 2]$



$\because A \subset B \Rightarrow A \cap B = A$

$\therefore P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$

but $P(A) = \int_{3/4}^{2} \frac{3}{8} x^2 dx = 0.947$

$P(B) = \int_{1/2}^{2} \frac{3}{8} x^2 dx = 0.984$

$\therefore \frac{P(A)}{P(B)} = 0.962$

* Continuous distribution function (cdf):

X is continuous RV having pdf as $f(x)$ then,
$F(x)$ is probability of :

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt, \quad -\infty < x < \infty$$

is called cummulative distribution func of RV x.

→ Properties :

(i) $0 \leq F(x) \leq 1$

(ii) $x_1 \leq x_2 \Rightarrow F(x_1) \leq F(x_2)$

(iii) $F'(x) = f(x)$

(iv) $P(a \leq x \leq b) = F(b) - F(a)$

Ex: cdf of continuous RV X is given by,

$$F(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

find : (i) Pdf
(ii) $P(\frac{1}{2} \leq X \leq 4/5)$

ans (i) Pdf : $f(x) = \begin{cases} 0, & x < 0 \\ 2x & 0 \leq x < 1 \\ 0 & x > 1 \end{cases}$

(ii) $P\left(\frac{1}{2} \leq X \leq \frac{4}{5}\right) = F(4/5) - F(1/2)$

$$= \left(\frac{4}{5}\right)^2 - \left(\frac{1}{2}\right)^2 = 0.39$$

or $$= \int_{1/2}^{4/5} f(x)\,dx = \int_{1/2}^{4/5} 2x\,dx = 0.39$$

BARFI Page No.
Date

Q Find the distribution function corresponding to the following pdf:

$$f(x) = \begin{cases} \frac{1}{2} x^2 e^{-x}, & 0 \leq x < \infty \\ 0, & o.w \end{cases}$$

ans: $F(x) = \int_0^x f(x)\,dx$

$$= \int_0^x \frac{1}{2} x^2 e^{-x}\,dx$$

$$= \frac{1}{2}\left[x^2\left(\frac{e^{-x}}{-1}\right) - (2x)\left(\frac{e^{-x}}{+1}\right) + (2)\left(\frac{e^{-x}}{-1}\right)\right]_0^x$$

$$= \frac{1}{2}\left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x} + 2\right]_0^x$$

$$F(x) = \begin{cases} 1 - \frac{e^{-x}}{2}(x^2+2x+2), & 0 \leq x < \infty \\ 0, & o.w \end{cases}$$

* **Expection and Variance:**

Descrete:

(1) If X is a discrete R.V then Expection is given by

$$E(X) = \sum x_i P_i = mean(X)$$
$$= \mu_1'$$

(2) Variance of x is :

$$Var(x) = E(x-\bar{x})^2$$
$$= E(x^2) - [E(x)]^2$$
$$= \sum x_i^2 p_i - [E(x)]^2$$
$$= \mu_2' - (\mu_1')^2$$

For Continuous Random Variable x :

(1) $E(x) = \int_{-\infty}^{+\infty} x\, f(x)\, dx = \mu_1'$

(2) $Var(x) = \int_{-\infty}^{\infty} x^2 p\, dx - [E(x)]^2$

$\quad p = f(x)$

$$= \int_{-\infty}^{\infty} x^2 f(x)\, dx - [E(x)]^2$$

Q A fair coin is tossed till head appears. What is the expectation the no. of tosses required.

ans:

	H	TH	TTH	. . . . .
x :	1	2	3	. . .
$P(x=x)$:	$\frac{1}{2}$	$\frac{1}{2}\times\frac{1}{2}$	$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$	. . . . . .

$E(x) \Rightarrow S = \frac{1}{2} + 2\times\frac{1}{2^2} + 3\times\frac{1}{2^3} + 4\times\frac{1}{2^4} \ldots$

$\sum (x_i p_i)$

Date

$$\frac{1}{2}S = \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{4}{2^5} \cdots$$

$$S - \frac{1}{2}S = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} \cdots$$

$$\cancel{\frac{1}{2}}S = \cancel{\frac{1}{2}}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} \cdots\right)$$

$$S = \frac{1}{1 - 1/2} = 2$$

$\therefore\ E(x) = 2$

$\therefore$ no. of tosses required = 2

Q Find the Expectation of :

(i) Sum of

(ii) Product of

the numbers appearing on throw of n dice

ans: $\overset{1}{\square}\ \overset{2}{\square}\ \overset{3}{\square} \cdots \overset{n}{\square}$

X_1 : $x_1 \quad x_2 \quad x_3 \quad x_n$

(i) $E(X_1 + X_2 + X_3 \ldots X_n) = E(\Sigma X_i) = \Sigma(E(X_i))$

(ii) $E(X_1 X_2 X_3 \ldots X_n) = E(\prod_{i=1}^{n} x_i) = \prod_{i=1}^{n}(E(x_i))$

(i) Let X_i = the number appearing on ith dice

X_i :	1	2	3	4	5	6
$P(X_i = x_i)$	1/6	1/6	1/6	1/6	1/6	1/6

$\therefore E(X_1) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6}$

$= \frac{1}{6}(21) = \frac{7}{2}$

$\therefore E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \left(E(X_i)\right)$

$= \sum_{i=1}^{n} 7/2 = \frac{n7}{2}$

(ii) $E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} E(X_i)$

$= \left(\frac{7}{2}\right)^n$

Q Find the Expectation of the number of failures precceding the first success in an infinite series of independent trials with probabilities P and Q of success and failure respectively.

ans:

$P \quad qP \quad qqP \quad \ldots$

$X: \quad 0 \quad 1 \quad 2 \quad \ldots$

$P(X=x): \quad P \quad qP \quad q^2P \quad \ldots$

$\therefore E(X) = qP + 2q^2P + 3q^3P + \ldots$

$= qP(1 + 2q + 3q^2 \ldots)$

$= qP(1-q)^{-2}$

$= qP \times P^{-2} = \frac{q}{P}$

Date / /

Q. Capital X is a continuous R.V with pdf:

$$f(x) = \begin{cases} kx^2(1-x^3) & \text{for } 0 \leq x < 1 \\ 0 & \text{o.w} \end{cases}$$

(i) find k

(ii) $P(0 \leq x \leq 1/2)$

(iii) mean & var of x

ans: (i) Since f is a pdf;

$$\int_0^1 f(x)\,dx = 1$$

$$\Rightarrow k\int_0^1 (x^2 - x^5)\,dx = 1$$

$$\Rightarrow k\left(\frac{x^3}{3} - \frac{x^6}{6}\right)_0^1 = 1$$

$$\Rightarrow k\left(\frac{1}{3} - \frac{1}{6}\right) = 1$$

$$\Rightarrow k\left(\frac{1}{6}\right) \quad \therefore k = 6$$

(ii) $P(0 \leq x \leq 1/2) = \int_0^{1/2} f(x)\,dx$

$$= 6\int_0^{1/2} (x^2 - x^5)\,dx$$

$$= 15/64$$

(iii) mean $= E(x)$

$$= \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= 6\int_0^1 (x^3 - x^6)\,dx$$

$$= 6\left(\frac{x^4}{4} - \frac{x^7}{7}\right)_0^1 = 6\left(\frac{1}{4} - \frac{1}{7}\right) = \frac{9}{14}$$

$\frac{3 \times}{28}$
14

$$\frac{\cancel{18}\,9}{\cancel{40}\,20}$$

~~Var~~ $E(x^2) = \int_0^1 x^2 f(x)$

$$= 6\int_0^1 (x^4 - x^7)\,dx$$

$$= 6\left(\frac{x^5}{5} - \frac{x^8}{8}\right)_0^1 = 6\left(\frac{1}{5} - \frac{1}{8}\right) = \frac{9}{20}$$

$$Var(x) = E(x^2) - (E(x))^2$$

$$= \frac{9}{20} - \left(\frac{9}{14}\right)^2 = \frac{9}{245} \,//$$

* Binomial Distribution :

Let x be a discrete R.V then, x follows a binomial distribution with parameter n and p.

If $P(X=r) = {}^nC_r\, p^r q^{n-r}$ where $q = 1-p$

→ "x follows a binomial distribution with parameters n and p" $\Rightarrow$ $x \sim B(n,p)$

Mean & Variance :

$$\text{Mean} = E(x) = \sum x_i p_i$$

$$= \sum_{x=0}^{n} x\, {}^nC_x\, p^x q^{n-x}$$

$$= {}^nC_1\, p q^{n-1} + 2\,{}^nC_2\, p^2 q^{n-2} + 3\,{}^nC_3\, p^3 q^{n-3} \ldots\ldots n\,{}^nC_n\, p^n q^0$$

$$= npq^{n-1} + \cancel{2} \times \frac{n(n-1)}{\cancel{2!}} p^2 q^{n-2} + \ldots\ldots np^n$$

$$= np\left[q^{n-1} + (n-1)pq^{n-2} + \frac{(n-1)(n-2)p^2q^{n-3}}{2} + \ldots\ldots + p^{n-1}\right]$$

$$= np\left[{}^{(n-1)}C_0\, p^0 q^{n-1} + {}^{(n-1)}C_1\, p^1 q^{(n-1)-1} + {}^{(n-1)}C_2\, p^2 q^{(n-1)-2} + \ldots\ldots {}^{(n-1)}C_{n-1}\, p^{n-1} q\right]$$

$$= np\left[\underbrace{(p+q)^{n-1}}_{=1}\right]$$

$$\boxed{E(x) = np}$$

Variance:

$$Var(x) = E(x^2) - [E(x)]^2$$

Now,

$$E(x^2) = \sum_n x_i^2 P_i$$

$$= \sum_{x=0} x^2 \, {}^nC_x p^x q^{n-x}$$

$$= \sum {}^nC_x p^x q^{n-x} [x + x(x-1)]$$

$$= \underbrace{\sum_{x=0}^{n} {}^nC_x p^x q^{n-x} x}_{= np} + \sum_{x=0}^{n} x(x-1) \, {}^nC_x p^x q^{n-x}$$

$$= np + 2 \, {}^nC_2 p^2 q^{n-2} + 3 \times 2 \, {}^nC_3 p^3 q^{n-3} + \cdots$$

$$= np + \left[\frac{\cancel{2}(n)(n-1)}{\cancel{2}} p^2 q^{n-2} + \frac{\cancel{3 \times 2} n(n-1)(n-2)}{\cancel{3!}} p^3 q^{n-3} + \cdots \right]$$

$$= np + n(n-1)p^2 [q^{n-2} + (n-2) p q^{n-3} + \cdots]$$

$$= np + n(n-1)p^2 [{}^{(n-2)}C_0 \, p^0 q^{n-2} + {}^{(n-2)}C_1 \, p^1 q^{(n-2)-1} + \cdots]$$

$$= np + n(n-1)p^2 \underbrace{(p+q)^{n-2}}_{=1}$$

$$= np + n(n-1)p^2$$

$$= np + n^2p^2 - np^2$$

$$= np[1 + np - p] = np[q + np]$$

$$= npq + n^2p^2$$

$Var(x) = E(x^2) - [E(x)]^2$
$= npq + n^2p^2 - (np)^2$
$= npq$

$\boxed{Var(X) = npq}$ //

Properties:

1. If $X_1 \sim B(n_1, p_1)$ and $X_2 \sim B(n_2, p_2)$ then:
$X_1 + X_2 \sim B(n_1+n_2, p)$
where, $p = p_1 = p_2$

2. mode is the value of x which has the maximum probability.

case (i):
If $(n+1)p \in Z$ say k then there are 2 modes k and k-1.

case (ii):
If $(n+1)p \notin Z$ then mode is an integral part of $(n+1)p$. Ex: $n = 16$, $p = 2/3$
$(n+1)p = 17 \times \frac{2}{3} = \frac{34}{3}$ $\therefore$ mode $= 11$

Q If X is binomially distributed with $E(x) = 2$ & $Var(x) = 4/3$. Find the probability distribution of x and and p(at least one success).

Ans:

$E(x) = 2 = np$

$Var(x) = \frac{4}{3} = npq$

We have $\frac{4}{3} = 2q \therefore q = \frac{2}{3}$

$\therefore p = 1 - \frac{2}{3} = 1/3$

but $np = 2 \therefore \frac{1}{3}n = 2 \therefore n = 6$

$\therefore p(x=x) = {}^nC_x p^x q^{n-x}$

$= {}^6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$

$\therefore x$:	0	1	2	3	4	5	6
$P(x=x)$:	$\frac{64}{729}$	$\frac{192}{729}$	$\frac{240}{729}$	$\frac{160}{729}$	$\frac{60}{729}$	$\frac{12}{729}$	$\frac{1}{729}$

P(at least one success) = 1 − P(no success)

$= 1 - \frac{64}{729} = 0.9$

Poisson:

$X \sim P(m)$, $P(x=x) = \frac{e^{-m} m^x}{x!}$

but $m = np$ is finite

$n \to \infty$

$p \to 0$

Uniform Distribution :

discrete : $P(X=x) = \frac{1}{n} \quad \forall x$

Continuous :

$$P(X=x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{o.w} \end{cases}$$

Exponential :

Continuous : $P(X=x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{o.w} \end{cases}$

Q Probability that man aged 60 will live upto 70 is 0.65 what is the probability out of 10 such men that at least 7 will upto 70.

ans: $n = 10$

$p = 0.65$

$q = 0.35$

X = no. of men who will live upto 70

To find : $P(X \geq 7) = \sum_{x=7}^{10} {}^{10}C_x (0.65)^x (0.35)^{10-x} = 0.51$

Q Ratio of Probability of 3 successes in 5 independent Trials to the probability of 2 successes in 5 independent trials is 1/4. Find the probability of 4 successes in 6 independent trials.

ans: X = no. of successes

$n=5, \ x=3$ then, $P(x=3) = {}^5C_3 p^3 q^2$ —(1)

$n=5, \ x=2$ then, $P(x=2) = {}^5C_2 p^2 q^3$ —(2)

(1) ÷ (2) = 1/4 (given)

i.e., $\frac{P(X=3)}{P(X=2)} = \frac{1}{4}$

$= \frac{\cancel{{}^5C_3} p^3 q^2}{\cancel{{}^5C_2} p^2 q^3} = \frac{1}{4} \Rightarrow \frac{p}{q} = \frac{1}{4}$

$\therefore 4p = q$ but $q = 1-p$

$\therefore p = 1/5$

$\therefore q = 4/5$

$\therefore$ For $n = 6$;

$P(x=4) = ?$

$= {}^6C_4 \, p^4 q^2$

$= {}^6C_4 \left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^2 = \frac{48}{3125} = 0.015$

Q A communication has n components, each of which functions independently with probability p. The total system will work efficiently if atleast half of its components are working. For what value of p, 5 component system will work more efficiently than a 3 component system.

ans

X = no of components

5 component system will work more effectively than a 3 component system if P of 5 component sys is greater than P of 3 component sys.

P(5 comp) > P(3 comp)

if $P(X=3,4,5) > P(X=2,3)$

if $\sum_{x=3}^{5} {}^5C_x p^x q^{5-x} > \sum_{x=2}^{3} {}^3C_x p^x q^{3-x}$

if, ${}^5C_3\, p^3 q^2 + {}^5C_4\, p^4 q + {}^5C_5\, p^5 > {}^3C_2\, p^2 q + {}^3C_3\, p^3$

if, $10p^3(1-p)^2 + 5p^4(1-p) + p^5 - 3p^2(1-p) - p^3 > 0$

if, $10p^3(1-2p+p^2) + 5p^4 - 5p^5 + p^5 - 3p^2 + 3p^3 - p^3 > 0$

if, $12p^3 - 15p^4 + 6p^5 - 3p^2 > 0$

if, $3p^2(4p - 5p^2 + 2p^3 - 1) > 0$

if, $3p^2(p-1)^2(2p-1) > 0$

if, $2p-1 > 0$

if $\underline{\underline{p > 1/2}}$.

1 is root if sum of all coefficients = 0

-1 is root if sum of even powers of p = sum of odd powers of p.

$2p^3 - 5p^2 + 4p - 1 = 0$

$$
\begin{array}{c|cccc}
1 & 2 & -5 & 4 & -1 \\
 & \downarrow & 2 & -3 & 1 \\
\hline
1 & 2 & -3 & 1 & \boxed{0} \\
 & \downarrow & 2 & -1 & \\
\hline
 & 2 & -1 & \boxed{0} & \\
\end{array}
$$

sum is 0.

linear eq^n.

$\therefore\ (p-1)^2(2p-1) = 0$

BARFI Page No.
Date

Q 7 dice are thrown 729 times. How many times do you expect atleast 4 dice to show 3 or 5.

ans: $n = 7$

$N = 729$

x = no. of dices showing 3 or 5

$p = \frac{1}{6} + \frac{1}{6} = \frac{1}{3} \quad \therefore q = \frac{2}{3}$

$\therefore P(x \geq 4) = \sum_{x=4}^{7} {}^{7}C_{x}\, p^{x} q^{x}$

$= \sum_{x=4}^{7} \left(\frac{1}{3}\right)^{x} \left(\frac{2}{3}\right)^{7-x} = \frac{379}{3^7}$

$\therefore$ In 729 times:

$= 729 \times \frac{379}{3^7} \sim 126$

Q Let X and Y be 2 independent binomial variates with parameters $n_1 = 6$, $P_1 = 1/2$ and $n_2 = 4$, $P_2 = 1/2$ respectively. Evaluate probability of $X + Y = 3$ ie $P(X+Y=3)$.

ans: let, $X + Y = Z$

then $Z \sim B(10, 1/2)$

$\therefore P(Z = 3) = {}^{10}C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^7$

$= 0.117$

Also,

$P(X+Y \geq 3) = P(Z \geq 3)$

$= 1 - [P(Z=0) + P(Z=1) + P(Z=2)]$

Date

$$= 1 - \left[{}^{10}C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{10} + {}^{10}C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^9 + {}^{10}C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^8 \right]$$

$$= 0.945$$

Q 3 fair coins are tossed 3000 times. Find the frequencies of distribution of heads and tales and tabulate the result. Also find mean & S.D. of the result.

ans:

$n = 3$

$N = 3000$

$p = 1/2$

$\therefore P(X = x) = {}^3C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{3-x}$

x :	0	1	2	3
$P(X = x)$ ~~NP~~ :				
$f = NP$:				

mean $= np$

$SD = \sqrt{npq}$

* Fitting of Binomial distribution:

$P(X = x) : {}^nC_x \, p^x q^{n-x}$

Q Seven coins are tossed and no. of heads obtained are noted. The experiment is repeated 128 times. Fit a binomial distribution if:

(i) coins are unbiased

(ii) nature of the coin is not known.

ans: (i) $n = 7$, $N = 127 = 2^7$

X = no. of heads

$p = 1/2 = q$

$P(X=x) : {}^7C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{7-x}$

$P(X=0) = 1/2^7$

$P(X=1) = 7/2^7$

$P(X=2) = 21/2^7$

$P(X=3) : 35/2^7$

$P(X=4) : 35/2^7$

$P(X=5) : 21/2^7$

$P(X=6) : 7/2^7$

$P(X=7) : 1/2^7$

The following frequency distribution:

Given

X :	0	1	2	3	4	5	6	7	Total
Freq :	7	6	19	35	30	23	7	1	128.

X :	0	1	2	3	4	5	6	7	
Exp freq NP :	1	7	21	35	35	21	7	1	= 128

(ii) the nature of the coin is unknown $\therefore$ p is not 1/2

$$mean = \bar{x} = \frac{\sum x_i f_i}{\sum f_i}$$

$$= \frac{0\times7 + 1\times6 + 2\times19 + 3\times35 + \cdots 7\times1}{128}$$

$$= 3.38$$
$$= np$$

$$\Rightarrow p = \frac{3.38}{7} = 0.48$$

$\therefore q = 1-p = 0.52$

$\therefore P(X=x) = {}^7C_x (0.48)^x (0.52)^{7-x}$

x:	0	1	2	3	4	5	6	7
freq = NP :	1.31	8.5	23.5	36.2	36.2	23.5	8.5	1.31
	= 1	9	24	36	36	24	9	

= 140

Poisson's distribution:

If $n \to \infty$
$p \to 0$
but np = finite

n: very big
p: very small

* Descrete R.V X is said to follow poisson's distribution with parameter m if $P(X=x) = \frac{e^{-m} m^x}{x!}$, $x = 0,1,2\ldots$

$$\text{mean}(x) = E(x) = \sum x_i P_i$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-m} m^x}{x!}$$

$$= \sum_{x=1}^{\infty} \frac{\cancel{x} e^{-m} m^x}{\cancel{x}(x-1)!}$$

$$= e^{-m} \sum_{x=1}^{\infty} \frac{m^x}{(x-1)!}$$

$$= e^{-m} m \sum_{x=1}^{\infty} \frac{m^{x-1}}{(x-1)!}$$

$$= e^{-m} m e^{m}$$

$$\boxed{E(x) = m}$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$E(x^2) = \sum x_i^2 P_i$$

$$= \sum_{x=0}^{\infty} x^2 \frac{e^{-m} m^x}{x!}$$

$$= \sum_{x=0}^{\infty} (x + x(x-1)) \frac{e^{-m} m^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{x e^{-m} m^x}{x!} + \sum_{x=0}^{\infty} x(x-1) \frac{e^{-m} m^x}{x!}$$

$$= m + \sum_{x=2}^{\infty} \frac{e^{-m} m^x}{(x-2)!}$$

$$= m + \cancel{m}\, e^{-m} m^2 \sum_{x=2}^{\infty} \frac{m^{x-2}}{(x-2)!}$$

$$= m + e^{-m} m^2 e^{m}$$

$$\therefore E(x^2) = m + m^2$$

$\therefore \ var(X) = E(X^2) + [E(X)]^2$

$= m + \cancel{m^2} - \cancel{(m)^2}$

$= \underline{\underline{m}}$

$\boxed{Var(X) = m}$

Note:

If $X_1, X_2, \dots X_n$ are poisson variates with parameters $m_1, m_2, m_3 \dots m_n$ then $Y = X_1 + X_2 + \dots X_n$ is also a poisson variate with parameter $m_1 + m_2 + \dots m_n$ ~~and~~ + But,

$Y = a_1X_1 + a_2X_2 \dots a_nX_n$ and $Y = X_1 - X_2 \dots - X_n$

are not poisson variates.

Ex: If mean of a poisson distribution is 4. find $P(m - 2\sigma < X < m + 2\sigma)$

ans: mean $= m = 4 =$ var

$\sigma = \sqrt{var}$

$= \sqrt{4} = 2$

$P(m - 2\sigma < X < m + 2\sigma)$

$= P(0 < X < 8)$

But, $P(X = x) = \dfrac{e^{-4} \, 4^x}{x!}$

$\therefore \ P(0 < X < 8) = \sum_{x=1}^{7} P(X = x)$

$= \underline{\underline{0.93}}$

Q Car Hire has 2 cars. The no. of demands for a car on each day is distributed ~~between~~ +5 as poisson variate with mean 1.5

Calculate the proportion that neither car is used and some demand is refused.

ans $X =$ no. of demands

$m = 1.5$

$$P(X=x) = \frac{e^{-1.5} (1.5)^x}{x!}$$

(i) $P(x=0) = e^{-1.5} = 0.22$

(ii) $P(x>2) = 1 - P(X \leq 2)$

$= 1 - [P(X=0) + P(X=1) + P(X=2)]$

$= 1 - [0.22 + 0.33 + 0.25]$

$= 0.19 \approx 0.2$

BARFI Page No.
Date

Q Car Hire has 2 cars. The no. of demands for a car on each day is distributed ~~between~~ ~~+5~~ as poisson variate with mean 1.5
Calculate the proportion that neither car is used and some demand is refused.

ans $X =$ no. of demands
$m = 1.5$
$$P(X=x) = \frac{e^{-1.5}(1.5)^x}{x!}$$

(i) $P(X=0) = e^{-1.5} = 0.22$

(ii) $P(X>2) = 1 - P(X \leq 2)$
$= 1 - [P(X=0) + P(X=1) + P(X=2)]$
$= 1 - [0.22 + 0.33 + 0.25]$
$= 0.19 \approx 0.2$

Q If X_1, X_2, X_3 be three dependent Possoin's variables with parameters $m_1 = 1$, $m_2 = 2$, $m_3 = 3$. Find:
(i) $P[(X_1+X_2+X_3) \geq 3]$
(ii) $P[(X_1=1)/(X_1+X_2+X_3)=3]$

ans: let $Z = X_1+X_2+X_3$ with parameter $m_1+m_2+m_3 = 6 = m$
$$\therefore P(X=x) = \frac{e^{-m}m^x}{x!} = \frac{e^{-6}6^x}{x!}$$

$P(Z \geq 3) = 1 - P(Z<3)$
$= 1 - P(Z=0,1,2)$
$$= 1 - e^{-6}\left[\sum_{z=0}^{2} \frac{6^z}{z!}\right]$$
$$= 1 - e^{-6}\left[1 + \frac{6}{1!} + \frac{6^2}{2!}\right] = \underline{0.94}$$

(ii) $P(A/B) = \frac{P(A \cap B)}{P(B)}$

$P(A \cap B) = P(A)P(B)$ when independent

$\frac{25 \times 6 \times 2}{216 \times 2} = \frac{25}{72}$

$$P[(X_1 = 1)/(X_1 + X_2 + X_3 = 3)]$$

$$= \frac{P[(X=1) \cap (X_1 + X_2 + X_3 = 3)]}{P[(X_1 + X_2 + X_3 = 3)]}$$

$$= \frac{P[(X_1 = 1) \cap (X_2 + X_3 = 2)]}{P(X_1 + X_2 + X_3 = 3)}$$

$$= \frac{P(X_1 = 1) \times P(X_2 + X_3 = 2)}{P(X_1 + X_2 + X_3 = 3)}$$

$$= \frac{\frac{e^{-1} \times 1^1}{1!} \times \frac{e^{-5} 5^2}{2!}}{\frac{e^{-6} 6^3}{3!}} = \frac{25}{72}$$

Q An insurance company found that only ~~10~~ 0.01% of the population meet with an accident every year. If its 1000 policy holders are selected randomly what is the probability that no more than 2 of its client meet with an accident next year.

ans:

$X =$

$P = 0.01\% = 0.0001$

$n = 1000$

$m = np$

$= 0.1$

$$P(X = x) = \frac{e^{-m} m^x}{x!} = \frac{e^{-0.1} 0.1^x}{x!}$$

BARFI Page No.
Date

We wand $P(X \leq 2) = P(X = 0, 1, 2)$

$$= e^{-0.1}\left[\sum_{x=0}^{2} \frac{(0.1)^x}{x!}\right]$$

$$= e^{-0.1}\left[1 + \frac{0.1}{1} + \frac{0.1^2}{2}\right] = 0.999$$

Q Using Poisson's distribution find the approx value of $^{300}C_2 (0.02)^2 (0.98)^{298} + {}^{300}C_3 (0.02)^3 (0.98)^{297}$

ans: We observe that they are Probabilities of Binomial distribution.

$n = 300$

$p = 0.02$

$q = 0.98$

$m = np = 6$

$\therefore P(X = x) = \frac{e^{-m} m^x}{x!} = \frac{e^{-6} 6^x}{x!}$

$\therefore$ Given exp $= \frac{e^{-6} 6^2}{2!} + \frac{e^{-6} 6^3}{3!}$

$= 0.13938$

Q In A certain factory turning out blades, there is a chance that 1/500 for a blade to be defective. Blades are supplied in a packet of 10. Calculate approx no. of packets containg
(i) no defective
(ii) 1 defective
(iii) 2 defective blades in a consignment of 10000 packets.

Date / /

ans: X = no. of defective blades (in 1 packet)

$p = 1/500$

$n = 10$

$m = np = 0.02$

$$P(X=x) = \frac{e^{-m} m^x}{x!} = \frac{e^{-0.02} \times 0.02^x}{x!}$$

$P(X=0) = e^{-0.02} = 0.98$

$$P(X=1) = \frac{e^{-0.02}(0.02)^1}{1!} = 0.0196$$

$P(X=2) = 0.000196$

$N = 10{,}000$

no. of packets having no defective blades $= NP(X=0) = 980$

" " " 1 " " $= NP(X=1) = 1960.3$

" " " 2 " " $= NP(X=2) = 19.6$

Q. Fit a poisson distribution to the following data:

x_i: no. of deaths:	0	1	2	3	4
f_i: Freq:	123	59	14	3	1

ans: $m = \frac{\Sigma x_i f_i}{N = \Sigma f_i} = \frac{100}{200} = 0.5$

$$P(X=x) = \frac{e^{-0.5}\, 0.5^x}{x!}$$

X:	0	1	2	3	4
$P(X=x)$:	0.606	0.303	0.075	0.0[illegible]12	0.00[illegible]1579
Exp freq: NP	121.2	60.6	15	2.4	0.3158

* ## Normal Distribution:

Continuous R.V X follows a normal distribution with parameter m(mean) & σ(s.d) if it's pdf is given by :

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \times e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \qquad -\infty < x < \infty$$

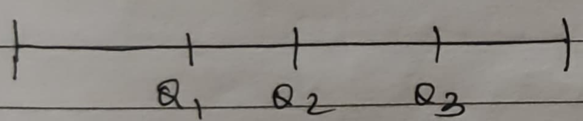
1. mean = mode = median = m

2. If $x_1 \cdots x_n$ are independent normal variables with mean $m_1, m_2 \cdots m_n$ & variance $\sigma_1^2, \sigma_2^2, \sigma_3^2 \cdots$ then $y = a_1x_1 + a_2x_n \cdots a_nx_n$ is also a normal variable with mean,

$$m = a_1m_1 + a_2m_2 \cdots a_nm_n$$

$$\& \ \sigma^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 \cdots a_n^2\sigma_n^2$$

3. Quartiles



$$Q_1 = m = -\frac{2\sigma}{3}, \qquad Q_3 = m + \frac{2\sigma}{3}$$

$$Q.D = \frac{Q_3 - Q_1}{2} = \frac{2\sigma}{3}.$$

4. If $x \sim N(m, \sigma)$ then $z = \frac{x-m}{\sigma}$ is std normal variate (SNV) is also a normal variate with $z \sim N(0,1)$.
mσ

Date / /

Ex:

1. Normal distribution w:

$m = 50, \sigma = 15$

find: (i) Q_1, Q_2

(ii) Q.D

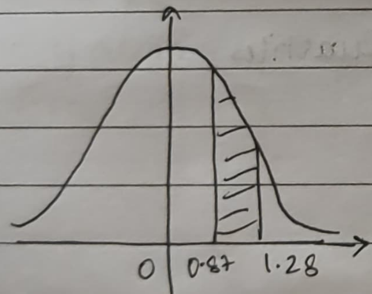
2. If $Q_1 = 36, Q_3 = 44$ find mean, std.

* $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \times e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$, $P(a \leq x \leq b) = ?$

$P(a \leq x \leq b) = \int_a^b f(x)\,dx$

$\frac{z-0}{1} = \frac{x-m}{\sigma}$

Q Find $P(0.87 < z < 1.28)$



$P(0.87 < z < 1.28) = P(0 < z < 1.28) - P(0 < z < 0.87)$

$= 0.3997 - 0.3078$

$= 0.0919.$

Q find c s.t (i) $P(-c < z < c) = 0.95$

(ii) $P(|z| < c) = 0.01$

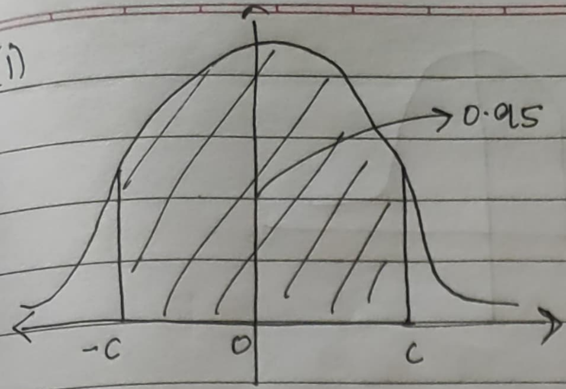
(iii) $P(x > c) = 0.02$

(iv) $P(x < c) = 0.05$

lim $m = 120$, $\sigma = 10$

Date

ans: (i)



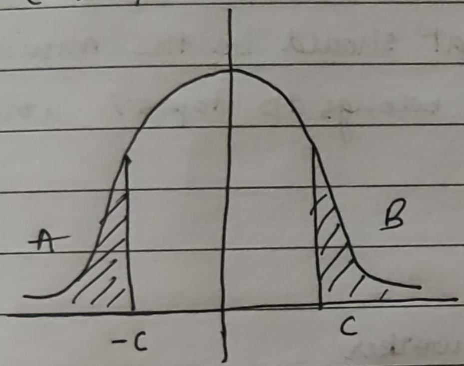
$P(0 < z < c) = \frac{0.95}{2} = 0.475$

$\therefore c = 1.9 + 0.06 = 1.96$

(ii) $P(|z| > c)$

$= P(z > c, z < -c) = 0.01$

$=$



$A + B = 0.01$

$\therefore$ Area $B = \frac{0.01}{2} = 0.005$

$\therefore P(0 < z < c) = 0.5 - 0.005$

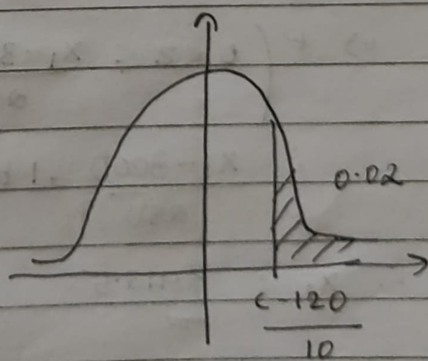
$= 0.495$

$\therefore c = 2.58$

(iii) $P(x > c) = 0.02$

$\Rightarrow P\left(\frac{x-m}{\sigma} > \frac{c-m}{\sigma}\right) = 0.02$

$\Rightarrow P\left(z > \frac{c-120}{10}\right) = 0.02$



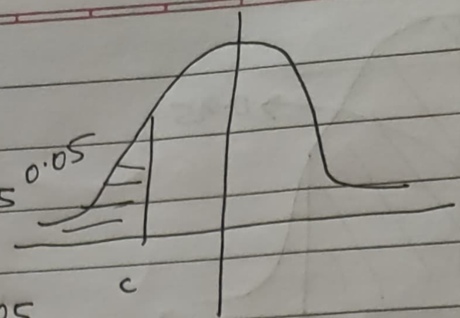
$P\left(0 < z < \frac{c-120}{10}\right) = 0.5 - 0.02$

$= 0.48$

$\frac{c-120}{10} = 2.06$

$\therefore c = 140.6$

(iv) $P(X < C) = 0.05$



$= P\left(\frac{X-m}{\sigma} < \frac{C-m}{\sigma}\right) = 0.05$

$= P\left(z < \frac{C-120}{10}\right) = 0.05$

$= P\left(0 < z < \frac{C-120}{10}\right) = 0.5 - 0.05$

$= 0.45$

$\therefore \frac{C-120}{10} = 1.65 \quad \therefore C = 136.5$

Q Monthly salary X is normally distributed between ~~3 thousand~~ 3000 Rs & $\sigma = 250$ Rs. What should be the minimum salary of a worker so that it belongs to top 5% workers.

ans: $m = 3000$

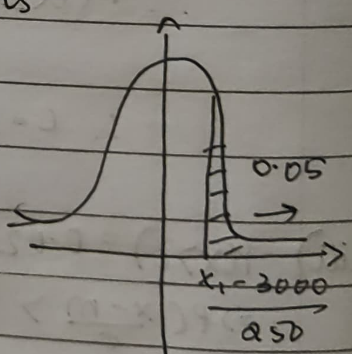
$\sigma = 250$

Let X = monthly salary of a worker

X_1 = minimum salary of top 5% workers

Given : $P(X \geq x_1) = 0.05$

$\Rightarrow P\left(z \geq \frac{X_1 - 3000}{250}\right) = 0.05$



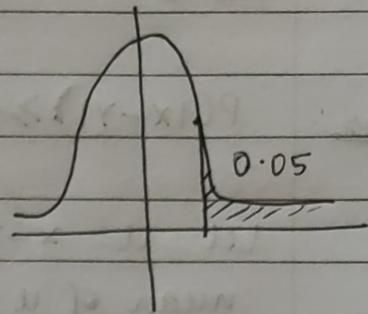
$\Rightarrow P\left(0 \leq z \leq \frac{X_1 - 3000}{250}\right) = 0.5 - 0.05$

$= 0.45$

$\therefore \frac{X_1 - 3000}{250} = 1.65$

$\therefore X_1 = 3412.5$

Q The diameter of tops of cans is normally distributed with standard deviation 0.05. At what mean diameter the machine should be set so that not more than 5% of the tops have dia exceeding 3 cm.



ans: X = diameter of the top

$P(X>3) = 5\% = 0.05$

$P\left(\frac{X-m}{\sigma} > \frac{3-m}{\sigma}\right) = 0.05$

$P\left(Z > \frac{3-m}{0.05}\right) = 0.05$

So.

$P\left(0 < Z < \frac{3-m}{0.05}\right) = 0.5 - 0.05 = 0.45$

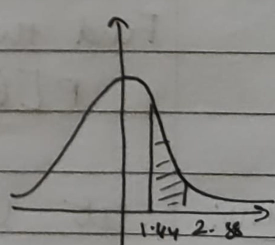
$\Rightarrow \frac{3-m}{0.05} = 1.65$

$\therefore m = 2.9175$

Q If x_1 & x_2 are two independent normal variates with mean 30 & 25 and var = 16 & 12. If $Y = 3x_1 - 2x_2$. Find prob of : $P(60 \leq Y \leq 80)$

ans: mean of $3x_1 - 2x_2$ is $= 3 \times 30 - 2 \times 25 = 40 = m$

var of $3x_1 - 2x_2$ is $= 9 \times 16 + 4 \times 12 = 192 = \sigma^2$



$\therefore P(60 \leq Y \leq 80) = P\left(\frac{60-m}{\sigma} \leq Z \leq \frac{80-m}{\sigma}\right)$

$= P(1.44 \leq Z \leq 2.88)$

$= P(0 \leq Z \leq 2.88) - P(0 \leq Z \leq 1.44)$

= [illegible]

$= 0.4980 - 0.4251$

$= 0.0729$

Q. If R.V X & Y are normally dist with m & std as (52, 3) and (50, 2) respectively. Find the prob that randomly choosen ~~values~~ pairs of X & Y differ by 1.7 or more.

ans: $P(|X-Y| \geq 1.7)$ ⇒ to find

let $U = X - Y$

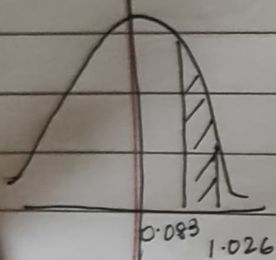
mean of $U = 52 - 50 = 2$

var of $U = 9 + 4 = 13$

$P(|U| \geq 1.7) = 1 - P(|U| < 1.7)$

$= 1 - P(-1.7 < U < 1.7)$

$= 1 - P\left(\frac{-1.7-2}{\sqrt{13}} < Z < \frac{1.7-2}{\sqrt{13}}\right)$



$= 1 - P(-1.026 < Z < -0.083)$

$= 1 - [P(0 < Z < 1.026) - P(0 < Z < 0.083)]$

$= 1 - [\cancel{0.3485}\ 0.3481 - 0.0319]$

0.3485

$= 0.6834$

Q. If X & Y are independent N.V with mean $m_1 = 8$, $m_2 = 12$ & $\sigma_1 = 2$, $\sigma_2 = 4\sqrt{3}$.

Find the value of alpha α such that.

$P[(2X - Y) \leq 2\alpha] = P[(X + 2Y) \geq 3\alpha]$

ans: let $U = 2X - Y$

$V = X + 2Y$

mean of $U = 4$ var of $U = 64$

mean of $V = 32$ var of $V = 196$

$\therefore P(U \leq 2\alpha) = P(V \geq 3\alpha)$

$\Rightarrow P\left(Z \leq \frac{2\alpha - 4}{8}\right) = P\left(Z \geq \frac{3\alpha - 32}{14}\right)$

$\Rightarrow P\left(Z \leq \frac{2\alpha - 4}{8}\right) = P\left(Z \leq -\left(\frac{3\alpha - 32}{14}\right)\right)$

$\Rightarrow \frac{2\alpha - 4}{\cancel{8}\,4} = -\left(\frac{3\alpha - 32}{\cancel{14}\,7}\right)$

$\Rightarrow 14\alpha - 28 = -12\alpha + 128$

$\Rightarrow 26\alpha = 128 + 28$

$\Rightarrow 26\alpha = 156$

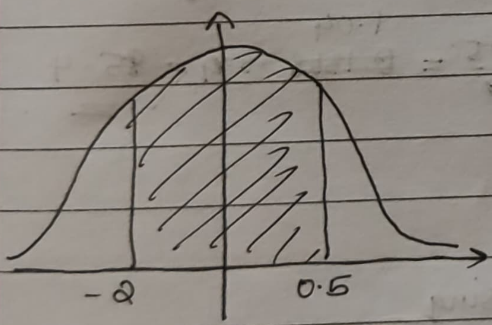
$\therefore \alpha = \frac{156}{26} = \underline{\underline{6}}$

Q Income of grp of 10,000 persons are normally distributed with mean 520 & s.d = 60. Find :

(i) no. of persons having income between 400 & 550.

(ii) lowest income of richest 500.

ans:



$m = 520$

$\sigma = 60$

find : (i) $P(400 \leq X \leq 550)$

$= P(-2 \leq Z \leq 0.5)$

$= P(0 \leq Z \leq 0.5) + P(0 \leq Z \leq 2)$

$= 0.6687$

no. of persons having income between 400 & 550 = NP

$= 10{,}000 \times 0.6687$

$= 6687$

P(Person selected at random among 500 from 10,000 persons) $= \frac{500}{10000} = \underline{\underline{0.05}}$

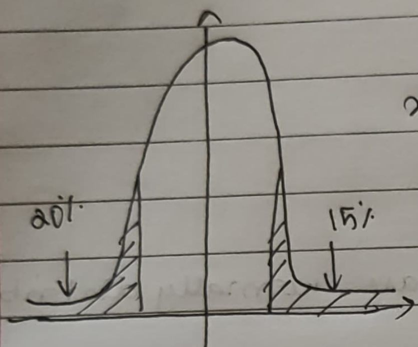
Date

x_1 = lowest income

$P(x > x_1) = 0.05$

Q In a competitive exam top 15% of students will get grade A, while bottom 20% will be declared failed. The grades are normally distributed with mean % of marks = 75 and SD = 10. Determine lowest % of marks to receive grade A and lowest % of marks that passes.

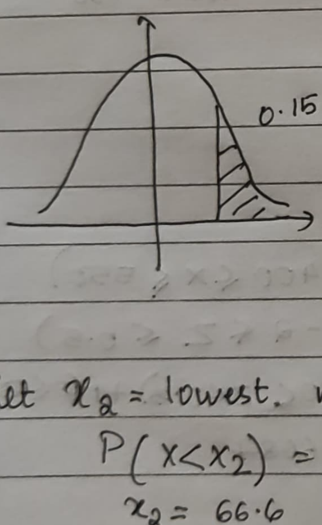
ans:



x_1 = marks req for grade A

$\Rightarrow P(X \geq x_1) = 0.15$

$\Rightarrow P\left(Z \geq \frac{x_1 - 75}{10}\right) = 0.15$



$P\left(0 \leq Z \leq \frac{x_1 - 75}{10}\right) = 0.5 - 0.15$

$= 0.35$

$\therefore \frac{x_1 - 75}{10} = \cancel{0.1368}\ 1.04 \Rightarrow x_1 = 85.4$

let x_2 = lowest marks for passing

$P(X < x_2) = 0.2$

$x_2 = 66.6$

$P\left(Y < \frac{x_2 - 75}{10}\right) = 0.2$

$P\left(0 < Z < \frac{x_2 - 75}{10}\right) = 0.5 - 0.2$

$= 0.3$

$\cancel{\frac{x_2 - 75}{10}}$

$\frac{-75 + x_2}{10} = -0.84$

$P\left(\frac{75 - x_2}{10} > Z > 0\right) = 0.3$

$\therefore x_2 = 66.6$

Q If the actual amount of coffee which a machine puts into 6 ounce jars is a random variable ~~wit~~ having normal distribution & $\sigma = 0.05$ and if only 3% contains less than 6 ounces of coffee, what must be the mean fill of these jars

ans: X = amount of coffee.

$P(X<6) = 3\%$.

$P\left(\frac{X-m}{\sigma} < \frac{6-m}{\sigma}\right) = 0.03$

i.e $P\left(Z < \frac{6-m}{\sigma}\right) = 0.03$

$\Rightarrow P\left(0 \le Z \le \frac{6-m}{\sigma}\right) = 0.5 - 0.03$

$= 0.47$

$\frac{6-m}{\sigma} = -1.88$

$\therefore m = 6.094$

$6 < \cancel{X}$

BARFI Page No.
Date

Q If the actual amount of coffee which a machine puts into 6 ounce jars is a random variable ~~wit~~ having normal distribution & $\sigma = 0.05$ and if only 3% contain less than 6 ounces of coffee, what must be the mean fill of these jars

ans: x = amount of coffee.

$P(x<6) = 3\%$

$P\left(\frac{x-m}{\sigma} < \frac{6-m}{\sigma}\right) = 0.03$

i.e $P\left(z < \frac{6-m}{\sigma}\right) = 0.03$

$\Rightarrow P\left(0 \le z \le \frac{6-m}{\sigma}\right) = 0.5 - 0.03$
$= 0.47$

$\frac{6-m}{\sigma} = -1.88$

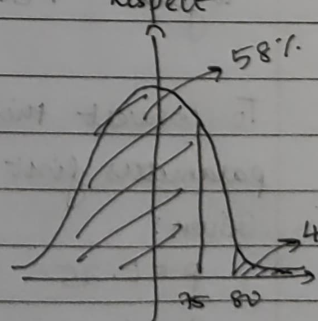
$\therefore m = 6.094$

$6 < \cancel{x}$

Q Find the mean & s.d of the normal distribution of marks where 58% obtained marks below 75, 4% obtained above 80 and rest between 75 & 80.

ans: x = marks obtained

let m and σ be mean & s.d.



$\Rightarrow P(m \le x \le 75) = 8\% = 0.08$

$\therefore P\left(\frac{m-m}{\sigma} \le z \le \frac{75-m}{\sigma}\right) = 0.08$

$\therefore P\left(0 \le z \le \frac{75-m}{\sigma}\right) = 0.08$

$\therefore \frac{75-m}{\sigma} = 0.2 \Rightarrow m + 0.2\sigma = 75$ - (1)

Date

Also,

$$P(X > 80) = 4\% = 0.04$$

$$\Rightarrow P\left(Z > \frac{80-m}{\sigma}\right) = 0.04$$

$$\Rightarrow P\left(0 < Z < \frac{80-m}{\sigma}\right) = 0.5 - 0.04 = 0.46$$

$$\therefore \frac{80-m}{\sigma} = 1.75$$

$$\therefore 80 = m + 1.75\sigma \quad -(2)$$

$\therefore$ From (1) and (2) we get,

$m = 74.36$

$\sigma = 3.225$

Q The prob than electronic component will fail in less than 1200 hours of continuous use is 0.25. Using normal approximation to Binomial dist find the prob that amongst 200 such components fewer than 45 will fail in less than 1200 hrs of continuous use.

ans: X = no. of components which will fail in less than 1200 hrs

To find: $P(X < 45)$

To convert this into normal dist we need to find the parameters first: m, σ

Given:

$p = 0.25$

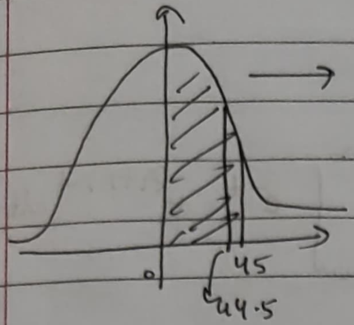
$q = 1 - p = 0.75$

$n = 200$

$m = np = 200 \times 0.25 = 50$

s.d (σ) $= \sqrt{npq} = \cancel{9.35}\ 6.123$

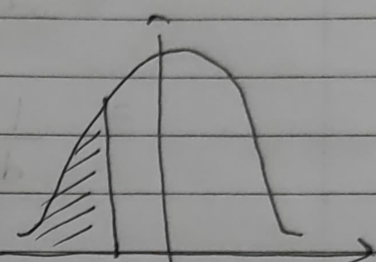
$z = \frac{x-m}{\sigma} = \frac{x-50}{6.123}$



$\longrightarrow$ ~~AE~~ Rule : subtract 0.5

$\therefore$ Req Prob is $P(X \leq 44.5)$

$= P(Z \leq \frac{44.5-50}{6.123})$



$= P(Z \leq -0.898) = 0.5 - P(0 < z < 0.898)$

$= 0.5 - 0.313$

$= 0.1867$

* **Exponential distⁿ :**

Continuous R.V X is ~~is~~ said to follow exponential dist withinparameter $\lambda > 0$ if it's pdf is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & o.w \end{cases}$$

N.B: $\int_0^\infty f(x)\,dx = 1$

mean & variance :

$$E(x^r) = \int_0^\infty x^r f(x)\,dx$$

$$= \int_0^\infty x^r \lambda e^{-\lambda x}\,dx$$

put $\lambda x = t$ $\quad \therefore dx = \frac{dt}{\lambda}$

$$\therefore \cancel{\lambda} \int_0^\infty \left(\frac{t}{\lambda}\right)^r e^{-t} \frac{dt}{\cancel{\lambda}}$$

$$= \frac{1}{\lambda^r} \int_0^\infty t^r e^{-t} dt = \frac{1}{\lambda^r} \int_0^\infty e^{-t} t^{(r+1)-1} dt$$

$$= \frac{1}{\lambda^r} \Gamma(r+1) = \frac{r!}{\lambda^r}$$

$$\therefore E(x^r) = \frac{r!}{\lambda^r}$$

$$\therefore \text{mean} = E(x) = \frac{1}{\lambda}$$

$$Var(x) = E(x^2) - (E(x))^2$$

$$= \frac{2!}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

note: $$P(X > k) = \int_k^\infty \lambda e^{-\lambda x} dx = e^{-\lambda k}$$

Q Mileage which car owners get is a R.V having an exponential dist ~~[illegible]~~ with mean 40,000 Km. Find the probability that one of these tires will last atleast 20,000 km & atmost 30,000 km.

ans: X = mileage of a car

$M = 40,000 = \frac{1}{\lambda} \quad \therefore \lambda = 2.5 \times 10^{-5}$

(i) $P(X \geq 20{,}000) = \int_{20K}^{\infty} \frac{1}{40{,}000} \times e^{-\frac{1}{40{,}000}x}\, dx$

(using note)

$= e^{-\frac{1}{40{,}000} \times 20{,}000}$

$= e^{-0.5} = 0.606$

(ii) $P(X \leq 30{,}000) = \int_{0}^{30{,}000} \frac{1}{40{,}000} e^{-\frac{1}{40{,}000}x}\, dx$

$= \frac{1}{40{,}000} \left[\frac{e^{-\frac{1}{40{,}000}x}}{-\frac{1}{40{,}000}} \right]_0^{30{,}000}$

$= -\left[e^{-0.75} - 1 \right] = 1 - e^{-0.75}$

note: ~~[illegible]~~

If x is exponentially distributed then $P(X > s+t / X > s) = P(X > t)$ for $s, t > 0$.

Q The length of a shower has expontential distribution with parameter 2 & ~~time~~ time being measured in mins then what is the prop. that shower will last more than 3 mins. If shower has lasted 2 mins then find the prob that it will last 1 more min.

ans: (i) $P(X > 3) = \int_{3}^{\infty} 2e^{-2x} dx = e^{-6}$

X = length of shower

Given $\lambda = 2$, $f(x) = 2e^{-2x}$

(ii) we want $P(X > 2+1 | X > 2) = P(X > 1)$

$$= \int_1^\infty 2e^{-2x}\,dx = e^{-2}$$

* uniform distribution:

Discrete R.V X is said to follow uniform distribution if $P(X = x_i) = \frac{1}{n}$ $\forall x_i = 1, 2 \ldots$ $1 \le i \le n$

Ex: when coin is tossed $P = 1/2$.

Mean: $E(X) = \sum p_i x_i$

$$= \sum x_i \frac{1}{n}$$

$$= 1\frac{1}{n} + 2\frac{1}{n} \ldots \ldots + n\frac{1}{n}$$

$$= \frac{1}{n}(1+2 \ldots n) = \frac{\cancel{n}(n+1)}{2} \times \frac{1}{\cancel{n}}$$

$$= \frac{n+1}{2}$$

Variance:

$$E(X^2) = \sum x_i^2 p_i$$

$$= 1^2\frac{1}{n} + 2^2\frac{1}{n} \ldots \ldots n^2\frac{1}{n}$$

$$= \frac{1}{n}(1^2 + 2^2 \ldots n^2) =$$

$$= \frac{1}{\cancel{n}} \times \frac{\cancel{n}(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6}$$

Date

Variance: $E(x^2) - [E(x)]^2$

$$= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{(n+1)}{12}\left(4n+2-3n-3\right) = \frac{n^2-1}{12}$$

Continuous Uniform Distribution:

A continuous R.V X follows uniform dist over the interval $[a,b]$ if:

$$f(x) = \begin{cases} k, & a \le x \le b \\ 0 & o.w \end{cases}$$

$$\because \int_a^b f(x)\,dx = 1$$

$$= \int_a^b k\,dx = 1 \quad \Rightarrow k(b-a) = 1$$

$$k = \frac{1}{b-a}.$$

$$\therefore f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & o.w \end{cases}$$

$$\text{mean} = \int_a^b x f(x)\,dx$$

$$= \int_a^b x \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left(\frac{x^2}{2}\right)_a^b = \frac{1}{2(b-a)}(b^2-a^2)$$

$$= \frac{b+a}{2}$$

$$E(x^2) = \int_a^b x^2 f(x)\, dx$$

$$= \int_a^b x^2 \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\left(\frac{x^3}{3}\right)_a^b = \frac{1}{3(b-a)}(b^3-a^3)$$

$$= \frac{b^3-a^3}{3(b-a)} = \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}}$$

$$= \frac{b^2+ab+a^2}{3}$$

$$\text{Variance} = E(x^2) - (E(x))^2$$

$$= \frac{b^2+ab+a^2}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{(b^2+a^2+2ab)}{4}$$

$$= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}.$$

$$E(x^2) = \int_a^b x^2 f(x)\,dx$$

$$= \int_a^b x^2 \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left(\frac{x^3}{3}\right)_a^b = \frac{1}{3(b-a)}(b^3-a^3)$$

$$= \frac{b^3-a^3}{3(b-a)} = \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}}$$

$$= \frac{b^2+ab+a^2}{3}$$

$$\text{Variance} = E(x^2) - (E(x))^2$$

$$= \frac{b^2+ab+a^2}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{b^2+ab+a^2}{3} - \frac{(b^2+a^2+2ab)}{4}$$

$$= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}.$$

* Baye's Theorem:

=> Let $E_1, E_2 \cdots E_m$ be mutually disjoint events with prob of $P(E_i) \neq 0$. Let A be any arbitaryary event such that ~~a case~~ $A \subseteq \bigcup_{i=1}^{n} E_i$, $P(A) > 0$

then, $$P(E_i/A) = \frac{P(E_i)\,P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\,P(A/E_i)}$$

$$P(E_i/A) = \frac{P(E_i)\,P(A/E_i)}{P(A)}$$

BARFI Page No.
Date

we have, $A = A \cap \left(\bigcup_{i=1}^{n} E_i \right)$

$= A \cap (E_1 \cup E_2 \cup \cdots E_n)$

$= (A \cap E_1) \cup (A \cap E_2) \cdots \cup (A \cap E_n)$

$P(A) = P(A \cap E_1) + P(A \cap E_2) \cdots P(A \cap E_n)$

$= \sum_{i=1}^{n} P(A \cap E_i)$

$= \sum_{i=1}^{n} P(A/E_i) P(E_i)$

* $P(A \cup B) = P(A) + P(B)$

$P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

Ex: A product is produced in 3 factories X, Y, Z. X produces ~~as~~ thrice as many times as Y. Y and Z produce same no. of items. Assume that 3% of items produced by each factory X and Z are defective. while 5% produced by Y are defective. An item is selected at random. (i) what is the prob that this item is defective. (ii) If the item is already known to be defective then what is the prob that is produced by X, Y or Z.

ans: let,

A = item is defective.

E_i = item produced by X, Y and Z respectively.

Date

Events E_i	Prior prob $P(E_i)$	Cond Prob $P(A/E_i)$	Joint prob $P(A \cap E_i)$ $= P(E_i)P(A/E_i)$	Posterior prob $P(E_i/A)$ $(5) = (4) \div P(A)$
(1)	(2)	(3)	(4) = (2) x (3)	
E_1	3/5	0.03	0.018	0.529
E_2	1/5	0.05	0.01	0.294
E_3	1/5	0.03	0.006	0.176
			$\underbrace{P(A) = 0.034}_{\text{Ans (i)}}$	

Ans (ii)

$\frac{25}{10000}$

$\frac{4 \times 9}{100}$ 20

Q factory produces certain types of outputs by 3 types of machine. Daily productions are as follows:

$M_1 = 3000$ units

$M_2 = 2500$

$M_3 = 4500$

It is already known that 1% of outputs by M_1 are defective, M_2: 1.2%, M_3 = 2%. An item drawn is already found to be defective. What is the prob that it is from M_1, M_2 or M_3.

BARFI Page No.
Date

ans: E_i = item produced by M_i

E_i	$P(E_i)$	$P(A/E_i)$	$P(A \cap E_i)$	$P(E_i/A)$
E_1	3/10 ~~0.01~~	0.01	0.003	0.2
E_2	1/4 ~~0.012~~	0.012	0.003	0.2
E_3	9/20 ~~0.02~~	0.02	0.009	0.6
			0.015	

$$P(E_i/A) = \frac{P(E_i)P(A/E_i)}{\underbrace{\Sigma P(E_i)\, P(A/E_i)}_{P(A)}}$$

Q A → n white + 2 black
B → 2 white + n black

1 bag is selected at random & 2 balls are drawn without replacement. If both the balls drawn are white & prob that A was used is 6/7. Find the value of n.

ans: E_1: bag A selected
E_2: bag B selected
A: Two balls drawn are white.

E_i	$P(E_i)$	$P(A/E_i)$	$P(A \cap E_i)$	$P(E_i/A)$
E_1	1/2	$\frac{{}^nC_2}{{}^{n+2}C_2}$		
E_2	1/2	$\frac{{}^2C_2}{{}^{n+2}C_2}$		

By Baey's theorem,
prob that 2 white balls are drawn from A :

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{\left(P(E_1)P(A/E_1) + P(E_2)P(A/E_2)\right)} = \frac{6}{7}$$

$$\Rightarrow \frac{\frac{1}{2} \times \frac{{}^{n}C_2}{{}^{n+2}C_2}}{\frac{1}{2}\frac{{}^{n}C_2}{{}^{n+2}C_2} + \frac{1}{2}\frac{{}^{2}C_2}{{}^{n+2}C_2}} = \frac{6}{7}$$

$$\Rightarrow \frac{\frac{n(n-1)}{2}}{\frac{n(n-1)}{2} + 1} = 6/7$$

$$\Rightarrow \frac{n(n-1)}{n(n-1)+2} = 6/7$$

$$\Rightarrow 7n(n-1) = 6(n(n-1)+2)$$

$$\Rightarrow 7n^2 - 7n = 6n^2 - 6n + 12$$

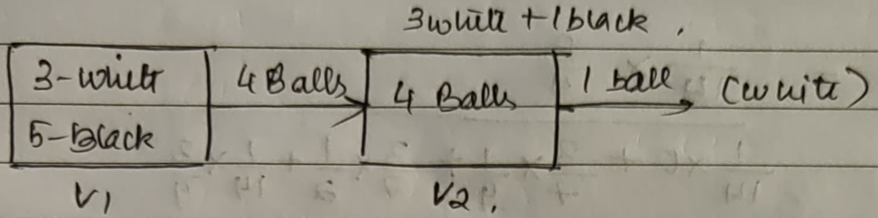
$$\Rightarrow n^2 - n - 12 = 0$$

$$\Rightarrow n = 4, -3$$

$$\therefore n = 4$$

8. A vessel containing 3-white, 5-black. 4 balls are transfered to an empty vessel. From this vessel a ball is drawn at random & is found to be white. what is the prob that out of 4 balls, 3 are white and 1 is black.

ans:



Define: E_1 = 0 white and 4 black
E_2 = 1 white & 3 black
E_3 = 2 white & 2 black
E_4 = 3 white and 1 black

A = a ball drawn is white

$P(E_1) = \dfrac{{}^3C_0 \times {}^5C_4}{{}^8C_4} = 1/14$

$P(E_2) = \dfrac{{}^3C_1 \times {}^5C_3}{{}^8C_4} = 3/7$

$P(E_3) = \dfrac{{}^3C_2 \times {}^5C_2}{{}^8C_4} = 3/7$

$P(E_4) = \dfrac{{}^3C_3 \times {}^5C_1}{{}^8C_4} = 1/14$

$P(E/E_1) = 0$ To find: $P(E_4/E)$
$P(E/E_2) = 1/4$
$P(E/E_3) = 1/2$
$P(E/E_4) = 3/4$

$$P(E_4/E) = \frac{P(E_4)\,P(E/E_4)}{\sum_{i=1}^{4} P(E_i)\,P(E/E_i)}$$

$$= \frac{\frac{1}{14} \times \frac{3}{4}}{\frac{1}{14}\times 0 + \frac{3}{7}\times\frac{1}{4} + \frac{3}{7}\times\frac{1}{2} + \frac{1}{14}\times\frac{3}{4}}$$

$$= 1/7 = 0.14$$

2D or Joint Conditional Prob:

$(X, Y) \rightarrow (x, y) \in R^2$

Let X and Y be 2 R.V defined on the sample space S then, a function (X, Y) which assigns a point in $\mathbb{R}^2$ is called 2 Dimensional R.V.

Joint prob mass function:

If X, Y is 2 Dimensional discrete RV with each possible outcome (x_i, y_j) with the prob P_{ij} satisfying the conditions,

(i) $P(x_i, y_j) \geq 0 \ \forall\, i, j$

(ii) $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} P(x_i, y_j) = 1$

the P is called joint pmf and joint prob distribution is given as follows:

X \ Y	y_1	y_2	...	y_m	Total
x_1	P_{11}	P_{12}	...	P_{1m}	P_1
x_2	P_{21}	P_{22}	...	P_{2m}	P_2
$\vdots$					$\vdots$
x_n	P_{n1}	P_{n2}	...	P_{nm}	P_n
Total	P_1'	P_2'	...	P_m'	

Marginal prob distribution of x is given by:

$$X: \quad x_1 \quad x_2 \quad \cdots \quad x_n$$
$$P(X): \quad P_1 \quad P_2 \quad \cdots \quad P_n$$

where $P_1 = \sum_{j=1}^{m} P_{1j}$

and, $P_n = \sum_{j=1}^{m} P_{nj}$

$$Y: \quad y_1 \quad y_2 \quad \cdots \quad y_m$$
$$P(Y): \quad P_1' \quad P_2' \quad \cdots \quad P_m'$$

where, $P_1' = \sum_{i=1}^{n} P_{i1}$

$P_m' = \sum_{i=1}^{n} P_{im}$

$$P(X=x_i, Y=y_i) = \frac{P(X=x_i, Y=y_i)}{P(Y=y_i)}$$

Q Joint Prob dist of X and Y is given by:

$$P(X=x, Y=y) = \frac{x+3y}{24} \; ; \; x,y = 1,2.$$

Find pmf and marginal prob dist of X and Y

ans:

Y \ X	1	2	Total
1	1/6	5/24	3/8
2	7/24	1/3	5/8
Total	11/24	13/24	1

X :	1	2	Total
P(X) :	11/24	13/24	1

Y :	1	2	Total
P(Y) :	3/8	5/8	1

Q Joint prob dist of X_1 & X_2 is given by,

$$P(X_1=x_1, X_2=x_2) = \frac{1}{27}(x_1+2x_2)$$

$$\Rightarrow x_1, x_2 = 0,1,2$$

Find Pmf of X_1 & X_2 ~~indepe~~ in terms of x_1 & x_2

ans:
$$P(X_1=x_1) = \sum_{x_2=0}^{n} P(x_1, x_2)$$

$$= \sum_{x_2=0}^{2} \frac{1}{27}(x_1+2x_2)$$

Date

$$= \frac{1}{27}\left[x_1 + (x_1+2) + (x_1+4)\right]$$

$$= \frac{1}{27}[3x_1+6] = \frac{x_1+2}{9}$$

$$P(X=x_2) = \sum_{x_1=0}^{2} P(x_1, x_2)$$

$$= \sum_{x_1=0}^{2} \frac{1}{27}(x_1+2x_2)$$

$$= \frac{1}{27}\left[2x_2 + (1+2x_2) + (2+2x_2)\right]$$

$$= \frac{1}{27}[6x_2+3] = \frac{2x_2+1}{9}$$

Q 3 balls are drawn at random from a box containing 4 black & 3 red. If X denotes no. of white balls, Y = no. of red balls. Find Joint P dist of X & Y

ans (i)

X \ Y	0	1	2	3	Total
0	1/21	3/14	1/7	1/84	35/84
1	1/7	2/7	1/14	0	42/84
2	1/21	1/28	0	0	7/84
Total:	5/21	15/28	3/14	1/84	1

2 - white
4 - black
3 - red

$$P(X=0, Y=0) = \frac{{}^4C_3}{{}^9C_3} = \frac{{}^4C_1}{{}^9C_6} = \frac{1}{21}$$

$$P(X=0, Y=1) = \frac{{}^3C_1 \times {}^4C_2}{{}^9C_3} = \frac{3}{14}$$

X = no. of white balls
Y = no. of red balls

76/84

(ii) $P(X \leq 1) = 35/84 + 42/84 = 77/84$

$P(X \leq 1, Y \leq 2) = P(X=0, Y \leq 2) + P(X=1, Y \leq 2) = \cancel{76/84}$ 76/84

$P(Y \leq 2 / X \leq 1) = \dfrac{P(X \leq 1, Y \leq 2)}{P(X \leq 1)} = 76/77$

$$P(X+Y \leq 2) = \sum_{j=0}^{2} P(X=0, Y=j) + \sum_{j=0}^{1} P(X=1, Y=j)$$

$$+ P(X=2, Y=0)$$

$$= 37/42.$$

(iii) check whether X & Y are independent,

$P(X,Y) = P(X)\,P(Y)$

$\left.\begin{array}{l} P(0,1) = 3/14 \\ P(0) \times P(1) = \dfrac{35}{84} \times \dfrac{15}{28} \end{array}\right\}$ not equal

hence, not independent.

* Continuous prob distribution:

Let (X, Y) be 2-D continuous R.V and let $f_{XY}(x,y)$ be a function of (X,Y) such that:

(i) $f_{XY}(x,y) \geq 0$

(ii) $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\,dy = 1$

(iii) $\displaystyle\int_{a}^{b}\int_{c}^{d} f_{XY}(x,y)\, dx\,dy = P(a \leq x \leq b, c \leq Y \leq d)$

Date / /

* Marginal pdf of x-axis : $f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x, y)\, dy$

$f_x(x) \geq 0$ and $\int_{-\infty}^{\infty} f_x(x)\, dx = 1$

* Marginal pdf of y axis : $f_y(y) = \int_{-\infty}^{\infty} f_{xy}(xy)\, dx$

$f_y(y) \geq 0$ & $\int_{-\infty}^{\infty} f_y(y)\, dy = 1$

Q 2-D R.V has joint pdf $f_{xy}(xy) = \begin{cases} 15e^{-3x-5y} & x>0,\ y>0 \\ 0 & \text{o.w} \end{cases}$

find : (i) $P(1<X<2,\ 0.2<Y<0.3)$
(ii) $P(X<2,\ Y>0.2)$
(iii) Marginal prob distribution of X & Y

ans (i) $P(1<X<2,\ 0.2<Y<0.3) = \int_{y=0.2}^{0.3}\int_{x=1}^{2} 15e^{-3x-5y}\, dx\, dy$

$= 15\int_{0.2}^{0.3} e^{-5y}\, dy \int_{1}^{2} e^{-3x}\, dx$

$= 15\left(\frac{e^{-5y}}{-5}\right)_{0.2}^{0.3} \times \left(\frac{e^{-3x}}{-3}\right)_{1}^{2}$

$= \left(e^{-5\times0.3} \cdot e^{-5\times0.2}\right) \times \left(e^{-6} - e^{-3}\right)$

$= 0.00684$

(ii) $P(X<2,\ Y>0.2) = \int_{y=0.2}^{\infty}\int_{x=0}^{2} 15e^{-3x-5y}\, dx\, dy = 0.367$

(iii) $f_x(x) = \int_{y=0}^{\infty} 15e^{-3x-5y}\,dy = 15e^{-3x}\left(\frac{e^{-5y}}{-5}\right)_0^{\infty}$

$= -3e^{-3x}(-1)$

$f_x(x) = \begin{cases} 3e^{-3x} & x>0 \\ 0 & o.w \end{cases}$

$f_y(y) = 15e^{-5y}\int_0^{\infty} e^{-3x}\,dx = \frac{-15}{3}e^{-5y}(-1)$

$f_y(y) = \begin{cases} 5e^{-5y} & y>0 \\ 0 & o.w \end{cases}$

Q $f_{xy}(x,y) = \begin{cases} x(x-y) & 0<x<2, \; -x<y<x \\ 0 & o.w \end{cases}$

find : (i) c
(ii) $f_x(x)$
(iii) $f_{y/x}(y/x)$

ans: we have $\int_{y=-x}^{2}\int_{0}^{2} f_{xy}(x,y)\,dx\,dy = 1$

$= c\int_{y=-x}^{x}\int_{x=0}^{1} x(x-y)\,dx\,dy = 1$

$= c\int_{x=0}^{1} x\left(xy - \frac{y^2}{2}\right)_{y=-x}^{x} dx = 1$

$= c\int_0^2 x\left(x^2 - \frac{x^2}{2} + x^2 + \frac{x^2}{2}\right) dx = 1$

Date

$$= 2c\left(\frac{x^4}{4}\right)_0^2 = 1 \quad \Rightarrow \frac{c}{2}(16) = 1 \quad \therefore c = \frac{1}{8}$$

(ii) $$f_x(x) = \int_{y=-x}^{x} f_{xy}(x,y)\,dy$$

$$= \frac{1}{8}\int_{y=-x}^{x} x(x-y)\,dy = \frac{x^3}{4}, \quad 0<x<2$$

(iii) $$f_{y/x}(y/x) = \frac{f_{xy}(x,y)}{f_x(x)}$$

$$= \frac{1/8\, x(x-y)}{x^3/4} = \frac{x-y}{2x^2}$$

Q $f_{xy}(x,y) = xy^2 + \frac{x^2}{8}$, $0 \le x \le 2$, $0 \le y \le 1$

find:

(i) $P(X>1)$

(ii) $P(Y<1/2)$

(iii) $P(X>1, Y<1/2)$

(iv) $P(Y<1/2 / X>1)$

(v) $P(X<Y)$

(vi) $P(X+Y\le 1)$

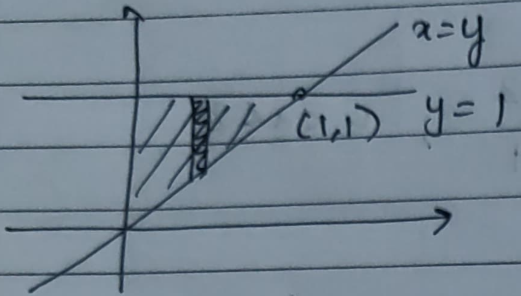
ans: (i) $$P(X>1) = \int_{y=0}^{1}\int_{x=1}^{2}\left(xy^2+\frac{x^2}{8}\right)dx\,dy = \frac{19}{24}$$

(ii) $$P(Y<1/2) = \int_{y=0}^{1/2}\int_{x=0}^{2}\left(xy^2+\frac{x^2}{8}\right)dx\,dy = \frac{1}{4}$$

(iii) $$\int_{y=0}^{1/2}\int_{x=0}^{2}\left(xy^2+\frac{x^2}{8}\right)dx\,dy = \frac{1}{4}$$

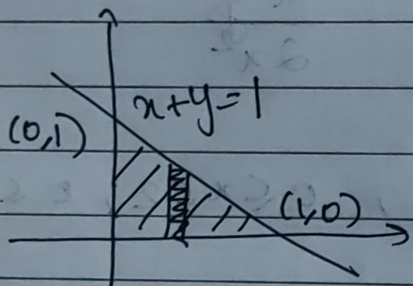
(iii) $P(X>1, Y<1/2) = \int_{y=0}^{1/2} \int_{x=1}^{2} \left(xy^2 + \frac{x^2}{8}\right) dx\,dy = \frac{5}{24}$

(iv) $P(Y<1/2 / X>1) = \frac{P(X>1, Y<1/2)}{P(Y<1/2)} = \frac{5/24}{1/4} = \frac{5}{6}$



(v) $P(X<Y) = \int_{x=8}^{1} \int_{y=x}^{1} \left(xy^2 + \frac{x^2}{8}\right) dy\,dx = \frac{53}{480}$

(vi) $P(X+Y \leq 1) =$



$= \int_{x=0}^{1} \int_{y=0}^{1-x} \left(xy^2 + \frac{x^2}{8}\right) dy\,dx = \frac{13}{480}$