

Thesis

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1 Introduction to SAG

Lemma 1.1. *R is local, i.e. if $x, y : R$ and $x \neq y$, then x is invertible or y is invertible.*

Lemma 1.2. *If $\text{char} \neq 2$, Let $\rho \neq 0$, then $x^2 = \rho^2$ implies $x = \rho$ or $x = -\rho$*

Proof. Indeed, as $\rho \neq -\rho$, one of them is invertible by 1.1 □

Example for zariski local choice

Example 1.3. *For some A and $g, g' : A$ define*

$$g|_A g' \equiv \|\{h : A \mid hg =_A g'\}\|$$

Claim: For any $g, g' : A$, we have

$$g|_A g' \leftrightarrow \forall x : \text{Spec } A, gx|_R g'x$$

Proof. \rightarrow is obvious using that the duality map is an algebra isomorphism.

\leftarrow . For any $x : \text{Spec } A$ we merely find some $h : R$ with $h \cdot g(x) = g'(x)$, i.e. we define our family of inhabited types as

$$Bx = \{h : R \mid h \cdot g(x) = g'(x)\}$$

By zariski local choice we merely find some principal open cover $\text{Spec } A = \bigcup_{i=1}^n D(f_i)$ and local sections

$$\begin{aligned} & \prod_{x:D(f_i)} \{h_i : R \mid h_i \cdot g(x) = g'(x)\} \\ & \stackrel{??}{\simeq} \{h_i : D(f_i) \rightarrow R \mid (h_i x) \cdot g(x) = g'(x)\} \\ & \stackrel{??}{\simeq} \left\{ h_i : A_{f_i} \mid h_i \cdot \frac{g}{1} =_{A_{f_i}} \frac{g'}{1} \right\} \end{aligned}$$

We can multiply h_i by high enough powers of f_i to obtain some $h_i : A$ with $h_i \cdot g = g' \cdot f_i^n$ for some $n : \mathbb{N}$. we may assume that n does not depend on $i = 1, \dots, n$ by taking the maximum and multiplying the h_i again with enough powers of f_i . Now use $??$ to write $1 = \sum_{i=1}^n \ell_i f_i^n$ for some $\ell_i : A$ and then

$$\left(\sum_i \ell_i h_i \right) \cdot g = \sum_i \ell_i f_i^n g' = 1 g' = g'$$

□

2 Preparation

Lemma 2.1 (Strong boundedness, NEEDED?). *Consider a sequence of embeddings of types*

$$X_0 \xrightarrow{\iota_0} X_1 \xrightarrow{\iota_1} X_2 \dots$$

Then any map $f : \text{Spec } A \rightarrow \text{colim}_n X_n \equiv: \bigcup_n X_n$ factors through some $\kappa_m : X_m \hookrightarrow \text{colim}_n X_n$.

Proof. For every term $x : \text{Spec } A$ consider the subset S_x of natural numbers n , such that $f(x) \in \text{im } \kappa_m$. Its a merely inhabited upwards closed subset. By the strong boundedness principle [ref?], the subset $\bigcap_{x:\text{Spec } A} S_x$ is merely inhabited. □

Lemma 2.2. *Let Y be a type, which admits a jointly surjective family of maps with smooth domain $X_i \rightarrow Y$ Then Y is formally smooth.*

Proof. $\sum_{n:\mathbb{N}} X_n \rightarrow Y$ is surjective with formally smooth domain, as \mathbb{N} is formally smooth. □

Corollary 2.3 (Monoid is smooth). *Let $(Y, +)$ be a magma, which is generated by a map with smooth domain $f : X \rightarrow Y$, i.e. every $a : Y$ can merely be written as a finite sum*

$$a = f(x_1) + \dots + f(x_n)$$

Then Y is formally smooth.

Lemma 2.4. *Let C be a class of types stable under \sum . Let $\mathbb{P} \subset \text{Aff}$ (in most cases $\mathbb{P} := \text{Aff}$) be any subclass of affines stable under finite limits. The class $\text{HasAtlas}_{\mathbb{P}}^{\mathbb{P}}$ of types Y which admit a map $\mathbb{P} \ni S \rightarrow Y$ fibered in C is stable under identity types.*

Proof. By assumption we can choose a map $\mathbb{P} \ni V \xrightarrow{p} Y$ fibered in C . Let $y, y' : Y$. Then we have the map

$$\begin{aligned} & (\text{fib}_p y) \times_V (\text{fib}_p y') \rightarrow y = y' \\ & (v, q : y = pv), (v', q' : y' = pv'), (h : v = v') \mapsto q \cdot h \cdot q'^{-1} \end{aligned}$$

The fiber over $j : y = y'$ looks like

$$\sum_v \underbrace{\left(\sum_{v'} (h : v = v') \right)}_{\text{isContr}} \times (q : y = pv) \times (q' : y' = pv') \times (q \cdot h \cdot q'^{-1} = j) \simeq \sum_v (v = py) \simeq \text{fib}_p y$$

Hence the map is fibered in C . It suffices to show, that $(\text{fib}_p y) \times_V (\text{fib}_p y')$ has an atlas, because then we can compose such an atlas with the above map to obtain an atlas of $y = y'$. By assumption the fibers of p have an atlas, so we can choose $q : W \rightarrow \text{fib}_p y, q' : W' \rightarrow \text{fib}_p y'$ atlases. Then $W \times_V W' \rightarrow (\text{fib}_p y) \times_V (\text{fib}_p y')$ is an atlas: The domain is a fiber product of types in \mathbb{P} , hence it belongs to \mathbb{P} . The fiber over (x, x') is equivalent to the product of fibers $(\text{fib}_q x) \times (\text{fib}_{q'} x')$ which is in C by stability under dependent sums (so in particular under finite products). \square

Lemma 2.5. *Let $\mathcal{U}' \subset \mathcal{U}$ be stable under dependent sums. Let X be a type with a map $p : U \rightarrow X$ fibered in \mathcal{U}' . For any $x : X$, let Y_x be a type and moreover for any $u : U$, we are given a map $q_u : V_u \rightarrow Y_{p(u)}$ fibered in \mathcal{U}' . Then the induced map*

$$p : \sum_{u:U} V_u \rightarrow \sum_{x:X} Y_x$$

is fibered in \mathcal{U}'

Proof. The fiber of p over some $(x, y) \in \sum_{x:X} Y_x$ is given by

$$\sum_{u:\text{fib}_p x} \text{fib}_{q_u}(y')$$

where $y' : Y_{p(u)}$ (depending on u) is the transport of $y : Y_x$ along $x = p(u)$. As \mathcal{U}' is stable under dependent sum those fibers are again in \mathcal{U}' . This shows the result. \square

3 Talk: Algebraic Spaces: (Counter-)examples

We assume today, that schemes are étale sheaves.

Notation.

$$\mathbb{T} = \{X : \text{Aff} \mid X \text{ formally étale} + \underbrace{\text{faithfully flat}}_{= \text{flat} + \neg\neg \text{ inhabited}}\}$$

\mathbb{T} contains 1 and is stable under \sum . A \mathbb{T} -cover is a map fibered in \mathbb{T} .

Definition 3.1. A naive algebraic space is an étale -sheaf X , that merely admits a \mathbb{T} -cover $\text{Spec } B \rightarrow F$.

We call it covering, if we can choose $\text{Spec } B \in \mathbb{T}$.

This is not good enough, because we can NOT prove the following

- all schemes are naive algebraic spaces
- naive algebraic spaces having descent, i.e. the type of them is an étale -stack.

Instead we have to do asking for an atlas twice.

Definition 3.2. An algebraic space is an étale -sheaf X , such that one of the following equivalent conditions holds:

- merely we find a map $\text{Spec } A \rightarrow X$ such that each fiber is a covering naive algebraic space.
- We merely can express it as the sheaf-quotient of an affine $\text{Spec } A$ by an equivalence relation R that is covering, i.e. for each $x : \text{Spec } A$, $[x] := \sum_{y : \text{Spec } A} R(x, y)$ is a covering naive algebraic space.

We call X covering, if we can choose $\text{Spec } A$ to lie in \mathbb{T} .

This class is stable under \sum and under quotients by covering equivalence relations

Theorem 3.3 (DESCENT). *The type of (covering) algebraic spaces is an étale -stack.*

Example 3.4. *Schemes are algebraic spaces!*

Proof. Open propositions are naive algebraic spaces. A merely inhabited finite sum of open propositions is covering. \square

Question 1. Can we find algebraic spaces that are not schemes?
Can we prove with them, that Schemes do not have descent?

3.1 Quotients by Group actions

Let $\ell \neq 0$ denote a prime. Consider $\mu_\ell = R[X]/(X^\ell - 1)$.

Example 3.5 (Non-free action). *If $2 \neq 0$, the sheaf-quotient of \mathbb{A}^1 by the non-free μ_2 action is not an algebraic space!*

This suggests that we need a free action. However

Example 3.6. *The quotient of \mathbb{A}^\times by the free μ_ℓ action gives a scheme.*

Having a free action on the whole space might be not good enough to cook up examples of algebraic spaces that are not schemes.

Construction. Given a formally étale + flat affine (e.g. μ_ℓ or finite) group G that acts on an affine $\text{Spec } A$. Assume G acts free on some open subset $U \subset \text{Spec } A$. Then we construct a covering equivalence relation R on $\text{Spec } A$, such that

- for any $x : U$ and $y : \text{Spec } A$

$$R(x, y) \leftrightarrow \sum_{g \in G} gx = y.$$

- for some $y : \text{Spec } A \setminus U$, we have $R(x, y) \leftrightarrow x = y$.

We write $\text{Spec } A/G|_U \equiv \text{Spec } A/R$ and call it the quotient of $\text{Spec } A$ by the G -action on U .

Proof.

$$R_G(x, y) \equiv (x = y) + (x \in U) \times \sum_{g \in G \setminus \{1\}} gx = y$$

This is covering: For any $x : \text{Spec } A$ we have

$$\sum_{y : X} x = y + (x \neq 0) \times \sum_{g \in G \setminus \{1\}} gx = y \simeq 1 + (x \in U) \times G \setminus \{1\}$$

$G \setminus \{1\} = \sum_{g \in G} g \neq 1$ is a \sum of formally étale + flat affines (recall that formally étale affines have decidable equality).

Indeed, the two conditions hold, using that G has decidable equality. \square

Example 3.7 (Free action). *Set $U = \text{Spec } A$. Then this construction yields the actual group quotient: $\text{Spec } A/G|_{\text{Spec } A} = \text{Spec } A/G$.*

Proof. Indeed, the equivalence relation is the same, using that G has decidable equality. \square

Notation. If $U = \text{Spec } A \setminus Z$ the complement of a closed subset we write

$$U \equiv Z^c$$

Example 3.8 (Quotient of the Line). *If $\ell \neq 0$ is prime, we have $R/\mu_\ell|_{0^c}$ is an algebraic space*

Example 3.9 (Quotient of the Cross). *Let μ_ℓ act on $X = \text{Spec } R[X, Y]/X^\ell - Y^\ell$ via multiplication on the second component. Then*

$$X/\mu_\ell|_{\{0,0\}^c}$$

is an algebraic space.

Are those schemes?

3.2 Not a scheme?

Definition 3.10. A pointed-free action of G on a pointed type $(X, 0)$ is a G -action that has 0 as a fixpoint such that if $gx = x$ for some $g \neq 1$, then $x = 0$.

Definition 3.11. A point $0 : \text{Spec } B$ is regular, if $\text{Spec } B \setminus \{0\} = D(p_1, \dots, p_n)$ for some $p_1, \dots, p_n : B$ jointly-regular, i.e. if $p_i^m \cdot b = 0$ for all $i = 1, \dots, n$ then $b = 0$.

If $0 : X$ is a point of a scheme, we call it regular, if one of the following equivalent conditions is satisfied

1. it admits some open affine neighborhood U such that $0 : U$ is regular.

2. It is a regular point of any open affine neighborhood.

Theorem 3.12. *Let $0 : \text{Spec } A$ be a regular point. Let G be a non-trivial formally étale flat affine group pointed- freely acting on the pointed affine $(\text{Spec } A, 0)$. Then $\text{Spec } A/G|_{0^c}$ from 4.19 is non-locally-separated, In particular not a scheme.*

Example 3.13 (Non locally-separated examples). *Assume $\ell \neq 0$ prime. Let μ_ℓ act on $\text{Spec } B$ in one of the following ways:*

1. Let μ_ℓ act on $\text{Spec } B = \mathbb{A}^1$.
2. Let μ_ℓ act on

$$\text{Spec } B \equiv \sum_{x,y:R} x^\ell = y^\ell$$

$$\text{via } g(x, y) = (x, gy)$$

Then $\text{Spec } B/\mu_\ell|_{0^c}$ is an algebraic space that is not a scheme.

Proof. $\neg\neg$ merely, μ_ℓ is finite ([ref?]) and $\mu_\ell \setminus \{1\}$ is inhabited by 4.11.

1. Pointed-Free action is clear. $0 : \mathbb{A}^1$ is a regular point by first projection.
2. Pointed-Free action is clear. The cross is regular pointed, witnessed by the first projection: It is regular vanishing at $(0,0)$ And for any point $(0, y) : \text{Spec } B$ we deduce $y^\ell = -0^\ell = 0$, hence $\neg\neg(x, y) = (0, 0)$.

□

3.3 Fiber Collapse!

An alternative approach to construct algebraic spaces is the fiber collapse away from the origin.

Definition 3.14. Let $Y_\bullet : R \rightarrow \text{Aff}$ be a dependent family of affines The fiber collapse of Y_\bullet away from the origin $-Y_\bullet-$ is the following type over R

$$\sum_{x:R} (x \neq 0) \star Y_x \rightarrow R$$

This space over R looks exactly like the line away from the origin and over an infinitesimal ε the fiber is Y_ε .

Lemma 3.15. *Assume that if $x \neq 0$, then Y_x is formally étale + flat. Then $-Y-$ is an algebraic space.*

Proof. Let $x : R$. Let $Y : \text{Aff}$ such that $x \neq 0$ implies that Y is formally étale + flat. We will show that $\eta : Y \rightarrow (x \neq 0) \star Y$ is the sheaf-quotient map of the relation on Y given by $y \sim y' \equiv (y = y') + (x \neq 0) \times y \neq y'$, which is enough by 4.22. We apply ??

- The map is \mathbb{T} -surjective: We have a term in

$$\prod_{y:Y} \|\text{fib}_\eta(\eta y)\|_{\mathbb{T}}$$

and each $\|\text{fib}_\eta \eta y\|_{\mathbb{T}}$ is modal, i.e. contractible if $x \neq 0$. Hence we get

$$\prod_{y:(x \neq 0) \star Y} \|\text{fib}_\eta y\|_{\mathbb{T}}$$

- Given $y, y' : Y$, we have

$$\begin{aligned} \eta(y') = \eta(y) &\simeq (x \neq 0) \star (y = y') & | \text{ closed modality is lex ([2] Example 3.1.4).} \\ &\simeq (y = y') \vee (x \neq 0) & | (x \neq 0) \rightarrow \text{HasDecEq}(Y) \\ &\simeq (y = y') + (x \neq 0) \times y \neq y', \end{aligned}$$

□

Example 3.16. —Bool— is the line with two origins.

$\text{—Spec } R[X]/(X^2 + 1)\text{—}$ is the twisted line with two origins, i.e. over the origin we have the roots of -1 .

$\text{—Spec } R[Y]/(Y^2 - \bullet)\text{—}$ is $\mathbb{A}^1/\mu_2|_{0^c}$ which looks like $\mathbb{D}(1)$ over 0 .

$\text{—Spec } R[Y]/(Y^2 - \bullet^2)\text{—}$ is the quotient of the cross that looks like $\mathbb{D}(1)$ over every $\varepsilon : \mathbb{D}(1)$.

$\text{—Spec } R[Y]/(\bullet Y)\text{—}$ is the affine plus.

Proposition 3.17. Let G be a formally $+$ flat affine group. Let $p : \tilde{Y} \rightarrow R$ such that the pullback to R^\times can be enhanced to a G torsor over R^\times . Write $Y_x \equiv \text{fib}_p x$. Then there is a canonical equivalence

$$\begin{array}{ccc} & \tilde{Y} & \\ \swarrow & & \searrow \\ \tilde{Y}/G|_{(Y_0)^c} & \xrightarrow{\simeq} & \text{—}Y\text{—} \end{array}$$

Proof. As every non-base fiber is merely equivalent to G , its formally étale $+$ flat. In between you can put $\sum_{x:R} Y_x/G|_{x \neq 0 \times Y_x}$. As all three maps defined on \tilde{Y} are \mathbb{T} -surjective, by ?? we may only check that the identity types coincide. For any $y, y' : \tilde{Y}$. Using that if $py \neq 0$ then the G action on Y_{py} is a G -torsor, We have an equivalence

$$(y = y') + y \notin Y_0 \wedge \underbrace{\sum_{g \neq 1} gy = y'}_{\simeq (py = py') \wedge y \neq y'} \simeq (py = py') \wedge ((y = y') + py \neq 0 \wedge y \neq y')$$

If we fix $x : R$ and put $y, y' : Y_x$ we have, writing $\eta : Y_x \rightarrow (x \neq 0) \star Y_x$

$$(y = y') + py \neq 0 \wedge y \neq y' \simeq \eta y = \eta y'$$

by the proof of 3.15. □

3.4 Schemes do not have descent

For this section, let $\rho : R \setminus \{0\}$ denote a term, e.g. $\rho = 1$. Set $C = R[T]/(T^2 + \rho)$.

Proposition 3.18. If $\text{—Spec } C\text{—}$ is a scheme, then $X^2 + \rho$ has a root.

Proof. Let $p : \text{—Spec } C\text{—} \rightarrow R$ be the first projection. We proceed as follows

1. There is no open affine subset of $L(\text{Spec } C)$ containing $\text{fib}_p(0)$.
2. Any inhabited cover of $\text{Spec } C$ by open subsets strictly smaller than $\text{Spec } C$ yields a root.

Proves:

1. Because we want to show \perp , we may assume $\neg \text{Spec } C = \neg \text{Bool}$. Assume there is an open affine subset $\text{fib}_p(0) \subset U \subset \neg \text{Bool}$. Then $p(U) \subset R$ is an open neighborhood of 0, as

$$x \in p(U) \leftrightarrow (x, N) \in U \vee (x, S) \in U$$

Claim: the map $R^{p(U)} \rightarrow R^U$ is an equivalence. If we have shown that: As U is affine we conclude that the map

$$\begin{aligned} U &\rightarrow \text{Spec}(R^{p(U)}) \\ x &\mapsto \phi \mapsto (\phi(px)) \end{aligned}$$

is an equivalence, which is a contradiction to the assumption, that U contains both origins.

Proof of claim: Injectivity: If two maps $f, g : p(U) \rightarrow R$ coincide after precomposing with $U \rightarrow p(U)$, then they coincide away from 0 so conclude by ??.

Surjectivity: Given a map $U \rightarrow R$, by pulling back along $p : R + R \rightarrow \neg \text{Bool}$ we can view it as a map $R + R \supset U' \rightarrow R$ defined at both origins, so in particular as a pair of maps to R defined on some open neighborhood of 0 of R . They coincide away from 0 so by 4.15 they are equal.

2. As $\text{Spec } C$ is an étale sheaf, its enough to show to construct a function $\|\text{Spec } C\| \rightarrow \text{Spec } C$.
Define

$$\begin{aligned} A : \text{Spec } C &\rightarrow \text{Prop}^{\text{Fin}(n)} \\ x &\mapsto \{j : x \in U_j\} \end{aligned}$$

where we transport $x : \text{Spec } C$ along $\text{Spec } C \xrightarrow{\sim} \text{fib}_p 0$. Observe

- (a) for any $x, x' : \text{Spec } C$,

$$\|Ax \cap Ax'\| \rightarrow \neg \neg (x = x') \xrightarrow{\text{DecEq}} x = x'$$

where the first implication follows like this : if $x \neq x'$ and $\text{fib}_p 0 = \{x, x'\} \subset U_j$ then we have a contradiction to the first point

- (b) For any $x : \text{Spec } C$, $\|Ax\|$.

Assume $\|\text{Spec } C\|$. Lets try to construct a term of the following type

$$\sum_{x : \text{Spec } C} \forall x' : \text{Spec } C, j : Ax, j' : Ax' \rightarrow j \leq j'$$

For this we may assume $\text{Spec } C = \text{Bool} = \{N, S\}$, as the above type is a proposition: If we have given two such minimal x_1, x_2 , we can set first $x' \equiv x_2$ and then $x' \equiv x_1$ respectively and then (by (b)) choosing $j : Ax_1, j' : Ax_2$ gives $j \leq j' \leq j$ so that $j : Ax_1 \cap Ax_2$ such that $x_1 = x_2$ by (a).

I will explain an algorithm to do the following: Given $n \geq 2$, and a pair of merely inhabited disjoint subsets A_N and A_S of $\text{Fin}(n)$, we can decide in which of the two we find the smaller number of $\text{Fin}(n)$.

Induction over n . If $n = 2$, then $\|A_N\|$, so we find a term in the proposition $(0 \in A_N) + (1 \in A_N)$. In the left case return S , in the right case N .

For $n \mapsto n + 1$, if $n \in A_N$ return S . If $n \in A_S$, return N . Otherwise both A_N and A_S are subset of $\text{Fin}(n)$, so conclude by induction.

□

Corollary 3.19. *Schemes do not have descent.*

Proof. If Schemes have descent, then $\text{---Spec } R[T]/(T^2 + \rho)\text{---} \in \mathbf{Sch}$ is a sheaf. As $\text{---Spec } R[T]/(T^2 + \rho)\text{---}$ is \mathbb{T} -merely a scheme, it is a scheme, so by the previous lemma $T^2 + \rho$ has a root. Contradiction to [1] A . 0.3.

□

4 Algebraic spaces

Theorem 4.1. *Let X be a modal set. The following are equivalent:*

1. X is a (covering) geometric 0-stack
2. X is merely of the form $L_{\mathbb{T}}(U/R)$ for some (covering) affine U and $R : U^2 \rightarrow \text{Prop}_{\circ}$ a covering equivalence relation.
3. there exists some map $S \rightarrow X$ with S (covering) affine whose fibers merely have \mathbb{T} -atlases.

We call this class (covering) algebraic spaces.

Proof.

2 \leftrightarrow 3 This is ??

2 \rightarrow 1 Choose a presentation $R : U^2 \rightarrow \text{Prop}$. It suffices to show, that the map $f : U \rightarrow L_{\mathbb{T}}(U/R)$ is a geometric (c)atlas. The map f is \mathbb{T} -surjective by the well-definedness of the bijection ?? . By descent we may just show, that the fibers $\text{fib}_f(f(s))$ for $s : U$ are covering 0-stacks. But by the bijection in ?? those are equivalent to the fibers R_s , which are covering 0-stacks as the equivalence relation is covering.

1 \rightarrow 2 This can be reformulated in the following way, using the recursion principle for (covering) geometric 0-stacks: Let X be a sheaf of sets. Let S be (covering-) affine and $f : S \rightarrow X$ be fibered in covering algebraic spaces. Then X is a (covering) algebraic space. This follows from the observation, that the equivalence relation determined by f is covering ?? , because the fibers of f are covering 0-stacks.f

□

Proposition 4.2. *For any $n \geq 1$, we have inclusions*

$$W_n \subset \mathbf{CS}_{n-1} \subset W_{n+1}$$

Proof. Induction. $n = 1$ gives

$$\mathbf{HasCatlas}_{\mathbb{T}} \subset \mathbf{CS}_0 \subset \text{types admitting a catlas fibered in } W_1$$

the latter inclusion is the previous theorem.

The induction step is obtained by 4.3

□

4.1 Schemes are algebraic Spaces for the Zariski Topology

Definition 4.3. A proposition U is open iff its merely of the form $f_1 \text{ inv} \vee \dots \vee f_n \text{ inv}$ for some $f_i : R$.

Lemma 4.4. *Given $f_1, \dots, f_n : R$ such that $\|D(f_1) + \dots + D(f_n)\|$ then $\sum_{i=1}^n D(f_i) \in \mathbf{Zar}$.*

Proposition 4.5. *Every Zariski-merely-inhabited type that is merely of the form $U_1 + \dots + U_n$ for open propositions U_i admits a Zar-catlas.*

Proof. By definition of openness, We can choose a surjection $\coprod_{j=1}^{n_i} D(f_{ij}) \rightarrow U_i$ for any i . We want to show, that the map

$$\coprod_{i,j} D(f_{ij}) \rightarrow U_1 + \dots + U_n$$

is a Zar-catlas.

- Let us first show that the fibers are in Zar. Assume U_i holds. So we find a term in $\coprod_j D(f_{ij})$. In particular we have $\|\coprod_j D(f_{ij})\|_{\text{Zar}}$. By the lemma we conclude, that the fiber $\sum_j D(f_{ij})$ belongs to Zar.
- The total space is in Zar: This follows as the surjection after propositional truncation becomes an equivalence. As we have $\|U_1 + \dots + U_n\|$, we can conclude by the lemma. \square

Warning. The converse does not hold! We want to apply ??, to the map

$$\text{Zar} \ni 1 + 1 \rightarrow \sum D(f)$$

- $\sum D(f)$ is separated as $D(f)$ is a sheaf.
- All the fibers are equivalent to $1 + X$, hence they are in the Zariski topology.

Lemma 4.6. *let X be a scheme. There merely exists some affine S map $S \rightarrow X$ whose fibers are merely inhabited finite sums of open propositions*

Corollary 4.7. *Every scheme is an algebraic space.*

Lemma 4.8. *If X is an algebraic space, then the global sections embed via a R -algebra homomorphisms into a finitely presented R -algebra.*

Proof. Choose an atlas $S \rightarrow X$, in particular \mathbb{T} -surjective. As \mathbb{T} is subcanonical the map $R^X \rightarrow R^S$ is an injection. \square

Question 2. Is it an open embedding of types?

The goal of this subsection is to construct algebraic spaces. The first example actually gives us a scheme:

Example 4.9. *Let $p \neq 0$ be a prime. You can let $\mu_p := \text{Spec}(R[X]/(X^p - 1))$ act on \mathbb{A}^\times via multiplication. Set $\mathbb{T} = \text{fppf}$. Then the p .th power map*

$$\text{pow} : \|\mathbb{A}^\times / \mu_p\|_0^{\mathbb{T}} \rightarrow \mathbb{A}^\times$$

is an equivalence.

- *It is an embedding: First note, that $\|\mathbb{A}^\times / \mu_p\|_0$ is \mathbb{T} -separated: as μ_p act freely on \mathbb{A}^\times , $\mathbb{A}^\times / \mu_p$ is already a set. Meaning that the identity types of the set-quotient are $\sum_{g:\mu_p} gx =_{\mathbb{A}^\times} y$, hence sheaves. On the other hand the map $\|\mathbb{A}^\times / \mu_p\|_0 \rightarrow \mathbb{A}^\times$ is an embedding, as for any $x, y : \mathbb{A}^\times$ the map $(\sum_{g:\mu_p} gx = y) \rightarrow (x^p = y^p)$ is an equivalence.*
- *It is \mathbb{T} -surjective, as for any $\lambda : \mathbb{A}^\times$, we find $S = \text{Spec } R[X]/(X^p - \lambda) \in \mathbb{T}$ with*

$$S \rightarrow \text{fib}_{\mathbb{A}^\times / \mu_p \rightarrow \mathbb{A}^\times}(\lambda)$$

hence

$$1 = \|S\|_{\mathbb{T}} \rightarrow \|\text{fib}_{\text{pow}}\|_0^{\mathbb{T}}$$

Example 4.10. Let P be the open proposition $x \neq 0$ for some $x : \mathbb{A}^1$. Then $H = 1 + P$ is an open subgroup of $\mathbb{Z}/2$. The sheaf quotient G/H is the scheme $\text{Susp}(x \neq 0)$.

Let $\ell \neq 0$ denote a prime. Consider $\mu_\ell = R[X]/(X^\ell - 1)$.

Lemma 4.11. Let $(G, 1)$ be a pointed formally étale flat affine type. Then $(G \setminus \{1\})$ is formally étale + flat affine.

In particular $\mu_\ell \setminus \{1\}$ is a covering stack.

Proof. $G \setminus \{1\} = \sum_{g:G} g \neq 1$ is a \sum of formally étale + flat affines (recall that formally étale affines have decidable equality).

To show, that $\mu_\ell \setminus \{1\}$ is a covering stack, by ??, we need to show it is $\neg\neg$ -inhabited. Indeed as we want to prove a contradiction we may assume a term in $g : \text{Spec } R[X]/(\sum_{i=0}^{\ell-1} X^i)$. But this type is equivalent to $\mu_\ell \setminus \{1\}$, using that $\sum_{i=0}^{\ell-1} X^i | X^\ell - 1$ and $\ell \neq 0$. \square

Lemma 4.12. Given a modal equivalence relation R on a sheaf X and a 1-stack T and a map $f : X \rightarrow T$ and term $p : \prod_{x,y:X} R(x,y) \rightarrow fx = fy$ such that $p(x,y) \cdot p(y,z) = p(x,z)$, where the witnesses for R are left implicit. Then f factors through the quotient.

Lemma 4.13. Put $\ell = 2$. If $\ell \neq 0$, the sheaf quotient of \mathbb{A}^1 by the μ_2 action is not an algebraic space.

Proof. Assume this it is an algebraic space.

Set $\mathbb{D}(1) = \text{Spec } R[X]/X^\ell$. Then $\sum_{x:\mathbb{A}^1/\mu_\ell} x^\ell =_{\mathbb{A}^1} 0 \simeq \mathbb{D}(1)/\mu_\ell$ is an algebraic space by \sum -stability.

Then we can choose a geometric atlas $p : \text{Spec } A \rightarrow \mathbb{D}(1)/\mu_\ell$. We proceed in the following steps

1. There is an equivalence $\text{Spec } A \simeq \text{fib}_p 0 \times \mathbb{D}(1)/\mu_\ell$.
2. The fiber over 0 is affine
3. $\mathbb{D}(1)/\mu_\ell$ is $\neg\neg$ affine
4. $\mathbb{D}(1)/\mu_\ell$ is \neg affine

Proofs

1. Let us denote $F : \mathbb{D}(1)/\mu_2 \rightarrow \mathbf{CS}_0$ the bundle of fibers of f , where we note that the fibers are indeed sets. As \mathbf{CS}_0 is formally étale ([ref?]), we have terms

$$\phi : \prod_{x:\mathbb{D}(1)} F[x] = F[0], \phi^- : \prod_{x:\mathbb{D}(1)} F[-x] = F(0)$$

that both evaluate at $x = 0$ to $\text{refl}_{F[0]}$.

The goal is to produce a term in

$$\prod_{x:\mathbb{D}(1)/\mu_2} Fx = F[0]$$

By the previous lemma, using that \mathbf{CS}_0 is a 1-stack, we need to show, that under the path $p_x : [x] = [-x]$ in the quotient we have

$$\text{ap}_{p_x} F \cdot \phi^- x = \phi x$$

This proposition is formally étale as \mathbf{CS}_0 is formally étale. Thus we may assume the closed dense proposition $x = 0$. Then $p_x = \text{refl}_{[0]}$ and $\phi^- 0 = \text{refl} = \phi 0$ by assumption.

2. Let us first show, that We may assume that our geometric cover factors through the \mathbb{T} -surjection $\text{Spec } A \xrightarrow{f} \mathbb{D}(1) \rightarrow \mathbb{D}(1)/\mu_\ell$. Proof: By \mathbb{T} -local choice applied to the \mathbb{T} -surjection $\mathbb{D}(1) \rightarrow \mathbb{D}(1)/\mu_\ell$, we find a \mathbb{T} -cover $\text{Spec } B \rightarrow \text{Spec } A$ and a factorization

$$\begin{array}{ccc} \exists \text{Spec } B & \dashrightarrow & \mathbb{D}(1) \\ \downarrow & & \downarrow \\ \text{Spec } A & \longrightarrow & \mathbb{D}(1)/\mu_\ell \end{array}$$

□(Claim)

Its enough to see that the map $\text{fib}_f 0 \rightarrow F$ is an equivalence. That follows because 0 is a fix point of the μ_ℓ action on $\mathbb{D}(1)$.

3. F is a covering stack, hence $\neg\neg$ -inhabited. As the goal is $\neg\neg$ -modal, we may pick a map $1 \rightarrow F$. Then, by step 1

$$\mathbb{D}(1)/\mu_\ell = 1 \times_F (F \times \mathbb{D}(1)/\mu_\ell) = 1 \times_F \text{Spec } A$$

is a fiber product of affines, hence affine.

4. Here we need that $\ell = 2$. The affinization map would be induced by

$$\begin{array}{ccc} \mathbb{D}(1) & & \\ \downarrow & \searrow z \mapsto z^\ell & \\ \mathbb{D}(1)/\mu_\ell & \dashrightarrow & \mathbb{D}(1) \end{array}$$

But the map is not an embedding: For any $\varepsilon : \text{Spec } R[X]/X^\ell$, we have $\varepsilon^\ell = 0^\ell$ but $\varepsilon =_{\mathbb{D}(1)/\mu_\ell} 0$ iff there \mathbb{T} -merely exists some $g : \mu_\ell$ with $g\varepsilon = 0$, but as g is invertible this is equivalent to $\varepsilon = 0$.

□

4.2 Non locally-seperated Examples

Definition 4.14. A point $0 : \text{Spec } B$ is regular, if $\text{Spec } B \setminus \{0\} = D(p_1, \dots, p_n)$ for some $p_1, \dots, p_n : B$ jointly-regular, i.e. if $p_i^m \cdot b = 0$ for all $i = 1, \dots, n$ then $b = 0$.

If $0 : X$ is a point of a scheme, we call it regular, if one of the following equivalent conditions is satisfied

1. it admits some open affine neighborhood U such that $0 : U$ is regular.
2. It is a regular point of any open affine neighborhood.

Proof. Consider an open affine neighborhood $0 : D(f) \subset U = \text{Spec } B$. We will show

1. If 0 is regular in $D(f)$, then it is regular in $\text{Spec } B$: Consider $g_1, \dots, g_n : B$ such that

$$B_f \rightarrow \prod_i B_{fg_i}$$

is injective. Define $g_0 := f - f(0)$, where $0 \notin D(g_0)$. Let us show, that g_0, \dots, g_n are jointly surjective in B . Let $b : B$ such that $g_i^n b = 0$ for all $0 \leq i \leq n$. Then in particular $b/1 =_{B_f} 0$. Thus b is in the kernel of $B \rightarrow B_{g_0} \times B_f$. But $D(g_0) \cup D(f)$ forms an open cover of $\text{Spec } B$ as (f, g_0) generate the unit ideals. Thus $b : \text{Spec } B \rightarrow R$ equals 0 on an open cover, thus its 0.

2. If 0 is regular in $\text{Spec } B$, then it is regular in $D(f)$: Assume $B \rightarrow \prod B_{g_i}$ is injective. Let $f : B \rightarrow B$. Let us show that $B_f \rightarrow \prod B_{g_i f}$ is injective. If $(g_i f)^n b = 0$, then $(g_i)^n (f^n b) = 0$, thus $f^n b = 0$ by assumption. Thus $b/1 =_{B/f} 0$ as desired.

□

Lemma 4.15. *If $0 : X$ is a regular point in a scheme, then $R^X \rightarrow R^{X \setminus \{0\}}$ is injective.*

Proof. Lets first reduce to the affine case. Choose an open affine neighborhood U of 0 such that $0 : U$ is regular. Then the surjection $U + X \setminus \{0\} \rightarrow X$ induces a vertical left injection

$$\begin{array}{ccc} R^{U+X \setminus \{0\}} & \hookrightarrow & R^{U \setminus \{0\} + X \setminus \{0\}} \\ \uparrow & & \uparrow \\ R^X & \longrightarrow & R^{X \setminus \{0\}} \end{array}$$

So we may assume that $X = \text{Spec } A$ is affine.

Let $p_1, \dots, p_n : A$ be jointly-regular, i.e. if $p_i^{m_i} \cdot a = 0$ for all $i = 1, \dots, n$ then $a = 0$. If $f : \text{Spec } A \rightarrow R$ such that $f(x) = 0$ for all $x \in D(p_1, \dots, p_n)$, then $f(x) = 0$ for all $x : \text{Spec } A$. f is in the kernel of the diagonal map

$$\begin{array}{ccc} A & \xlongequal{\quad} & R^{\text{Spec } A} \\ \downarrow & & \downarrow \\ \prod_{i=1}^n A_{p_i} & \xlongequal{\quad} & R^{\coprod_i D(p_i)} \hookrightarrow R^{\cup D(p_i)} \end{array}$$

which is injective, as p_1, \dots, p_n are jointly-regular in A .

Thus $f = 0$ in A .

□

Question 3. What has this todo with separatedness?

Proposition 4.16. *Consider an affine S and an open subset $U \subset S$. Consider a \mathbb{T} -flat irreflexive relation \sharp on U , i.e.*

1. *Irreflexivity:* $\neg(x \sharp x)$
2. *\mathbb{T} -flatness.* For all $x : U$, the fiber $\sum_{y:S} x \sharp y$ is \mathbb{T} -flat.

Define a relation on S as

$$R_{\sharp}(x, y) = (x = y) + (x \in U \wedge y \in U) \times (x \sharp y)$$

(Abuse of notation: where the \times is secretly a \sum) Then the sheaf quotient S/R_{\sharp} is an algebraic space.

Proof. • This is a proposition: First note, that both summands are propositions and if both summands are inhabited we get a contradiction.

- The relation is covering: Furthermore, for any $x : S$ we have

$$\sum_{y:S} (x = y) + (x, y \in U \times x \sharp y) = 1 + (x, y \in U \times \sum_y x \sharp y) \in \mathbb{T}$$

as \sharp was assumed to be \mathbb{T} -flat on U : we can write the binary product as $\sum_{p:x \in U} \sum_y x \sharp y$, a \sum of \mathbb{T} -flat geometric stacks.

□

4.3 Group quotients

For this section let G denote a group that is a covering 0-stack. Let X be a sheaf equipped with a G action.

Lemma 4.17. $\mu_p = \text{Spec } R[X]/(X^p - 1)$ is covering for $p \neq 0$ prime.

Proof. It is fppf + étale as $X^p - 1$ is monic separable. TODO □

Definition 4.18. A G action on X is free, if for all $x, y : X$ the type

$$\sum_{g:G} gx = y$$

is a proposition.

Example 4.19. Given a formally étale + flat affine (e.g. μ_ℓ or finite) group that acts on an affine $\text{Spec } A$. Assume G acts free on some open subset U .

Then put $x \sharp y = \sum_{g:G \setminus \{1\}} gx = y$.

This provides a covering equivalence relation $R_{G,U}$ on $\text{Spec } A$, such that

- for any $x : U$

$$[x] = \sum_{y:\text{Spec } A} \sum_{g:G} gy = x.$$

- for some $y \notin U$, we have $R(x, y) \leftrightarrow x = y$.

By abuse of notation we write $\text{Spec } A/G|_U \equiv \text{Spec } A/R_G$ and call it the quotient of $\text{Spec } A$ by the G -action.

Proof. • It is irreflexive: If $x : U$ then $gx \neq x$, by freeness.

- We have $G \setminus \{1\}$ is flat affine using 4.11.

□

Notation. If $U = \text{Spec } A \setminus Z$ the complement of a closed subset we write

$$U \equiv Z^c$$

Example 4.20 (Free action). Set $U = \text{Spec } A$. Then this construction yields the actual group quotient. The quotient of \mathbb{A}^\times by the free μ_ℓ action gives a scheme.

Lemma 4.21. Algebraic spaces are stable by free quotients of covering group 0-stacks.

Proof. The map $X \rightarrow L_T(X/G)$ is fibered in covering 0-stacks, so in particular covering 0-stacks. As X is a geometric 0-stack, the quotient is a geometric 0-stack as well, This follows by the description in , choosing a geometric atlas of X and postcomposing this to get a geometric atlas of the quotient. □

Example 4.22. If $p : \sum_{r:R} S_r \rightarrow R$ be a map between formally étale + flat affine into R whose fibers, except possibly the fiber over 0, are formally étale + flat. Define $U = (x \neq 0) \times S_x \subset S_x$. $y \sharp y' \equiv y \neq y'$ is an irreflexive \mathbb{T} -flat relation on S_x . From this we obtain the algebraic space

$$\sum_{x:R} Y_x / R_\sharp$$

which we will later recognize as a fiber collapse.

Proof. \sharp is a modal irreflexive relation. By assumption we have given \mathbb{T} -flatness of S_x if $x \neq 0$. □

Lemma 4.23 (Not needed). *Let Y be affine. Let $X \hookrightarrow Y$ be a map fibered in locally closed propositions. Then its factors as the composite of a closed and then an open embedding*

Proof. By zariski local choice we find $Y = \bigcup Y_i$ and factorizations of the basechanges $X_i \rightarrow Z_i \rightarrow Y_i$. Then $\bigcup X_i \rightarrow \bigcup Z_i \rightarrow \bigcup Y_i = Y$ is a global factorization. \square

Proposition 4.24. *If $0 : \text{Spec } B$ is regular, Then the subtype $\{0\} + D(0) \subset \text{Spec } B$ is not locally closed.*

Proof. We have $x \neq 0 \leftrightarrow g(x) \neq 0$ We proceed by proving $1 \rightarrow 2 \rightarrow 3$.

1. $R^{\text{Spec } B} \rightarrow R^{\text{Spec } B \setminus \{0\}}$ is injective: by 4.15.
2. The infinitesimal neighborhood of 0 is not an open subtype: If it would, it would be principal open $D(g)$, as 0 admits a principal open neighborhood, which however already contains the whole infinitesimal one.
Then for any $x \neq 0$, we have $\neg \neg g(x) = 0$. As $\text{Spec } B \setminus \{0\}$ is a scheme, it admits a boundedness principle, thus we find some n , such that $g^n(x) = 0$ for all $x \neq 0$.
By the first point we deduce $g^n = 0$, hence $D(g) = D(g^n) = \emptyset$ contradiction.
3. The subtype $\{0\} + D(0) \subset \text{Spec } B$ is not locally closed. Let $U, C \subset \text{Spec } B$ be an open subset and a closed subset respectively, such that $(x \neq 0) + (x \neq 0) \leftrightarrow x \in U \wedge x \in C$. Then, for any $x : U$,

$$(x = 0) + (x \neq 0) = x \in C$$

is a closed proposition. Thus the decidable subtype $x \neq 0$ is a closed proposition. To contradict the assumption, we may convince ourself that the right vertical map

$$\begin{array}{ccc} \sum_{x:U} \neg \neg x = 0 & \xrightarrow{\sim} & \sum_{x:\text{Spec } B} \neg \neg x = 0 \\ \downarrow & & \downarrow \\ U & \hookrightarrow & \text{Spec } B \end{array}$$

is an open embedding

where the upper horizontal map is indeed an equivalence as for any $x : \text{Spec } B$, $x \in U$ is $\neg \neg$ -stable, but $\neg \neg x = 0$ and $0 \in U$, thus $x \in U$.

\square

Proposition 4.25. *Let S be affine with a regular point $*$. Assume we have function $g : S \rightarrow S$ such that $*$ is the unique fixpoint $*$ (e.g. if $(S, *)$ admits a pointed-free action of a nontrivial group) Let \sharp be an irreflexive \mathbb{T} -flat relation on $U \equiv \sum_{x:S} x \neq *$, such that for all $y : U$, we have $gy : U$ and $y \sharp gy$. Then the algebraic space S/R_\sharp is non locally separated, in particular not a scheme.*

Proof. It is an algebraic space by the previous prop.

A pointed-free action of a non-trivial group yields such a map g : If $\neg(G = \{1\})$, then $\neg \neg(G \setminus \{1\})$ by decidable equality of G . As we want to prove a contradiction, we may assume $g : G \setminus \{1\}$, this yields a map $S \rightarrow S$ such that

- $*$ is the unique fixpoint by the pointed-freeness
- If $y \neq *$, then $gy \neq *$ and $y \sharp gy$

We have that every scheme X is locally-separated, i.e. its identity types are locally closed. Indeed, this follows from the proof of Foundations Prop 5.5.2 .

Let us show that R is not valued in locally closed propositions. Recall

$$y \in U \rightarrow y \sharp gy \quad (1)$$

$$y \in U \leftrightarrow y \neq * \quad (2)$$

We have for any $y : S$

$$R_{\sharp}(y, gy) \simeq (y = gy) + (y \in U) \wedge y \sharp gy \stackrel{(1)}{\simeq} (y = *) + (y \in U) \stackrel{(2)}{\simeq} (y = *) + (y \neq *)$$

But if this is locally closed for all $y : S$, we have a contradiction to 4.24. \square

Corollary 4.26. *Let $Y : R \rightarrow \text{Aff}$ be formally étale + flat affine away from the origin, such that $p : \tilde{Y} = \sum_{x:R} Y_x \rightarrow R$ is regular and Y_0 is infinitesimal. If you find a map $g : \tilde{Y} \rightarrow \tilde{Y}$ over p with a unique fixpoint, which lies over 0, then the algebraic space Y_{\bullet} — is non-locally-separated, In particular not a scheme.*

Proof. Lets call the unique fix point $*$, i.e. we have

$$gy = y \leftrightarrow y = *$$

Note that $* : \tilde{Y}$ is a regular point, as $p : \tilde{Y} \rightarrow R$ is a regular section with Y_0 infinitesimal. \square

Definition 4.27. A pointed-free action of G on a pointed type $(X, 0)$ is a G -action with fixpoint 0, such that if $g\varepsilon = \varepsilon$ for some $g \neq 1$, then $\varepsilon = 0$.

Lemma 4.28. *Let G be a group with decidable equality acting pointed free on a pointed type $(X, 0)$. Then G acts free away from zero.*

Proof. let $x, y \neq 0$. We need to show, that $\sum_g gx = y$ is a proposition. Let $g, g' : G$ such that $gx = y$. as G has decidable equality, we may show $\neg\neg(g = g')$. If $g^{-1}g' \neq 1$, then by pointed-freeness applied to $g^{-1}g'x = x$, we have $x = 0$. Contradiction. \square

Corollary 4.29. *Let $0 : \text{Spec } A$ be a regular point. Let G be a nontrivial formally étale flat affine group acting pointed- freely on the pointed affine $(\text{Spec } A, 0)$. Then the pointed-free quotient of $\text{Spec } A$ by G from 4.19 is non-locally-separated, In particular not a scheme.*

Example 4.30 (Non locally-separated examples). Assume $\ell \neq 0$ prime. Let μ_{ℓ} act on $(\text{Spec } B, 0)$ in one of the following ways:

1. Let μ_{ℓ} act on $\text{Spec } B = \mathbb{A}^1$

2. Let μ_{ℓ} act on

$$\text{Spec } B \equiv \sum_{x,y:R} x^{\ell} = y^{\ell}$$

$$\text{via } g(x, y) = (x, gy)$$

Then $\text{Spec } B / \mu_{\ell}|_{0^c}$ is an algebraic space that is not a scheme.

Proof. $\neg\neg$ merely, μ_{ℓ} is finite ([ref?]) and $\mu_{\ell} \setminus \{1\}$ is inhabited by 4.11.

1. Pointed-Free action is clear. $0 : \mathbb{A}^1$ is regular by first projection.

2. Pointed-Free action is clear. The cross midpoint regular, witnessed by the first projection: It is regular vanishing at $(0, 0)$ And for any point $(0, y) : \text{Spec } B$ we deduce $y^{\ell} = -0^{\ell} = 0$, hence $\neg\neg(x, y) = (0, 0)$. \square

Question 4. If μ_{ℓ} acts on Y some affine, does every μ_{ℓ} -invariant $\phi : Y \rightarrow R$ is invariant on a ℓ -neighborhood?

4.4 Obsolete

Proposition 4.31. *Let $Y : R \rightarrow \text{Aff}$ be formally étale + flat affine away from the origin. If you find two sections $y, y' : \prod_{x:R} Y_x$ such that $y_x = y'_x \leftrightarrow x = 0$, then the algebraic space $-Y_\bullet-$ is non-locally-separated, In particular not a scheme.*

Proof. It is an algebraic space by the previous prop.

We have that every scheme X is locally-separated, i.e. its identity types are locally closed. Indeed, this follows from the proof of Foundations Prop 5.5.2 .

Let us show that R is not valued in locally closed propositions. We have

$$\prod_{x:R} \eta y_x = \eta y'_x \simeq \prod_{x:R} y_x = y'_x + (x \neq 0) \times y_x \neq y'_x \simeq (x = 0) + (x \neq 0)$$

but if this is locally closed for all $x : R$, we have a contradiction to 4.24. □

Lemma 4.32 (Not needed). *For an algebraic space X , we have implications $1 \Rightarrow 2 \Rightarrow 3$*

1. X admits an separated open cover.
2. For any covering equivalence relation $R : U^2 \rightarrow \text{Prop}$ on an affine U such that $X = U/R$, F is valued in locally closed propositions
3. We find such a presentation such that R is valued in locally closed propositions.

Proof $1 \Rightarrow 2$ Let $X' \rightarrow X$ be a map fibered in merely inhabited finite sums of open propositions with X' a separated algebraic space. Then any geometric atlas $U \rightarrow X'$ will be fibered in closed subtypes of U . We need to show, that the fibers of $U \rightarrow X' \rightarrow X$ are locally closed subtypes of U . Let $x : X$. the fiber in X' is of the form $U_1 + \dots + U_n$. Thus the fiber in U is a finite sums of \sum of $U_i \rightarrow (U \rightarrow \text{ClosedProp})$, which is enough.

$3 \Rightarrow 1$ Let $x : X$. □

Lemma 4.33 (Not needed). *Let $\text{char} \neq 2$. Let $p : R[X]$ be such that $0 \in D(p)$ and $x \in D(p)$ implies $-x \in D(p)$. If $f : R[X]$ is a polynomial such that $f(x) = f(-x)$ for all $x : D(p) \setminus \{0\}$, then f is even i.e. in the image of $R[X^2] \hookrightarrow R[X]$.*

Proof. We splitting f into $f_1 + Xf_2$ for $f_i : R[X^2] \subset R[X]$. I claim, that $f_2 = 0$ in $R[X]$. realizing that $(Xf_2)(x) = (Xf_2)(-x)$ implies $2f_2(x)x = 0$, thus $f_2(x)x = 0$ for all $x : D(p) \setminus 0 = D(pX)$, thus by the previous lemma $X \cdot f_2 = 0$ in $R[X]$, hence $f_2 = 0$. □

Lemma 4.34. *Let G be a finite group whose cardinality is invertible in R . Let G act on an affine scheme equipped with a fixpoint 0 . Let U be an open neighborhood of 0 , such that $g(U) = U$ for all $g : G$. Then we find some G -invariant p such that $0 \in D(p) \subset U$.*

Proof. Choose a principal open neighborhood $0 \in D(p) \subset U$. G acts on $R[X]$, via $(g.p)(x) = p(gx)$. Then

$$p' = \sum_{g:G} g.p : R[X]$$

is a G -invariant polynomial, in particular $D(p)$ is G -invariant. Moreover $0 \in D(p')$ as

$$p'(0) = \sum_{g:G} p(g(0)) = \sum_{g:G} p(0) = |G| \cdot p(0)$$

is invertible, as $|G|$ and $p(0)$ are both invertible. Furthermore, as U was G invariant and contained $D(p)$ it also has to contain $D(p')$: Indeed

$$D(p') \subset \bigcup_g D(g.p) \subset U$$

□

Lemma 4.35. *Let G be a formally étale + flat affine group, such that $\neg\neg$ its finite, with cardinality invertible in R and $G \setminus \{1\}$ inhabited. Let it act on an affine scheme $\text{Spec } A$ with a fixpoint 0 . Let R be a relation on $\text{Spec } A$ such that*

- $R(x, y)$ implies that there merely is some g with $y = gx$.
- $\neg\neg R(x, gx)$

Assume that for all $p : A^G$ with $0 \in D(p)$, $D(p)/R$ is not an affine scheme. Then $\text{Spec } A/R$ is not a scheme.

Proof. Assume 0 admits a open affine neighborhood U in $\text{Spec } A/R$. The preimage along the quotient map obtained from the relation induces a open neighborhood V of 0 in $\text{Spec } A$. As we want to prove a contradiction we may assume that μ_ℓ consists of ℓ many elements, where $\ell \neq 0$ in R . Note that V is G -invariant: For any $x \in V, g : G$, the goal $gx \in V$ as an open proposition is $\neg\neg$ -stable, thus we may assume $R(x, gx)$.

We apply the previous lemma to V to obtain an invariant principal open neighborhood $0 \in D(p) \subset V \subset \text{Spec } A$. As p is G -invariant, $p : \text{Spec } A \rightarrow R$ descends to $X \rightarrow R$. Restricting to U' yields a map $p' : U \rightarrow R$, such that setting $U' \equiv D(p')$ yields $q^{-1}(U') = q^{-1}(D(p')) = D(p' \circ q) = D(p)$. We are now in the following situation

$$\begin{array}{ccccc} D(p) & \hookrightarrow & V & \hookrightarrow & \text{Spec } A \\ \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow q \\ U' & \hookrightarrow & U & \hookrightarrow & X \end{array}$$

where U' is an open affine neighborhood of 0 .

By assumption $U = D(p)/\sim'$ cannot be affine. Contradiction. □

Proposition 4.36 (Not needed). *Let $\ell \neq 0$ be prime. Let μ_ℓ act on $\text{Spec } B$ with fixpoint 0 . . Let V be an infinitesimal neighborhood of 0 , i.e. a subtype $0 \in V \subset \text{Spec } B$ such that $\neg\neg x = 0$ for every $x : V$. Assume*

Strong freeness We find some $0 \in V' \subsetneq V$ for any $\varepsilon : \text{Spec } B, g \neq 1, g\varepsilon = \varepsilon$ implies $\varepsilon \in V'$

checking away from 0 For any $p : B$ and any $\phi : R^{D(p)}$ such that $\phi|_{D(p) \setminus \{0\}} = 0$, we have that $\phi|_V = 0$.

The sheaf quotient of $\text{Spec } B$ by the relation as above is an algebraic space but not an affine scheme.

Proof. • Let us check the conditions on the relation

- If $R(x, y)$ then either $x = y$ putting $g = 1$ or in the second case we get some g such that $gx = y$
- Let $x : X, g : G$. Assume $\neg R(x, gx)$, i.e. $x \neq gx$ and $\neg\neg x = 0$. But 0 was assumed to be a fixpoint, hence $\neg\neg gx = x$.

- Let $p : B$ be as above. We have to show that the quotient of $D(p)$ is not affine. The conditions on p give $p(0) \neq 0$ and $p(x) \neq 0 \rightarrow p(gx) \neq 0$ for all $g : \mu_\ell$. Lets call this quotient X .

Define

$$A = \{ \phi : R^{D(p)} \mid \phi|_{D(p) \setminus \{0\}} \text{ is } \mu_\ell\text{-invariant} \}$$

This is an R -subalgebra: for any $r : R$, $r : B_p$ is μ_ℓ -invariant. μ_ℓ -invariant functions are stable under addition and multiplication .

Claim: The affinization map of X is the induced dashed map $f : X \rightarrow \text{Spec } A$ in

$$\begin{array}{ccc} D(p) & \xlongequal{\quad} & \text{Spec } R^{D(p)} \\ \downarrow q & & \downarrow q' \\ X & \xrightarrow{\quad \exists! f \quad} & \text{Spec } A \end{array}$$

Proof: A function $\phi : D(p) \rightarrow R$ factors through q iff $\phi|_{D(p) \setminus \{0\}}$ is μ_ℓ -invariant. Thus the embedding (using that R is a sheaf) $R^X \hookrightarrow R^{D(p)}$ has image A \square (Claim).

Proof that X is not an affine: Assume that X were affine. Then the map f would be in particular an embedding. We may assume a term $g : \mu_\ell \setminus \{1\}$: Indeed, as we want to prove a contradiction we may assume a term in $g : \text{Spec } R[X]/(\sum_{i=0}^{\ell-1} X^i)$. But this type is equivalent to $\mu_\ell \setminus \{1\}$, using that $\sum_{i=0}^{\ell-1} X^i | X^\ell - 1$ and $\ell \neq 0$.

The given infinitesimal neighborhood V satisfies $V \subset D(p)$, using that invertibility is $\neg \neg$ stable. Then for any $\varepsilon : V$ we have

$$(q\varepsilon =_X q(g\varepsilon)) \stackrel{??}{=} (\varepsilon = g\varepsilon) + (\varepsilon \neq 0 \wedge \sum_{h \neq 1} \varepsilon = hg\varepsilon) = (\varepsilon = g\varepsilon) = (\varepsilon \in V')$$

where the last step comes from pointed-freeness. But we have

$$(q'\varepsilon =_{\text{Spec } A} q'(g\varepsilon)) = \left(\prod_{\phi:A} \phi(q'\varepsilon) = \phi(q'(g\varepsilon)) \right) = \prod_{\substack{\phi:R^{D(p)} \\ \phi \in A}} \phi(\varepsilon) = \phi(g\varepsilon),$$

The right hand side is inhabited: For any $\phi : D(p) \rightarrow R$ such that $\psi := \phi - g.\phi$ satisfies $\psi|_{D(p) \setminus \{0\}} = 0$ we have $\psi|_V = 0$ by 'checking away from 0', inparticular $\psi(\varepsilon) = 0$. So we conclude the the embedding $V' \hookrightarrow V$ is an equivalence. But we asked $V' \subsetneq V$ to be a proper subset.

\square

Example 4.37. Let μ_ℓ act on $\text{Spec } B = \mathbb{A}^1$.

Proof. 1. Put $V \equiv \text{Spec } R[X]/X^n$ for some $n > 1$.

2. As $(g-1)$ is invertible, $((g-1)\varepsilon = 0)$ gives us $\varepsilon \in \{0\} \equiv V' \subsetneq V$. Note that indeed V is non contractible, because $R[X]/X^n \rightarrow R[X]/X$ is not an algebra isomorphism
3. We have to show, that then ϕ is μ_ℓ invariant. We can apply 4.15, observing $\phi - g.\phi = 0$ on $D(X/1) \subset \text{Spec } B_p$, where $X/1 : B_p$ is regular, because X is regular in B . TODO as each ϕ satisfies the cond. \square (Claim)

\square

Example 4.38. Assume $2 \neq 0$. Let μ_2 act on

$$\mathrm{Spec} B \equiv \sum_{x,y:R} xy = 0$$

via the swap. Then $\mathrm{Spec} B/R$ is an algebraic space but not a scheme.

Proof. 1. Put $V = \mathrm{Spec} R[X]/X^k \subset \mathrm{Spec} B$, $k > 2$.

2. If $(x, y) = (y, x)$ but $xy = 0$ we get $x \in V' \equiv \mathrm{Spec} R[X]/X^2$.

3. Let $\phi : D(p) \rightarrow R$ be 0 everywhere except near the origin. Then we get a restricted map $\phi' : D(p') \rightarrow R$ where $D(p') \subset V(X)$ is given by the intersection $D(p) \cap V(X)$. Indeed : Put $p' : R[X]$ the image of $p : R[X, Y]/(XY)$ and the map induced by evaluating Y at 0.

Here we can apply 4.15, getting that ϕ' is 0 everywhere in particular in $V \subset V(X)$. \square

4.5 Locally seperated examples

Lemma 4.39 (not needed). Given a map $P : \mathrm{Susp}(Q) \rightarrow \mathrm{Prop}$, such that $P(N)$ and $P(S)$ hold, then $\prod_{t:\mathrm{Susp}(Q)} P(t)$

Lemma 4.40 (not needed). Assume $2 \neq 0$. For any $x : R$, the map

$$\begin{aligned} \mathrm{Susp}(x \neq 0) &\rightarrow \sum_{y:R/x} y^2 = 1 \\ N &\mapsto 1 \\ S &\mapsto -1 \end{aligned}$$

is well-defined and an equivalence.

Proof. The following maps are mutually inverse

$$\begin{aligned} \sum_{y:R/x} y^2 = 1 &\simeq \sum_{e:R/x} e^2 = e \\ y &\mapsto (y - 1)/2 \\ 2e - 1 &\mapsto e \end{aligned}$$

So it remains to show that the map

$$\begin{aligned} f : \mathrm{Susp}(x \neq 0) &\rightarrow \sum_{e:R/x} e^2 = e \\ N &\mapsto 1 \\ S &\mapsto 0 \end{aligned}$$

is a bijection.

- It is injective, i.e. for all $p, q : \mathrm{Susp}(x \neq 0)$, if $f(p) = f(q)$, then $p = q$. As the latter is a proposition, we may assume p, q beeing combinations of north and south poles. The interesting case is if wlog $p = N, q = S$. Then assuming $0 =_{R/x} 1$ means $R/x = 0$, i.e. $x \neq 0$, thus $N = S$ in $\mathrm{Susp}(x \neq 0)$.
- It is surjective: Choose $e : R$, such that $e^2 = e$ in R/x . By locality in R , e or $1 - e$ is invertible in R , thus in R/x . By $e^2 = e$ we deduce $e = 0$ or $e = 1$ in R/x , both lie in the image of f .

□

Example 4.41 (Not needed). Let $L = \sum_{x:\mathbb{A}^1} \text{Susp}(x \neq 0) = \sum_{x:\mathbb{A}^1} \sum_{y:R/x} y^2 =_{R/x} 1$ be the line with two origins.

Lemma 4.42 (Not needed). Let $2 \neq 0$. Let $y, y' : A$ be two elements of an fp-algebra, whose squares coincide and such that y is invertible. Then $y =_A y'$ is formally étale

Proof. We may assume that $A = R$, as equality in A can be checked pointwise and formally étale is a modality. We may show its $\neg\neg$ -stable. Assume $\neg\neg(y =_R y')$, i.e. $y - y'$ being nilpotent in A . So pick n large enough, such that $(y - y')^{2^n} = 0$. Proof by induction over n . If $n = 0$, then it's fine. Induction step $n \mapsto n+1$. Let $(y - y')^{2^{n+1}} =_R 0$, then $(2y^2 - 2yy')^{2^n} = 0$, or $(y(y - y'))^{2^n} = 0$, as y is invertible, $(y - y')^{2^n} = 0$, so by induction hypothesis $y = y'$. □

4.6 FiberCollaps away from the origin

Definition 4.43. Let $Y : R \rightarrow \text{Aff}$ be a dependent family of affines, such that $(Y \in \text{EF})^{x \neq 0}$. The fiber collapse of Y away from the origin $-Y-$ is the space over R

$$\sum_{x:R} (x \neq 0) \star Y_x \rightarrow R$$

This space over R looks exactly like the line away from the origin and over an infinitesimal ε the fiber is Y_ε .

Lemma 4.44. $-Y-$ is an algebraic space.

Proof. Let $x : R$. Let $Y : \text{Aff}$ such that $x \neq 0$ implies that Y is formally étale + flat. We will show that $\eta : Y \rightarrow (x \neq 0) \star Y$ is the sheaf-quotient map of the relation on Y given by $y \sim y' \equiv (y = y') + (x \neq 0) \times y \neq y'$, which is enough by 4.22. We apply ??

- The map is \mathbb{T} -surjective: We have a term in

$$\prod_{y:Y} \|\text{fib}_\eta(\eta y)\|_{\mathbb{T}}$$

and each $\|\text{fib}_\eta \eta y\|_{\mathbb{T}}$ is modal, i.e. contractible if $x \neq 0$. Hence we get a (unique term in) filler in

$$\prod_{y:(x \neq 0) \star Y} \|\text{fib}_\eta y\|_{\mathbb{T}}$$

- Given $y, y' : Y$, we have

$$\begin{aligned} \eta(y') &= \eta(y) \simeq (x \neq 0) \star (y = y') & | \text{ closed modality is lex ([2] Example 3.1.4).} \\ &\simeq L_{\mathbb{T}}((y = y') \vee (x \neq 0)) & | (x \neq 0) \rightarrow \text{HasDecEq}(Y) \\ &\simeq (y = y') + (x \neq 0) \times y \neq y', \end{aligned}$$

□

Example 4.45. $-Bool-$ is the line with two origins.

$-\text{Spec } R[X]/(X^2 + 1)-$ is the twisted line with two origins, i.e. over the origin we have the roots of -1 .

$-\text{Spec } R[Y]/(Y^2 - \bullet^2)-$ is the quotient of the cross, that looks like $\mathbb{D}(1)$ over the origin.

$-\text{Spec } R[Y]/(\bullet Y)-$ is the affine Plus.

4.7 Schemes do not have descent

For this section, let $\rho : R \setminus \{0\}$ denote a term, e.g. $\rho = 1$. Set $C = R[T]/(T^2 + \rho)$.

Proposition 4.46. *If $\text{---Spec } C\text{---}$ is a scheme, then $X^2 + \rho$ has a root.*

Proof. Let $p : \text{---Spec } C\text{---} \rightarrow R$ be the first projection. We proceed as follows

1. There is no open affine subset of $L(\text{Spec } C)$ containing $\text{fib}_p(0)$.
2. Any cover of $\text{Spec } C$ by open subsets strictly smaller than $\text{Spec } C$ yields a root.

Proves:

1. Because we want to show a sheaf, we may assume $L(\text{Spec } C) = L(2)$. Assume there is an open affine subset $\text{fib}_p(0) \subset U \subset L(2)$. Then $p(U) \subset R$ is an open neighborhood of 0, as

$$x \in p(U) \leftrightarrow (x, N) \in U \vee (x, S) \in U$$

Claim: the map $R^{p(U)} \rightarrow R^U$ is an equivalence. If we have shown that: As U is affine we conclude that the map

$$\begin{aligned} U &\rightarrow \text{Spec}(R^{p(U)}) \\ x &\mapsto \phi \mapsto (\phi(px)) \end{aligned}$$

is an equivalence, which is a contradiction to the assumption, that U contains both origins.

Proof of claim: Injectivity: If two maps $f, g : p(U) \rightarrow R$ coincide after precomposing with $U \rightarrow p(U)$, then they coincide away from 0 so conclude by 4.15.

Surjectivity: Given a map $U \rightarrow R$, by pulling back along $p : R + R \rightarrow L(2)$ we can view it as a map $R + R \supset U' \rightsquigarrow R$ defined at both origins, so in particular as a pair of maps to R defined on some open neighborhood of 0 of R . They coincide away from 0 so by 4.15 they are equal.

2. As $\text{Spec } C$ is an étale sheaf, its enough to show to construct a function $\|\text{Spec } C\| \rightarrow \text{Spec } C$.
Define

$$\begin{aligned} A : \text{Spec } C &\rightarrow \text{Prop}^{\text{Fin}(n)} \\ x &\mapsto \{j : x \in U_j\} \end{aligned}$$

where we transport $x : \text{Spec } C$ along $\text{Spec } C \xrightarrow{\sim} \text{fib}_p 0$. Observe

- (a) for any $x, x' : \text{Spec } C$,

$$\|Ax \cap Ax'\| \rightarrow \neg\neg(x = x') \xrightarrow{\text{DecEq}(\text{Spec } C)} x = x'$$

where the first implication follows like this : if $x \neq x'$ and $\text{fib}_p 0 = \{x, x'\} \in U_j$ then we have a contradiction to the first point

- (b) For any $x : \text{Spec } C$, $\|Ax\|$.

Assume $\|\text{Spec } C\|$. Lets try to construct a term of the following type

$$\sum_{x : \text{Spec } C} \forall x' : \text{Spec } C, j : Ax, j' : Ax' \rightarrow j \leq j'$$

For this we may assume $\text{Spec } C = \text{Bool} \equiv \{N, S\}$, as the above type is a proposition: If we have given two such minimal x_1, x_2 , we can set first $x' \equiv x_2$ and then $x' \equiv x_1$

respectively and then (by (b)) choosing $j : A_{x_1}, j' : A_{x_2}$ gives $j \leq j' \leq j$ so that $j : A_{x_1} \cap A_{x_2}$ such that $x_1 = x_2$ by (a).

I will explain an algorithm to do the following: Given $n \geq 2$, and a pair of merely inhabited disjoint subsets A_N and A_S of $\text{Fin}(n)$, we can decide in which of the two we find the smaller number of $\text{Fin}(n)$.

Induction over n . If $n = 2$, then $\|A_N\|$, so we find a term in the proposition $(0 \in A_N) + (1 \in A_N)$. In the left case return S , in the right case N .

For $n \mapsto n + 1$, if $n \in A_N$ return S . If $n \in A_S$, return N . Otherwise both A_N and A_S are subset of $\text{Fin}(n)$, so conclude by induction.

□

Corollary 4.47. *Schemes do not have descent.*

Proof. If Schemes have descent, then $\text{---Spec } R[T]/(T^2 + \rho)\text{---} \in \mathbf{Sch}$ is a sheaf. As $\text{---Spec } R[T]/(T^2 + \rho)\text{---}$ is \mathbb{T} -merely a scheme, it is a scheme, so by the previous lemma $T^2 + \rho$ has a root. Contradiction to [1] A . 0.3.

□

4.8 Gluing in an affine on the line

Definition 4.48. Let Y be an affine. The n .th order gluing of Y on the line is given by the sheaf

$$L_n(X) = \sum_{x:R} Y^{x^n=0}$$

Lemma 4.49. *If $Y = \text{Spec } R[T]/f$, we have*

$$L_n(X) = \sum_{x:R} \sum_{y:R/x^n} f(y) =_{R/x^n} 0$$

Proof. For any R -algebra A (e.g. R/x^n) we have by the universal property of $R[T]/f$

$$\sum_{y:A} f(y) =_A 0 = \text{Hom}_R(R[T]/f, A) = Y^{\text{Spec } A}$$

□

Lemma 4.50. *If Y is formally étale , then the map over R*

$$\begin{array}{ccc} R \times Y & \xrightarrow{\quad} & L_n(Y) \\ & \searrow & \swarrow \\ & R & \end{array}$$

pulls back to an equivalence over $\mathcal{N}_\infty(0)$.

If Y is formally unramified, then $L_n(x)$ is locally seperated.

Proof. Indeed, the diagonal map

$$Y \rightarrow Y^{x^n=0}$$

is an equivalence, as for any $\neg\neg x = 0$, $x^n = 0$ is a closed dense proposition and Y is formally étale .

If Y is formally unramified, then the identity types look like

$$(x, y) =_{L_n(Y)} (x', y') \simeq (x = x') \times (x^n = 0 \rightarrow Q)$$

where Q is an open proposition such that for any $p : x^n = 0$ we have $Q \equiv yp = y'p$. Indeed by the proof of ?? we can find a filler of $y_\bullet = y'_\bullet : P \rightarrow \mathbf{Open}$. By [1](4.2.11) this proposition is locally closed.

□

Question 5. Is the map $\sum_{y:R/x^3} y^2 = 0 \rightarrow \sum_{y:R/x^2} y^2 =_0$ surjective? This is how i understand [David Madore](#).

Lemma 4.51. *For $\varepsilon : \mathcal{N}_\infty(0)$, the affine $\text{Ann}(\varepsilon) = \{x : R \mid x\varepsilon = 0\}$ is not $_\varepsilon$ formally smooth. In particular $R \rightarrow R/\varepsilon$ is not $_\varepsilon$ a geometric cover.*

Proof. We have the map $1 : (\varepsilon = 0) \rightarrow \text{Ann}(\varepsilon)$. Assume there is a filler $x : \text{Ann}(\varepsilon)$, i.e. $(\varepsilon = 0) \rightarrow x = 1$. Then not not, $x = 1$, i.e. $(x - 1)^n = 0$ for n large enough. Hence

$$0 = \varepsilon(x - 1)^n = \varepsilon x(\dots) + (-1)^n \varepsilon = (-1)^n \varepsilon$$

as desired. \square

Lemma 4.52 (TODO). *If Y is formally étale + flat affine, then $L_1(Y)$ is an algebraic space.*

Proof. Recall the closed modality associated to a proposition P , given by $P \star _$. We can define a map

$$\begin{aligned} f : (x \neq 0) \star Y &\rightarrow Y^{x=0} \\ y &\mapsto \Delta(y) \end{aligned}$$

where we check, that if $x \neq 0$ holds, then indeed $Y^{x=0}$ is contractible.

f is a bijection:

- injectivity: Given two terms of the domain, as the map out of Y is \mathbb{T} -surjective (and the goal is a sheaf), we may assume that they are of the form $\text{inl}(y), \text{inl}(y')$ for $y, y' : Y$. Then if $\Delta(y) = \Delta(y')$ we have $(x = 0) \rightarrow (y = y')$. As $y = y'$ is open, we have $(x \neq 0) \vee (y = y')$. If $x \neq 0$, then $\text{inl}(y) = \text{inl}(y')$ by the construction of the join.

- surjectivity: TODO

\square

Question 6. Is $L_2(\mathbb{D}(1))$ an algebraic space or fppf-geometric 0-stack? For this: Is

$$\begin{aligned} (\text{Spec } R[X, Y]/X^2 - Y^2)/\sim &\rightarrow L_2(\mathbb{D}(1)) = \sum_{x:R} \sum_{y:R/x^2} y^2 = 0 \\ (x, y) &\mapsto (x, [y]) \end{aligned}$$

an equivalence? Here we mod out the relation generated by $(x, -x) \sim (x, x) \forall x \neq 0$.

This is equivalent to : For any $x : R$, is the map

$$(x \neq 0) \star \text{Spec } R[Y]/(Y^2 - x^2) \rightarrow \mathbb{D}(1)^{x^2=0}$$

an equivalence?

Example 4.53. *I suggest a new definition of fppf topology: We take the topology generated by the Zariski topology and algebras of the form $R[X]/f$ where one of coefficients of f is invertible (non necessarily the leading coefficient). This is still a free module hence fppf.*

4.9 Weakly-flat stacks

Definition 4.54. We call a geometric stack X weakly-flat iff one of the following conditions is satisfied

1. $\|X\|_{-1}^{\mathbb{T}} \rightarrow X \in \mathbf{CS}$
2. For any geometric atlas $W \rightarrow X$, W is weakly-flat, i.e $\|W\|^{\mathbb{T}} \rightarrow W \in \mathbb{T}$.

Proof.

$1 \Rightarrow 2$ Choose a geometric atlas $W \rightarrow X$. In particular its \mathbb{T} -surjective, hence we have $\|W\|^{\mathbb{T}}$, so by assumption $W \in \mathbb{T}$. So $X \in \mathbf{CS}$.

$2 \Rightarrow 1$

$$\|W\|^{\mathbb{T}} \rightarrow \|X\|^{\mathbb{T}} \rightarrow X \in \mathbf{CS} \stackrel{??}{\rightarrow} W \in \mathbb{T}$$

□

They behave bad as they are not stable under \sum (and not under id-types, although this holds for affines).

Lemma 4.55. For any weakly-flat geometric stack X , $\|X\|_{-1}^{\mathbb{T}}$ is a geometric stack.

Proof. $X \rightarrow \|X\|_{-1}^{\mathbb{T}}$ is a geometric cover. □

Proposition 4.56. We may define X to be 0-wf-seperated, iff its weakly flat and $n+1$ -wf-seperated, iff identity types of X are n -wf-seperated.

For X a geometric stack, TFAE

1. X is $n+1$ -wf-seperated, i.e. all $n+1$ -fold identity types of X are weakly-flat.
2. For any x , $\Omega^{n+1}(X, x)$ is covering.
3. For any $x : X$, $x = x$ is n -wf-seperated, i.e. n -fold identity types of $x = x$ are weakly flat.

Proof.

$1 \Rightarrow 3 \Rightarrow 2$ ez

$3 \Rightarrow 1$ We prove this by induction. $n = 0$. To show that $x =_X y$ is weakly-flat, by descent we may assume that $x = y$. Then we have $(x = y) \simeq (x =_X x)$. By assumption this is weakly flat.

Assume now, that for any $x : X$, that $x = x$ is n -wf-seperated. Let $x, y : X$. We want to show that $x = y$ is n -wf-seperated. By induction we may just prove that for any $p : x = y$, $p = p$ is $n-1$ -wf-seperated. Applying $p \cdot _-$ induces an equivalence $\mathbf{refl}_x = \mathbf{refl}_x \simeq p = p$. But $x = x$ is n -wf-seperated, hence $\mathbf{refl}_x = \mathbf{refl}_x$ is $n-1$ -wf-seperated.

$2 \Rightarrow 3$ Induction. $n = 0$ is fine. Let $x : X$. To show that $\Omega(X, x)$ is n -wf-seperated, just use that $\Omega^n(\Omega(X, x))$ is covering, hence by the inductive statement $2 \Rightarrow 3 \Rightarrow 1$, we now that $\Omega(X, x)$ is n -wf-seperated. □

5 Omega-stability and gerbes

Definition 5.1. A geometric stack X is an n -gerbe iff the map $\eta_n^{\mathbb{T}} : X \rightarrow \|X\|_n^{\mathbb{T}}$ is a geometric cover.

Example 5.2. If G is a covering group sheaf, then BG is a 0-gerbe.

Example 5.3. It may happen, that $\|X\|_n^{\mathbb{T}}$ is a geometric n -stack for X a geometric stack, although X is not an n -gerbe. Indeed: Put $n = 0$ and X any pointed \mathbb{T} -connected geometric stack that is not covering, like $\text{Susp}(1 + x = 0)$ for some

Theorem 5.4. Assume that Covering stacks are Ω -stable, Then every geometric stack is a 1-gerbe.

Proof. By ??, we need to show that for any $x : X$, $\Omega^2(X, x)$ is covering. choose an geometric atlas $f : S \rightarrow X$. by descent we may only show that $\Omega^2(X, fs)$ for $s : S$ is covering.

$$\Omega\left(\sum_{t:S} ft = fs\right) \simeq \left(\sum_{p:\Omega(S,s)} \text{tp}_p(\text{refl}_{fs}) = \text{refl}_{fs}\right) \simeq \text{refl}_{fs=fs} \text{refl}$$

where the last equivalence is obtained, as $\Omega(S, s)$ is contractible with center refl_s . So $\Omega^2(X, fs)$ is the loop space of a covering stack, hence by assumption covering. \square

Corollary 5.5. Any Deligne Mumford Stack is a 1-gerbe

Proof. Use that étale topology is lex-flattened and ??. \square

Proposition 5.6. This proposition seems only interesting for $n = 0$ by the previous theorem. Assume that covering stacks are Ω -stable. Then X is an n -gerbe iff $\Omega^{n+1}(X, x)$ is covering for all $x : X$

Proof. One direction is ??. The other follows
By applying iteratively ??

$$\begin{aligned} \Omega^{n+1}(\text{fib}(\eta_n^{\mathbb{T}} X)|x|) &\simeq \Omega^n \text{fib}(\eta_{n-1}^{\mathbb{T}}(\Omega(X, x)))pt \simeq \dots \\ &\simeq \Omega^{n-k} \text{fib}(\eta_{n-k-1}^{\mathbb{T}} \Omega^{k+1}(X, x))pt \simeq \dots \\ &\simeq \text{fib}(\eta_{-1}^{\mathbb{T}} \Omega^{n+1}(X, x))pt \\ &\simeq \Omega^{n+1}(X, x) \end{aligned}$$

The LHS is covering by Ω -stability. \square

We can reprove ?? by just observing that \mathbb{T} -flat geometric stacks have covering loop spaces.

Remark 1. Put \mathbb{T} the étale topology. Observe, that we have an analogous statement if we replace covering stack by formally étale :

1. $\eta_0^{\mathbb{T}} X : X \rightarrow \|X\|_0^{\mathbb{T}}$ is formally étale
2. $X \rightarrow \|X\|_0^{\mathbb{T}}$ is formally unramified
3. for every $x : X$, $\Omega(X, x)$ is formally étale .

Proof 1 \Leftrightarrow 2 Observe that the map $\eta_0^{\mathbb{T}}$ is \mathbb{T} -smooth.

2 \Rightarrow 3 okay as the fibers of $\eta_0^{\mathbb{T}}$ embed into X .

3 \Rightarrow 2 Let $x, y : X$ be \mathbb{T} -merely equal. The goal is $\text{FormallyEtale}(x = y)$ is a sheaf, so we may assume that $x = y$. \square

Corollary 5.7. If covering stacks are Ω -stable, then identity types of geometric stacks are 0-gerbes.

Proof. We need to check, that identity types of a 1-gerbe X are 0-gerbes. So assume $p : x = y$. Then

$$\Omega(x = y, p) = \Omega(x = x, \text{refl}) = \Omega^2(X, x)$$

which is covering as X is a 1-gerbe. \square

6 Questions // TODO

Theorem 6.1 (TODO). *An Artin stack X is Deligne Mumford iff one of the following conditions is satisfied:*

1. *There exists a geometric atlas $W \rightarrow X$*
2. *The identity types of X are \mathbb{P} -separated*

Proof. $\Rightarrow 2$. ??

2. $\Rightarrow 1$ Residual ??? [06MF]

□

Prove ??!!!

Question 7. if $\mathbb{T} \subset \mathbb{T}'$ do we have that for each $X : \mathbf{GS}_{\mathbb{T}} \rightarrow \mathbf{L}_{\mathbb{T}'} X \in \mathbf{GS}_{\mathbb{T}'}$?

Theorem 6.2 (TODO). *The class of flat affines is stable under \sum . Moreover flatness can be defined fiberwise.*

7 Not clear where to put that

Lemma 7.1 (Not needed). *Open subtypes of \mathbb{A}^1 are $\neg\neg$ principal open.*

Proof. • An open affine subscheme of \mathbb{A}^1 is $\neg\neg$ principal open: Let $D(f_1, \dots, f_n) \subset \mathbb{A}^1$ be an arbitrary open subset. We may assume that each $f_i : R[X] \rightarrow R$ is non constant (in particular non zero). By [ref?], $\neg\neg$ -merely each $D(f_i) \subset R$ is cofinite. Thus $\neg\neg$ -merely, the finite union $\bigcup_{i=1}^n D(f_i) \subset R$ is cofinite as well, hence principal open.

□

Proposition 7.2. *Assume covering stacks are Ω -stable. A truncated stack (e.g. geometric stack) is covering iff $\pi_0^{\mathbb{T}} X := \|X\|_0^{\mathbb{T}}$ and all higher homotopy groups*

$$\pi_i^{\mathbb{T}}(X, x) = \|\Omega^i(X, x)\|_0^{\mathbb{T}}, i \geq 1$$

are covering algebraic spaces.

Proof. Let X be an n -stack. If X is covering, then by Ω -stability all the $\pi_i^{\mathbb{T}}$ are covering 5.6 Now the converse. Consider the postnikov tower

$$X = \|X\|_n^{\mathbb{T}} \rightarrow \|X\|_{n-1}^{\mathbb{T}} \rightarrow \dots \rightarrow \|X\|_1^{\mathbb{T}} \rightarrow \|X\|_0^{\mathbb{T}}$$

As $\|X\|_0^{\mathbb{T}}$ is covering, by quotient stability of covering stacks we may show that all the maps are geometric covers. Let $1 \leq k \leq n$ and consider the map $f_k^X : \|X\|_k^{\mathbb{T}} \rightarrow \|X\|_{k-1}^{\mathbb{T}}$. By descent for covering stacks, we may only consider the fiber over $|x|$, as the $\eta_{k-1}^{\mathbb{T}}$ is \mathbb{T} -surjective. It suffices to show, that the fiber is given by $B_{\mathbb{T}}^k \pi_k^{\mathbb{T}}(X, x)$ as deloopings of covering stacks are covering ??.

We apply ??. First observe that $\Omega^k(\text{fib}(f_k^X)|x|) = \text{fib}(\Omega^k(f_k^X, x))$ is equivalent to the basefiber of

$$\pi_k^{\mathbb{T}}(X, x) \equiv \|\Omega^k X\|_0^{\mathbb{T}} \simeq \Omega^k(\|X\|_k^{\mathbb{T}}) \rightarrow \Omega^k\|X\|_{k-1}^{\mathbb{T}} \simeq 1$$

So it suffices to show by induction over k , that for all pointed stacks (X, x) , $\text{fib}(f_k^X)|x|$ is \mathbb{T} - k -connected.

This is definitely \mathbb{T} -connected by using that any term $(y, p) : \text{fib}(f_k^X)|x| = \sum_{y : \|X\|_n^{\mathbb{T}}} \|x - y\|_n^{\mathbb{T}}$ yields a witness of $\|x - y\|_n^{\mathbb{T}}$. Then $\Omega(\text{fib}(f_k^X)|x|) = \text{fib}(\Omega(f_k^X, x)) = \text{fib}(f_{k-1}^{\Omega(X, x)})$ which is \mathbb{T} - $k-1$ -connected by induction. □

7.1 Remarks about weakly flat affines

Lemma 7.3. *The proposition $\|X\|_{\mathbb{T}}$ is geometric iff there exists a map from a weakly flat affine $\text{Spec } B \rightarrow X$ such that $\|\text{Spec } B\|_{\mathbb{T}} \rightarrow \|X\|_{\mathbb{T}}$ is an equivalence.*

Proof. ' \leftarrow ' is clear.

' \rightarrow '. Choose $\text{Spec } B'$ weakly flat such that $\|X\|_{\mathbb{T}} = \|\text{Spec } B'\|_{\mathbb{T}}$. As the map $X \rightarrow \|X\|_{\mathbb{T}}$ is \mathbb{T} -surjective, by \mathbb{T} -local choice we find a \mathbb{T} -cover $\text{Spec } B \rightarrow \text{Spec } B'$ and a commutative diagram

$$\begin{array}{ccc} \exists \text{Spec } B & \dashrightarrow & X \\ \downarrow & & \downarrow \\ \text{Spec } B' & \longrightarrow & \|X\|_{\mathbb{T}} \end{array}$$

As $\text{Spec } B'$ was weakly flat and the left vertical map is a \mathbb{T} -cover, $\text{Spec } B$ is weakly flat. \square

Lemma 7.4 (DM). *If $\text{Spec } A + \text{Spec } B$ is weakly flat affine, then $\text{Spec } A$ is weakly flat.*

Proof. Indeed

$$\|X\|_{\mathbb{T}} \rightarrow \|X + Y\|_{\mathbb{T}} \rightarrow X + Y \in \mathbb{T} \rightarrow X \in \mathbb{P}$$

but $\|X\|_{\mathbb{T}} \wedge X \in \mathbb{P} \rightarrow X \in \mathbb{T}$. \square

Lemma 7.5. *if the topology is saturated Bering weakly-flat descends along \mathbb{T} -covers.*

Lemma 7.6 (DM). *If $\|P + Q\|_{\mathbb{T}}$ is a geometric prop, then TODO*

Proof. By the previous two lemma and we find a map out of a weakly flat affine $\text{Spec } B \rightarrow P + Q$ that induces an equivalence on \mathbb{T} -truncations, but it splits into two map out of a weakly affine $\text{Spec } B_1 \rightarrow P, \text{Spec } B_2 \rightarrow Q$. \square

Notation. For $P : (\varepsilon : \mathcal{N}_{\infty}(0)) \rightarrow X \rightarrow \text{Prop}$, let $\varepsilon : \mathcal{N}_{\infty}(0) \vdash x : X$. We say x is $\text{not}_{\varepsilon} P$, if $\forall \varepsilon, P_{\varepsilon} x \rightarrow \varepsilon = 0$. Observe, if x is $\text{not}_{\varepsilon} P$ for any $\varepsilon^2 = 0$, then x is not P .

Remark 2. If $2 \neq 0$. Let $\varepsilon, \varepsilon' : \mathcal{N}_{\infty}(0)$. $\varepsilon = \varepsilon' + \varepsilon = -\varepsilon'$ is not_{ε} weakly-flat

Proof. We prove that once its \mathbb{T} -merely inhabited, then its not_{ε} covering, which is enough as $\neg\neg(\varepsilon = \varepsilon' + \varepsilon = -\varepsilon')$. As the goal is a stack we may assume $\varepsilon = \varepsilon' + \varepsilon = -\varepsilon'$. wlog the first case. Then assume $1 + (\varepsilon = -\varepsilon) \simeq 1 + \varepsilon = 0$ is covering. Then $\varepsilon = 0$ is formally étale, thus inhabited as a formally étale closed dense proposition. \square

Example 7.7 (Obsolete). *The map $q : \mathbb{A}^1 \rightarrow \mathbb{A}^1/\mu_{\ell}$ is not a geometric cover.*

Proof. The map factors through the geometric cover $\mathbb{A}^1 \rightarrow \mathbb{A}^1/\mu_{\ell}$. Thus its enough to show that \mathbb{A}^1/μ_{ℓ} is not a 0-gerbe, or that not every loop space is covering. Let us show that, $\Omega(\mathbb{A}^1/\mu_{\ell}, \varepsilon)$ is not_{ε} covering. Assume it is covering for some $\varepsilon \in \mathcal{N}_{\infty}(0)$. As μ_{ℓ} has decidable equality,

$$\begin{aligned} \Omega(\mathbb{A}^1/\mu_{\ell}, \varepsilon) &= \left(\sum_{g:\mu_{\ell}} g\varepsilon = \varepsilon \right) \\ &= (\varepsilon = \varepsilon) + \sum_{g:\mu_{\ell} \setminus \{1\}} (g-1)\varepsilon = 0 \\ &= 1 + \mu_{\ell} \setminus \{1\} \times (\varepsilon = 0) \end{aligned}$$

Thus $(\varepsilon = 0) \times (\mu_{\ell} \setminus \{1\})$ is an étale -flat geometric stack. Moreover $(\mu_{\ell} \setminus \{1\})$ is a covering stack by 4.11. Thus $\varepsilon = 0$ is an affine étale -flat geometric stack, thus formally étale + flat affine by saturatedness of the étale topology ?? . So as a formally étale + closed dense proposition, $\varepsilon = 0$ holds as desired. \square

References

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