

# Thesis

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## 1 Introduction to SAG

**Lemma 1.1.**  *$R$  is local, i.e. if  $x, y : R$  and  $x \neq y$ , then  $x$  is invertible or  $y$  is invertible.*

**Lemma 1.2.** *If  $\text{char} \neq 2$ , Let  $\rho \neq 0$ , then  $x^2 = \rho^2$  implies  $x = \rho$  or  $x = -\rho$*

*Proof.* Indeed, as  $\rho \neq -\rho$ , one of them is invertible by 1.1 □

Example for zariski local choice

**Example 1.3.** For some  $A$  and  $g, g' : A$  define

$$g \mid_A g' \equiv \|\{h : A \mid hg =_A g'\}\|$$

*Claim:* For any  $g, g' : A$ , we have

$$g \mid_A g' \leftrightarrow \forall x : \text{Spec } A, gx \mid_R g'x$$

*Proof.*  $\rightarrow$  is obvious using that the duality map is an algebra isomorphism.

$\leftarrow$ . For any  $x : \text{Spec } A$  we merely find some  $h : R$  with  $h \cdot g(x) = g'(x)$ , i.e. we define our family of inhabited types as

$$Bx = \{h : R \mid h \cdot g(x) = g'(x)\}$$

By zariski local choice we merely find some principal open cover  $\text{Spec } A = \bigcup_{i=1}^n D(f_i)$  and local sections

$$\begin{aligned} & \prod_{x:D(f_i)} \{h_i : R \mid h_i \cdot g(x) = g'(x)\} \\ & \stackrel{??}{\simeq} \{h_i : D(f_i) \rightarrow R \mid (h_i x) \cdot g(x) = g'(x)\} \\ & \stackrel{??}{\simeq} \left\{ h_i : A_{f_i} \mid h_i \cdot \frac{g}{1} =_{A_{f_i}} \frac{g'}{1} \right\} \end{aligned}$$

We can multiply  $h_i$  by high enough powers of  $f_i$  to obtain some  $h_i : A$  with  $h_i \cdot g = g' \cdot f_i^n$  for some  $n : \mathbb{N}$ . we may assume that  $n$  does not depend on  $i = 1, \dots, n$  by taking the maximum and multiplying the  $h_i$  again with enough powers of  $f_i$ . Now use ?? to write  $1 = \sum_{i=1}^n \ell_i f_i^n$  for some  $\ell_i : A$  and then

$$\left( \sum_i \ell_i h_i \right) \cdot g = \sum_i \ell_i f_i^n g' = 1g' = g'$$

□

## 2 Topology

**Lemma 2.1.** Let  $f, g : X \rightarrow Y$  be two functions into a seperated scheme where  $X = \text{Spec } A$  for  $A$  a reduced ring. If  $f$  and  $g$  coincide on a dense subset, then  $f = g$ .

*Proof.* The equalizer

$$Z \equiv \sum_{x:X} fx =_Y gx$$

is a closed subset of  $X$ , as  $Y$  is seperated. As its open complement does not intersect the given dense subset, its empty. In other words  $\neg \neg Z = X$ . Writing  $I \subset A$  as the ideal of functions that vanish on  $Z$ , By [2], we have

$$\text{Spec } A = \neg \neg Z = \bigcup_n \text{Spec } A/I^n$$

But by the strong boundedness principle, we find some  $n$  such that  $\text{Spec } A = \text{Spec } A/I^n$ , in other words,  $I^n = 0$ . As  $A$  is a reduced ring, we conclude  $I = 0$ , so  $Z = \text{Spec } A/I = \text{Spec } A$ . □

**Definition 2.2.** A point  $0 : \text{Spec } B$  is regular, if  $\text{Spec } B \setminus \{0\} = D(p_1, \dots, p_n)$  for some  $p_1, \dots, p_n : B$  jointly-regular, i.e. if  $p_i^m \cdot b = 0$  for all  $i = 1, \dots, n$  then  $b = 0$ .

If  $0 : X$  is a point of a scheme, we call it regular, if one of the following equivalent conditions is satisfied

1. it admits some open affine neighborhood  $U$  such that  $0 : U$  is regular.

2. It is a regular point of any open affine neighborhood.

*Proof.* Consider an open affine neighborhood  $0 : D(f) \subset U = \text{Spec } B$ . We will show

1. If  $0$  is regular in  $D(f)$ , then it is regular in  $\text{Spec } B$ : Consider  $g_1, \dots, g_n : B$  such that

$$B_f \rightarrow \prod_i B_{fg_i}$$

is injective. Define  $g_0 := f - f(0)$ , where  $0 \notin D(g_0)$ . Let us show, that  $g_0, \dots, g_n$  are jointly surjective in  $B$ . Let  $b : B$  such that  $g_i^n b = 0$  for all  $0 \leq i \leq n$ . Then in particular  $b/1 =_{B_f} 0$ . Thus  $b$  is in the kernel of  $B \rightarrow B_{g_0} \times B_f$ . But  $D(g_0) \cup D(f)$  forms an open cover of  $\text{Spec } B$  as  $(f, g_0)$  generate the unit ideals. Thus  $b : \text{Spec } B \rightarrow R$  equals 0 on an open cover, thus its 0.

2. If  $0$  is regular in  $\text{Spec } B$ , then it is regular in  $D(f)$ : Assume  $B \rightarrow \prod B_{g_i}$  is injective. Let  $f : B$ . Let us show that  $B_f \rightarrow \prod B_{fg_i}$  is injective. If  $(g_i f)^n b = 0$ , then  $(g_i)^n (f^n b) = 0$ , thus  $f^n b = 0$  by assumption. Thus  $b/1 =_{B/f} 0$  as desired.

□

**Lemma 2.3.** *If  $0 : X$  is a regular point in a scheme, then both holds:*

1.  $X \setminus \{0\}$  is dense

2.  $R^X \rightarrow R^{X \setminus \{0\}}$  is injective.

*Proof.* 1. We write  $A \perp B$  for  $A \cap B = \emptyset$ . We reduce to affine case: Let  $0 \in \text{Spec } B \subset X$ . Let  $U \subset X$  be open such that  $U \perp X \setminus \{0\}$ . Then  $U \perp X \setminus \{0\} \Rightarrow U \perp \text{Spec } B \setminus \{0\} \Rightarrow U \perp \text{Spec } B$  so  $U \perp (\text{Spec } B \cup X \setminus \{0\}) = X$ , thus  $U = \emptyset$ .

So we may assume that  $X = \text{Spec } B$  is affine: Then by [2], an open subset of  $\text{Spec } B$  is dense iff it is of the form  $D(g_1, \dots, g_n)$  for nilregular functions  $g_i : B$ . Conclude, as regular implies nilregular [2].

2. Lets first reduce to the affine case. Choose an open affine neighborhood  $U$  of  $0$  such that  $0 : U$  is regular. Then the surjection  $U + X \setminus \{0\} \rightarrow X$  induces a vertical left injection

$$\begin{array}{ccc} R^{U+X \setminus \{0\}} & \hookrightarrow & R^{U \setminus \{0\} + X \setminus \{0\}} \\ \uparrow & & \uparrow \\ R^X & \longrightarrow & R^{X \setminus \{0\}} \end{array}$$

So we may assume that  $X = \text{Spec } A$  is affine.

Let  $p_1, \dots, p_n : A$  be jointly-regular, i.e. if  $p_i^m \cdot a = 0$  for all  $i = 1, \dots, n$  then  $a = 0$ . If  $f : \text{Spec } A \rightarrow R$  such that  $f(x) = 0$  for all  $x \in D(p_1, \dots, p_n)$ , then  $f(x) = 0$  for all  $x : \text{Spec } A$ .  $f$  is in the kernel of the diagonal map

$$\begin{array}{ccc} A & \xlongequal{\quad} & R^{\text{Spec } A} \\ \downarrow & & \downarrow \\ \prod_{i=1}^n A_{p_i} & \xlongequal{\quad} & R \prod_i D(p_i) \hookrightarrow R^{\cup D(p_i)} \end{array}$$

which is injective, as  $p_1, \dots, p_n$  are jointly-regular in  $A$ .

Thus  $f = 0$  in  $A$ .

□

**Remark 1.** If  $A$  is an algebra that is reduced as a ring, then for  $X = \text{Spec } A$ , 1. implies 2. by 2.1

**Proposition 2.4.** *the subtype  $\{0\} + 0^c \subset \text{Spec } B$  is not locally closed whenever one of the following conditions is satisfied:*

1.  $\text{Spec } B \setminus \{0\}$  is dense
2.  $R^{\text{Spec } B} \rightarrow R^{\text{Spec } B \setminus \{0\}}$  is injective

*Proof.* Let us first show, that the infinitesimal neighborhood of 0 is not open.

1. If  $0^c \subset \text{Spec } A$  is dense) : The non-empty open  $\mathcal{N}_\infty$  does not intersect the dense subset  $0^c$ .
2. If  $R^{\text{Spec } B} \rightarrow R^{\text{Spec } B \setminus \{0\}}$  is injective: If it would, we find a principal open smaller neighborhood  $0 \in D(g) \subset \mathcal{N}_\infty(0)$ , which however already contains the whole infinitesimal one, thus  $\mathcal{N}_\infty(0) = D(g)$   
Then for any  $x \neq 0$ , we have  $\neg\neg g(x) = 0$ . As  $\text{Spec } B \setminus \{0\}$  is a scheme, it admits a boundedness principle, thus we find some  $n$ , such that  $g^n(x) = 0$  for all  $x \neq 0$ .  
by 2.3 we have that  $R^{\text{Spec } B} \rightarrow R^{\text{Spec } B \setminus \{0\}}$  is injective, so we deduce  $g^n = 0$ , hence  $D(g) = D(g^n) = \emptyset$  contradiction.

Just assuming that the infinitesimal neighborhood is not open, The subtype  $\{0\} + 0^c \subset \text{Spec } B$  is not locally closed. Let  $U, C \subset \text{Spec } B$  be an open subset and a closed subset respectively, such that  $(x \neq 0) + (x \neq 0) \leftrightarrow x \in U \wedge x \in C$ . Then, for any  $x : U$ ,

$$(x = 0) + (x \neq 0) = x \in C$$

is a closed proposition. Thus the decidable subtype  $x \neq 0$  is a closed proposition. To contradict the assumption, we may convince ourself that the right vertical map

$$\begin{array}{ccc} \sum_{x:U} \neg\neg x = 0 & \xrightarrow{\sim} & \sum_{x:\text{Spec } B} \neg\neg x = 0 \\ \downarrow & & \downarrow \\ U & \hookrightarrow & \text{Spec } B \end{array}$$

is an open embedding

where the upper horizontal map is indeed an equivalence as for any  $x : \text{Spec } B$ ,  $x \in U$  is  $\neg\neg$ -stable, but  $\neg\neg x = 0$  and  $0 \in U$ , thus  $x \in U$ . □

### 3 Preparation

**Lemma 3.1** (Strong boundedness, NEEDED?). *Consider a sequence of embeddings of types*

$$X_0 \xrightarrow{\iota_0} X_1 \xrightarrow{\iota_1} X_2 \dots$$

*Then any map  $f : \text{Spec } A \rightarrow \text{colim}_n X_n \equiv: \bigcup_n X_n$  factors through some  $\kappa_m : X_m \hookrightarrow \text{colim}_n X_n$ .*

*Proof.* For every term  $x : \text{Spec } A$  consider the subset  $S_x$  of natural numbers  $n$ , such that  $f(x) \in \text{im } \kappa_m$ . Its a merely inhabited upwards closed subset. By the strong boundedness principle [ref?], the subset  $\bigcap_{x:\text{Spec } A} S_x$  is merely inhabited. □

**Lemma 3.2.** *Let  $Y$  be a type, which admits a jointly surjective family of maps with smooth domain  $X_i \rightarrow Y$  Then  $Y$  is formally smooth.*

*Proof.*  $\sum_{n:\mathbb{N}} X_n \rightarrow Y$  is surjective with formally smooth domain, as  $\mathbb{N}$  is formally smooth.  $\square$

**Corollary 3.3** (Monoid is smooth). *Let  $(Y, +)$  be a magma, which is generated by a map with smooth domain  $f : X \rightarrow Y$ , i.e. every  $a : Y$  can merely be written as a finite sum*

$$a = f(x_1) + \dots + f(x_n)$$

*Then  $Y$  is formally smooth.*

**Lemma 3.4.** *Let  $C$  be a class of types stable under  $\sum$ . Let  $\mathbb{P} \subset \text{Aff}$  (in most cases  $\mathbb{P} := \text{Aff}$ ) be any subclass of affines stable under finite limits. The class  $\text{HasAtlas}_C^{\mathbb{P}}$  of types  $Y$  which admit a map  $\mathbb{P} \ni S \rightarrow Y$  fibered in  $C$  is stable under identity types.*

*Proof.* By assumption we can choose a map  $\mathbb{P} \ni V \xrightarrow{p} Y$  fibered in  $C$ . Let  $y, y' : Y$ . Then we have the map

$$\begin{aligned} & (\text{fib}_p y) \times_V (\text{fib}_p y') \rightarrow y = y' \\ & (v, q : y = pv), (v', q' : y' = pv'), (h : v = v') \mapsto q \cdot h \cdot q'^{-1} \end{aligned}$$

The fiber over  $j : y = y'$  looks like

$$\sum_v \underbrace{\left( \sum_{v'} (h : v = v') \right)}_{\text{isContr}} \times (q : y = pv) \times (q' : y' = pv') \times (q \cdot h \cdot q'^{-1} = j) \simeq \sum_v (v = py) \simeq \text{fib}_p y$$

Hence the map is fibered in  $C$ . It suffices to show, that  $(\text{fib}_p y) \times_V (\text{fib}_p y')$  has an atlas, because then we can compose such an atlas with the above map to obtain an atlas of  $y = y'$ . By assumption the fibers of  $p$  have an atlas, so we can choose  $q : W \rightarrow \text{fib}_p y, q' : W' \rightarrow \text{fib}_p y'$  atlases. Then  $W \times_V W' \rightarrow (\text{fib}_p y) \times_V (\text{fib}_p y')$  is an atlas: The domain is a fiber product of types in  $\mathbb{P}$ , hence it belongs to  $\mathbb{P}$ . The fiber over  $(x, x')$  is equivalent to the product of fibers  $(\text{fib}_q x) \times (\text{fib}_{q'} x')$  which is in  $C$  by stability under dependent sums (so in particular under finite products).  $\square$

**Lemma 3.5.** *Let  $\mathcal{U}' \subset \mathcal{U}$  be stable under dependent sums. Let  $X$  be a type with a map  $p : U \rightarrow X$  fibered in  $\mathcal{U}'$ . For any  $x : X$ , let  $Y_x$  be a type and moreover for any  $u : U$ , we are given a map  $q_u : V_u \rightarrow Y_{p(u)}$  fibered in  $\mathcal{U}'$ . Then the induced map*

$$p : \sum_{u:U} V_u \rightarrow \sum_{x:X} Y_x$$

*is fibered in  $\mathcal{U}'$*

*Proof.* The fiber of  $p$  over some  $(x, y) \in \sum_{x:X} Y_x$  is given by

$$\sum_{u:\text{fib}_p x} \text{fib}_{q_u}(y')$$

where  $y' : Y_{p(u)}$  (depending on  $u$ ) is the transport of  $y : Y_x$  along  $x = p(u)$ . As  $\mathcal{U}'$  is stable under dependent sum those fibers are again in  $\mathcal{U}'$ . This shows the result.  $\square$

## 4 Talk: Algebraic Spaces vs Schemes

### 4.1 Recap: Sheaftheory

**Notation.**

$$\mathbb{T} = \sum_{X:\mathbf{Aff}} X \text{ formally étale} \wedge \underbrace{\text{faithfully flat}}_{= \text{flat} + \neg\neg \text{ inhabited}}$$

Any  $X : \mathbb{T}$  has decidable equality!  $\mathbb{T}$  contains 1 and is stable under  $\sum$ . A  $\mathbb{T}$ -cover is a map fibered in  $\mathbb{T}$ .

**Definition 4.1.** A type  $X$  is an étale-stack, if for any  $\text{Spec } A : \mathbb{T}$  and any map  $\|\text{Spec } A\| \rightarrow X$ , there exists a unique filler

$$\begin{array}{ccc} \|\text{Spec } A\| & \xrightarrow{\forall} & X \\ \downarrow & \nearrow \exists! & \\ 1 & & \end{array}$$

An étale -sheaf is a 0-type that is an étale -stack.

Being an étale -stack determines a lex-modality, that we write  $L_{\mathbb{T}}$ .

Given an equivalence relation  $R$  on a sheaf  $X$  valued in propositional sheaves, we can form the sheaf-quotient  $p : X \rightarrow X/R \coloneqq L_{\mathbb{T}}\|X//R\|_0$  that is uniquely characterized by the following properties

- for any  $x, y : X$ ,  $p(x) =_{X/R} p(y) \leftrightarrow R(x, y)$
- The map  $p$  is  $\mathbb{T}$ -surjective, i.e. for any  $z : X/R$ ,  $\|\text{fib}_p(z)\|_{\mathbb{T}}$  (' $\mathbb{T}$ -merely the proposition  $\|\text{fib}_p(0)\|$  holds').

**Lemma 4.2.** *Given a subtype  $\mathcal{C}$  of stacks the following are equivalent*

- *the type  $\mathcal{C}$  is a stack.*
- *If  $X$  is a stack, then the proposition  $X \in \mathcal{C}$  is a sheaf.*

*In this case we say that  $\mathcal{C}$  has descent.*

### 4.2 Algebraic Spaces

We assume today, that schemes are étale sheaves. Algebraic Spaces are meant as a generalization of schemes that satisfy descent.

**Definition 4.3.** A naive algebraic space is an étale -sheaf  $X$ , that merely admits a  $\mathbb{T}$ -cover  $\text{Spec } B \rightarrow X$ .

We call it covering, if we can choose  $\text{Spec } B \in \mathbb{T}$ .

This is not good enough, because we can NOT prove the following

- all schemes are naive algebraic spaces
- naive algebraic spaces having descent, i.e. the type of them is an étale -stack.

Instead we have to repeat the process asking for an atlas twice.

**Definition 4.4.** An algebraic space is an étale -sheaf  $X$ , such that one of the following equivalent conditions holds:

- merely we find a map  $\text{Spec } A \rightarrow X$  such that each fiber is a covering naive algebraic space.

- We merely can express it as the sheaf-quotient of an affine  $\text{Spec } A$  by an equivalence relation  $R$  that is covering, i.e. for each  $x : \text{Spec } A$ ,  $\sum_{y : \text{Spec } A} R(x, y)$  is a covering naive algebraic space.

We call  $X$  covering, if we can choose  $\text{Spec } A$  to lie in  $\mathbb{T}$ .

This class is stable under  $\sum$  and under quotients by covering equivalence relations

**Theorem 4.5.** *(Covering) Algebraic spaces have descent.*

**Example 4.6.** *Schemes are algebraic spaces!*

*Proof.* 1. Merely inhabited finite sums of principal open propositions belong to  $\mathbb{T}$ .

2. open propositions are naive algebraic spaces.

3. If  $X$  is a scheme, merely we find a map  $\coprod_i \text{Spec } A_i \rightarrow X$  whose fibers are merely inhabited finite sums of open propositions, so the fibers are covering.  $\square$

**Question 1.** Can we find algebraic spaces that are not schemes?  
Can we prove with them, that Schemes do not have descent?

### 4.3 Quotients by Group actions

Let  $\ell \neq 0$  denote a prime. Consider  $\mu_\ell = R[X]/(X^\ell - 1)$ .

**Example 4.7** (Non-free action). *If  $2 \neq 0$ , the sheaf-quotient of  $\mathbb{A}^1$  by the non-free  $\mu_2$  action is not an algebraic space!*

This suggests that we need a free action. On the other hand, classically, the quotient of a quasi-projective scheme by a finite free group action is a scheme.

**Construction.** Given a formally étale + flat affine (e.g.  $\mu_\ell$  or finite) group  $G$  that acts on an affine  $\text{Spec } A$ . Assume  $G$  acts free on some open subset  $U \subset \text{Spec } A$ . Then we construct a covering equivalence relation  $R$  on  $\text{Spec } A$ , such that

- On  $U$  we are just taking the quotient by the  $G$ -action: for any  $x : U$  and  $y : \text{Spec } A$

$$R(x, y) \leftrightarrow \sum_{g : G} gx = y.$$

- On the complement  $U^c \equiv \text{Spec } A \setminus U$  we do nothing: for some  $x : U^c, y : \text{Spec } A$ , we have  $R(x, y) \leftrightarrow x = y$ .

We write  $\text{Spec } A/_U G \equiv \text{Spec } A/R$  and call it the quotient of  $\text{Spec } A$  by the  $G$ -action on  $U$ .

*Proof.* To show, that its an algebraic space, one checks, that  $G$  acts free on  $0^c$ .

$$R_G(x, y) \equiv (x = y) + (x \in U) \times \sum_{g : G \setminus \{1\}} gx = y$$

This is covering: For any  $x : \text{Spec } A$  we have

$$\sum_{y : X} x = y + (x \neq 0) \times \sum_{g : G \setminus \{1\}} gx = y \simeq 1 + (x \in U) \times G \setminus \{1\}$$

which is a covering naive algebraic space: For that use, that  $G \setminus \{1\} = \sum_{g : G} g \neq 1$  is a  $\sum$  of formally étale + flat affines (recall that formally étale affines have decidable equality). Indeed, the two conditions hold, using that  $G$  has decidable equality.  $\square$

**Example 4.8** (Free action). *Let  $G$  act freely on the whole space  $U \equiv \text{Spec } A$ . Then this construction yields the actual group quotient:  $\text{Spec } A /_{\text{Spec } A} G = \text{Spec } A / G$ .*

*Proof.* Indeed, the equivalence relation is the same, using that  $G$  has decidable equality.  $\square$

**Example 4.9** (Quotient of the Line). *If  $\ell \neq 0$  is prime, we have  $R /_{0^c} \mu_\ell$  is an algebraic space*

**Example 4.10** (Quotient of the Cross). *Let  $\mu_\ell$  act on  $X = \text{Spec } R[X, Y] / X^\ell - Y^\ell$  via multiplication on the second component. Then*

$$X /_{\{0,0\}^c} \mu_\ell$$

*is an algebraic space.*

Are those schemes?

**Proposition 4.11.** *Let  $0 : \text{Spec } A$  be a regular point, i.e. we can write  $\text{Spec } B \setminus \{0\} = D(p_1, \dots, p_n)$  for some  $p_1, \dots, p_n : B$  jointly-regular, i.e.  $B \rightarrow \prod_{j=1}^n B_{p_j}$  is injective. Let  $G$  be a non-trivial formally étale flat affine group acting on  $\text{Spec } A$ , such that*

- $0$  is a fixpoint
- if  $gx = x$  for some  $g \neq 1$ , then  $x = 0$ .

*Then  $\text{Spec } A /_{0^c} G$  from 5.18 is an algebraic spaces that is not a scheme.*

*Proof.* As  $\neg\neg(G \setminus \{1\})$  and our goal is  $\neg\neg$ -stable, we may choose some  $g : G \setminus \{1\}$ . Then for all  $y : \text{Spec } A$

$$R_{\sharp}(y, gy) \simeq (y = gy) + (y \neq 0) \wedge \sum_{h \neq 1} hy = gy \simeq (y = 0) + (y \neq 0) \simeq (y = 0) + (y \neq 0)$$

Regularity of  $0$  gives us that  $\{0\} + D(0) \subset \text{Spec } A$  is not a locally closed subtype, i.e a closed subset of an open subset. But the identity types of a scheme are locally closed propositions (subsets of the point).  $\square$

**Example 4.12.** *Assume  $\ell \neq 0$  prime. Let  $\mu_\ell$  act on  $\text{Spec } B$  in one of the following ways:*

1. *Let  $\mu_\ell$  act on  $\text{Spec } B = \mathbb{A}^1$ .*
2. *Let  $\mu_\ell$  act on*

$$\text{Spec } B \equiv \sum_{x,y:R} x^\ell = y^\ell$$

$$\text{via } g(x, y) = (x, gy)$$

*Then  $\text{Spec } B /_{0^c} \mu_\ell$  is an algebraic space that is not a scheme.*

*Proof.*  $\neg\neg$  merely,  $\mu_\ell$  is finite ([ref?]) and  $\mu_\ell \setminus \{1\}$  is inhabited by 5.12.

1. Pointed-Free action is clear.  $0 : \mathbb{A}^1$  is a regular point by first projection.
2. Pointed-Free action is clear. The cross is regular pointed, witnessed by the first projection: It is regular vanishing at  $(0,0)$  And for any point  $(0, y) : \text{Spec } B$  we deduce  $y^\ell = -0^\ell = 0$ , hence  $\neg\neg(x, y) = (0, 0)$ .

$\square$



## 4.4 Fiber Collapse!

An alternative approach to construct algebraic spaces is the fiber collapse away from the origin.

**Definition 4.13.** Given a sheaf proposition  $P$ , there is a closed modality  $\text{Cl}_P$  where a type  $X$  is modal, if it is a sheaf and  $P \rightarrow \text{isContr}(X)$ . We have to stackify to belong to the sheaf topos:

$$\begin{aligned} \mathcal{U} &\rightarrow \mathcal{U} \\ X &\mapsto P \star X \equiv L_{\mathbb{T}}(P \sqcup_{P \times X} X) \end{aligned}$$

where we need to stackify the pushout. This determines a lex modality.

**Definition 4.14.** Let  $Y_{\bullet} : R \rightarrow \text{Aff}$  be a dependent family of affines. The fiber collapse of  $Y_{\bullet}$  away from the origin  $-Y_{\bullet}-$  is the following type over  $R$

$$\sum_{x:R} (x \neq 0) \star Y_x \rightarrow R$$

- The *infinitesimal fibers* over  $\varepsilon : \mathcal{N}_{\infty}$  are  $\text{fib}_p(\varepsilon) = Y_{\varepsilon}$ .
- In particular the *basefiber*  $\text{fib}_p(0)$  is equivalent to  $Y_0$ ,
- The *other fibers*  $\text{fib}_p(x)$ ,  $x \neq 0$ , are contractible.

So  $-Y_{\bullet}-$  is obtained from  $\sum_{x:R} Y_x \rightarrow R$  by keeping only the infinitesimal fibers and collapsing all the other fibers. This space over  $R$  looks exactly like the line away from the origin.

**Lemma 4.15.** Assume that if  $x \neq 0$ , then  $Y_x \in \mathbb{T}$ . Then  $-Y-$  is an algebraic space.

*Proof.* Let  $x : R$ . Let  $Y : \text{Aff}$  such that  $x \neq 0$  implies that  $Y$  is formally étale + flat. We will show that  $\eta : Y \rightarrow (x \neq 0) \star Y$  is the sheaf-quotient map of the relation on  $Y$  given by  $y \sim y' \equiv (y = y') + (x \neq 0) \times y \neq y'$ , which is enough by 5.21. We apply ??

- The map is  $\mathbb{T}$ -surjective: We have a  $\mathbb{T}$ -surjection  $(x \neq 0) + Y \rightarrow (x \neq 0) \star Y$ . In case  $x \neq 0$ , the map of interest is  $Y \rightarrow 1$ , which is  $\mathbb{T}$ -surjective, as then  $Y \in \mathbb{T}$ .
- Given  $y, y' : Y$ , we have

$$\begin{aligned} \eta(y') = \eta(y) &\simeq (x \neq 0) \star (y = y') && | \text{ closed modality is lex ([4] Example 3.1.4).} \\ &\simeq L_{\mathbb{T}}((y = y') \vee (x \neq 0)) && | (x \neq 0) \rightarrow \text{HasDecEq}(Y) \\ &\simeq (y = y') + (x \neq 0) \times y \neq y', \end{aligned}$$

□

**Example 4.16.**  $-Bool-$  is the line with two origins.

$-\text{Spec } R[X]/(X^2 + 1)-$  is the twisted line with two origins, i.e. over the origin we have the roots of  $-1$ .

$-\text{Spec } R[Y]/(Y^2 - \bullet)-$  is  $\mathbb{A}^1/_{0^c} \mu_2$  which looks like  $\mathbb{D}(1)$  over  $0$ .

$-\text{Spec } R[Y]/(Y^2 - \bullet^2)-$  is the quotient of the cross that looks like  $\mathbb{D}(1)$  over every  $\varepsilon : \mathbb{D}(1)$ .

$-\text{Spec } R[Y]/(\bullet Y)-$  is the affine plus.

**Proposition 4.17.** *Let  $G$  be a formally  $+$  flat affine group. Let  $p : \tilde{Y} \rightarrow R$  such that the pullback to  $R^\times$  can be enhanced to a  $G$  torsor over  $R^\times$ . Write  $Y_x \equiv \text{fib}_p x$ . Then there is a canonical equivalence*

$$\begin{array}{ccc} & \tilde{Y} & \\ \swarrow & & \searrow \\ \tilde{Y}/_{Y_0^c} G & \xrightarrow{\cong} & -Y- \\ \searrow & & \swarrow \\ & R & \end{array}$$

*Proof.* As every *other* fiber is merely equivalent to  $G$ , its formally étale  $+$  flat. In between you can put for  $U_x \equiv x \neq 0 \times Y_x$

$$\tilde{Y} \rightarrow \sum_{x:R} Y_x/U_x G$$

As all three maps defined on  $\tilde{Y}$  are  $\mathbb{T}$ -surjective, by ?? we may only check that the identity types coincide. For any  $y, y' : \tilde{Y}$ . Using that if  $py \neq 0$  then the  $G$  action on  $Y_{py}$  is a  $G$ -torsor, We have an equivalence

$$(y = y') + y \notin Y_0 \wedge \underbrace{\sum_{g \neq 1} gy = y'}_{\simeq (py=py') \wedge y \neq y'} \simeq (py = py') \wedge ((y = y') + py \neq 0 \wedge y \neq y')$$

If we fix  $x : R$  and put  $y, y' : Y_x$  we have, writing  $\eta : Y_x \rightarrow (x \neq 0) \star Y_x$

$$(y = y') + py \neq 0 \wedge y \neq y' \simeq \eta y = \eta y'$$

by the proof of 4.15. □

## 4.5 Schemes do not have descent

**Remark 2.** Whenever we want to show a proposition that is an étale -sheaf, we may assume a term in  $i : \text{Spec } C \subset R$ . Because  $i \neq -i$ , this determines an embedding

$$\begin{aligned} \text{Bool} &\rightarrow \text{Spec } C \\ \text{true} &\mapsto i \\ \text{false} &\mapsto -i \end{aligned}$$

But any embedding  $\text{Bool} \hookrightarrow \text{Spec } C$  is already an equivalence (\*), as for any  $x : R$ ,  $x - i$  or  $x + i$  is invertible, so if  $(x - i)(x + i) = 0$  we know that one of the factors is zero.

**Proposition 4.18.** *let  $\rho : R \setminus \{0\}$  (e.g.  $\rho = 1$ ). Set  $C = R[T]/(T^2 + \rho)$ . If  $-\text{Spec } C-$  is a scheme, then  $X^2 + \rho$  merely has a root.*

*Proof.* Let  $p : -\text{Spec } C- \rightarrow R$  be the first projection. We proceed as follows

1. There is no open affine subset of  $-\text{Spec } C-$  containing  $\text{fib}_p(0)$ .
2. Any open subset of  $\text{Spec } C$ , that is strictly smaller than  $\text{Spec } C$ , is an open proposition

Any finite open affine cover of  $-\text{Spec } C-$  can be restricted to a finite open affine cover  $\text{Spec } C = \bigcup_{j=0}^n U_j$  of the basefiber  $\text{Spec } C$  consisting of strictly smaller open subsets by point 1. Then the goal is

$$\|\text{Spec } C\| = \left\| \bigcup_{j=0}^n U_j \right\| = \bigvee_j U_j$$

an open proposition by point 2., thus an étale sheaf, as open propositions are  $\neg\neg$ -stable. So we can conclude. Proofs:

1. Because we want to show  $\perp$ , we may assume  $-\text{Spec } C- = -\text{Bool}-$ . Assume there is an open affine subset  $\text{fib}_p(0) \subset U \subset -\text{Bool}-$ . Then  $p(U) \subset R$  is an open neighborhood of 0, as

$$x \in p(U) \leftrightarrow (x, N) \in U \vee (x, S) \in U$$

Claim: the map  $R^{p(U)} \rightarrow R^U$  is an equivalence. If we have shown that: As  $U$  is affine we conclude that the map

$$\begin{aligned} U &\rightarrow \text{Spec}(R^{p(U)}) \\ x &\mapsto \phi \mapsto (\phi(px)) \end{aligned}$$

is an equivalence, which is a contradiction to the assumption, that  $U$  contains both origins.

Proof of claim: In words: As  $U$  is a subset of a quotient of  $R + R$ , the function  $U \rightarrow R$  determines two (partially defined on open domain) functions to  $R$  that coincide away from the origin, which is a regular point. Thus by 2.3 they coincide everywhere. Injectivity: If two maps  $f, g : p(U) \rightarrow R$  coincide after precomposing with  $U \rightarrow p(U)$ , then they coincide away from 0 so conclude by 2.3.

Surjectivity: Given a map  $U \rightarrow R$ , by pulling back along  $p : R + R \rightarrow -\text{Bool}-$  we can view it as a map  $R + R \supset U' \rightarrow R$  defined at both origins, so in particular as a pair of maps to  $R$  defined on some open neighborhood of 0 of  $R$ . They coincide away from 0 so by 2.3 they are equal.

2. Any strictly smaller open subset  $U \subset \text{Spec } C$  is an open proposition: Note, that  $U$  is a proposition: If  $x, x' : U$ , then  $x = x' \simeq \neg\neg(x = x')$  by decidable equality of  $U$ , but if  $x \neq x'$ , then  $\{x, x'\} \hookrightarrow \text{Spec } C$  is an embedding, so by (\*) an equivalence. But then  $U = \text{Spec } C$ , contradiction.

We first reduce to the case where  $U$  is a principal open of  $\text{Spec } C$ . By [1] we find  $f_1, \dots, f_n : C$  such that  $U = D(f_1, \dots, f_n)$ . As the left hand side is a proposition we have

$$U \leftrightarrow \bigvee D(f_i)$$

so we may show, that each  $D(f_i)$  is an open proposition.

Let  $f : C$  such that  $D(f)$  is a proposition. Choose a representative  $a + bT : R[T]$ .

Let us show  $(2a \neq 0) \leftrightarrow D(f)$ , which is an étale -sheaf and a proposition, so we may assume  $x : \text{Spec } C$ . Using that  $D(f)$  is a proposition we have

$$D(f) = (a + bx \neq 0) + (a - bx \neq 0) \xrightarrow{\sim} (a + bx \neq 0) \vee (a - bx \neq 0)$$

We may show both implications  $2a \neq 0 \leftrightarrow (a + bx \neq 0) \vee (a - bx \neq 0)$ .

$' \rightarrow'$   $(a + bx) + (a - bx)$  is invertible, so by locality one of the summands is invertible.  $' \leftarrow'$  by symmetry wlog  $a + bx \neq 0$ . Then as  $D(f)$  is a prop,  $\neg\neg(a - bx = 0)$ . Thus  $\neg\neg(a + a = a + bx \neq 0)$ , hence  $2a \neq 0$ .

□

**Corollary 4.19.** *Schemes do not have descent.*

*Proof.* If Schemes have descent, then  $\text{---Spec } R[T]/(T^2 + \rho)\text{---} \in \mathbf{Sch}$  is a sheaf. As  $\text{---Spec } R[T]/(T^2 + \rho)\text{---}$  is  $\mathbb{T}$ -merely a scheme, it is a scheme, so by the previous lemma  $T^2 + \rho$  has a root. As  $\rho : R \setminus \{0\}$  was arbitrary, we get a contradiction to [1] A . 0.3.  $\square$

## 5 Algebraic spaces

**Theorem 5.1.** *Let  $X$  be a modal set. The following are equivalent:*

1.  $X$  is a (covering) geometric 0-stack
2.  $X$  is merely of the form  $L_{\mathbb{T}}(U/R)$  for some (covering) affine  $U$  and  $R : U^2 \rightarrow \text{Prop}_{\circ}$  a covering equivalence relation.
3. there exists some map  $S \rightarrow X$  with  $S$  (covering) affine whose fibers merely have  $\mathbb{T}$ -atlases.

We call this class (covering) algebraic spaces.

*Proof.*

2  $\leftrightarrow$  3 This is ??

2  $\rightarrow$  1 Choose a presentation  $R : U^2 \rightarrow \text{Prop}$ . It suffices to show, that the map  $f : U \rightarrow L_{\mathbb{T}}(U/R)$  is a geometric (c)atlas. The map  $f$  is  $\mathbb{T}$ -surjective by the well-definedness of the bijection ?? . By descent we may just show, that the fibers  $\text{fib}_f(f(s))$  for  $s : U$  are covering 0-stacks. But by the bijection in ?? those are equivalent to the fibers  $R_s$ , which are covering 0-stacks as the equivalence relation is covering.

1  $\rightarrow$  2 This can be reformulated in the following way, using the recursion principle for (covering) geometric 0-stacks: Let  $X$  be a sheaf of sets. Let  $S$  be (covering-) affine and  $f : S \rightarrow X$  be fibered in covering algebraic spaces. Then  $X$  is a (covering) algebraic space. This follows from the observation, that the equivalence relation determined by  $f$  is covering ?? , because the fibers of  $f$  are covering 0-stacks.f

$\square$

**Proposition 5.2.** *For any  $n \geq 1$ , we have inclusions*

$$W_n \subset \mathbf{CS}_{n-1} \subset W_{n+1}$$

*Proof.* Induction.  $n = 1$  gives

$$\mathbf{HasCatlas}_{\mathbb{T}} \subset \mathbf{CS}_0 \subset \text{types admitting a atlas fibered in } W_1$$

the latter inclusion is the previous theorem.

The induction step is obtained by 5.4  $\square$

### 5.1 Schemes are algebraic Spaces for the Zariski Topology

**Definition 5.3.** A proposition  $U$  is open iff its merely of the form  $f_1 \text{ inv} \vee \dots \vee f_n \text{ inv}$  for some  $f_i : R$ .

**Lemma 5.4.** *Given  $f_1, \dots, f_n : R$  such that  $\|D(f_1) + \dots + D(f_n)\|$  then  $\sum_{i=1}^n D(f_i) \in \mathbf{Zar}$ .*

**Proposition 5.5.** *Every Zariski-merely-inhabited type that is merely of the form  $U_1 + \dots + U_n$  for open propositions  $U_i$  admits a Zar-atlas.*

*Proof.* By definition of openness, We can choose a surjection  $\coprod_{j=1}^{n_i} D(f_{ij}) \twoheadrightarrow U_i$  for any  $i$ . We want to show, that the map

$$\coprod_{i,j} D(f_{ij}) \twoheadrightarrow U_1 + \dots + U_n$$

is a Zar-catlas.

- Let us first show that the fibers are in Zar. Assume  $U_i$  holds. So we find a term in  $\coprod_j D(f_{ij})$ . In particular we have  $\|\coprod_j D(f_{ij})\|_{\text{Zar}}$ . By the lemma we conclude, that the fiber  $\sum_j D(f_{ij})$  belongs to Zar.
- The total space is in Zar: This follows as the surjection after propositional truncation becomes an equivalence. As we have  $\|U_1 + \dots + U_n\|$ , we can conclude by the lemma.

□

**Warning.** The converse does not hold! We want to apply ??, to the map

$$\text{Zar} \ni 1 + 1 \rightarrow \sum D(f)$$

- $\sum D(f)$  is separated as  $D(f)$  is a sheaf.
- All the fibers are equivalent to  $1 + X$ , hence they are in the Zariski topology.

**Lemma 5.6.** *let  $X$  be a scheme. There merely exists some affine  $S$  map  $S \rightarrow X$  whose fibers are merely inhabited finite sums of open propositions*

**Corollary 5.7.** *Every scheme is an algebraic space.*

**Lemma 5.8.** *If  $X$  is an algebraic space, then the global sections embed via a  $R$ -algebra homomorphisms into a finitely presented  $R$ -algebra.*

*Proof.* Choose an atlas  $S \rightarrow X$ , in particular  $\mathbb{T}$ -surjective. As  $\mathbb{T}$  is subcanonical the map  $R^X \rightarrow R^S$  is an injection. □

**Question 2.** Is it an open embedding of types?

**Example 5.9.** *Using descent for Affines (compare to [3]), not every scheme admits a  $\mathbb{T}$ -cover out of an affine: Let us show that for any open proposition  $P$ : If  $\text{Susp}(P)$  admits a  $\mathbb{T}$ -cover  $\text{Spec } S \rightarrow \text{Susp}(P)$  then  $P$  is affine.*

*Then by Zariski Local choice, we may assume that it factors through  $\text{Bool} \rightarrow \text{Susp}(P)$ , so then we may call the two fibers of the map  $\text{Spec } S \rightarrow \text{Bool}$ ,  $\text{Spec } A$  and  $\text{Spec } B$ . Then the two fibers are  $F_1 \equiv \text{Spec } A + P \times \text{Spec } B$  and  $F_2 \equiv \text{Spec } B + P \times \text{Spec } A$ , which by assumption are in  $\mathbb{T}$ .  $P$  being affine is a sheaf, thus we may show  $F_2 \rightarrow \text{isAff}(P)$  So pick a point of  $F_2$ . If it comes from  $P \times \text{Spec } A$ , then  $P$  so  $P$  is affine. If we have  $t : \text{Spec } B$ ,  $P$  is a fiber of  $P \times \text{Spec } B \rightarrow \text{Spec } B$ , but the domain is affine using that  $F_2 \in \mathbb{T}$ . So  $P$  is affine.*

## 5.2 Examples

The goal of this subsection is to construct algebraic spaces. The first example actually gives us a scheme:

**Example 5.10.** *Let  $p \neq 0$  be a prime. You can let  $\mu_p := \text{Spec}(R[X]/(X^p - 1))$  act on  $\mathbb{A}^\times$  via multiplication. Set  $\mathbb{T} = \text{fppf}$ . Then the  $p$ .th power map*

$$\text{pow} : \|\mathbb{A}^\times / \mu_p\|_0^\mathbb{T} \rightarrow \mathbb{A}^\times$$

*is an equivalence.*

- *It is an embedding: First note, that  $\|\mathbb{A}^\times/\mu_p\|_0$  is  $\mathbb{T}$ -seperated:*  
as  $\mu_p$  act freely on  $\mathbb{A}^\times$ ,  $\mathbb{A}^\times/\mu_p$  is already a set. Meaning that the identity types of the set-quotient are  $\sum_{g:\mu_p} gx =_{\mathbb{A}^\times} y$ , hence sheaves.  
On the other hand the map  $\|\mathbb{A}^\times/\mu_p\|_0 \rightarrow \mathbb{A}^\times$  is an embedding, as for any  $x, y : \mathbb{A}^\times$  the map  $(\sum_{g:\mu_p} gx = y) \rightarrow (x^p = y^p)$  is an equivalence.
- *It is  $\mathbb{T}$ -surjective, as for any  $\lambda : \mathbb{A}^\times$ , we find  $S = \text{Spec } R[X]/(X^p - \lambda) \in \mathbb{T}$  with*

$$S \rightarrow \text{fib}_{\mathbb{A}^\times/\mu_p \rightarrow \mathbb{A}^\times}(\lambda)$$

hence

$$1 = \|S\|_{\mathbb{T}} \rightarrow \|\text{fib}_{\text{pow}}\|_0^{\mathbb{T}}$$

**Example 5.11.** Let  $P$  be the open proposition  $x \neq 0$  for some  $x : \mathbb{A}^1$ . Then  $H = 1 + P$  is an open subgroup of  $\mathbb{Z}/2$ . The sheaf quotient  $G/H$  is the scheme  $\text{Susp}(x \neq 0)$ .

Let  $\ell \neq 0$  denote a prime. Consider  $\mu_\ell = R[X]/(X^\ell - 1)$  acting on  $\mathbb{A}^1$  by multiplication.

**Lemma 5.12.** Let  $(G, 1)$  be a pointed formally étale flat affine type. Then  $(G \setminus \{1\})$  is formally étale + flat affine.

In particular  $\mu_\ell \setminus \{1\}$  is a covering stack.

*Proof.*  $G \setminus \{1\} = \sum_{g:G} g \neq 1$  is a  $\sum$  of formally étale + flat affines (recall that formally étale affines have decidable equality).

To show, that  $\mu_\ell \setminus \{1\}$  is a covering stack, by ??, we need to show it is  $\neg\neg$ -inhabited. Indeed as we want to prove a contradiction we may assume a term in  $g : \text{Spec } R[X]/(\sum_{i=0}^{\ell-1} X^i)$ . But this type is equivalent to  $\mu_\ell \setminus \{1\}$ , using that  $\sum_{i=0}^{\ell-1} X^i | X^\ell - 1$  and  $\ell \neq 0$ .  $\square$

**Lemma 5.13.** Given a modal equivalence relation  $R$  on a sheaf  $X$  and a 1-stack  $T$  and a map  $f : X \rightarrow T$  and term  $p : \prod_{x,y:X} R(x,y) \rightarrow fx = fy$  such that  $p(x,y) \cdot p(y,z) = p(x,z)$ , where the witnesses for  $R$  are left implicit. Then  $f$  factors through the quotient.

**Lemma 5.14.** Put  $\ell = 2$  If  $\ell \neq 0$ , the sheaf quotient of  $\mathbb{A}^1$  by the  $\mu_2$  action is not an algebraic space.

*Proof.* Assume this it is an algebraic space.

Set  $\mathbb{D}(1) = \text{Spec } R[X]/X^\ell$ . Then  $\sum_{x:\mathbb{A}^1/\mu_\ell} x^\ell =_{\mathbb{A}^1} 0 \simeq \mathbb{D}(1)/\mu_\ell$  is an algebraic space by  $\sum$ -stability.

Then we can choose a geometric atlas  $p : \text{Spec } A \rightarrow \mathbb{D}(1)/\mu_\ell$ . We proceed in the following steps

1. There is an equivalence  $\text{Spec } A \simeq \text{fib}_p 0 \times \mathbb{D}(1)/\mu_\ell$ .
2. The fiber over 0 is affine
3.  $\mathbb{D}(1)/\mu_\ell$  is  $\neg\neg$  affine
4.  $\mathbb{D}(1)/\mu_\ell$  is  $\neg$  affine

Proofs

1. Let us denote  $F : \mathbb{D}(1)/\mu_2 \rightarrow \mathbf{CS}_0$  the bundle of fibers of  $f$ , where we note that the fibers are indeed sets. As  $\mathbf{CS}_0$  is formally étale ([ref?]), we have terms

$$\phi : \prod_{x:\mathbb{D}(1)} F[x] = F[0], \phi^- : \prod_{x:\mathbb{D}(1)} F[-x] = F(0)$$

that both evaluate at  $x = 0$  to  $\text{refl}_{F[0]}$ .  
The goal is to produce a term in

$$\prod_{x:\mathbb{D}(1)/\mu_2} Fx = F[0]$$

By the previous lemma, using that  $\mathbf{CS}_0$  is a 1-stack, we need to show, that under the path  $p_x : [x] = [-x]$  in the quotient we have

$$\mathbf{ap}_{p_x} F \cdot \phi^- x = \phi x$$

This proposition is formally étale as  $\mathbf{CS}_0$  is formally étale . Thus we may assume the closed dense proposition  $x = 0$ . Then  $p_x = \text{refl}_{[0]}$  and  $\phi^- 0 = \text{refl} = \phi 0$  by assumption.

2. Let us first show, that We may assume that our geometric cover factors through the  $\mathbb{T}$ -surjection  $\text{Spec } A \xrightarrow{f} \mathbb{D}(1) \rightarrow \mathbb{D}(1)/\mu_\ell$ . Proof: By  $\mathbb{T}$ -local choice applied to the  $\mathbb{T}$ -surjection  $\mathbb{D}(1) \rightarrow \mathbb{D}(1)/\mu_\ell$ , we find a  $\mathbb{T}$ -cover  $\text{Spec } B \rightarrow \text{Spec } A$  and a factorization

$$\begin{array}{ccc} \exists \text{Spec } B & \dashrightarrow & \mathbb{D}(1) \\ \downarrow & & \downarrow \\ \text{Spec } A & \longrightarrow & \mathbb{D}(1)/\mu_\ell \end{array}$$

□(Claim)

Its enough to see that the map  $\text{fib}_f 0 \rightarrow F$  is an equivalence. That follows because 0 is a fix point of the  $\mu_\ell$  action on  $\mathbb{D}(1)$ .

3.  $F$  is a covering stack, hence  $\neg\neg$ -inhabited. As the goal is  $\neg\neg$ -modal, we may pick a map  $1 \rightarrow F$ . Then, by step 1

$$\mathbb{D}(1)/\mu_\ell = 1 \times_F (F \times \mathbb{D}(1)/\mu_\ell) = 1 \times_F \text{Spec } A$$

is a fiber product of affines, hence affine.

4. Here we need that  $\ell = 2$  . The affinization map would be induced by

$$\begin{array}{ccc} \mathbb{D}(1) & & \\ \downarrow & \searrow z \mapsto z^\ell & \\ \mathbb{D}(1)/\mu_\ell & \dashrightarrow & \mathbb{D}(1) \end{array}$$

But the map is not an embedding: For any  $\varepsilon : \text{Spec } R[X]/X^\ell$ , we have  $\varepsilon^\ell = 0^\ell$  but  $\varepsilon =_{\mathbb{D}(1)/\mu_\ell} 0$  iff there  $\mathbb{T}$ -merely exists some  $g : \mu_\ell$  with  $g\varepsilon = 0$ , but as  $g$  is invertible this is equivalent to  $\varepsilon = 0$ .

□

### 5.3 Non locally-separated Examples

**Proposition 5.15.** *Consider an affine  $S$  and an open subset  $U \subset S$ . Consider a  $\mathbb{T}$ -flat irreflexive relation  $\sharp$  on  $U$  , i.e.*

1. *Irreflexivity:*  $\neg(x\sharp x)$
2.  *$\mathbb{T}$ -flatness.* For all  $x : U$  , the fiber  $\sum_{y:S} x\sharp y$  is  $\mathbb{T}$ -flat.

Define a relation on  $S$  as

$$R_{\sharp}(x, y) = (x = y) + (x \in U \wedge y \in U) \times (x \sharp y)$$

(Abuse of notation: where the  $\times$  is secretly a  $\sum$ ) Then the sheaf quotient  $S/R_{\sharp}$  is an algebraic space.

*Proof.* • This is a proposition: First note, that both summands are propositions and if both summands are inhabited we get a contradiction.

- The relation is covering: Furthermore, for any  $x : S$  we have

$$\sum_{y:S} (x = y) + (x, y \in U \times x \sharp y) = 1 + (x, y \in U \times \sum_y x \sharp y) \in \mathbb{T}$$

as  $\sharp$  was assumed to be  $\mathbb{T}$ -flat on  $U$ : we can write the binary product as  $\sum_{p:x \in U} \sum_y x \sharp y$ , a  $\sum$  of  $\mathbb{T}$ -flat geometric stacks. □

## 5.4 Group quotients

For this section let  $G$  denote a group that is a covering 0-stack. Let  $X$  be a sheaf equipped with a  $G$  action.

**Lemma 5.16.**  $\mu_p = \text{Spec } R[X]/(X^p - 1)$  is covering for  $p \neq 0$  prime.

*Proof.* It is fppf + étale as  $X^p - 1$  is monic separable. TODO □

**Definition 5.17.** A  $G$  action on  $X$  is free, if for all  $x, y : X$  the type

$$\sum_{g:G} gx = y$$

is a proposition.

**Example 5.18.** Given a formally étale + flat affine (e.g.  $\mu_\ell$  or finite) group that acts free on some open subset  $U \subset \text{Spec } A$  of some affine.

Then put  $x \sharp y = \sum_{g:G \setminus \{1\}} gx = y$ .

This provides a covering equivalence relation  $R_{G,U}$  on  $\text{Spec } A$ , such that

- for any  $x : U$

$$[x] = \sum_{y:\text{Spec } A} \sum_{g:G} gy = x.$$

- for some  $y \notin U$ , we have  $R(x, y) \leftrightarrow x = y$ .

By abuse of notation we write  $\text{Spec } A/_U G \equiv \text{Spec } A/R_G$  and call it the quotient of  $\text{Spec } A$  by the  $G$ -action.

*Proof.* • It is irreflexive: If  $x : U$  then  $gx \neq x$ , by freeness.

- We have  $G \setminus \{1\}$  is flat affine using 5.12. □

**Notation.** If  $U = \text{Spec } A \setminus Z$  the complement of a closed subset we write

$$U \equiv Z^c$$



**Example 5.19** (Free action). Set  $U = \text{Spec } A$ . Then this construction yields the actual group quotient. The quotient of  $\mathbb{A}^\times$  by the free  $\mu_\ell$  action gives a scheme.

**Lemma 5.20.** Algebraic spaces are stable by free quotients of covering group 0-stacks.

*Proof.* The map  $X \rightarrow L_T(X/G)$  is fibered in covering 0-stacks, so in particular covering 0-stacks. As  $X$  is a geometric 0-stack, the quotient is a geometric 0-stack as well. This follows by the description in , choosing a geometric atlas of  $X$  and postcomposing this to get a geometric atlas of the quotient.  $\square$

**Example 5.21.** If  $p : \sum_{r:R} S_r \rightarrow R$  be a map between formally étale + flat affine into  $R$  whose fibers, except possibly the fiber over 0, are formally étale + flat. Define  $U = (x \neq 0) \times S_x \subset S_x$ .  $y \sharp y' \equiv y \neq y'$  is an irreflexive  $\mathbb{T}$ -flat relation on  $S_x$ . From this we obtain the algebraic space

$$\sum_{x:R} Y_x / R_\sharp$$

which we will later recognize as a fiber collapse.

*Proof.*  $\sharp$  is a modal irreflexive relation. By assumption we have given  $\mathbb{T}$ -flatness of  $S_x$  if  $x \neq 0$ .  $\square$

**Lemma 5.22** (Not needed). Let  $Y$  be affine. Let  $X \hookrightarrow Y$  be a map fibered in locally closed propositions. Then its factors as the composite of a closed and then an open embedding

*Proof.* By zariski local choice we find  $Y = \bigcup Y_i$  and factorizations of the basechanges  $X_i \rightarrow Z_i \rightarrow Y_i$ . Then  $\bigcup X_i \rightarrow \bigcup Z_i \rightarrow \bigcup Y_i = Y$  is a global factorization.  $\square$

**Proposition 5.23.** Let  $S$  be affine with a regular point  $*$ . Assume we have function  $g : S \rightarrow S$  such that  $*$  is the unique fixpoint  $*$  (e.g. if  $(S, *)$  admits a pointed-free action of a nontrivial group) Let  $\sharp$  be an irreflexive  $\mathbb{T}$ -flat relation on  $U \equiv \sum_{x:S} x \neq *$ , such that for all  $y : U$ , we have  $gy : U$  and  $y \sharp gy$ . Then the algebraic space  $S/R_\sharp$  is non locally separated, in particular not a scheme.

*Proof.* It is an algebraic space by the previous prop.

A pointed-free action of a non-trivial group yields such a map  $g$ : If  $\neg(G = \{1\})$ , then  $\neg\neg(G \setminus \{1\})$  by decidable equality of  $G$ . As we want to prove a contradiction, we may assume  $g : G \setminus \{1\}$ , this yields a map  $S \rightarrow S$  such that

- $*$  is the unique fixpoint by the pointed-freeness
- If  $y \neq *$ , then  $gy \neq *$  and  $y \sharp gy$

We have that every scheme  $X$  is locally-separated, i.e. its identity types are locally closed. Indeed, this follows from the proof of Foundations Prop 5.5.2 .

Let us show that  $R$  is not valued in locally closed propositions. Recall

$$y \in U \rightarrow y \sharp gy \tag{1}$$

$$y \in U \leftrightarrow y \neq * \tag{2}$$

We have for any  $y : S$

$$R_\sharp(y, gy) \simeq (y = gy) + (y \in U) \wedge y \sharp gy \stackrel{(1)}{\simeq} (y = *) + (y \in U) \stackrel{(2)}{\simeq} (y = *) + (y \neq *)$$

But if this is locally closed for all  $y : S$ , we have a contradiction to 2.4.  $\square$

**Corollary 5.24.** *Let  $Y : R \rightarrow \text{Aff}$  be formally étale + flat affine away from the origin, such that  $p : \tilde{Y} = \sum_{x:R} Y_x \rightarrow R$  is regular and  $Y_0$  is infinitesimal. If you find a map  $g : \tilde{Y} \rightarrow \tilde{Y}$  over  $p$  with a unique fixpoint, which lies over 0, then the algebraic space  $-Y_\bullet-$  is non-locally-separated, In particular not a scheme.*

*Proof.* Lets call the unique fix point  $*$ , i.e. we have

$$gy = y \leftrightarrow y = *$$

Note that  $* : \tilde{Y}$  is a regular point, as  $p : \tilde{Y} \rightarrow R$  is a regular section with  $Y_0$  infinitesimal.  $\square$

**Definition 5.25.** A pointed-free action of  $G$  on a pointed type  $(X, 0)$  is a  $G$ -action with fixpoint 0, such that if  $g\varepsilon = \varepsilon$  for some  $g \neq 1$ , then  $\varepsilon = 0$ .

**Lemma 5.26.** *Let  $G$  be a group with decidable equality acting pointed free on a pointed type  $(X, 0)$ . Then  $G$  acts free away from zero.*

*Proof.* let  $x, y \neq 0$ . We need to show, that  $\sum_g gx = y$  is a proposition. Let  $g, g' : G$  such that  $gx = y = g'x$ . as  $G$  has decidable equality, we may show  $\neg\neg(g = g')$ . If  $g^{-1}g' \neq 1$ , then by pointed-freeness applied to  $g^{-1}g'x = x$ , we have  $x = 0$ . Contradiction.  $\square$

**Corollary 5.27.** *Let  $0 : \text{Spec } A$  be a regular point. Let  $G$  be a nontrivial formally étale flat affine group acting pointed- freely on the pointed affine  $(\text{Spec } A, 0)$ . Then the pointed-free quotient of  $\text{Spec } A$  by  $G$  from 5.18 is non-locally-separated, In particular not a scheme.*

**Example 5.28** (Non locally-separated examples). Assume  $\ell \neq 0$  prime. Let  $\mu_\ell$  act on  $(\text{Spec } B, 0)$  in one of the following ways:

1. Let  $\mu_\ell$  act on  $\text{Spec } B = \mathbb{A}^1$
2. Let  $\mu_\ell$  act on

$$\text{Spec } B \equiv \sum_{x, y: R} x^\ell = y^\ell$$

$$\text{via } g(x, y) = (x, gy)$$

Then  $\text{Spec } B /_{0^c \mu_\ell}$  is an algebraic space that is not a scheme.

*Proof.*  $\neg\neg$  merely,  $\mu_\ell$  is finite ([ref?]) and  $\mu_\ell \setminus \{1\}$  is inhabited by 5.12.

1. Pointed-Free action is clear.  $0 : \mathbb{A}^1$  is regular by first projection.
2. Pointed-Free action is clear. The cross midpoint regular, witnessed by the first projection: It is regular vanishing at  $(0, 0)$  And for any point  $(0, y) : \text{Spec } B$  we deduce  $y^\ell = -0^\ell = 0$ , hence  $\neg\neg(x, y) = (0, 0)$ .

$\square$

**Question 3.** If  $\mu_\ell$  acts on  $Y$  some affine, does every  $\mu_\ell$ -invariant  $\phi : Y \rightarrow R$  is invariant on a  $\ell$ -neighborhood?

## 5.5 Obsolete

**Proposition 5.29.** *Let  $Y : R \rightarrow \text{Aff}$  be formally étale + flat affine away from the origin. If you find two sections  $y, y' : \prod_{x:R} Y_x$  such that  $y_x = y'_x \leftrightarrow x = 0$ , then the algebraic space  $-Y_\bullet-$  is non-locally-separated, In particular not a scheme.*

*Proof.* It is an algebraic space by the previous prop.

We have that every scheme  $X$  is locally-separated, i.e. its identity types are locally closed. Indeed, this follows from the proof of Foundations Prop 5.5.2 .

Let us show that  $R$  is not valued in locally closed propositions. We have

$$\prod_{x:R} \eta y_x = \eta y'_x \simeq \prod_{x:R} y_x = y'_x + (x \neq 0) \times y_x \neq y'_x \simeq (x = 0) + (x \neq 0)$$

but if this is locally closed for all  $x : R$ , we have a contradiction to 2.4. □

**Lemma 5.30** (Not needed). *For an algebraic space  $X$ , we have implications  $1 \Rightarrow 2 \Rightarrow 3$*

1.  $X$  admits an seperated open cover.
2. For any covering equivalence relation  $R : U^2 \rightarrow \text{Prop}$  on an affine  $U$  such that  $X = U/R$ ,  $F$  is valued in locally closed propositions
3. We find such a presentation such that  $R$  is valued in locally closed propositions.

*Proof*  $1 \Rightarrow 2$  Let  $X' \rightarrow X$  be a map fibered in merely inhabited finite sums of open propositions with  $X'$  a seperated algebraic space. Then any geometric atlas  $U \rightarrow X'$  will be fibered in closed subtypes of  $U$ . We need to show, that the fibers of  $U \rightarrow X' \rightarrow X$  are locally closed subtypes of  $U$ . Let  $x : X$ . the fiber in  $X'$  is of the form  $U_1 + \dots + U_n$ . Thus the fiber in  $U$  is a finite sums of  $\sum$  of  $U_i \rightarrow (U \rightarrow \text{ClosedProp})$ , which is enough.

$3 \Rightarrow 1$  Let  $x : X$ . □

**Lemma 5.31** (Not needed). *Let  $\text{char} \neq 2$ . Let  $p : R[X]$  be such that  $0 \in D(p)$  and  $x \in D(p)$  implies  $-x \in D(p)$ . If  $f : R[X]$  is a polynomial such that  $f(x) = f(-x)$  for all  $x : D(p) \setminus \{0\}$ , then  $f$  is even i.e. in the image of  $R[X^2] \hookrightarrow R[X]$ .*

*Proof.* We splitting  $f$  into  $f_1 + X f_2$  for  $f_i : R[X^2] \subset R[X]$ . I claim, that  $f_2 = 0$  in  $R[X]$ . realizing that  $(X f_2)(x) = (X f_2)(-x)$  implies  $2 f_2(x) x = 0$ , thus  $f_2(x) x = 0$  for all  $x : D(p) \setminus 0 = D(pX)$ , thus by the previous lemma  $X \cdot f_2 = 0$  in  $R[X]$ , hence  $f_2 = 0$ . □

**Lemma 5.32.** *Let  $G$  be a finite group whose cardinality is invertible in  $R$ . Let  $G$  act on an affine scheme equipped with a fixpoint  $0$ . Let  $U$  be an open neighborhood of  $0$ , such that  $g(U) = U$  for all  $g : G$ . Then we find some  $G$ -invariant  $p$  such that  $0 \in D(p) \subset U$ .*

*Proof.* Choose a principal open neighborhood  $0 \in D(p) \subset U$ .  $G$  acts on  $R[X]$ , via  $(g.p)(x) = p(gx)$ . Then

$$p' = \sum_{g:G} g.p : R[X]$$

is a  $G$ -invariant polynomial, in particular  $D(p)$  is  $G$ -invariant. Moreover  $0 \in D(p')$  as

$$p'(0) = \sum_{g:G} p(g(0)) = \sum_{g:G} p(0) = |G| \cdot p(0)$$

is invertible, as  $|G|$  and  $p(0)$  are both invertible. Furthermore, as  $U$  was  $G$  invariant and contained  $D(p)$  it also has to contain  $D(p')$ : Indeed

$$D(p') \subset \bigcup_g D(g.p) \subset U$$

□

**Lemma 5.33.** *Let  $G$  be a formally étale + flat affine group, such that  $\neg\neg$  its finite, with cardinality invertible in  $R$  and  $G \setminus \{1\}$  inhabited. Let it act on an affine scheme  $\text{Spec } A$  with a fixpoint  $0$ . Let  $R$  be a relation on  $\text{Spec } A$  such that*

- $R(x, y)$  implies that there merely is some  $g$  with  $y = gx$ .
- $\neg\neg R(x, gx)$

*Assume that for all  $p : A^G$  with  $0 \in D(p)$ ,  $D(p)/R$  is not an affine scheme. Then  $\text{Spec } A/R$  is not a scheme.*

*Proof.* Assume  $0$  admits a open affine neighborhood  $U$  in  $\text{Spec } A/R$ . The preimage along the quotient map obtained from the relation induces a open neighborhood  $V$  of  $0$  in  $\text{Spec } A$ . As we want to prove a contradiction we may assume that  $\mu_\ell$  consists of  $\ell$  many elements, where  $\ell \neq 0$  in  $R$ . Note that  $V$  is  $G$ -invariant: For any  $x \in V, g : G$ , the goal  $gx \in V$  as an open proposition is  $\neg\neg$ -stable, thus we may assume  $R(x, gx)$ .

We apply the previous lemma to  $V$  to obtain an invariant principal open neighborhood  $0 \in D(p) \subset V \subset \text{Spec } A$ . As  $p$  is  $G$ -invariant,  $p : \text{Spec } A \rightarrow R$  descends to  $X \rightarrow R$ . Restricting to  $U'$  yields a map  $p' : U \rightarrow R$ , such that setting  $U' \equiv D(p')$  yields  $q^{-1}(U') = q^{-1}(D(p')) = D(p' \circ q) = D(p)$ . We are now in the following situation

$$\begin{array}{ccccc} D(p) & \hookrightarrow & V & \hookrightarrow & \text{Spec } A \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow q \\ U' & \hookrightarrow & U & \hookrightarrow & X \end{array}$$

where  $U'$  is an open affine neighborhood of  $0$ .

By assumption  $U = D(p)/\sim'$  cannot be affine. Contradiction.  $\square$

**Proposition 5.34** (Not needed). *Let  $\ell \neq 0$  be prime. Let  $\mu_\ell$  act on  $\text{Spec } B$  with fixpoint  $0$ . . Let  $V$  be an infinitesimal neighborhood of  $0$ , i.e. a subtype  $0 \in V \subset \text{Spec } B$  such that  $\neg\neg x = 0$  for every  $x : V$ . Assume*

*Strong freeness* We find some  $0 \in V' \subsetneq V$  for any  $\varepsilon : \text{Spec } B, g \neq 1, g\varepsilon = \varepsilon$  implies  $\varepsilon \in V'$

*checking away from  $0$*  For any  $p : B$  and any  $\phi : R^{D(p)}$  such that  $\phi|_{D(p) \setminus \{0\}} = 0$ , we have that  $\phi|_V = 0$ .

*The sheaf quotient of  $\text{Spec } B$  by the relation as above is an algebraic space but not an affine scheme.*

*Proof.* • Let us check the conditions on the relation

- If  $R(x, y)$  then either  $x = y$  putting  $g = 1$  or in the second case we get some  $g$  such that  $gx = y$
- Let  $x : X, g : G$ . Assume  $\neg R(x, gx)$ , i.e.  $x \neq gx$  and  $\neg\neg x = 0$ . But  $0$  was assumed to be a fixpoint, hence  $\neg\neg gx = x$ .

- Let  $p : B$  be as above. We have to show that the quotient of  $D(p)$  is not affine. The conditions on  $p$  give  $p(0) \neq 0$  and  $p(x) \neq 0 \rightarrow p(gx) \neq 0$  for all  $g : \mu_\ell$ .

Lets call this quotient  $X$ .

Define

$$A = \{ \phi : R^{D(p)} \mid \phi|_{D(p) \setminus \{0\}} \text{ is } \mu_\ell\text{-invariant} \}$$

This is an  $R$ -subalgebra: for any  $r : R, r : B_p$  is  $\mu_\ell$ -invariant.  $\mu_\ell$ -invariant functions are stable under addition and multiplication.

Claim: The affinization map of  $X$  is the induced dashed map  $f : X \rightarrow \text{Spec } A$  in

$$\begin{array}{ccc} D(p) & \xlongequal{\quad} & \text{Spec } R^{D(p)} \\ \downarrow q & & \downarrow q' \\ X & \dashrightarrow^{\exists! f} & \text{Spec } A \end{array}$$

Proof: A function  $\phi : D(p) \rightarrow R$  factors through  $q$  iff  $\phi|_{D(p) \setminus \{0\}}$  is  $\mu_\ell$ -invariant. Thus the embedding (using that  $R$  is a sheaf)  $R^X \hookrightarrow R^{D(p)}$  has image  $A$   $\square$ (Claim).

Proof that  $X$  is not an affine: Assume that  $X$  were affine. Then the map  $f$  would be in particular an embedding. We may assume a term  $g : \mu_\ell \setminus \{1\}$ : Indeed, as we want to prove a contradiction we may assume a term in  $g : \text{Spec } R[X]/(\sum_{i=0}^{\ell-1} X^i)$ . But this type is equivalent to  $\mu_\ell \setminus \{1\}$ , using that  $\sum_{i=0}^{\ell-1} X^i | X^\ell - 1$  and  $\ell \neq 0$ . The given infinitesimal neighborhood  $V$  satisfies  $V \subset D(p)$ , using that invertibility is  $\neg \neg$  stable. Then for any  $\varepsilon : V$  we have

$$(q\varepsilon =_X q(g\varepsilon)) \stackrel{??}{=} (\varepsilon = g\varepsilon) + (\varepsilon \neq 0 \wedge \sum_{h \neq 1} \varepsilon = hg\varepsilon) = (\varepsilon = g\varepsilon) = (\varepsilon \in V')$$

where the last step comes from pointed-freeness. But we have

$$(q'\varepsilon =_{\text{Spec } A} q'(g\varepsilon)) = \left( \prod_{\phi:A} \phi(q'\varepsilon) = \phi(q'(g\varepsilon)) \right) = \prod_{\substack{\phi:R^{D(p)} \\ \phi \in A}} \phi(\varepsilon) = \phi(g\varepsilon),$$

The right hand side is inhabited: For any  $\phi : D(p) \rightarrow R$  such that  $\psi := \phi - g.\phi$  satisfies  $\psi|_{D(p) \setminus \{0\}} = 0$  we have  $\psi|_V = 0$  by 'checking away from 0', inparticular  $\psi(\varepsilon) = 0$ . So we conclude the the embedding  $V' \hookrightarrow V$  is an equivalence. But we asked  $V' \subsetneq V$  to be a proper subset.  $\square$

**Example 5.35.** Let  $\mu_\ell$  act on  $\text{Spec } B = \mathbb{A}^1$ .

*Proof.* 1. Put  $V \equiv \text{Spec } R[X]/X^n$  for some  $n > 1$ .

2. As  $(g - 1)$  is invertible,  $((g - 1)\varepsilon = 0)$  gives us  $\varepsilon \in \{0\} \equiv V' \subsetneq V$ . Note that indeed  $V$  is non contractible, because  $R[X]/X^n \rightarrow R[X]/X$  is not an algebra isomorphism
3. We have to show, that then  $\phi$  is  $\mu_\ell$  invariant. We can apply 2.3, observing  $\phi - g.\phi = 0$  on  $D(X/1) \subset \text{Spec } B_p$ , where  $X/1 : B_p$  is regular, because  $X$  is regular in  $B$ . TODO as each  $\phi$  satisfies the cond.  $\square$ (Claim)

$\square$

**Example 5.36.** Assume  $2 \neq 0$ . Let  $\mu_2$  act on

$$\text{Spec } B \equiv \sum_{x,y:R} xy = 0$$

via the swap. Then  $\text{Spec } B/R$  is an algebraic space but not a scheme.

*Proof.* 1. Put  $V = \text{Spec } R[X]/X^k \subset \text{Spec } B$ ,  $k > 2$ .

2. If  $(x, y) = (y, x)$  but  $xy = 0$  we get  $x \in V' \equiv \text{Spec } R[X]/X^2$ .

3. Let  $\phi : D(p) \rightarrow R$  be 0 everywhere except near the origin. Then we get a restricted map  $\phi' : D(p') \rightarrow R$  where  $D(p') \subset V(X)$  is given by the intersection  $D(p) \cap V(X)$ . Indeed : Put  $p' : R[X]$  the image of  $p : R[X, Y]/(XY)$  and the map induced by evaluating  $Y$  at 0. Here we can apply 2.3, getting that  $\phi'$  is 0 everywhere in particular in  $V \subset V(X)$ .  $\square$

## 5.6 Locally seperated examples

**Lemma 5.37** (not needed). *Given a map  $P : \text{Susp}(Q) \rightarrow \text{Prop}$ , such that  $P(N)$  and  $P(S)$  hold, then  $\prod_{t: \text{Susp}(Q)} P(t)$*

**Lemma 5.38** (not needed). *Assume  $2 \neq 0$ . For any  $x : R$ , the map*

$$\begin{aligned} \text{Susp}(x \neq 0) &\rightarrow \sum_{y: R/x} y^2 = 1 \\ N &\mapsto 1 \\ S &\mapsto -1 \end{aligned}$$

*is well-defined and an equivalence.*

*Proof.* The following maps are mutually inverse

$$\begin{aligned} \sum_{y: R/x} y^2 = 1 &\simeq \sum_{e: R/x} e^2 = e \\ y &\mapsto (y - 1)/2 \\ 2e - 1 &\mapsto e \end{aligned}$$

So it remains to show that the map

$$\begin{aligned} f : \text{Susp}(x \neq 0) &\rightarrow \sum_{e: R/x} e^2 = e \\ N &\mapsto 1 \\ S &\mapsto 0 \end{aligned}$$

is a bijection.

- It is injective, i.e. for all  $p, q : \text{Susp}(x \neq 0)$ , if  $f(p) = f(q)$ , then  $p = q$ . As the latter is a proposition, we may assume  $p, q$  beeing combinations of north and south poles. The interesting case is if wlog  $p = N, q = S$ . Then assuming  $0 =_{R/x} 1$  means  $R/x = 0$ , i.e.  $x \neq 0$ , thus  $N = S$  in  $\text{Susp}(x \neq 0)$ .
- It is surjective: Choose  $e : R$ , such that  $e^2 = e$  in  $R/x$ . By locality in  $R$ ,  $e$  or  $1 - e$  is invertible in  $R$ , thus in  $R/x$ . By  $e^2 = e$  we deduce  $e = 0$  or  $e = 1$  in  $R/x$ , both lie in the image of  $f$ .  $\square$

**Example 5.39** (Not needed). *Let  $L = \sum_{x: \mathbb{A}^1} \text{Susp}(x \neq 0) = \sum_{x: \mathbb{A}^1} \sum_{y: R/x} y^2 =_{R/x} 1$  be the line with two origins.*

**Lemma 5.40** (Not needed). *Let  $2 \neq 0$ . Let  $y, y' : A$  be two elements of an fp-algebra, whose squares coincide and such that  $y$  is invertible. Then  $y =_A y'$  is formally étale*

*Proof.* We may assume that  $A = R$ , as equality in  $A$  can be checked pointwise and formally étale is a modality. We may show its  $\neg\neg$ -stable. Assume  $\neg\neg(y =_R y')$ , i.e.  $y - y'$  beeing nilpotent in  $A$ . So pick  $n$  large enough, such that  $(y - y')^{2^n} = 0$ . Proof by induction over  $n$ . If  $n = 0$ , then its fine. Induction step  $n \mapsto n+1$ . Let  $(y - y')^{2^{n+1}} =_R 0$ , then  $(2y^2 - 2yy')^{2^n} = 0$ , or  $(y(y - y'))^{2^n} = 0$ , as  $y$  is invertible,  $(y - y')^{2^n} = 0$ , so by induction hypothesis  $y = y'$ .  $\square$

## 5.7 FiberCollaps away from the origin

OUTDATED!

**Example 5.41.** *—Bool— is the line with two origins.*

*—Spec  $R[X]/(X^2 + 1)$ — is the twisted line with two origins, i.e. over the origin we have the roots of  $-1$ .*

*—Spec  $R[Y]/(Y^2 - \bullet^2)$ — is the quotient of the cross, that looks like  $\mathbb{D}(1)$  over the origin.*

*—Spec  $R[Y]/(\bullet Y)$ — is the affine Plus.*

## 5.8 Schemes do not have descent

For this section, let  $\rho : R \setminus \{0\}$  denote a term, e.g.  $\rho = 1$ . Set  $C = R[T]/(T^2 + \rho)$ .

**Proposition 5.42** (OUTDATED! Copy from talk!). *If —Spec  $C$ — is a scheme, then  $X^2 + \rho$  has a root.*

**Corollary 5.43.** *Schemes do not have descent.*

*Proof.* If Schemes have descent, then  $\text{—Spec } R[T]/(T^2 + \rho)\text{—} \in \mathbf{Sch}$  is a sheaf. As  $\text{—Spec } R[T]/(T^2 + \rho)\text{—}$  is  $\mathbb{T}$ -merely a scheme, it is a scheme, so by the previous lemma  $T^2 + \rho$  has a root. Contradiction to [1] A . 0.3.  $\square$

## 5.9 Gluing in an affine on the line

**Definition 5.44.** Let  $Y$  be an affine. The  $n$ .th order gluing of  $Y$  on the line is given by the sheaf

$$L_n(X) = \sum_{x:R} Y^{x^n=0}$$

**Lemma 5.45.** *If  $Y = \text{Spec } R[T]/f$ , we have*

$$L_n(X) = \sum_{x:R} \sum_{y:R/x^n} f(y) =_{R/x^n} 0$$

*Proof.* For any  $R$ -algebra  $A$  (e.g.  $R/x^n$ ) we have by the universal property of  $R[T]/f$

$$\sum_{y:A} f(y) =_A 0 = \text{Hom}_R(R[T]/f, A) = Y^{\text{Spec } A}$$

$\square$

**Lemma 5.46.** *If  $Y$  is formally étale , then the map over  $R$*

$$\begin{array}{ccc} R \times Y & \xrightarrow{\quad} & L_n(Y) \\ & \searrow & \swarrow \\ & R & \end{array}$$

*pulls back to an equivalence over  $\mathcal{N}_\infty(0)$ .*

*If  $Y$  is formally unramified, then  $L_n(x)$  is locally seperated.*

*Proof.* Indeed, the diagonal map

$$Y \rightarrow Y^{x^n=0}$$

is an equivalence, as for any  $\neg\neg x = 0$ ,  $x^n = 0$  is a closed dense proposition and  $Y$  is formally étale.

If  $Y$  is formally unramified, then the identity types look like

$$(x, y) =_{L_n(Y)} (x', y') \simeq (x = x') \times (x^n = 0 \rightarrow Q)$$

where  $Q$  is an open proposition such that for any  $p : x^n = 0$  we have  $Q \equiv yp = y'p$ . Indeed by the proof of ?? we can find a filler of  $y_\bullet = y'_\bullet : P \rightarrow \mathbf{Open}$ . By [1](4.2.11) this proposition is locally closed.  $\square$

**Question 4.** Is the map  $\sum_{y:R/x^3} y^2 = 0 \rightarrow \sum_{y:R/x^2} y^2 =_0$  surjective? This is how i understand David Madore.

**Lemma 5.47.** *For  $\varepsilon : \mathcal{N}_\infty(0)$ , the affine  $\text{Ann}(\varepsilon) = \{x : R \mid x\varepsilon = 0\}$  is not $_\varepsilon$  formally smooth. In particular  $R \rightarrow R/\varepsilon$  is not $_\varepsilon$  a geometric cover.*

*Proof.* We have the map  $1 : (\varepsilon = 0) \rightarrow \text{Ann}(\varepsilon)$ . Assume there is a filler  $x : \text{Ann}(\varepsilon)$ , i.e.  $(\varepsilon = 0) \rightarrow x = 1$ . Then not not,  $x = 1$ , i.e.  $(x - 1)^n = 0$  for  $n$  large enough. Hence

$$0 = \varepsilon(x - 1)^n = \varepsilon x(\dots) + (-1)^n \varepsilon = (-1)^n \varepsilon$$

as desired.  $\square$

**Lemma 5.48 (TODO).** *If  $Y$  is formally étale + flat affine, then  $L_1(Y)$  is an algebraic space.*

*Proof.* Recall the closed modality associated to a proposition  $P$ , given by  $P \star \_$ . We can define a map

$$\begin{aligned} f : (x \neq 0) \star Y &\rightarrow Y^{x=0} \\ y &\mapsto \Delta(y) \end{aligned}$$

where we check, that if  $x \neq 0$  holds, then indeed  $Y^{x=0}$  is contractible.

$f$  is a bijection:

- injectivity: Given two terms of the domain, as the map out of  $Y$  is  $\mathbb{T}$ -surjective (and the goal is a sheaf), we may assume that they are of the form  $\text{inl}(y), \text{inl}(y')$  for  $y, y' : Y$ . Then if  $\Delta(y) = \Delta(y')$  we have  $(x = 0) \rightarrow (y = y')$ . As  $y = y'$  is open, we have  $(x \neq 0) \vee (y = y')$ . If  $x \neq 0$ , then  $\text{inl}(y) = \text{inl}(y')$  by the construction of the join.

- surjectivity: TODO

$\square$

**Question 5.** Is  $L_2(\mathbb{D}(1))$  an algebraic space or fppf-geometric 0-stack? For this: Is

$$\begin{aligned} (\text{Spec } R[X, Y]/X^2 - Y^2)/\sim &\rightarrow L_2(\mathbb{D}(1)) = \sum_{x:R} \sum_{y:R/x^2} y^2 = 0 \\ (x, y) &\mapsto (x, [y]) \end{aligned}$$

an equivalence? Here we mod out the relation generated by  $(x, -x) \sim (x, x) \forall x \neq 0$ .

This is equivalent to : For any  $x : R$ , is the map

$$(x \neq 0) \star \text{Spec } R[Y]/(Y^2 - x^2) \rightarrow \mathbb{D}(1)^{x^2=0}$$

an equivalence?



**Example 5.49.** *I suggest a new definition of fppf topology: We take the topology generated by the Zariski topology and algebras of the form  $R[X]/f$  where one of coefficients of  $f$  is invertible (non necessarily the leading coefficient). This is still a free module hence fppf.*

## 5.10 Weakly-flat stacks

**Definition 5.50.** We call a geometric stack  $X$  weakly-flat iff one of the following conditions is satisfied

1.  $\|X\|_{-1}^{\mathbb{T}} \rightarrow X \in \mathbf{CS}$
2. For any geometric atlas  $W \rightarrow X$ ,  $W$  is weakly-flat, i.e  $\|W\|^{\mathbb{T}} \rightarrow W \in \mathbb{T}$ .

*Proof.*

1  $\Rightarrow$  2 Choose a geometric atlas  $W \rightarrow X$ . In particular its  $\mathbb{T}$ -surjective, hence we have  $\|W\|^{\mathbb{T}}$ , so by assumption  $W \in \mathbb{T}$ . So  $X \in \mathbf{CS}$ .

2  $\Rightarrow$  1

$$\|W\|^{\mathbb{T}} \rightarrow \|X\|^{\mathbb{T}} \rightarrow X \in \mathbf{CS} \stackrel{??}{\rightarrow} W \in \mathbb{T}$$

□

They behave bad as they are not stable under  $\sum$  (and not under id-types, although this holds for affines).

**Lemma 5.51.** For any weakly-flat geometric stack  $X$ ,  $\|X\|_{-1}^{\mathbb{T}}$  is a geometric stack.

*Proof.*  $X \rightarrow \|X\|_{-1}^{\mathbb{T}}$  is a geometric cover. □

**Proposition 5.52.** We may define  $X$  to be 0-wf-seperated, iff its weakly flat and  $n+1$ -wf-seperated, iff identity types of  $X$  are  $n$ -wf-seperated.

For  $X$  a geometric stack, TFAE

1.  $X$  is  $n+1$ -wf-seperated, i.e. all  $n+1$ -fold identity types of  $X$  are weakly-flat.
2. For any  $x$ ,  $\Omega^{n+1}(X, x)$  is covering.
3. For any  $x : X$ ,  $x = x$  is  $n$ -wf-seperated, i.e.  $n$ -fold identity types of  $x = x$  are weakly flat.

*Proof.*

1  $\Rightarrow$  3  $\Rightarrow$  2 ez

3  $\Rightarrow$  1 We prove this by induction.  $n = 0$ . To show that  $x =_X y$  is weakly-flat, by descent we may assume that  $x = y$ . Then we have  $(x = y) \simeq (x =_X x)$ . By assumption this is weakly flat.

Assume now, that for any  $x : X$ , that  $x = x$  is  $n$ -wf-seperated. Let  $x, y : X$ . We want to show that  $x = y$  is  $n$ -wf-seperated. By induction we may just prove that for any  $p : x = y$ ,  $p = p$  is  $n-1$ -wf-seperated. Applying  $p \cdot _-$  induces an equivalence  $\mathbf{refl}_x = \mathbf{refl}_x \simeq p = p$ . But  $x = x$  is  $n$ -wf-seperated, hence  $\mathbf{refl}_x = \mathbf{refl}_x$  is  $n-1$ -wf-seperated.

2  $\Rightarrow$  3 Induction.  $n = 0$  is fine. Let  $x : X$ . To show that  $\Omega(X, x)$  is  $n$ -wf-seperated, just use that  $\Omega^n(\Omega(X, x))$  is covering, hence by the inductive statement 2  $\Rightarrow$  3  $\Rightarrow$  1, we now that  $\Omega(X, x)$  is  $n$ -wf-seperated. □

## 6 Omega-stability and gerbes

**Definition 6.1.** A geometric stack  $X$  is an  $n$ -gerbe iff the map  $\eta_n^{\mathbb{T}} : X \rightarrow \|X\|_n^{\mathbb{T}}$  is a geometric cover.

**Example 6.2.** If  $G$  is a covering group sheaf, then  $BG$  is a 0-gerbe.

**Example 6.3.** It may happen, that  $\|X\|_n^{\mathbb{T}}$  is a geometric  $n$ -stack for  $X$  a geometric stack, although  $X$  is not an  $n$ -gerbe. Indeed: Put  $n = 0$  and  $X$  any pointed  $\mathbb{T}$ -connected geometric stack that is not covering, like  $\text{Susp}(1 + x = 0)$  for some

**Theorem 6.4.** Assume that Covering stacks are  $\Omega$ -stable, Then every geometric stack is a 1-gerbe.

*Proof.* By ??, we need to show that for any  $x : X$ ,  $\Omega^2(X, x)$  is covering. choose an geometric atlas  $f : S \rightarrow X$ . by descent we may only show that  $\Omega^2(X, fs)$  for  $s : S$  is covering.

$$\Omega\left(\sum_{t:S} ft = fs\right) \simeq \left(\sum_{p:\Omega(S,s)} \text{tp}_p(\text{refl}_{fs}) = \text{refl}_{fs}\right) \simeq \text{refl}_{fs=fs} \text{refl}$$

where the last equivalence is obtained, as  $\Omega(S, s)$  is contractible with center  $\text{refl}_s$ . So  $\Omega^2(X, fs)$  is the loop space of a covering stack, hence by assumption covering.  $\square$

**Corollary 6.5.** Any Deligne Mumford Stack is a 1-gerbe

*Proof.* Use that étale topology is lex-flattened and ??.  $\square$

**Proposition 6.6.** This proposition seems only interesting for  $n = 0$  by the previous theorem. Assume that covering stacks are  $\Omega$ -stable. Then  $X$  is an  $n$ -gerbe iff  $\Omega^{n+1}(X, x)$  is covering for all  $x : X$

*Proof.* One direction is ??. The other follows  
By applying iteratively ??

$$\begin{aligned} \Omega^{n+1}(\text{fib}(\eta_n^{\mathbb{T}} X)|x|) &\simeq \Omega^n \text{fib}(\eta_{n-1}^{\mathbb{T}}(\Omega(X, x)))pt \simeq \dots \\ &\simeq \Omega^{n-k} \text{fib}(\eta_{n-k-1}^{\mathbb{T}} \Omega^{k+1}(X, x))pt \simeq \dots \\ &\simeq \text{fib}(\eta_{-1}^{\mathbb{T}} \Omega^{n+1}(X, x))pt \\ &\simeq \Omega^{n+1}(X, x) \end{aligned}$$

The LHS is covering by  $\Omega$ -stability.  $\square$

We can reprove ?? by just observing that  $\mathbb{T}$ -flat geometric stacks have covering loop spaces.

**Remark 3.** Put  $\mathbb{T}$  the étale topology. Observe, that we have an analogous statement if we replace covering stack by formally étale :

1.  $\eta_0^{\mathbb{T}} X : X \rightarrow \|X\|_0^{\mathbb{T}}$  is formally étale
2.  $X \rightarrow \|X\|_0^{\mathbb{T}}$  is formally unramified
3. for every  $x : X$ ,  $\Omega(X, x)$  is formally étale .

*Proof* 1  $\Leftrightarrow$  2 Observe that the map  $\eta_0^{\mathbb{T}}$  is  $\mathbb{T}$ -smooth.

2  $\Rightarrow$  3 okay as the fibers of  $\eta_0^{\mathbb{T}}$  embed into  $X$ .

3  $\Rightarrow$  2 Let  $x, y : X$  be  $\mathbb{T}$ -merely equal. The goal is  $\text{FormallyEtale}(x = y)$  is a sheaf, so we may assume that  $x = y$ .  $\square$

**Corollary 6.7.** If covering stacks are  $\Omega$ -stable, then identity types of geometric stacks are 0-gerbes.

*Proof.* We need to check, that identity types of a 1-gerbe  $X$  are 0-gerbes. So assume  $p : x = y$ . Then

$$\Omega(x = y, p) = \Omega(x = x, \text{refl}) = \Omega^2(X, x)$$

which is covering as  $X$  is a 1-gerbe.  $\square$

## 7 Questions // TODO

**Theorem 7.1** (TODO). *An Artin stack  $X$  is Deligne Mumford iff one of the following conditions is satisfied:*

1. *There exists a geometric atlas  $W \rightarrow X$*
2. *The identity types of  $X$  are  $\mathbb{P}$ -separated*

*Proof.*  $\Rightarrow 2$ . ??

2.  $\Rightarrow 1$  Residual ??? [06MF]

□

Prove ???!!

**Question 6.** if  $\mathbb{T} \subset \mathbb{T}'$  do we have that for each  $X : \mathbf{GS}_{\mathbb{T}} L_{\mathbb{T}'} X \in \mathbf{GS}_{\mathbb{T}'}$ ?

**Theorem 7.2** (TODO). *The class of flat affines is stable under  $\sum$ . Moreover flatness can be defined fiberwise.*

## 8 Not clear where to put that

**Lemma 8.1.** *Let  $\rho \neq 0$ .  $\mathrm{Spec} R[T]/(T^2 + 1)$  is compact.*

*Proof.* Let  $U \subset \mathrm{Spec} C$  be open. Then we find  $f_1, \dots, f_n : C$ , such that  $U = D(f_1, \dots, f_n)$ . Choose representatives  $f_i = a_i + b_i T \pmod{T^2 + \rho}$ . Then consider the following numbers

$$r_{ij} = \begin{cases} a_i b_j - a_j b_i & , i \neq j \\ a_i^2 + \rho b_j^2 & , i = j \end{cases}$$

We will show that  $D((r_{ij})_{i,j}) \leftrightarrow (\mathrm{Spec} C \subset U)$ .

' $\rightarrow$ '

Assume  $r_{ij} \neq 0$ . If  $i = j$ , then  $\mathrm{Spec} C \subset D(f_i) \subset U$ .

If  $i \neq j$ , then  $\mathrm{Spec} C \subset D(f_i, f_j) \subset U$ .

' $\leftarrow$ '

Because this statement is a propositional sheaf, we may assume a term  $x : \mathrm{Spec} C$ . Choose  $i, j$ , s.th.  $x \in D(f_i)$ ,  $-x \in D(f_j)$ . In both cases  $i = j$  and  $i \neq j$ , then  $r_{ij} \neq 0$ . □

**Lemma 8.2** (Not needed). *Open subtypes of  $\mathbb{A}^1$  are  $\neg\neg$  principal open.*

*Proof.* • An open affine subscheme of  $\mathbb{A}^1$  is  $\neg\neg$  principal open: Let  $D(f_1, \dots, f_n) \subset \mathbb{A}^1$  be an arbitrary open subset. We may assume that each  $f_i : R[X]$  is non constant (in particular non zero). By [ref?],  $\neg\neg$ -merely each  $D(f_i) \subset R$  is cofinite. Thus  $\neg\neg$ -merely, the finite union  $\bigcup_{i=1}^n D(f_i) \subset R$  is cofinite as well, hence principal open. □

**Proposition 8.3.** *Assume covering stacks are  $\Omega$ -stable. A truncated stack (e.g. geometric stack) is covering iff  $\pi_0^{\mathbb{T}} X := \|X\|_0^{\mathbb{T}}$  and all higher homotopy groups*

$$\pi_i^{\mathbb{T}}(X, x) = \|\Omega^i(X, x)\|_0^{\mathbb{T}}, i \geq 1$$

*are covering algebraic spaces.*

*Proof.* Let  $X$  be an  $n$ -stack. If  $X$  is covering, then by  $\Omega$ -stability all the  $\pi_i^{\mathbb{T}}$  are covering 6.6. Now the converse. Consider the postnikov tower

$$X = \|X\|_n^{\mathbb{T}} \rightarrow \|X\|_{n-1}^{\mathbb{T}} \rightarrow \dots \rightarrow \|X\|_1^{\mathbb{T}} \rightarrow \|X\|_0^{\mathbb{T}}$$

As  $\|X\|_0^{\mathbb{T}}$  is covering, by quotient stability of covering stacks we may show that all the maps are geometric covers. Let  $1 \leq k \leq n$  and consider the map  $f_k^X : \|X\|_k^{\mathbb{T}} \rightarrow \|X\|_{k-1}^{\mathbb{T}}$ . By descent for covering stacks, we may only consider the fiber over  $|x|$ , as the  $\eta_{k-1}^{\mathbb{T}}$  is  $\mathbb{T}$ -surjective. It suffices to show, that the fiber is given by  $B_{\mathbb{T}}^k \pi_k^{\mathbb{T}}(X, x)$  as deloopings of covering stacks are covering ??.

We apply ??. First observe that  $\Omega^k(\text{fib}(f_k^X)|x|) = \text{fib}(\Omega^k(f_k^X, x))$  is equivalent to the basefiber of

$$\pi_k^{\mathbb{T}}(X, x) \equiv \|\Omega^k X\|_0^{\mathbb{T}} \simeq \Omega^k(\|X\|_k^{\mathbb{T}}) \rightarrow \Omega^k\|X\|_{k-1}^{\mathbb{T}} \simeq 1$$

So it suffices to show by induction over  $k$ , that for all pointed stacks  $(X, x)$ ,  $\text{fib}(f_k^X)|x|$  is  $\mathbb{T}$ - $k$ -connected.

This is definitely  $\mathbb{T}$ -connected by using that any term  $(y, p) : \text{fib}(f_k^X)|x| = \sum_{y: \|X\|_n^{\mathbb{T}}} \|x = y\|_{n-1}^{\mathbb{T}}$  yields a witness of  $\|x - y\|^{\mathbb{T}}$ . Then  $\Omega(\text{fib}(f_k^X)|x|) = \text{fib}(\Omega(f_k^X, x)) = \text{fib}(f_{k-1}^{\Omega(X, x)})$  which is  $\mathbb{T}$ - $k-1$ -connected by induction.  $\square$

## 8.1 Remarks about weakly flat affines

**Lemma 8.4.** *The proposition  $\|X\|_{\mathbb{T}}$  is geometric iff there exists a map from a weakly flat affine  $\text{Spec } B \rightarrow X$  such that  $\|\text{Spec } B\|_{\mathbb{T}} \rightarrow \|X\|_{\mathbb{T}}$  is an equivalence.*

*Proof.* ' $\leftarrow$ ' is clear.

' $\rightarrow$ '. Choose  $\text{Spec } B'$  weakly flat such that  $\|X\|_{\mathbb{T}} = \|\text{Spec } B'\|_{\mathbb{T}}$ . As the map  $X \rightarrow \|X\|_{\mathbb{T}}$  is  $\mathbb{T}$ -surjective, by  $\mathbb{T}$ -local choice we find a  $\mathbb{T}$ -cover  $\text{Spec } B \rightarrow \text{Spec } B'$  and a commutative diagram

$$\begin{array}{ccc} \exists \text{Spec } B & \dashrightarrow & X \\ \downarrow & & \downarrow \\ \text{Spec } B' & \longrightarrow & \|X\|_{\mathbb{T}} \end{array}$$

As  $\text{Spec } B'$  was weakly flat and the left vertical map is a  $\mathbb{T}$ -cover,  $\text{Spec } B$  is weakly flat.  $\square$

**Lemma 8.5 (DM).** *If  $\text{Spec } A + \text{Spec } B$  is weakly flat affine, then  $\text{Spec } A$  is weakly flat.*

*Proof.* Indeed

$$\|X\|_{\mathbb{T}} \rightarrow \|X + Y\|_{\mathbb{T}} \rightarrow X + Y \in \mathbb{T} \rightarrow X \in \mathbb{P}$$

but  $\|X\|_{\mathbb{T}} \wedge X \in \mathbb{P} \rightarrow X \in \mathbb{T}$ .  $\square$

**Lemma 8.6.** *if the topology is saturated Bering weakly-flat descends along  $\mathbb{T}$ -covers.*

**Lemma 8.7 (DM).** *If  $\|P + Q\|_{\mathbb{T}}$  is a geometric prop, then TODO*

*Proof.* By the previous two lemma and we find a map out of a weakly flat affine  $\text{Spec } B \rightarrow P + Q$  that induces an equivalence on  $\mathbb{T}$ -truncations, but it splits into two map out of a weakly affine  $\text{Spec } B_1 \rightarrow P, \text{Spec } B_2 \rightarrow Q$ .  $\square$

**Notation.** For  $P : (\varepsilon : \mathcal{N}_{\infty}(0)) \rightarrow X \rightarrow \text{Prop}$ , let  $\varepsilon : \mathcal{N}_{\infty}(0) \vdash x : X$ . We say  $x$  is  $\text{not}_{\varepsilon} P$ , if  $\forall \varepsilon, P_{\varepsilon} x \rightarrow \varepsilon = 0$ . Observe, if  $x$  is  $\text{not}_{\varepsilon} P$  for any  $\varepsilon^2 = 0$ , then  $x$  is not  $P$ .

**Remark 4.** If  $2 \neq 0$ . Let  $\varepsilon, \varepsilon' : \mathcal{N}_{\infty}(0)$ .  $\varepsilon = \varepsilon' + \varepsilon = -\varepsilon'$  is  $\text{not}_{\varepsilon}$  weakly-flat

*Proof.* We prove that once its  $\mathbb{T}$ -merely inhabited, then its  $\text{not}_\varepsilon$  covering, which is enough as  $\neg\neg(\varepsilon = \varepsilon' + \varepsilon = -\varepsilon')$ . As the goal is a stack we may assume  $\varepsilon = \varepsilon' + \varepsilon = -\varepsilon'$ . wlog the first case. Then assume  $1 + (\varepsilon = -\varepsilon) \simeq 1 + \varepsilon = 0$  is covering. Then  $\varepsilon = 0$  is formally étale, thus inhabited as a formally étale closed dense proposition.  $\square$

**Example 8.8** (Obsolete). *The map  $q : \mathbb{A}^1 \rightarrow \mathbb{A}^1/\mu_\ell$  is not a geometric cover.*

*Proof.* The map factors through the geometric cover  $\mathbb{A}^1 \rightarrow \mathbb{A}^1/\mu_\ell$ . Thus its enough to show that  $\mathbb{A}^1/\mu_\ell$  is not a 0-gerbe, or that not every loop space is covering. Let us show that,  $\Omega(\mathbb{A}^1/\mu_\ell, \varepsilon)$  is  $\text{not}_\varepsilon$  covering. Assume it is covering for some  $\varepsilon \in \mathcal{N}_\infty(0)$ . As  $\mu_\ell$  has decidable equality,

$$\begin{aligned} \Omega(\mathbb{A}^1/\mu_\ell, \varepsilon) &= \left( \sum_{g:\mu_\ell} g\varepsilon = \varepsilon \right) \\ &= (\varepsilon = \varepsilon) + \sum_{g:\mu_\ell \setminus \{1\}} (g - 1)\varepsilon = 0 \\ &= 1 + \mu_\ell \setminus \{1\} \times (\varepsilon = 0) \end{aligned}$$

Thus  $(\varepsilon = 0) \times (\mu_\ell \setminus \{1\})$  is an étale -flat geometric stack. Moreover  $(\mu_\ell \setminus \{1\})$  is a covering stack by 5.12. Thus  $\varepsilon = 0$  is an affine étale -flat geometric stack, thus formally étale + flat affine by saturatedness of the étale topology ?? . So as a formally étale + closed dense proposition,  $\varepsilon = 0$  holds as desired.  $\square$

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