Thesis

Tim Lichtnau

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1 Preparation

Lemma 1.1. *Let C be a class of types stable under $\sum$. The class $\mathsf{HasAtlas}_C$ of types Y which admit a map $\operatorname{Spec} A \to Y$ fibered in C is stable under identity types.*

Proof. By assumption we can choose a map $p : V \to Y$ out of an affine fibered in C. Let $y, y' : Y$. Then we have the map

$$\begin{gathered}(\mathrm{fib}_p\, y) \times_V (\mathrm{fib}_p\, y') \to y = y' \\ (v, q : y = pv), (v', q' : y' = pv'), (h : v = v') \mapsto q \cdot h \cdot q'^{-1}\end{gathered}$$

The fiber over $j : y = y'$ looks like

$$\sum_v (\underbrace{\sum_{v'} (h : v = v')}_{\mathsf{isContr}}) \times (q : y = pv) \times (q' : y' = pv') \times (q \cdot h \cdot q'^{-1} = j) \simeq \sum_v (v = py) \simeq \mathrm{fib}_p\, y$$

Hence the map is fibered in C. It suffices to show, that $(\mathrm{fib}_p\, y) \times_V (\mathrm{fib}_p\, y')$ has an atlas, because then we can compose such an atlas with the above map to obtain an atlas of $y = y'$. By assumption the fibers of p have an atlas, so we can choose $q : W \to \mathrm{fib}_p\, y, q' : W' \to \mathrm{fib}_p\, y'$ atlasses. Then $W \times_V W' \to (\mathrm{fib}_p\, y) \times_V (\mathrm{fib}_p\, y')$ is an atlas: The domain is a fiber product of affines, hence affine. The fiber over (x, x') is equivalent to the product of fibers $(\mathrm{fib}_q\, x) \times (\mathrm{fib}_{q'}\, x')$ which is in C by stability under dependent sums (so in particular under finite products).

□

2 Covering stacks

Fix $\mathbb{T}$ a topology, which we call the covering-affines.

Definition 2.1. Covering geometric stacks are the smallest intermediate class containing $\mathbb{T}$ such that: If Y is a sheaf and $\mathbb{T} \ni S \to Y$ is fibered in covering geometric stacks, then Y is a covering geometric stack.

We call such map $X \to Y$ whose fibers are covering $\mathcal{V}$-stacks a $\mathcal{V}$-cover. If X is affine we call it an $\mathcal{V}$-atlas. If X is in $\mathbb{T}$ we call it a $\mathcal{V}$-catlas. In Case of $\mathcal{V} = \mathcal{U}_{\mathbb{T}}$ the sheaves we call it a geometric cover / geometric atlas / geometric catlas.

Proposition 2.2 (Recursion principle for covering stacks). *Let $P : \mathcal{V} \to \mathrm{Prop}$ be a property of types in $\mathcal{V}$. Assume*

- *Every covering affine has P*
- *If $\mathbb{T} \ni S \to Y$ is fibered in P then Y has P*

Then every covering $\mathcal{V}$-stack has P.

Proof. Replace P by $P \wedge \mathsf{is}-\mathsf{covering}-\mathsf{stack}$. Then usual induction $\square$

Lemma 2.3. *This class is $\sum$-stable.*

Proof. Define the predicate PX as every family $B : X \to \mathsf{CS}_{\mathcal{V}}$ of covering $\mathcal{V}$-stacks indexed over X satisfies $\sum_{x:X} Bx \in \mathsf{CS}_{\mathcal{V}}$. If X is a covering affine, by choice of X we can choose $\mathcal{V}$-catlasses $S_x \to Bx$ for all $x : X$. Then $\sum_{x:X} S_x \to \sum_x Bx$ is a $\mathcal{V}$-catlas.
If $f : S \to X$ is a map fibered in P with $S \in T$, then let $B : X \to \mathsf{CS}_{\mathcal{V}}$. By choice of S we can choose $\mathcal{V}$-catlasses $\tilde{B}s \to B(fs)$ for all $s : S$. Then consider $\sum_{s:S} \tilde{B}s \to \sum_{x:X} Bx$. Its domain is in $\mathbb{T}$. It remains to show, that the fiber over (x, t) is a covering stack. It is a dependent sum over $\mathrm{fib}_f\, x$, which by induction satisfies P that lets us conclude by definition of P. $\square$

Lemma 2.4. *$\mathcal{V}$-covers are stable under composition.*

Proof. covering $\mathcal{V}$-stacks are stable under $\sum$. $\square$

TODO same prop for geometric stack as well?

Proposition 2.5. *Every covering $\mathcal{V}$-stack X merely admits a $\mathcal{V}$-catlas, i.e. a $\mathcal{V}$-cover $Y \to X$ with $Y \in \mathbb{T}$.*

Proof. We apply the recursion principle of covering stacks

- If X is covering affine, then $X \to X$ is a $\mathcal{V}$-catlas with covering domain.

- If X is obtained as a quotient then it already is equipped with a $\mathcal{V}$-atlas.

$\square$

Proposition 2.6. *The class of covering $\mathcal{V}$-stacks is stable under quotients: If $X \to Y$ is fibered in covering $\mathcal{V}$-stacks and X is a covering $\mathcal{V}$-stack and $Y \in \mathcal{V}$, then Y is a covering $\mathcal{V}$-stack.*

Proof. Choose an $\mathcal{V}$-catlas of X. Then the composition with the map $X \to Y$ is a $\mathcal{V}$-cover by 2.4. Surely its a $\mathcal{V}$-catlas. $\square$

Now we want to show that the clash of terminology regarding 'covering' is reasonable:

Proposition 2.7. *Let $\mathbb{T}$ be saturated. A covering stack X is affine iff its a covering affine.*

Proof. The converse is clear. The direct direction follows by the recursion principle. choosing a $\mathcal{V}$-catlas $S \to X$. As both S and X are affine the fibers are affine. By induction the fibers are covering affines. By saturatedness of the topology X is covering affine. $\square$

Lemma 2.8. *Let $\mathbb{T}$ be saturated. Let X be a covering $\mathcal{V}$-stack. Let $f : \operatorname{Spec} A \to X$ be a $\mathcal{V}$-atlas. Then* $\operatorname{Spec} A \in \mathbb{T}$

Proof. As $\operatorname{Spec} A \simeq \sum_{x:X} \mathrm{fib}_f\, x$ is a dependent sum of covering $\mathcal{V}$-stacks, it is a covering $\mathcal{V}$-stack again. We conclude by 2.7. $\square$

2.1 Geometric stacks

Definition 2.9. We call X a geometric stack if it merely has a geometric atlas, i.e some $\operatorname{Spec} A \to X$ fibered in covering geometric stacks.

Lemma 2.10. *geometric $\mathcal{V}$-stacks are closed under* id*-types.*

Proof. This is 1.1, using that covering stacks are closed under $\sum$. $\square$

warning. The previous lemma does not hold for covering stacks: Identity types of things in $\mathbb{T}$ could be empty.

Proposition 2.11 (Recursion principle for geometric stacks)**.** *Let $P : \mathsf{GS} \to \mathrm{Prop}$ be a property of types. Assume*

- *Every affine has P*
- *If $S \to Y$ is fibered in covering stacks that have P then Y has P*

Then every $\mathcal{V}$-stack has P.

Proof. One could explain geometric stacks as the smallest class containing all affines and if $\operatorname{Spec} A \to X$ is fibered in geometric stacks that happens to be covering, then X is a geometric stack. $\square$

2.2 About the smallest class in a subuniverse

Definition 2.12. Let $\mathcal{V} \supset \mathbb{T}$ be a superclass stable under $\sum$. covering stacks are the smallest intermediate class $\mathbb{T} \subset \mathsf{CS}_{\mathcal{V}} \subset \mathcal{V}$ such that: If $X : \mathbb{T}\ Y : \mathcal{V}$ and $X \to Y$ is fibered in $\mathsf{CS}_{\mathcal{V}}$, then $Y \in \mathsf{CS}_{\mathcal{V}}$

Example 2.13. *covering Aff-stacks = saturation of $\mathbb{T}$. Indeed: By definition, the saturation of $\mathbb{T}$ is is obtained by quotients of $\mathbb{T}$ by $\mathbb{T}$-covers. We have shown, that its closed under covers between affines.*

Proposition 2.14. *Let $\mathcal{V}$ be stable under finite limits and containing (covering) affines. X is a (covering) $\mathcal{V}$-stack iff it is in $\mathcal{V}$ and a (covering) geometric stack.*

Proof. The direct direction is clear. For the converse we apply the recursion principle to the property '$X \in \mathcal{V}$ implies X is a (covering) $\mathcal{V}$-stack'. If $X \in \mathbb{T}$, its clear. Otherwise its equipped with a geometric (c)atlas. The fibers are in $\mathcal{V}$, as they can be written as a fiberproduct of $S, X, 1 \in \mathcal{V}$. By induction all fibers are covering $\mathcal{V}$-stacks. $\square$

Proposition 2.15. *A sheaf X merely admits some affine $\operatorname{Spec} A \to X$ fibered in covering $\mathcal{V}$-stacks, iff its a geometric stack whose identity types are $\mathcal{V}$-stacks.*

Proof. The direct direction: By 1.1 the identity types are $\mathcal{V}$-stacks.
The converse direction: Choose a $\mathcal{V}$-atlas $S \to X$. As each fiber is in V and is a covering $\mathcal{V}$-stack, its a covering $\mathcal{V}$-stack by 2.14. $\square$

Definition 2.16. Let $n \geq -2$. A (covering) geometric n-stack is a (covering) geometric stack that is an n-type.

Proposition 2.17. *Let X be a sheaf. For all $n \geq 0$, the following are equivalent:*

1. *X is a (covering) geometric $n+1$-stack*
2. *X merely admits some map $S \to X$ out of a (covering) affine fibered in covering n-stacks*

3. *X merely admits some (covering) geometric n-stack $Y \to X$ fibered in covering n-stacks.*

Proof.

1 . $\Leftrightarrow$ 2. X is a (covering) geometric $n+1$ stack iff (2.10) its a (covering) geometric stack whose identity types are geometric n-stack iff (2.15) 2.

2 . $\Rightarrow$ 3. S is a (covering) geometric n-stack

3. $\Rightarrow$ 2 Y admits a map $S \to Y$ fibered in covering n-stacks with S (covering) affine, so the composition $S \to X$ will have the same property by 2.4.

□

2.3 Truncatedness

In this subsection we want to prove

Theorem 2.18. *Every geometric stack is n-truncated for some $n : \mathbb{N}$.*

Lemma 2.19. *Every covering $\mathcal{V}$-stack X is $\mathbb{T}$-merely inhabited.*

Proof.

- If X is in $\mathbb{T}$ then its clear.
- If X is obtained by a quotient, we have a map $\operatorname{Spec} A \to X$ with domain in $\mathbb{T}$. Now use that we get a map on $\mathbb{T}$-propositional-truncations and that Spec A is T-merely inhabited.

□

Lemma 2.20. *Let X be an $n+1$-type and Y a sheaf. If $X \to Y$ is a n-truncated $\mathbb{T}$-surjective map, then Y is an $n+1$-type.*

Proof. Use that $\mathsf{is-n-truncated}(y = y')$ is a sheaf for $y, y' : Y$. □

Proof. of the theorem. We apply the recursion principle for geometric stacks.

- If Y is affine its clear with $n = 0$.
- Assume Y is equipped with a $\mathcal{V}$-atlas $f : S \to Y$, such that every fiber in n-truncated for some n. f is $\mathbb{T}$-surjetive by 2.19. We apply 2.20. So it remains to find an n such that all fibers are n-truncated. For any $x : S$, By induction $\mathrm{fib}_f(fx)$ is n-truncated for some n. By projectivity of S, we find some n such that $\mathrm{fib}_f(fx)$ is n-truncated for all $x : S$. For general $y : Y$, using that is-n-truncated $\mathrm{fib}_f\, y$ is a sheaf , we can conclude by $\mathbb{T}$-surjectivity of f.

□

2.4 Descent

For this subsection lets assume $\mathcal{V}$ a subuniverse (stable under $\sum$), that satisfies: If $Y \in \mathcal{V}$, then $L_{\mathbb{T}}Y \in \mathcal{V}$. St a class of sheaves, such that $\mathbb{T}$ is contained in it and for any map $X \to Y$ of sheaves in $\mathcal{V}$ fibered in $\mathbb{T}$, $X \in \mathsf{St}$ iff $Y \in \mathsf{St}$. We call types in this class stacky.

Lemma 2.21. *Let $\mathbb{T}$ satisfy descent, i.e. beeing affine in the topology is a sheaf. If Y admits a $\mathbb{T}$-cover $f : X \to Y$ where $Y \in \mathcal{V}$ is seperated, then there is a $\mathbb{T}$-cover $X \to \bullet Y$.*

Proof. Consider $X \xrightarrow{f} Y \xrightarrow{\eta} \bullet Y$. As beeing affine in $\mathbb{T}$ is a sheaf, we may just show that for all $y : Y$, the fibers over $\eta y : \bullet Y$ are in $\mathbb{T}$. As η is a monomorphism by 4.4 , η restricts to an equivalence

$$\mathrm{fib}_f\, y \to \mathrm{fib}_{\eta f}(\eta y)$$

But the left hand side is in $\mathbb{T}$ by assumption. □

Lemma 2.22. *Assume $\mathbb{T}$ have descent. Let $X \in \mathsf{St}$ and $Y \in \mathcal{V}$. Let $f : X \twoheadrightarrow Y$ be fibered in $\mathbb{T}$ and surjective. Then $\bullet Y$ is stacky.*

Proof. As X is stacky, it suffices to show, that $\bullet Y$ admits a $\mathbb{T}$-cover. We want to apply 2.21. So it remains to show, that Y is seperated. By surjectivity of f we may only show that for any $x : X, y : Y$, the type $fx =_Y y$ is a sheaf. If we define U to be the fiber over y, it is in $\mathbb{T}$ by assumption. But then $fx =_Y y$ is the outer pullback

$$\begin{tikzcd}
fx = y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near start] & U \in \mathbb{T} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near start] & 1 \arrow[d, "y"'] \\
1 \arrow[r, "x"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

of stacky types, in particular sheaves. □(Claim)

□

Theorem 2.23. *Assume $\mathbb{T}$ have descent. Then St is a sheaf.*

Proof. St is seperated: This follows from the embedding St into the seperated (TODO) type of sheaves.
Let $U \in \mathbb{T}$ and $P : \|U\| \to \mathsf{St}$. We want to construct a filler

$$\begin{tikzcd}
\|U\| \arrow[r, "P"] \arrow[d] & \mathsf{St} \\
1 \arrow[ur, dashed] &
\end{tikzcd}$$

Claim: $\bullet(\sum_{x:\|U\|} Px)$ is stacky.

Proof. of the claim. We want to apply the previous lemma to the map

$$\sum_{x:U} P|x| \to \sum_{x:\|U\|} Px$$

The domain is in St by stability under $\sum$. The fibers are equivalent to $U \in \mathbb{T} \subset \mathsf{St}$. □

The claim provides the map $1 \to \mathsf{St}$. The diagram commutes: Assuming $x : \|\operatorname{Spec} A\|$ we wish to show $Px = \sum_{x:\|U\|} Px$. Using univalence, we may show that the maps

$$Px \to \sum_{x:\|U\|} Px \xrightarrow{\eta} \bullet \sum_{x:\|U\|} Px$$

are both equivalences. The first one is an equivalence as $\|U\|$ is contractible. Hence the middle term is a sheaf, thus the unit map is an equivalence as well.

□

Corollary. *(covering) geometric stacks satisfy descent.*

Lemma 2.24 (TODO). *If Y is an n type , then $L_{\mathbb{T}} Y$ is an n-type.*

Corollary. *For all $n : \mathbb{N}$, the class of (covering) (n-)stacks has descent.*

Proof. We set $\mathcal{V}$ as the n-truncated-types which is fine by the lemma.: □

3 Saturated Topologies

Consider a topology $\mathbb{T}$ finer than the Zariski topology.

Definition 3.1. A covering atlas of X is some $\hat{X} \in \mathbb{T}, \hat{X} \to X$ $\mathbb{T}$-cover

Definition 3.2. $\mathbb{T}$ is saturated if Beeing in the topology descents along $\mathbb{T}$-covers between affines, i.e. every affine schemes that has a covering atlas lies itself in $\mathbb{T}$.
The saturated closure of a topology $\mathbb{T}$ is the topology $\mathbb{T}'$ defined by (todo finite sums of?)

$$X \in \mathbb{T}' \text{ iff } X \text{ is affine} \wedge \exists \text{ covering atlas of } X$$

Lemma 3.3. *Using ZLC, this is the smallest saturated topology containing* $\mathbb{T}$.

Proof. Obviously $1 \in \mathbb{T}'$. Types which have a covering atlas are stable by dependent sums by the proof of **??**. For the saturatedness consider some $\mathbb{T}'$-cover $\mathbb{T}' \ni X' \to X$. By replacing X' with some covering atlas, we may assume that $X' \in \mathbb{T}$. As every fiber $X'_x \in \mathbb{T}'$, we merely find a covering atlas $\tilde{X}'_x \to X'_x$. Then by Zariski local choice there exists a Zariski atlas $\hat{X} \to X$ and a commutative diagram

$$\begin{tikzcd}
Y \equiv \sum_{x:\hat{X}} \tilde{X}'_x \arrow[r] \arrow[d] & \sum_{x:X} X'_x = X' \arrow[d] \\
\hat{X} \arrow[r, "Zar"'] & X
\end{tikzcd}$$

As $X' \in \mathbb{T}$ and $Y \to X'$ is fibered in $\mathbb{T}$ (6.3) we have $Y \in \mathbb{T}$. But $Y \to \hat{X}$ is a $\mathbb{T}$-cover and $\hat{X} \to X$ is a $\mathbb{T}$-cover, $Y \to X$ is a $\mathbb{T}$-cover. Hence $X \in \mathbb{T}'$. $\square$

Lemma 3.4. *A type T is a sheaf wrt to $\mathbb{T}'$ iff it is a sheaf wrt to* $\mathbb{T}$

Proof. As $\mathbb{T} \subset \mathbb{T}'$ the $\to$ direction is clear. Now, let $X \in \mathbb{T}'$. We have to show that $T \to T^{\|X\|}$ is an equivalence. Choose $\mathbb{T} \ni Y \to X$. Then we have a commutative diagram

$$\begin{tikzcd}
T \arrow[r] \arrow[dr, "\simeq"'] & T^{\|X\|} \arrow[d] \\
 & T^{\|Y\|}
\end{tikzcd}$$

So $T \to T^{\|X\|}$ has a left-inverse. Thus it suffices to show that any $f : T^{\|X\|}$ has a preimage. Choose $t : T$, s.th. cnst_t^Y is the composite $\|Y\| \to \|X\| \xrightarrow{f} T$. We have $\|Y\| \to (\mathrm{cnst}_t^X = f)$. But as $Y \in \mathbb{T}$ and $\Delta_t = f$ is a sheaf (as an identitytype in the sheaf $T^{\|X\|}$) we are done. $\square$

Remark 1. We never used that we only talk about $\mathbb{T}$-covers.

Lemma 3.5. *Every saturated affine (i.e.* $\operatorname{Spec} A \in \mathbb{T}'$*) it $\mathbb{T}$-merely inhabited.*

Proof. We have $\|X\| \to \|\operatorname{Spec} A\|$ for some covering atlas $\mathbb{T} \ni X \to \operatorname{Spec} A$. $\square$

Question 1. Does the converse hold, i.e. is every $\mathbb{T}$-merely inhabited affine saturated?

4 Lex Modalities

Lemma 4.1 (Stability resuls)**.** *Lex Modalities are stable under*

1. *Conjunction*

2. *Composition*

Lemma 4.2. *Let $\bigcirc$ be a lex-modality. Let X be $\bigcirc$-modal and $B : X \to \mathcal{U}_{\bigcirc}$ be a family of modal types. Then $\sum_{x:X} B_x$ is $\bigcirc$-modal*

Lemma 4.3. *Let $B : \bullet X \to \mathcal{U}$. Then $\bullet(\sum_{x:X} B(\eta x)) = \sum_{x:\bullet X} \bullet Bx$*

Proof. Observe that

$$\sum_{x:X} B\eta x \to \sum_{x:\bullet X} Bx$$

is a $\bullet$-equivalence, because for all modal types T, the type $Bx \to T$ is modal for any $x : \bullet X$. Then it follows by [ref?]. $\square$

Lemma 4.4. *For a type X the following are equivalent:*

- *the identity types of X are sheaves*

- *the unit $X \to \bullet X$ is a monomorphism*

In this case we call X seperated

5 Atlas

Definition 5.1. A $\mathbb{T}$-atlas of X is a $\mathbb{T}$-cover $\operatorname{Spec} A \to X$ out of an affine scheme.

Remark 2. Any good enough TODO scheme has a Zariski atlas. If $\mathbb{T}$ is finer than the Zariski-topology then in the definition we may replace affine scheme by good enough scheme, if its just about the question whether a type admits an atlas.

Example 5.2. *Let X be a (1-)type. X has a Zariski-atlas, iff there exists some $f : \operatorname{Spec} A \to X$ fibered in types of the form $\operatorname{Spec}(R_{f_1} \times \ldots \times R_{f_n})$ for $(f_1, \ldots, f_n) \in Um(R)$.*

Remark 3. If one applies ZLC to an affine scheme $\operatorname{Spec} A$ the resulting principal open cover $D(f_i), f_i \in A$ will induce indeed a zariski atlas $\bigsqcup D(f_i) \to \operatorname{Spec} A$, because the fiber over $x : \operatorname{Spec} A$ is $\bigsqcup D(f_i(x))$.

Question: Does every zariski atlas of $\operatorname{Spec} A$ have this form? **??**

Example 5.3. *$\mathbb{P}^n$ has a zariski atlas given by the standart homogeneous principal opens $\sum_{i=0}^{n} D_+(x_i)$. The fiber over a point $[y_0 : \ldots . y_n]$ is $D(y_0) + \ldots D(y_n)$ where $(y_1, \ldots, y_n) \in Um(R)$.*

6 Local Choice

In this section let $\mathbb{T}$ denote a topology finer than the zariski topology.

Definition 6.1. Let *Cov* be a class of morphisms (which we think of n-atlasses of some n), containing $\mathbb{T}$-atlas, (stable under pullback NECESSARY TODO?) A type S has *local choice* wrt *Cov* if for any $\mathbb{T}$ -surjective map $X \to Y$ and any map $f : S \to Y$ there exists a map $p' : S' \to S$ in *Cov* and a commutative diagram

$$\begin{tikzcd} S' \arrow[r, dashed] \arrow[d] & X \arrow[d, "p"] \\ S \arrow[r, "f"] & Y \end{tikzcd}$$

Proposition 6.2. *Assume that* Cov *is stable under composition.*

- *If $\hat{S} \to S$ is a* Cov*er and $\hat{S}$ has $\mathbb{T}$-local choice, then S has $\mathbb{T}$-local choice.*
- *Affine schemes have $\mathbb{T}$-local choice.*
- *Any type admitting a* Cov *- Atlas* $\operatorname{Spec} A \to S$ *has $\mathbb{T}$-local choice.*

Proof. The first point follows from stability under composition of *Cov* . the third point follows from the second. By the first point, we may assume that S is affine. As p is $\mathbb{T}$-surjective, for any $x : S$ there merely is a $\operatorname{Spec} B_x \in T$ and a map $\operatorname{Spec} B_x \to \| \operatorname{fib}_p(x)\|$. As S is projective, we have a term in

$$\prod_{x:S} \sum_{\operatorname{Spec} B_x \in T} \operatorname{Spec} B_x \to \| \operatorname{fib}_p(fx)\|$$

By setting

$$(S' := \sum_{x:S} \operatorname{Spec} B_x) \xrightarrow{\pi} S$$

the projection, we are now in the situation that for any $t : S'$ we merely have a point in $\operatorname{fib}_p((p'(t)))$ and $S' \to S$ is a $\mathbb{T}$-cover, thus it is in *Cov* . Moreover, S' is affine, as it is a dependent sum of affines. Hence again we now can find a lift $S' \to X$ making

$$\begin{tikzcd} S' \arrow[r] \arrow[d, "p'"'] & Y \arrow[d, "p"] \\ S \arrow[r, "f"] & X \end{tikzcd}$$

commute. ☐

The next lemma shows, that the class of types equipped with a $\mathbb{T}$-atlas is stable under dependent sums.

Lemma 6.3. *Let $\mathcal{U}' \subset \mathcal{U}$ be stable under dependent sums (e.g. $\mathbb{T}$-inhabited types) Let X be a type with a map $p : U \to X$ fibered in $\mathcal{U}'$. For any $x : X$, let Y_x be a type and moreover for any $u : U$, we are given a map $q_u : V_u \to Y_{p(u)}$ fibered in $\mathcal{U}'$. Then the induced map*

$$p : \sum_{u:U} V_u \to \sum_{x:X} Y_x$$

is fibered in $\mathcal{U}'$

Proof. The fiber of p over some $(x, y) \in \sum_{x:X} Y_x$ is given by

$$\sum_{u:\mathrm{fib}_p\, x} \mathrm{fib}_{q_u}(y')$$

where $y' : Y_{p(u)}$ (depending on u) is the transport of $y : Y_x$ along $x = p(u)$. As $\mathcal{U}'$ is stable under dependent sum those fibers are again in $\mathcal{U}'$. This shows the result. $\square$

Theorem 6.4. *Let $\mathcal{U}'$ be a class stable under dependent sums. The class of types admitting a $\mathcal{U}'$-atlas is closed under dependent sums. If $\mathbb{T}$ is a topology, the same holds for $\mathcal{U}'$-atlasses with domain in $\mathbb{T}$.*

Proof. Let us construct some atlas $\mathrm{Spec}\, A \to \sum_{x:X} B_x$ For any $x : X$ we merely have an atlas $V_x \to B_x$, i.e. with V_x affine. X has local choice wrt atlasses by (6.2) using $\mathcal{U}'$ is $\sum$-stable (we use the trivial topology).
If additionally, all the B_x and X are smooth n-stacks, just observe that we can choose the affine V_{pu} to lie in $\mathbb{T}$, Accordingly $\sum_{u:U} V_{pu} \in T$ as $\mathbb{T}$ is stable under Σ.
By Local choice for X, we merely find U affine, an atlas $p : U \to X$ with

$$\prod_{u:U} \sum_{V_{p(u)} \in T} (q : V_{p(u)} \to B_{p(u)}) \times (q \text{ fibered in smooth } n \text{ stacks })$$

Now the desired map is $\sum_{u:U} V_{pu} \to \sum_{x:X} B_x$, because it is an atlas by 6.3

$\square$

7 Fundamental Theorem of algebraic spaces

7.1 For groupoids

Lemma 7.1. *If $R \twoheadrightarrow X \to X$ is a $\mathbb{T}$-htpy-coequalizer diagram of two $\mathbb{T}$-covers between affines, then X is a 1-stack.*

7.2 For sets

Lemma 7.2. *Denote $\mathbb{T}Set$ for the sets that are $\mathbb{T}$-sheaves. Assume given a $\mathbb{T}set$ X then the following maps are mutually inverse*

$$\sum_{R:X\to X\to \mathbb{T}\,\mathrm{Prop}} R \text{ equivalence relation} \simeq \sum_{Y:\mathbb{T}\mathsf{Set}} \sum_{p:X\to Y} p \ \mathbb{T}\text{surjective}$$
$$R \mapsto (X/R, [\_])$$
$$\lambda x, y.(p(x) = p(y)) \leftarrow\!\shortmid (Y, p)$$

where X/R is defined by applying $L_T\|\_\|_0$ at the higher inductive type $X//R$.

Proof.

- Well-definedness: The map $[\_] : X \to \|X//R\|_0 \to L_T\|X//R\|_0$ is the composition of a surjective with a $\mathbb{T}$-surjective map [ref?], hence its $\mathbb{T}$-surjective.
  Conversely given (Y, p) as Y is a sheaf, we have for all $x, y : X$ that $p(x) =_Y p(y)$ is a sheaf.

- If $x, y : X$ then we have a chain of equivalences

  $$R(x, y) \simeq (\bar{x} =_{\|X//R\|_0} \bar{y}) \to ([x] =_{L_T\|X//R\|_0} [y])$$

  where the first map is plain HoTT and the second map is ap, i.e. the unit of the modality [ref?], but as the $\bar{x} =_{\|X//R\|_0} \bar{y}$ is already a sheaf, it is an isomorphism as well.

- Let (Y, p) be in the RHS. Let $R(x, y) = (p(x) = p(y)) : \mathbb{T}\,\mathrm{Prop}$. By plain HoTT, There is a map $\eta : X//R \to Y$ (defined by the universal property of the set truncation and by induction on the higher inductive type $X//R$ on canonical terms through the map $p : X \to Y$). I claim η exhibits Y as the localization for $\mathbb{T}\,\mathsf{Set}$-modality of $X//R$. Let T be another $\mathbb{T}\,\mathsf{Set}$ equipped with a map $X//R \to T$. By precomposition we obtain a map $X \to T$. Claim: it factors uniquely through $p : X \to Y$.

  $$\begin{tikzcd} X \arrow[r] \arrow[dr] & X//R \arrow[r] & T \\ & Y \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

  Proof:
  Existence: We want to define a map $Y \to T$. Let $y : Y$. As p is $\mathbb{T}$-surjective and T is a sheaf, we may assume we merely have some element in the fiber of p over y. Now push this element through

  $$\|\operatorname{fib}_p y\| \to \|X//R\|_0 \to T$$

  where the first map is by Plain HoTT and the second one is induced from $X//R \to T$ by assumption and the fact that T is a set.. One can easily check this makes the diagram commute. Uniqueness follows from $X \to Y$ beeing $\mathbb{T}$-surjective and the following Fact: Two parellel maps $Y \rightrightarrows T$ into a $\mathbb{T}\,\mathsf{Set}$ T are already equal if the become equal after

precomposition with a $\mathbb{T}$-surjection $X \to Y$.
Proof of the fact : Let $y : Y$. The goal is an identity type of a $\mathbb{T}\,\mathsf{Set}$, hence a $\mathbb{T}\,\mathrm{Prop}$. Hence As the fiber over y in X is $\mathbb{T}$-merely inhabited, we may assume an actual term in the fiber. As $X \to Y$ equalizes the arrows, this term allows us to conclude. □(fact) □(Claim)
We apply the fact to the ($\mathbb{T}$-)surjectivity of $X \to X//R$ to get a unique factorization

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[dr] & X//R \arrow[r] \arrow[d] & T \\ & Y \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

making the right triangle commute. This is what we wanted to show.

□

Definition 7.3. An equivalence relation R on a type X is called:

- redundant if for all $x, y : X$ the proposition $R(x, y)$ is a -1-stack.
- covering if its and for any $y : X$ its fibers:

$$R_y :\equiv \sum_{x:X} R(x, y)$$

are affine in $\mathbb{T}$.

Lemma 7.4. *Assume that $\mathbb{T}$ satisfies descent for propositions and for sets* **??**, *i.e. that a modal proposition being a (-1)-stack is a sheaf. Assume that a modal set beeing affine in $\mathbb{T}$ is a sheaf. Assume given a $\mathbb{T}$set X, then the following types are equivalent:*

- *The type of redundant covering equivalence relations over X.*
- *The type of $\mathbb{T}$sets Y with identity types beeing stacks and an -1-atlas X to Y (in V2 a $\mathbb{T}$-cover).*

Proof. By the equivalence in 7.2, it is enough to check that:

- The identity types in X/R are (-1)-stacks if and only if the relation R is redundant . For any $x, y : X$ we know that:

$$R(x, y) \simeq [x] =_{X/R} [y]$$

so the direct direction is immediate. For the converse we use the assumption that a modal proposition being a (-1)-stack is a sheaf and that the map $[\_] : X \to X/R$ is $\mathbb{T}$-surjective.

- The fibers of:

$$[\_] : X \to X/R$$

are affine in $\mathbb{T}$ if and only if the relation R is covering. For any $y : X$ we have that:

$$\sum_{x:X} R(x, y) \simeq \mathrm{fib}_{[\_]}([y])$$

so the direct direction is immediate. Here as well the converse follows from $\mathbb{T}$-surjectivity of $[\_]$ and that the topology has descent.

□

Corollary. *Assume $\mathbb{T}$ satisfies descent for propositions and for sets. A type is a 0-stack iff its merely the $\mathbb{T}$-quotient of an affine scheme by a covering equivalence relation.*

Theorem 7.5. *Assume $\mathbb{T}$ satisfies descent for propositions. The quotient of a* 0*-stack* $X \in \mathbb{T}\,\mathsf{Set}$ *by an* 0*-covering equivalence relation* R *is a* 0*-stack. TODO*

Proof. The identity types in X/R are propositional 0-stacks, hence (-1)-$\mathbb{T}$runcations of -1-stacks by **??** as desired.
How to find an atlas: todo. How to proceed, if we could choose all atlasses we want at the same time?

□

Remark 4. This is equivalent to saying that 1-stacks that are 0-types are geomeric 0-stacks: One direction we prove later. If R is a 0-covering equivalence relation on a 0-stack X, then X/R is a 1-stack by observing that any -1-atlas $X' \to X$ gives a 0-atlas $X' \to X \to X/R$. Moreover, X/R is a 0-type, hence by assumption a 0-stack.

Example 7.6. *There are open affine subschemes U of affine schemes* $\operatorname{Spec} A$*, which are not (disjoints unions of) principal open*

Proof. Consider $A = R[x,y,u,v]/(xy + ux^2 + vy^2), X = \operatorname{Spec} A$ and consider the open $U = D(x,y)$.
We cant expect U to be a disjoint union of principal opens (todo). However, $D(x,y)$ is affine: We have maps $U \to R$ given by $f = -v/x = (y+ux)/y^2, g = -u/y = (x+vy)/x^2$. Then $D(f) \cup D(g) = \operatorname{Spec} R^X$, as $yf + xg = 1$ in R^U. Taking preimages under the affinization map, $U_f \cup U_g = X$ and one checks this defines an open affine cover (for example : $U_f \simeq \operatorname{Spec} R[x,u,f^{\pm 1},g]/(xy + ux^2 + uy^2)$ with $y := (1-gx)/f$.) But on both of this open subsets the affinization map is an isomorphism hence the affinization of X is an isomorphism. compare (Hartshorne II.2.17) □

Lemma 7.7. *Let $f : X \to Y$ be surjective. There exists a Zariski Cover $X' \to X$ such that $X' \to Y$ is a Zariski cover iff there exists a Zariski Cover $X' \to X$, some $n : \mathbb{N}$ and an open affine embedding $X' \hookrightarrow Y^n$ over Y.*

8 Algebraic Space

Recall the notion of (covering) 0-stacks. it is the smallest pair of classes that satisfies the following

- Stability under $\sum$ **??**
- (covering) affines are (covering) algebraic spaces.
- stable under covering quotients: If X is an algebraic space, Y modal 0-type and $X \to Y$ is fibered in covering algebraic spaces, then Y is an algebraic space. Additionally, if X is covering, then Y is covering.

8.1 Geometric propositions

Definition 8.1. An affine Scheme U is called geometric, if

$$\|U\|_{\mathbb{T}} \to (U \in \mathbb{T})$$

Lemma 8.2. *The converse holds always*

Proof. because things in $\mathbb{T}$ are automatically $\mathbb{T}$-merely inhabited □

Recall the definition of $\mathbb{T}$-atlas 5.1

Definition 8.3. We call a modal proposition geometric, if one of the equivalent conditions is satisfied:

1. its merely of the form $\|U\|_{\mathbb{T}}$ for some geometric affine U.
2. There is a $\mathbb{T}$-surjective map out of a geometric affine U.
3. It has a $\mathbb{T}$-atlas.

Proof.

$1 \Leftrightarrow 2$ Clear.

$1 \Rightarrow 3$ we show that $U \to \|U\|_{\mathbb{T}}$ is a $\mathbb{T}$-atlas. Every fiber is in $\mathbb{T}$, because U is geometric.

$3 \Rightarrow 1$ Let $V \to P$ be a $\mathbb{T}$-atlas. have to show TFAE $\|V\|_{\mathbb{T}} \to P \to (V \in \mathbb{T}) \overset{8.2}{\to} \|V\|_{\mathbb{T}}$. Proof: $\|V\|_{\mathbb{T}} \to P$ as P is modal prop. Secondly, because $V \to P$ is a $\mathbb{T}$-cover.
Hence P is a geometric proposition.

□

Lemma 8.4. *geometric propositions are algebraic spaces.*

Proof. We have $U \to \|U\|_{\mathbb{T}}$ where U is affine, hence an algebraic space and the fibers are in $\mathbb{T}$ by geometricness of U, hence they are covering algebraic spaces. By stability under quotients, our geometric proposition is an algebraic space. □

8.2 Algebraic spaces

Definition 8.5. Consider a modal equivalence relation $R : U^2 \to \mathsf{GeomProp}$ on an affine U. We call it covering if one of the following equivalent conditions

- every fiber $R_s \equiv \sum_{t:S} Rst$ admits a $\mathbb{T}$-catlas.
- every fiber $R_s \equiv \sum_{t:S} Rst$ is a covering 0-stack.

Proof. Every type admitting a $\mathbb{T}$-catlas is a covering 0-stack. Conversely: if the fibers are covering 0-stacks. For all $t : S$ we can choose a geometric atlas $\operatorname{Spec} A_t \to Rst$ by 8.3. Then

$$\sum_{t:S} \operatorname{Spec} A_t \to \sum_{t:S} Rst$$

is a $\mathbb{T}$-atlas. As $\sum_{t:S} Rst$ is a covering 0-stack by assumption, the map has to be a $\mathbb{T}$-catlas by 2.8. □

Definition 8.6. A modal set X is an algebraic space iff it is merely of the form $L_{\mathbb{T}}(U/R)$ for some affine U and $R : U^2 \to \text{Prop}$ a covering equivalence relation. Equivalently there exists some map $U \to X$ whose fibers merely have $\mathbb{T}$-catlasses. We call X covering if U can be choosen to be in $\mathbb{T}$.

Lemma 8.7. *Every (covering) algebraic space is a (covering) geometric 0-stack.*

Proof. Choose a presentation $R : U^2 \to \text{Prop}$. It suffices to show, that the map $f : U \to L_{\mathbb{T}}(U/R)$ is a geometric (c)atlas. The map f is $\mathbb{T}$-surjective by the well-definedness of the bijection 7.2. By descent we may just show, that the fibers $\operatorname{fib}_f(f(s))$ for $s : U$ are covering 0-stacks. But by the bijection in 7.2 those are equivalent to the fibers R_s, which are covering 0-stacks as the equivalence relation is covering. □

Corollary. *The identity types of algebraic spaces are geometric propositions.*

Proof. By the previous lemma and 2.10 □

Lemma 8.8. *Let P be a sheaf and a proposition that admits a map $\operatorname{Spec} A \to P$ fibered in covering algebraic spaces. Then P is a geometric proposition.*

Proof. The fibers are covering algebraic spaces and affine, hence covering affine. By 8.3 we conclude. □

Theorem 8.9. *Let X be a sheaf of sets. Let S be (covering-) affine and $f : S \to X$ be fibered in covering algebraic spaces. Then X is a (covering) algebraic space.*

Proof. The identity types of X admit a map fibered in covering algebraic spaces (todo check stability under $\sum$) out of an affine by 1.1. by 8.8 they are geometric propositions. The equivalence relation determined by f is covering 8.5 , because the fibers of f are covering 0-stacks. □

9 Schemes are algebraic Spaces for the Zariski Topology

Definition 9.1. A proposition U is open iff its merely of the form $f_1\ inv \vee \ldots f_n inv$ for some $f_i : R$.

Definition 9.2. A Zariski sheaf X is a scheme if there merely exists some affine S map $S \to X$ whose fibers are Zariski-merely inhabited finite sums of open propositions

Lemma 9.3. *Given $f_1, \ldots, f_n : R$ such that $\|D(f_1) + \ldots + D(f_n)\|_{Zar}$ then $\sum_{i=1}^{n} D(f_i) \in Zar$.*

Proof. We have to show that $(f_1, \ldots, f_n) = 1$. Claim: $(f_1, \ldots, f_n) = 1$ is a sheaf. Proof: Recall that the Zariski topology is subcanonical ([ref?]). Now the type is equivalent to the statement that the map between the two sheaves $\operatorname{Spec} 0 \to \operatorname{Spec} R/(f_1, \ldots, f_n)$ is an equivalence. This is a sheaf [ref?]. $\square$

Proposition 9.4. *Every Zariski-merely-inhabited type that is merely of the form $U_1 + \ldots + U_n$ for open propositions U_i admits a zariski-catlas.*

Proof. By definition of openness, We can choose a surjection $\coprod_{j=1}^{n_i} D(f_{ij}) \twoheadrightarrow U_i$ for any i. We want to show, that the map

$$\coprod_{i,j} D(f_{ij}) \twoheadrightarrow U_1 + \ldots U_n$$

is a Zariski-catlas.

- Let us first show that the fibers are in Zar. Assume U_i holds. So we find a term in $\coprod_j D(f_{ij})$. In particular we have $\|\coprod_j D(f_{ij})\|_{Zar}$. By the lemma we conclude, that the fiber $\sum_j D(f_{ij})$ belongs to Zar.

- The total space is in Zar: This follows as the surjection after $\mathbb{T}$-truncation becomes an equivalence. As we have $\|U_1 + \ldots + U_n\|_{\mathbb{T}}$, we can conclude by the lemma.

$\square$

warning. The converse does not hold! Apply 2.21 to the map

$$Zar \ni 1 + 1 \to \sum D(f)$$

$\sum D(f)$ is seperated as $D(f)$ is a sheaf. All the fibers are equivalent to $1 + X$, hence they are in the Zariski topology. Use that beeing in the Zariski topology has Zariski-descent.

Corollary. *Every scheme is an algebraic space for the Zariski topology.*

Question 2. Is every algebraic space for the zariski topology a scheme?

10 Stability under Quotients

Definition 10.1. A morphism between n-stacks is covering if it is fibered in

- $\mathbb{T}$ if $n \leq 0$
- covering n-stacks if $n > 0$.

Theorem 10.2. *Let $f : X \to Y$ be a $\mathbb{T}$-surjective covering morphism between modal n-types. If X is a (covering) stack , then Y a (covering) stack.*

(*) This can only hold if we define -1-stacks to be modal propositions with a -2-atlas $\operatorname{Spec} A \to P$, i.e. algebraic propositions 8.3

Proof. Induction. For $n = -2$ its clear. Let X be a n-stack. Lets first construct the $n-1$-atlas of Y. We merely find a $V \twoheadrightarrow X$ which is an $n-1$-atlas. Then $V \to X \to Y$ is an n-atlas because it is $\mathbb{T}$-surjective and is fibered in the correct $\sum$-stable class of types, i.e. $\mathbb{T}$ if $n \leq 1$ and covering $n-1$-stacks for $n > 1$. Hence Y is an $n+1$-stack. As Y is an n-type, Y is an n-stack **??**.
If additionally X is assumed to be covering, then V can be assumed to lie in $\mathbb{T}$ which directly gives us that Y has a covering atlas.
It remains to show that the identity types of Y are $n-1$-stacks. As Y has an $n-1$-atlas, by 1.1 we find some $n-1$-atlas $p : W \to y = y'$. The map is covering. If $n = 0$, $y = y'$ is a -1-stack by (*). If $n > 0$, W is an $n-1$-stack and p is covering, so by induction $y = y'$ is an $n-1$-stack.

∎

Remark 5 (Using descent but not induction)**.** Hugo suggested an alternative argument proving that the identity types of Y are $n-1$-stacks, which presumable avoids **??** but uses descent for n-stacks: For $x : X, y : Y$ we have that

$$(f(x) = y) \simeq (1 \times_X \operatorname{fib}_f y)$$

is an n-stack by stability under $\sum$. Because it is an $n-1$-type, it is a $n-1$-stack by **??**. Now conclude that every identity type of Y is an $n-1$-stack by using descent for $n-1$-stacks and $\mathbb{T}$-surjectivity of f.

11 Local properties

Definition 11.1. Let *Cov* be the property of morphisms of n-stacks defined by asking that the morphism is $\mathbb{T}$-surjective and fibered in covering n-stacks. Its stable under basechange. A property of n-stacks is local if $P(1)$ holds, P is stable by dependent sums and given a *Cov*er $X \to Y$ we have PX iff PY.

Example 11.2. *beeing covering n-stack is a local property of stacks.*

Proof. We have to show: If $f : X \to Y$ is a $\mathbb{T}$-surjective map fibered in covering n-stacks between n-stacks, then X is a covering n-stack iff Y is a covering n-stack. The only if is clear by stability under dependent sums. The other direction is 10.2.

∎

Definition 11.3. A property of morphisms between n-stacks is local, if it is satisfied by identities, stable under composition and basechange/descent along *Cov* -maps, precomposition/right cancellability with *Cov* -maps.

Lemma 11.4. *Given a local property of types P. Then beeing fibered in P is a local property of morphisms.*

Lemma 11.5 ([ref?])**.** *Given a local property P of morphisms of n-stacks, a morphism $f : X \to Y$ has P if there exists an n-atlas of f having P.*

Example 11.6. *A morphism of n-stacks is covering iff there exists an n-atlas of f*

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "\tilde{f}"'] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

such that $\tilde{f}$ is a $\mathbb{T}$-cover.

The previous lemma tells us that we have the correct notion of covering morphisms between n-stacks for $n = 0, 1$.