Thesis

Tim Lichtnau

May 2024

Inhalt

1 Introduction to SAG 1

2 Topology 2

3 Preparation 4

4 Talk: Algebraic Spaces: (Counter-)examples 6
- 4.1 Recap: Sheaftheory . . . 6
- 4.2 Algebraic Spaces . . . 6
- 4.3 Quotients by Group actions . . . 7
- 4.4 Fiber Collapse! . . . 9
- 4.5 Schemes do not have descent . . . 10

5 Algebraic spaces 12
- 5.1 Schemes are algebraic Spaces for the Zariski Topology . . . 13
- 5.2 Non locally-seperated Examples . . . 16
- 5.3 Group quotients . . . 16
- 5.4 Obsolete . . . 19
- 5.5 Locally seperated examples . . . 22
- 5.6 FiberCollaps away from the origin . . . 23
- 5.7 Schemes do not have descent . . . 23
- 5.8 Gluing in an affine on the line . . . 23
- 5.9 Weakly-flat stacks . . . 26

6 Omega-stability and gerbes 26

7 Questions // TODO 28

8 Not clear where to put that 28
- 8.1 Remarks about weakly flat affines . . . 29

1 Introduction to SAG

Lemma 1.1. *R is local, i.e. if $x, y : R$ and $x \neq y$, then x is invertible or y is invertible.*

Lemma 1.2. *If char $\neq 2$, Let $\rho \neq 0$, then $x^2 = \rho^2$ implies $x = \rho$ or $x = -\rho$*

Proof. Indeed, as $\rho \neq -\rho$, one of them is invertible by 1.1 □

Example for zariski local choice

Example 1.3. *For some A and $g, g' : A$ define*

$$g \mid_A g' \equiv \|\{h : A \mid hg =_A g'\}\|$$

Claim: For any $g, g' : A$, we have

$$g|_A g' \leftrightarrow \forall x : \operatorname{Spec} A, gx|_R g'x$$

Proof. $\to$ is obvious using that the duality map is an algebra isomorphism.
$\leftarrow$. For any $x : \operatorname{Spec} A$ we merely find some $h : R$ with $h \cdot g(x) = g'(x)$, i.e. we define our family of inhabited types as

$$Bx = \{h : R \mid h \cdot g(x) = g'(x)\}$$

By zariski local choice we merely find some principal open cover $\operatorname{Spec} A = \bigcup_{i=1}^n D(f_i)$ and local sections

$$\begin{aligned} &\prod_{x:D(f_i)} \{h_i : R \mid h_i \cdot g(x) = g'(x)\} \\ &\overset{??}{\simeq} \{h_i : D(f_i) \to R \mid (h_i x) \cdot g(x) = g'(x)\} \\ &\overset{??}{\simeq} \left\{ h_i : A_{f_i} \mid h_i \cdot \frac{g}{1} =_{A_{f_i}} \frac{g'}{1} \right\} \end{aligned}$$

We can multiply h_i by high enough powers of f_i to obtain some $h_i : A$ with $h_i \cdot g = g' \cdot f_i^n$ for some $n : \mathbb{N}$. we may assume that n does not depend on $i = 1, \ldots, n$ by taking the maximum and multiplying the h_i again with enough powers of f_i. Now use **??** to write $1 = \sum_{i=1}^n \ell_i f_i^n$ for some $\ell_i : A$ and then

$$(\sum_i \ell_i h_i) \cdot g = \sum_i \ell_i f_i^n g' = 1g' = g'$$

□

2 Topology

Lemma 2.1. *Let $f, g : X \to Y$ be two functions into a seperated scheme where $X = \operatorname{Spec} A$ for A a reduced ring. If f and g coincide on a dense subset, then $f = g$.*

Proof. The equalizer

$$Z \equiv \sum_{x:X} fx =_Y gx$$

is a closed subset of X, as Y is seperated. As its open complement does not intersect the given dense subset, its empty. In other words $\neg\neg Z = X$. Writing $I \subset A$ as the ideal of functions that vanish on Z, By [2], we have

$$\operatorname{Spec} A = \neg\neg Z = \bigcup_n \operatorname{Spec} A/I^n$$

But by the strong boundedness principle, we find some n such that $\operatorname{Spec} A = \operatorname{Spec} A/I^n$, in other words, $I^n = 0$. As A is a reduced ring, we conclude $I = 0$, so $Z = \operatorname{Spec} A/I = \operatorname{Spec} A$. □

Definition 2.2. A point $0 : \operatorname{Spec} B$ is regular, if $\operatorname{Spec} B \setminus \{0\} = D(p_1, \ldots, p_n)$ for some $p_1, \ldots, p_n : B$ jointly-reguar, i.e. if $p_i^m \cdot b = 0$ for all $i = 1, \ldots, n$ then $b = 0$.
If $0 : X$ is a point of a scheme, we call it regular, if one of the following equivalent conditions is satisfied

1. it admits some open affine neigbhorhood U such that $0 : U$ is regular.

2. It is a regular point of any open affine neigbhorhood.

Proof. Consider an open affine neigbhorhood $0 : D(f) \subset U = \operatorname{Spec} B$. We will show

1. If 0 is regular in $D(f)$, then it is regular in $\operatorname{Spec} B$: Consider $g_1, \ldots, g_n : B$ such that
$$B_f \to \prod_i B_{fg_i}$$
is injective. Define $g_0 := f - f(0)$, where $0 \notin D(g_0)$. Let us show, that $g_0, \ldots, g_n$ are jointly surjective in B. Let $b : B$ such that $g_i^n b = 0$ for all $0 \leq i \leq n$. Then in particular $b/1 =_{B_f} 0$. Thus b is in the kernel of $B \to B_{g_0} \times B_f$. But $D(g_0) \cup D(f)$ forms an open cover of $\operatorname{Spec} B$ as (f, g_0) generate the unit ideals. Thus $b : \operatorname{Spec} B \to R$ equals 0 on an open cover, thus its 0.

2. If 0 is regular in $\operatorname{Spec} B$, then it is regular in $D(f)$: Assume $B \to \prod B_{g_i}$ is injective. Let $f : B$. Let us show that $B_f \to \prod B_{g_i f}$ is injective. If $(g_i f)^n b = 0$, then $(g_i)^n (f^n b) = 0$, thus $f^n b = 0$ by assumption. Thus $b/1 =_{B/f} 0$ as desired.

□

Lemma 2.3. *If $0 : X$ is a regular point in a scheme, then both holds:*

1. *$X \setminus \{0\}$ is dense*

2. *$R^X \to R^{X \setminus \{0\}}$ is injective.*

Proof. 1. We write $A \perp B$ for $A \cap B = \varnothing$. We reduce to affine case: Let $0 \in \operatorname{Spec} B \subset X$. Let $U \subset X$ be open such that $U \perp X \setminus \{0\}$. Then $U \perp X \setminus \{0\} \Rightarrow U \perp \operatorname{Spec} B \setminus \{0\} \Rightarrow U \perp \operatorname{Spec} B$ so $U \perp (\operatorname{Spec} B \cup X \setminus \{0\}) = X$, thus $U = \varnothing$.
So we may assume that $X = \operatorname{Spec} B$ is affine: Then by [2], an open subset of $\operatorname{Spec} B$ is dense iff it is of the form $D(g_1, \ldots, g_n)$ for nilregular functions $g_i : B$. Conclude, as regular implies nilregular [2].

2. Lets first reduce to the affine case. Choose an open affine neibhorhood U of 0 such that $0 : U$ is regular. Then the surjection $U + X \setminus \{0\} \twoheadrightarrow X$ induces a vertical left injection
$$\begin{tikzcd} R^{U+X\setminus\{0\}} \arrow[r, hook] & R^{U\setminus\{0\}+X\setminus\{0\}} \\ R^X \arrow[u, hook] \arrow[r] & R^{X\setminus\{0\}} \arrow[u] \end{tikzcd}$$
So we may assume that $X = \operatorname{Spec} A$ is affine.
Let $p_1, \ldots, p_n : A$ be jointly-reguar, i.e. if $p_i^m \cdot a = 0$ for all $i = 1, \ldots, n$ then $a = 0$. If $f : \operatorname{Spec} A \to R$ such that $f(x) = 0$ for all $x \in D(p_1, \ldots, p_n)$, then $f(x) = 0$ for all $x : \operatorname{Spec} A$. f is in the kernel of the diagonal map
$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, hook] & R^{\operatorname{Spec} A} \arrow[d] & \\ \prod_{i=1}^n A_{p_i} \arrow[r, equal] & R^{\coprod_i D(p_i)} \arrow[r, hook] & R^{\cup D(p_i)} \end{tikzcd}$$
which is injective, as $p_1, \ldots, p_n$ are jointly-regular in A.
Thus $f = 0$ in A.

□

Remark 1. If A is an algebra that is reduced as a ring, then for $X = \operatorname{Spec} A$, 1. implies 2. by 2.1

Proposition 2.4. *the subtype* $\{0\} + 0^c \subset \operatorname{Spec} B$ *is not locally closed whenever one of the following conditions is satisfied:*

1. $\operatorname{Spec} B \setminus \{0\}$ *is dense*
2. $R^{\operatorname{Spec} B} \to R^{\operatorname{Spec} B \setminus \{0\}}$ *is injective*

Proof. Let us first show, that the infinitesimal neigbhorhood of 0 is not open.

1. If $0^c \subset \operatorname{Spec} A$ is dense) : The non-empty open $\mathcal{N}_\infty$ does not intersect the dense subset 0^c.
2. If $R^{\operatorname{Spec} B} \to R^{\operatorname{Spec} B \setminus \{0\}}$ is injective: If it would, we find a principal open smaller neighborhood$0 \in D(g) \subset \mathcal{N}_\infty(0)$, which however already cotains the whole infinitesimal one, thus $\mathcal{N}_\infty(0) = D(g)$
   Then for any $x \neq 0$, we have $\neg\neg g(x) = 0$. As $\operatorname{Spec} B \setminus \{0\}$ is a scheme, it admits a boundedness principle, thus we find some n, such that $g^n(x) = 0$ for all $x \neq 0$.
   by 2.3 we have that $R^{\operatorname{Spec} B} \to R^{\operatorname{Spec} B \setminus \{0\}}$ is injective, so we deduce $g^n = 0$, hence $D(g) = D(g^n) = \varnothing$ contradiction.

Just assumg that the infinitesimal neighborhood is not open, The subtype $\{0\}+0^c \subset \operatorname{Spec} B$ is not locally closed. Let $U, C \subset \operatorname{Spec} B$ be an open subset and a closed subset respectively, such that $(x \neq 0) + (x \neq 0) \leftrightarrow x \in U \wedge x \in C$. Then, for any $x : U$,

$$(x = 0) + (x \neq 0) = x \in C$$

is a closed proposition. Thus the decidable subtype $x \neq 0$ is a closed proposition. To contradict the assumption, we may convince ourself that the right vertical map

$$\begin{tikzcd}
\sum_{x:U} \neg\neg x = 0 \arrow[r, "\sim"'] \arrow[d, hook] & \sum_{x:\operatorname{Spec} B} \neg\neg x = 0 \arrow[d] \\
U \arrow[r, hook] & \operatorname{Spec} B
\end{tikzcd}$$

is an open embedding
where the upper horizontal map is indeed an equivalence as for any $x : \operatorname{Spec} B$, $x \in U$ is $\neg\neg$-stable, but $\neg\neg x = 0$ and $0 \in U$, thus $x \in U$.

□

3 Preparation

Lemma 3.1 (Strong boundedness, NEEDED?)**.** *Consider a sequence of embeddings of types*

$$X_0 \overset{\iota_0}{\hookrightarrow} X_1 \overset{\iota_1}{\hookrightarrow} X_2 \ldots$$

Then any map $f : \operatorname{Spec} A \to \operatorname*{colim}_n X_n \equiv: \bigcup_n X_n$ *factors through some* $\kappa_m : X_m \hookrightarrow \operatorname*{colim}_n X_n$.

Proof. For every term $x : \operatorname{Spec} A$ consider the subset S_x of natural numbers n, such that $f(x) \in \operatorname{im} \kappa_m$. Its a merely inhabited upwards closed subset. By the strong boundedness principle [ref?], the subset $\bigcap_{x:\operatorname{Spec} A} S_x$ is merely inhabited. □

Lemma 3.2. *Let* Y *be a type, which admits a jointly surjective family of maps with smooth domain* $X_i \to Y$ *Then* Y *is formally smooth.*

Proof. $\sum_{n:\mathbb{N}} X_n \to Y$ is surjective with formally smooth domain, as $\mathbb{N}$ is formally smooth. $\square$

Corollary 3.3 (Monoid is smooth)**.** *Let $(Y, +)$ be a magma, which is generated by a map with smooth domain $f : X \to Y$, i.e. every $a : Y$ can merely be written as a finite sum*

$$a = f(x_1) + \ldots + f(x_n)$$

Then Y is formally smooth.

Lemma 3.4. *Let C be a class of types stable under $\sum$. Let $\mathbb{P} \subset \mathsf{Aff}$ (in most cases $\mathbb{P} := \mathsf{Aff}$) be any subclass of affines stable under finite limits. The class $\mathsf{HasAtlas}_C^{\mathbb{P}}$ of types Y which admit a map $\mathbb{P} \ni S \to Y$ fibered in C is stable under identity types.*

Proof. By assumption we can choose a map $\mathbb{P} \ni V \xrightarrow{p} Y$ fibered in C. Let $y, y' : Y$. Then we have the map

$$\begin{gathered}(\mathrm{fib}_p\, y) \times_V (\mathrm{fib}_p\, y') \to y = y' \\ (v, q : y = pv), (v', q' : y' = pv'), (h : v = v') \mapsto q \cdot h \cdot q'^{-1}\end{gathered}$$

The fiber over $j : y = y'$ looks like

$$\sum_v \underbrace{(\sum_{v'} (h : v = v'))}_{\mathsf{isContr}} \times (q : y = pv) \times (q' : y' = pv') \times (q \cdot h \cdot q'^{-1} = j) \simeq \sum_v (v = py) \simeq \mathrm{fib}_p\, y$$

Hence the map is fibered in C. It suffices to show, that $(\mathrm{fib}_p\, y) \times_V (\mathrm{fib}_p\, y')$ has an atlas, because then we can compose such an atlas with the above map to obtain an atlas of $y = y'$. By assumption the fibers of p have an atlas, so we can choose $q : W \to \mathrm{fib}_p\, y, q' : W' \to \mathrm{fib}_p\, y'$ atlasses. Then $W \times_V W' \to (\mathrm{fib}_p\, y) \times_V (\mathrm{fib}_p\, y')$ is an atlas: The domain is a fiber product of types in $\mathbb{P}$, hence it belongs to $\mathbb{P}$. The fiber over (x, x') is equivalent to the product of fibers $(\mathrm{fib}_q\, x) \times (\mathrm{fib}_{q'}\, x')$ which is in C by stability under dependent sums (so in particular under finite products). $\square$

Lemma 3.5. *Let $\mathcal{U}' \subset \mathcal{U}$ be stable under dependent sums. Let X be a type with a map $p : U \to X$ fibered in $\mathcal{U}'$. For any $x : X$, let Y_x be a type and moreover for any $u : U$, we are given a map $q_u : V_u \to Y_{p(u)}$ fibered in $\mathcal{U}'$. Then the induced map*

$$p : \sum_{u:U} V_u \to \sum_{x:X} Y_x$$

is fibered in $\mathcal{U}'$

Proof. The fiber of p over some $(x, y) \in \sum_{x:X} Y_x$ is given by

$$\sum_{u:\mathrm{fib}_p\, x} \mathrm{fib}_{q_u}(y')$$

where $y' : Y_{p(u)}$ (depending on u) is the transport of $y : Y_x$ along $x = p(u)$. As $\mathcal{U}'$ is stable under dependent sum those fibers are again in $\mathcal{U}'$. This shows the result. $\square$

4 Talk: Algebraic Spaces: (Counter-)examples

4.1 Recap: Sheaftheory

Notation.

$$\mathbb{T} = \sum_{X:\mathsf{Aff}} X \text{ formally étale } \wedge \underbrace{\text{faithfully flat}}_{=\text{ flat } + \neg\neg \text{ inhabited}}$$

Any $X : \mathbb{T}$ has decidable equality! $\mathbb{T}$ contains 1 and is stable under $\sum$. A $\mathbb{T}$-cover is a map fibered in $\mathbb{T}$.

Definition 4.1. A type X is an etale-stack, if for any $\operatorname{Spec} A : \mathbb{T}$ and any map $\|\operatorname{Spec} A\| \to X$, there exists a unique filler

$$\begin{tikzcd} \|\operatorname{Spec} A\| \arrow[r, "\forall"] \arrow[d] & X \\ 1 \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

An étale -sheaf is a 0-type that is an étale -stack.

Beeing an étale -stack determines a lex-modality, that we write $L_{\mathbb{T}}$.
Given an equivalence relation R on a sheaf X valued in propositional sheaves, we can form the sheaf-quotient $p : X \to X/R :\equiv L_{\mathbb{T}}\|X//R\|_0$ that is uniquely characterized by the following properties

- for any $x, y : X$, $p(x) =_{X/R} p(y) \leftrightarrow R(x, y)$
- The map p is $\mathbb{T}$-surjective, i.e. for any $z : X/R$, $\|\operatorname{fib}_p(0)\|_{\mathbb{T}}$ ('$\mathbb{T}$-merely the proposition $\|\operatorname{fib}_p(0)\|$ holds').

4.2 Algebraic Spaces

We assume today, that schemes are étale sheaves.

Definition 4.2. A naive algebraic space is an étale -sheaf X, that merely admits a $\mathbb{T}$-cover $\operatorname{Spec} B \to F$.
We call it covering, if we can choose $\operatorname{Spec} B \in \mathbb{T}$.

This is not good enough, because we can NOT prove the following

- all schemes are naive algebraic spaces
- naive algebraic spaces having descent, i.e. the type of them is an étale -stack.

Instead we have to repeat the process asking for an atlas twice.

Definition 4.3. An algebraic space is an étale -sheaf X, such that one of the following equivalent conditions holds:

- merely we find a map $\operatorname{Spec} A \to X$ such that each fiber is a covering naive algebraic space.
- We merely can express it as the sheaf-quotient of an affine $\operatorname{Spec} A$ by an equivalence relation R that is covering, i.e. for each $x : \operatorname{Spec} A$, $\sum_{y:\operatorname{Spec} A} R(x, y)$ is a covering naive algebraic space.

We call X covering, if we can choose $\operatorname{Spec} A$ to lie in $\mathbb{T}$.

This class is stable under $\sum$ and under quotients by covering equivalence relations

Theorem 4.4 (DESCENT)**.** *The type of (covering) algebraic spaces is an étale -stack.*

Example 4.5. *Schemes are algebraic spaces!*

Proof.
1. Merely inhabited finite sums of principal open propositions belong to $\mathbb{T}$.
2. open propositions are naive algebraic spaces.
3. If X is a scheme, merely we find a map $\coprod_i \operatorname{Spec} A_i \to X$ whose fibers are merely inhabited finite sums of open propositions, which are covering.

□

Question 1. Can we find algebraic spaces that are not schemes?
Can we prove with them, that Schemes do not have descent?

4.3 Quotients by Group actions

Let $\ell \neq 0$ denote a prime. Consider $\mu_\ell = R[X]/(X^\ell - 1)$.

Example 4.6 (Non-free action)**.** *If $2 \neq 0$, the sheaf-quotient of $\mathbb{A}^1$ by the non-free μ_2 action is not an algebraic space!*

This suggests that we need a free action. On the other hand, classically, the quotient of a quasi-projective scheme by a finite free group action is a scheme.

Construction. Given a formally étale + flat affine (e.g. μ_ℓ or finite) group G that acts on an affine $\operatorname{Spec} A$. Assume G acts free on some open subset $U \subset \operatorname{Spec} A$.
Then we construct a covering equivalence relation R on $\operatorname{Spec} A$, such that

- On U we are just taking the quotient by the G-action: for any $x : U$ and $y : \operatorname{Spec} A$

$$R(x, y) \leftrightarrow \sum_{g:G} gx = y.$$

- On the complement $U^c \equiv \operatorname{Spec} A \setminus U$ we do nothing: for some $x : U^c, y : \operatorname{Spec} A$, we have $R(x, y) \leftrightarrow x = y$.

We write $\operatorname{Spec} A/_U G \equiv \operatorname{Spec} A/R$ and call it the quotient of $\operatorname{Spec} A$ by the G-action on U .

Proof.

$$R_G(x, y) \equiv (x = y) + (x \in U) \times \sum_{g:G\setminus\{1\}} gx = y$$

This is covering: For any $x : \operatorname{Spec} A$ we have

$$\sum_{y:X} x = y + (x \neq 0) \times \sum_{g:G\setminus\{1\}} gx = y \simeq 1 + (x \in U) \times G \setminus \{1\}$$

$G \setminus \{1\} = \sum_{g:G} g \neq 1$ is a $\sum$ of formally étale + flat affines (recall that formally étale affines have decidable equality).
Indeed, the two conditions hold, using that G has decidable equality. □

Example 4.7 (Free action)**.** *Let G act freely on the whole space $U \equiv \operatorname{Spec} A$. Then this construction yields the actual group quotient:* $\operatorname{Spec} A/_{\operatorname{Spec} A} G = \operatorname{Spec} A/G$.

Proof. Indeed, the equivalence relation is the same, using that G has decidable equality. □

Example 4.8 (Quotient of the Line). *If $\ell \neq 0$ is prime, we have $R/_{0^c}\mu_\ell$ is an algebraic space*

Example 4.9 (Quotient of the Cross). *Let μ_ℓ act on $X = \operatorname{Spec} R[X,Y]/X^\ell - Y^\ell$ via multiplication on the second component. Then*

$$X/_{\{0,0\}^c}\mu_\ell$$

is an algebraic space.

Are those schemes?

Proposition 4.10. *Let $0 : \operatorname{Spec} A$ be a regular point, i.e. we can write $\operatorname{Spec} B \setminus \{0\} = D(p_1, \ldots, p_n)$ for some $p_1, \ldots, p_n : B$ jointly-reguar, i.e. $B \to \prod_{j=1}^n B_{p_j}$ is injective.*
Let G be a non-trivial formally étale flat affine group acting on $\operatorname{Spec} A$, such that

- *0 is a fixpoint*
- *if $gx = x$ for some $g \neq 1$, then $x = 0$.*

Then $\operatorname{Spec} A/_{0^c}G$ from 5.17 is an algebraic spaces that is not a scheme.

Proof. As $\neg\neg(G \setminus \{1\})$ and our goal is $\neg\neg$-stable, we may choose some $g : G \setminus \{1\}$. Then for all $y : \operatorname{Spec} A$

$$R_\sharp(y, gy) \simeq (y = gy) + (y \neq 0) \wedge \sum_{h \neq 1} hy = gy \simeq (y = 0) + (y \neq 0) \simeq (y = 0) + (y \neq 0)$$

Regularity of 0 gives us that $\{0\} + D(0) \subset \operatorname{Spec} A$ is not a locally closed subtype, i.e a closed subset of an open subset. But the identity types of a scheme are locally closed propositions (subsets of the point). □

Example 4.11. *Assume $\ell \neq 0$ prime. Let μ_ℓ act on $\operatorname{Spec} B$ in one of the following ways:*

1. *Let μ_ℓ act on $\operatorname{Spec} B = \mathbb{A}^1$.*
2. *Let μ_ℓ act on*

$$\operatorname{Spec} B \equiv \sum_{x,y:R} x^\ell = y^\ell$$

via $g(x, y) = (x, gy)$

Then $\operatorname{Spec} B/_{0^c}\mu_\ell$ is an algebraic space that is not a scheme.

Proof. $\neg\neg$ merely, μ_ℓ is finite ([ref?]) and $\mu_\ell \setminus \{1\}$ is inhabited by 5.11.

1. Pointed-Free action is clear. $0 : \mathbb{A}^1$ is a regular point by first projection.
2. Pointed-Free action is clear. The cross is regular pointed, witnessed by the first projection: It is regular vanishing at $(0, 0)$ And for any point $(0, y) : \operatorname{Spec} B$ we deduce $y^\ell = -0^\ell = 0$, hence $\neg\neg(x, y) = (0, 0)$.

□

4.4 Fiber Collapse!

An alternative approach to construct algebraic spaces is the fiber collapse away from the origin.

Definition 4.12. Given a sheaf proposition P, there is a closed modality Cl_P where a type X is modal, if it is a sheaf and $P \to \text{isContr}(X)$. We have to stackify to belong to the sheaf topos:

$$\begin{aligned} &\mathcal{U} \to \mathcal{U} \\ &X \mapsto P \star X :\equiv L_{\mathbb{T}}(P \sqcup_{P \times X} X) \end{aligned}$$

where we need to stackify the pushout. This determines a lex modality.

Definition 4.13. Let $Y_\bullet : R \to \mathsf{Aff}$ be a dependent family of affines The fiber collapse of $Y_\bullet$ away from the origin —$Y_\bullet$— is the following type over R

$$\sum_{x:R} (x \neq 0) \star Y_x \to R$$

- The *infinitesimal fibers* over $\varepsilon : \mathcal{N}_\infty$ are $\text{fib}_p(\varepsilon) = Y_\varepsilon$.
- In particular the *basefiber* $\text{fib}_p(0)$ is equivalent to Y_0,
- The *other* fibers $\text{fib}_p(x)$, $x \neq 0$, are contractible.

So —$Y_\bullet$— is obtained by keeping only the infinitesimal fibers and collapsing all the other fibers. This space over R looks exactly like the line away from the origin.

Lemma 4.14. *Assume that if $x \neq 0$, then $Y_x \in \mathbb{T}$. Then —Y— is an algebraic space.*

Proof. Let $x : R$. Let $Y : \mathsf{Aff}$ such that $x \neq 0$ implies that Y is formally étale + flat. We will show that $\eta : Y \to (x \neq 0) \star Y$ is the sheaf-quotient map of the relation on Y given by $y \sim y' \equiv (y = y') + (x \neq 0) \times y \neq y'$¸, which is enough by 5.20. We apply **??**

- The map is $\mathbb{T}$-surjective: We have a $\mathbb{T}$-surjection $(x \neq 0) + Y \to (x \neq 0) \star Y$. In case $x \neq 0$, the map of interest is $Y \to 1$, which is $\mathbb{T}$-surjective, as then $Y \in \mathbb{T}$.
- Given $y, y' : Y$, we have

$$\begin{aligned} \eta(y') = \eta(y) &\simeq (x \neq 0) \star (y = y') && \mid \text{ closed modality is lex ([3] Example 3.1.4).} \\ &\simeq L_{\mathbb{T}}\left((y = y') \vee (x \neq 0)\right) && \mid (x \neq 0) \to \mathsf{HasDecEq}(Y) \\ &\simeq (y = y') + (x \neq 0) \times y \neq y'¸ \end{aligned}$$

□

Example 4.15. *—Bool— is the line with two origins.*
—Spec $R[X]/(X^2+1)$— is the twisted line with two origins, i.e. over the origin we have the roots of -1.
—Spec $R[Y]/(Y^2 - \bullet)$— is $\mathbb{A}^1/_{0^c}\mu_2$ which looks like $\mathbb{D}(1)$ over 0.
—Spec $R[Y]/(Y^2 - \bullet^2)$— is the quotient of the cross that looks like $\mathbb{D}(1)$ over every $\varepsilon : \mathbb{D}(1)$.

—Spec $R[Y]/(\bullet Y)$— is the affine plus.

Proposition 4.16. *Let G be a formally + flat affine group. Let $p : \tilde{Y} \to R$ such that the pullback to $R^\times$ can be enhanced to a G torsor over $R^\times$. Write $Y_x \equiv \mathrm{fib}_p\, x$. Then there is a canonical equivalence*

$$\begin{tikzcd}
 & \tilde{Y} \arrow[dl] \arrow[dr] & \\
\tilde{Y}/_{Y_0^c} G \arrow[rr, "\simeq"] \arrow[dr] & & -Y- \arrow[dl] \\
 & R &
\end{tikzcd}$$

Proof. As every *other* fiber is merely equivalent to G, its formally étale + flat. In between you can put for $U_x \equiv x \neq 0 \times Y_x$

$$\tilde{Y} \to \sum_{x:R} Y_x/_{U_x} G$$

As all three maps defined on $\tilde{Y}$ are $\mathbb{T}$-surjective, by **??** we may only check that the identity types coincide. For any $y, y' : \tilde{Y}$. Using that if $py \neq 0$ then the G action on Y_{py} is a G-torsor, We have an equivalence

$$(y = y') + y \notin Y_0 \wedge \underbrace{\sum_{g \neq 1} gy = y'}_{\simeq (py = py') \wedge y \neq y'} \simeq (py = py') \wedge ((y = y') + py \neq 0 \wedge y \neq y')$$

If we fix $x : R$ and put $y, y' : Y_x$ we have, writing $\eta : Y_x \to (x \neq 0) \star Y_x$

$$(y = y') + py \neq 0 \wedge y \neq y' \simeq \eta y = \eta y'$$

by the proof of 4.14. ∎

4.5 Schemes do not have descent

Remark 2. Whenever we want to show a proposition that is an étale -sheaf, we may assume a term in $i : \mathrm{Spec}\, C \subset R$. Because $i \neq -i$, this determines an embedding

$$\begin{aligned} Bool &\to \mathrm{Spec}\, C \\ + &\mapsto i \\ - &\mapsto -i \end{aligned}$$

But any embedding $Bool \hookrightarrow \mathrm{Spec}\, C$ is already an equivalence (*), as for any $x : R$, $x - i$ or $x + i$ is invertible, so if $(x - i)(x + i) = 0$ we know that one of the factors is zero.

Lemma 4.17. *Let $\rho \neq 0$. $\mathrm{Spec}\, R[T]/(T^2 + 1)$ is compact.*

Proof. Let $U \subset \mathrm{Spec}\, C$ be open. Then we find $f_1, \ldots, f_n : C$, such that $U = D(f_1, \ldots, f_n)$. Choose representatives $f_i = a_i + b_i T \mod T^2 + \rho$. Then consider the following numbers

$$r_{ij} = \begin{cases} a_i b_j - a_j b_i & , i \neq j \\ a_i^2 + b_j^2 & , i = j \end{cases}$$

We will show that $D((r_{ij})_{i,j}) \leftrightarrow (\mathrm{Spec}\, C \subset U)$. Because this statement is a sheaf, we may assume $\mathrm{Spec}\, C = Bool$.

'$\to$'
Assume $r_{ij} \neq 0$. If $i = j$, then $\operatorname{Spec} C \subset D(f_i) \subset U$.
If $i \neq j$, then $\operatorname{Spec} C \subset D(f_i, f_j) \subset U$.
'$\leftarrow$'
Let $x : \operatorname{Spec} C$. Choose i, j, s.th. $x \in D(f_i), -x \in D(f_j)$. In both cases $i = j$ and $i \neq j$, then $r_{ij} \neq 0$. □

Proposition 4.18. *let $\rho : R \setminus \{0\}$ (e.g. $\rho = 1$). Set $C = R[T]/(T^2 + \rho)$. If —Spec C— is a scheme, then $X^2 + \rho$ merely has a root.*

Proof. Let $p : —\operatorname{Spec} C— \to R$ be the first projection. We proceed as follows

1. There is no open affine subset of $—\operatorname{Spec} C—$ containing $\mathrm{fib}_p(0)$.
2. Any inhabited cover of $\operatorname{Spec} C$ by open subsets strictly smaller than $\operatorname{Spec} C$ yields a root.

Any finite open affine cover of —Spec C— can be restricted to a finite open affine cover of the basefiber $\operatorname{Spec} C$, that satisfies the condition in 2. Proofs:

1. Because we want to show $\perp$, we may assume —Spec C— = —*Bool*—. Assume there is an open affine subset $\mathrm{fib}_p(0) \subset U \subset$ —*Bool*—. Then $p(U) \subset R$ is an open neigbhorhood of 0, as
$$x \in p(U) \leftrightarrow (x, N) \in U \vee (x, S) \in U$$
Claim: the map $R^{p(U)} \to R^U$ is an equivalence. If we have shown that: As U is affine we conclude that the map
$$\begin{aligned} U &\to \operatorname{Spec}(R^{p(U)}) \\ x &\mapsto \phi \mapsto (\phi(px)) \end{aligned}$$
is an equivalence, which is a contradiction to the assumption, that U contains both origins.
Proof of claim: In words: As U is a subset of a quotient of $R + R$, the function $U \to R$ determines two (partially defined on open domain) functions to R that coincide away from the origin, which is a regular point. Thus by 2.3 they coincide everywhere.
Injectivity: If two maps $f, g : p(U) \to R$ coincide after precomposing with $U \to p(U)$, then they coincide away from 0 so conclude by 2.3.
Surjectivity: Given a map $U \to R$, by pulling back along $p : R + R \to$ —*Bool*— we can view it as a map $R + R \supset U' \to R$ defined at both origins, so in particular as a pair of maps to R defined on some open neigbhorhood of 0 of R. They coincide away from 0 so by 2.3 they are equal.
2. Let $\operatorname{Spec} C = \bigcup_{j=1}^{n} U_j$ be an open cover of strictly smaller subsets of $\operatorname{Spec} C$.
Claim: Any open subset $U \subset \operatorname{Spec} C$ that is a proposition, is an open proposition.
Proof: By [1] we find $f_1, \ldots, f_n : C$ such that $U = D(f_1, \ldots, f_n)$. As the left hand side is a proposition we have
$$U \leftrightarrow \bigvee D(f_i)$$
so we may show, that each $D(f_i)$ is open.
Let $f : C$ such that $D(f)$ is a proposition. Choose a representative $a + bT : R[T]$. Then for any $x : \operatorname{Spec} C$, $(a + bx)(a - bx) = a^2 + b^2$ is not not zero. Then, for any $x : \operatorname{Spec} C$, $a \neq 0 \leftrightarrow (a + bx \neq 0) \vee (a - bx \neq 0) = D(f)$. But $(a \neq 0) \leftrightarrow D(f)$ is an étale -sheaf and a proposition, so conclude. □(Claim)

For $0 \leq k \leq n$, let us prove

$$\left(\operatorname{Spec} C \subset \bigcup_{j=0}^{k+1} U_j\right) \to U_{k+1} \vee \left(\operatorname{Spec} C \subset \bigcup_{j=0}^{k} U_j\right)$$

The right hand side is an open proposition: U_{k+1} is open by the claim and the right hand side is open by compactness of $\operatorname{Spec} C$. Thus it is an étale -sheaf. Thus we may assume $\operatorname{Spec} C = Bool$. Proof by induction over k If $k = 0$, then the right summand is empty, so we have to show U_1. But this is clear
For $k \mapsto k+1$, we merely find j, j' such that $i \in U_j, -i \in U_{j'}$. Now, exploiting decidable equality in $\mathsf{Fin}\ (n+1)$: If $k = j$ return i. If $k = j'$, return $-i$. Otherwise, $\operatorname{Spec} C \hookleftarrow \{i, -i\} \subset U_j \cup U_{j'} \subset \bigcup_{j=0}^{k} U_j$.

□

Corollary 4.19. *Schemes do not have descent.*

Proof. If Schemes have descent, then —$\operatorname{Spec} R[T]/(T^2 + \rho)$— $\in \mathsf{Sch}$ is a sheaf. As —$\operatorname{Spec} R[T]/(T^2 + \rho)$— is $\mathbb{T}$-merely a scheme, it is a scheme, so by the previous lemma $T^2 + \rho$ has a root. As $\rho : R \setminus \{0\}$ was arbitrary, we get a contradiction to [1] A . 0.3.

□

5 Algebraic spaces

Theorem 5.1. *Let X be a modal set. The following are equivalent:*

1. *X is a (covering) geometric 0-stack*
2. *X is merely of the form $L_{\mathbb{T}}(U/R)$ for some (covering) affine U and $R : U^2 \to \mathrm{Prop}_{\bigcirc}$ a covering equivalence relation.*
3. *there exists some map $S \to X$ with S (covering) affine whose fibers merely have $\mathbb{T}$-catlasses.*

We call this class (covering) algebraic spaces.

Proof.

$2 \leftrightarrow 3$ This is **??**

$2 \to 1$ Choose a presentation $R : U^2 \to \mathrm{Prop}$. It suffices to show, that the map $f : U \to L_{\mathbb{T}}(U/R)$ is a geometric (c)atlas. The map f is $\mathbb{T}$-surjective by the well-definedness of the bijection **??**. By descent we may just show, that the fibers $\mathrm{fib}_f(f(s))$ for $s : U$ are covering 0-stacks. But by the bijection in **??** those are equivalent to the fibers R_s, which are covering 0-stacks as the equivalence relation is covering.

$1 \to 2$ This can be reformulated in the following way, using the recursion principle for (covering) geometric 0-stacks: Let X be a sheaf of sets. Let S be (covering-) affine and $f : S \to X$ be fibered in covering algebraic spaces. Then X is a (covering) algebraic space. This follows from the observation, that the equivalence relation determined by f is covering **??** , because the fibers of f are covering 0-stacks.f

□

Proposition 5.2. *For any $n \geq 1$, we have inclusions*

$$W_n \subset \mathsf{CS}_{n-1} \subset W_{n+1}$$

Proof. Induction. $n = 1$ gives

$$\mathsf{HasCatlas}_{\mathbb{T}} \subset \mathsf{CS}_0 \subset \text{ types admitting a catlas fibered in } W_1$$

the latter inclusion is the previous theorem.
The induction step is obtained by 5.3 □

5.1 Schemes are algebraic Spaces for the Zariski Topology

Definition 5.3. A proposition U is open iff its merely of the form $f_1 \; inv \vee \ldots f_n inv$ for some $f_i : R$.

Lemma 5.4. *Given $f_1, \ldots, f_n : R$ such that $\|D(f_1) + \ldots + D(f_n)\|$ then $\sum_{i=1}^{n} D(f_i) \in \mathsf{Zar}$.*

Proposition 5.5. *Every Zariski-merely-inhabited type that is merely of the form $U_1 + \ldots + U_n$ for open propositions U_i admits a Zar-catlas.*

Proof. By definition of openness, We can choose a surjection $\coprod_{j=1}^{n_i} D(f_{ij}) \twoheadrightarrow U_i$ for any i. We want to show, that the map

$$\coprod_{i,j} D(f_{ij}) \twoheadrightarrow U_1 + \ldots U_n$$

is a Zar-catlas.

- Let us first show that the fibers are in Zar. Assume U_i holds. So we find a term in $\coprod_j D(f_{ij})$. In particular we have $\| \coprod_j D(f_{ij})\|_{\mathsf{Zar}}$. By the lemma we conclude, that the fiber $\sum_j D(f_{ij})$ belongs to Zar.

- The total space is in Zar: This follows as the surjection after propositional truncation becomes an equivalence. As we have $\|U_1 + \ldots + U_n\|$, we can conclude by the lemma.

□

Warning. The converse does not hold! We want to apply **??**, to the map

$$\mathsf{Zar} \ni 1 + 1 \to \sum D(f)$$

- $\sum D(f)$ is seperated as $D(f)$ is a sheaf.
- All the fibers are equivalent to $1 + X$, hence they are in the Zariski topology.

Lemma 5.6. *let X be a scheme. There merely exists some affine S map $S \to X$ whose fibers are merely inhabited finite sums of open propositions*

Corollary 5.7. *Every scheme is an algebraic space.*

Lemma 5.8. *If X is an algebraic space, then the global sections embed via a R-algebra homomorphisms into a finitely presented R-algebra.*

Proof. Choose an atlas $S \to X$, in particular $\mathbb{T}$-surjective. As $\mathbb{T}$ is subcanonical the map $R^X \to R^S$ is an injection. □

Question 2. Is it an open embedding of types?

The goal of this subsection is to construct algebraic spaces. The first example actually gives us a scheme:

Example 5.9. *Let $p \neq 0$ be a prime. You can let $\mu_p := \operatorname{Spec}(R[X]/(X^p - 1))$ act on $\mathbb{A}^\times$ via multiplication. Set $\mathbb{T} = fppf$. Then the p.th power map*

$$pow : \|\mathbb{A}^\times//\mu_p\|_0^{\mathbb{T}} \to \mathbb{A}^\times$$

is an equivalence.

- *It is an embedding: First note, that $\|\mathbb{A}^\times//\mu_p\|_0$ is $\mathbb{T}$-seperated:*

  as μ_p act freely on $\mathbb{A}^\times$, $\mathbb{A}^\times//\mu_p$ is already a set. Meaning that the identity types of the set-quotient are $\sum_{g:\mu_p} gx =_{\mathbb{A}^\times} y$, hence sheaves.
  On the other hand the map $\|\mathbb{A}^\times//\mu_p\|_0 \to \mathbb{A}^\times$ is an embedding, as for any $x, y : \mathbb{A}^\times$ the map $(\sum_{g:\mu_p} gx = y) \to (x^p = y^p)$ is an equivalence.

- *It is $\mathbb{T}$-surjective, as for any $\lambda : \mathbb{A}^\times$, we find $S = \operatorname{Spec} R[X]/(X^p - \lambda) \in \mathbb{T}$ with*

  $$S \to \operatorname{fib}_{\mathbb{A}^\times/\mu_p \to \mathbb{A}^\times}(\lambda)$$

  hence

  $$1 = \|S\|_{\mathbb{T}} \to \|\operatorname{fib}_{pow}\|_0^{\mathbb{T}}$$

Example 5.10. *Let P be the open proposition $x \neq 0$ for some $x : \mathbb{A}^1$. Then $H = 1 + P$ is an open subgroup of $\mathbb{Z}/2$. The sheaf quotient G/H is the scheme $\mathsf{Susp}(x \neq 0)$.*

Let $\ell \neq 0$ denote a prime. Consider $\mu_\ell = R[X]/(X^\ell - 1)$.

Lemma 5.11. *Let $(G, 1)$ be a pointed formally étale flat affine type. Then $(G \setminus \{1\})$ is formally étale + flat affine.*
In particular $\mu_\ell \setminus \{1\}$ is a covering stack.

Proof. $G \setminus \{1\} = \sum_{g:G} g \neq 1$ is a $\sum$ of formally étale + flat affines (recall that formally étale affines have decidable equality).
To show, that $\mu_\ell \setminus \{1\}$ is a covering stack, by **??**, we need to show it is $\neg\neg$-inhabited. Indeed as we want to prove a contradiction we may assume a term in $g : \operatorname{Spec} R[X]/(\sum_{i=0}^{\ell-1} X^i)$. But this type is equivalent to $\mu_\ell \setminus \{1\}$, using that $\sum_{i=0}^{\ell-1} X^i | X^\ell - 1$ and $\ell \neq 0$.

□

Lemma 5.12. *Given a modal equivalence relation R on a sheaf X and a 1-stack T and a map $f : X \to T$ and term $p : \prod_{x,y:X} R(x, y) \to fx = fy$ such that $p(x, y) \cdot p(y, z) = p(x, z)$, where the witnesses for R are left implicit. Then f factors through the quotient.*

Lemma 5.13. *Put $\ell = 2$ If $\ell \neq 0$, the sheaf quotient of $\mathbb{A}^1$ by the μ_2 action is not an algebraic space.*

Proof. Assume this it is an algebraic space.
Set $\mathbb{D}(1) = \operatorname{Spec} R[X]/X^\ell$. Then $\sum_{x:\mathbb{A}^1/\mu_\ell} x^\ell =_{\mathbb{A}^1} 0 \simeq \mathbb{D}(1)/\mu_\ell$ is an algebraic space by $\sum$-stability.
Then we can choose a geometric atlas $p : \operatorname{Spec} A \to \mathbb{D}(1)/\mu_\ell$. We proceed in the following steps

1. There is an equivalence $\operatorname{Spec} A \simeq \operatorname{fib}_p 0 \times \mathbb{D}(1)/\mu_\ell$.
2. The fiber over 0 is affine
3. $\mathbb{D}(1)/\mu_\ell$ is $\neg\neg$ affine
4. $\mathbb{D}(1)/\mu_\ell$ is $\neg$ affine

Proofs

1. Let us denote $F : \mathbb{D}(1)/\mu_2 \to \mathsf{CS}_0$ the bundle of fibers of f, where we note that the fibers are indeed sets. As CS_0 is formally étale ([ref?]), we have terms

$$\phi : \prod_{x:\mathbb{D}(1)} F[x] = F[0], \phi^- : \prod_{x:\mathbb{D}(1)} F[-x] = F(0)$$

that both evaluate at $x = 0$ to $\mathsf{refl}_{F[0]}$.
The goal is to produce a term in

$$\prod_{x:\mathbb{D}(1)/\mu_2} Fx = F[0]$$

By the previous lemma, using that CS_0 is a 1-stack, we need to show, that under the path $p_x : [x] = [-x]$ in the quotient we have

$$\mathsf{ap}_{p_x} F \cdot \phi^- x = \phi x$$

This proposition is formally étale as CS_0 is formally étale . Thus we may assume the closed dense proposition $x = 0$. Then $p_x = \mathsf{refl}_{[0]}$ and $\phi^- 0 = \mathsf{refl} = \phi 0$ by assumption.

2. Let us first show, that We may assume that our geometric cover factors through the $\mathbb{T}$-surjection $\operatorname{Spec} A \xrightarrow{f} \mathbb{D}(1) \to \mathbb{D}(1)/\mu_\ell$. Proof: By $\mathbb{T}$-local choice applied to the $\mathbb{T}$-surjection $\mathbb{D}(1) \to \mathbb{D}(1)/\mu_\ell$, we find a $\mathbb{T}$-cover $\operatorname{Spec} B \to \operatorname{Spec} A$ and a factorization

$$\begin{tikzcd} \exists \operatorname{Spec} B \arrow[r, dashed] \arrow[d, dashed] & \mathbb{D}(1) \arrow[d, two heads] \\ \operatorname{Spec} A \arrow[r] & \mathbb{D}(1)/\mu_\ell \end{tikzcd}$$

□(Claim)
Its enough to see that the map $\operatorname{fib}_f 0 \to F$ is an equivalence. That follows because 0 is a fix point of the μ_ℓ action on $\mathbb{D}(1)$.

3. F is a covering stack, hence $\neg\neg$-inhabited. As the goal is $\neg\neg$-modal, we may pick a map $1 \to F$. Then, by step 1

$$\mathbb{D}(1)/\mu_\ell = 1 \times_F (F \times \mathbb{D}(1)/\mu_\ell) = 1 \times_F \operatorname{Spec} A$$

is a fiber product of affines, hence affine.

4. Here we need that $\ell = 2$. The affinization map would be induced by

$$\begin{tikzcd} \mathbb{D}(1) \arrow[d] \arrow[dr, "z \mapsto z^\ell"] & \\ \mathbb{D}(1)/\mu_\ell \arrow[r, dashed] & \mathbb{D}(1) \end{tikzcd}$$

But the map is not an embedding: For any $\varepsilon : \operatorname{Spec} R[X]/X^\ell$, we have $\varepsilon^\ell = 0^\ell$ but $\varepsilon =_{\mathbb{D}(1)/\mu_\ell} 0$ iff there $\mathbb{T}$-merely exists some $g : \mu_\ell$ with $g\varepsilon = 0$, but as g is invertible this is equivalent to $\varepsilon = 0$.

□

5.2 Non locally-seperated Examples

Proposition 5.14. *Consider an affine S and an open subset $U \subset S$. Consider a $\mathbb{T}$-flat irreflexive relation $\sharp$ on U , i.e.*

1. *Irreflexivity:* $\neg(x \sharp x)$
2. *$\mathbb{T}$-flatness. For all $x : U$, the fiber $\sum_{y:S} x \sharp y$ is $\mathbb{T}$-flat.*

Define a relation on S as

$$R_\sharp(x, y) = (x = y) + (x \in U \wedge y \in U) \times (x \sharp y)$$

(Abuse of notation: where the $\times$ is secretly a $\sum$) Then the sheaf quotient $S/R_\sharp$ is an algebraic space.

Proof.

- This is a proposition: First note, that both summands are propositions and if both summands are inhabited we get a contradiction.
- The relation is covering: Furthermore, for any $x : S$ we have

$$\sum_{y:S} (x = y) + (x, y \in U \times x \sharp y) = 1 + (x, y \in U \times \sum_y x \sharp y) \in \mathbb{T}$$

as $\sharp$ was assumed to be $\mathbb{T}$-flat on U: we can write the binary product as $\sum_{p:x \in U} \sum_y x \sharp y$, a $\sum$ of of $\mathbb{T}$-flat geometric stacks.

□

5.3 Group quotients

For this section let G denote a group that is a covering 0-stack. Let X be a sheaf equipped with a G action.

Lemma 5.15. $\mu_p = \operatorname{Spec} R[X]/(X^p - 1)$ *is covering for $p \neq 0$ prime.*

Proof. It is fppf + étale as $X^p - 1$ is monic seperable. TODO □

Definition 5.16. A G action on X is free, if for all $x, y : X$ the type

$$\sum_{g:G} gx = y$$

is a proposition.

Example 5.17. *Given a formally étale + flat affine (e.g. μ_ℓ or finite) group that acts on an affine* $\operatorname{Spec} A$*. Assume G acts free on some open subset U.*
Then put $x \sharp y = \sum_{g:G \setminus \{1\}} gx = y$.
This provides a covering equivalence relation $R_{G,U}$ on $\operatorname{Spec} A$*, such that*

- *for any $x : U$*

$$[x] = \sum_{y:\operatorname{Spec} A} \sum_{g:G} gy = x.$$

- *for some $y \notin U$, we have $R(x, y) \leftrightarrow x = y$.*

By abuse of notation we write $\operatorname{Spec} A/_U G \equiv \operatorname{Spec} A/R_G$ *and call it the quotient of* $\operatorname{Spec} A$ *by the G-action .*

Proof.

- It is irreflexive: If $x : U$ then $gx \neq x$, by freeness.

- We have $G \setminus \{1\}$ is flat affine using 5.11.

$\square$

Notation. If $U = \operatorname{Spec} A \setminus Z$ the complement of a closed subset we write

$$U \equiv Z^c$$

Example 5.18 (Free action)**.** *Set $U = \operatorname{Spec} A$. Then this construction yields the actual group quotient. The quotient of $\mathbb{A}^\times$ by the free μ_ℓ action gives a scheme.*

Lemma 5.19. *Algebraic spaces are stable by free quotients of covering group 0-stacks.*

Proof. The map $X \to L_T(X/G)$ is fibered in covering 0-stacks, so in particular covering 0-stacks. As X is a geometric 0-stack, the quotient is a geometric 0-stack as well, This follows by the description in , choosing a geometric atlas of X and postcomposing this to get a geometric atlas of the quotient. $\square$

Example 5.20. *If $p : \sum_{r:R} S_r \to R$ be a map between formally étale + flat affine into R whose fibers, except possibly the fiber over 0, are formally étale + flat. Define $U = (x \neq 0) \times S_x \subset S_x$. $y \sharp y' \equiv y \neq y'$ is an irreflexive $\mathbb{T}$-flat relation on S_x. From this we obtain the algebraic space*

$$\sum_{x:R} Y_x / R_\sharp$$

which we will later recognize as a fiber collapse.

Proof. $\sharp$ is a modal irreflexive relation. By assumption we have given $\mathbb{T}$-flatness of S_x if $x \neq 0$. $\square$

Lemma 5.21 (Not needed)**.** *Let Y be affine. Let $X \hookrightarrow Y$ be a map fibered in locally closed propositions. Then its factors as the composite of a closed and then an open embedding*

Proof. By zariski local choice we find $Y = \bigcup Y_i$ and factorizations of the basechanges $X_i \to Z_i \to Y_i$. Then $\bigcup X_i \to \bigcup Z_i \to \bigcup Y_i = Y$ is a global factorization. $\square$

Proposition 5.22. *Let S be affine with a regular point $*$. Assume we have function $g : S \to S$ such that $*$ is the unique fixpoint $*$ (e.g. if $(S, *)$ admits a pointed-free action of a nontrivial group) Let $\sharp$ be an irreflexive $\mathbb{T}$-flat relation on $U \equiv \sum_{x:S} x \neq *$, such that for all $y : U$, we have $gy : U$ and $y \sharp gy$. Then the algebraic space $S/R_\sharp$ is non locally seperated, in particular not a scheme.*

Proof. It is an algebraic space by the previous prop.

A pointed-free action of a non-trivial group yields such a map g: If $\neg(G = \{1\})$, then $\neg\neg(G \setminus \{1\})$ by decidable equality of G. As we want to prove a contradiction, we may assume $g : G \setminus \{1\}$, this yields a map $S \to S$ such that

- $*$ is the unique fixpoint by the pointed-freeness
- If $y \neq *$, then $gy \neq *$ and $y \sharp gy$

We have that every scheme X is locally-seperated, i.e. its identity types are locally closed. Indeed, this follows from the proof of Foundations Prop 5.5.2 .
Let us show that R is not valued in locally closed propositions. Recall

$$y \in U \to y \sharp gy \tag{1}$$
$$y \in U \leftrightarrow y \neq * \tag{2}$$

We have for any $y : S$

$$R_\sharp(y, gy) \simeq (y = gy) + (y \in U) \wedge y\sharp gy \overset{(1)}{\simeq} (y = *) + (y \in U) \overset{(2)}{\simeq} (y = *) + (y \neq *)$$

But if this is locally closed for all $y : S$, we have a contradiction to 2.4.

□

Corollary 5.23. *Let $Y : R \to \mathsf{Aff}$ be formally étale + flat affine away from the origin , such that $p : \tilde{Y} = \sum_{x:R} Y_x \to R$ is regular and Y_0 is infinitesimal. If you find a map $g : \tilde{Y} \to \tilde{Y}$ over p with a unique fixpoint, which lies over 0, then the algebraic space—$Y_\bullet$— is non-locally-seperated, In particular not a scheme.*

Proof. Lets call the unique fix point $*$, i.e. we have

$$gy = y \leftrightarrow y = *$$

Note that $* : \tilde{Y}$ is a regular point , as $p : \tilde{Y} \to R$ is a regular section with Y_0 infinitesimal. □

Definition 5.24. A pointed-free action of G on a pointed type $(X, 0)$ is a G-action with fixpoint 0, such that if $g\varepsilon = \varepsilon$ for some $g \neq 1$, then $\varepsilon = 0$.

Lemma 5.25. *Let G be a group with decidable equality acting pointed free on a pointed type $(X, 0)$. Then G acts free away from zero.*

Proof. let $x, y \neq 0$. We need to show, that $\sum_g gx = y$ is a proposition. Let $g, g' : G$ such that $gx = y$. as G has decidable equality, we may show $\neg\neg(g = g')$. If $g^{-1}g' \neq 1$, then by pointed-freeness applied to $g^{-1}g'x = x$, we have $x = 0$. Contradiction. □

Corollary 5.26. *Let $0 : \operatorname{Spec} A$ be a regular point. Let G be a nontrivial formally étale flat affine group acting pointed- freely on the pointed affine $(\operatorname{Spec} A, 0)$. Then the pointed-free quotient of $\operatorname{Spec} A$ by G from 5.17 is non-locally-seperated, In particular not a scheme.*

Example 5.27 (Non locally-separated examples)**.** *Assume $\ell \neq 0$ prime. Let μ_ℓ act on $(\operatorname{Spec} B, 0)$ in one of the following ways:*

1. *Let μ_ℓ act on $\operatorname{Spec} B = \mathbb{A}^1$*
2. *Let μ_ℓ act on*

$$\operatorname{Spec} B \equiv \sum_{x,y:R} x^\ell = y^\ell$$

   via $g(x, y) = (x, gy)$

Then $\operatorname{Spec} B/_{0^c}\mu_\ell$ is an algebraic space that is not a scheme.

Proof. $\neg\neg$ merely, μ_ℓ is finite ([ref?]) and $\mu_\ell \setminus \{1\}$ is inhabited by 5.11.

1. Pointed-Free action is clear. $0 : \mathbb{A}^1$ is regular by first projection.
2. Pointed-Free action is clear. The cross middlepoint regular, witnessed by the first projection: It is regular vanishing at $(0, 0)$ And for any point $(0, y) : \operatorname{Spec} B$ we deduce $y^\ell = -0^\ell = 0$, hence $\neg\neg(x, y) = (0, 0)$.

□

Question 3. If μ_ℓ acts on Y some affine, does every μ_ℓ-invariant $\phi : Y \to R$ is invariant on a ℓ-neigborhood?

5.4 Obsolete

Proposition 5.28. *Let $Y : R \to \mathsf{Aff}$ be formally étale + flat affine away from the origin If you find two sections $y, y' : \prod_{x:R} Y_x$ such that $y_x = y'_x \leftrightarrow x = 0$, then then the algebraic space—$Y_\bullet$— is non-locally-seperated, In particular not a scheme.*

Proof. It is an algebraic space by the previous prop.

We have that every scheme X is locally-seperated, i.e. its identity types are locally closed. Indeed, this follows from the proof of Foundations Prop 5.5.2 .
Let us show that R is not valued in locally closed propositions. We have

$$\prod_{x:R} \eta y_x = \eta y'_x \simeq \prod_{x:R} y_x = y'_x + (x \neq 0) \times y_x \neq y'_x \simeq (x = 0) + (x \neq 0)$$

but if this is locally closed for all $x : R$, we have a contradiction to 2.4.

□

Lemma 5.29 (Not needed)**.** *For an algebraic space X, we have implications $1 \Rightarrow 2 \Rightarrow 3$*

1. *X admits an seperated open cover.*
2. *For any covering equivalence relation $R : U^2 \to \mathrm{Prop}$ on an affine U such that $X = U/R$, F is valued in locally closed propositions*
3. *We find such a presentation such that R is valued in locally closed propositions.*

Proof$1 \Rightarrow 2$ Let $X' \to X$ be a map fibered in merely inhabited finite sums of open propositions with X' a seperated algebraic space. Then any geometric atlas $U \to X'$ will be fibered in closed subtypes of U. We need to show, that the fibers of $U \to X' \to X$ are locally closed subtypes of U. Let $x : X$. the fiber in X' is of the form $U_1 + \ldots + U_n$. Thus the fiber in U is a finite sums of $\sum$ of $U_i \to (U \to \mathsf{ClosedProp})$, which is enough.

$3 \Rightarrow 1$ Let $x : X$.

□

Lemma 5.30 (Not needed)**.** *Let char $\neq 2$. Let $p : R[X]$ be such that $0 \in D(p)$ and $x \in D(p)$ implies $-x \in D(p)$. If $f : R[X]$ is a polynomial such that $f(x) = f(-x)$ for all $x : D(p) \setminus \{0\}$, then f is even i.e. in the image of $R[X^2] \hookrightarrow R[X]$.*

Proof. We splitting f into $f_1 + Xf_2$ for $f_i : R[X^2] \subset R[X]$. I claim, that $f_2 = 0$ in $R[X]$. realizing that $(Xf_2)(x) = (Xf_2)(-x)$ implies $2f_2(x)x = 0$, thus $f_2(x)x = 0$ for all $x : D(p) \setminus 0 = D(pX)$, thus by the previous lemma $X \cdot f_2 = 0$ in $R[X]$, hence $f_2 = 0$. □

Lemma 5.31. *Let G be a finite group whose cardinality is invertible in R. Let G act on an affine scheme equipped with a fixpoint 0. Let U be an open neighborhood of 0, such that $g(U) = V$ for all $g : G$. Then we find some G-invariant p such that $0 \in D(p) \subset V$.*

Proof. Choose a principal open neighborhood $0 \in D(p) \subset U$. G acts on $R[X]$, via $(g.p)(x) = p(gx)$.Then

$$p' = \sum_{g:G} g.p : R[X]$$

is a G-invariant polynomial, in particular $D(p)$ is G-invariant. Moreover $0 \in D(p')$ as

$$p'(0) = \sum_{g:G} p(g(0)) = \sum_{g:G} p(0) = |G| \cdot p(0)$$

is invertible, as $|G|$ and $p(0)$ are both invertible. Furthermore, as U was G invariant and contained $D(p)$ it also has to contain $D(p')$: Indeed

$$D(p') \subset \bigcup_g D(g.p) \subset U$$

□

Lemma 5.32. *Let G be a formally étale + flat affine group, such that $\neg\neg$ its finite, with cardinality invertible in R and $G \setminus \{1\}$ inhabited. Let it act on an affine scheme* $\operatorname{Spec} A$ *with a fixpoint 0. Let R be a relation on* $\operatorname{Spec} A$ *such that*

- *$R(x, y)$ implies that there merely is some g with $y = gx$.*
- *$\neg\neg R(x, gx)$*

Assume that for all $p : A^G$ with $0 \in D(p)$, $D(p)/R$ is not an affine scheme. Then $\operatorname{Spec} A/R$ *is not a scheme.*

Proof. Assume 0 admits a open affine neibhorhood U in $\operatorname{Spec} A/R$. The preimage along the quotient map obtained from the relation induces a open neigbhorhood V of 0 in $\operatorname{Spec} A$. As we want to prove a contradiction we may assume that μ_ℓ consists of ℓ many elements, where $\ell \neq 0$ in R. Note that V is G-invariant: For any $x \in V, g : G$, the goal $gx \in V$ as an open proposition is $\neg\neg$-stable , thus we may assume $R(x, gx)$.
We apply the previous lemma to V to obtain an invariant principal open neigborhood $0 \in D(p) \subset V \subset \operatorname{Spec} A$. As p is G-invariant, $p : \operatorname{Spec} A \to R$ descends to $X \to R$. Restricting to U' yields a map $p' : U \to R$, such that setting $U' \equiv D(p')$ yields $q^{-1}(U') = q^{-1}(D(p')) = D(p' \circ q) = D(p)$. We are now in the following situation

$$\begin{tikzcd}
D(p) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & V \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \operatorname{Spec} A \arrow[d, "q"] \\
U' \arrow[r, hook] & U \arrow[r, hook] & X
\end{tikzcd}$$

where U' is an open affine neighborhood of 0.
By assumption $U = D(p)/\sim'$ cannot be affine. Contradiction. □

Proposition 5.33 (Not needed)**.** *Let $\ell \neq 0$ be prime. Let μ_ℓ act on* $\operatorname{Spec} B$ *with fixpoint* 0. *. Let V be an infinitesimal neigborhood of* 0*, i.e. a subtype* $0 \in V \subset \operatorname{Spec} B$ *such that* $\neg\neg x = 0$ *for every* $x : V$*. Assume*

Strong freeness We find some $0 \in V' \subsetneq V$ for any $\varepsilon : \operatorname{Spec} B, g \neq 1$, $g\varepsilon = \varepsilon$ implies $\varepsilon \in V'$

checking away from 0 For any $p : B$ and any $\phi : R^{D(p)}$ such that $\phi|_{D(p)\setminus\{0\}} = 0$, we have that $\phi|_V = 0$.

The sheaf quotient of $\operatorname{Spec} B$ *by the relation as above is an algebraic space but not an affine scheme.*

Proof. • Let us check the conditions on the relation

- If $R(x, y)$ then either $x = y$ putting $g = 1$ or in the second case we get some g such that $gx = y$
- Let $x : X, g : G$. Assume $\neg R(x, gx)$, i.e. $x \neq gx$ and $\neg\neg x = 0$.But 0 was assumed to be a fixpoint, hence $\neg\neg gx = x$.

- Let $p : B$ be as above. We have to show that the quotient of $D(p)$ is not affine. The conditions on p give $p(0) \neq 0$ and $p(x) \neq 0 \to p(gx) \neq 0$ for all $g : \mu_\ell$.

  Lets call this quotient X.

  Define
  $$A = \{\phi : R^{D(p)} \mid \phi|_{D(p)\setminus\{0\}} \text{ is } \mu_\ell\text{-invariant } \}$$

  This is an R-subalgebra: for any $r : R$, $r : B_p$ is μ_ℓ-invariant. μ_ℓ-invariant functions are stable under addition and multiplication .

  Claim: The affinization map of X is the induced dashed map $f : X \to \operatorname{Spec} A$ in

  $$\begin{tikzcd} D(p) \arrow[r, equal] \arrow[d, "q"] & \operatorname{Spec} R^{D(p)} \arrow[d, "q'"] \\ X \arrow[r, dashed, "\exists! f"] & \operatorname{Spec} A \end{tikzcd}$$

  Proof: A function $\phi : D(p) \to R$ factors through q iff $\phi|_{D(p)\setminus\{0\}}$ is μ_ℓ-invariant. Thus the embedding (using that R is a sheaf) $R^X \hookrightarrow R^{D(p)}$ has image A $\quad\square$(Claim).

  Proof that X is not an affine: Assume that X were affine. Then the map f would be in particular an embedding. We may assume a term $g : \mu_\ell \setminus \{1\}$: Indeed, as we want to prove a contradiction we may assume a term in $g : \operatorname{Spec} R[X]/(\sum_{i=0}^{\ell-1} X^i)$. But this type is equivalent to $\mu_\ell \setminus \{1\}$, using that $\sum_{i=0}^{\ell-1} X^i | X^\ell - 1$ and $\ell \neq 0$.
  The given infinitesimal neigbhorhood V satisfies $V \subset D(p)$, using that invertibility is $\neg\neg$ stable. Then for any $\varepsilon : V$ we have

  $$(q\varepsilon =_X q(g\varepsilon)) \stackrel{??}{=} (\varepsilon = g\varepsilon) + (\varepsilon \neq 0 \wedge \sum_{h\neq 1} \varepsilon = hg\varepsilon) = (\varepsilon = g\varepsilon) = (\varepsilon \in V')$$

  where the last step comes from pointed-freeness. But we have

  $$(q'\varepsilon =_{\operatorname{Spec} A} q'(g\varepsilon)) = \left(\prod_{\phi : A} \phi(q'\varepsilon) = \phi(q'(g\varepsilon))\right) = \prod_{\substack{\phi : R^{D(p)} \\ \phi \in A}} \phi(\varepsilon) = \phi(g\varepsilon),$$

  The right hand side is inhabited: For any $\phi : D(p) \to R$ such that $\psi := \phi - g.\phi$ satisfies $\psi|_{D(p)\setminus\{0\}} = 0$ we have $\psi|_V = 0$ by 'checking away from 0', inparticular $\psi(\varepsilon) = 0$. So we conclude the the embedding $V' \hookrightarrow V$ is an equivalence. But we asked $V' \subsetneq V$ to be a proper subset.

$\square$

Example 5.34. *Let* μ_ℓ *act on* $\operatorname{Spec} B = \mathbb{A}^1$*.*

Proof.

1. Put $V :\equiv \operatorname{Spec} R[X]/X^n$ for some $n > 1$.
2. As $(g-1)$ is invertible, $((g-1)\varepsilon = 0)$ gives us $\varepsilon \in \{0\} \equiv V' \subsetneq V$. Note that indeed V is non contractible, because $R[X]/X^n \to R[X]/X$ is not an algebra isomorphism
3. We have to show, that then ϕ is μ_ℓ invariant. We can apply 2.3, observing $\phi - g.\phi = 0$ on $D(X/1) \subset \operatorname{Spec} B_p$, where $X/1 : B_p$ is regular, because X is regular in B. TODO as each ϕ satisfies the cond. $\square$(Claim)

$\square$

Example 5.35. *Assume* $2 \neq 0$*. Let* μ_2 *act on*

$$\operatorname{Spec} B \equiv \sum_{x,y:R} xy = 0$$

via the swap. Then $\operatorname{Spec} B/R$ *is an algebraic space but not a scheme.*

Proof.

1. Put $V = \operatorname{Spec} R[X]/X^k \subset \operatorname{Spec} B$, $k > 2$.
2. If $(x, y) = (y, x)$ but $xy = 0$ we get $x \in V' \equiv \operatorname{Spec} R[X]/X^2$.
3. Let $\phi : D(p) \to R$ be 0 everywhere except near the origin. Then we get a restricted map $\phi' : D(p') \to R$ where $D(p') \subset V(X)$ is given by the intersection $D(p) \cap V(X)$. Indeed : Put $p' : R[X]$ the image of $p : R[X, Y]/(XY)$ und the map induced by evaluating Y at 0.
   Here we can apply 2.3, getting that ϕ' is 0 everywhere in particular in $V \subset V(X)$.

□

5.5 Locally seperated examples

Lemma 5.36 (not needed)**.** *Given a map* $P : \mathsf{Susp}(Q) \to \mathrm{Prop}$*, such that* $P(N)$ *and* $P(S)$ *hold, then* $\prod_{t:\mathsf{Susp}(Q)} P(t)$

Lemma 5.37 (not needed)**.** *Assume* $2 \neq 0$*. For any* $x : R$*, the map*

$$\begin{aligned} \mathsf{Susp}(x \neq 0) &\to \sum_{y:R/x} y^2 = 1 \\ N &\mapsto 1 \\ S &\mapsto -1 \end{aligned}$$

is well-defined and an equivalence.

Proof. The following maps are mutually inverse

$$\begin{aligned} \sum_{y:R/x} y^2 = 1 &\simeq \sum_{e:R/x} e^2 = e \\ y &\mapsto (y-1)/2 \\ 2e - 1 &\mapsfrom e \end{aligned}$$

So it remains to show that the map

$$\begin{aligned} f : \mathsf{Susp}(x \neq 0) &\to \sum_{e:R/x} e^2 = e \\ N &\mapsto 1 \\ S &\mapsto 0 \end{aligned}$$

is a bijection.

- It is injective, i.e. for all $p, q : \mathsf{Susp}(x \neq 0)$, if $f(p) = fq$, then $p = q$. As the latter is a proposition, we may assume p, q beeing combinations of north and south poles. The interesting case is if wlog $p = N, q = S$. Then assuming $0 =_{R/x} 1$ means $R/x = 0$, i.e. $x \neq 0$, thus $N = S$ in $\mathsf{Susp}(x \neq 0)$.
- It is surjective: Choose $e : R$, such that $e^2 = e$ in R/x. By locality in R, e or $1 - e$ is invertible in R, thus in R/x. By $e^2 = e$ we deduce $e = 0$ or $e = 1$ in R/x, both lie in the image of f.

□

Example 5.38 (Not needed). *Let* $L = \sum_{x:\mathbb{A}^1} \mathsf{Susp}(x \neq 0) = \sum_{x:\mathbb{A}^1} \sum_{y:R/x} y^2 =_{R/x} 1$ *be the line with two origins.*

Lemma 5.39 (Not needed). *Let* $2 \neq 0$. *Let* $y, y' : A$ *be two elements of an fp-algebra, whose squares coincide and such that* y *is invertible. Then* $y =_A y'$ *is formally étale*

Proof. We may assume that $A = R$, as equality in A can be checked pointwise and formally étale is a modality. We may show its $\neg\neg$-stable. Assume $\neg\neg(y =_R y')$, i.e. $y - y'$ beeing nilpotent in A. So pick n large enough, such that $(y - y')^{2^n} = 0$. Proof by induction over n If $n = 0$, then its fine. Induction step $n \mapsto n+1$. Let $(y-y')^{2^{n+1}} =_R 0$, then $(2y^2-2yy')^{2^n} = 0$, or $(y(y-y'))^{2^n} = 0$, as y is invertible, $(y-y')^{2^n} = 0$, so by induction hypothesis $y = y'$. □

5.6 FiberCollaps away from the origin

OUTDATED!

Example 5.40. *—Bool— is the line with two origins.*
—Spec $R[X]/(X^2+1))$— *is the twisted line with two origins, i.e. over the origin we have the roots of* -1.
—Spec $R[Y]/(Y^2 - \bullet^2)$— *is the quotient of the cross, that looks like* $\mathbb{D}(1)$ *over the origin.*
—Spec $R[Y]/(\bullet Y)$— *is the affine Plus.*

5.7 Schemes do not have descent

For this section, let $\rho : R \setminus \{0\}$ denote a term, e.g. $\rho = 1$. Set $C = R[T]/(T^2 + \rho)$.

Proposition 5.41 (OUTDATED! Copy from talk!). *If —Spec* C*— is a scheme, then* $X^2+\rho$ *has a root.*

Corollary 5.42. *Schemes do not have descent.*

Proof. If Schemes have descent, then —Spec $R[T]/(T^2+\rho)$— $\in \mathsf{Sch}$ is a sheaf. As —Spec $R[T]/(T^2+\rho)$— is $\mathbb{T}$-merely a scheme, it is a scheme, so by the previous lemma $T^2+\rho$ has a root. Contradiction to [1] A . 0.3.

□

5.8 Gluing in an affine on the line

Definition 5.43. Let Y be an affine. The n.th order gluing of Y on the line is given by the sheaf

$$L_n(X) = \sum_{x:R} Y^{x^n=0}$$

Lemma 5.44. *If* $Y = \operatorname{Spec} R[T]/f$, *we have*

$$L_n(X) = \sum_{x:R} \sum_{y:R/x^n} f(y) =_{R/x^n} 0$$

Proof. For any R-algebra A (e.g. R/x^n) we have by the universal property of $R[T]/f$

$$\sum_{y:A} f(y) =_A 0 = \operatorname{Hom}_R(R[T]/f, A) = Y^{\operatorname{Spec} A}$$

□

Lemma 5.45. *If Y is formally étale , then the map over R*

$$\begin{tikzcd} R \times Y \arrow[rr] \arrow[dr] & & L_n(Y) \arrow[dl] \\ & R & \end{tikzcd}$$

pulls back to an equivalence over $\mathcal{N}_\infty(0)$.
If Y is formally unramified, then $L_n(x)$ is locally seperated.

Proof. Indeed, the diagonal map

$$Y \to Y^{x^n=0}$$

is an equivalence, as for any $\neg\neg x = 0$, $x^n = 0$ is a closed dense proposition and Y is formally étale .
If Y is formally unramified, then the identity types look like

$$(x, y) =_{L_n(Y)} (x', y') \simeq (x = x') \times (x^n = 0 \to Q)$$

where Q is an open proposition such that for any $p : x^n = 0$ we have $Q \equiv yp = y'p$. Indeed by the proof of **??** we can find a filler of $y_\bullet = y'_\bullet : P \to \mathsf{Open}$. By [1](4.2.11) this proposition is locally closed. $\square$

Question 4. Is the map $\sum_{y:R/x^3} y^2 = 0 \to \sum_{y:R/x^2} y^2 =_0$ surjective? This is how i understand David Madore.

Lemma 5.46. *For $\varepsilon : \mathcal{N}_\infty(0)$, the affine $Ann(\varepsilon) = \{x : R \mid x\varepsilon = 0\}$ is not$_\varepsilon$ formally smooth. In particular $R \to R/\varepsilon$ is not$_\varepsilon$ a geometric cover.*

Proof. We have the map $1 : (\varepsilon = 0) \to \mathsf{Ann}(\varepsilon)$. Assume there is a filler $x : \mathsf{Ann}(\varepsilon)$, i.e. $(\varepsilon = 0) \to x = 1$. Then not not, $x = 1$, i.e. $(x-1)^n = 0$ for n large enough. Hence

$$0 = \varepsilon(x-1)^n = \varepsilon x(\ldots) + (-1)^n\varepsilon = (-1)^n\varepsilon$$

as desired. $\square$

Lemma 5.47 (TODO)**.** *If Y is formally étale + flat affine, then $L_1(Y)$ is an algebraic space.*

Proof. Recall the closed modality associated to a proposition P, given by $P \star \_$. We can define a map

$$\begin{aligned} f : (x \neq 0) \star Y &\to Y^{x=0} \\ y &\mapsto \Delta(y) \end{aligned}$$

where we check, that if $x \neq 0$ holds, then indeed $Y^{x=0}$ is contractible.

f is a bijection:

- injectivity: Given two terms of the domain, as the map out of Y is $\mathbb{T}$-surjective (and the goal is a sheaf), we may assume that they are of the form $inl(y), inl(y')$ for $y, y' : Y$. Then if $\Delta(y) = \Delta(y')$ we have $(x = 0) \to (y = y')$. As $y = y'$ is open, we have $(x \neq 0) \vee (y = y')$. If $x \neq 0$, then $inl(y) = inl(y')$ by the construction of the join.
- surjectivity: TODO

$\square$

Question 5. Is $L_2(\mathbb{D}(1))$ an algebraic space or fppf-geometric 0-stack? For this: Is

$$(\operatorname{Spec} R[X,Y]/X^2 - Y^2)/\sim \to L_2(\mathbb{D}(1)) = \sum_{x:R} \sum_{y:R/x^2} y^2 = 0$$

$$(x,y) \mapsto (x,[y])$$

an equivalence? Here we mod out the relation generated by $(x,-x) \sim (x,x) \forall x \neq 0$.

This is equivalent to : For any $x : R$, is the map

$$(x \neq 0) \star \operatorname{Spec} R[Y]/(Y^2 - x^2) \to \mathbb{D}(1)^{x^2=0}$$

an equivalence?

Example 5.48. *I suggest a new definition of fppf topology: We take the topology generated by the Zariski topology and algebras of the form $R[X]/f$ where one of coefficients of f is invertible (non necessarily the leading coefficient). This is still a free module hence fppf.*

5.9 Weakly-flat stacks

Definition 5.49. We call a geometric stack X weakly-flat iff one of the following conditions is satisfied

1. $\|X\|^{\mathbb{T}}_{-1} \to X \in \mathsf{CS}$
2. For any geometric atlas $W \to X$, W is weakly-flat, i.e $\|W\|^{\mathbb{T}} \to W \in \mathbb{T}$.

Proof.

$1 \Rightarrow 2$ Choose a geometric atlas $W \to X$. In particular its $\mathbb{T}$-surjective, hence we have $\|W\|^{\mathbb{T}}$, so by assumption $W \in \mathbb{T}$. So $X \in CS$.

$2 \Rightarrow 1$

$$\|W\|^{\mathbb{T}} \to \|X\|^{\mathbb{T}} \to X \in \mathsf{CS} \overset{??}{\to} W \in \mathbb{T}$$

□

They behave bad as they are not stable under $\sum$ (and not under id-types, although this holds for affines).

Lemma 5.50. *For any weakly-flat geometric stack X, $\|X\|^{\mathbb{T}}_{-1}$ is a geometric stack.*

Proof. $X \to \|X\|^{\mathbb{T}}_{-1}$ is a geometric cover. □

Proposition 5.51. *We may define X to be 0-wf-seperated, iff its weakly flat and $n+1$-wf-seperated, iff identity types of X are n-wf-seperated.*
For X a geometric stack, TFAE

1. *X is $n+1$-wf-seperated, i.e. all $n+1$-fold identity types of X are weakly-flat.*
2. *For any x, $\Omega^{n+1}(X,x)$ is covering.*
3. *For any $x : X$, $x = x$ is n-wf-seperated, i.e. n-fold identity types of $x = x$ are weakly flat.*

Proof.

$1 \Rightarrow 3 \Rightarrow 2$ ez

$3 \Rightarrow 1$ We prove this by induction. $n = 0$. To show that $x =_X y$ is weakly-flat, by descent we may assume that $x = y$. Then we have $(x = y) \simeq (x =_X x)$. By assumption this is weakly flat.
Assume now, that for any $x : X$, that $x = x$ is n-wf-seperated. Let $x, y : X$. We want to show that $x = y$ is n-wf-seperated. By induction we may just prove that for any $p : x = y$, $p = p$ is $n-1$-wf-seperated. Applying $p \cdot \_$ induces an equivalence $\mathsf{refl}_x = \mathsf{refl}_x \simeq p = p$. But $x = x$ is n-wf-seperated, hence $\mathsf{refl}_x = \mathsf{refl}_x$ is $n-1$-wf-seperated.

$2 \Rightarrow 3$ Induction. $n = 0$ is fine. Let $x : X$. To show that $\Omega(X,x)$ is n-wf-seperated, just use that $\Omega^n(\Omega(X,x))$ is covering, hence by the inductive statement $2 \Rightarrow 3 \Rightarrow 1$, we now that $\Omega(X,x)$ is n-wf-seperated. □

6 Omega-stability and gerbes

Definition 6.1. A geometric stack X is an n-gerbe iff the map $\eta^{\mathbb{T}}_n : X \to \|X\|^{\mathbb{T}}_n$ is a geometric cover.

Example 6.2. *If G is a covering group sheaf, then BG is a 0-gerbe.*

Example 6.3. *It may happen, that $\|X\|_n^{\mathbb{T}}$ is a geometric n-stack for X a geometric stack, although X is not an n-gerbe. Indeed: Put $n = 0$ and X any pointed $\mathbb{T}$-connected geometric stack that is not covering, like $\mathsf{Susp}(1 + x = 0)$ for some*

Theorem 6.4. *Assume that Covering stacks are Ω-stable, Then every geometric stack is a 1-gerbe.*

Proof. By **??**, we need to show that for any $x : X$, $\Omega^2(X, x)$ is covering. choose an geometric atlas $f : S \to X$. by descent we may only show that $\Omega^2(X, fs)$ for $s : S$ is covering.

$$\Omega(\sum_{t:S} ft = fs) \simeq \left(\sum_{p:\Omega(S,s)} \mathrm{tp}_p(\mathsf{refl}_{fs}) = \mathsf{refl}_{fs} \right) \simeq \mathsf{refl} =_{fs=fs} \mathsf{refl}$$

where the last equivalence is obtained, as $\Omega(S, s)$ is contractible with center refl_s.
So $\Omega^2(X, fs)$ is the loop space of a covering stack, hence by assumption covering. □

Corollary 6.5. *Any Deligne Mumford Stack is a 1-gerbe*

Proof. Use that étale topology is lex-flattened and **??**. □

Proposition 6.6. *This proposition seems only interesting for $n = 0$ by the previous theorem. Assume that covering stacks are Ω-stable. Then X is an n-gerbe iff $\Omega^{n+1}(X, x)$ is covering for all $x : X$*

Proof. One direction is **??**. The other follows
By applying iteratively **??**

$$\begin{aligned}
\Omega^{n+1}(\mathrm{fib}(\eta_n^{\mathbb{T}} X)|x|) &\simeq \Omega^n \,\mathrm{fib}(\eta_{n-1}^{\mathbb{T}}(\Omega(X, x)))pt \simeq \dots \\
&\simeq \Omega^{n-k} \,\mathrm{fib}(\eta_{n-k-1}^{\mathbb{T}} \Omega^{k+1}(X, x))pt \simeq \dots \\
&\simeq \mathrm{fib}(\eta_{-1}^{\mathbb{T}} \Omega^{n+1}(X, x))pt \\
&\simeq \Omega^{n+1}(X, x)
\end{aligned}$$

The LHS is covering by Ω-stability. □

We can reprove **??** by just observing that $\mathbb{T}$-flat geometric stacks have covering loop spaces.

Remark 3. Put $\mathbb{T}$ the étale topology. Observe, that we have an analogous statement if we replace covering stack by formally étale :

1. $\eta_0^{\mathbb{T}} X : X \to \|X\|_0^{\mathbb{T}}$ is formally étale
2. $X \to \|X\|_0^{\mathbb{T}}$ is formally unramified
3. for every $x : X$, $\Omega(X, x)$ is formally étale .

Proof. 1 $\Leftrightarrow$ 2 Observe that the map $\eta_0^{\mathbb{T}}$ is $\mathbb{T}$-smooth.

2 $\Rightarrow$ 3 okay as the fibers of $\eta_0^{\mathbb{T}}$ embed into X.

3 $\Rightarrow$ 2 Let $x, y : X$ be $\mathbb{T}$-merely equal. The goal $\mathsf{isFormallyEtale}(x = y)$ is a sheaf, so we may assume that $x = y$.

□

Corollary 6.7. *If covering stacks are Ω-stable, then identity types of geometric stacks are 0-gerbes.*

Proof. We need to check, that identity types of a 1-gerbe X are 0-gerbes. So assume $p : x = y$. Then

$$\Omega(x = y, p) = \Omega(x = x, \mathsf{refl}) = \Omega^2(X, x)$$

which is covering as X is a 1-gerbe. □

7 Questions // TODO

Theorem 7.1 (TODO)**.** *An Artin stack X is Deligne Mumford iff one of the following conditions is satisfied:*

1. *There exists a geometric atlas $W \to X$*
2. *The identity types of X are $\mathbb{P}$-seperated*

Proof. 1. $\Rightarrow$2. **??**

2. $\Rightarrow$ 1 Residual ??? [06MF]

□

Prove **??**!!!

Question 6. if $\mathbb{T} \subset \mathbb{T}'$ do we have that for each $X : \mathsf{GS}_{\mathbb{T}}$ $L_{\mathbb{T}'} X \in \mathsf{GS}_{\mathbb{T}'}$?

Theorem 7.2 (TODO)**.** *The class of flat affines is stable under $\sum$. Moreover flatness can be defined fiberwise.*

8 Not clear where to put that

Lemma 8.1 (Not needed)**.** *Open subtypes of $\mathbb{A}^1$ are $\neg\neg$ principal open.*

Proof.
- An open affine subscheme of $\mathbb{A}^1$ is $\neg\neg$ principal open: Let $D(f_1, \ldots, f_n) \subset \mathbb{A}^1$ be an arbitrary open subset. We may assume that each $f_i : R[X]$ is non constant (in particular non zero). By [ref?], $\neg\neg$-merely each $D(f_i) \subset R$ is cofinite. Thus $\neg\neg$-merely, the finite union $\bigcup_{i=1}^n D(f_i) \subset R$ is cofinite as well, hence principal open.

□

Proposition 8.2. *Assume covering stacks are Ω-stable. A truncated stack (e.g. geometric stack) is covering iff $\pi_0^{\mathbb{T}} X := \|X\|_0^{\mathbb{T}}$ and all higher homotopy groups*

$$\pi_i^{\mathbb{T}}(X, x) = \|\Omega^i(X, x)\|_0^{\mathbb{T}}, i \geq 1$$

are covering algebraic spaces.

Proof. Let X be an n-stack. If X is covering, then by Ω-stability all the $\pi_i^{\mathbb{T}}$ are covering 6.6 Now the converse. Consider the postnikov tower

$$X = \|X\|_n^{\mathbb{T}} \to \|X\|_{n-1}^{\mathbb{T}} \to \ldots \to \|X\|_1^{\mathbb{T}} \to \|X\|_0^{\mathbb{T}}$$

As $\|X\|_0^{\mathbb{T}}$ is covering, by quotient stability of covering stacks we may show that all the maps are geometric covers. Let $1 \leq k \leq n$ and consider the map $f_k^X : \|X\|_k^{\mathbb{T}} \to \|X\|_{k-1}^{\mathbb{T}}$. By descent for covering stacks, we may only consider the fiber over $|x|$, as the $\eta_{k-1}^{\mathbb{T}}$ is $\mathbb{T}$-surjective. It suffices to show, that the fiber is given by $B_{\mathbb{T}}^k \pi_k^{\mathbb{T}}(X, x)$ as deloopings of covering stacks are covering **??**.
We apply **??**. First observe that $\Omega^k(\mathrm{fib}(f_k^X)|x| = \mathrm{fib}(\Omega^k(f_k^X, x))$ is equivalent to the basefiber of

$$\pi_k^{\mathbb{T}}(X, x) \equiv \|\Omega^k X\|_0^{\mathbb{T}} \simeq \Omega^k(\|X\|_k^{\mathbb{T}}) \to \Omega^k \|X\|_{k-1}^{\mathbb{T}} \simeq 1$$

So it suffices to show by induction over k, that for all pointed stacks (X, x), $\mathrm{fib}(f_k^X)|x|$ is $\mathbb{T}$-k-connected.
This is definitely $\mathbb{T}$-connected by using that any term $(y, p) : \mathrm{fib}(f_k^X)|x| = \sum_{y:\|X\|_n^{\mathbb{T}}} \|x = y\|_{n-1}^{\mathbb{T}}$ yields a witness of $\|x - y\|^{\mathbb{T}}$. Then $\Omega(\mathrm{fib}(f_k^X)|x| = \mathrm{fib}(\Omega(f_k^X, x)) = \mathrm{fib}(f_{k-1}^{\Omega(X,x)})$ which is $\mathbb{T}$-$k-1$-connected by induction. □

8.1 Remarks about weakly flat affines

Lemma 8.3. *The proposition $\|X\|_{\mathbb{T}}$ is geometric iff there exists a map from a weakly flat affine* $\operatorname{Spec} B \to X$ *such that* $\|\operatorname{Spec} B\|_{\mathbb{T}} \to \|X\|_{\mathbb{T}}$ *is an equivalence.*

Proof. '$\leftarrow$' is clear.
'$\to$'. Choose $\operatorname{Spec} B'$ weakly flat such that $\|X\|_{\mathbb{T}} = \|\operatorname{Spec} B'\|_{\mathbb{T}}$. As the map $X \to \|X\|_{\mathbb{T}}$ is $\mathbb{T}$-surjective, by $\mathbb{T}$-local choice we find a $\mathbb{T}$-cover $\operatorname{Spec} B \to \operatorname{Spec} B'$ and a commutative diagram

$$\begin{tikzcd} \exists \operatorname{Spec} B \arrow[r, dashed] \arrow[d, dashed] & X \arrow[d] \\ \operatorname{Spec} B' \arrow[r] & \|X\|_{\mathbb{T}} \end{tikzcd}$$

As $\operatorname{Spec} B'$ was weakly flat and the left vertical map is a $\mathbb{T}$-cover, $\operatorname{Spec} B$ is weakly flat. ☐

Lemma 8.4 (DM)**.** *If* $\operatorname{Spec} A + \operatorname{Spec} B$ *is weakly flat affine, then* $\operatorname{Spec} A$ *is weakly flat.*

Proof. Indeed

$$\|X\|_{\mathbb{T}} \to \|X + Y\|_{\mathbb{T}} \to X + Y \in \mathbb{T} \to X \in \mathbb{P}$$

but $\|X\|_{\mathbb{T}} \wedge X \in \mathbb{P} \to X \in \mathbb{T}$.

☐

Lemma 8.5. *if the topology is saturated Beeing weakly-flat descends along $\mathbb{T}$-covers.*

Lemma 8.6 (DM)**.** *If $\|P + Q\|_{\mathbb{T}}$ is a geometric prop, then TODO*

Proof. By the previous two lemma and we find a map out of a weakly flat affine $\operatorname{Spec} B \to P + Q$ that induces an equivalence on $\mathbb{T}$-truncations, but it splits into two map out of a weakly affine $\operatorname{Spec} B_1 \to P, \operatorname{Spec} B_2 \to Q$. ☐

Notation. For $P : (\varepsilon : \mathcal{N}_\infty(0)) \to X \to \mathrm{Prop}$, let $\varepsilon : \mathcal{N}_\infty(0) \vdash x : X$. We say x is not$_\varepsilon$ P, if $\forall \varepsilon$, $P_\varepsilon x \to \varepsilon = 0$. Observe, if x is not$_\varepsilon$ P for any $\varepsilon^2 = 0$, then x is not P.

Remark 4. If $2 \neq 0$. Let $\varepsilon, \varepsilon' : \mathcal{N}_\infty(0)$. $\varepsilon = \varepsilon' + \varepsilon = -\varepsilon'$ is not$_\varepsilon$ weakly-flat

Proof. We prove that once its $\mathbb{T}$-merely inhabited, then its not$_\varepsilon$ covering, which is enough as $\neg\neg(\varepsilon = \varepsilon' + \varepsilon = -\varepsilon')$.As the goal is a stack we may assume $\varepsilon = \varepsilon' + \varepsilon = -\varepsilon'$. wlog the first case. Then assume $1 + (\varepsilon = -\varepsilon) \simeq 1 + \varepsilon = 0$ is covering. Then $\varepsilon = 0$ is formally étale , thus inhabited as a formally étale closed dense proposition. ☐

Example 8.7 (Obsolete)**.** *The map $q : \mathbb{A}^1 \to \mathbb{A}^1/\mu_\ell$ is not a geometric cover.*

Proof. The map factors through the geometric cover $\mathbb{A}^1 \to \mathbb{A}^1//\mu_\ell$. Thus its enough to show that $\mathbb{A}^1//\mu_\ell$ is not a 0-gerbe, or that not every loop space is covering. Let us show that, $\Omega(\mathbb{A}^1//\mu_\ell, \varepsilon)$ is not$_\varepsilon$ covering. Assume it is covering for some $\varepsilon \in \mathcal{N}_\infty(0)$. As μ_ℓ has decidable equality,

$$\begin{aligned} \Omega(\mathbb{A}^1//\mu_\ell, \varepsilon) &= \left(\sum_{g:\mu_\ell} g\varepsilon = \varepsilon \right) \\ &= (\varepsilon = \varepsilon) + \sum_{g:\mu_\ell \setminus \{1\}} (g-1)\varepsilon = 0 \\ &= 1 + \mu_\ell \setminus \{1\} \times (\varepsilon = 0) \end{aligned}$$

Thus $(\varepsilon = 0) \times (\mu_\ell \setminus \{1\})$ is an étale -flat geometric stack. Moreover $(\mu_\ell \setminus \{1\})$ is a covering stack by 5.11. Thus $\varepsilon = 0$ is an affine étale -flat geometric stack, thus formally étale + flat affine by saturatedness of the étale topology **??**. So as a formally étale + closed dense proposition, $\varepsilon = 0$ holds as desired. ☐

References

[1] Felix Cherubini, Thierry Coquand, and Matthias Hutzler. *A Foundation for Synthetic Algebraic Geometry*. 2023. arXiv: 2307.00073 [math.AG]. URL: https://arxiv.org/abs/2307.00073.

[2] Hugo ; Felix Wellen Moeneclay. *Topology in SAG*. URL: felix-cherubini.de/topology.pdf.

[3] Egbert Rijke, Michael Shulman, and Bas Spitters. “Modalities in homotopy type theory”. In: *Logical Methods in Computer Science* Volume 16, Issue 1, 2 (Jan. 2020). ISSN: 1860-5974. DOI: 10.23638/LMCS-16(1:2)2020. URL: https://lmcs.episciences.org/3826.