Dynamic Programming

Module 2

Sem IV

AoA Even 2022-23

Introduction

- Richard Bellman invented dynamic programming technique in 1950s
- It is an algorithm design technique used for solving optimization problems.
- Similar to Divide and Conquer technique as it divides original problem into several subproblems.
- This technique starts with finding optimal solution for a smallest problem and then gradually enlarges the problem.
- In dynamic programming, same subproblems are not solved once and results are stored in a table for later use.

Example

Fibonacci Series:

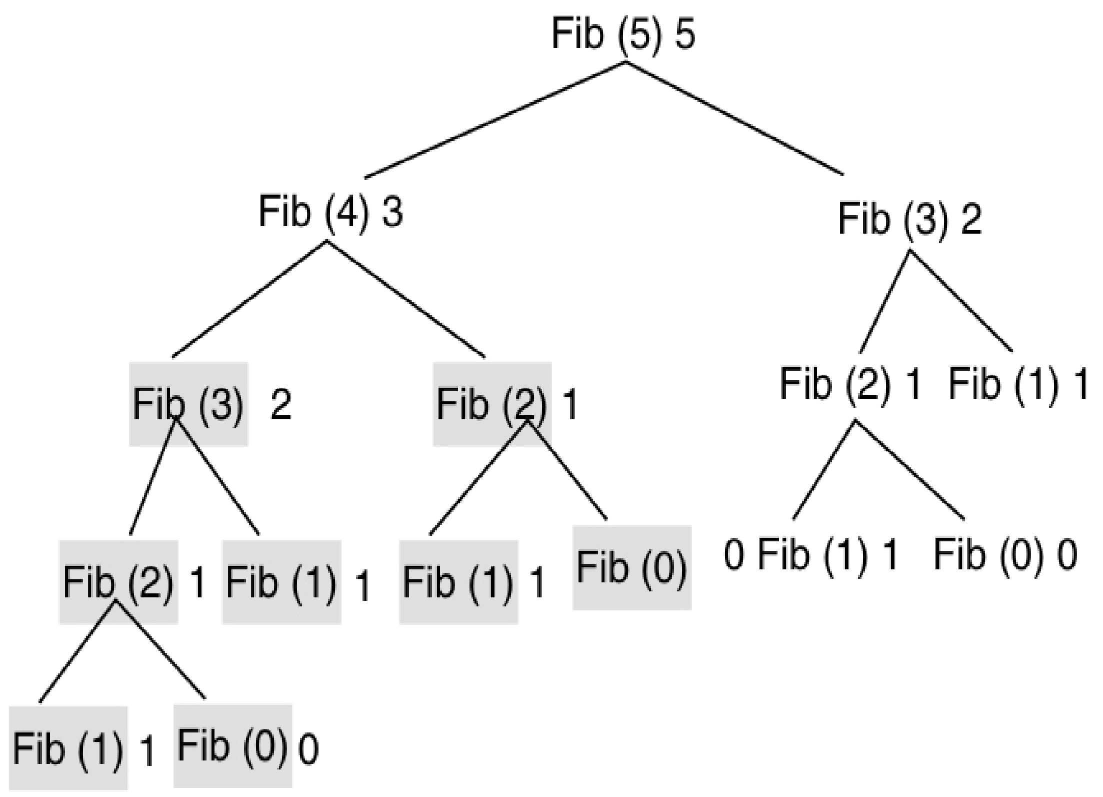
- Recursive formula for calculating nth element of the Fibonacci series:

$$fib(n) = fib(n-1) + fib(n-2)$$

- When solving using D&C same functions are called again and again.
- Problem can be solved using Dynamic programming by storing the result of intermediate problems in table.
- Here many sub problems such as *fib(2)*, *fib(1)* and *fib(0)*, which are overlapping.
- Here, dynamic programming technique follows bottom-up approach.

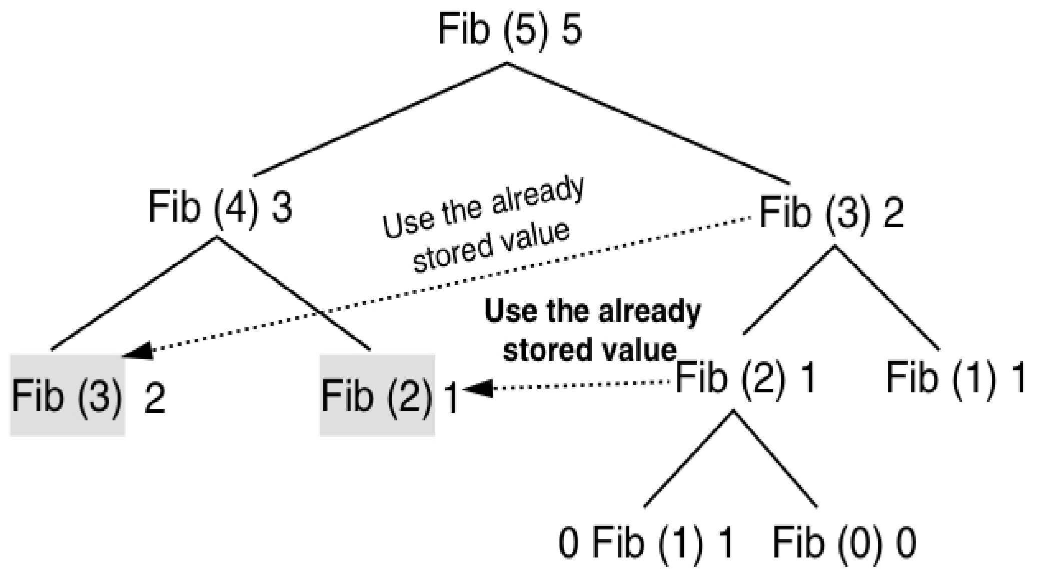
Example

Divide-and-Conquer Technique



(a)

Dynamic Programming



(b)

Example

Fibonacci Series: Recursive Algorithm

```
fib(int n)
 {
     if ((n==0) || (n==1))
        return n;
     else
        return fib(n-1) + fib(n-2);
}
```

Example

Fibonacci Series:

- Dynamic Programming improves time complexity
- In dynamic programming, values are stored in a table for later use.

Fibonacci Series: Iterative Algorithm

```
fib(int a[],int n)
 {
     int n;
        if ((n==0) || (n==1))
          return n;
        a[0]=0; a[1]=1;
     for(i=2; i<=n; i++)
         a[i] = a[i-1] + a[i-2];
         return a[n];
}
```

The time complexity of Iterative algorithm is O(n), since loop runs n-1 times.

Characteristics of Dynamic Programming

1. Optimal Substructure:

- Problems can be decomposed into several subproblems.
- Optimal solutions to subproblems are found and solution of original problem is expressed in terms of solutions for smaller problems.
- Optimal solution contains optimal subsolution, then problem exhibits optimal substructure.

2. Overlapping Subproblems:

- When recursive algorithm revisits same subproblem over and over, the problem is said to have overlapping subproblem

Characteristics of Dynamic Programming

Principle of Optimality:

- Dynamic Programming works on a principle of optimality.
- Sequence of decision to be made at every stage in dynamic programming starting from smallest subproblem.
- The principle of optimality states that

  "*the optimal solution of a problem is a combination of optimal solutions to some of its subproblems*".

Applications of Dynamic Programming

- Longest Common Subsequence(LCS)
- 0/1 Knapsack Problem
- Travelling Salesman Problem
- Single Source shortest path (Bellman Ford Algorithm)
- All pair shortest path (Floyd Warshall Algorithm)
- Matrix chain Multiplication
- Multistage graphs

Homogeneous Recurrences

- Let $T(n)$ is denoted by t_n
- Consider homogeneous recurrences with constant coefficients i.e., recurrence of form

$$a_0\, t_n + a_1\, t_{n-1} + a_2\, t_{n-2} + \cdots + a_k\, t_{n-k} = 0 \qquad \text{-----} \textcircled{1}$$

- This recurrence relation satisfies following properties:
  - It is linear, because it does not contain term of the form t_{n-i} x t_{n-j} , ${t_{n-i}}^2$ etc.
  - It is homogeneous because the linear combination of t_{n-i} is zero.
  - It has constant coefficients since all a_i's are constant.

Homogeneous Recurrences

- For example, consider following recurrence of Fibonacci series

$$t_n = t_{n-1} + t_{n-2}$$

$$t_n - t_{n-1} - t_{n-2} = 0$$

This is homogeneous recurrence with

$$k=2,\ a_o = 1,\ a_1 = 1,\ a_2 = -1$$

Property of Homogeneous Recurrences

- If *f(n)* and *g(n)* are solutions to eq. ① then, their linear combination is also a solution to eq. ①

$$a_0 f_n + a_1 f_{n-1} + a_2 f_{n-2} + \cdots + a_k f_{n-k} = 0$$

$$a_0 g_n + a_1 g_{n-1} + a_2 g_{n-2} + \cdots + a_k g_{n-k} = 0$$

Proof: Let

$t_n = cf_n + dg_n$ be solution of recurrence relation.

substituting t_n in eq. ①

Property of Homogeneous Recurrences

$t_n = a_0 (cf_n + dg_n) + a_1 (cf_{n-1} + dg_{n-1}) + a_2 (cf_{n-2} + dg_{n-2}) + \cdots + a_k (cf_{n-k} + dg_{n-k})$

$= c(a_0 f_n + a_1 f_{n-1} + a_2 f_{n-2} + \cdots + a_k f_{n-k}) +$

$\quad \mathrm{d}(a_0 g_n + a_1 g_{n-1} + a_2 g_{n-2} + \cdots + a_k g_{n-k})$

$= c(0) + d(0)$

$= 0$

Hence, linear combination of solutions to eq. ① is also a solution to eq. ①

Homogeneous Recurrences

To find solution of eq. ① homogeneous recurrence

Let $t_n = x^n$

So, substituting in eq. ①

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_k x^{n-k} = 0$$

Multistage Graph

- A Multistage Graph G = {V,E} is a directed weighted graph.
- Vertices V of Graph G are partitioned into k(≥2) disjoint sets

  $I_1, I_2, \ldots I_i \ldots. I_k$ such that $1 \le i \le k$

- If (u,v) is an edge E then $u \in V_i$ and $v \in Vi + 1$ *for some* $1 \le i \le k$ and c[i][j] is the cost of edge (i,j)
- Set $| I_1| = | I_k| = 1$

  Source Vertex(S) Sink Vertex(T)

- There are set of vertices in each stage in a multistage graph

Multistage Graph

- Cost of path from S to T is calculated as the sum of weights along the edge on the path.
- The multistage graph problem is to find minimum cost path from a given source node to destination node.
- The minimum path in multistage graph is obtained by taking the minimum path at each current stage by considering the path length of each vertex obtained in earlier stage.
- The multistage graph problem holds the principle of optimality.
- Application: used to represent resource allocation

Multistage Graph

<u>Use Recurrence Formula:</u>

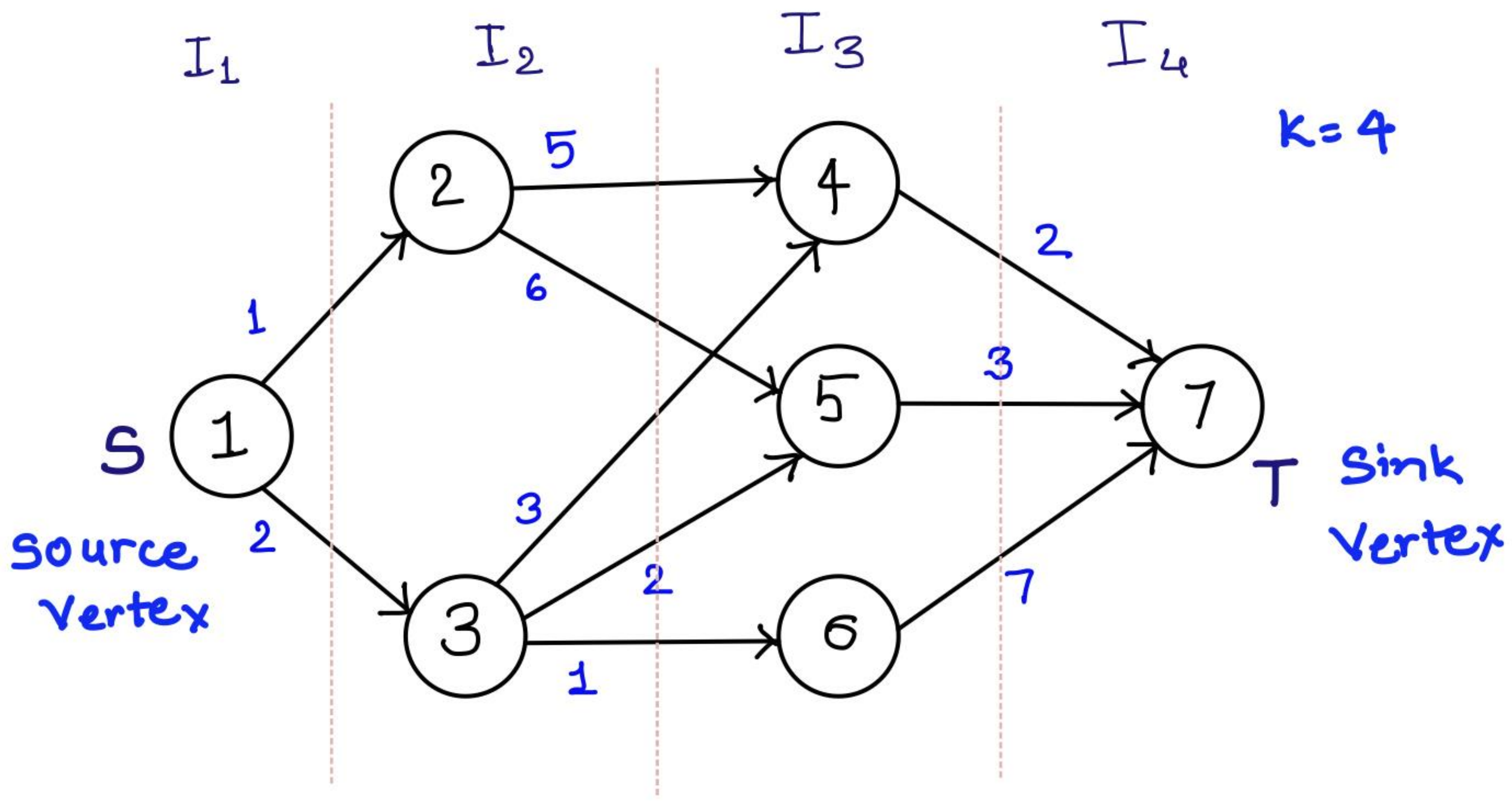
- If cost[i][j] is cost of path from vertex j in stage I to destination vertex T, then cost of each vertex on every stage can be calculated using following recurrence formula.

$$cost[i][j] = c[j][intermediate] + cost[i+1][intermediate]$$

OR

$$cost[i][j] = min\{c[j][intermediate] + cost[i+1][intermediate]\}$$

Multistage Graph: Example



Multistage Graph: Methods to Solve

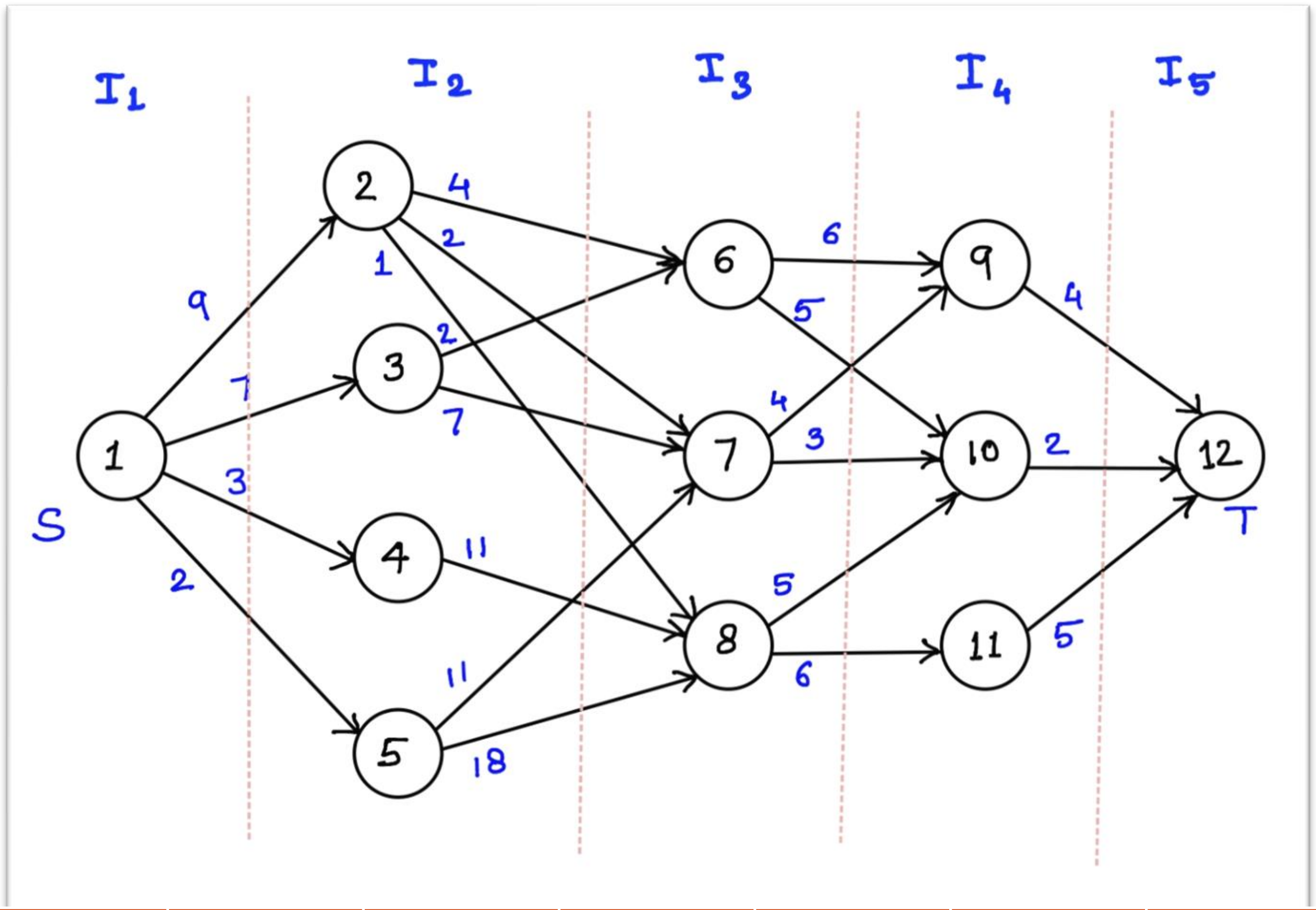
- **Forward Approach:**
  - here we start from destination vertex(i.e. Sink)

$cost[i][j] = min\{c[j][intermediate] + cost[i+1][intermediate]\}$

- **Backward Approach:**
  - Here we start from source vertex

$cost[i][j] = min\{c[intermediate][j] + cost[i-1][intermediate]\}$

Multistage Graph: (Forward Approach)



Vertex	1	2	3	4	5	6	7	8	9	10	11	12
cost												0
d												

Multistage Graph: (Forward Approach)

Stage 5:

i=5; Vertices(j) = {12}

cost[5][12]=0

Stage 4:

i=4; vertices(j) ={9,10,11}

cost[4][9]= c[9][12]+cost[5][12]

cost[4][9]=(4+0)=4

Similarly, cost[4][10] = 2

cost[4][11]=5

Stage 3:

i=3; Vertices(j) = {6,7,8}

cost[3][6]=min{(c[6][9] + cost [4][9]), (c[6][10] + cost [4][10])}

=min{(6+4),(5+2)} = 7

cost[3][7]= min{(c[7][9] + cost [4][9]), (c[7][10] + cost [4][10])}

=min{(4+4),(3+2)} = 5

cost[3][8]=min{(c[8][10] + cost [4][10]),(c[8][11] + cost [4][11])}

=min{(5+2),(6+5)} = 7

Vertex	1	2	3	4	5	6	7	8	9	10	11	12
cost						7	5	7	4	2	5	0
d						10	10	10	12	12	12	12

Multistage Graph: (Forward Approach)

Stage 2:

i=2; Vertices(j) = {2, 3, 4, 5}

cost[2][2]=min{(c[2][6] + cost [3][6]), (c[2][7] + cost [3][7]), (c[2][8] + cost [3][8]) }

=min{(4+7), (2+5), (1+7)} = min{11,7,8} = 7

cost[2][3]= min{(c[3][6] + cost [3][6]), (c[3][7] + cost [3][7])}

=min{(2+7),(7+5)} = min{9,11} = 9

cost[2][4]=min{(c[4][8] + cost [3][8]) }

=min{(11+7)} = 18

cost[2][5]=min{(c[5][7] + cost [3][7]),(c[5][8] + cost [3][8])}

=min{(11+5),(8+7)} = min{16,15} = 15

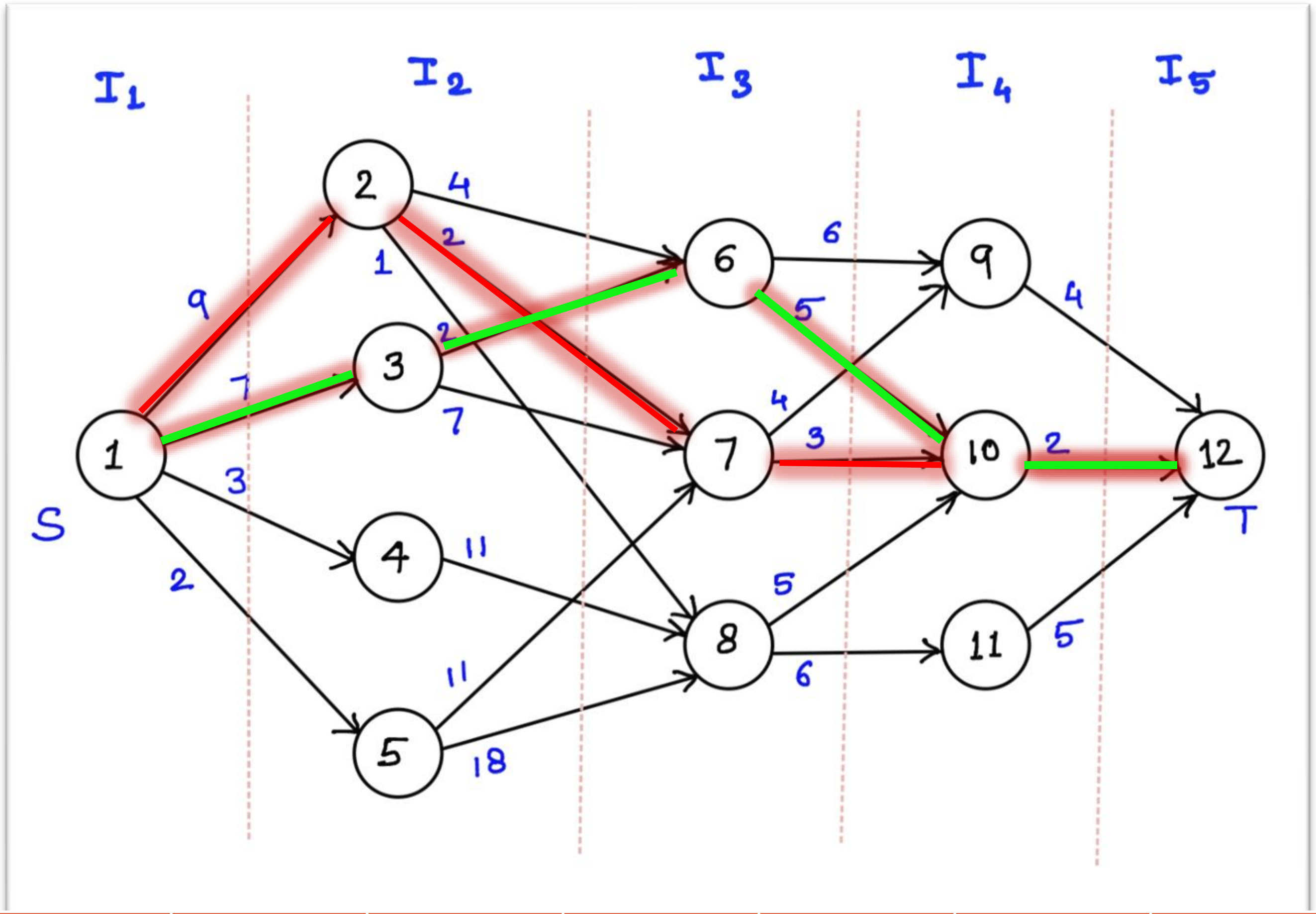
Stage 1:

i=1; Vertices(j) = {1}= Source

cost[1][1] = min{(c[1][2] + cost [2][2]),
(c[1][3] + cost [2][3]),
(c[1][4] + cost [2][4]),
(c[1][5] + cost [2][5])}
= min{(9+7), (7+9), (3+18) +(2+15)}
= min{16, 16, 21, 17} = 16

Vertex	1	2	3	4	5	6	7	8	9	10	11	12
cost	16	7	9	18	15	7	5	7	4	2	5	0
d	2 or 3	7	6	8	8	10	10	10	12	12	12	12

Multistage Graph: (Forward Approach)

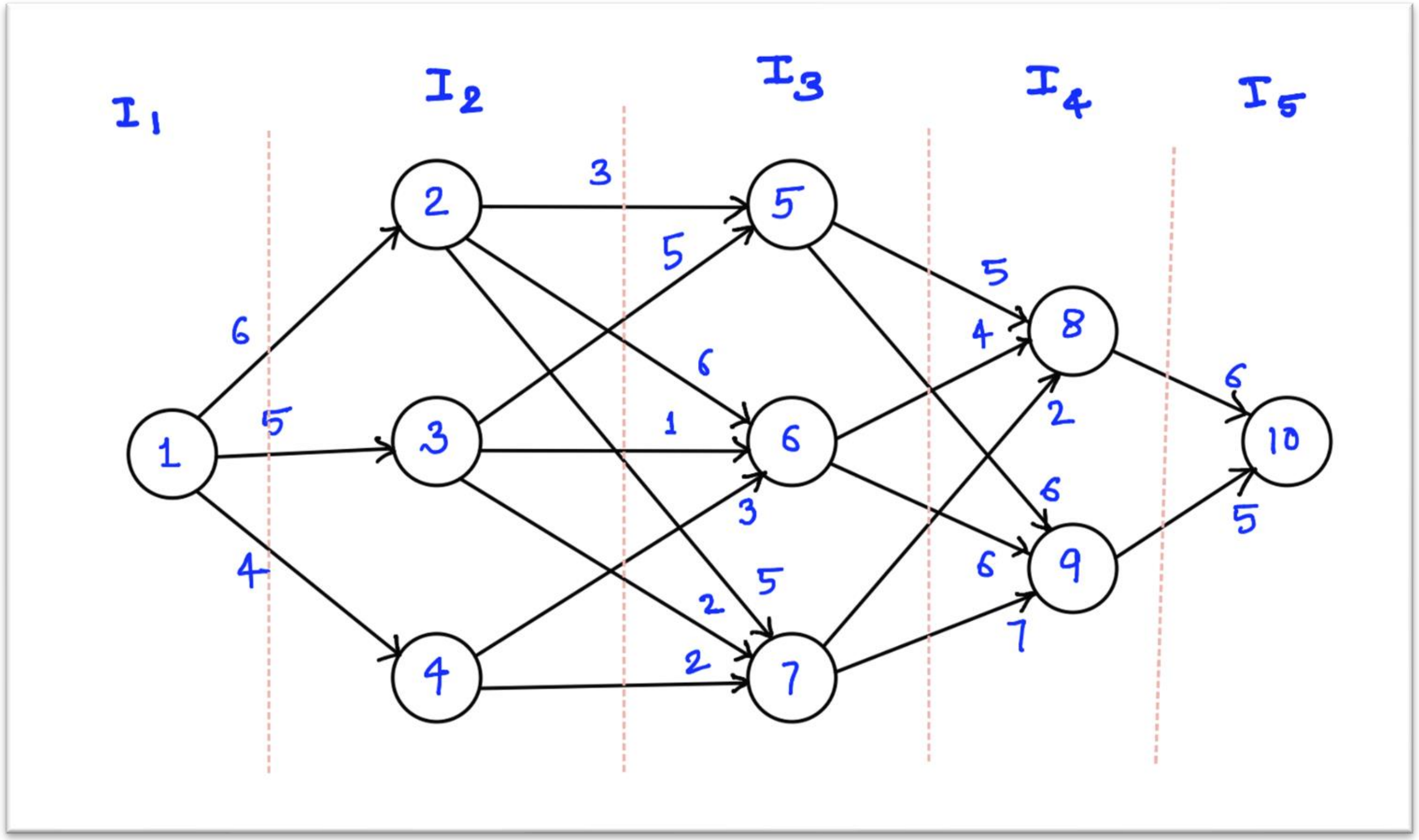


Shortest Path 1:
1 – 2 – 7 – 10 – 12

Shortest Path 2:
1 – 3 – 6 – 10 – 12

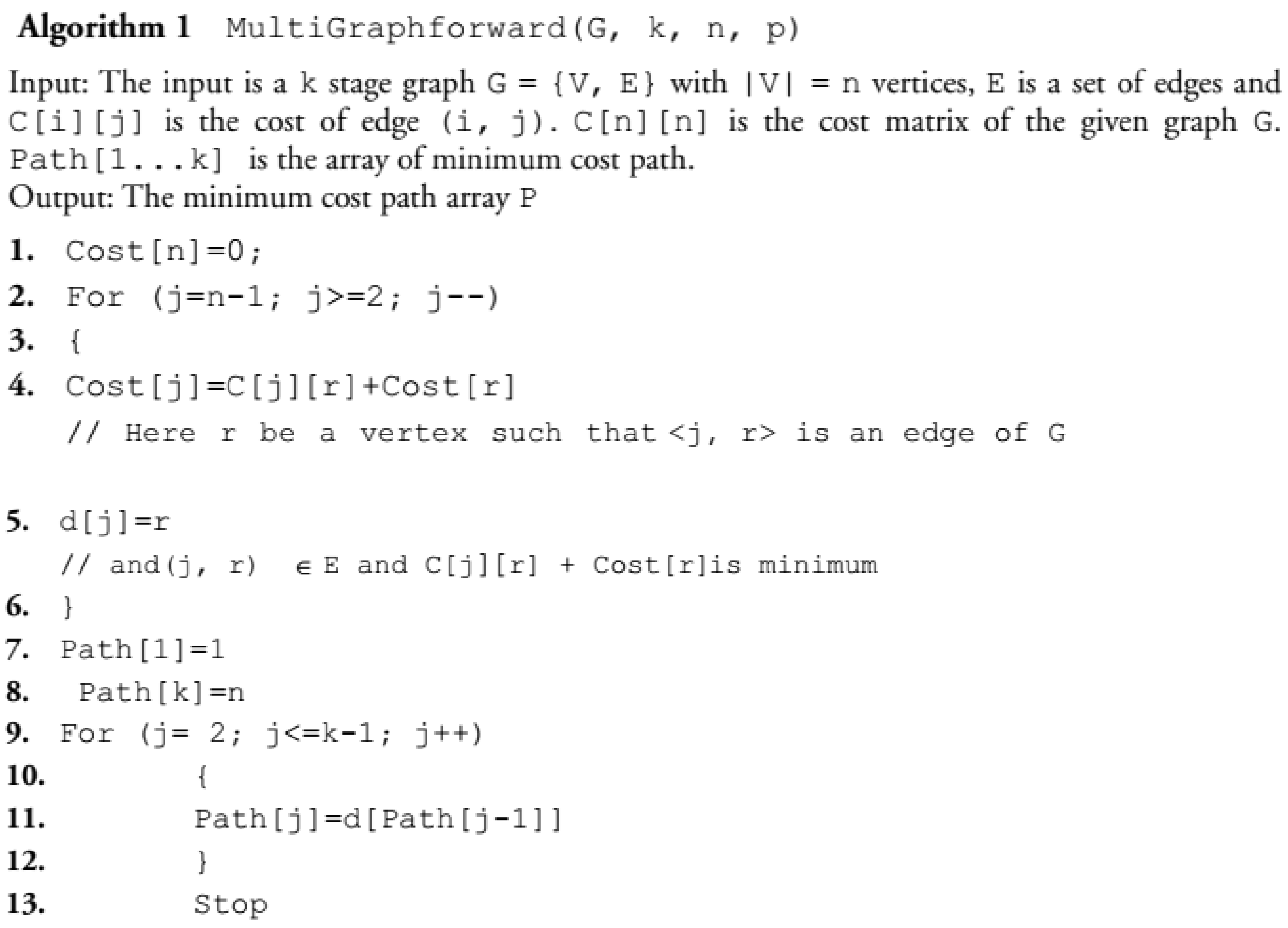
Vertex	1	2	3	4	5	6	7	8	9	10	11	12
cost	16	7	9	18	15	7	5	7	4	2	5	0
d	2 or 3	7	6	8	8	10	10	10	12	12	12	12

Multistage Graph: (Backward Approach)



Vertex	1	2	3	4	5	6	7	8	9	10
cost										
d										

Algorithm: (Forward Approach)



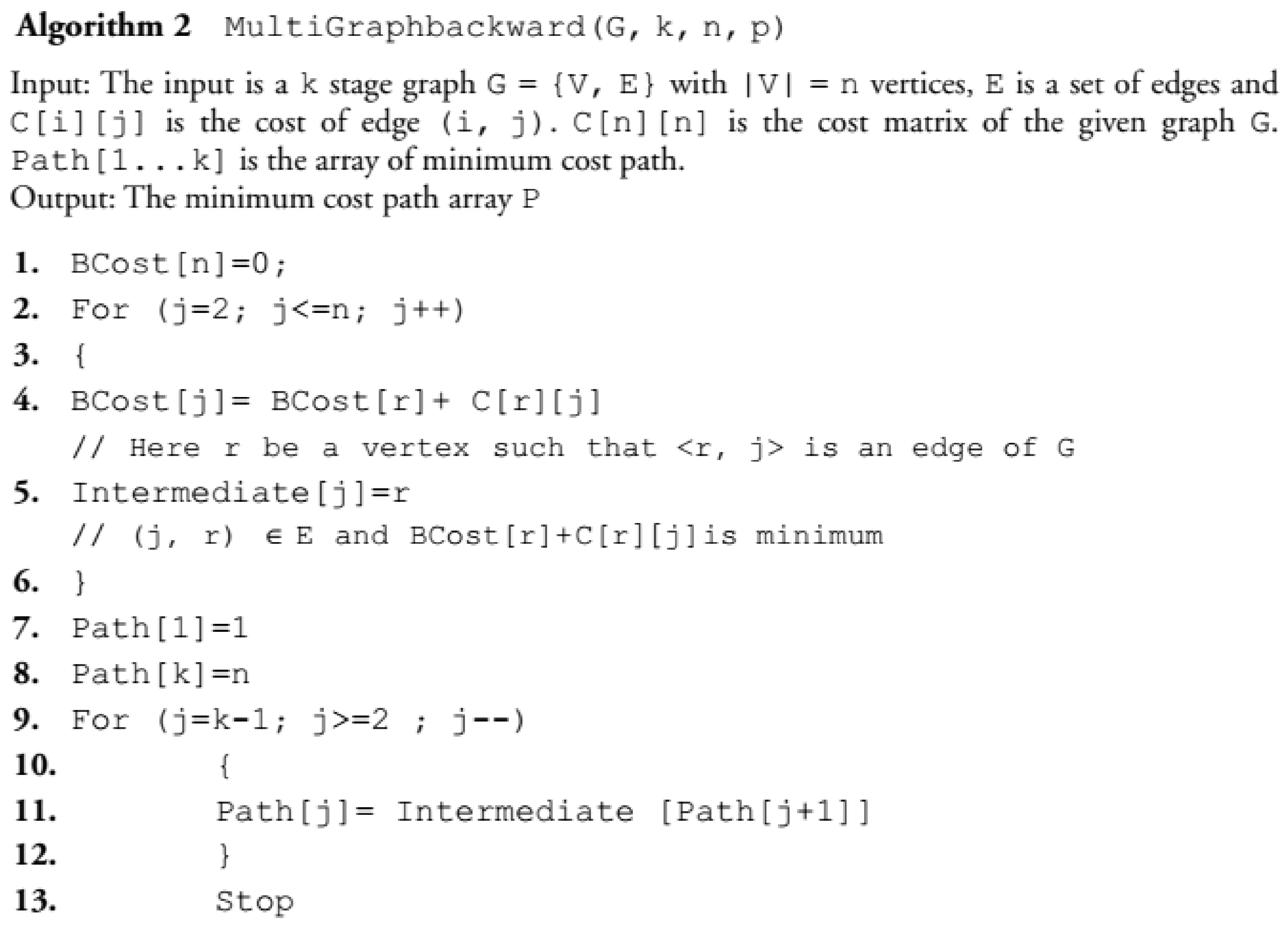
Algorithm 1 `MultiGraphforward(G, k, n, p)`

Input: The input is a k stage graph G = {V, E} with |V| = n vertices, E is a set of edges and C[i][j] is the cost of edge (i, j). C[n][n] is the cost matrix of the given graph G. Path[1...k] is the array of minimum cost path.
Output: The minimum cost path array P

```
1.  Cost[n]=0;
2.  For (j=n-1; j>=2; j--)
3.  {
4.  Cost[j]=C[j][r]+Cost[r]
    // Here r be a vertex such that <j, r> is an edge of G

5.  d[j]=r
    // and(j, r)  ∈ E and C[j][r] + Cost[r]is minimum
6.  }
7.  Path[1]=1
8.   Path[k]=n
9.  For (j= 2; j<=k-1; j++)
10.         {
11.         Path[j]=d[Path[j-1]]
12.         }
13.         Stop
```

Algorithm: (Backward Approach)



Algorithm 2 `MultiGraphbackward(G, k, n, p)`

Input: The input is a k stage graph G = {V, E} with |V| = n vertices, E is a set of edges and C[i][j] is the cost of edge (i, j). C[n][n] is the cost matrix of the given graph G. Path[1...k] is the array of minimum cost path.
Output: The minimum cost path array P

```
1.  BCost[n]=0;
2.  For (j=2; j<=n; j++)
3.  {
4.  BCost[j]= BCost[r]+ C[r][j]
    // Here r be a vertex such that <r, j> is an edge of G
5.  Intermediate[j]=r
    // (j, r) ∈ E and BCost[r]+C[r][j]is minimum
6.  }
7.  Path[1]=1
8.  Path[k]=n
9.  For (j=k-1; j>=2 ; j--)
10.         {
11.         Path[j]= Intermediate [Path[j+1]]
12.         }
13.         Stop
```

0/1 Knapsack Problem

- The objective of knapsack problem is to fill the knapsack with items to maximize the total value/profit, subject to its capacity.
- It is known as 0/1 knapsack problem , because we put one item into the knapsack or we will not include that item into the knapsack.
- Knapsack problem shows both overlapping subproblems and optimal substructure.
- The recursive definition for the optimum is given as follows:

$$S[k,u] \begin{cases} = 0 & \text{if } k=0 \text{ or } u=0 \\ = S[k-1,u] & \text{if } u<w_k \\ = \max\{S[k-1,u], S[k-1,u-w_k]+v_k\} & \text{if } u\geq w_k \end{cases}$$

0/1 Knapsack Problem: Example

Item (k)	**1**	**2**	**3**	**4**
Weight(w_k)	2	3	4	5
Value(v_k)	4	8	9	11

W = 5

Weight(u) → / **Item(k)**	**0**	**1**	**2**	**3**	**4**	**5**
0						
1						
2						
3						
4						

0/1 Knapsack Problem: Example

Weight(u) → Item(k)	0	1	2	3	4	5
0	0	0	0	0	0	0
1						
2						
3						
4						

Step 1: k = 0; u = {0…5}
k=0 , u = 0
S[k,u] = 0 for u={1,2,3,4,5}

0/1 Knapsack Problem: Example

Step 2: k = 1; u = {0…5}
$k=1$, $w_1=2$, $u = 0$;
$u<w_{1,}$
$S[k,u] = S[k-1,u]$
$S[1,0] = S[0,0] = 0$

Step 2: k = 1; u = {0…5}
$k=1$, $w_1=2$, $u = 1$;
$u<w_{1,}$
$S[k,u] = S[k-1,u]$
$S[1,1] = S[0,1] = 0$

Step 2: k = 1; u = {0…5}
$k=1$, $w_1=2$, $u = 2$;
$u=w_{1,}$
$S[k,u] = \max\{S[k-1,u], S[k-1, u-w_k] + v_k\}$
$S[1,2] = \max\{S[0,2], S[0, 0] + 4\} = \max\{0,0+4\} = 4$

Step 2: k = 1; u = {0…5}
$k=1$, $w_1=2$, $u = 3$;
$u>w_{1,}$
$S[k,u] = \max\{S[k-1,u], S[k-1, u-w_k] + v_k\}$
$S[1,3] = \max\{S[0,3], S[0,1] + 4\} = \max\{0,0+4\} = 4$

Step 2: k = 1; u = {0…5}
$k=1$, $w_1=2$, $u = 4$;
$u>w_{1,}$
$S[k,u] = \max\{S[k-1,u], S[k-1, u-w_k] + v_k\}$
$S[1,4] = \max\{S[0,4], S[0,2] + 4\} = \max\{0,0+4\} = 4$

Step 2: k = 1; u = {0…5}
$k=1$, $w_1=2$, $u = 5$;
$u>w_{1,}$
$S[k,u] = \max\{S[k-1,u], S[k-1, u-w_k] + v_k\}$
$S[1,5] = \max\{S[0,5], S[0,3] + 4\} = \max\{0,0+4\} = 4$

0/1 Knapsack Problem: Example

Weight(u) → Item(k)	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	4	4	4	4
2						
3						
4						

0/1 Knapsack Problem: Example

Step 3: k = 2; u = {0…5}
k=2, w_2=3, u = 0;
u<$w_{2,}$
S[k,u] = S[k-1,u]
S[2,0] = S[1,0] = 0

Step 3: k = 2; u = {0…5}
k=2, w_2=3, u = 1;
u<$w_{2,}$
S[k,u] = S[k-1,u]
S[2,1] = S[1,1] = 0

Step 3: k = 2; u = {0…5}
k=2, w_2=3, u = 2;
u<$w_{2,}$
S[k,u] = S[k-1,u]
S[2,2] = S[1,2]= 4

Step 3: k = 2; u = {0…5}
k=2, w_2=3, u = 3;
u=$w_{1,}$
S[k,u] = max{S[k-1,u], S[k-1, u-w_k] + v_k}
S[2,3] = max{S[1,3], S[1,0] + 8} = max{4,0+8} = 8

Step 3: k = 2; u = {0…5}
k=2, w_2=3, u = 4;
u>$w_{1,}$
S[k,u] = max{S[k-1,u], S[k-1, u-w_k] + v_k}
S[2,4] = max{S[1,4], S[1,1] + 8} = max{4,0+8} = 8

Step 3: k = 2; u = {0…5}
k=2, w_2=3, u = 5;
u>$w_{1,}$
S[k,u] = max{S[k-1,u], S[k-1, u-w_k] + v_k}
S[2,5] = max{S[1,5], S[1,2] + 8} = max{4,4+8} = 12

0/1 Knapsack Problem: Example

Weight(u) → / Item(k)	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	4	4	4	4
2	0	0	4	8	8	12
3						
4						

0/1 Knapsack Problem: Example

Step 4: k = 3; u = {0…5}
k=3, w_3=4, u = 0;
$u<w_3$,
S[k,u] = S[k-1,u]
S[3,0] = S[2,0] = 0

Step 4: k = 3; u = {0…5}
k=3, w_3=4, u = 1;
$u<w_3$,
S[k,u] = S[k-1,u]
S[3,1] = S[2,1] = 0

Step 4: k = 3; u = {0…5}
k=3, w_3=4, u = 2;
$u<w_3$,
S[k,u] = S[k-1,u]
S[3,2] = S[2,2]= 4

Step 4: k = 3; u = {0…5}
k=3, w_3=4, u = 3;
$u<w_3$,
S[k,u] = S[k-1,u]
S[3,3] = S[2,3] = 8

Step 4: k = 3; u = {0…5}
k=3, w_3=4, u = 4;
$u=w_3$,
S[k,u] = max{S[k-1,u], S[k-1, u-w_k] + v_k}
S[3,4] = max{S[2,4], S[2,0] + 8} = max{8,0+9} = 9

Step 4: k = 3; u = {0…5}
k=3, w_3=4, u = 5;
$u>w_3$,
S[k,u] = max{S[k-1,u], S[k-1, u-w_k] + v_k}
S[3,5] = max{S[2,5], S[2,1] +11 } = max{12,0+11} = 12

0/1 Knapsack Problem: Example

Weight(u) → Item(k)	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	4	4	4	4
2	0	0	4	8	8	12
3	0	0	4	8	9	12
4						

0/1 Knapsack Problem: Example

Step 5: k = 4; u = {0…5}
k=4, w_4=5, u = 0;
u<$w_{4,}$
S[k,u] = S[k-1,u]
S[4,0] = S[3,0] = 0

Step 5: k = 4; u = {0…5}
k=4, w_4=5, u = 1;
u<$w_{4,}$
S[k,u] = S[k-1,u]
S[4,1] = S[3,1] = 0

Step 5: k =4; u = {0…5}
k=4, w_4=5, u = 2;
u<$w_{4,}$
S[k,u] = S[k-1,u]
S[4,2] = S[3,2]= 4

Step 5: k = 4; u = {0…5}
k=4, w_4=5, u = 3;
u<$w_{4,}$
S[k,u] = S[k-1,u]
S[4,3] = S[3,3] = 8

Step 5: k = 4; u = {0…5}
k=4, w_4=5, u = 4;
u<$w_{4,}$
S[k,u] = S[k-1,u]
S[4,4] = S[3,4] = 9

Step 5: k = 4; u = {0…5}
k=4, w_4=5, u = 5;
u=$w_{4,}$
S[k,u] = max{S[k-1,u], S[k-1, u-w_k] + v_k}
S[4,5] = max{S[3,5], S[3,0] +11 } = max{12,0+11} = 12

0/1 Knapsack Problem: Example

Weight(u) → Item(k)	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	4	4	4	4
2	0	0	4	8	8	12
3	0	0	4	8	9	12
4	0	0	4	8	9	12

optimal profit of the given knapsack problem is 12

0/1 Knapsack Problem

To find the items of the optimal cost:

```
if S[k][u] ≠ S[k - 1][u],            //S[k][u] and S[k - 1][u] are
                                     different
then, kth item is selected i.e.      //k=n and u=W(Capacity of Knapsack)
        print k and set $u = u - w_k$
k = k–1
if S[k][u] = S[k - 1][u] then,
      set k = k–1
```

0/1 Knapsack Problem: Example

Weight(u) → Item(k)	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	4	4	4	4
2	0	0	4	8	8	12
3	0	0	4	8	9	12
4	0	0	4	8	9	12

optimal profit

S[4][5]= S[3][5]
So, k=4-1=3

S[3][5]= S[2][5]
So, k=3-1=2

S[2][5]≠ S[1][5]
So, k=2-1=1 & $u=u-w_2$ =5-3=2

S[1][2]≠ S[0][2]
So, k=1-1=0 & $u=u-w_2$ =2-2=0

I = { 1 , 2 , 3 , 4 }

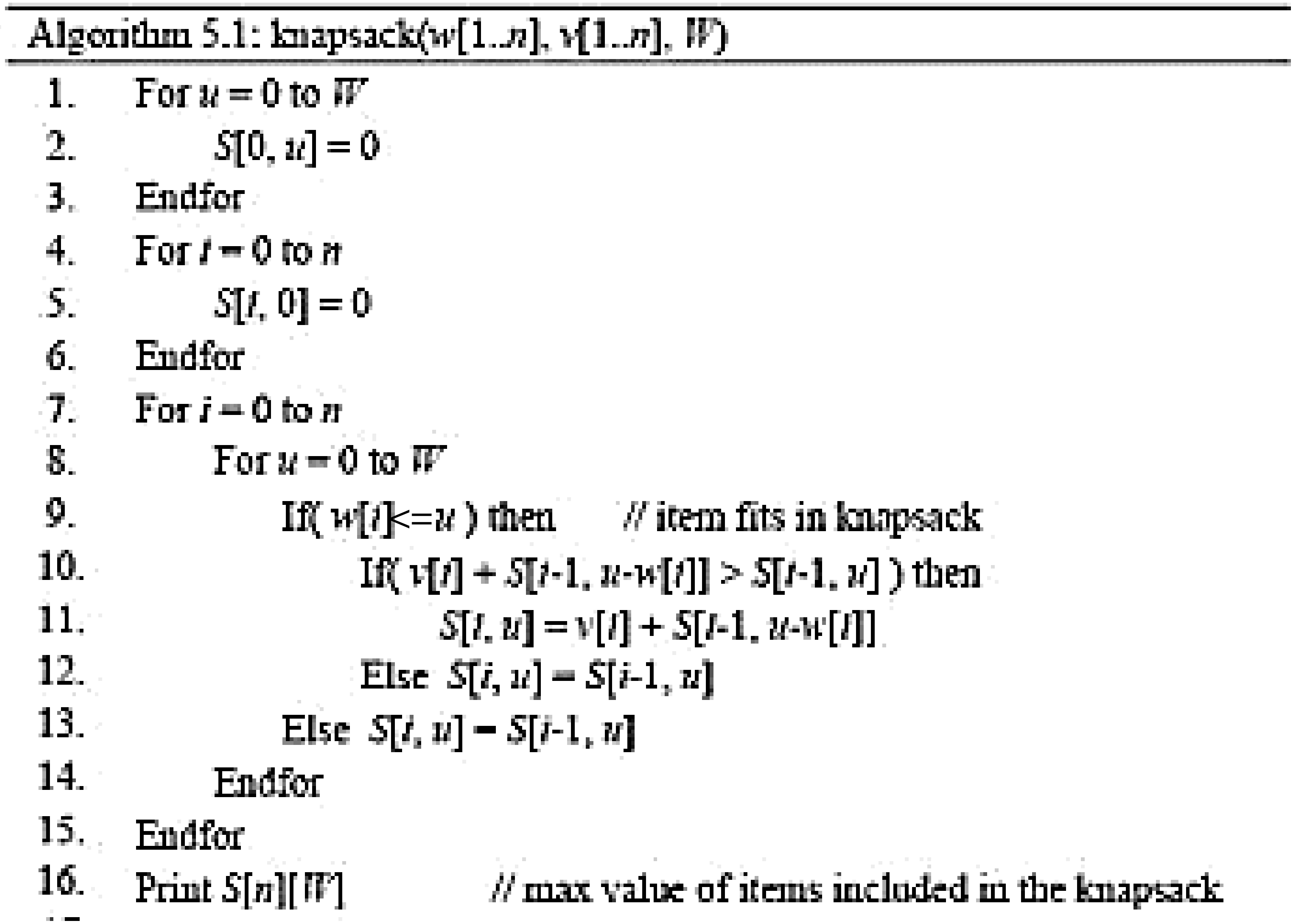
I = { 0 , 0 , 0 , 0 }

I = { 0 , 0 , 0 , 0 }

I = { 0 , 1 , 0 , 0 }

I = { 1 , 1 , 0 , 0 }

0/1 Knapsack Problem: Algorithm



Algorithm 5.1: knapsack($w[1..n]$, $v[1..n]$, W)

```
1.  For $u = 0$ to $W$
2.      $S[0, u] = 0$
3.  Endfor
4.  For $i = 0$ to $n$
5.      $S[i, 0] = 0$
6.  Endfor
7.  For $i = 0$ to $n$
8.      For $u = 0$ to $W$
9.          If( $w[i] <= u$ ) then      // item fits in knapsack
10.             If( $v[i] + S[i-1, u-w[i]] > S[i-1, u]$ ) then
11.                 $S[i, u] = v[i] + S[i-1, u-w[i]]$
12.             Else  $S[i, u] = S[i-1, u]$
13.         Else  $S[i, u] = S[i-1, u]$
14.     Endfor
15. Endfor
16. Print $S[n][W]$             // max value of items included in the knapsack
```

0/1 Knapsack Problem: Algorithm

```
// Find knapsack items
$i = n,\ k = W$
While( $i > 0$ and $k > 0$)
    If( $S[i][k] \neq S[i-1][k]$ )
        Print $i$                // item $i$ is in the knapsack
        $k = k - w[i]$
    Endif
    $i = i - 1$
Endwhile
```

All Pair Shortest Path

Floyd-Warshall Algorithm:

- Floyd’s Algorithm is for finding shortest path between every pair of vertices of a graph.
- Works for both directed and undirected graph.
- The graph may contain negative edges, but it should not contain negative cycles.
- Floyd’s algorithm requires a weighted graph
- Floyd’s algorithm computes distance matrix of weighted graph.
- It takes dynamic programming approach and runs in $O(n^3)$ times.

All Pair Shortest Path

Floyd-Warshall Algorithm:

- Distance matrix of a weighted graph is computed with n vertices through a series of $n \times n$ matrices

  D^0, D^1 , D^2 , D^3 ,….. D^k,…… D^{n-1}

where,

D^0 is weighted matrix with no intermediate vertex in its path between source i to destination j.

D^k is weighted matrix with k as intermediate vertex between i and j, k vary from $1\ to\ k-1$, and

D^n is last matrix contains length of shortest paths among all paths that can use all vertices as intermediate

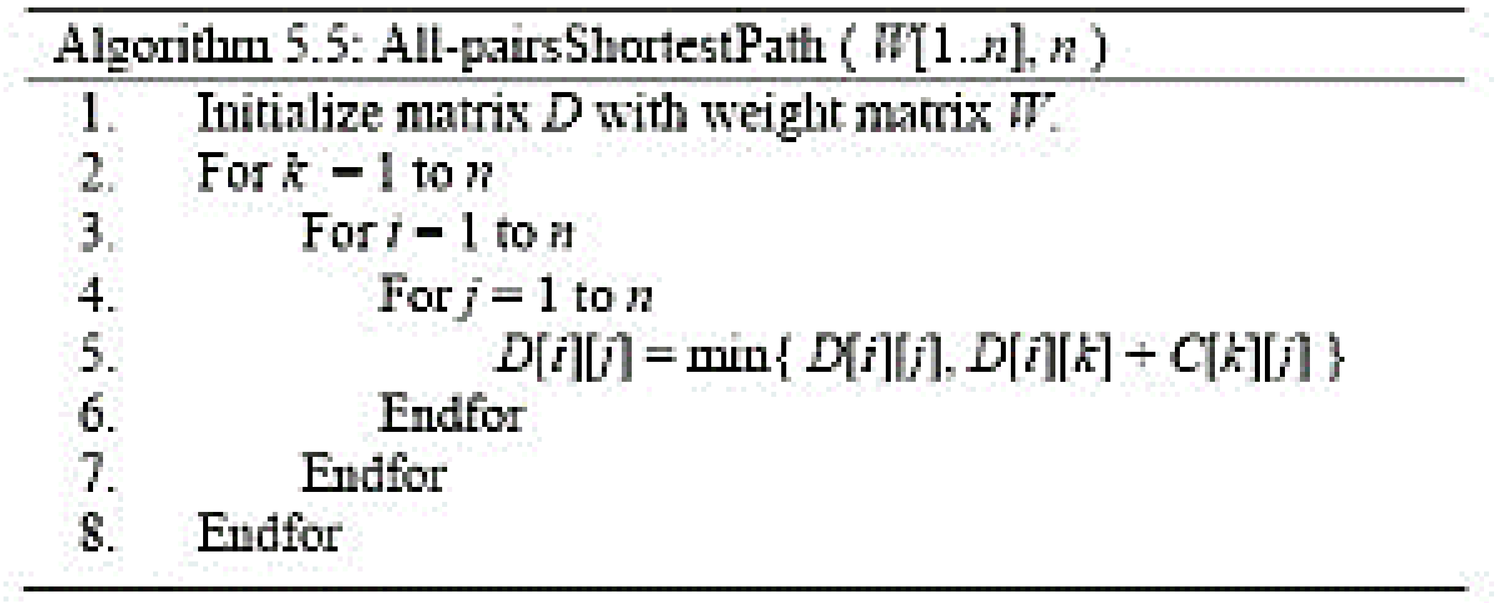
All Pair Shortest Path: Algorithm

Floyd-Warshall Algorithm:

Recursive formula:

$$D^k[i][j] = min\{D^{k-1}[i][j], D^{k-1}[i][k] + D^{k-1}[k][j]\}$$

Here, old values of D^{k-1} is overwritten by new values of D^k



Algorithm 5.5: All-pairsShortestPath ($W[1..n]$, n)

```
1.  Initialize matrix $D$ with weight matrix $W$.
2.  For $k = 1$ to $n$
3.      For $i = 1$ to $n$
4.          For $j = 1$ to $n$
5.              $D[i][j] = \min\{ D[i][j], D[i][k] + C[k][j] \}$
6.          Endfor
7.      Endfor
8.  Endfor
```

All Pair Shortest Path: Example

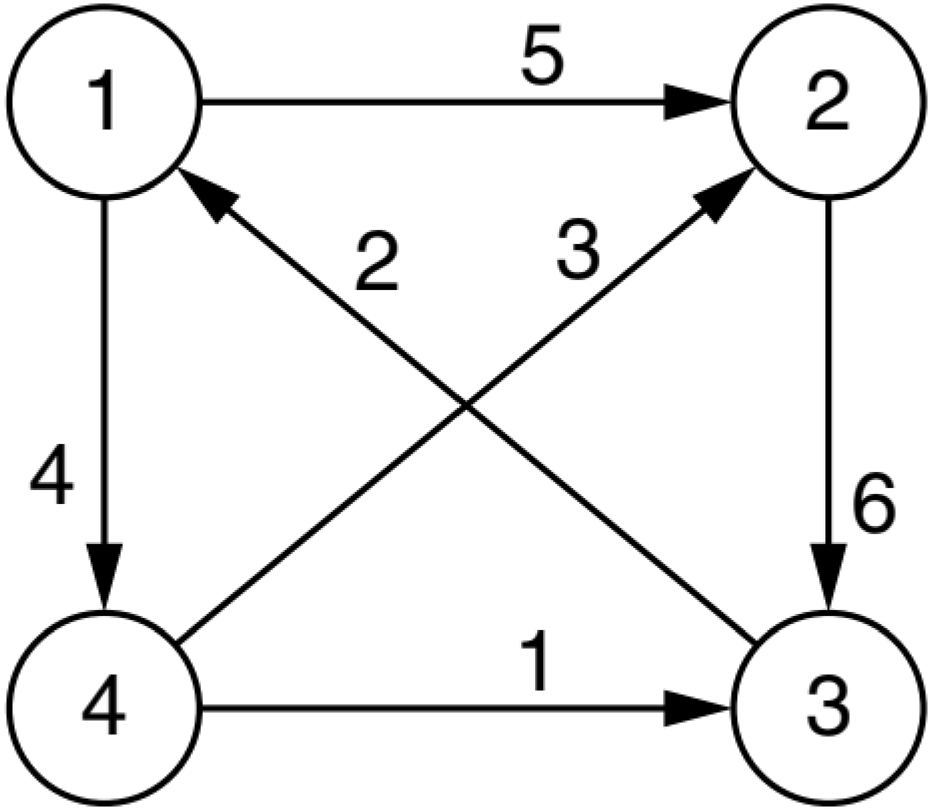
Floyd-Warshall Algorithm:

Recursive formula:

$$D^k[i][j] = min\{D^{k-1}[i][j], D^{k-1}[i][k] + D^{k-1}[k][j]\}$$

Here, old values of D^{k-1} is overwritten by new values of D^k

$$D^0 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & d_{11} & d_{12} & d_{13} & d_{14} \\ 2 & d_{21} & d_{22} & d_{23} & d_{24} \\ 3 & d_{31} & d_{32} & d_{33} & d_{34} \\ 4 & d_{41} & d_{42} & d_{43} & d_{44} \end{array} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 5 & \infty & 4 \\ 2 & \infty & 0 & 6 & \infty \\ 3 & 2 & \infty & 0 & \infty \\ 4 & \infty & 3 & 1 & 0 \end{array}$$

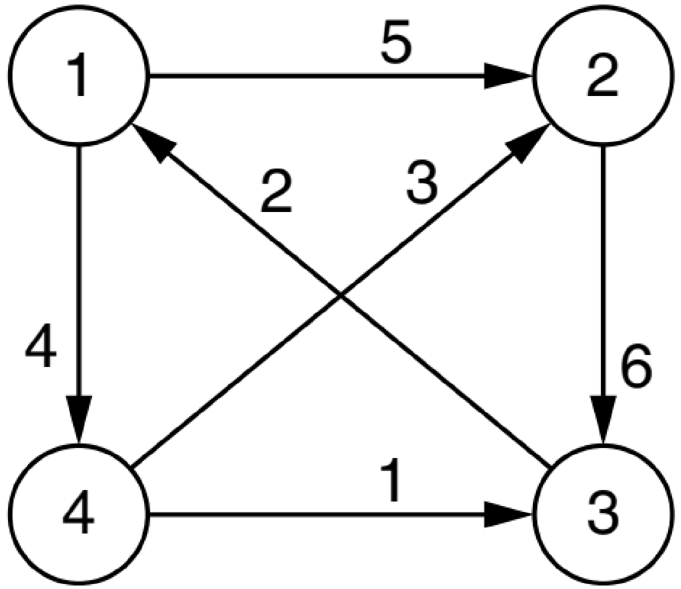


All Pair Shortest Path: Example

Floyd-Warshall Algorithm:

Path matrix P is also initialized to NULL (0) entries.

$$P^0 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$



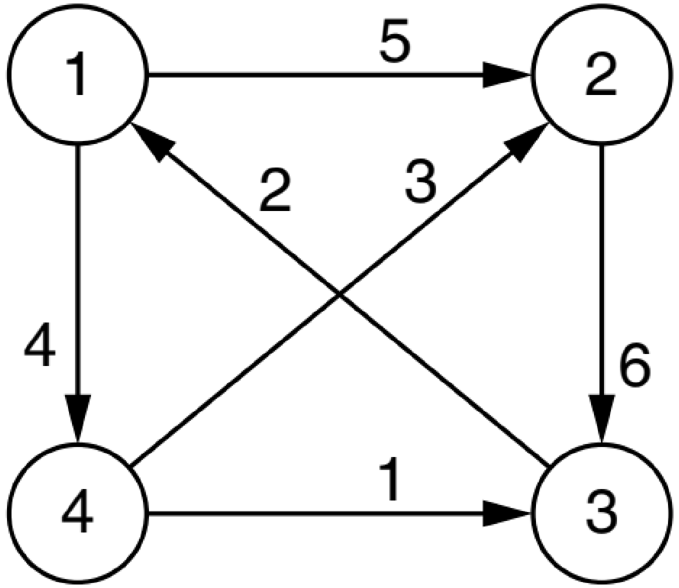
All Pair Shortest Path: Example

Floyd-Warshall Algorithm: D¹ : Node 1 as intermediate node

$$P^1 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

$$D^0 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ 1 & d_{11} & d_{12} & d_{13} & d_{14} \\ 2 & d_{21} & d_{22} & d_{23} & d_{24} \\ 3 & d_{31} & d_{32} & d_{33} & d_{34} \\ 4 & d_{41} & d_{42} & d_{43} & d_{44} \end{array} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ 1 & 0 & 5 & \infty & 4 \\ 2 & \infty & 0 & 6 & \infty \\ 3 & 2 & \infty & 0 & \infty \\ 4 & \infty & 3 & 1 & 0 \end{array}$$

$$D^1 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ 1 & d_1{}^{(1)}{}_1 & d_1{}^{(1)}{}_2 & d_1{}^{(1)}{}_3 & d_1{}^{(1)}{}_4 \\ 2 & d_2{}^{(1)}{}_1 & d_2{}^{(1)}{}_2 & d_2{}^{(1)}{}_3 & d_2{}^{(1)}{}_4 \\ 3 & d_3{}^{(1)}{}_1 & d_3{}^{(1)}{}_2 & d_3{}^{(1)}{}_3 & d_3{}^{(1)}{}_4 \\ 4 & d_4{}^{(1)}{}_1 & d_4{}^{(1)}{}_2 & d_4{}^{(1)}{}_3 & d_4{}^{(1)}{}_4 \end{array} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ 1 & 0 & 5 & \infty & 4 \\ 2 & \infty & 0 & 6 & \infty \\ 3 & 2 & 7 & 0 & \mathbf{6} \\ 4 & \infty & 3 & 1 & 0 \end{array}$$



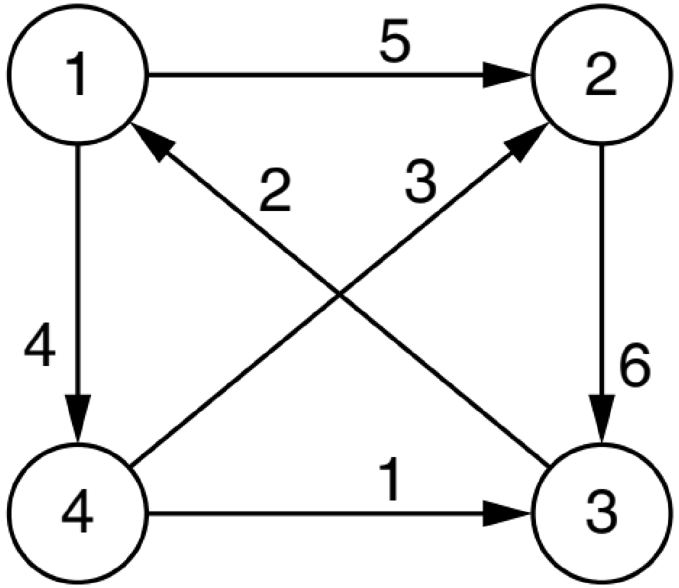
All Pair Shortest Path: Example

Floyd-Warshall Algorithm:

D^2 : Node 2 as intermediate node

$$P^2 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & \mathbf{2} & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & \mathbf{1} & 0 & \mathbf{1} \\ \mathbf{4} & 0 & 0 & 0 & 0 \end{array}$$

$$D^2 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & d_{1\ \ 1}^{(1,2)} & d_{1\ \ 2}^{(1,2)} & d_{1\ \ 3}^{(1,2)} & d_{1\ \ 4}^{(1,2)} \\ 2 & d_{2\ \ 1}^{(1,2)} & d_{2\ \ 2}^{(1,2)} & d_{2\ \ 3}^{(1,2)} & d_{2\ \ 4}^{(1,2)} \\ 3 & d_{3\ \ 1}^{(1,2)} & d_{3\ \ 2}^{(1,2)} & d_{3\ \ 3}^{(1,2)} & d_{3\ \ 4}^{(1,2)} \\ 4 & d_{4\ \ 1}^{(1,2)} & d_{4\ \ 2}^{(1,2)} & d_{4\ \ 3}^{(1,2)} & d_{4\ \ 4}^{(1,2)} \end{array} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 5 & \mathbf{11} & 4 \\ 2 & \infty & 0 & 6 & \infty \\ 3 & 2 & 7 & 0 & 6 \\ 4 & \infty & 3 & 1 & 0 \end{array}$$



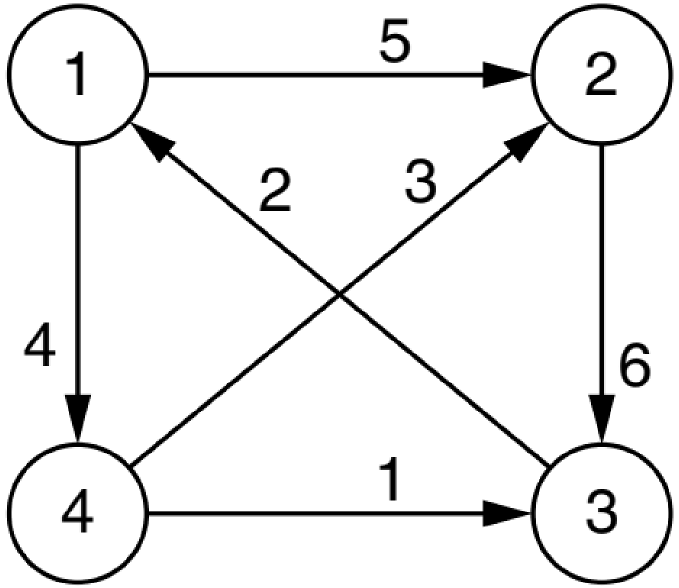
All Pair Shortest Path: Example

Floyd-Warshall Algorithm:

D³ : Node 3 as intermediate node

$$P^3 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & \mathbf{2} & 0 \\ 2 & \mathbf{3} & 0 & 0 & \mathbf{3} \\ 3 & 0 & \mathbf{1} & 0 & \mathbf{1} \\ 4 & \mathbf{3} & 0 & 0 & 0 \end{array}$$

$$D^3 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & d_1^{(1,2,3)}{}_1 & d_1^{(1,2,3)}{}_2 & d_1^{(1,2,3)}{}_3 & d_1^{(1,2,3)}{}_4 \\ 2 & d_2^{(1,2,3)}{}_1 & d_2^{(1,2,3)}{}_2 & d_2^{(1,2,3)}{}_3 & d_2^{(1,2,3)}{}_4 \\ 3 & d_3^{(1,2,3)}{}_1 & d_3^{(1,2,3)}{}_2 & d_3^{(1,2,3)}{}_3 & d_3^{(1,2,3)}{}_4 \\ 4 & d_4^{(1,2,3)}{}_1 & d_4^{(1,2,3)}{}_2 & d_4^{(1,2,3)}{}_3 & d_4^{(1,2,3)}{}_4 \end{array} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 5 & 11 & 4 \\ 2 & \mathbf{8} & 0 & 6 & \mathbf{12} \\ 3 & 2 & 7 & 0 & 6 \\ 4 & \mathbf{3} & 3 & 1 & 0 \end{array}$$

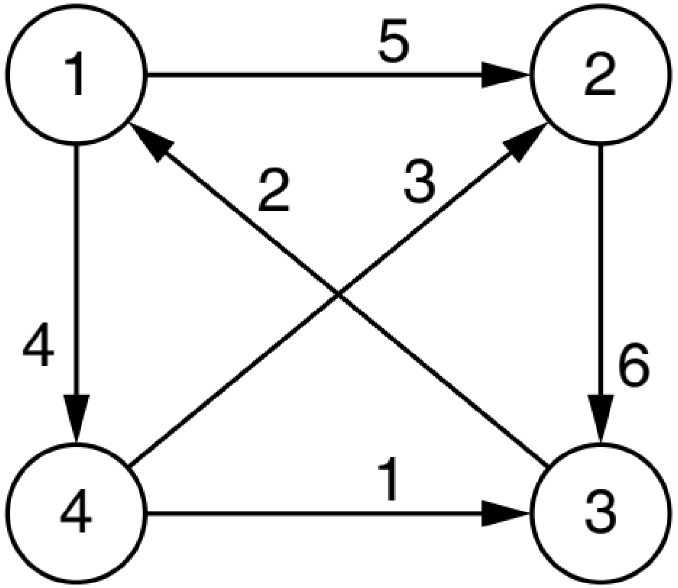


All Pair Shortest Path: Example

D4 : Node 4 as intermediate node

$$P^4 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & \mathbf{4} & 0 \\ 2 & \mathbf{3} & 0 & 0 & \mathbf{3} \\ 3 & 0 & 1 & 0 & 1 \\ 4 & \mathbf{3} & 0 & 0 & 0 \end{array}$$

$$D^4 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & d_1{}^{(1,2,3,4)}{}_1 & d_1{}^{(1,2,3,4)}{}_2 & d_1{}^{(1,2,3,4)}{}_3 & d_1{}^{(1,2,3,4)}{}_4 \\ 2 & d_2{}^{(1,2,3,4)}{}_1 & d_2{}^{(1,2,3,4)}{}_2 & d_2{}^{(1,2,3,4)}{}_3 & d_2{}^{(1,2,3,4)}{}_4 \\ 3 & d_3{}^{(1,2,3,4)}{}_1 & d_3{}^{(1,2,3,4)}{}_2 & d_3{}^{(1,2,3,4)}{}_3 & d_3{}^{(1,2,3,4)}{}_4 \\ 4 & d_4{}^{(1,2,3,4)}{}_1 & d_4{}^{(1,2,3,4)}{}_2 & d_4{}^{(1,2,3,4)}{}_3 & d_4{}^{(1,2,3,4)}{}_4 \end{array} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 5 & \mathbf{5} & 4 \\ 2 & 8 & 0 & 6 & 12 \\ 3 & 2 & 7 & 0 & 6 \\ 4 & 3 & 3 & 1 & 0 \end{array}$$



Single Source Shortest Path

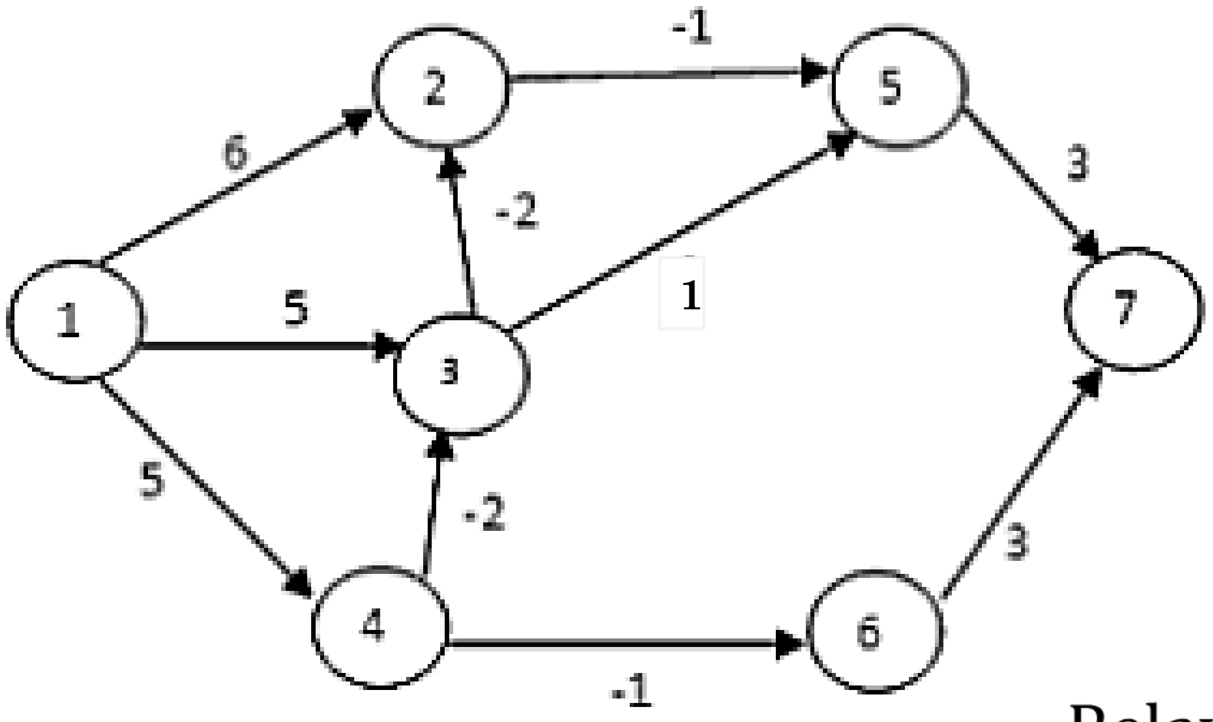
Bellman Ford Algorithm:

- If graph has negative edge cost, then Dijkstra’s Algorithm does not work.
- Bellman ford proposed an algorithm, which is combination of Dijkstra’s algorithm and unweighted algorithms.
- Slower than Dijkstra’s but more versatile.
- If there exists negative cycle in a graph, then there will be no shortest path.
- Relaxation Formula:

```
If (d[v]> d[u]+cost[u,v])
    then, d[v]= d[u]+cost[u,v]
```

Single Source Shortest Path

Bellman Ford Algorithm: Example



EdgeList={(1,2), (1,3), (1,4), (4,3), (3,2), (2,5), (3,5), (4,6), (5,7), (6,7) }

- Relaxation Formula:

```
If (d[v]> d[u]+cost[u,v])
    then, d[v]= d[u]+cost[u,v]
```

Single Source Shortest Path

Bellman Ford Algorithm: Example

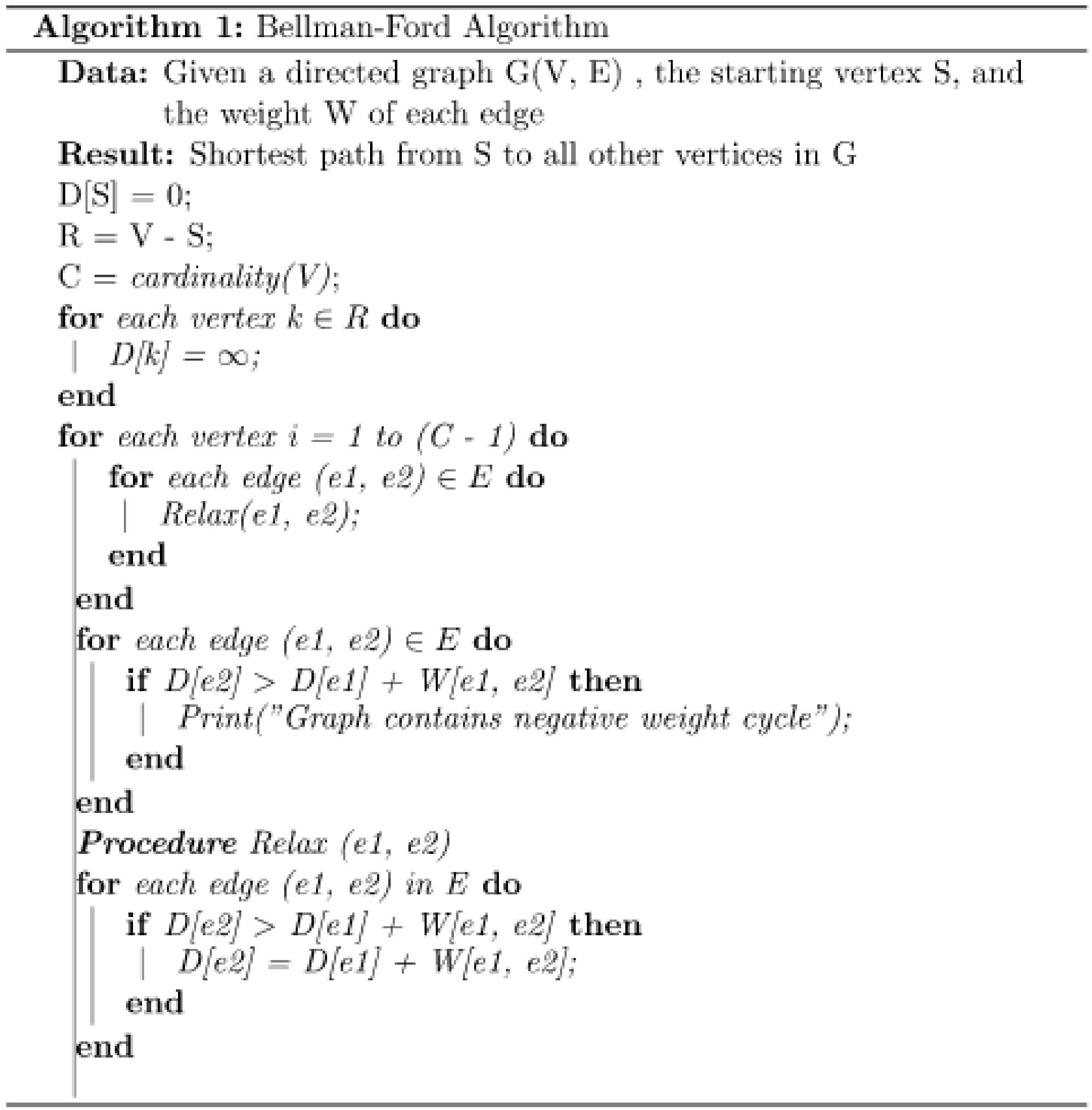
EdgeList={(1,2), (1,3), (1,4), (4,3), (3,2), (2,5), (3,5), (4,6), (5,7), (6,7) }

		1	2	3	4	5	6	7
Initialization	0	0	∞	∞	∞	∞	∞	∞
k=	1	0	**6**	**5**	**5**	∞	∞	∞
	2	0	**3**	**3**	5	**5**	**4**	∞
	3	0	**1**	3	5	**2**	4	**5**
	4	0	1	3	5	**0**	4	**3**
	5	0	1	3	5	0	4	3
	6	0	1	3	5	0	4	3

Path Length (k) = |V|-1 = 7-1 = 6

Single Source Shortest Path

Bellman Ford Algorithm:



Algorithm 1: Bellman-Ford Algorithm

```
Data: Given a directed graph G(V, E) , the starting vertex S, and
      the weight W of each edge
Result: Shortest path from S to all other vertices in G
D[S] = 0;
R = V - S;
C = cardinality(V);
for each vertex k ∈ R do
  | D[k] = ∞;
end
for each vertex i = 1 to (C - 1) do
  | for each edge (e1, e2) ∈ E do
  |   | Relax(e1, e2);
  | end
 end
 for each edge (e1, e2) ∈ E do
  | if D[e2] > D[e1] + W[e1, e2] then
  |   | Print("Graph contains negative weight cycle");
  | end
 end
 Procedure Relax (e1, e2)
 for each edge (e1, e2) in E do
  | if D[e2] > D[e1] + W[e1, e2] then
  |   | D[e2] = D[e1] + W[e1, e2];
  | end
 end
```

The Travelling Salesman Problem

- Trying all the possibilities – **Brute Force Approach**.
- There are **n** cities
- Salesman has to visit all cities, starting from a city.
- He is not allowed to visit any city more than once and come back to starting city.
- Cost of travelling cities is given.
- We have to find tour of salesman with lowest cost.

The Travelling Salesman Problem

- Let **G(V,E)** be the directed graph such that $|V|=n$.
- Vertices of a graph represent cities
- Weight associated with each edge gives cost of traversal(represented by cost matrix) between the cities connected by the edge.
- Let starting vertex is 1.
- Tour starts at vertex 1 and goes to $k \in V - \{1\}$
- From vertex k, there is tour to vertex 1, which goes through vertex $V - \{1, k\}$, only once.
- Tour from vertex k to 1 must be optimal.
- This problem satisfies optimal substructure property.

The Travelling Salesman Problem

- Recursive Formula:

$$\mathbf{SP(i, S)} = \min_{\boldsymbol{k \in S}} \left[\boldsymbol{d(i, k)} + \boldsymbol{SP(k, S - \{k\})} \right]$$

where,

V= set of all vertices

S = set of remaining or unvisited vertices →subset of V

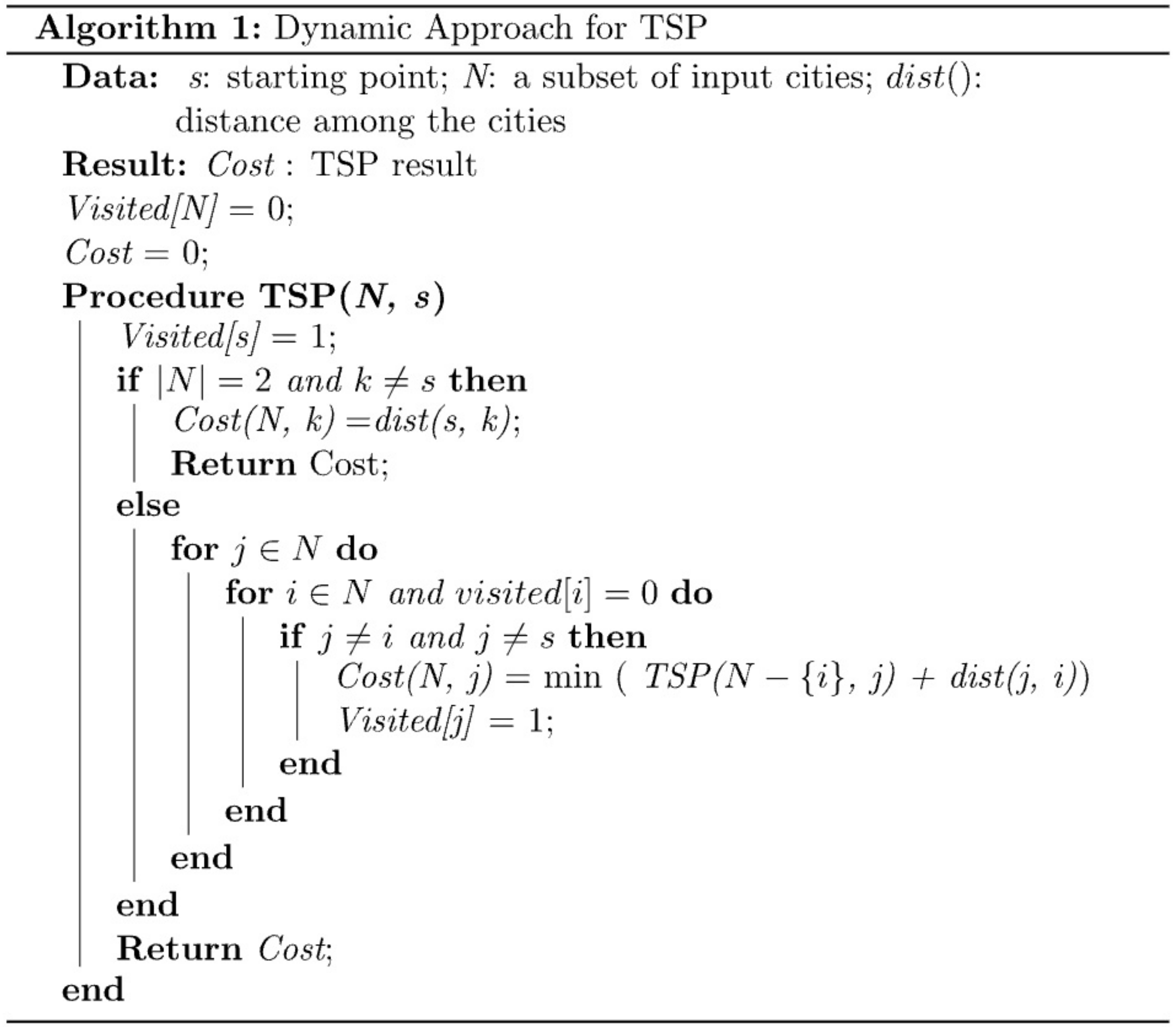
i = source vertex

k = intermediate vertex

and, SP(i, S) = shortest path from i to all other unvisited vertices.

The Travelling Salesman Problem

- Algorithm:

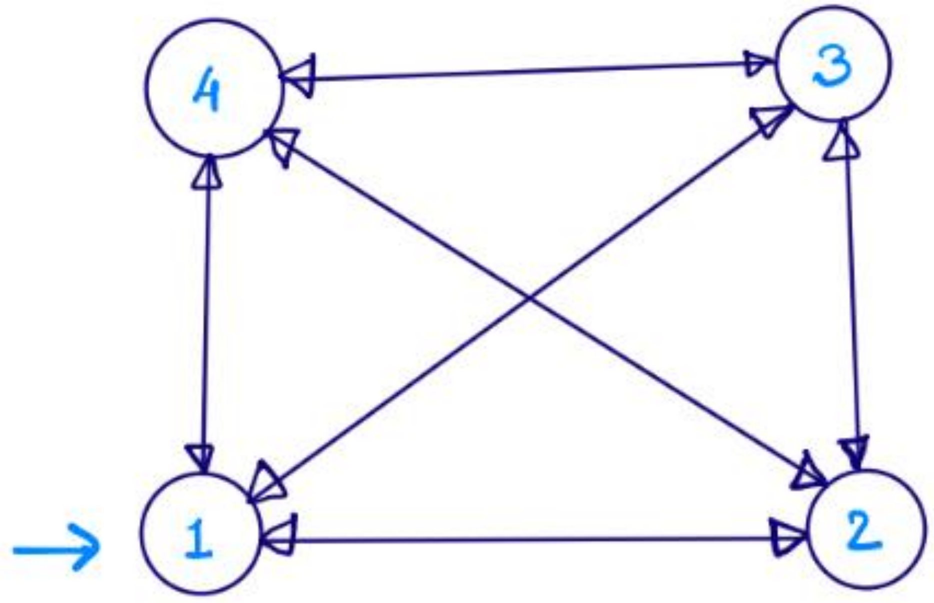


Algorithm 1: Dynamic Approach for TSP

```
Data: $s$: starting point; $N$: a subset of input cities; $dist()$:
      distance among the cities
Result: $Cost$ : TSP result
$Visited[N] = 0$;
$Cost = 0$;
Procedure TSP($N$, $s$)
    $Visited[s] = 1$;
    if $|N| = 2$ and $k \neq s$ then
        $Cost(N, k) = dist(s, k)$;
        Return Cost;
    else
        for $j \in N$ do
            for $i \in N$ and $visited[i] = 0$ do
                if $j \neq i$ and $j \neq s$ then
                    $Cost(N, j) = \min\ (\ TSP(N - \{i\}, j) + dist(j, i))$
                    $Visited[j] = 1$;
                end
            end
        end
    end
    Return $Cost$;
end
```

The Travelling Salesman Problem

- Example:

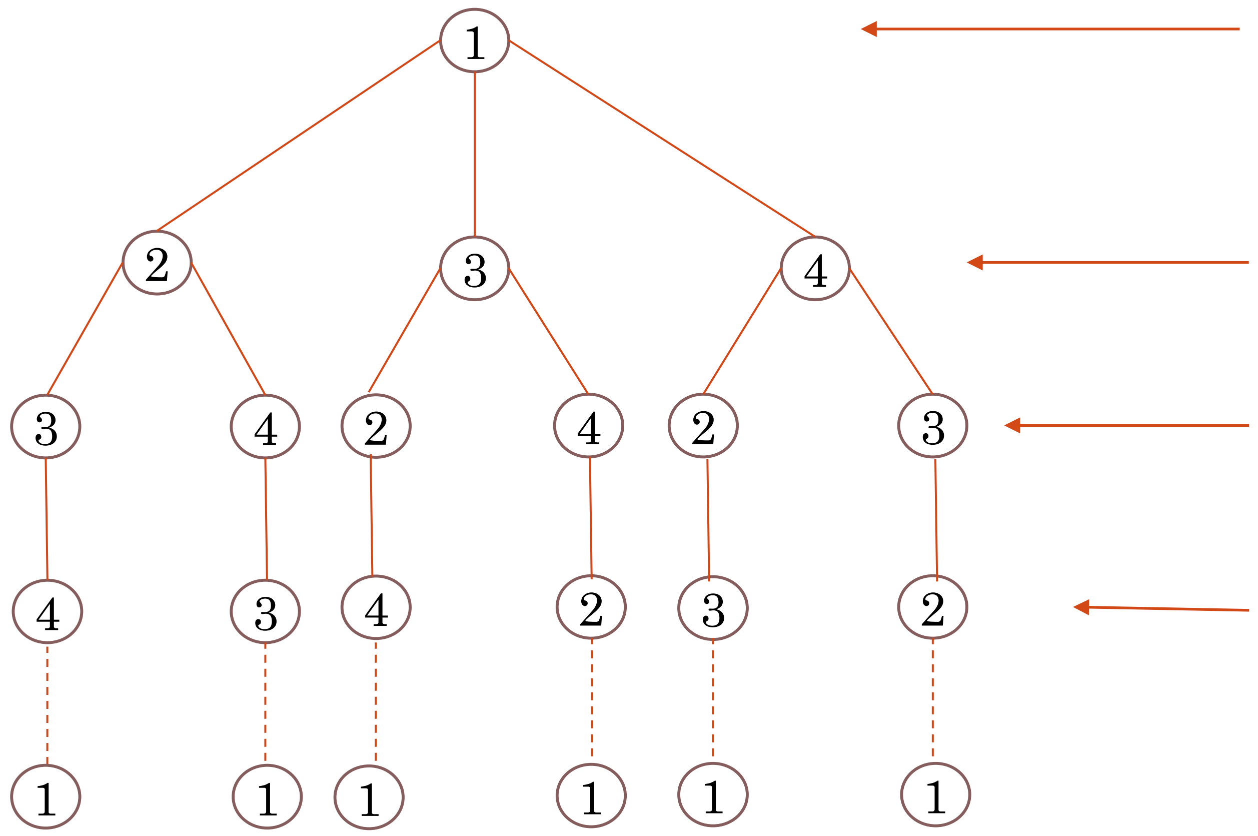


Directed Graph

	1	2	3	4
1	0	5	10	15
2	6	0	8	12
3	8	10	0	14
4	7	9	20	0

Cost Matrix

The Travelling Salesman Problem



Level 4 → S{2,3,4}

Level 3→ S{2,3}, S{3,2} etc.

Level 2 → S{2}, S{3} or S{4}

Level 1 → S=∅

The Travelling Salesman Problem

Solution: Recursive Formula:

$$\mathbf{SP}(\mathbf{i},\mathbf{S}) = \min_{k\in S}[d(i,k) + SP(k, S - \{k\})]$$

Level 4:

Here, i = 1; S = {2,3,4} and k∈ {2,3,4}

$$SP(1,\{2,3,4\}) = \min[d(1,2) + SP(2, \{3,4\}), \; d(1,3) + SP(3, \{2,4\}), \; d(1,4) + SP(4, \{2,3\})]$$

Terms SP(2, {3,4}), SP(3, {2,4}), SP(4, {2,3}) are unknow for now

Level 3:

for i = 2 S = {3,4} and k∈ {3,4}

$$SP(2, \{3,4\}) = \min[d(2,3) + SP(3, \{4\}), \; d(2,4) + SP(4, \{3\})]$$

for i = 3 S = {2,4} and k∈ {2,4}

$$SP(3, \{2,4\}) = \min[d(3,2) + SP(2, \{4\}), \; d(3,4) + SP(4, \{2\})]$$

for i = 4 S = {2,4} and k∈ {2,3}

$$SP(4, \{2,3\}) = \min[d(4,2) + SP(2, \{3\}), \; d(4,3) + SP(3, \{2\})]$$

Terms SP(3, {4}), SP(4, {3}), SP(2, {4}), SP(4, {2}), SP(2,{3}) and SP(3,{2}) are unknow for now

The Travelling Salesman Problem

Solution: Recursive Formula:

$$\mathbf{SP(i,S)} = \min_{k \in S}[d(i,k) + SP(k, S - \{k\})]$$

Level 2:

$$SP(2,\{3\}) = \min[d(2,3) + SP(3, \emptyset)]$$
$$SP(2,\{4\}) = \min[d(2,4) + SP(4, \emptyset)]$$
$$SP(3,\{2\}) = \min[d(3,2) + SP(2, \emptyset)]$$
$$SP(3,\{4\}) = \min[d(3,4) + SP(4, \emptyset)]$$
$$SP(4,\{2\}) = \min[d(4,2) + SP(2, \emptyset)]$$
$$SP(4,\{3\}) = \min[d(4,3) + SP(3, \emptyset)]$$

Terms $SP(2, \emptyset)$, $SP(3, \emptyset)$, $SP(4, \emptyset)$, are unknow for now

Level 1:

$$SP(2, \emptyset) = d(2,1) = 6$$
$$SP(3, \emptyset) = d(3,1) = 8$$
$$SP(4, \emptyset) = d(4,1) = 7$$

Substitute above values in previous levels

The Travelling Salesman Problem

Level 1:

$SP(2, \emptyset) = d(2,1) = 6$
$SP(3, \emptyset) = d(3,1) = 8$
$SP(4, \emptyset) = d(4,1) = 7$

Substitute above values in previous levels

Level 2:

$SP(2,\{3\}) = \min[d(2,3) + SP(3, \emptyset)] = 8+8 = 16$
$SP(2,\{4\}) = \min[d(2,4) + SP(4, \emptyset)] = 12+7 = 19$
$SP(3,\{2\}) = \min[d(3,2) + SP(2, \emptyset)] = 10+6 = 16$
$SP(3,\{4\}) = \min[d(3,4) + SP(4, \emptyset)] = 14+7 = 21$
$SP(4,\{2\}) = \min[d(4,2) + SP(2, \emptyset)] = 9+6 = 15$
$SP(4,\{3\}) = \min[d(4,3) + SP(3, \emptyset)] = 20+8 = 28$

Level 3:

$SP(2, \{3,4\}) = \min[d(2,3) + SP(3, \{4\}), d(2,4) + SP(4, \{3\})]$
$= \min[(\underline{8+21}),(12+28)] = 29$
$SP(3, \{2,4\}) = \min[d(3,2) + SP(2, \{4\}), d(3,4) + SP(4, \{2\})]$
$= \min[(10+19),(14+15)] = 29$
$SP(4, \{2,3\}) = \min[d(4,2) + SP(2, \{3\}), d(4,3) + SP(3, \{2\})]$
$= \min[(\underline{9+16}), (20+16)] = 25$

Level 4:

$SP(1,\{2,3,4\}) = \min[d(1,2) + SP(2, \{3,4\}), d(1,3) + SP(3, \{2,4\}), d(1,4) + SP(4, \{2,3\})]$

$= \min[(\underline{5+29}), (10+29), (15+25)] = 34$

$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$

The Travelling Salesman Problem

Time Complexity:

$$\mathbf{SP(i, S)} = \min_{k \in S}[d(i,k) + SP(k, S - \{k\})] \ldots\ldots\ldots \text{①}$$

- Let M be number of **SP(i,S)**'s that have to be computed (i.e. no. of subproblems) before we can compute **SP(1, V-{1}).**

- For each value of |S| there are n-1 choices for I .

- No. of distinct set of S of size k not including 1 and is $\binom{n-2}{k}$.

- Hence , $\mathbf{M} = \sum_{k=0}^{n-2}(n-1)\binom{n}{k} = (n-1)2^{n-2}$

The Travelling Salesman Problem

Time Complexity:

- The computation of SP(i, S) with |S|=k requires (k-1) comparisons when solving equation ① .
- Time T required to find an optimal tour

$$\mathbf{T} = \sum_{k=1}^{n-2}(k-1)(n-1)\binom{n-2}{k}$$

$$= (n-1)\sum_{k=1}^{n-2}(k-1)\binom{n-2}{k}$$

But,

$$\sum_{k=1}^{n-2}(k-1)\binom{n-2}{k} = \sum_{k=1}^{n-2}(k\binom{n-2}{k} - \sum_{k=1}^{n-2}\binom{n-2}{k}$$

$$= \sum_{k=1}^{n-2}(n-2)\boxed{\binom{n-3}{k-1}} - \sum_{k=1}^{n-2}\binom{n-2}{k}$$

$$\boxed{\because \textit{for integer } k, \quad k\binom{r}{k} = \mathrm{r}.\binom{r-1}{k-1}}$$

The Travelling Salesman Problem

Time Complexity:

$$= (n-2)\sum_{k=1}^{n-2}\binom{n-3}{k-1} - (2^{n-2}-1)$$

$$= (n-2)\ (2^{n-3}) - (2^{n-2}-1)$$

$$\boxed{since \sum_{k=0}^{k=n}\binom{n}{k} = 2^n}$$

Hence, $T = (n-1)[(n-2)\ (2^{n-3}) - (2^{n-2}-1)]$

$$=(n-1)[(n-2)(2^n)/8 - (2^n/4-1)]$$

$$=(n-1)(n-2)(2^n)/8 - (n-1)(2^n/4 + (n-1)]$$

$$=O(n^2 2^n)$$

Matrix Chain Multiplication

- It is an optimization problem that can be solved using dynamic programming.
- Given a problem, we want to find most efficient way to multiply the given matrices together.
- Problem is not to perform multiplications, but to decide in which order to perform the multiplications.
- Suppose we want to multiply four matrices $\mathrm{A} \times B \times \mathrm{C} \times \mathrm{D}$ of dimensions 50×20, 20×1, 1×10, 10×100 respectively.
- This will involve iteratively multiplying two matrices at a time.
- Matrix multiplication is not commutative i.e. $\mathrm{A} \times B \neq B \times A$, but it is associative i.e. $(A \times B) \times C = A \times (B \times C)$.
- So, $A \times B \times C \times D$ can be done in different ways depending upon how parenthesis is done.

Matrix Chain Multiplication

- *A is having dimensions (50x20)*
- *B is having dimension (20x1)*
- *C is having dimension (1x10)*
- *D is having dimension (10x100)*

$$A \times B \times C \times D = A \times (B \times C) \times D = \big(A \times (B \times C)\big) \times D = ((A \times (B \times C)) \times D)$$

Generalised Recursive Formula:

$$C\left[i,j\right] = \min_{i \le k < j} \{C\left[i,k\right] + C\left[k+1,j\right] + d_{i-1} * d_k * d_j\}$$

Matrix Chain Multiplication

Example:

Find optimal multiplication of the chain of 4 matrices

A1, A2, A3, A4 with dimension 3x2, 2x4, 4x2 and 2x5

respectively.

So, here $P[d_0, d_1, d_2, d_3, d_4] = P[3, 2, 4, 2, 5]$

Matrix Chain Multiplication

Algorithm: To find value of an optimal solution.

```
Matrix-Chain($p, n$)
{   for ($i = 1$ to $n$) $m[i, i] = 0$;
    for ($l = 2$ to $n$)
    {
        for ($i = 1$ to $n - l + 1$)
        {
            $j = i + l - 1$;
            $m[i, j] = \infty$;
            for ($k = i$ to $j - 1$)
            {
                $q = m[i, k] + m[k + 1, j] + p[i - 1] * p[k] * p[j]$;
                if ($q < m[i, j]$)
                {
                    $m[i, j] = q$;
                    $s[i, j] = k$;
                }
            }
        }
    }
    return $m$ and $s$; (Optimum in $m[1, n]$)
}
```

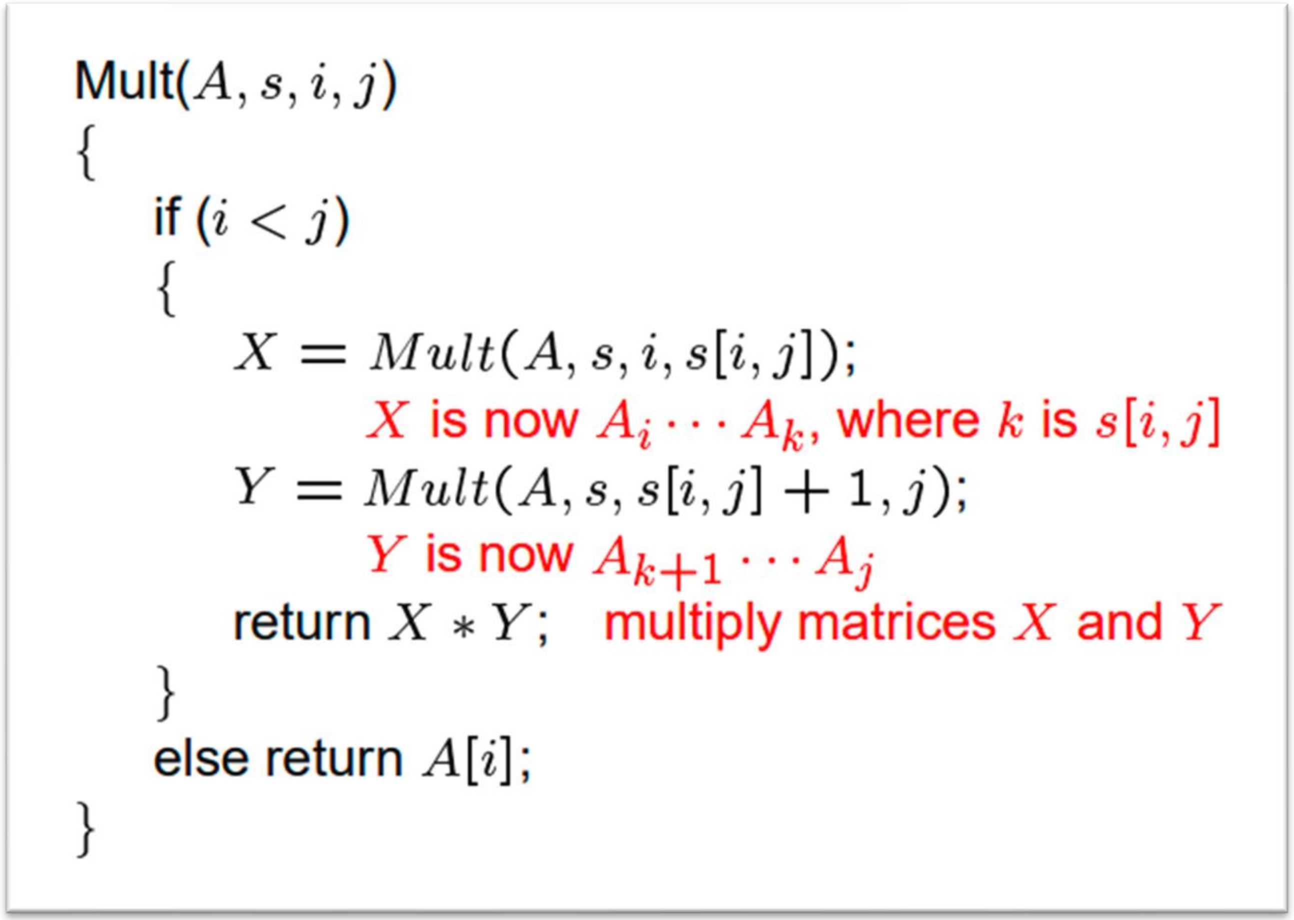
Complexity: The loops are nested three deep.

Each loop index takes on $\leq n$ values.

Hence the time complexity is $O(n^3)$. Space complexity $\Theta(n^2)$.

Matrix Chain Multiplication

Algorithm: Construct an optimal solution



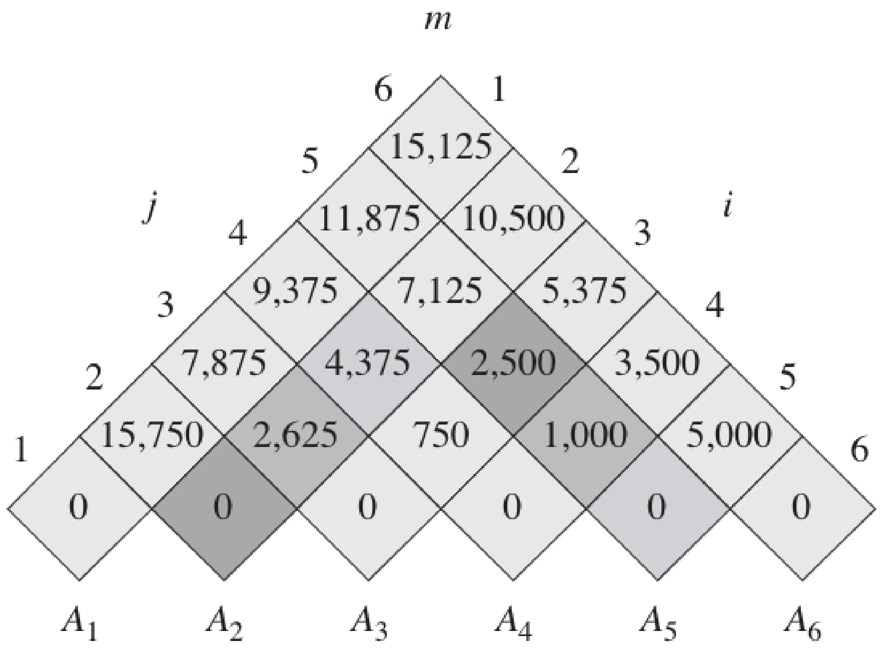
```
Mult($A, s, i, j$)
{
    if ($i < j$)
    {
        $X = Mult(A, s, i, s[i, j])$;
                $X$ is now $A_i \cdots A_k$, where $k$ is $s[i, j]$
        $Y = Mult(A, s, s[i, j] + 1, j)$;
                $Y$ is now $A_{k+1} \cdots A_j$
        return $X * Y$;    multiply matrices $X$ and $Y$
    }
    else return $A[i]$;
}
```

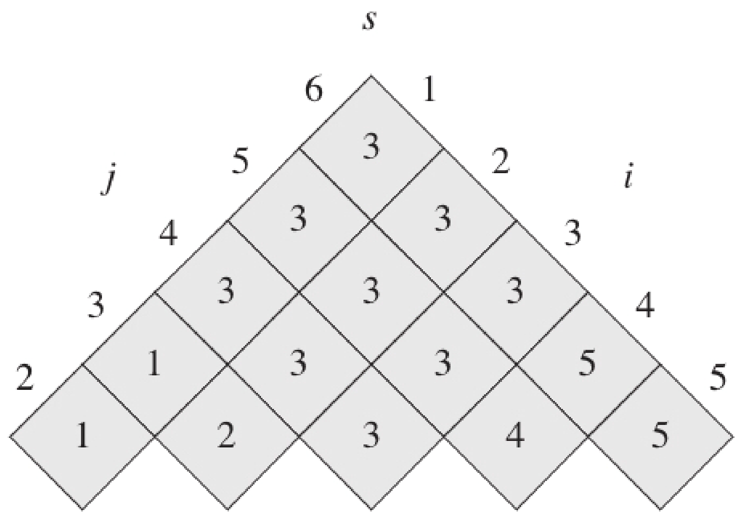
To compute $A_1 A_2 \cdots A_n$, call Mult($A, s, 1, n$).

Matrix Chain Multiplication

Example 2:

matrix	A_1	A_2	A_3	A_4	A_5	A_6
dimension	30×35	35×15	15×5	5×10	10×20	20×25





Longest Common Subsequence(LCS)

- A Subsequence of a sequence is the same sequence with some elements (possibly none) left out.
- Given a sequence X = $(x_1, x_2, x_3 \ldots, x_m)$, the subsequence of X is $(x_{i_1}, x_{i_2}, x_{i_3} \ldots, x_{i_k})$ where $i_1 < i_2 < i_3 < \ldots < i_k$.
- For example, given the sequence X=(B C D B A C G H),

  (B D A C) and (C A G H) are some of the subsequence of X

Longest Common Subsequence(LCS)

- Given two sequences X and Y, a sequence Z is a common subsequence of X and Y
- For Example, X= (A D C B E J M) and Y = (B A C D E B M), then the sequence (D B M) having length 3 is common subsequence of both X and Y.
- However, (D B M) is not longest common subsequence of X and Y.
- Subsequence (A D B M) of length 4 common to both X and Y.
- Longest Common subsequence problem is stated below:

"Given two sequence X=(x_1, x_2, x_3..., x_m) and Y=(y_1, y_2, y_3..., y_n), find a maximum length common subsequence of X and Y."

Longest Common Subsequence(LCS)

Recursive Formula:

$$L[i,j] = \begin{cases} 0, & \text{if } i = 0 \text{ or } j = 0 \\ L[i-1][j-1]+1, & \text{if } i,j > 0 \text{ and } x_i = y_j \\ \max(L[i-1][j], L[i][j-1]), & \text{if } i,j > 0 \text{ and } x_i \neq y_j \end{cases}$$

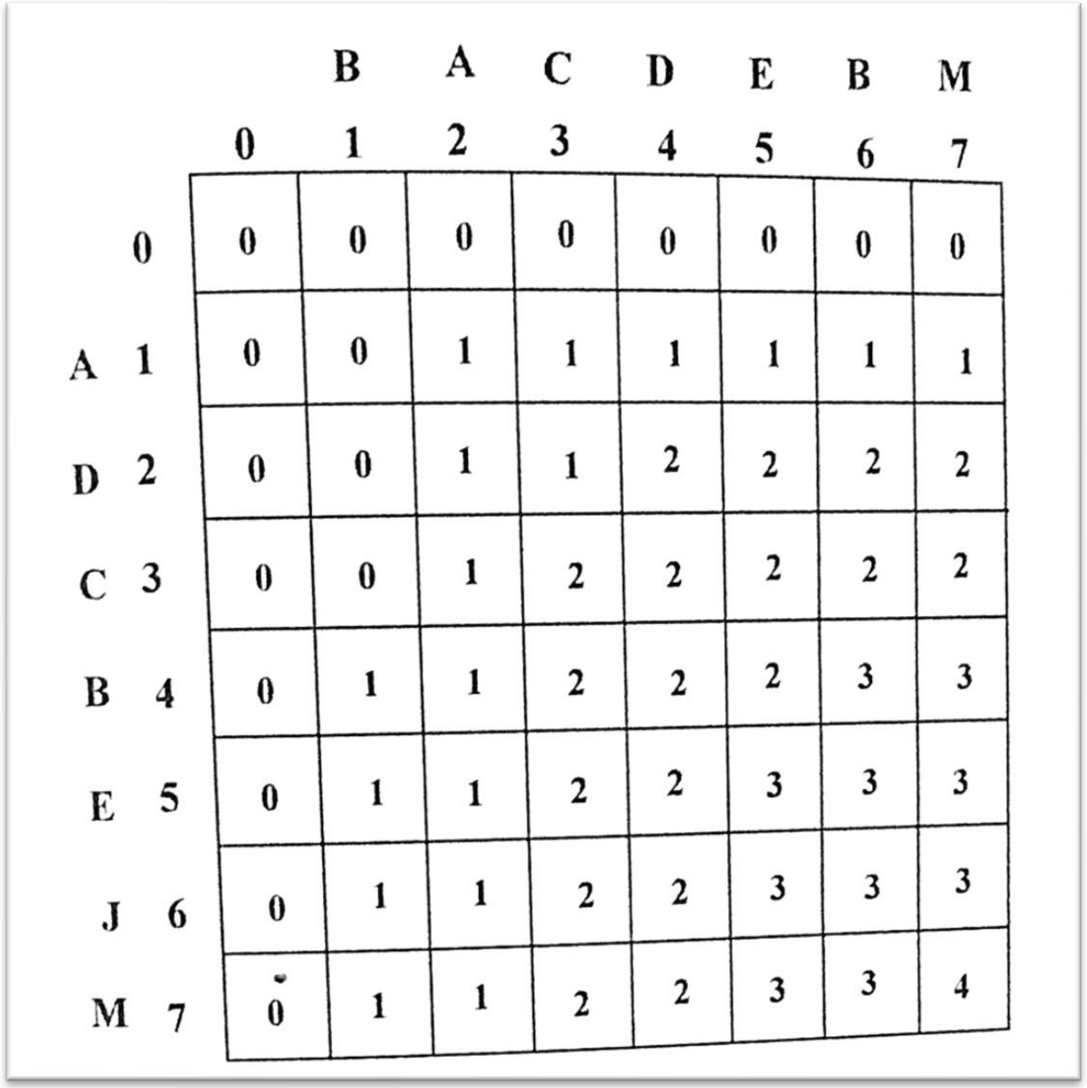
Longest Common Subsequence(LCS)

Example: 1

Find the Longest Common Subsequence of the sequences

X = (A D C B E J M) and

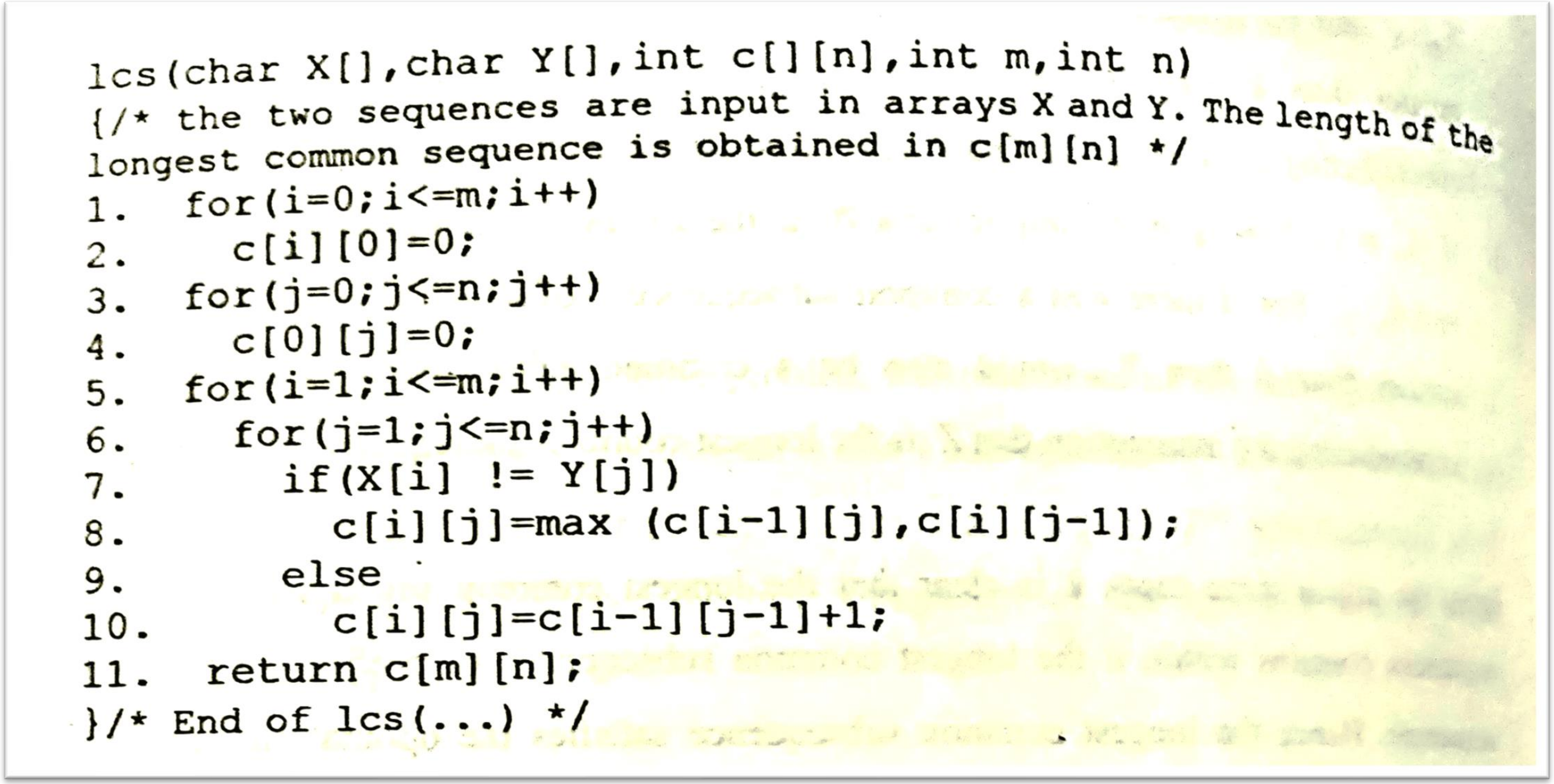
Y = (B A C D E B M)



			B	A	C	D	E	B	M
		0	1	2	3	4	5	6	7
	0	0	0	0	0	0	0	0	0
A	1	0	0	1	1	1	1	1	1
D	2	0	0	1	1	2	2	2	2
C	3	0	0	1	2	2	2	2	2
B	4	0	1	1	2	2	2	3	3
E	5	0	1	1	2	2	3	3	3
J	6	0	1	1	2	2	3	3	3
M	7	0	1	1	2	2	3	3	4

Longest Common Subsequence(LCS)

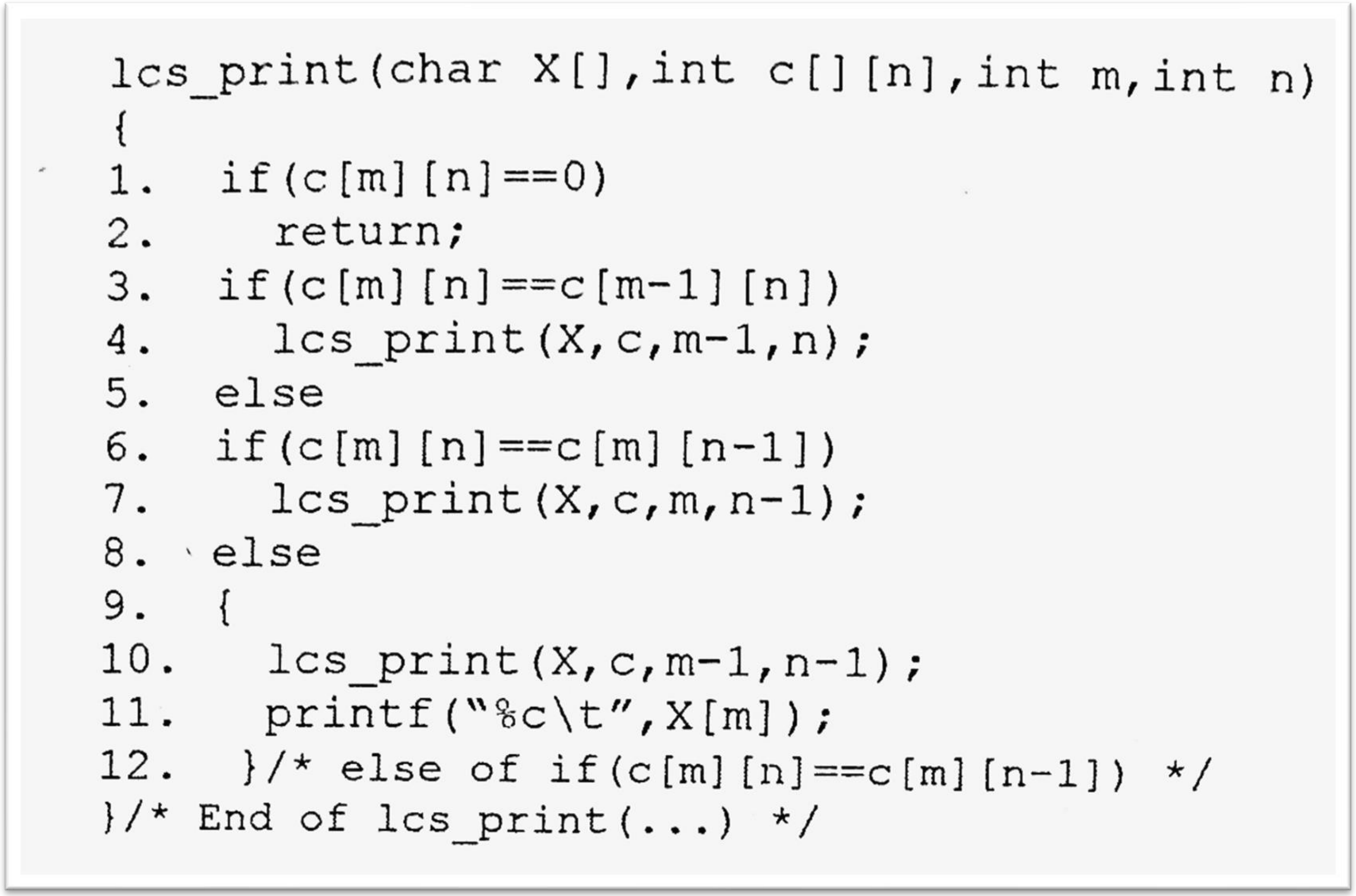
Algorithm:



```
lcs(char X[],char Y[],int c[][n],int m,int n)
{/* the two sequences are input in arrays X and Y. The length of the
longest common sequence is obtained in c[m][n] */
1.   for(i=0;i<=m;i++)
2.     c[i][0]=0;
3.   for(j=0;j<=n;j++)
4.     c[0][j]=0;
5.   for(i=1;i<=m;i++)
6.     for(j=1;j<=n;j++)
7.       if(X[i] != Y[j])
8.         c[i][j]=max (c[i-1][j],c[i][j-1]);
9.       else
10.        c[i][j]=c[i-1][j-1]+1;
11.   return c[m][n];
}/* End of lcs(...) */
```

Longest Common Subsequence(LCS)

Algorithm:



```
lcs_print(char X[],int c[][n],int m,int n)
{
    if(c[m][n]==0)
      return;
    if(c[m][n]==c[m-1][n])
      lcs_print(X,c,m-1,n);
    else
    if(c[m][n]==c[m][n-1])
      lcs_print(X,c,m,n-1);
    else
    {
      lcs_print(X,c,m-1,n-1);
      printf("%c\t",X[m]);
    }/* else of if(c[m][n]==c[m][n-1]) */
}/* End of lcs_print(...) */
```