Geometric stacks in Synthetic Algebraic Geometry

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Abstract

This is meant as a short summary of the progress of the Master Thesis of Tim Lichtnau so far. We work in Synthetic Algebraic Geometry, i.e. Homotopy Type Theory + 3 Axioms. A type gets interpreted as a Zariski-sheaf on the site given by the opposite category of finitely presented algebras over a fixed ring [CCH23]. Affine types get interpreted as the representable sheaves.

Definition 1.1. A Grothendieck topology $\mathbb{T}$ is a subclass of affine schemes, such that

- $1 \in \mathbb{T}$
- $\mathbb{T}$ is $\sum$-stable, i.e. if $X : \mathbb{T}, B : X \to \mathbb{T}$, then $\sum_{x:X} Bx$ belongs to $\mathbb{T}$.

A map is a $\mathbb{T}$-cover iff its fibered in $\mathbb{T}$. Relating to the classical definition, we could call a family of maps of affines $\{U_i \to U\}_{i=1}^n$ covering iff

$$\sum_{i=1}^{n} U_i \to U$$

is a $\mathbb{T}$-cover.
We fix a topology $\mathbb{T}$, for which we want to define the notion of (geometric) stack.

Definition 1.2. A type X is a (higher) stack iff its $\|\operatorname{Spec} A\|$-local for any $\operatorname{Spec} A \in \mathbb{T}$. An n-stack is a stack that is an n-type.

This exactly captures the expected classical notion of a stack for a Grothendieck topology [Moe24b], e.g. a 0-type X is a stack, if for any $\mathbb{T}$-cover $A \to B$,

$$X^B \to X^A \rightrightarrows X^{A \times_B A}$$

is an equalizer diagram.

Example 1.3 ([Moe24a])**.**

- *The fppf topology is given by the faithfully flat affine schemes.*
- *The étale topology is given by formaly-étale + faithfully flat affine schemes.*
- *The smooth topology is given by smooth + faithfully flat affine schemes.*

We start by defining the relative setting (this corresponds to fibers of smooth morphism of geometric stacks in [Sim96]). There is a short inductive definition:

Definition 1.4. A stack X is covering, whenever inductively

- $X \in \mathbb{T}$ or
- X is equipped with a map $\mathbb{T} \ni \operatorname{Spec} A \to X$ fibered in covering stacks.

We call a map $X \to Y$ fibered in covering stacks a geometric cover (That corresponds to smooth morphisms in [Sim96])

Definition 1.5. A stack X is geometric iff it merely admits a geometric cover $\operatorname{Spec} A \to X$.

We have the following classical labels associated to geometric stacks depending on the topologies:

$\mathbb{T}$	Geometric stacks for $\mathbb{T}$
étale	(Higher) Deligne Mumford Stacks
smooth	(Higher) Artin stacks
fppf	Something similar to Artin stacks ?

Example 1.6. *Every stack that is a scheme is geometric.*

Theorem 1.7 (Stability Results)**.**

- *The class of covering / geometric stacks is $\sum$-stable.*
- *The class of covering / geometric stacks is closed under quotients: If $X \to Y$ is a geometric cover with X covering / geometric stack, then Y is covering / geometric*
- *Geometric stacks are closed under taking identity types.*
- *Every geometric stack is a geometric n-stack for some n*
- *Covering / Geometric stacks have descent: Both types* GeometricStack *and* CoveringStack *are a stack.*

It is maybe worth mentioning, that proving descent was surprisingly easy.

Under some very mild condition on the topology [1] (e.g. satisfied by fppf), which is equivalent to saying that every geometric cover between affines is a $\mathbb{T}$-cover we have the following explicit description depending on the truncation level n:

Theorem 1.8. *An n-stack X is geometric if and only if*

$(n = 0)$*: there merely exists a map* $\operatorname{Spec} A \to X$ *whose fibers F merely admit a $\mathbb{T}$-cover* $\mathbb{T} \ni \operatorname{Spec} B \to F$. [2]

$(n \geq 1)$*: there merely exist a map* $\operatorname{Spec} A \to X$ *whose fibers are covering $(n-1)$-stacks. Additionally X is covering iff we can choose* $\operatorname{Spec} A$ *to lie in $\mathbb{T}$.*

For the étale topology we have the following notable results

Theorem 1.9.

- *Every Deligne-Mumford stack is a 1-gerbe, i.e. $X \to \|X\|_1^{\mathbb{T}}$ is a geometric cover, where the latter means the $\mathbb{T}$-sheafification of the 1-truncation of X.*
- *A Deligne-Mumford stack X is covering iff $\pi_0^{\mathbb{T}} X := \|X\|_0^{\mathbb{T}}$ and all higher homotopy groups*
$$\pi_i^{\mathbb{T}}(X, x) = \|\Omega^i(X, x)\|_0^{\mathbb{T}}, i \geq 1$$
are covering algebraic spaces for the étale topology.

[1] We can always enforce this condition without changing the notion of (covering / geometric) stack

[2] One can reformulate also as taking a quotient of $\operatorname{Spec} A$ by an equivalence relation satisfying a certain property.

References

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