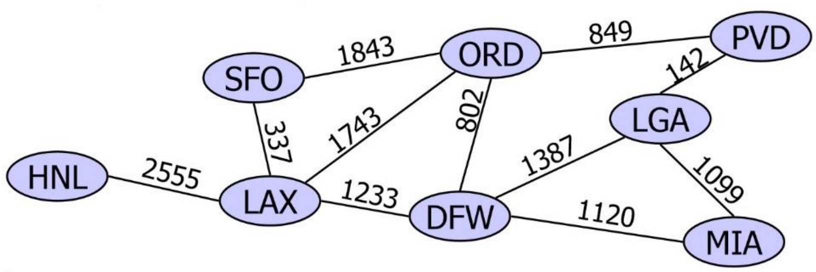
Graphs and Applications of Matrices

From: http://www.math.utah.edu/~gustafso/s2019/2270/labs/lab3-adjacency.pdf
And https://www.eecs.yorku.ca/course_archive/2006-07/W/2011/Notes/graphs_part1.pdf

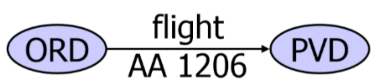
Graphs

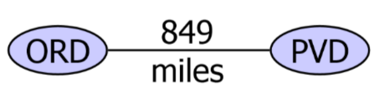
A graph is a set of points, called vertices or nodes, and edges that connect them. For example, a vertex represents an airport with an airport code, and an edge represents a flight route between two airports and stores the mileage of the route



Edge Types and Terminology

Directed (represented with arrow) and undirected. A directed graph is one in which all edges are directed, e.g. a flight network. A route network would be an undirected graph.

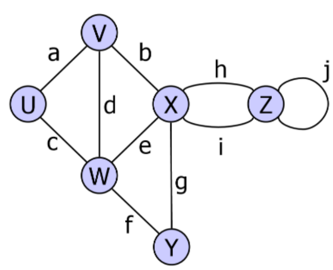




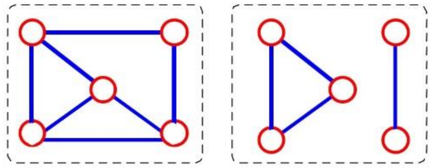
Adjacent vertices are connected by an edge. U and V are adjacent

Degree of a vertex refers to how many end vertices of an edge are incident on a vertex. W has degree 4

A *loop* is an edge with both ends incident on the same vertex, e.g. J



A graph is *connected* if there is a path from every vertex to every other vertex



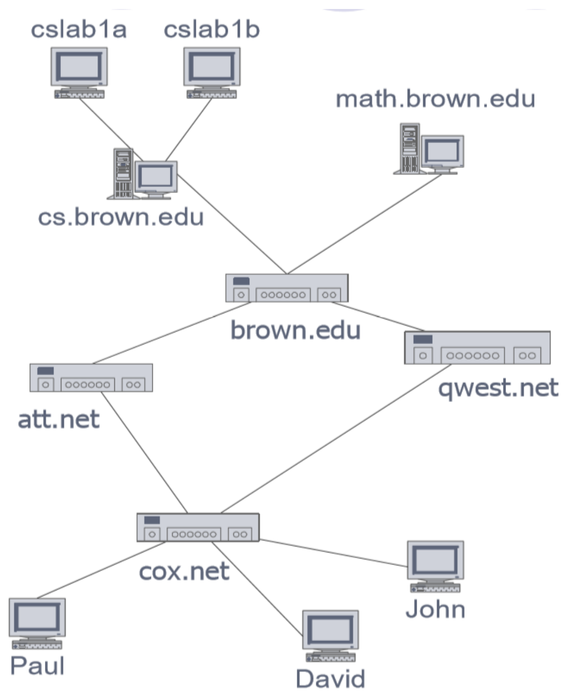
connected   not connected

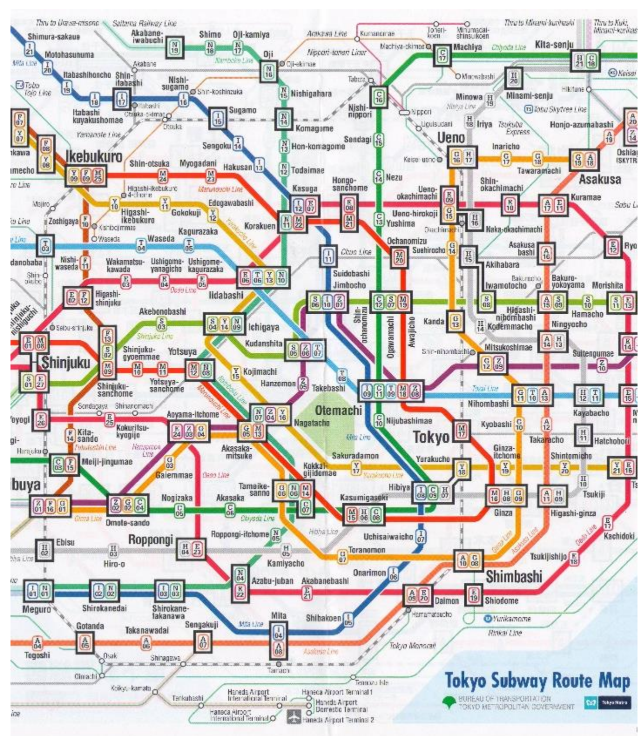
Sample Graph Applications

Electronic circuits: printed circuit boards; integrated circuit

Transportation networks: Highway network; flight network

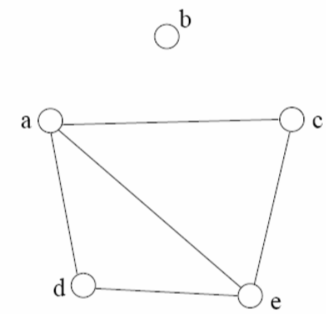
Computer networks: Local area network; internet





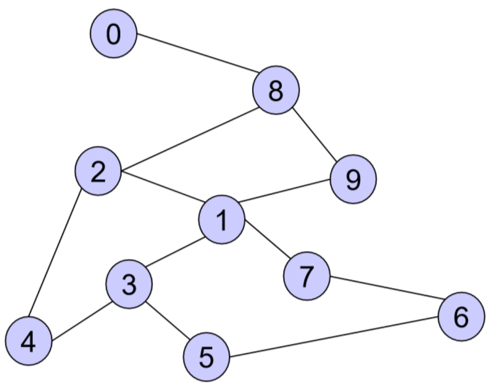
Representations of Graphs: Adjacency Matrices

A 2D matrix (or array) of size *n x n*, where *n* is the number of vertices in the graph, may be used to represent a graph. Recall that two vertices connected by an edge are said to be adjacent. Consider matrix A below. $A_{i,j} = 1$ if there is an edge connecting vertices I and j. Otherwise, $A_{i,j} = 0$



	a	b	c	d	e
a	0	0	1	1	1
b	0	0	0	0	0
c	1	0	0	0	1
d	1	0	0	0	1
e	1	0	1	1	0

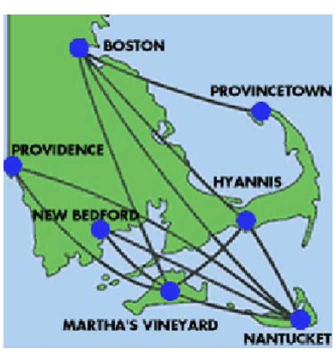
Adjacency Matrix Example:



	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	0	0	0	0	0	0	0	1	0
1	0	0	1	1	0	0	0	1	0	1
2	0	1	0	0	1	0	0	0	1	0
3	0	1	0	0	1	1	0	0	0	0
4	0	0	1	1	0	0	0	0	0	0
5	0	0	0	1	0	0	1	0	0	0
6	0	0	0	0	0	1	0	1	0	0
7	0	1	0	0	0	0	1	0	0	0
8	1	0	1	0	0	0	0	0	0	1
9	0	1	0	0	0	0	0	0	1	0

Applications of Adjacency Matrices

Airline or railway route maps may be represented by adjacency matrices. Below is a northern route map for Cape Air from May 2001. Here the vertices are the cities to which Cape Air flies, and two vertices are connected if a direct flight exists between them.



It might be important to know if two vertices are connected by a sequence of two edges, even if they are not connected by a single edge. Notice that A and C are connected by a two-edge sequence (actually, there are four distinct ways to go from A to C in two steps). In the route map, Provincetown and Hyannis are connected by a two-edge sequence, meaning that a passenger would have to stop in Boston while flying between those cities on Cape Air.

If the vertices in the Cape Air graph respectively correspond to Boston, Hyannis, Martha's Vineyard, Nantucket, New Bedford, Providence, and Provincetown, then the adjacency matrix, *F*, for Cape Air is

$$F = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Which vertices are connected by a two-edge sequence? How many different two-edge sequences connect each pair of vertices? Given matrix *F* above, represent matrix F^2 and F^3 below. What does each represent?

$$F^2 = \begin{bmatrix} 4 & 2 & 2 & 2 & 2 & 2 & 0 \\ 2 & 3 & 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 5 & 4 & 1 & 1 & 1 \\ 2 & 2 & 4 & 5 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 2 & 2 & 0 \\ 2 & 2 & 1 & 1 & 2 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad F^3 = \begin{bmatrix} 6 & 8 & 12 & 12 & 4 & 4 & 4 \\ 8 & 6 & 11 & 11 & 4 & 4 & 2 \\ 12 & 11 & 10 & 11 & 9 & 9 & 2 \\ 12 & 11 & 11 & 10 & 9 & 9 & 2 \\ 4 & 4 & 9 & 9 & 2 & 2 & 2 \\ 4 & 4 & 9 & 9 & 2 & 2 & 2 \\ 4 & 2 & 2 & 2 & 2 & 2 & 0 \end{bmatrix}$$

What would be easy to see from a small graph is harder to see from the adjacency matrix. However, the opposite is true of very large graphs.

Other Matrix Applications

Encryption; Representing systems of equations; Transforming 3D images; colour filters.