① Geometric random variable (几何随机变量)

X = Bernoulli trials required to obtain the 1st success

p = probability of success of each Bernoulli trial

$$P(X=k) = (1-p)^{k-1} p$$

1> $E(X) = \frac{1}{p}$

2> $P(X=m+n \mid X>m) = P(X=n)$

3> $P(X>m+n \mid X>m) = P(X>n)$

4> $P(X\leq m+n \mid X>m) = P(X\leq n)$

例: $X \to$ No. of operations a tool will last till breakdown

$p \to 0.01$

(1) $E(X) = ?$ $\quad E(X) = \frac{1}{p} = 100$

(2) tool has already lasted 10 ops, probability it will last 30 more

$$P(X>10+30 \mid X>10) = P(X>30) = (1-p)^{30} = 0.99^{30}$$

(3) still working after m (<40) ops. Probability still work after 40 ops

$$P(X>40 \mid X>m) = P(X>40-m) = (1-p)^{40-m}$$

$$= 0.99^{40-m}$$

5> $P(G_1 < G_2) = \dfrac{p_1(1-p_2)}{p_1+p_2-p_1p_2}$

证明: $G_1 = k \quad G_2 > k \quad P(G_1 < G_2) = \sum_{k=1}^{\infty} P(G_1 = k, G_2 > k)$

$$P(G_1 = k, G_2 > k) = P(G_1 = k)P(G_2 > k) = (1-p_1)^{k-1}p_1 \cdot (1-p_2)^k$$

$$\Rightarrow P(G_1 < G_2) = \sum_{k=1}^{\infty}(1-p_1)^{k-1}p_1 \cdot (1-p_2)^k = \underbrace{\sum_{k=1}^{\infty}\left((1-p_1)(1-p_2)\right)^{k-1}}_{=\frac{1}{1-(1-p_1)(1-p_2)}} p_1(1-p_2)$$

$$= \frac{p_1(1-p_2)}{1-(1-p_1)(1-p_2)} = \frac{p_1(1-p_2)}{p_1+p_2-p_1p_2}$$

② Exponential Random variable (指数)

$X = e^{\lambda}$ λ is rate of X

概率密度 (pdf):

假设 Δt 内坏的概率为 $\lambda \Delta t$ 设 $S(t) = P(X > t)$ 表示 t 时未坏

则 $S(t+\Delta t) = S(t)(1 - \lambda \Delta t)$

$$\Rightarrow \frac{dS(t)}{dt} = \frac{S(t+\Delta t) - S(t)}{\Delta t} = \frac{-\lambda \Delta t\, S(t)}{\Delta t} = -\lambda S(t)$$

解为 $S(t) = e^{-\lambda t} \Rightarrow$ pdf 为 $f(t) = -\frac{d}{dt} S(t) = \lambda e^{-\lambda t}$

1> pdf of $X = f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \le 0 \end{cases}$

$\Downarrow$

累积分布 (cdf):

$$F_X(x) = P(X \le x) = \int_0^x f_X(t)\,dt = -e^{-\lambda t}\Big|_0^x = 1 - e^{-\lambda x}$$

☆ 2> cdf of $X = F_X(x) = P(X \le x) = 1 - e^{-\lambda x}$ $0 \le x < \infty$

3> $P(X > x) = e^{-\lambda x}$

4> $E(X) = \frac{1}{\lambda}$

5> $P(X > x+y \mid X > x) = P(X > y)$

6> $P(X \le x+y \mid X > x) = P(X \le y)$

例: $\lambda = 0.01/\text{hour}$ $X \rightarrow$ exponentially distribution (failure time)

(1) Mean time between failures $E(X) = \frac{1}{0.01} = 100$ hours

(2) $P(X>50) = e^{-\lambda(50)} = e^{-0.5}$

(3) $P(X>50 \mid X>40) = P(X>10) = e^{-\lambda(10)} = e^{-0.1}$

(4) $P(X\le 15 \mid X>10) = P(X\le 5) = 1-e^{-\lambda(5)} = 1-e^{-0.05}$

7> $P(X_1<X_2) = \dfrac{\lambda_1}{\lambda_1+\lambda_2}$

证明:

$$P(X_1<X_2) = \int_0^\infty P(X_2>x_1)\, f_{X_1}(x_1)\,dx_1$$
$$= \int_0^\infty e^{-\lambda_2 x_1}\cdot \lambda_1 e^{-\lambda_1 x_1}\,dx_1 = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

8> $X = \min(X_1, X_2) \sim Exp(\lambda_1+\lambda_2)$

证明:

$$P(X>x) = P(X_1>x, X_2>x) = P(X_1>x)\,P(X_2>x)$$
$$= e^{-\lambda_1 x} e^{-\lambda_2 x}$$
$$= e^{-(\lambda_1+\lambda_2)x}$$

$$F_X(x) = P(X\le x) = 1-e^{-(\lambda_1+\lambda_2)x}$$

$$f_X(x) = \frac{d}{dx}F_X(x) = (\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)x}$$

$\Rightarrow \quad X = \min(X_1, X_2) \sim Exp(\lambda_1+\lambda_2)$

☆ Geometric (几何) 是唯一满足无记忆性 in discrete

Exponential (指数) 是唯一满足无记忆性 in continuous

③ Stochastic Progress (随机)

1> discrete time, discrete state space (discrete time chain)

2> discrete time, continuous state space

3> continuous time, discrete state space (continuous time chain)

4> continuous time, continuous state space

④ Poisson process (泊松)

1> $p(X(t)=k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$

2> 在 $[0,t]$ 区间无到达概率 $p(X(t)=0) = e^{-\lambda t}$

$\Downarrow$

3> 设 T 为两次到达间隔 则 $p(T>t) = e^{-\lambda t}$

和前面 $p(X>x) = e^{-\lambda x}$ 相似

4> $p(T \le t) = 1 - e^{-\lambda t}$

5> 时间间隔 $[0,t]$ 到达人数是 Poisson 分布，mean 为 λt

☆ 指数分布和 Poisson 分布区别

分布	随机变量含义	参数含义	均值
Exp	两次事件间时间间隔	λ = 事件频率	$\frac{1}{\lambda}$
Poisson	固定时间发生次数	λ = 每单位时间事件数	λt

例：每分钟有 $\lambda = 2$ 车经过路口

1> 5分钟经过车数 $N(5) \sim$ Poisson (2×5) $E(N(5)) = 10$

2> 每2辆车平均间距 $T \sim Exp(2)$ $E(T) = \frac{1}{2}$

6> Superposition (叠加) $\boxed{\sum_{i=1}^{n} \lambda_i}$

7> Decomposition (分裂)   输入为 Poisson (λ)

分裂后 $\lambda_1 = p_1\lambda$ $\lambda_2 = p_2\lambda$ ⋯ $\boxed{\sum_{i=1}^{n} p_i = 1}$

⑤ Discrete Time Markov Chain (DTMC)

状态转移 1> $P_{ij}(m, m+n) = P(X_{m+n}=j \mid X_m = i) \xlongequal[\text{齐次}]{\text{homogeneous}} P_{ij}(n)$

步数

Sojourn Time 停留时间

T_i — sojourn time：DTMC在转换到不同状态前驻留状态 i 的时间

2> $P(T_i > n) = P_{ii}^{n}$ $\quad P(T_i = n) = P(T_i > n-1) - P(T_i > n)$

P_{ii}是停留概率，$1-P_{ii}$是成功概率 $= P_{ii}^{n-1}(1-P_{ii})$

3> $E(T_i) = \dfrac{1}{1-P_{ii}}$

⑥ Chapman–Kolmogorov Equation (C-K Equation)

1> $P(m+n) = P(m)P(n)$ $\quad p_{ij}(m+n) = \sum_{k\in S} p_{ik}(m)p_{kj}(n)$

2> Let $P(n) = [P_{ij}(n)]$ $\quad P = P_{ij}$ 则 $\boxed{P(n) = P^n}$

3> $p_j(n) = \sum_{i\in S} p_i(0)p_{ij}(n)$

Let $\pi(n) = [p_0(n) \ \ p_1(n) \cdots]$ $\Rightarrow$ $\pi(n) = \pi(0)P(n)$

4> Steady state Analysis

Let $Y = \lim_{n\to\infty} \pi(n) = [y_0 \ \ y_1 \ \ y_2 \cdots]$

$\pi(n) = \pi(0)P^n = \pi(0)P^{n-1}P \Rightarrow \boxed{\lim_{n\to\infty}\pi(n) = \lim_{n\to\infty}\pi(n-1)P}$

$\Rightarrow Y = YP$ and $\sum_j y_j = 1$

例：

$Y = [y_0 \ \ y_1]$ $\quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

from $Y = YP \Rightarrow [y_0 \ \ y_1] = [y_0 \ \ y_1]\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\Rightarrow \begin{matrix} y_0 = y_1 \\ y_1 = y_0 \end{matrix}$

又 $y_1 + y_0 = 1 \Rightarrow y_1 = y_0 = 0.5$

⑦ CTMC 和 C-K eqn

⑧ Kolmogorov Differential Equation

forward K equation

$$\frac{\partial H(s,t)}{\partial t} = H(s,t)\,Q(t)$$

backward K equation

$$\frac{\partial H(s,t)}{\partial s} = -Q(s)\,H(s,t)$$

State Probabilities

$$\pi(t) = [\,p_0(t)\ \ p_1(t)\ \ p_2(t)\cdots]$$

$$= \pi(0)\,H(t)$$

$$= \pi(0)\exp(Qt)$$

$$\frac{d\pi(t)}{dt} = \pi(0)\,Q\exp(Qt)$$

$$= \pi(t)\,Q$$

Steady State Analysis

$$\pi_j = \lim_{t \to \infty} P_j(t)$$

① $\sum_j \pi_j = 1$

② $\pi(t) Q = \pi Q = 0$ Q为转移矩阵

③ $\pi_j \geq 0$

注意 P_{ii} 放在最后算