Thesis

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Inhalt

1 Atlas

Definition 1.1. Given $\mathcal{V} \subset \mathcal{U}$ a subclass stable under $\sum$, a $\mathcal{V}$-cover is a map fibered in $\mathcal{V}$. A $\mathcal{V}$-atlas of X is a $\mathbb{T}$-cover $\operatorname{Spec} A \to X$ out of an affine scheme.
In the context of a topology $\mathbb{T}$, We call a $\mathcal{V}$-atlas $\operatorname{Spec} A \to X$ a $\mathcal{V}$-catlas, if the domain $\operatorname{Spec} A$ belongs to $\mathbb{T}$.

Example 1.2. *Let X be a (1-)type. X has a* Zar*-atlas, iff there exists some $f : \operatorname{Spec} A \to X$ fibered in types of the form $\operatorname{Spec}(R_{f_1} \times \ldots \times R_{f_n})$ for $(f_1, \ldots, f_n) \in Um(R)$.*

Remark 1. If one applies ZLC to an affine scheme $\operatorname{Spec} A$ the resulting principal open cover $D(f_i), f_i \in A$ will induce indeed a zariski atlas $\bigsqcup D(f_i) \to \operatorname{Spec} A$, because the fiber over $x : \operatorname{Spec} A$ is $\bigsqcup D(f_i(x))$.

Question: Does every zariski atlas of $\operatorname{Spec} A$ have this form? **??**

Example 1.3. *$\mathbb{P}^n$ has a zariski atlas given by the standart homogeneous principal opens $\sum_{i=0}^{n} D_+(x_i)$. The fiber over a point $[y_0 : \ldots . y_n]$ is $D(y_0) + \ldots D(y_n)$ where $(y_1, \ldots, y_n) \in Um(R)$.*

2 HoTT

2.1 Subtypes

3 Introduction to SAG

Lemma 3.1. *R is local, i.e. if $x, y : R$ and $x \neq y$, then x is invertible or y is invertible.*

Lemma 3.2. *If char $\neq 2$, Let $\rho \neq 0$, then $x^2 = \rho^2$ implies $x = \rho$ or $x = -\rho$*

Proof. Indeed, as $\rho \neq -\rho$, one of them is invertible by 3.1 ☐

Example for zariski local choice

Example 3.3. *For some A and $g, g' : A$ define*

$$g \mid_A g' \equiv \|\{h : A \mid hg =_A g'\}\|$$

Claim: For any $g, g' : A$, we have

$$g|_A g' \leftrightarrow \forall x : \operatorname{Spec} A, gx|_R g'x$$

Proof. $\rightarrow$ is obvious using that the duality map is an algebra isomorphism.
$\leftarrow$. For any $x : \operatorname{Spec} A$ we merely find some $h : R$ with $h \cdot g(x) = g'(x)$, i.e. we define our family of inhabited types as

$$Bx = \{h : R \mid h \cdot g(x) = g'(x)\}$$

By zariski local choice we merely find some principal open cover $\operatorname{Spec} A = \bigcup_{i=1}^n D(f_i)$ and local sections

$$\begin{aligned} &\prod_{x:D(f_i)} \{h_i : R \mid h_i \cdot g(x) = g'(x)\} \\ &\overset{??}{\simeq} \{h_i : D(f_i) \to R \mid (h_i x) \cdot g(x) = g'(x)\} \\ &\overset{??}{\simeq} \left\{h_i : A_{f_i} \mid h_i \cdot \frac{g}{1} =_{A_{f_i}} \frac{g'}{1}\right\} \end{aligned}$$

We can multiply h_i by high enough powers of f_i to obtain some $h_i : A$ with $h_i \cdot g = g' \cdot f_i^n$ for some $n : \mathbb{N}$. we may assume that n does not depend on $i = 1, \dots, n$ by taking the maximum and multiplying the h_i again with enough powers of f_i. Now use **??** to write $1 = \sum_{i=1}^n \ell_i f_i^n$ for some $\ell_i : A$ and then

$$(\sum_i \ell_i h_i) \cdot g = \sum_i \ell_i f_i^n g' = 1g' = g'$$

☐

4 Topology

Lemma 4.1. *Let $f, g : X \to Y$ be two functions into a seperated scheme where $X = \operatorname{Spec} A$ for A a reduced ring. If f and g coincide on a dense subset, then $f = g$.*

Proof. The equalizer

$$Z \equiv \sum_{x:X} fx =_Y gx$$

is a closed subset of X, as Y is seperated. As its open complement does not intersect the given dense subset, its empty. In other words $\neg\neg Z = X$. Writing $I \subset A$ as the ideal of functions that vanish on Z, By [2], we have

$$\operatorname{Spec} A = \neg\neg Z = \bigcup_n \operatorname{Spec} A/I^n$$

But by the strong boundedness principle, we find some n such that $\operatorname{Spec} A = \operatorname{Spec} A/I^n$, in other words, $I^n = 0$. As A is a reduced ring, we conclude $I = 0$, so $Z = \operatorname{Spec} A/I = \operatorname{Spec} A$. □

Definition 4.2. A point $0 : \operatorname{Spec} B$ is regular, if $\operatorname{Spec} B \setminus \{0\} = D(p_1, \dots, p_n)$ for some $p_1, \dots, p_n : B$ jointly-reguar, i.e. if $p_i^m \cdot b = 0$ for all $i = 1, \dots, n$ then $b = 0$.
If $0 : X$ is a point of a scheme, we call it regular, if one of the following equivalent conditions is satisfied

1. it admits some open affine neigbhorhood U such that $0 : U$ is regular.

2. It is a regular point of any open affine neigbhorhood.

Proof. Consider an open affine neigbhorhood $0 : D(f) \subset U = \operatorname{Spec} B$. We will show

1. If 0 is regular in $D(f)$, then it is regular in $\operatorname{Spec} B$: Consider $g_1, \dots, g_n : B$ such that

$$B_f \to \prod_i B_{fg_i}$$

is injective. Define $g_0 := f - f(0)$, where $0 \notin D(g_0)$. Let us show, that $g_0, \dots, g_n$ are jointly surjective in B. Let $b : B$ such that $g_i^n b = 0$ for all $0 \leq i \leq n$. Then in particular $b/1 =_{B_f} 0$. Thus b is in the kernel of $B \to B_{g_0} \times B_f$. But $D(g_0) \cup D(f)$ forms an open cover of $\operatorname{Spec} B$ as (f, g_0) generate the unit ideals. Thus $b : \operatorname{Spec} B \to R$ equals 0 on an open cover, thus its 0.

2. If 0 is regular in $\operatorname{Spec} B$, then it is regular in $D(f)$: Assume $B \to \prod B_{g_i}$ is injective. Let $f : B$. Let us show that $B_f \to \prod B_{g_i f}$ is injective. If $(g_i f)^n b = 0$, then $(g_i)^n (f^n b) = 0$, thus $f^n b = 0$ by assumption. Thus $b/1 =_{B/f} 0$ as desired.

□

Lemma 4.3. *If $0 : X$ is a regular point in a scheme, then both holds:*

1. *$X \setminus \{0\}$ is dense*

2. *$R^X \to R^{X\setminus\{0\}}$ is injective.*

Proof. 1. We write $A \perp B$ for $A \cap B = \varnothing$. We reduce to affine case: Let $0 \in \operatorname{Spec} B \subset X$. Let $U \subset X$ be open such that $U \perp X \setminus \{0\}$. Then $U \perp X \setminus \{0\} \Rightarrow U \perp \operatorname{Spec} B \setminus \{0\} \Rightarrow U \perp \operatorname{Spec} B$ so $U \perp (\operatorname{Spec} B \cup X \setminus \{0\}) = X$, thus $U = \varnothing$.
So we may assume that $X = \operatorname{Spec} B$ is affine: Then by [2], an open subset of $\operatorname{Spec} B$ is dense iff it is of the form $D(g_1, \dots, g_n)$ for nilregular functions $g_i : B$. Conclude, as regular implies nilregular [2].

2. Lets first reduce to the affine case. Choose an open affine neibhorhood U of 0 such that $0 : U$ is regular. Then the surjection $U + X \setminus \{0\} \twoheadrightarrow X$ induces a vertical left injection

$$\begin{tikzcd}
R^{U+X\setminus\{0\}} \arrow[r, hook] & R^{U\setminus\{0\}+X\setminus\{0\}} \\
R^X \arrow[u, hook] \arrow[r] & R^{X\setminus\{0\}} \arrow[u]
\end{tikzcd}$$

So we may assume that $X = \operatorname{Spec} A$ is affine.
Let $p_1, \dots, p_n : A$ be jointly-reguar, i.e. if $p_i^m \cdot a = 0$ for all $i = 1, \dots, n$ then $a = 0$.
If $f : \operatorname{Spec} A \to R$ such that $f(x) = 0$ for all $x \in D(p_1, \dots, p_n)$, then $f(x) = 0$ for all $x : \operatorname{Spec} A$. f is in the kernel of the diagonal map

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, hook] & R^{\operatorname{Spec} A} \arrow[d] & \\ \prod_{i=1}^n A_{p_i} \arrow[r, equal] & R^{\coprod_i D(p_i)} & R^{\cup D(p_i)} \arrow[l, hook'] \end{tikzcd}$$

which is injective, as $p_1, \dots, p_n$ are jointly-regular in A.
Thus $f = 0$ in A.

□

Remark 2. If A is an algebra that is reduced as a ring, then for $X = \operatorname{Spec} A$, 1. implies 2. by 4.1

Proposition 4.4. *the subtype* $\{0\} + 0^c \subset \operatorname{Spec} B$ *is not locally closed whenever one of the following conditions is satisfied:*

1. $\operatorname{Spec} B \setminus \{0\}$ *is dense*
2. $R^{\operatorname{Spec} B} \to R^{\operatorname{Spec} B \setminus \{0\}}$ *is injective*

Proof. Let us first show, that the infinitesimal neigbhorhood of 0 is not open.

1. If $0^c \subset \operatorname{Spec} A$ is dense) : The non-empty open $\mathcal{N}_\infty$ does not intersect the dense subset 0^c.
2. If $R^{\operatorname{Spec} B} \to R^{\operatorname{Spec} B \setminus \{0\}}$ is injective: If it would, we find a principal open smaller neighborhood$0 \in D(g) \subset \mathcal{N}_\infty(0)$, which however already cotains the whole infinitesimal one, thus $\mathcal{N}_\infty(0) = D(g)$
   Then for any $x \neq 0$, we have $\neg\neg g(x) = 0$. As $\operatorname{Spec} B \setminus \{0\}$ is a scheme, it admits a boundedness principle, thus we find some n, such that $g^n(x) = 0$ for all $x \neq 0$.
   by 4.3 we have that $R^{\operatorname{Spec} B} \to R^{\operatorname{Spec} B \setminus \{0\}}$ is injective, so we deduce $g^n = 0$, hence $D(g) = D(g^n) = \varnothing$ contradiction.

Just assumg that the infinitesimal neighborhood is not open, The subtype $\{0\} + 0^c \subset \operatorname{Spec} B$ is not locally closed. Let $U, C \subset \operatorname{Spec} B$ be an open subset and a closed subset respectively, such that $(x \neq 0) + (x \neq 0) \leftrightarrow x \in U \wedge x \in C$. Then, for any $x : U$,

$$(x = 0) + (x \neq 0) = x \in C$$

is a closed proposition. Thus the decidable subtype $x \neq 0$ is a closed proposition. To contradict the assumption, we may convince ourself that the right vertical map

$$\begin{tikzcd} \sum_{x:U} \neg\neg x = 0 \arrow[r, "\sim"'] \arrow[d, hook] & \sum_{x:\operatorname{Spec} B} \neg\neg x = 0 \arrow[d] \\ U \arrow[r, hook] & \operatorname{Spec} B \end{tikzcd}$$

is an open embedding
where the upper horizontal map is indeed an equivalence as for any $x : \operatorname{Spec} B$, $x \in U$ is $\neg\neg$-stable, but $\neg\neg x = 0$ and $0 \in U$, thus $x \in U$.

□

5 Preparation

Lemma 5.1 (Strong boundedness, NEEDED?)**.** *Consider a sequence of embeddings of types*

$$X_0 \overset{\iota_0}{\hookrightarrow} X_1 \overset{\iota_1}{\hookrightarrow} X_2 \dots$$

Then any map $f : \operatorname{Spec} A \to \operatorname*{colim}_n X_n \equiv: \bigcup_n X_n$ factors through some $\kappa_m : X_m \hookrightarrow \operatorname*{colim}_n X_n$.

Proof. For every term $x : \operatorname{Spec} A$ consider the subset S_x of natural numbers n, such that $f(x) \in \operatorname{im} \kappa_m$. Its a merely inhabited upwards closed subset. By the strong boundedness principle [ref?], the subset $\bigcap_{x:\operatorname{Spec} A} S_x$ is merely inhabited. $\square$

Lemma 5.2. *Let Y be a type, which admits a jointly surjective family of maps with smooth domain $X_i \to Y$ Then Y is formally smooth.*

Proof. $\sum_{n:\mathbb{N}} X_n \to Y$ is surjective with formally smooth domain, as $\mathbb{N}$ is formally smooth. $\square$

Corollary 5.3 (Monoid is smooth)**.** *Let $(Y, +)$ be a magma, which is generated by a map with smooth domain $f : X \to Y$, i.e. every $a : Y$ can merely be written as a finite sum*

$$a = f(x_1) + \ldots + f(x_n)$$

Then Y is formally smooth.

Lemma 5.4. *Let C be a class of types stable under $\sum$. Let $\mathbb{P} \subset \mathsf{Aff}$ (in most cases $\mathbb{P} := \mathsf{Aff}$) be any subclass of affines stable under finite limits. The class $\mathsf{HasAtlas}_C^{\mathbb{P}}$ of types Y which admit a map $\mathbb{P} \ni S \to Y$ fibered in C is stable under identity types.*

Proof. By assumption we can choose a map $\mathbb{P} \ni V \overset{p}{\to} Y$ fibered in C. Let $y, y' : Y$. Then we have the map

$$\begin{gathered}(\operatorname{fib}_p y) \times_V (\operatorname{fib}_p y') \to y = y' \\ (v, q : y = pv), (v', q' : y' = pv'), (h : v = v') \mapsto q \cdot h \cdot q'^{-1}\end{gathered}$$

The fiber over $j : y = y'$ looks like

$$\sum_v \underbrace{(\sum_{v'} (h : v = v'))}_{\mathsf{isContr}} \times (q : y = pv) \times (q' : y' = pv') \times (q \cdot h \cdot q'^{-1} = j) \simeq \sum_v (v = py) \simeq \operatorname{fib}_p y$$

Hence the map is fibered in C. It suffices to show, that $(\operatorname{fib}_p y) \times_V (\operatorname{fib}_p y')$ has an atlas, because then we can compose such an atlas with the above map to obtain an atlas of $y = y'$. By assumption the fibers of p have an atlas, so we can choose $q : W \to \operatorname{fib}_p y, q' : W' \to \operatorname{fib}_p y'$ atlasses. Then $W \times_V W' \to (\operatorname{fib}_p y) \times_V (\operatorname{fib}_p y')$ is an atlas: The domain is a fiber product of types in $\mathbb{P}$, hence it belongs to $\mathbb{P}$. The fiber over (x, x') is equivalent to the product of fibers $(\operatorname{fib}_q x) \times (\operatorname{fib}_{q'} x')$ which is in C by stability under dependent sums (so in particular under finite products). $\square$

Lemma 5.5. *Let $\mathcal{U}' \subset \mathcal{U}$ be stable under dependent sums. Let X be a type with a map $p : U \to X$ fibered in $\mathcal{U}'$. For any $x : X$, let Y_x be a type and moreover for any $u : U$, we are given a map $q_u : V_u \to Y_{p(u)}$ fibered in $\mathcal{U}'$. Then the induced map*

$$p : \sum_{u:U} V_u \to \sum_{x:X} Y_x$$

is fibered in $\mathcal{U}'$

Proof. The fiber of p over some $(x, y) \in \sum_{x:X} Y_x$ is given by

$$\sum_{u:\mathrm{fib}_p\, x} \mathrm{fib}_{q_u}(y')$$

where $y' : Y_{p(u)}$ (depending on u) is the transport of $y : Y_x$ along $x = p(u)$. As $\mathcal{U}'$ is stable under dependent sum those fibers are again in $\mathcal{U}'$. This shows the result. $\square$

6 (Lex) Modalities

Lemma 6.1 (Stability resuls)**.** *Modalities are stable under*

1. *Conjunction*

2. *Composition*

Lemma 6.2. *Let $\bigcirc$ be a modality. Let X be $\bigcirc$-modal and $B : X \to \mathcal{U}_{\bigcirc}$ be a family of modal types. Then $\sum_{x:X} B_x$ is $\bigcirc$-modal*

Lemma 6.3. *Let $B : \bullet X \to \mathcal{U}$. Then $\bullet(\sum_{x:X} B(\eta x)) = \sum_{x:\bullet X} \bullet Bx$*

Proof. Observe that

$$\sum_{x:X} B\eta x \to \sum_{x:\bullet X} Bx$$

is a $\bullet$-equivalence, because for all modal types T, the type $Bx \to T$ is modal for any $x : \bullet X$. Then it follows by [ref?]. $\square$

Lemma 6.4. *Let $\bullet$ be a lex modality. Let $x, y : X$. The map*

$$\bullet(x = y) \to \eta x =_{\bullet X} \eta y$$

induced by $ap_\eta : x = y \to \eta x =_{\bullet X} \eta y$ is an equivalence

Proof. By Modalities Theorem 3.1 [ix]. $\square$

Definition 6.5. Let $\bullet$ be a lex modality. we call a type X $\bullet$-seperated if one of the following equivalent conditions hold

- the identity types of X are modal
- the unit $X \to \bullet X$ is an embedding

In this case

Proof. by 6.4 the vertical map in the commutative diagram

$$\begin{tikzcd} x =_X y \arrow[r, "\eta_{x=y}"] \arrow[dr, "ap_{\eta_X}"'] & L(x = y) \arrow[d, "\simeq"] \\ & \eta x =_{LX} \eta y \end{tikzcd}$$

is an equivalence. So $x = y$ is a sheaf if $\eta_{x=y}$ is an equivalence iff η_X is an embedding. $\square$

Lemma 6.6. *If $\bullet$ is a lex modality, then the universe of modal types $\mathcal{U}_\bullet$ is modal.*

7 Local Choice

In this section let $\mathbb{T}$ denote a topology finer than the zariski topology.

Definition 7.1. Let *Cov* be a class of morphisms (which we think of n-atlasses of some n), containing $\mathbb{T}$-atlas, (stable under pullback NECESSARY TODO?) A type S has *local choice* wrt *Cov* if for any $\mathbb{T}$ -surjective map $X \to Y$ and any map $f : S \to Y$ there exists a map $p' : S' \to S$ in *Cov* and a commutative diagram

$$\begin{tikzcd} S' \arrow[r, dashed] \arrow[d] & X \arrow[d, "p"] \\ S \arrow[r, "f"] & Y \end{tikzcd}$$

Proposition 7.2. *Assume that* Cov *is stable under composition and that Zariski-covers are in* Cov *.* S *has* $\mathbb{T}$*-local choice wrt* Cov *if it has an* Cov*-atlas, i.e. there exists an affine scheme* $\hat{S}$ *with a map* $g : \hat{S} \to S$ *in* Cov *.*

Proof. By stability under composition of *Cov* , We may assume that $g : \hat{S} \to S$ is the identity. As p is $\mathbb{T}$-surjective, for any $x : S$ there merely is a $\operatorname{Spec} B_x \in T$ and a map $\operatorname{Spec} B_x \to \| \operatorname{fib}_p(x)\|$. By ZLC (1), there exists a Zariski atlas $S' \xrightarrow{p'} S$ and a term in

$$\prod_{x:S'} \sum_{\operatorname{Spec} B_x \in T} \operatorname{Spec} B_x \to \| \operatorname{fib}_p(fp'x)\|$$

By setting

$$(S'' := \sum_{x:S'} \operatorname{Spec} B_x) \xrightarrow{\pi} S'$$

the projection, we are now in the situation that for any $t : S''$ we merely have a point in $\operatorname{fib}_p((p''(t)))$ and $S'' \to S'$ is a $\mathbb{T}$-cover, thus it is in *Cov* . Moreover, S'' is affine, as it is a dependent sum of affines. By replacing S'' again with a Zariski cover we find a lift $S'' \to X$ making

$$\begin{tikzcd} S'' \arrow[r] \arrow[d] & X \arrow[dd, "p"] \\ S' \arrow[d, "p'"] & \\ S \arrow[r, "\mathrm{id}"] & S \end{tikzcd}$$

commute. Now $S'' \to S' \to S$ as the composition of Zariski-covers and *Cov*er is a *Cov*er [...]as desired. ☐

The next lemma shows, that the class of types equipped with a $\mathbb{T}$-atlas is stable under dependent sums.

Theorem 7.3. *Let* $\mathbb{T}$ *be a topology. Let* $\mathcal{U}'$ *be a class stable under dependent sums containing* Zar*. The class of types merely admitting a* $\mathcal{U}'$*-atlas is closed under dependent sums. The same holds for* $\mathcal{U}'$*-catlasses.*

Proof. For any $x : X$ we merely have an atlas $V_x \to B_x$, i.e. with V_x affine. X has $\mathbb{T}$-local choice wrt atlasses by (7.2) using $\mathcal{U}'$ is $\sum$-stable (we use the trivial topology).
If additionally, all the B_x and X are smooth n-stacks, just observe that we can choose the affine V_{pu} to lie in $\mathbb{T}$, Accordingly $\sum_{u:U} V_{pu} \in T$ as $\mathbb{T}$ is stable under Σ.
By Local choice for X, we merely find U affine, an atlas $p : U \to X$ with

$$\prod_{u:U} \sum_{V_{p(u)} \in T} (q : V_{p(u)} \to B_{p(u)}) \times (q \text{ fibered in } \mathcal{U}')$$

Now the desired map is $\sum_{u:U} V_{pu} \to \sum_{x:X} B_x$, because it is an atlas by 5.5

□

Corollary 7.4. *The class of stacks merely admitting a $\mathbb{T}$-catlas has descent.*

Proof. The class is stable under $\sum$ and contains $\mathbb{T}$. If $X \to Y$ is a $\mathbb{T}$-cover and X admits a $\mathbb{T}$-catlas then Y admits a $\mathbb{T}$-catlas. □

8 Covering stacks

Fix $\mathbb{T}$ a topology, which we call the covering-affines.

Definition 8.1. covering stacks are the smallest class containing contractible Types such that: If Y is a stack and $\mathbb{T} \ni S \to Y$ is fibered in covering stacks, then Y is a covering stack.

We call such map $X \to Y$ whose fibers are covering stacks a geometric cover. If X is affine we call it a geometric atlas. If X is in $\mathbb{T}$ we call it a geometric catlas.

Definition 8.2. We call X a geometric stack if it merely has a geometric atlas, i.e some $\operatorname{Spec} A \to X$ fibered in covering stacks.

Proposition 8.3 (Recursion principle for covering / geometric stacks)**.** *Let P be a property of covering / geometric stacks. Assume*

- *contractibles have P*
- *If S is (covering) affine and $S \to Y$ is fibered in covering stacks having P then Y has P*

Then every covering / geometric stack has P.

Why I did it this way. Should P be defined more generally for all sheaves? No, because we want for the recursion principle for geometric stacks, that the fibers are covering stacks (proof of truncatedness).

Proposition 8.4. *Every covering stack X merely admits a geometric catlas.*

Proof.
- If X is covering affine, then $X \to X$ is a geometric catlas.
- If X is obtained as a quotient then it already is equipped with a catlas.

□

8.1 Needing finitely many steps

In this subsection we want to prove that one could equivalently define covering stacks just by induction over the natural numbers, e.g. the truncation level or over the number of needed constructors.

Lemma 8.5. *Every covering stack X is $\mathbb{T}$-merely inhabited.*

Proof.
- If X is in $\mathbb{T}$ then its clear.
- If X is obtained by a quotient, we have a map $\operatorname{Spec} A \to X$ with domain in $\mathbb{T}$. Now use that we get a map on $\mathbb{T}$-propositional-truncations and that Spec A is T-merely inhabited.

□

Proposition 8.6. *Given a geometric stack Y and a family $M : Y \to (\mathbb{N} \to \mathrm{Prop}_{\bigcirc})$ be a family of upwards closed merely inhabited subsets of $\mathbb{N}$. Then there exists some n, such that Myn for all $y : Y$.*

Proof. Write $M_n = \{y : Y \mid Myn\}$. Choose a geometric atlas $f : S \to Y$. For any $x : S$, $M(fx)n$ for some n. By foundations Prop 3.3.5, we merely find some n such that $f(x) \in M_n$ for all $x : S$. Let us show that for general $y : Y$ we have $y \in M_n$. Using that $y \in M_n$ is modal , we can conclude by $\mathbb{T}$-surjectivity of f, which follows from 8.5

□

Proposition 8.7. *Let $W : \mathsf{GS} \to (\mathbb{N} \to \mathrm{Prop}_{\bigcirc})$ be a family of upwards closed subsets of $\mathbb{N}$. Assume*

- *$W1$ is merely inhabited*
- *whenever there is some $n : \mathbb{N}$ and a geometric atlas $S \to X$ fibered in covering stacks F satisfying $WFn \equiv: W_n F$, then $W_{n+1} X$.*

Then for any $X \in \mathsf{GS}$, WX is merely inhabited.

Proof. We apply the recursion principle for geometric stacks.

- If Y is contractible its clear by assumption
- Assume Y is equipped with a geometric atlas $f : S \to Y$, such that every fiber has W_n for some n. Apply 8.6 to $Myn = W_n(\mathrm{fib}_f\, y)$ to find some n such that $W_n(\mathrm{fib}_f\, y)$ for all $y : Y$. Then we can conclude by applying the assumption.

□

Definition 8.8. Define

$$
\begin{aligned}
W_0 &\equiv \mathbb{T} \\
W_{n+1} &\equiv \{X \textit{ stack} \mid X \text{ has a } W_n - \textit{catlas}\}
\end{aligned}
$$

Why I did it this way. $W0$ is not defined as isContr, because for $\sum$ stability later, we want to apply 7.3, so we need that Zariski covers are allowed covers.

Lemma 8.9. *W is monotone*

Proof. We prove $\forall n. Wn \subset W(n+1)$. Induction. $n = 0$. For any $X : \mathbb{T}$, $X \to X$ is a W_0-catlas, as $1 \in \mathbb{T} = W_0$. If $X \in W_n$, it admits a W_{n-1} catlas. By induction this is a W_n catlas. So $X \in W_{n+1}$. □

Lemma 8.10. *For all $n : \mathbb{N}$, W_n covering stacks are $\sum$-stable.*

Proof. Induction over n. If $n = 0$, then this is the stability under $\sum$ of $\mathbb{T}$
If we wish to prove the statement for $n+1$, we may assume that W_n covering stacks are $\sum$-stable. We have $\mathsf{Zar} \subset \mathbb{T} \subset W_n$. So we can apply 7.3.

□

Proposition 8.11. *Every covering stack has W_n for some n.*

Proof. The idea is to apply 8.7. We need that $X \in W_n$ is a sheaf for X a stack.
Let $T \in \mathbb{T}$ such that $T \to \exists(\mathbb{T} \ni S \to X\ W_n\text{-atlas})$. We want to construct a W_n-catlas of X. By Zariski local choice we find a Zariski atlas $T' \to T$ with a term in

$$\prod_{t:T'} \sum_{S_t : \mathbb{T}} W_n \mathsf{atlas}(S_t, X)$$

From this we obtain a map

$$\sum_{t:T'} S_t \to T' \times X \to X$$

. As $T' \in \mathbb{T} \subset W_n$ by $\sum$-stability of $\mathbb{T}$, both maps are W_n-covers. By 8.10 the composite is a W_n-cover. Its domain is in $\mathbb{T}$ by $\sum$-stability of $\mathbb{T}$. This is what we wanted to show. □

8.2 Stability

Theorem 8.12. *The class of covering / geometric stacks is $\sum$-stable.*

Proof. The geometric case follows from the covering case by 7.3. Let X be a covering stack and $B : X \to \mathsf{CS}$ a family of covering stacks. We apply 8.6 to the predicate ' X belongs to Wn for some n', which holds definitely for some n by 8.11. So we merely find an $n : \mathbb{N}$ such that $Bx \in W_n$ for all $x : X$. By making n larger, we may assume X has Wn for some n. Conclude by 8.10

□

Lemma 8.13. *geometric covers are stable under composition.*

Proof. covering stacks are stable under $\sum$. □

Proposition 8.14. *The class of covering / geometric stacks is stable under quotients: If $X \to Y$ is fibered in covering stacks and X is a (covering) stack and Y is a stack then Y is a covering / geometric stack.*

Proof. Choose a geometric (c)atlas of X. Then the composition with the map $X \to Y$ is a cover by 8.13. As the domain is (covering) affine, its a geometric (c)atlas. □

Now we want to show that the clash of terminology regarding 'covering' is reasonable:

Proposition 8.15. *Let $\mathbb{T}$ be saturated. A covering stack X is affine iff its a covering affine.*

Proof. The converse is clear. The direct direction follows by the recursion principle. choosing a geometric catlas $S \to X$. As both S and X are affine the fibers are affine. By induction the fibers are covering affines. By saturatedness of the topology X is covering affine. □

Lemma 8.16. *Let $\mathbb{T}$ be saturated. Let X be a covering stack. Let $f : \operatorname{Spec} A \to X$ be a geometric atlas. Then* $\operatorname{Spec} A \in \mathbb{T}$

Proof. As $\operatorname{Spec} A \simeq \sum_{x:X} \operatorname{fib}_f x$ is a dependent sum of covering stacks, it is a covering stack again by 8.12. We conclude by 8.15. □

Lemma 8.17. *geometric stacks are stable under finite sums. If $\mathbb{T}$ is finer than the zariski topology, then this holds for covering stacks as well*

Proof. We have to show that finite sums of geometric (c)atlasses are geometric (c)atlasses. For the geometric case just use that affines are stable under finite sums. For the covering case use that $1 + \ldots + 1 \in \mathsf{Zar} \subset \mathbb{T}$, hence the topology is stable under finite sums. □

Lemma 8.18. *geometric stacks are closed under* id*-types.*

Proof. This is 5.4, using that covering stacks are closed under $\sum$ (8.12) □

Warning. The previous lemma does not hold for covering stacks: Identity types of things in $\mathbb{T}$ could be empty.

8.3 About the covering stacks in a subuniverse

Definition 8.19. Let $\mathcal{V} \supset \mathsf{Aff}$ be a superclass stable under $\sum$. covering $\mathcal{V}$ stacks are the smallest intermediate class $\mathbb{T} \subset \mathsf{CS}_{\mathcal{V}} \subset \mathcal{V}$ such that: If $X : \mathbb{T}$, $Y : \mathcal{V}$ and $X \to Y$ is fibered in $\mathsf{CS}_{\mathcal{V}}$, then $Y \in \mathsf{CS}_{\mathcal{V}}$.
X is a geometric $\mathcal{V}$-stack if its in $\mathcal{V}$ and it merely admits a map $\operatorname{Spec} A \to X$ fibered in $\mathsf{CS}_{\mathcal{V}}$.

Definition 8.20. We define the saturation of $\mathbb{T}$ as the class of covering Aff-stacks. We call a topology $\mathbb{T}$ saturated if it coincides with its saturation, or more concretely: Every affine schemes that has a catlas lies itself in $\mathbb{T}$.

In a further chapter we will develop this theory further.

Proposition 8.21. *Let $\mathcal{V}$ be stable under finite limits and containing (covering) affines. X is a (covering) $\mathcal{V}$-stack iff it is in $\mathcal{V}$ and a covering / geometric stack.*

Proof. The direct direction is clear. For the converse we apply the recursion principle to the property '$X \in \mathcal{V}$ implies X is a (covering) $\mathcal{V}$-stack'. If X is contractible, its clear. Otherwise its equipped with a geometric (c)atlas. The fibers are in $\mathcal{V}$, as they can be written as a fiberproduct of $S, X, 1 \in \mathcal{V}$. By induction all fibers are covering $\mathcal{V}$-stacks (we may show the covering part of the proposition first). □

Proposition 8.22. *(covering) $\mathcal{V}$-stacks are stable under dependent sums. In particular the saturation of a topology defines a topology.*

Proof. Both the classes $\mathcal{V}$ and (covering) stacks are stable under dependent sums. Hence the intersection of them is $\sum$-stable as well.
The saturation is a class of affines, that in particular contains $1 \in \mathbb{T}$. We have argued its stable under $\sum$. □

Proposition 8.23. *A stack X merely admits some map $S \to X$ out of a (covering) affine fibered in covering $\mathcal{V}$-stacks, iff its a covering / geometric stack whose identity types are in $\mathcal{V}$.*

Proof. The direct direction: By 5.4 the identity types are geometric $\mathcal{V}$-stacks.
The converse direction: Choose a geometric (c)atlas $f : S \to X$. As each fiber $\sum_{s:S} fs =_X x$ is in V by $\sum$-stability of $\mathcal{V}$ and is a covering stack, its a covering $\mathcal{V}$-stack by 8.21. □

Definition 8.24. Let $n \geq -2$. A covering / geometric n-stack is a covering / geometric stack that is an n-type.

Proposition 8.25. *Let X be a stack. For all $n \geq 0$, the following are equivalent:*

1. *X is a covering / geometric $n+1$-stack*
2. *X merely admits some map $S \to X$ out of a (covering) affine fibered in covering n-stacks*
3. *X merely admits some covering / geometric n-stack Y and a map $Y \to X$ fibered in covering n-stacks.*

Proof.

1 . ⇔ 2. X is a covering / geometric $n+1$ stack iff its a covering / geometric stack whose identity types are n-types. But this is equivalent to 2. by 8.23.

2 . ⇒ 3. S is a covering / geometric n-stack

3. ⇒ 2 Y admits a map $S \to Y$ fibered in covering n-stacks with S (covering) affine, so the composition $S \to X$ will have the same property by 8.13.

□

Proposition 8.26. *We have inclusions*

$$W_n \subset \mathsf{CS}_n \subset W_{n+1}$$

Proof. □

8.4 Truncatedness

Lemma 8.27. *Let X be an $n+1$-type and Y a stack. If $X \to Y$ is a n-truncated $\mathbb{T}$-surjective map, then Y is an $n+1$-type.*

Proof. Use that $\mathsf{is} - \mathsf{n} - \mathsf{truncated}(y = y')$ is a stack for $y, y' : Y$. $\square$

Corollary 8.28. *Every geometric stack is n-truncated for some $n : \mathbb{N}$.*

Proof. Apply the prop 8.7. Use 8.27. For a stack X, is-n-truncated X is indeed a stack. $\square$

8.5 Descent

Definition 8.29. A class of types C has descent if the type C is a stack.

Lemma 8.30. *A class of stacks C has descent iff for any stack X, the proposition $X \in C$ is a sheaf.*

Theorem 8.31. *Let $\mathbb{T}$ be subcanonical. Consider a class of stacks St stable under $\sum$ such that $\mathbb{T} \subset \mathsf{St}$ and whenever you have a $\mathbb{T}$-cover $X \to Y$ between stacks then $X \in \mathsf{St}$ implies $Y \in \mathsf{St}$. Then St has descent.*

Proof. St is separated: This follows from the embedding GS into the separated type of sheaves 6.6.
Let $U \in \mathbb{T}$ and $P : \|U\| \to \mathsf{St}$. We want to construct a filler

$$\begin{tikzcd} \|U\| \arrow[r, "P"] \arrow[d] & \mathsf{GS} \\ 1 \arrow[ur, dashed] & \end{tikzcd}$$

Given $U \in \mathbb{T}$ and a map $P : \|U\| \to \mathsf{St}$. Claim: $L_T \sum_{x:\|U\|} Px \in \mathsf{St}$. If the claim is proven, the diagram commutes: Assuming $x : \|U\|$ we wish to show $Px = L_T \sum_{x:\|U\|} Px$. Using univalence, we may show that the maps

$$Px \to \sum_{x:\|U\|} Px \xrightarrow{\eta} L_{\mathbb{T}} \sum_{x:\|U\|} Px$$

are both equivalences. The first one is an equivalence as $\|U\|$ is contractible. Hence the middle term is a stack, thus the unit map is an equivalence as well.
Proof of the claim:
We introduce notation

$$\sum_{x:U} Px \xrightarrow{f} \sum_{x:\|U\|} Px \equiv: Y \overset{\eta}{\hookrightarrow} L_T Y.$$

Claim: For any $y : L_T Y$, the map $\mathrm{fib}_{\eta f}\, y \hookrightarrow \sum_{x:U} Px \to U$ is an equivalence.
Proof: To ask that a map between stack is an equivalence is a stack, hence we may replace y by $\eta y'$ with $y' : Y$. Consider the following commutative diagram

$$\begin{tikzcd} \mathrm{fib}_{\eta f}\, \eta y' \arrow[r] & \sum_{x:U} Px \arrow[d] \\ \mathrm{fib}_f\, y' \arrow[u] \arrow[r, "\simeq"'] & U \end{tikzcd}$$

The left vertical map is an equivalence, as $\sum_{x:\|U\|} Px$ is seperated (the geometric stacks Px are stacks, so in particular seperated).
As $U \in \mathsf{St}$ and St is $\sum$-stable, $\sum_{x:U} Px \in \mathsf{St}$ By the assumption of the theorem $L_T Y \in \mathsf{St}$ ☐

In the proof we have learned the following:

Lemma 8.32. *If Y is seperated and admits some $U \in \mathbb{T}$ and a map $f : X \to Y$ such that every fiber is equivalent to U, then there is a $\mathbb{T}$-cover $X \to L_{\mathbb{T}} Y$.*

Corollary 8.33. *(covering) geometric stacks satisfy descent.*

Corollary 8.34. *For all $n : \mathbb{N}$, the class of (covering) (n-)stacks has descent.*

Proof. The class of (covering) geometric n-stacks is the intersection of (covering) geometric stacks and n-truncated stacks. Both have descent. ☐

9 Saturated Topologies

Definition 9.1. Consider the partial order

$$\mathsf{Top} = \{\mathbb{T} : \mathrm{Prop}^{\mathsf{Aff}} \mid 1 \in \mathbb{T} \wedge \mathbb{T} \sum -stable\}$$

ordered by inclusion. An inflation P on Top is a monotone endofunction such that $X \subset PX$.
P is stack-preserving if for any topology $\mathbb{T}$, $P\mathbb{T} \subset \mathbb{T}$-merely inhabited types.
it is covering-stack-preserving if for any $X : P\mathbb{T}$, X is a $\mathbb{T}$-covering stack.

Note that covering-stack-preserving implies stack-preserving, as $\mathbb{T}$-covering stacks are $\mathbb{T}$-merely inhabited.

Proposition 9.2. *Given a stack-preserving inflation P. Then for any topology $\mathbb{T}$, A Type Y is a stack wrt to $P\mathbb{T}$ iff it is a stack wrt to $\mathbb{T}$.*
If P is idempotent, then the class $P\mathbb{T}$ is the smallest P-fixpoint topology containing $\mathbb{T}$.
If P is covering-stack preserving, $\mathbb{T}$ and $P\mathbb{T}$ will induce the same covering stacks.

Proof. $\mathbb{T} \subset P\mathbb{T}$ by inflationarity. Regarding Stacks: As $\mathbb{T} \subset \mathbb{T}'$ the $\to$ direction is clear. Now, let $X \in \mathbb{T}'$. We have

$$\begin{tikzcd}
\|X\| \arrow[r, "\forall"] \arrow[d] & Y \\
\|X\|_{\mathbb{T}} \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

by the stack-preserving-property $\|X\|_{\mathbb{T}} \simeq 1$. Hence T is $\|X\|$-local If P is idempotent, every other fixpoint X containg $\mathbb{T}$ satisfies $PT \subset PX = X$ by monotonicity.
If P is covering-stack-preserving, notice that every $\mathbb{T}$-covering stack is also a $P\mathbb{T}$-covering stack as $\mathbb{T} \subset P\mathbb{T}$. For the converse we use the recursion principle: For X a $P\mathbb{T}$-covering stack, consider the predicate 'is $P\mathbb{T}$-covering'. 1 has it. If $P\mathbb{T} \ni \operatorname{Spec} A \to X$ is a $\mathbb{T}$-geometric atlas, i.e. whose fibers are $\mathbb{T}$-covering stacks, as $\operatorname{Spec} A$ is a $\mathbb{T}$-covering stack by the covering-stack-preservation, by quotient stability of $\mathbb{T}$-covering stacks X is a $\mathbb{T}$-covering stack as well ∎

Definition 9.3. A catlas of X is some $\hat{X} \in \mathbb{T}, \hat{X} \to X$ $\mathbb{T}$-cover

Proposition 9.4. *The assignment*

$$\begin{aligned} \mathsf{Top} &\to \mathsf{Top} \\ \mathbb{T} &\mapsto \mathbb{T}' \equiv \{X \in \mathsf{Aff} \mid \exists \ catlas \ of \ X\} \end{aligned}$$

covering-stack-preserving idempotent Monad, called the saturation monad.
$\mathbb{T}'$ is the class of covering Aff-stacks.

Proof.
- $\mathbb{T}'$ is $\sum$-stable by 7.3.
- $\mathbb{T} \subset \mathbb{T}'$ is clear.
- Monotonicity clear
- Idempotentency: consider some $\mathbb{T}'$-cover $\mathbb{T}' \ni X' \to X$. By replacing X' with some smooth atlas, we may assume that $X' \in \mathbb{T}$. As every fiber $X'_x \in \mathbb{T}'$, we merely find a smooth atlas $\tilde{X}'_x \to X'_x$. Then by Zariski local choice there exists a Zariski atlas $\hat{X} \to X$ and a commutative diagram

$$\begin{tikzcd}
Y \equiv \sum_{x:\hat{X}} \tilde{X}'_x \arrow[r] \arrow[d] & \sum_{x:X} X'_x = X' \arrow[d] \\
\hat{X} \arrow[r, "Zar"'] & X
\end{tikzcd}$$

As $X' \in \mathbb{T}$ and $Y \to X'$ is fibered in $\mathbb{T}$ (5.5) we have $Y \in \mathbb{T}$. But $Y \to \hat{X}$ is a $\mathbb{T}$-cover and $\hat{X} \to X$ is a $\mathbb{T}$-cover, $Y \to X$ is a $\mathbb{T}$-cover. Hence $X \in \mathbb{T}'$.

- covering-stack-preserving: For any $\operatorname{Spec} A : \mathbb{T}'$ we merely have some $\mathbb{T}$-catlas $\mathbb{T} \ni X \to \operatorname{Spec} A$, witnessing that $\operatorname{Spec} A$ is a covering stack.

For the last claim, just observe that $\mathbb{T}'$ is definitely contained in covering Aff-stacks. ☐

Lemma 9.5. *if* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is faithfully flat and* $\operatorname{Spec} B$ *is flat, then* $\operatorname{Spec} A$ *is flat.*

Proof. Consider an injection of R-modules $M \hookrightarrow N$. We wish to show, that $A \otimes_R M \to A \otimes_R N$ is injective. As B is faithfully flat over A it suffices to show, that $B \otimes_R M \cong B \otimes_A A \otimes_R N \to B \otimes_A A \otimes_R N = B \otimes_R N$ is injective. This follows as B is flat over R. ☐

Example 9.6. *The fppf-Topology is saturated.*

Proof. Given a faithfully flat algebra homomorphism $A \to B$ with B faithfully flat, we want to show, that A is faithfully flat. First observe, that A is flat by the previous lemma. Then if $M \otimes_R A = 0$ for some R-module M, then $M \otimes_R B = M \otimes_R A \otimes_A B = 0$. As B is faithfully flat over R, we conclude $M = 0$. ☐

Example 9.7. *The unramified-topology (unramified + fppf) is saturated.*

Proof. Let $\operatorname{Spec} B \to \operatorname{Spec} A$ be unramified + fppf and $\operatorname{Spec} B$ unramified + fppf. We have to show that $\operatorname{Spec} A$ is unramified (fppf is the above example). For this, we may show that identity types $x = y$ are $\neg\neg$-stable. So assume $\neg\neg(x = y)$.
As $\operatorname{Spec} A$ admits a faithfully flat map with flat affine domain, the identity type $x = y$ admits such a map $\operatorname{Spec} B' \to x = y$ as well. As its fibers are $\neg\neg$-inhabited, we can conclude that the flat $\operatorname{Spec} B'$ is $\neg\neg$-inhabited, hence fppf. But now $x = y$ is a fppf-covering -1-stack, hence contractible 10.5. ☐

Lemma 9.8. *The étale topology is saturated*

Proof. fppf is clear by saturatedness of the fppf topology. Conclude By 14.16 ☐

10 Geometric propositions

Definition 10.1. $U : \mathsf{Aff}$ is called weakly-flat, if

$$\|U\|_{\mathbb{T}} \to (U \in \mathbb{T})$$

Lemma 10.2. *The converse holds always*

Proof. because things in $\mathbb{T}$ are automatically $\mathbb{T}$-merely inhabited $\square$

Example 10.3. *Examples of weakly-flat affines for the Zariski topology*

- *finite sums of principal opens*
- *Closed propositions*

for the fppf topology: flat affines .
For the étale topology: formally étale affines

Recall the definition of $\mathbb{T}$-atlas 1.1

Definition 10.4. Let $\mathbb{T}$ be saturated. We call a modal proposition geometric, if one of the equivalent conditions is satisfied:

1. its merely of the form $\|U\|_{\mathbb{T}}$ for some geometric affine U.
2. It is a geometric stack
3. It has a $\mathbb{T}$-atlas.

Proof.

$1 \Rightarrow 2$ we show that $U \to \|U\|_{\mathbb{T}}$ is a geometric atlas. Every fiber is in $\mathbb{T}$, because U is geometric. A $\mathbb{T}$-atlas is a geometric atlas.

$2 \Rightarrow 3$ If P is a geometric -1-stack, then we may choose $U \to P$ a geometric atlas. This is a $\mathbb{T}$-atlas by 8.15.

$3 \Rightarrow 1$ Let $V \to P$ be a $\mathbb{T}$-atlas. have to show TFAE $\|V\|_{\mathbb{T}} \to P \to (V \in \mathbb{T}) \overset{10.2}{\to} \|V\|_{\mathbb{T}}$. Proof: $\|V\|_{\mathbb{T}} \to P$ as P is modal prop. Secondly, because $V \to P$ is a $\mathbb{T}$-cover.
Hence P is a geometric proposition.

$\square$

Lemma 10.5. *Even Without any saturatedness condition, Covering -1-stacks X are contractible.*

Proof. Choose a geometric catlas $\mathbb{T} \ni \operatorname{Spec} A \to X$. By the same trick as in the previous lemma, this induces an equivalence $1 \simeq \|\operatorname{Spec} A\|_{\mathbb{T}} \xrightarrow{\sim} X$. $\square$

Example 10.6. *Open / Closed Propositions are geometric.*

Question 1. Is every geometric proposition a scheme?

It is an algebraic space that embeds into an affine, so it suffices to reproduce the statement from the presheaf model.

11 Algebraic Space

Recall the notion of (covering) geometric 0-stacks, which we call (covering) Algebraic Spaces. it is the smallest pair of classes that satisfies the following

- (covering) affines are (covering) algebraic spaces.
- stable under covering quotients: If X is an algebraic space, Y modal 0-type and $X \to Y$ is fibered in covering algebraic spaces, then Y is an algebraic space. Additionally, if X is covering, then Y is covering.

11.1 Equivalence relations vs Surjections

Lemma 11.1. *Denote $\mathbb{T}Set$ for the sets that are $\mathbb{T}$-sheaves. Assume given a $\mathbb{T}set$ X then the following maps are mutually inverse*

$$\mathrm{EqRel}(X, \mathbb{T}\,\mathrm{Prop}) \equiv \sum_{R:X\to X\to\mathbb{T}\,\mathrm{Prop}} R \text{ equivalence relation} \simeq \sum_{Y:\mathbb{T}\mathsf{Set}} \sum_{p:X\to Y} p \text{ } \mathbb{T}\text{surjective}$$

$$R \mapsto (L_{\mathbb{T}}\|X//R\|_0, [\_])$$

$$\lambda x, y.(p(x) = p(y)) \leftarrow\!\shortmid (Y, p)$$

Question 2. Do we actually need to set-truncate? Do we want to also mod out relations which are not given as an equivalence relation?

Proof.

- Well-definedness: The map $[\_] : X \to \|X//R\|_0 \to L_T\|X//R\|_0$ is the composition of a surjective with a $\mathbb{T}$-surjective map [ref?], hence its $\mathbb{T}$-surjective.
  Conversely given (Y, p) as Y is a sheaf, we have for all $x, y : X$ that $p(x) =_Y p(y)$ is a sheaf.
- If $x, y : X$ then we have a chain of equivalences

$$R(x, y) \simeq (\bar{x} =_{\|X//R\|_0} \bar{y}) \overset{\mathsf{ap}_\eta}{\to} ([x] =_{L_T\|X//R\|_0} [y])$$

  where the first map is plain HoTT, meaning that $\|X//R\|_0$ is seperated. The second map is an equivalence by 6.5.
- Let (Y, p) be in the RHS. Let $R(x, y) = (p(x) = p(y)) : \mathbb{T}\,\mathrm{Prop}$. By plain HoTT, There is a map $\eta : X//R \to Y$ (defined by the universal property of the set truncation and by induction on the higher inductive type $X//R$ on canonical terms through the map $p : X \to Y$). I claim η exhibits Y as the localization for $\mathbb{T}\,\mathsf{Set}$-modality of $X//R$. Let T be another $\mathbb{T}\,\mathsf{Set}$ equipped with a map $X//R \to T$. By precomposition we obtain a map $X \to T$. Claim: it factors uniquely through $p : X \to Y$.

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X//R \arrow[r] & T \\ & Y \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

  Proof:
  Existence: We want to define a map $Y \to T$. Let $y : Y$. As p is $\mathbb{T}$-surjective and T is a sheaf, we may assume we merely have some element in the fiber of p over y. Now push this element through

$$\|\,\mathrm{fib}_p\, y\| \to \|X//R\|_0 \to T$$

  where the first map is by Plain HoTT and the second one is induced from $X//R \to T$ by assumption and the fact that T is a set.. One can easily check this makes the diagram

commute. Uniqueness follows from $X \to Y$ beeing $\mathbb{T}$-surjective and the following Fact: Two parellel maps $Y \rightrightarrows T$ into a $\mathbb{T}\,\mathsf{Set}$ T are already equal if the become equal after precomposition with a $\mathbb{T}$-surjection $X \to Y$.
Proof of the fact : Let $y : Y$. The goal is an identity type of a $\mathbb{T}\,\mathsf{Set}$, hence a $\mathbb{T}\,\mathrm{Prop}$. Hence As the fiber over y in X is $\mathbb{T}$-merely inhabited, we may assume an actual term in the fiber. As $X \to Y$ equalizes the arrows, this term allows us to conclude. □(fact) □(Claim)
We apply the fact to the ($\mathbb{T}$-)surjectivity of $X \to X//R$ to get a unique factorization

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[dr] & X//R \arrow[r] \arrow[d] & T \\ & Y \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

making the right triangle commute. This is what we wanted to show.

□

Definition 11.2. Let S be a geometric stack. A relation R on S is called covering, if every fiber $R_s \equiv \sum_{t:S} R(s,t)$ is a covering stack.

Lemma 11.3. *If R is covering on S, then the propsotions $R(x,y)$ are geometric propositions, in particular sheaves.*

Proof. For all $s,t : S$, $R(s,t)$ is a geometric stack: $R(s,t)$ is the fiber of the projection $\sum_{t:S} R(s,t) \to S$ between geometric stacks, which are stable under finite limits.

□

Lemma 11.4. *If S is affine, then a modal equivalence relation on S is covering iff every fiber $R_s \equiv \sum_{t:S} R(s,t)$ merely admits a $\mathbb{T}$-catlas.*

Proof. Every sheaf admitting a $\mathbb{T}$-catlas is a covering 0-stack. Conversely: Let $s : S$ such that the fiber R_s is a covering 0-stacks. We want to construct at $\mathbb{T}$-catlas of R_s. The $R(s,t)$ are geometric propositions by 11.3. For all $t : S$ we there merely is a geometric atlas $\operatorname{Spec} A_t \to R(s,t)$ by 10.4. By Zariski Local choice we find a Zariski cover $f : S' \to S$ equipped with a Geometric atlas $\operatorname{Spec} A_{t'} \to R(s, f(t'))$ for all $t' : S$. Then

$$\sum_{t:S'} \operatorname{Spec} A_{t'} \to \sum_{t:S} R(s,t)$$

is a $\mathbb{T}$-atlas by 5.5. As $\sum_{t:S} R(s,t)$ is a covering 0-stack by assumption, the map has to be a $\mathbb{T}$-catlas by 8.16. □

Lemma 11.5. *Given an affine X, the following types are equivalent:*

- *The type of covering equivalence relations on X.*
- *The type of $\mathbb{T}$sets Y equipped with a map $X \to Y$ fibered in types admitting a $\mathbb{T}$-catlas.*

Proof. By the equivalence in **??** it is enough to check that The fibers of:

$$[\_] : X \to L_{\mathbb{T}} \| X//R \|_0$$

merely admit a $\mathbb{T}$-catlas if and only if the relation R is covering. For any $y : X$ we have that:

$$\sum_{x:X} R(x,y) \simeq \operatorname{fib}_{[\_]}([y])$$

so the direct direction is immediate. The converse follows from $\mathbb{T}$-surjectivity of $[\_]$ and from 7.4. □

12 Algebraic spaces

Theorem 12.1. *Let X be a modal set. The following are equivalent:*

1. *X is a (covering) geometric 0-stack*
2. *X is merely of the form $L_{\mathbb{T}}(U/R)$ for some (covering) affine U and $R : U^2 \to \mathrm{Prop}_{\bigcirc}$ a covering equivalence relation.*
3. *there exists some map $S \to X$ with S (covering) affine whose fibers merely have $\mathbb{T}$-catlasses.*

We call this class (covering) algebraic spaces.

Proof.

$2 \leftrightarrow 3$ This is 11.5

$2 \to 1$ Choose a presentation $R : U^2 \to \mathrm{Prop}$. It suffices to show, that the map $f : U \to L_{\mathbb{T}}(U/R)$ is a geometric (c)atlas. The map f is $\mathbb{T}$-surjective by the well-definedness of the bijection **??**. By descent we may just show, that the fibers $\mathrm{fib}_f(f(s))$ for $s : U$ are covering 0-stacks. But by the bijection in **??** those are equivalent to the fibers R_s, which are covering 0-stacks as the equivalence relation is covering.

$1 \to 2$ This can be reformulated in the following way, using the recursion principle for (covering) geometric 0-stacks: Let X be a sheaf of sets. Let S be (covering-) affine and $f : S \to X$ be fibered in covering algebraic spaces. Then X is a (covering) algebraic space. This follows from the observation, that the equivalence relation determined by f is covering 11.2 , because the fibers of f are covering 0-stacks.f

□

Proposition 12.2. *For any $n \geq 1$, we have inclusions*

$$W_n \subset \mathsf{CS}_{n-1} \subset W_{n+1}$$

Proof. Induction. $n = 1$ gives

$$\mathsf{HasCatlas}_{\mathbb{T}} \subset \mathsf{CS}_0 \subset \text{ types admitting a catlas fibered in } W_1$$

the latter inclusion is the previous theorem.
The induction step is obtained by 12.4 □

12.1 Schemes are algebraic Spaces for the Zariski Topology

Definition 12.3. A proposition U is open iff its merely of the form $f_1\ inv \vee \ldots f_n inv$ for some $f_i : R$.

Lemma 12.4. *Given $f_1, \ldots, f_n : R$ such that $\|D(f_1) + \ldots + D(f_n)\|$ then $\sum_{i=1}^{n} D(f_i) \in \mathsf{Zar}$.*

Proposition 12.5. *Every Zariski-merely-inhabited type that is merely of the form $U_1 + \ldots + U_n$ for open propositions U_i admits a Zar-catlas.*

Proof. By definition of openness, We can choose a surjection $\coprod_{j=1}^{n_i} D(f_{ij}) \twoheadrightarrow U_i$ for any i. We want to show, that the map

$$\coprod_{i,j} D(f_{ij}) \twoheadrightarrow U_1 + \ldots U_n$$

is a Zar-catlas.

- Let us first show that the fibers are in Zar. Assume U_i holds. So we find a term in $\coprod_j D(f_{ij})$. In particular we have $\| \coprod_j D(f_{ij})\|_{\mathsf{Zar}}$. By the lemma we conclude, that the fiber $\sum_j D(f_{ij})$ belongs to Zar.

- The total space is in Zar: This follows as the surjection after propositional truncation becomes an equivalence. As we have $\|U_1 + \ldots + U_n\|$, we can conclude by the lemma.

$\square$

Warning. The converse does not hold! We want to apply 8.32, to the map

$$\mathsf{Zar} \ni 1 + 1 \to \sum D(f)$$

- $\sum D(f)$ is seperated as $D(f)$ is a sheaf.
- All the fibers are equivalent to $1 + X$, hence they are in the Zariski topology.

Lemma 12.6. *let X be a scheme. There merely exists some affine S map $S \to X$ whose fibers are merely inhabited finite sums of open propositions*

Corollary 12.7. *Every scheme is an algebraic space.*

Lemma 12.8. *If X is an algebraic space, then the global sections embed via a R-algebra homomorphisms into a finitely presented R-algebra.*

Proof. Choose an atlas $S \to X$, in particular $\mathbb{T}$-surjective. As $\mathbb{T}$ is subcanonical the map $R^X \to R^S$ is an injection. $\square$

Question 3. Is it an open embedding of types?

Example 12.9. *Using descent for Affines (compare to [3]), not every scheme admits a $\mathbb{T}$-cover out of an affine: Let us show that for any open proposition P: If $\mathsf{Susp}(P)$ admits a $\mathbb{T}$-cover $\operatorname{Spec} S \to \mathsf{Susp}(P)$ then P is affine.*
Then by Zariski Local choice, we may assume that it factors through Bool $\to \mathsf{Susp}(P)$, so then we may call the two fibers of the map $\operatorname{Spec} S \to$ Bool, $\operatorname{Spec} A$ and $\operatorname{Spec} B$. Then the two fibers are $F_1 \equiv \operatorname{Spec} A + P \times \operatorname{Spec} B$ and $F_2 \equiv \operatorname{Spec} B + P \times \operatorname{Spec} A$, which by assumption are in $\mathbb{T}$. P beeing affine is a sheaf, thus we may show $F_2 \to \mathsf{isAff}(P)$ So pick a point of F_2. If it comes from $P \times \operatorname{Spec} A$, then P so P is affine. If we have $t : \operatorname{Spec} B$, P is a fiber of $P \times \operatorname{Spec} B \to \operatorname{Spec} B$, but the domain is affine using that $F_2 \in \mathbb{T}$. So P is affine.

12.2 Examples

The goal of this subsection is to construct algebraic spaces. The first example actually gives us a scheme:

Example 12.10. *Let $p \neq 0$ be a prime. You can let $\mu_p := \operatorname{Spec}(R[X]/(X^p - 1))$ act on $\mathbb{A}^\times$ via multiplication. Set $\mathbb{T} = fppf$. Then the p.th power map*

$$pow : \|\mathbb{A}^\times // \mu_p\|_0^{\mathbb{T}} \to \mathbb{A}^\times$$

is an equivalence.

- *It is an embedding: First note, that $\|\mathbb{A}^\times // \mu_p\|_0$ is $\mathbb{T}$-seperated:*

  as μ_p act freely on $\mathbb{A}^\times$, $\mathbb{A}^\times // \mu_p$ is already a set. Meaning that the identity types of the set-quotient are $\sum_{g:\mu_p} gx =_{\mathbb{A}^\times} y$, hence sheaves.
  On the other hand the map $\|\mathbb{A}^\times // \mu_p\|_0 \to \mathbb{A}^\times$ is an embedding, as for any $x, y : \mathbb{A}^\times$ the map $(\sum_{g:\mu_p} gx = y) \to (x^p = y^p)$ is an equivalence.

- *It is $\mathbb{T}$-surjective, as for any $\lambda : \mathbb{A}^\times$, we find $S = \operatorname{Spec} R[X]/(X^p - \lambda) \in \mathbb{T}$ with*

$$S \to \operatorname{fib}_{\mathbb{A}^\times/\mu_p \to \mathbb{A}^\times}(\lambda)$$

  hence

$$1 = \|S\|_{\mathbb{T}} \to \| \operatorname{fib}_{pow} \|_0^{\mathbb{T}}$$

Example 12.11. *Let P be the open proposition $x \neq 0$ for some $x : \mathbb{A}^1$. Then $H = 1 + P$ is an open subgroup of $\mathbb{Z}/2$. The sheaf quotient G/H is the scheme $\mathsf{Susp}(x \neq 0)$.*

Let $\ell \neq 0$ denote a prime. Consider $\mu_\ell = R[X]/(X^\ell - 1)$ acting on $\mathbb{A}^1$ by multiplication.

Lemma 12.12. *Let $(G, 1)$ be a pointed formally étale flat affine type. Then $(G \setminus \{1\})$ is formally étale + flat affine.*
In particular $\mu_\ell \setminus \{1\}$ is a covering stack.

Proof. $G \setminus \{1\} = \sum_{g:G} g \neq 1$ is a $\sum$ of formally étale + flat affines (recall that formally étale affines have decidable equality).
To show, that $\mu_\ell \setminus \{1\}$ is a covering stack, by 17.1, we need to show it is $\neg\neg$-inhabited. Indeed as we want to prove a contradiction we may assume a term in $g : \operatorname{Spec} R[X]/(\sum_{i=0}^{\ell-1} X^i)$. But this type is equivalent to $\mu_\ell \setminus \{1\}$, using that $\sum_{i=0}^{\ell-1} X^i | X^\ell - 1$ and $\ell \neq 0$. □

Lemma 12.13. *Given a modal equivalence relation R on a sheaf X and a 1-stack T and a map $f : X \to T$ and term $p : \prod_{x,y:X} R(x, y) \to fx = fy$ such that $p(x, y) \cdot p(y, z) = p(x, z)$, where the witnesses for R are left implicit. Then f factors through the quotient.*

Lemma 12.14. *Put $\ell = 2$ If $\ell \neq 0$, the sheaf quotient of $\mathbb{A}^1$ by the μ_2 action is not an algebraic space.*

Proof. Assume this it is an algebraic space.
Set $\mathbb{D}(1) = \operatorname{Spec} R[X]/X^\ell$. Then $\sum_{x:\mathbb{A}^1/\mu_\ell} x^\ell =_{\mathbb{A}^1} 0 \simeq \mathbb{D}(1)/\mu_\ell$ is an algebraic space by $\sum$-stability.
Then we can choose a geometric atlas $p : \operatorname{Spec} A \to \mathbb{D}(1)/\mu_\ell$. We proceed in the following steps

1. There is an equivalence $\operatorname{Spec} A \simeq \operatorname{fib}_p 0 \times \mathbb{D}(1)/\mu_\ell$.
2. The fiber over 0 is affine
3. $\mathbb{D}(1)/\mu_\ell$ is $\neg\neg$ affine
4. $\mathbb{D}(1)/\mu_\ell$ is $\neg$ affine

Proofs

1. Let us denote $F : \mathbb{D}(1)/\mu_2 \to \mathsf{CS}_0$ the bundle of fibers of f, where we note that the fibers are indeed sets. As CS_0 is formally étale ([ref?]), we have terms

$$\phi : \prod_{x:\mathbb{D}(1)} F[x] = F[0], \phi^- : \prod_{x:\mathbb{D}(1)} F[-x] = F(0)$$

   that both evaluate at $x = 0$ to $\mathsf{refl}_{F[0]}$.
   The goal is to produce a term in

$$\prod_{x:\mathbb{D}(1)/\mu_2} Fx = F[0]$$

By the previous lemma, using that CS_0 is a 1-stack, we need to show, that under the path $p_x : [x] = [-x]$ in the quotient we have

$$\mathsf{ap}_{p_x} F \cdot \phi^- x = \phi x$$

This proposition is formally étale as CS_0 is formally étale . Thus we may assume the closed dense proposition $x = 0$. Then $p_x = \mathsf{refl}_{[0]}$ and $\phi^- 0 = \mathsf{refl} = \phi 0$ by assumption.

2. Let us first show, that We may assume that our geometric cover factors through the $\mathbb{T}$-surjection $\operatorname{Spec} A \xrightarrow{f} \mathbb{D}(1) \to \mathbb{D}(1)/\mu_\ell$. Proof: By $\mathbb{T}$-local choice applied to the $\mathbb{T}$-surjection $\mathbb{D}(1) \to \mathbb{D}(1)/\mu_\ell$, we find a $\mathbb{T}$-cover $\operatorname{Spec} B \to \operatorname{Spec} A$ and a factorization

$$\begin{tikzcd}
\exists \operatorname{Spec} B \arrow[r, dashed] \arrow[d, dashed] & \mathbb{D}(1) \arrow[d, two heads] \\
\operatorname{Spec} A \arrow[r] & \mathbb{D}(1)/\mu_\ell
\end{tikzcd}$$

$\square$(Claim)

Its enough to see that the map $\operatorname{fib}_f 0 \to F$ is an equivalence. That follows because 0 is a fix point of the μ_ℓ action on $\mathbb{D}(1)$.

3. F is a covering stack, hence $\neg\neg$-inhabited. As the goal is $\neg\neg$-modal, we may pick a map $1 \to F$. Then, by step 1

$$\mathbb{D}(1)/\mu_\ell = 1 \times_F (F \times \mathbb{D}(1)/\mu_\ell) = 1 \times_F \operatorname{Spec} A$$

is a fiber product of affines, hence affine.

4. Here we need that $\ell = 2$. The affinization map would be induced by

$$\begin{tikzcd}
\mathbb{D}(1) \arrow[d] \arrow[dr, "z \mapsto z^\ell"] & \\
\mathbb{D}(1)/\mu_\ell \arrow[r, dashed] & \mathbb{D}(1)
\end{tikzcd}$$

But the map is not an embedding: For any $\varepsilon : \operatorname{Spec} R[X]/X^\ell$, we have $\varepsilon^\ell = 0^\ell$ but $\varepsilon =_{\mathbb{D}(1)/\mu_\ell} 0$ iff there $\mathbb{T}$-merely exists some $g : \mu_\ell$ with $g\varepsilon = 0$, but as g is invertible this is equivalent to $\varepsilon = 0$.

$\square$

12.3 Non locally-seperated Examples

Proposition 12.15. *Consider an affine S and an open subset $U \subset S$. Consider a $\mathbb{T}$-flat irreflexive relation $\sharp$ on U , i.e.*

1. *Irreflexivity: $\neg(x \sharp x)$*
2. *$\mathbb{T}$-flatness. For all $x : U$, the fiber $\sum_{y:S} x \sharp y$ is $\mathbb{T}$-flat.*

Define a relation on S as

$$R_\sharp(x, y) = (x = y) + (x \in U \land y \in U) \times (x \sharp y)$$

(Abuse of notation: where the $\times$ is secretly a $\sum$) Then the sheaf quotient $S/R_\sharp$ is an algebraic space.

Proof.
- This is a proposition: First note, that both summands are propositions and if both summands are inhabited we get a contradiction.
- The relation is covering: Furthermore, for any $x : S$ we have

$$\sum_{y:S}(x = y) + (x, y \in U \times x\sharp y) = 1 + (x, y \in U \times \sum_{y} x\sharp y) \in \mathbb{T}$$

as $\sharp$ was assumed to be $\mathbb{T}$-flat on U: we can write the binary product as $\sum_{p:x\in U} \sum_y x\sharp y$, a $\sum$ of of $\mathbb{T}$-flat geometric stacks.

□

12.4 Group quotients

For this section let G denote a group that is a covering 0-stack. Let X be a sheaf equipped with a G action.

Lemma 12.16. *$\mu_p = \operatorname{Spec} R[X]/(X^p - 1)$ is covering for $p \neq 0$ prime.*

Proof. It is fppf + étale as $X^p - 1$ is monic seperable. TODO □

Definition 12.17. A G action on X is free, if for all $x, y : X$ the type

$$\sum_{g:G} gx = y$$

is a proposition.

Example 12.18. *Given a formally étale + flat affine (e.g. μ_ℓ or finite) group that acts free on some open subset $U \subset \operatorname{Spec} A$ of some affine.*
Then put $x\sharp y = \sum_{g:G\setminus\{1\}} gx = y$.
This provides a covering equivalence relation $R_{G,U}$ on $\operatorname{Spec} A$, such that

- *for any $x : U$*

$$[x] = \sum_{y:\operatorname{Spec} A} \sum_{g:G} gy = x.$$

- *for some $y \notin U$, we have $R(x, y) \leftrightarrow x = y$.*

By abuse of notation we write $\operatorname{Spec} A/_U G \equiv \operatorname{Spec} A/R_G$ and call it the quotient of $\operatorname{Spec} A$ by the G-action .

Proof.
- It is irreflexive: If $x : U$ then $gx \neq x$, by freeness.
- We have $G \setminus \{1\}$ is flat affine using 12.12.

□

Notation. If $U = \operatorname{Spec} A \setminus Z$ the complement of a closed subset we write

$$U \equiv Z^c$$

Example 12.19 (Free action)**.** *Set $U = \operatorname{Spec} A$. Then this construction yields the actual group quotient. The quotient of $\mathbb{A}^\times$ by the free μ_ℓ action gives a scheme.*

Lemma 12.20. *Algebraic spaces are stable by free quotients of covering group 0-stacks.*

Proof. The map $X \to L_T(X/G)$ is fibered in covering 0-stacks, so in particular covering 0-stacks. As X is a geometric 0-stack, the quotient is a geometric 0-stack as well, This follows by the description in , choosing a geometric atlas of X and postcomposing this to get a geometric atlas of the quotient. □

Example 12.21. *If $p : \sum_{r:R} S_r \to R$ be a map between formally étale + flat affine into R whose fibers, except possibly the fiber over 0, are formally étale + flat. Define $U = (x \neq 0) \times S_x \subset S_x$. $y \sharp y' \equiv y \neq y'$ is an irreflexive $\mathbb{T}$-flat relation on S_x. From this we obtain the algebraic space*

$$\sum_{x:R} Y_x / R_\sharp$$

which we will later recognize as a fiber collapse.

Proof. $\sharp$ is a modal irreflexive relation. By assumption we have given $\mathbb{T}$-flatness of S_x if $x \neq 0$. $\square$

Lemma 12.22 (Not needed)**.** *Let Y be affine. Let $X \hookrightarrow Y$ be a map fibered in locally closed propositions. Then its factors as the composite of a closed and then an open embedding*

Proof. By zariski local choice we find $Y = \bigcup Y_i$ and factorizations of the basechanges $X_i \to Z_i \to Y_i$. Then $\bigcup X_i \to \bigcup Z_i \to \bigcup Y_i = Y$ is a global factorization. $\square$

Proposition 12.23. *Let S be affine with a regular point $*$. Assume we have function $g : S \to S$ such that $*$ is the unique fixpoint $*$ (e.g. if $(S, *)$ admits a pointed-free action of a nontrivial group) Let $\sharp$ be an irreflexive $\mathbb{T}$-flat relation on $U \equiv \sum_{x:S} x \neq *$, such that for all $y : U$, we have $gy : U$ and $y \sharp gy$. Then the algebraic space $S/R_\sharp$ is non locally seperated, in particular not a scheme.*

Proof. It is an algebraic space by the previous prop.

A pointed-free action of a non-trivial group yields such a map g: If $\neg(G = \{1\})$, then $\neg\neg(G \setminus \{1\})$ by decidable equality of G. As we want to prove a contradiction, we may assume $g : G \setminus \{1\}$, this yields a map $S \to S$ such that

- $*$ is the unique fixpoint by the pointed-freeness
- If $y \neq *$, then $gy \neq *$ and $y \sharp gy$

We have that every scheme X is locally-seperated, i.e. its identity types are locally closed. Indeed, this follows from the proof of Foundations Prop 5.5.2 .
Let us show that R is not valued in locally closed propositions. Recall

$$y \in U \to y \sharp gy \tag{1}$$
$$y \in U \leftrightarrow y \neq * \tag{2}$$

We have for any $y : S$

$$R_\sharp(y, gy) \simeq (y = gy) + (y \in U) \wedge y \sharp gy \overset{(1)}{\simeq} (y = *) + (y \in U) \overset{(2)}{\simeq} (y = *) + (y \neq *)$$

But if this is locally closed for all $y : S$, we have a contradiction to 4.4. $\square$

Corollary 12.24. *Let $Y : R \to \mathsf{Aff}$ be formally étale + flat affine away from the origin , such that $p : \tilde{Y} = \sum_{x:R} Y_x \to R$ is regular and Y_0 is infinitesimal. If you find a map $g : \tilde{Y} \to \tilde{Y}$ over p with a unique fixpoint, which lies over 0, then the algebraic space—Y— is non-locally-seperated, In particular not a scheme.*

Proof. Lets call the unique fix point $*$, i.e. we have

$$gy = y \leftrightarrow y = *$$

Note that $* : \tilde{Y}$ is a regular point , as $p : \tilde{Y} \to R$ is a regular section with Y_0 infinitesimal. $\square$

Definition 12.25. A pointed-free action of G on a pointed type $(X, 0)$ is a G-action with fixpoint 0, such that if $g\varepsilon = \varepsilon$ for some $g \neq 1$, then $\varepsilon = 0$.

Lemma 12.26. *Let G be a group with decidable equality acting pointed free on a pointed type $(X, 0)$. Then G acts free away from zero.*

Proof. let $x, y \neq 0$. We need to show, that $\sum_g gx = y$ is a proposition. Let $g, g' : G$ such that $gx = y = g'x$. as G has decidable equality, we may show $\neg\neg(g = g')$. If $g^{-1}g' \neq 1$, then by pointed-freeness applied to $g^{-1}g'x = x$, we have $x = 0$. Contradiction. $\square$

Corollary 12.27. *Let $0 : \operatorname{Spec} A$ be a regular point. Let G be a nontrivial formally étale flat affine group acting pointed- freely on the pointed affine $(\operatorname{Spec} A, 0)$. Then the pointed-free quotient of $\operatorname{Spec} A$ by G from 12.18 is non-locally-seperated, In particular not a scheme.*

Example 12.28 (Non locally-separated examples)**.** *Assume $\ell \neq 0$ prime. Let μ_ℓ act on $(\operatorname{Spec} B, 0)$ in one of the following ways:*

1. *Let μ_ℓ act on $\operatorname{Spec} B = \mathbb{A}^1$*
2. *Let μ_ℓ act on*
$$\operatorname{Spec} B \equiv \sum_{x,y:R} x^\ell = y^\ell$$
via $g(x, y) = (x, gy)$

Then $\operatorname{Spec} B/_{0^c}\mu_\ell$ is an algebraic space that is not a scheme.

Proof. $\neg\neg$ merely, μ_ℓ is finite ([ref?]) and $\mu_\ell \setminus \{1\}$ is inhabited by 12.12.

1. Pointed-Free action is clear. $0 : \mathbb{A}^1$ is regular by first projection.
2. Pointed-Free action is clear. The cross middlepoint regular, witnessed by the first projection: It is regular vanishing at $(0, 0)$ And for any point $(0, y) : \operatorname{Spec} B$ we deduce $y^\ell = -0^\ell = 0$, hence $\neg\neg(x, y) = (0, 0)$.

$\square$

Question 4. If μ_ℓ acts on Y some affine, does every μ_ℓ-invariant $\phi : Y \to R$ is invariant on a ℓ-neigborhood?

12.5 Obsolete

Proposition 12.29. *Let $Y : R \to \mathsf{Aff}$ be formally étale + flat affine away from the origin If you find two sections $y, y' : \prod_{x:R} Y_x$ such that $y_x = y'_x \leftrightarrow x = 0$, then then the algebraic space—Y— is non-locally-seperated, In particular not a scheme.*

Proof. It is an algebraic space by the previous prop.

We have that every scheme X is locally-seperated, i.e. its identity types are locally closed. Indeed, this follows from the proof of Foundations Prop 5.5.2 .
Let us show that R is not valued in locally closed propositions. We have

$$\prod_{x:R} \eta y_x = \eta y'_x \simeq \prod_{x:R} y_x = y'_x + (x \neq 0) \times y_x \neq y'_x \simeq (x = 0) + (x \neq 0)$$

but if this is locally closed for all $x : R$, we have a contradiction to 4.4.

$\square$

Lemma 12.30 (Not needed)**.** *For an algebraic space X, we have implications $1 \Rightarrow 2 \Rightarrow 3$*

1. *X admits an seperated open cover.*

2. *For any covering equivalence relation $R : U^2 \to \mathrm{Prop}$ on an affine U such that $X = U/R$, F is valued in locally closed propositions*

3. *We find such a presentation such that R is valued in locally closed propositions.*

*Proof.*1 $\Rightarrow$ 2 Let $X' \to X$ be a map fibered in merely inhabited finite sums of open propositions with X' a seperated algebraic space. Then any geometric atlas $U \to X'$ will be fibered in closed subtypes of U. We need to show, that the fibers of $U \to X' \to X$ are locally closed subtypes of U. Let $x : X$. the fiber in X' is of the form $U_1 + \ldots + U_n$. Thus the fiber in U is a finite sums of $\sum$ of $U_i \to (U \to \mathsf{ClosedProp})$, which is enough.

3 $\Rightarrow$ 1 Let $x : X$.

□

Lemma 12.31 (Not needed)**.** *Let char $\neq 2$. Let $p : R[X]$ be such that $0 \in D(p)$ and $x \in D(p)$ implies $-x \in D(p)$. If $f : R[X]$ is a polynomial such that $f(x) = f(-x)$ for all $x : D(p) \setminus \{0\}$, then f is even i.e. in the image of $R[X^2] \hookrightarrow R[X]$.*

Proof. We splitting f into $f_1 + Xf_2$ for $f_i : R[X^2] \subset R[X]$. I claim, that $f_2 = 0$ in $R[X]$. realizing that $(Xf_2)(x) = (Xf_2)(-x)$ implies $2f_2(x)x = 0$, thus $f_2(x)x = 0$ for all $x : D(p) \setminus 0 = D(pX)$, thus by the previous lemma $X \cdot f_2 = 0$ in $R[X]$, hence $f_2 = 0$. □

Lemma 12.32. *Let G be a finite group whose cardinality is invertible in R. Let G act on an affine scheme equipped with a fixpoint* 0. *Let U be an open neighborhood of 0, such that $g(U) = V$ for all $g : G$. Then we find some G-invariant p such that $0 \in D(p) \subset V$.*

Proof. Choose a principal open neighborhood $0 \in D(p) \subset U$. G acts on $R[X]$, via $(g.p)(x) = p(gx)$.Then

$$p' = \sum_{g:G} g.p : R[X]$$

is a G-invariant polynomial, in particular $D(p)$ is G-invariant. Moreover $0 \in D(p')$ as

$$p'(0) = \sum_{g:G} p(g(0)) = \sum_{g:G} p(0) = |G| \cdot p(0)$$

is invertible, as $|G|$ and $p(0)$ are both invertible. Furthermore, as U was G invariant and contained $D(p)$ it also has to contain $D(p')$: Indeed

$$D(p') \subset \bigcup_g D(g.p) \subset U$$

□

Lemma 12.33. *Let G be a formally étale + flat affine group, such that $\neg\neg$ its finite, with cardinality invertible in R and $G \setminus \{1\}$ inhabited. Let it act on an affine scheme* $\operatorname{Spec} A$ *with a fixpoint 0. Let R be a relation on* $\operatorname{Spec} A$ *such that*

- *$R(x, y)$ implies that there merely is some g with $y = gx$.*
- *$\neg\neg R(x, gx)$*

Assume that for all $p : A^G$ with $0 \in D(p)$, $D(p)/R$ is not an affine scheme. Then $\operatorname{Spec} A/R$ *is not a scheme.*

Proof. Assume 0 admits a open affine neibhorhood U in $\operatorname{Spec} A/R$. The preimage along the quotient map obtained from the relation induces a open neigbhorhood V of 0 in $\operatorname{Spec} A$. As we want to prove a contradiction we may assume that μ_ℓ consists of ℓ many elements, where $\ell \neq 0$ in R. Note that V is G-invariant: For any $x \in V, g : G$, the goal $gx \in V$ as an open proposition is $\neg\neg$-stable , thus we may assume $R(x, gx)$.
We apply the previous lemma to V to obtain an invariant principal open neigborhood $0 \in D(p) \subset V \subset \operatorname{Spec} A$. As p is G-invariant, $p : \operatorname{Spec} A \to R$ descends to $X \to R$. Restricting to U' yields a map $p' : U \to R$, such that setting $U' \equiv D(p')$ yields $q^{-1}(U') = q^{-1}(D(p')) = D(p' \circ q) = D(p)$. We are now in the following situation

$$\begin{tikzcd}
D(p) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & V \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \operatorname{Spec} A \arrow[d, "q"] \\
U' \arrow[r, hook] & U \arrow[r, hook] & X
\end{tikzcd}$$

where U' is an open affine neighborhood of 0.
By assumption $U = D(p)/\sim'$ cannot be affine. Contradiction. □

Proposition 12.34 (Not needed)**.** *Let $\ell \neq 0$ be prime. Let μ_ℓ act on* $\operatorname{Spec} B$ *with fixpoint* 0. . *Let V be an infinitesimal neigborhood of* 0*, i.e. a subtype $0 \in V \subset \operatorname{Spec} B$ such that $\neg\neg x = 0$ for every $x : V$. Assume*

Strong freeness We find some $0 \in V' \subsetneq V$ for any $\varepsilon : \operatorname{Spec} B, g \neq 1$, $g\varepsilon = \varepsilon$ implies $\varepsilon \in V'$

checking away from 0 For any $p : B$ and any $\phi : R^{D(p)}$ such that $\phi|_{D(p)\setminus\{0\}} = 0$, we have that $\phi|_V = 0$.

The sheaf quotient of $\operatorname{Spec} B$ *by the relation as above is an algebraic space but not an affine scheme.*

Proof.
- Let us check the conditions on the relation
  - If $R(x, y)$ then either $x = y$ putting $g = 1$ or in the second case we get some g such that $gx = y$
  - Let $x : X, g : G$. Assume $\neg R(x, gx)$, i.e. $x \neq gx$ and $\neg\neg x = 0$.But 0 was assumed to be a fixpoint, hence $\neg\neg gx = x$.
- Let $p : B$ be as above. We have to show that the quotient of $D(p)$ is not affine.
  The conditions on p give $p(0) \neq 0$ and $p(x) \neq 0 \to p(gx) \neq 0$ for all $g : \mu_\ell$.

  Lets call this quotient X.

  Define
  $$A = \{\phi : R^{D(p)} \mid \phi|_{D(p)\setminus\{0\}} \text{ is } \mu_\ell\text{-invariant } \}$$
  This is an R-subalgebra: for any $r : R$, $r : B_p$ is μ_ℓ-invariant. μ_ℓ-invariant functions are stable under addition and multiplication .

  Claim: The affinization map of X is the induced dashed map $f : X \to \operatorname{Spec} A$ in

  $$\begin{tikzcd}
  D(p) \arrow[r, equal] \arrow[d, "q"] & \operatorname{Spec} R^{D(p)} \arrow[d, "q'"] \\
  X \arrow[r, dashed, "\exists! f"] & \operatorname{Spec} A
  \end{tikzcd}$$

  Proof: A function $\phi : D(p) \to R$ factors through q iff $\phi|_{D(p)\setminus\{0\}}$ is μ_ℓ-invariant. Thus the embedding (using that R is a sheaf) $R^X \hookrightarrow R^{D(p)}$ has image A □(Claim).

  Proof that X is not an affine: Assume that X were affine. Then the map f would be

in particular an embedding. We may assume a term $g : \mu_\ell \setminus \{1\}$: Indeed, as we want to prove a contradiction we may assume a term in $g : \operatorname{Spec} R[X]/(\sum_{i=0}^{\ell-1} X^i)$. But this type is equivalent to $\mu_\ell \setminus \{1\}$, using that $\sum_{i=0}^{\ell-1} X^i | X^\ell - 1$ and $\ell \neq 0$.
The given infinitesimal neigbhorhood V satisfies $V \subset D(p)$, using that invertibility is $\neg\neg$ stable. Then for any $\varepsilon : V$ we have

$$(q\varepsilon =_X q(g\varepsilon)) \overset{??}{=} (\varepsilon = g\varepsilon) + (\varepsilon \neq 0 \wedge \sum_{h \neq 1} \varepsilon = hg\varepsilon) = (\varepsilon = g\varepsilon) = (\varepsilon \in V')$$

where the last step comes from pointed-freeness. But we have

$$(q'\varepsilon =_{\operatorname{Spec} A} q'(g\varepsilon)) = \left(\prod_{\phi : A} \phi(q'\varepsilon) = \phi(q'(g\varepsilon)) \right) = \prod_{\substack{\phi : R^{D(p)} \\ \phi \in A}} \phi(\varepsilon) = \phi(g\varepsilon),$$

The right hand side is inhabited: For any $\phi : D(p) \to R$ such that $\psi := \phi - g.\phi$ satisfies $\psi|_{D(p)\setminus\{0\}} = 0$ we have $\psi|_V = 0$ by 'checking away from 0', inparticular $\psi(\varepsilon) = 0$. So we conclude the the embedding $V' \hookrightarrow V$ is an equivalence. But we asked $V' \subsetneq V$ to be a proper subset.

□

Example 12.35. *Let μ_ℓ act on* $\operatorname{Spec} B = \mathbb{A}^1$.

Proof. 1. Put $V :\equiv \operatorname{Spec} R[X]/X^n$ for some $n > 1$.

2. As $(g-1)$ is invertible, $((g-1)\varepsilon = 0)$ gives us $\varepsilon \in \{0\} \equiv V' \subsetneq V$. Note that indeed V is non contractible, because $R[X]/X^n \to R[X]/X$ is not an algebra isomorphism

3. We have to show, that then ϕ is μ_ℓ invariant. We can apply 4.3, observing $\phi - g.\phi = 0$ on $D(X/1) \subset \operatorname{Spec} B_p$, where $X/1 : B_p$ is regular, because X is regular in B. TODO as each ϕ satisfies the cond. □(Claim)

□

Example 12.36. *Assume* $2 \neq 0$. *Let μ_2 act on*

$$\operatorname{Spec} B \equiv \sum_{x,y:R} xy = 0$$

via the swap. Then $\operatorname{Spec} B/R$ *is an algebraic space but not a scheme.*

Proof. 1. Put $V = \operatorname{Spec} R[X]/X^k \subset \operatorname{Spec} B$, $k > 2$.

2. If $(x, y) = (y, x)$ but $xy = 0$ we get $x \in V' \equiv \operatorname{Spec} R[X]/X^2$.

3. Let $\phi : D(p) \to R$ be 0 everywhere except near the origin. Then we get a restricted map $\phi' : D(p') \to R$ where $D(p') \subset V(X)$ is given by the intersection $D(p) \cap V(X)$. Indeed : Put $p' : R[X]$ the image of $p : R[X,Y]/(XY)$ und the map induced by evaluating Y at 0.
Here we can apply 4.3, getting that ϕ' is 0 everywhere in particular in $V \subset V(X)$.

□

12.6 Locally seperated examples

Lemma 12.37 (not needed)**.** *Given a map $P : \mathsf{Susp}(Q) \to \mathrm{Prop}$, such that $P(N)$ and $P(S)$ hold, then $\prod_{t:\mathsf{Susp}(Q)} P(t)$*

Lemma 12.38 (not needed)**.** *Assume $2 \neq 0$. For any $x : R$, the map*

$$\begin{aligned}\mathsf{Susp}(x \neq 0) &\to \sum_{y:R/x} y^2 = 1\\ N &\mapsto 1\\ S &\mapsto -1\end{aligned}$$

is well-defined and an equivalence.

Proof. The following maps are mutually inverse

$$\begin{aligned}\sum_{y:R/x} y^2 = 1 &\simeq \sum_{e:R/x} e^2 = e\\ y &\mapsto (y-1)/2\\ 2e - 1 &\mapsfrom e\end{aligned}$$

So it remains to show that the map

$$\begin{aligned}f : \mathsf{Susp}(x \neq 0) &\to \sum_{e:R/x} e^2 = e\\ N &\mapsto 1\\ S &\mapsto 0\end{aligned}$$

is a bijection.

- It is injective, i.e. for all $p, q : \mathsf{Susp}(x \neq 0)$, if $f(p) = fq$, then $p = q$. As the latter is a proposition, we may assume p, q beeing combinations of north and south poles. The interesting case is if wlog $p = N, q = S$. Then assuming $0 =_{R/x} 1$ means $R/x = 0$, i.e. $x \neq 0$, thus $N = S$ in $\mathsf{Susp}(x \neq 0)$.
- It is surjective: Choose $e : R$, such that $e^2 = e$ in R/x. By locality in R, e or $1 - e$ is invertible in R, thus in R/x. By $e^2 = e$ we deduce $e = 0$ or $e = 1$ in R/x, both lie in the image of f.

□

Example 12.39 (Not needed)**.** *Let $L = \sum_{x:\mathbb{A}^1} \mathsf{Susp}(x \neq 0) = \sum_{x:\mathbb{A}^1} \sum_{y:R/x} y^2 =_{R/x} 1$ be the line with two origins.*

Lemma 12.40 (Not needed)**.** *Let $2 \neq 0$. Let $y, y' : A$ be two elements of an fp-algebra, whose squares coincide and such that y is invertible. Then $y =_A y'$ is formally étale*

Proof. We may assume that $A = R$, as equality in A can be checked pointwise and formally étale is a modality. We may show its $\neg\neg$-stable. Assume $\neg\neg(y =_R y')$, i.e. $y - y'$ beeing nilpotent in A. So pick n large enough, such that $(y - y')^{2^n} = 0$. Proof by induction over n If $n = 0$, then its fine. Induction step $n \mapsto n+1$. Let $(y-y')^{2^{n+1}} =_R 0$, then $(2y^2 - 2yy')^{2^n} = 0$, or $(y(y - y'))^{2^n} = 0$, as y is invertible, $(y - y')^{2^n} = 0$, so by induction hypothesis $y = y'$. □

12.7 FiberCollaps away from the origin

OUTDATED!

Example 12.41. *—Bool— is the line with two origins.*
—Spec $R[X]/(X^2+1))$— is the twisted line with two origins, i.e. over the origin we have the roots of -1.
—Spec $R[Y]/(Y^2 - \bullet^2)$— is the quotient of the cross, that looks like $\mathbb{D}(1)$ over the origin.
—Spec $R[Y]/(\bullet Y)$— is the affine Plus.

12.8 Schemes do not have descent

For this section, let $\rho : R \setminus \{0\}$ denote a term, e.g. $\rho = 1$. Set $C = R[T]/(T^2 + \rho)$.

Proposition 12.42. *let $\rho : R \setminus \{0\}$ (e.g. $\rho = 1$). Set $C = R[T]/(T^2 + \rho)$. If —Spec C— is a scheme, then $X^2 + \rho$ merely has a root.*

Proof. Let $p :$ —Spec C— $\to R$ be the first projection. We proceed as follows

1. There is no open affine subset of —Spec C— containing $\mathrm{fib}_p(0)$.
2. Any open subset of $\operatorname{Spec} C$, that is strictly smaller than $\operatorname{Spec} C$, is an open proposition

Any finite open affine cover of —Spec C— can be restricted to a finite open affine cover $\operatorname{Spec} C = \bigcup_{j=0}^n U_j$ of the basefiber $\operatorname{Spec} C$ consisting of strictly smaller open subsets by point 1. Then the goal is

$$\|\operatorname{Spec} C\| = \|\bigcup_{j=0}^{n} U_j\| = \bigvee_j U_j$$

an open propoosition by point 2., thus an étale sheaf, as open propositions are $\neg\neg$-stable. So we can conclude. Proofs:

1. Because we want to show $\perp$, we may assume —Spec C— = —*Bool*—. Assume there is an open affine subset $\mathrm{fib}_p(0) \subset U \subset$ —*Bool*—. Then $p(U) \subset R$ is an open neigbhorhood of 0, as
$$x \in p(U) \leftrightarrow (x, N) \in U \vee (x, S) \in U$$
Claim: the map $R^{p(U)} \to R^U$ is an equivalence. If we have shown that: As U is affine we conclude that the map
$$\begin{aligned} U &\to \operatorname{Spec}(R^{p(U)}) \\ x &\mapsto \phi \mapsto (\phi(px)) \end{aligned}$$
is an equivalence, which is a contradiction to the assumption, that U contains both origins.
Proof of claim: In words: As U is a subset of a quotient of $R+R$, the function $U \to R$ determines two (partially defined on open domain) functions to R that coincide away from the origin, which is a regular point. Thus by 4.3 they coincide everywhere.
Injectivity: If two maps $f, g : p(U) \to R$ coincide after precomposing with $U \to p(U)$, then they coincide away from 0 so conclude by 4.3.
Surjectivity: Given a map $U \to R$, by pulling back along $p : R + R \to$ —*Bool*— we can view it as a map $R + R \supset U' \to R$ defined at both origins, so in particular as a pair of maps to R defined on some open neigbhorhood of 0 of R. They coincide away from 0 so by 4.3 they are equal.
2. Any strictly smaller open subset $U \subset \operatorname{Spec} C$ is an open proposition: Note, that U is a proposition: If $x, x' : U$, then $x = x' \simeq \neg\neg(x = x')$ by decidable equality of U, but if $x \neq x'$, then $\{x, x'\} \hookrightarrow \operatorname{Spec} C$ is an embedding, so by (*) an equivalence. But then

$U = \operatorname{Spec} C$, contradiction.
We first reduce to the case where U is a principal open of $\operatorname{Spec} C$. By [1] we find $f_1, \ldots, f_n : C$ such that $U = D(f_1, \ldots, f_n)$. As the left hand side is a proposition we have

$$U \leftrightarrow \bigvee D(f_i)$$

so we may show, that each $D(f_i)$ is an open proposition.
Let $f : C$ such that $D(f)$ is a proposition. Choose a representative $a + bT : R[T]$.

Let us show $(2a \neq 0) \leftrightarrow D(f)$, which is an étale -sheaf and a proposition, so we may assume $x : \operatorname{Spec} C$. Using that $D(f)$ is a proposition we have

$$D(f) = (a + bx \neq 0) + (a - bx \neq 0) \xrightarrow{\sim} (a + bx \neq 0) \vee (a - bx \neq 0)$$

We may show both implications $2a \neq 0 \leftrightarrow (a + bx \neq 0) \vee (a - bx \neq 0)$.
$' \rightarrow'$ $(a + bx) + (a - bx)$ is invertible, so by locality one of the summands is invertible.
$' \leftarrow'$ by symmetry wlog $a + bx \neq 0$. Then as $D(f)$ is a prop, $\neg\neg(a - bx = 0)$. Thus $\neg\neg(a + a = a + bx \neq 0)$, hence $2a \neq 0$.

□

Corollary 12.43. *Schemes do not have descent.*

Proof. If Schemes have descent, then —$\operatorname{Spec} R[T]/(T^2 + \rho)$— $\in$ Sch is a sheaf. As —$\operatorname{Spec} R[T]/(T^2 + \rho)$— is $\mathbb{T}$-merely a scheme, it is a scheme, so by the previous lemma $T^2 + \rho$ has a root. As $\rho : R \setminus \{0\}$ was arbitrary, we get a contradiction to [1] A . 0.3.

□

12.9 Gluing in an affine on the line

Definition 12.44. Let Y be an affine. The n.th order gluing of Y on the line is given by the sheaf

$$L_n(X) = \sum_{x:R} Y^{x^n=0}$$

Lemma 12.45. *If $Y = \operatorname{Spec} R[T]/f$, we have*

$$L_n(X) = \sum_{x:R} \sum_{y:R/x^n} f(y) =_{R/x^n} 0$$

Proof. For any R-algebra A (e.g. R/x^n) we have by the universal property of $R[T]/f$

$$\sum_{y:A} f(y) =_A 0 = \operatorname{Hom}_R(R[T]/f, A) = Y^{\operatorname{Spec} A}$$

□

Lemma 12.46. *If Y is formally étale , then the map over R*

$$\begin{tikzcd} R \times Y \arrow[rr] \arrow[dr] & & L_n(Y) \arrow[dl] \\ & R & \end{tikzcd}$$

pulls back to an equivalence over $\mathcal{N}_\infty(0)$.
If Y is formally unramified, then $L_n(x)$ is locally seperated.

Proof. Indeed, the diagonal map

$$Y \to Y^{x^n=0}$$

is an equivalence, as for any $\neg\neg x = 0$, $x^n = 0$ is a closed dense proposition and Y is formally étale .

If Y is formally unramified, then the identity types look like

$$(x, y) =_{L_n(Y)} (x', y') \simeq (x = x') \times (x^n = 0 \to Q)$$

where Q is an open proposition such that for any $p : x^n = 0$ we have $Q \equiv yp = y'p$. Indeed by the proof of 17.6 we can find a filler of $y = y' : P \to \mathsf{Open}$. By [1](4.2.11) this proposition is locally closed. □

Question 5. Is the map $\sum_{y:R/x^3} y^2 = 0 \to \sum_{y:R/x^2} y^2 =_0$ surjective? This is how i understand David Madore.

Lemma 12.47. *For $\varepsilon : \mathcal{N}_\infty(0)$, the affine $Ann(\varepsilon) = \{x : R \mid x\varepsilon = 0\}$ is not$_\varepsilon$ formally smooth. In particular $R \to R/\varepsilon$ is not$_\varepsilon$ a geometric cover.*

Proof. We have the map $1 : (\varepsilon = 0) \to \mathsf{Ann}(\varepsilon)$. Assume there is a filler $x : \mathsf{Ann}(\varepsilon)$, i.e. $(\varepsilon = 0) \to x = 1$. Then not not, $x = 1$, i.e. $(x - 1)^n = 0$ for n large enough. Hence

$$0 = \varepsilon(x - 1)^n = \varepsilon x(\ldots) + (-1)^n\varepsilon = (-1)^n\varepsilon$$

as desired. □

Lemma 12.48 (TODO)**.** *If Y is formally étale + flat affine, then $L_1(Y)$ is an algebraic space.*

Proof. Recall the closed modality associated to a proposition P, given by $P \star \_$. We can define a map

$$\begin{aligned} f : (x \neq 0) \star Y &\to Y^{x=0} \\ y &\mapsto \Delta(y) \end{aligned}$$

where we check, that if $x \neq 0$ holds, then indeed $Y^{x=0}$ is contractible.

f is a bijection:

- injectivity: Given two terms of the domain, as the map out of Y is $\mathbb{T}$-surjective (and the goal is a sheaf), we may assume that they are of the form $inl(y), inl(y')$ for $y, y' : Y$. Then if $\Delta(y) = \Delta(y')$ we have $(x = 0) \to (y = y')$. As $y = y'$ is open, we have $(x \neq 0) \vee (y = y')$. If $x \neq 0$, then $inl(y) = inl(y')$ by the construction of the join.
- surjectivity: TODO

□

Question 6. Is $L_2(\mathbb{D}(1))$ an algebraic space or fppf-geometric 0-stack? For this: Is

$$\begin{aligned} (\operatorname{Spec} R[X, Y]/X^2 - Y^2)/\sim &\to L_2(\mathbb{D}(1)) = \sum_{x:R} \sum_{y:R/x^2} y^2 = 0 \\ (x, y) &\mapsto (x, [y]) \end{aligned}$$

an equivalence? Here we mod out the relation generated by $(x, -x) \sim (x, x) \forall x \neq 0$.

This is equivalent to : For any $x : R$, is the map

$$(x \neq 0) \star \operatorname{Spec} R[Y]/(Y^2 - x^2) \to \mathbb{D}(1)^{x^2=0}$$

an equivalence?

Example 12.49. *I suggest a new definition of fppf topology: We take the topology generated by the Zariski topology and algebras of the form $R[X]/f$ where one of coefficients of f is invertible (non necessarily the leading coefficient). This is still a free module hence fppf.*

13 Deloopings and Truncations

We denote $\| \cdot \|_n^{\mathbb{T}} := L_{\mathbb{T}} \| \cdot \|_n$, which is a modality. We denote

$$\eta_n^{\mathbb{T}} X : X \to \|X\|_n^{\mathbb{T}}$$

Definition 13.1. A pointed stack (X, x) is called $\mathbb{T}$-1-connected (or $\mathbb{T}$-connected) if for any $y : X$ we have $\|x = y\|_{\mathbb{T}}$.
Inductively, (X, x) is called $\mathbb{T}$-$n+1$-connected if its $\mathbb{T}$-connected and ΩX is $\mathbb{T}$-n-connected.

Definition 13.2. Let G be a stack. A delooping stack of G is a pointed $\mathbb{T}$-connected stack BG equipped with an equivalence $\Omega BG \simeq G$.

Lemma 13.3. *For X, Y pointed stacks, to construct an equivalence $X \simeq B^k Y$ we may show that X is $\mathbb{T}$-k-connected and construct an equivalence $\Omega^k X \simeq Y$.*

Proof. If $k = 1$ its fine. Then $X \simeq B^{k+1} Y$ iff X is $\mathbb{T}$-connected and $\Omega X \simeq B^k Y$. By induction the latter is equivalent to ΩX beeing $\mathbb{T}$-k-connected and $\Omega^{k+1} X \simeq Y$. □

Lemma 13.4. *Let G be a covering stack, that admits a delooping stack BG. Then BG is a covering stack.*

Proof. Now assume G is a covering stack. To show, that BG is a covering stack, we may show that the map $\mathbb{T} \ni 1 \to BG$ is a geometric atlas. As BG is $\mathbb{T}$-connected, every point is $\mathbb{T}$-merely equal to the basepoint. By descent for covering stacks, we may just show that the fiber over the basepoint is a covering stack But this is equivalent to $\Omega BG \simeq G$. □

Corollary 13.5. *If G is a covering group 0-stack, that admits an n-fold delooping stack $B^n G$, then this will be a covering n-stack.*

Lemma 13.6. *The fiber of $\eta_n^{\mathbb{T}} X : X \to \|X\|_n^{\mathbb{T}}$ over $|x|$ is $\sum_{y:X} \|x = y\|_{(n-1)\mathbb{T}}$*

Proof. For any $x : X$, we may show that the type family

$$\begin{aligned} B : \|X\|_n^{\mathbb{T}} &\to \mathcal{U}_{n-1}^{\mathbb{T}} \\ \|y\|_n &\mapsto \|x = y\|_{n-1}^{\mathbb{T}} \end{aligned}$$

defined using the n truncatedness of the stack $\mathcal{U}_{n-1,\mathbb{T}}$, is a unary identity system of $\|X\|_n^{\mathbb{T}}$ at $|x|$. By the fundamental system of identity types its enough to construct for all $y : \|X\|_n^{\mathbb{T}}$, a section of the map $|x| = y \to By$ induced by path induction.
As the space of sections of a map between n-stacks is in particular an n-stack, we may just for all $y : X$ construct a section of the map

$$\mathrm{ind} : |x| =_{\|X\|_n^{\mathbb{T}}} |y| \to \|x = y\|_{n-1}^{\mathbb{T}}$$

But $|x| = |y|$ is an $n-1$-stack, so there is a unique dashed map σ such that the above triangle

$$\begin{tikzcd} & x =_X y \arrow[dl, "\mathsf{ap}"'] \arrow[d, "\eta"] \\ {|x| = |y|} \arrow[dr, bend right, "\mathrm{ind}"'] & \|x =_X y\|_n^{\mathbb{T}} \arrow[l, dashed, "\exists! \sigma"] \arrow[d, "\mathrm{id}"] \\ & \|x =_X y\|_n^{\mathbb{T}} \end{tikzcd}$$

commutes. This is indeed a section of the above map, because the maps $\mathrm{ind} \circ \sigma$ and id targeting an n-stack become equal after postcomposition with the unit η of the modality $L_{\mathbb{T}} \| \cdot \|_n$.

□

Lemma 13.7. *For any X and any $n \geq -1$, the map $\eta_n^{\mathbb{T}} X : X \to \|X\|_n^{\mathbb{T}}$ is $\mathbb{T}$-surjective.*

Proof. It factors as $X \to \|X\|_n \to L_{\mathbb{T}}\|X\|_n$ where the latter map is $\mathbb{T}$-surjective. So it sufficess to show, that the former map is surjective. As $X \to \|X\|_0$ is surjective it suffices to show, that ap of the map $\|X\|_n \to \|X\|_0$ is surjective. TODO $\square$

Notation. Given a map $f : X \to Y$ and some $x : X$ we denote $\mathrm{fib}\, f x$ for the pointed type

$$\mathrm{fib}\, f x \equiv (\mathrm{fib}_f(f x), (x, \mathsf{refl}))$$

and f, x for the pointed map

$$(f, \mathsf{refl}_{fx}) : (X, x) \to (Y, f(x))$$

Lemma 13.8. *If (X, x) is a pointed stack, the looping of the fiber of $X \to \|X\|_n^{\mathbb{T}}$ over $|x|$ is the basefiber of $\Omega X \to \|\Omega X\|_{n-1}^{\mathbb{T}}$.*

$$\Omega(\mathrm{fib}(\eta_n^{\mathbb{T}} X)(x)) \simeq \mathrm{fib}(\eta_{n-1}^{\mathbb{T}} \Omega(X, x))(pt)$$

Proof. We have to understand the loop space of $\sum_{y:X} \|x = y\|_{(k-1)\mathbb{T}}$. It is given by

$$(p : \Omega X) \times \left(\mathrm{tp}_p\, r =_{\|\Omega X\|_{k-1,\mathbb{T}}} r\right),$$

where $r = |\mathsf{refl}|$. we calculate $\mathrm{tp}_p\, r = |p|$, so it is the fiber of

$$\Omega X \to \|\Omega X\|_{k-1,\mathbb{T}}$$

over the basepoint $|\mathsf{refl}|$.
Alternative proof

$$\begin{tikzcd}
\Omega(X,x) \arrow[r, "\Omega(\eta_n^{\mathbb{T}} X, x)"] \arrow[dr, "\eta_{n-1}^{\mathbb{T}}(\Omega(X,x))"'] & \Omega(\|X\|_n^{\mathbb{T}}, |x|) \arrow[d, "\simeq"] \\
 & \|\Omega(X,x)\|_{n-1}^{\mathbb{T}}
\end{tikzcd}$$

$$\Omega(\mathrm{fib}(\eta_n^{\mathbb{T}} X)(x)) = \mathrm{fib}(\Omega(\eta_n^{\mathbb{T}} X, x))pt = \mathrm{fib}(\eta_{n-1}^{\mathbb{T}} \Omega(X, x))pt$$

$\square$

Proposition 13.9. *Let $n \geq 0$, X be a geometric stack, such that for all $x : X$, $\Omega^{n+1}(X, x)$ is a covering stack for all $x : X$. Then $\eta_n^{\mathbb{T}} X : X \to \|X\|_n^{\mathbb{T}}$ is a geometric cover. In particular, $\|X\|_n^{\mathbb{T}}$ is a geometric n-stack.*

Proof. Let us show by induction over $k = -1, \ldots, n$ that

$$\eta_k^{\mathbb{T}}(\Omega^{n-k} X) : \Omega^{n-k} X \to \|\Omega^{n-k} X\|_k^{\mathbb{T}}$$

is a geometric cover.
$k = -1$ is okay as $\Omega^{n+1} X$ is a covering stack and $\mathbb{T}$-truncations of covering stacks are contractible.
For the induction step $k - 1 \mapsto k$: Set $X' := \Omega^{n-k} X$, so we want to show that $X' \to \|X'\|_k^{\mathbb{T}}$ is a geometric cover. Every fiber is modal so the fiber beeing a covering stack has descent, so we may just show that the fiber over the image of some $x : X$ is a covering stack. The fiber $\sum_{y:X} \|x = y\|_{(k-1)\mathbb{T}}$ is $\mathbb{T}$-connected, so its a delooping stack of the basefiber of

$$\Omega X \to \|\Omega X\|_{k-1,\mathbb{T}}$$

by 13.8 and 13.4 we conclude. $\square$

Definition 13.10. A higher group stack is a pointed $\mathbb{T}$-connected stack.

Let BG be a higher group stack and X be a geometric stack equipped with an action $\rho : BG \to \mathsf{GS}$. We use the standart notation

$$X//G :\equiv \sum_{BG} \rho$$

Lemma 13.11. *If G is covering, then $X//G$ is a geometric stack*

Proof. BG is a covering stack, as G is a covering stack 13.4. Hence $X//G :\equiv \sum_{BG} \rho$ is a geometric stack. $\square$

Proposition 13.12. *If $X//G$ is a geometric stack (e.g. if G is covering) and the isotropy stacks $\sum_{g:G} gx = x$ are covering stacks, then $\|X//G\|_0^{\mathbb{T}}$ is an algebraic space.*

Proof. To apply the prop, we have to show, that for all $x : X//G$, $\Omega(X//G, x)$ is a covering stack. As $X \to X//G$ is $\mathbb{T}$-surjective (todo details), we may just show this for $x : X$.

$$\Omega(X//G, [x]) \simeq \sum_{g:G} gx = x$$

which was covering by assumption $\square$

Corollary 13.13. *Let G be a covering group sheaf (e.g. finite group), acting on a geometric stack X with $\mathbb{T}$-flat identity types. Then $L_{\mathbb{T}}(X/G)$ is a geometric stack.*

Proof. The isotropy stacks are covering by 16.7, as they are $\sum$ of $\mathbb{T}$-flat geometric stacks and they are $\mathbb{T}$-merely inhabited $\square$

We can also reprove 12.20: G is a finite type by assumption, hence covering. The isotropy stacks are assumed to be propositions, but they are inhabited, so they are covering $\square$(lemma)

TODO: Find a good example of a non covering G.

14 Local properties

Definition 14.1. Let $\mathcal{V} \subset \mathcal{U}$ a subclass of types be stable under finite limits. We call a property P of morphisms of types in $\mathcal{V}$ $\mathbb{T}$-local, if

1. its satisfied by identities in $\mathcal{V}$,
2. stable under composition
3. Given a commutative triangle in $\mathcal{V}$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

with $X \to Y$ a geometric cover (wrt to $\mathbb{T}$). Then h has P iff g has P

Definition 14.2. P has descent along geometric covers: Given $X, Y, Z, W \in \mathcal{V}$. if $Y \to W$ is a geometric cover , then

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[d, "f"] \\ Y \arrow[r] & W \end{tikzcd}$$

If f has P then f' has P.

Lemma 14.3. *If P is local, then*

- *geometric covers have P*
- *in descent, The statement 'If f' has P then f has P' is automatic by Point 3.*

Lemma 14.4. *Beeing a geometric cover is local.*

Proof. Reduce to the case of $Z = 1$. If $X \to Y$ is a geometric cover, then X is a covering stack iff Y is a covering stack by stability under quotients and under sums. If both are coverings stacks, then the fibers $\square$

Lemma 14.5. *Let P be a local property of morphisms of geometric stacks. For A morphism between geometric stacks $f : X \to Y$ TFAE*

1. *f has P*
2. *For any Atlas $\operatorname{Spec} A \to Y$ and any atlas $S \to X \times_Y \operatorname{Spec} A$ the composite $S \to \operatorname{Spec} A$ has P*
3. *f has an atlas that has P.*

Proof. 1 $\Rightarrow$ 2 Given a geometric atlas $\operatorname{Spec} A \to Y$ and taking the pullback

$$\begin{tikzcd} X \times_Y \operatorname{Spec} A \arrow[r, "f'"] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

f' has P as a basechange of f along a geometric cover. Given a geometric atlas $S \to X \times_Y \operatorname{Spec} A$, it will have P, the composition $S \to \operatorname{Spec} A$ will be in P.

2 $\Rightarrow$ 3 Y is a geometric stack, hence admits some geom atlas $\operatorname{Spec} A \to Y$. Again, $X \times_Y \operatorname{Spec} A$ is a geometric stack hence admits a geometric atlass.

$3 \Rightarrow 1$ If we have an atlas $\tilde{f} : \tilde{X} \to \tilde{Y}$, then $\tilde{X} \to \tilde{Y} \to Y$ has P by stability under composition. Then by (4) $X \to Y$ has P, as $\tilde{X} \to X$ is a geometric cover

□

So we may extend algebraic notions of maps to all geometric stacks:

Definition 14.6. Let P be a property of morphisms $\mathbb{T}$-local in affine schemes.
We define a morphism of geometric stacks $f : X \to Y$ to have P iff there exist atlasses and a P-map on affines

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, dashed, "\hat{f}"] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Lemma 14.7. *Let P be a local property of affine schemes. The induced property of morphisms of geometric stacks is local. Additionally descent is inherited.*

Proof. 1. Ok

2. Ok

3. geometric covers have P and we have proven point 2., so one direction is clear. Now assume

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

Such that f is a geometric cover and h has P.
We first reduce to the case where Z is affine. Choose a geometric atlas $\tilde{Z} \to Z$. Then take the pullbacks

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \tilde{Z} \arrow[d] \\ X \arrow[r] \arrow[rr, bend right, "P"'] & Y \arrow[r] & Z \end{tikzcd}$$

$X' \to Y'$ is a geometric cover and By 3. $X' \to \tilde{Z}$ has P.
So we may assume that Z is affine. Then take a geometric atlas $\tilde{X} \to X$. The map $Y \to Z$ has the atlas $\tilde{X} \to X \to Z$ which has P by stability under composition. Hence $Y \to Z$ has P.

4. We show also descent: By 14.3 we only need to show stability under basechange. Let $Z \to W$ have P, Given $Y \to W$ a geometric cover. We want to show that a basechagne $Y \times_W Z \to Y$ has P. The idea is to construct a common atlas of $Z \to W$ and its basechange. Choose an atlas $\tilde{Y} \to Y$. Then $\tilde{Y} \times_W Z \to Y \times_W Z$ is a geometric cover: It is a basechange of $\tilde{Y} \to Y$, because the outer diagram is a pullback

$$\begin{tikzcd} \tilde{Y} \times_W Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Y \times_W Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[d] \\ \tilde{Y} \arrow[r] & Y \arrow[r] & W \end{tikzcd}$$

Now choose any geometric atlas $S \to \tilde{Y} \times_W Z$. By composition this induce a map $S \to \tilde{Y}$: It is both an atlas of the P-map $Z \to W$ and of $Y \times_W Z \to Y$. So by 14.5 $S \to \tilde{Y}$ has P and thus $Y \times_W Z \to Y$ has P.

□

14.1 Local properties of stacks

Definition 14.8. Let $\mathcal{V} \subset \mathcal{U}$ be a subclass of types stable under finite limits. A property P of types in $\mathcal{V}$ is local if

1. $1 \in P$
2. P is $\sum$-stable
3. If $X \to Y$ is a geometric cover between types in $\mathcal{V}$, then X has P iff Y has P.

We say P has descent if for all $X : \mathcal{V}$, then X having P is a $\mathbb{T}$-sheaf.

Lemma 14.9. *Every local property of types in $\mathcal{V}$ induces a local property of morphisms of types in $\mathcal{V}$, by asking the property fiberwise.*

Proof. Use descent for the descent along a geometric cover ($\mathbb{T}$-surjective!). □

Lemma 14.10. *Let P be a $\sum$-stable-property of affines containg $\mathbb{T}$. The induced property of geometric stacks is $\mathbb{T}$-local.*

Proof. The $\sum$-stability is 7.3. Covering stacks have P, as $\mathbb{T} \subset P$. The quotient stability is straightforward. □

14.2 Seperatedness

Definition 14.11. Let P be a $\mathbb{T}$-local property of stacks. We call a stack P-seperated, iff its identity types are P stacks.

Lemma 14.12. *Let P be a $\mathbb{T}$-local property of stacks. Then beeing P-seperated is a $\mathbb{T}$-local property*

Proof. Let $f : X \to Y$ be a geometric cover with X beeing P-seperated. Let $x, y : Y$. Then by the construction in 5.4 the map

$$\mathrm{fib}_f\, x \times_X \mathrm{fib}_f\, y \to x = y$$

is a geometric cover, whose domain has P as X is P-seperated and P is stable under $\sum$. As P is local, $x = y$ has P.

□

Lemma 14.13. *If $\sum$-stable property of affine schemes containing $\mathbb{T}$ is stable under identity types, then the induced $\mathbb{T}$-local property of geometric stacks is as well.*

Proof. Let X be a P geometric stack. Let $x, y : X$ we want to show that $x =_X y$ has P. Choose a geometric atlas $P \ni S \xrightarrow{f} X$. By assumption S is P-seperated. We have a geometric atlas $\mathrm{fib}_f\, x \times_S \mathrm{fib}_f\, y \to x = y$. The domain is a $\sum$ of types in $\mathbb{T}$ and identity types of S, which have P by stability under identity types for the affine S. Hence $x = y$ has P. □

14.3 Formally étale TODO

Lemma 14.14 (TODO)**.** *If $Y \to X$ is a formally étale $\mathbb{T}$-surjective map between stacks and Y is formally étale , then X is formally étale .*

Proof. Take L to be the modality which nullifies the propositions $\|\operatorname{Spec} A\|$ for $\operatorname{Spec} A$ étale + fppf and all close dense propositions. The square

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\sim"] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d] \\ L(Y) \arrow[r] & L(X) \end{tikzcd}$$

is a pullback as L is lex. We want to show, the right map is an equivalence. Every type occuring is an étale -stack. As the lower map is étale -surjective (TODO!!!), and the left vertical map is an equivalence, we can conclude. □

Lemma 14.15. *Covering stacks are formally étale .*

Proof. We apply the recursion principle. Contractible types are formally étale . If $\operatorname{Spec} B \to X$ is a formally étale geometric cover and $\operatorname{Spec} B \in \mathbb{T}$ then X is formally étale by the previous lemma.

□

Lemma 14.16. *formally étale is an étale -local property of geometric stacks.*

Proof. geometric covers are formally étale by 14.15, so conclude by 14.14 □

14.4 Weakly-flat stacks

Definition 14.17. We call a geometric stack X weakly-flat iff one of the following conditions is satisfied

1. $\|X\|_{-1}^{\mathbb{T}} \to X \in \mathsf{CS}$

2. For any geometric atlas $W \to X$, W is weakly-flat, i.e $\|W\|^{\mathbb{T}} \to W \in \mathbb{T}$.

Proof.

$1 \Rightarrow 2$ Choose a geometric atlas $W \to X$. In particular its $\mathbb{T}$-surjective, hence we have $\|W\|^{\mathbb{T}}$, so by assumption $W \in \mathbb{T}$. So $X \in CS$.

$2 \Rightarrow 1$

$$\|W\|^{\mathbb{T}} \to \|X\|^{\mathbb{T}} \to X \in \mathsf{CS} \overset{8.16}{\to} W \in \mathbb{T}$$

□

They behave bad as they are not stable under $\sum$ (and not under id-types, although this holds for affines).

Lemma 14.18. *For any weakly-flat geometric stack X, $\|X\|_{-1}^{\mathbb{T}}$ is a geometric stack.*

Proof. $X \to \|X\|_{-1}^{\mathbb{T}}$ is a geometric cover. □

Proposition 14.19. *We may define X to be 0-wf-seperated, iff its weakly flat and $n+1$-wf-seperated, iff identity types of X are n-wf-seperated.*
For X a geometric stack, TFAE

1. *X is $n+1$-wf-seperated, i.e. all $n+1$-fold identity types of X are weakly-flat.*

2. *For any x, $\Omega^{n+1}(X,x)$ is covering.*

3. *For any $x : X$, $x = x$ is n-wf-seperated, i.e. n-fold identity types of $x = x$ are weakly flat.*

Proof.

$1 \Rightarrow 3 \Rightarrow 2$ ez

$3 \Rightarrow 1$ We prove this by induction. $n = 0$. To show that $x =_X y$ is weakly-flat, by descent we may assume that $x = y$. Then we have $(x = y) \simeq (x =_X x)$. By assumption this is weakly flat.
Assume now, that for any $x : X$, that $x = x$ is n-wf-seperated. Let $x, y : X$. We want to show that $x = y$ is n-wf-seperated. By induction we may just prove that for any $p : x = y$, $p = p$ is $n-1$-wf-seperated. Applying $p \cdot \_$ induces an equivalence $\mathsf{refl}_x = \mathsf{refl}_x \simeq p = p$. But $x = x$ is n-wf-seperated, hence $\mathsf{refl}_x = \mathsf{refl}_x$ is $n-1$-wf-seperated.

$2 \Rightarrow 3$ Induction. $n = 0$ is fine. Let $x : X$. To show that $\Omega(X,x)$ is n-wf-seperated, just use that $\Omega^n(\Omega(X,x))$ is covering, hence by the inductive statement $2 \Rightarrow 3 \Rightarrow 1$, we now that $\Omega(X,x)$ is n-wf-seperated. □

15 Omega-stability and gerbes

Definition 15.1. A geometric stack X is an n-gerbe iff the map $\eta_n^{\mathbb{T}} : X \to \|X\|_n^{\mathbb{T}}$ is a geometric cover.

Example 15.2. *If G is a covering group sheaf, then BG is a 0-gerbe.*

Example 15.3. *It may happen, that $\|X\|_n^{\mathbb{T}}$ is a geometric n-stack for X a geometric stack, although X is not an n-gerbe. Indeed: Put $n = 0$ and X any pointed $\mathbb{T}$-connected geometric stack that is not covering, like* $\mathsf{Susp}(1 + x = 0)$ *for some*

Theorem 15.4. *Assume that Covering stacks are Ω-stable, Then every geometric stack is a 1-gerbe.*

Proof. By 13.9, we need to show that for any $x : X$, $\Omega^2(X, x)$ is covering. choose an geometric atlas $f : S \to X$. by descent we may only show that $\Omega^2(X, fs)$ for $s : S$ is covering.

$$\Omega(\sum_{t:S} ft = fs) \simeq \left(\sum_{p:\Omega(S,s)} \mathrm{tp}_p(\mathsf{refl}_{fs}) = \mathsf{refl}_{fs} \right) \simeq \mathsf{refl} =_{fs=fs} \mathsf{refl}$$

where the last equivalence is obtained, as $\Omega(S, s)$ is contractible with center refl_s.
So $\Omega^2(X, fs)$ is the loop space of a covering stack, hence by assumption covering. □

Corollary 15.5. *Any Deligne Mumford Stack is a 1-gerbe*

Proof. Use that étale topology is lex-flattened and **??**. □

Proposition 15.6. *This proposition seems only interesting for $n = 0$ by the previous theorem. Assume that covering stacks are Ω-stable. Then X is an n-gerbe iff $\Omega^{n+1}(X, x)$ is covering for all $x : X$*

Proof. One direction is 13.9. The other follows
By applying iteratively 13.8

$$\begin{aligned}
\Omega^{n+1}(\mathrm{fib}(\eta_n^{\mathbb{T}} X)|x|) &\simeq \Omega^n \,\mathrm{fib}(\eta_{n-1}^{\mathbb{T}}(\Omega(X, x)))pt \simeq \dots \\
&\simeq \Omega^{n-k} \,\mathrm{fib}(\eta_{n-k-1}^{\mathbb{T}} \Omega^{k+1}(X, x))pt \simeq \dots \\
&\simeq \mathrm{fib}(\eta_{-1}^{\mathbb{T}} \Omega^{n+1}(X, x))pt \\
&\simeq \Omega^{n+1}(X, x)
\end{aligned}$$

The LHS is covering by Ω-stability. □

We can reprove 16.12 by just observing that $\mathbb{T}$-flat geometric stacks have covering loop spaces.

Remark 3. Put $\mathbb{T}$ the étale topology. Observe, that we have an analogous statement if we replace covering stack by formally étale :

1. $\eta_0^{\mathbb{T}} X : X \to \|X\|_0^{\mathbb{T}}$ is formally étale
2. $X \to \|X\|_0^{\mathbb{T}}$ is formally unramified
3. for every $x : X$, $\Omega(X, x)$ is formally étale .

*Proof*1 $\Leftrightarrow$ 2 Observe that the map $\eta_0^{\mathbb{T}}$ is $\mathbb{T}$-smooth.

2 $\Rightarrow$ 3 okay as the fibers of $\eta_0^{\mathbb{T}}$ embed into X.

3 $\Rightarrow$ 2 Let $x, y : X$ be $\mathbb{T}$-merely equal. The goal $\mathsf{isFormallyEtale}(x = y)$ is a sheaf, so we may assume that $x = y$.

□

Corollary 15.7. *If covering stacks are Ω-stable, then identity types of geometric stacks are 0-gerbes.*

Proof. We need to check, that identity types of a 1-gerbe X are 0-gerbes. So assume $p : x = y$. Then

$$\Omega(x = y, p) = \Omega(x = x, \mathsf{refl}) = \Omega^2(X, x)$$

which is covering as X is a 1-gerbe. $\square$

16 Flat

Definition 16.1. Denote Top the topologies containing Bool, e.g. finer than the Zariski-topology. Let FLAT consists of all the classes of affines $\mathbb{P}$ containing $1, \perp$ stable under $\sum$. Given $\mathbb{P} : \mathsf{FLAT}, \mathbb{T} : \mathsf{Top}$ we say $\mathbb{P}$ flattens $\mathbb{T}$ iff ($\mathbb{T} \subset \mathbb{P}$ and)

$$\mathbb{T} = \{X : \mathbb{P} \mid \|X\|_{\mathbb{T}}\}$$

The goal of this section is to prove the following theorem

Theorem 16.2.

1. *There is at most one $\mathbb{P}$ that flattens a topology. Then we say, the topology is flatten.*
2. *A topology can be idempotently flattened without changing the stacks*
3. *For any $\mathbb{P} : \mathsf{FLAT}$ and any Lavwere Tierney Operator j, $\{X : \mathbb{P} \mid \|X\|_j\}$ is flattened by $\mathbb{P}$.*

We first want to show the power of this theorem.

Example 16.3. *finite sums of principal opens flatten the Zariski topology.*

Example 16.4. *flat affines flatten the fppf topology.*

Proof. Indeed we can either put $j = \neg\neg$ or j the fppf sheafification, because TFAE

1. X is flat and fppf-merely inhabited
2. X is flat and $\neg\neg$-inhabited
3. X is fppf

Proof. $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$ [ref?] □

□

Example 16.5. *formally étale + flat affines flatten the étale topology. For the etale topology = formally étale + fppf, we can put $\mathbb{P}$ = formally étale + flats.*

Proof. By the same argument as above.

□

Lemma 16.6. *Assume $\mathbb{T}$ is flatten. If X is $\mathbb{T}$-flat geometric stack, then $\|X\|_{\mathbb{T}}$ is a geometric prop.*

Proof. If $\operatorname{Spec} A$ is $\mathbb{T}$-flat, then $\operatorname{Spec} A$ is weakly-flat, i.e $\|\operatorname{Spec} A\|_{\mathbb{T}}$ is a geometric prop. □

Lemma 16.7. *Assume $\mathbb{T}$ is flatten. A stack is covering iff it it a $\mathbb{T}$-flat geometric stack and $\mathbb{T}$-merely inhabited.*

Lemma 16.8. *Assume $\mathbb{T}$ is flatten. If X is a covering stack and Y a $\mathbb{T}$-flat geometric stack, then $X + Y$ is covering*

Proof. Let $\mathbb{P}$ flatten $\mathbb{T}$. Let $\mathbb{P} \ni \tilde{X} \to X, \tilde{Y} \to Y$ be geometric atlasses. Then $\tilde{X} + \tilde{Y}$ is $\mathbb{P}$ and $\mathbb{T}$-merely inhabited, hence in the topology. □

16.1 Lex flatten Topologies

Definition 16.9. A saturated topology $\mathbb{T}$ is lex-flatten, if its flattened by some lex $\mathbb{P}$.

Note that $\bot = (left =_{1+1} right) \in \mathbb{P}$ is automatic as $\mathbb{P}$ is lex.

Example 16.10. *The étale topology is lex-flatten: formally étale + flat affines are stable under identity types , as formally étale seperated schemes have decidable equality.*

Proposition 16.11. *Let $\mathbb{T}$ be lex-flatten. Then covering stacks are Ω-stable.*

Proof. If X is a covering stack then ΩX is a $\mathbb{T}$-flat geometric stack 14.13 and $\mathbb{T}$-merely inhabited. Conclude by 16.7. □

Lemma 16.12. *Assume that $\mathbb{T}$ is lex-flattened. Then any $\mathbb{T}$-flat geometric stack is a 0-gerbe.*

Proof. I give a second proof of this at 15.6 without using the language of lex-flattened. By descent, we may only show that the fiber $\sum_{y:X} \|x = y\|_{-1}^{\mathbb{T}}$ of $\eta_0^{\mathbb{T}}$ over $|x|$ is a covering stack. Note that $x = y$ has $\mathbb{P}$ by id-stability of $\mathbb{P}$ 14.13. The $\mathbb{T}$-truncation of a $\mathbb{P}$-geometric stack is a $\mathbb{P}$ geometric stack 16.6. by $\sum$-stability of $\mathbb{P}$ the fiber is $\mathbb{P}$, but its $\mathbb{T}$-merely inhabited. by 16.7 its covering. □

16.2 Proof of the theorem

Observe that if $X + Y$ is affine, then X is affine, as $X \to X + Y$ is an affine map.
Let $\mathbb{T}$ be a topology containing 2.

Definition 16.13. $\mathcal{P}_{\mathbb{T}}$ is the smallest topology containing $\mathbb{T} \cup \{\bot\}$

Lemma 16.14. *Let $\mathbb{P}$ be $\sum$ stable containing $1, \bot$. Then its stable under decidable subtypes, i.e. If $X + Y \in \mathbb{P}$ then $X \in \mathbb{P}$.*

Proof. Given $X + Y \in \mathbb{P}$, we can define $(1, \bot) : X + Y \to \mathbb{P}$ Its $\sum$ will be X. □

Proposition 16.15. *Assume that $\mathbb{T}$ is saturated.*

$$\mathcal{P}_{\mathbb{T}} = \{X \mid \exists Y, X + Y \in \mathbb{T}\}$$

Proof. By 16.14 and as $\mathbb{T} \subset \mathcal{P}_{\mathbb{T}}$, we have $' \supset'$. So it remains to show that the RHS, lets call it $\mathbb{P}$, is a topology containing $\mathbb{T}, \bot$.

1. $\mathcal{P}_{\mathbb{T}} \subset \mathsf{Aff}$.
2. $\bot \in \mathbb{P}$
3. $\mathbb{T} \subset \mathbb{P}$
4. Assume $\mathbb{T}$ is saturated. Whenever $\mathbb{P} \ni S \to X \in \mathsf{Aff}$ is a $\mathbb{T}$-cover, then $X \in \mathbb{P}$. Indeed : choose $S + Y \in \mathbb{T}$, Then $\mathbb{T} \ni S + Y \to X + Y$ is a $\mathbb{T}$-cover, hence by saturatedness $X + Y \in \mathbb{T}$. Thus $X \in \mathbb{P}$.
5. If $\mathbb{T}$ is saturated, then $\mathbb{P}$ is stable under $\sum$. Proof: Let $\mathbb{P} \ni X \xrightarrow{B} \mathbb{P}$. Lets first handle the special case, where $Bx \in \mathbb{T}$ for any $x : X$. Choose Y such that $X + Y \in \mathbb{T}$. Then $\sum_{x:X} Bx + \sum_{y:Y} 1$ can be expressed as $\sum_{x:X+Y}(B + \mathrm{cnst}_1)x$, which belongs to $\mathbb{T}$.
   Now the general case. By Zariski local choice we find a Zariski cover $p : X' \to X$ with
   $$\prod_{x':X'} \sum_{Y_{x'}} B(px) + Y_{x'} \in \mathbb{T}$$
   Then $\sum_{x':X'} Y_{x'} + \sum_{x':X'} B(px) \in \mathbb{P}$, hence by 16.14 $\sum_{x':X'} B(px) \in \mathbb{P}$. As $\sum_{x':X'} B(px) \to \sum_{x:X} Bx \in \mathsf{Aff}$ is a $\mathbb{T}$-cover, we conclude by (4.)

$\square$

Definition 16.16. $\mathbb{T}$ is decompostable if for any type X

$$(\|X\|_{\mathbb{T}} \wedge \exists Y, X + Y \in \mathbb{T}) \to X \in \mathbb{T}.$$

Proposition 16.17. *Let $\mathbb{T}$ be saturated. There exists a smallest decompostable topology $\tilde{\mathbb{T}}$ containing $\mathbb{T}$. Moreover the stacks coincide.*

Proof. Define

$$\begin{aligned} &\mathsf{Top} \to \mathsf{Top} \\ &\mathbb{T} \mapsto \tilde{\mathbb{T}} \equiv \{X \mid \|X\|_{\mathbb{T}} \wedge \exists Y, X + Y \in \mathbb{T}\} \end{aligned}$$

We apply 9.2.

- The class is stable under $\sum$ as $\mathcal{P}_{\mathbb{T}}$ and $\mathbb{T}$-merely inhabited types are both $\sum$-stable.
- Monotonicity clear.
- Inflationarity clear
- stack-preservation is clear by construction.
- idempotency: Let X be a type such that $\|X\|_{\tilde{\mathbb{T}}}$ and there exists a Y with $X + Y \in \tilde{\mathbb{T}}$. By the first assumption, we have $\|X\|_{\mathbb{T}}$ as the stacks coincide by 9.2.
  The latter means in particular that we find Z with $X + Y + Z \in \mathbb{T}$. But this witnesses that $X \in \tilde{\mathbb{T}}$.

$\square$

Lemma 16.18. *Let $\mathbb{T}$ be a topology, such that any $X : \mathcal{P}_{\mathbb{T}}$ is $(\mathbb{T}-1)$-seperated, i.e. that the identity types of X belong to $\mathbb{T} - 1 \equiv \{X \mid X + 1 \in \mathbb{T}\}$.. Then we have for all X*

$$(\exists Y : \mathbb{T} - 1, X + Y \in \mathbb{T}) = (X \in (\mathbb{T} - 1)) \to (\|X\|_{\mathbb{T}} \to X \in CS)$$

Proof. For the first equality notice that $X + Y \to X + 1$ is a $\mathbb{T}$-cover. For the last implication, by descent for covering stacks we may choose a map $1 \to X$. Then $\mathbb{T} \ni X + 1 \to X$ is a $\mathbb{T}$-cover by assumption. $\square$

Warning. In general, the $\tilde{\cdot}$ -construction is presumably not covering-stack preserving: In the above lemma we would need

$$X \in \mathbb{P} \to (\|X\|_{\mathbb{T}} \to X \in CS)$$

Example 16.19. *If any type in $\mathbb{P}$ has decidable equality, then any type in $\mathcal{P}$ is $(\mathbb{T} - 1)$seperated.*

Proposition 16.20. *Let $\mathbb{T}$ be saturated. TFAE*

1. *$\mathbb{T}$ is decompostable, i.e. for any $X \in \mathcal{P}_{\mathbb{T}}$ we have $\|X\|_{\mathbb{T}} \to X \in \mathbb{T}$.*
2. *$\mathcal{P}_{\mathbb{T}}$ flattens $\mathbb{T}$, i.e. $\mathbb{T} = \{X : \mathcal{P}_{\mathbb{T}} \mid \|X\|_{\mathbb{T}}\}$*

In this case we have 3.$\mathcal{P}_{\mathbb{T}} = \mathbb{T} - 1$. If $\mathcal{P}_{\mathbb{T}} \subset (\mathbb{T} - 1)-$seperated and $\mathbb{T}$ is saturated. , then the converse holds.

Proof.

1 $\Leftrightarrow$ 2 We have
$$\{X \in \mathcal{P}_{\mathbb{T}} \mid \|X\|_{\mathbb{T}}\} = \{X \mid \|X\|_{\mathbb{T}} \wedge \exists Y, X+Y \in \mathbb{T}\}$$
which coincides with $\mathbb{T}$ iff $\mathbb{T}$ is decompostable.

1 $\Rightarrow$ 3 For the second observe $\mathbb{T}-1 \subset \mathcal{P}_{\mathbb{T}}$. Then If $X+Y \in \mathbb{T}$, then $1+X+Y \in \mathbb{T}$ as $\mathbb{T}$ is stable under $+$. By decompostability $1+X \in \mathbb{T}$. Hence $X \in \mathbb{T}-1$.

3 $\Rightarrow$ 1 By the above lemma and saturatedness of the topology.

□

Lemma 16.21. *For any $\mathbb{P} : \mathsf{FLAT}$ and any Lavwere Tierney operator j,*
$$\mathcal{T}^j_{\mathbb{P}} := \{X \in \mathbb{P} \mid j\|X\|\}$$
is flattened by $\mathbb{P}$. Furthermore
$$\mathbb{P} = \mathcal{P}_{\mathcal{T}^j_{\mathbb{P}}}.$$

Proof. This is indeed a topology as $\mathbb{P}$ and j are $\sum$-stable We need to show, that for any $X \in \mathbb{P}$, we have $\|X\|_{\mathcal{T}^j_{\mathbb{P}}} = j\|X\|$. Note
$$\|X\|_{\mathcal{T}^j_{\mathbb{P}}} = \exists Y \in \mathcal{T}^j_{\mathbb{P}} : \|Y\| \to \|X\|$$
If $j\|X\|$, then put $Y := X$. Conversely, given $Y \in \mathcal{T}^j_{\mathbb{P}}$ such that $\|Y\| \to \|X\|$, we deduce from $j\|Y\|$ that $j\|X\|$.
Furthermore,
$$\{X \mid \exists Y, X+Y \in \mathbb{P} \wedge j\|X+Y\|\} = \{X \mid X \in \mathbb{P}\}$$
by Summand-stability on $\mathbb{P}$ we have $' \subset'$. if $X \in \mathbb{P}$, then use $Y := 1$: $X+1 \in \mathbb{P}$ and $j\|X+1\|$.

□

Proof of theorem 16.2:

1. and 2. Assume that $\mathbb{P} : \mathsf{FLAT}$ flattens $\mathbb{T}$, i.e. $\mathcal{T}^{\|\cdot\|_{\mathbb{T}}}_{\mathbb{P}} = \mathbb{T}$. We want to show that then $\mathbb{T}$ is decompostable and $\mathbb{P} = \mathcal{P}_{\mathcal{T}}$. First observe that $\mathcal{P}_{\mathbb{T}} \subset \mathbb{P}$ as $\{\perp\} \cup \mathbb{T} \subset \mathbb{P}$ For decompostability we apply 16.20. Observe
$$\mathcal{T}^{\|\cdot\|_{\mathbb{T}}}_{\mathcal{P}_{\mathbb{T}}} \subset \mathcal{T}^{\|\cdot\|_{\mathbb{T}}}_{\mathbb{P}} = \mathbb{T}$$
The other inclusion is automatic. This shows decompostability. Note
$$\mathcal{P}_{\mathbb{T}} = \mathcal{P}_{\mathcal{T}^{\|\cdot\|_{\mathbb{T}}}_{\mathbb{P}}} \overset{16.21}{=} \mathbb{P}$$

3. By the first point and 16.21.

Question 7. If $\mathbb{T}$ is flattened, what is the difference between Ω-stability for covering stacks and lex $\mathbb{P}$?
Are 0-gerbes $\mathbb{T}$-flat ?

17 Geometric covers are formally étale

TODO rename standart étale to basic étale . In this section we want to prove, that covering stacks are formally étale .

17.1 The étale topology is saturated

Let P denote a closed dense proposition.

Lemma 17.1. *An étale -flat DM-stack that is* $\neg\neg$*-inhabited is covering.*

Proof. If X is an étale -flat geometric stack, we may choose a geometric atlas $W \to X$ with W formally étale + flat. Using that the fibers $W \to X$ are $\neg\neg$-inhabited, we have

$$\begin{aligned}\neg\neg X &\to \neg\neg W \\ &\to W \in \mathbb{T} \\ &\to X \in \mathsf{CS}\end{aligned}$$

□

Lemma 17.2. *For $X \in EF$, $X \to X^P$ is a map fibered in weakly-flat stacks iff for any $x, y : X$, $(x = y)^P$ is étale -flat.*

Proof.

$\leftarrow$ By descent for covering stacks we may only show this for the fiber over Δx for some $x : X$ (Indeed let $z : X^P$. Assume $\| \sum_x \Delta x = z \|_{\mathbb{T}}$. By descent we may replace z by Δx for some $x : X$) But then the fiber is $\sum_y (x = y)^P$, a $\sum$ of étale -flat geometric stacks which is merely inhabited, hence covering.

$\rightarrow$ The fiber-inclusion over Δx is $(\sum_{y:X} (x = y)^P) \to X$ as calculated above. By stability under finite limits of EF the fiber $(x = y)^P$ over y is an étale -flat geometric stack.

□

Lemma 17.3. *Let X be étale -flat geometric stack and P a closed dense proposition. Then TFAE*

1. *$X^P \in \mathsf{EF}$.*
2. *$X \to X^P$ is an EF-cover, i.e. a map fibered in étale -flat stacks.*
3. *$X \to X^P$ is a geometric cover*
4. *$\Delta(X) : X \to X^P$ is an equivalence.*
5. *For any étale -flat geometric stack W such that $\Delta(W)$ is an equivalence and for any geometric cover $W \to X$, $W^P \to X^P$ is fibered in EF-stacks.*
6. *The same as 5 but 'exists W' instead of 'for all'.*

Proof.

$1 \Rightarrow 2$ EF is stable under finite limits

$2 \Rightarrow 3$ 17.1

$4 \Rightarrow 3 \Rightarrow 1$ obvious

$1 \Rightarrow 4$ we do induction over the truncation level. Contractible types are okay. Now let X be an étale -flat geometric stack. It suffices to show that $X \to X^P$ is an embedding by assumption (3) and 10.5. As $X \to X^P$ is in particular a map fibered in weakly flat stacks, for any $x, y : X$, $(x = y)^P \in \mathsf{EF}$ by 17.2. $x = y$ is an étale -flat geometric stack of one truncation level lower, we may apply the induction hypothesis thus $x = y \to (x = y)^P = (\Delta X =_{X^P} \Delta Y)$ is an equivalence. This map is ap_Δ.

$6 \Rightarrow 2 \Rightarrow 5$ Let $W \to X$ be a geometric cover with ΔW beeing an equivalence. Consider the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\sim} & W^P \\ \downarrow & & \downarrow \\ X & \longrightarrow & X^P \end{array}$$

As beeing fibered in étale -flat geometric stacks is a local property of morphisms of geometric stacks, $W^P \to X^P$ is fibered in étale -flat stacks iff $X \to X^P$ is an EF-cover, which is condition 2.

$5 \Rightarrow 6$ just use the definition of EF: X admits an atlas with EF-domain.

□

Lemma 17.4. *étale -flat geometric propositions are formally étale .*

Proof. Let X be such a proposition. We may show, that X is $\neg\neg$ stable, as $\neg\neg$-stable propositions are formally étale . If we assume $\neg\neg X$, then X is covering by 17.1, thus by 10.5 its contractible. □

Corollary 17.5 (of 17.4.)**.** *Covering sheaves are formally unramified.*

Lemma 17.6. *The type of open propositons* OPEN *is smooth.*

Proof. Apply 5.3 to the map into a magma

$$\begin{aligned} R &\to (\mathsf{OPEN}, \vee) \\ f &\mapsto \mathrm{isInv} f \end{aligned}$$

, □

The next lemma will follow also from the next subsection in which we prove more generally that all étale flat geometric stacks are formally étale .

Lemma 17.7. *Let X be a scheme that is étale -flat as a GS. Then X is formally étale .*

Proof. By 17.3 we may show, that $X \to X^P$ is an étale -flat cover. As X is unramified by 17.4, $x_p = y$ is an open proposition depending on $p : P$. But as the type of open propositions is smooth we find an open proposition Q such that $\forall p, (x_p = y) = Q$. Then, using that Q is formally étale ,

$$(\prod_p x_p = y) = (P \to Q) = Q$$

But Q is an open proposition, hence a formally étale + flat scheme, thus an étale -flat geometric stack.

□

Corollary 17.8. *The étale topology is saturated.*

Proof. Every affine covering stack is a scheme that is étale -flat as a GS, by 17.7 its formally étale . Its also fppf by saturatedness of fppf. □

17.2 Formally étale subuniverses

Definition 17.9. A formally étale subuniverse is a subtype $\mathbb{F} \subset FET$, such that one of the following equivalent conditions is satisfied

1. $\mathbb{F}$ is formally étale .
2. $\mathbb{F}$ is formally smooth
3. For any $X : FET$, $X \in \mathbb{F}$ is smooth.

Proof. Use that FET is formally unramified.

$2 \Leftrightarrow 3$ Study the fibers of $\mathbb{F} \to FET$ and use that beeing formaly étale is stable under finite limits.

$1 \Leftrightarrow 2$ The map $\mathbb{F} \to FET$ is an inclusion, thus $\mathbb{F}$ is formally unramified.

□

Lemma 17.10. *Let $S : P \to \mathbb{F}$, Then $\prod_p S_p : 1 \to \mathbb{F}$ is the unique filler.*

Proof. note that there exists a unique filler $\tilde{Y} : 1 \to \mathbb{F}$, as $\mathbb{F}$ is formally étale . On the other hand the filler $1 \to FET$ is given by $\prod_p Y_p$. But $\mathbb{F} \hookrightarrow FET$ is an embedding.

$$\begin{tikzcd} P \arrow[r, "Y"] \arrow[d] & \mathbb{F} \arrow[r, hook] & FET \\ 1 \arrow[ur, dashed, "\exists!"] \arrow[urr, "\prod_p Y_p"'] & & \end{tikzcd}$$

□

Proposition 17.11. *For any modality $\bigcirc$, there is a formally étale subuniverse cut out by the $\bigcirc$-modal types $FET \cap \mathcal{U}_{\bigcirc}$*

Proof. We only need to show that $FET \cap \mathcal{U}_{\bigcirc}$ is formally smooth. Here you use that $\bigcirc$-modal types are stable by depedent products over arbitrary indexing types. □

Example 17.12. *The following form formally étale subuniverses:*

- *The class $FET \cap \mathsf{St}$ of formally étale étale -stacks*
- *The class of étale propositions, i.e. propositions that are formally étale étale -sheaves.*

17.3 Standart étale

If A is an fp R-algebra, Alg_A denotes fp A- algebras.

Definition 17.13. Let A be a f.p. R algebra. The type of standart étale A-algebra $\mathsf{StdEtAlg}_A$ is the type of f.p. flat A-algebras which are merely of the form

$$(A[X_1, \ldots, X_n]/(P_1, \ldots, P_m))_G$$

such that $\det(\mathrm{Jac}(P_1, \ldots, P_n))$ divides G in $A[X_1, \ldots, X_n]/(P_1, \ldots, P_m)$.
We define StdEt_R as the class of types which merely is of the form

$$\mathrm{Spec}\, A_1 + \ldots + \mathrm{Spec}\, A_n$$

for $A_1, \ldots, A_n : \mathsf{StdEtAlg}_R$.

Question 8. Is standart étale stable under finite sums?

Definition 17.14. Let $A : \mathrm{Alg}_R$. The type of Presentations of f.p. algebras over R is

$$\mathsf{Pres}_A = \sum_{n,m} A[X_1, \dots, X_n]^m$$

We have a presentation forgetting map

$$\begin{aligned} \mathrm{fgt} : \mathsf{Pres}_A &\to \mathrm{Alg}_A \\ (n, m, P_1, \dots, P_m) &\mapsto A[X_1, \dots, X_n]/(P_1, \dots, P_m) \end{aligned}$$

Construction. For any map of R-algebras $A \to B$ there is an evident pushforward map on type of presentations, which we call by abuse of notation the same as on algebras:

$$\begin{tikzcd} \mathsf{Pres}_A \arrow[r, dashed, "-\otimes_A B"] \arrow[d] & \mathsf{Pres}_B \arrow[d] \\ \mathrm{Alg}_A \arrow[r, "-\otimes_A B"] & \mathrm{Alg}_B \end{tikzcd}$$

making the diagram commute.
It is given

$$\mathsf{Pres}_A \simeq \sum_{n,m} A[X_1, \dots, X_n]^m \to \sum_{n,m} A[X_1, \dots, X_n]^m \otimes_A B \simeq \mathsf{Pres}_B$$
$$(n, m, P) \mapsto (n, m, P \otimes 1)$$

Lemma 17.15.
- *For any n the type $R[X_1, \dots, X_n]$ is formally smooth.*
- *The type Pres_R is smooth.*

Proof. The second point follows from the first by $\sum$-stability of formally smooth types, as $\mathbb{N}$ is formally smooth. Let $n : \mathbb{N}$.However we can write it as a sequential union

$$R[X_1, \dots, X_n] = \bigcup_k R[X_1, \dots, X_n]_{\leq k}$$

where $R[X_1, \dots, X_n]_{\leq k}$ is the finite free R-submodule generated by monomials with degree $\leq k$. In particular it is a smooth type. Conclude by 5.2. $\square$

Proposition 17.16. *Let $R \twoheadrightarrow A$ be a quotient algebra. duality for fp algebras restricts to a bijection*

$$\mathsf{StdEtAlg}_A \cong (\operatorname{Spec} A \to \mathsf{StdEtAlg}_R)$$

Proof. Duality enhances to a bijection for pointed algebras

$$\mathrm{Alg}_{A,*} \to (\operatorname{Spec} A \to \mathrm{Alg}_{R,*})$$

Moreover, flatness is preserved under duality by 19.2. Def: A pointed A-algebra (B, G) admits an apropriate presentation, if there exists a presentation $B = R[X_1, \dots, X_n]/(P_1, \dots, P_n)$ such that $\det(Jac(P_1, \dots, P_n))$ divides G in B.

Let (B, G) be a pointed A-algebra. We need to show, that (B, G) admits an approriate presentation iff it admits that pointwise.

- If B, G admits an appropriate presentation F , then for any $\mathfrak{p} : \operatorname{Spec} A$, $F \otimes_A \mathfrak{p}$ is an appropriate presentation of $B \otimes_A \mathfrak{p}$.

- Assume that for $\mathfrak{p} : \operatorname{Spec} A$ we merely find an appropriate presentation of $B \otimes_A \mathfrak{p}$ as an R-algebra. Denote the proposition $P \equiv \operatorname{Spec} A$. In particular we have a solid arrow commutative diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \mathsf{Pres}_R \arrow[r] \arrow[d, "-\otimes_R A"] & \mathrm{Alg}_R \arrow[d, "-\otimes_R A"] \\ 1 \arrow[ur, dashed, "(1)"] \arrow[r, dashed, "(2)"'] \arrow[rr, bend right, "B"'] & \mathsf{Pres} A \arrow[r] & \mathrm{Alg}_A \end{tikzcd}$$

By smoothness of Pres_R , we find (1) an actual presentation $F : \mathsf{Pres}_R$, which presents $B\mathfrak{p}$ whenever $\mathfrak{p} : P$. The other half of the diagram still commutes using [ref?]

$$B \equiv \prod_{\mathfrak{p}:P} B_{\mathfrak{p}} = (\operatorname{Spec} A \to \mathrm{fgt} F) = \mathrm{fgt} F \otimes_R A$$

We may use (2) the presentation $F \otimes_R A : \mathsf{Pres}_A$ of B as an A algebra.

It remains to show, that $F \otimes_R A$ is an appropriate presentation of the A-algebra (B, G). However this is encoded by divisibility in B which can be checked on points of $\operatorname{Spec} B$ 3.3. Each such a point allows us to assume $\operatorname{Spec} A$.

□

we call a type X P-smooth, if $X \twoheadrightarrow X^P$. P-smooth types are stable under $\sum$ by the following diagram

$$\begin{tikzcd} \sum_{x:A} Bx \arrow[d, two heads] & \\ \sum_{x:A} (Bx)^P \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d, two heads] \\ (\sum_{x:A} Bx)^P \arrow[r] & A^P \end{tikzcd}$$

Lemma 17.17. *Let P be closed dense. Let A be a P-unramified R-algebra. Then A beeing quasi flat is P-smooth.*

Proof. A is flat iff any strong syzygy is explained in A, i.e. if for any $L \in R^{1\times n}, X \in A^{n\times 1}$ such that $L \neq 0$ and $LX = 0$, there merely is a term in

$$\sum_{Y \in A^{m\times 1}} \{G \in R^{n\times m} \mid (LG = 0)\} \times (GY = X).$$

Its enough to see, that this type is P-smooth.

- The type A^m is even formally smooth, as A merely admits a surjection out of a polynomial ring 17.15.

- The R-module $\{G \in R^{n\times m} \mid (LG = 0)\} \simeq (L^\perp)^{\oplus m}$ is finite free, as $L \neq 0$, thus formally smooth. Here we denote the hyperplane $L^\perp = \{X : R^{n\times 1} \mid LX = 0\}$.
- $GY =_{A^{n\times 1}} X$ is P-smooth: $A^{n\times 1}$ is P-unramified, as A is P-unramified .

□

Example 17.18. *Let $B : P \to \mathsf{QuasiFlatAlg}_R$. Then $A \equiv \prod_{\mathfrak{p}:P} B_{\mathfrak{p}}$ is quasi-flat as an R-algebra. As it is P-merely quasiflat, we only have to check P-unramifiedness. given $x, y : \prod_{\mathfrak{p}:P} B_{\mathfrak{p}}$,*

$$(x = y)^P = \left(\prod_{\mathfrak{p}:P} x\mathfrak{p} = y\mathfrak{p}\right)^P = \prod_{\mathfrak{p}:P} x\mathfrak{p} = y\mathfrak{p} = (x = y)$$

Lemma 17.19 (TODO FLATNESS)**.** *Then StdEt_R is a formally étale subuniverse.*

Proof.

- For any $A : \mathsf{StdEtAlg}_R$, $\operatorname{Spec} A$ is formally étale [ref?]. Thus a standart étale type, as a finite sum of formally étale types, is formally étale
- formally smoothness: Apply 5.3 to the map into a magma

$$\begin{aligned} \mathsf{StdEtAlg}_R &\to (\mathsf{StdEt}_R, +) \\ A &\mapsto \operatorname{Spec} A \end{aligned}$$

So it remains to show, that $\mathsf{StdEtAlg}_R$ is smooth. Let $I^2 = 0$. Let $\operatorname{Spec} R/I \to \mathsf{StdEtAlg}_R$. By 17.16 This corresponds to a unique $\mathsf{StdEtAlg}_{R/I}$, where $B' = R/I \otimes_R B$. choose a presentation

$$T = (R/I[X_1, \ldots, X_n]/(P_1, \ldots, P_m))_G$$

and some $H : R/I[X_1, \ldots, X_n]/(P_1, \ldots, P_m)$ such that

$$\det(\operatorname{Jac}(P_1, \ldots, P_n)) \cdot H = G.$$

Then choose lifts $\tilde{P}_1, \ldots, \tilde{P}_m \in R[X_1, \ldots, X_n]$ of the P_i, then a lift

$$\tilde{H} : R[X_1, \ldots, X_n]/(\tilde{P}_1, \ldots, \tilde{P}_n)$$

of H. Then define

$$\tilde{G} :\equiv \det(\operatorname{Jac}(\tilde{P}_1, \ldots, \tilde{P}_n)) \cdot \tilde{H}$$

I claim, that

$$\hat{T} \equiv \left(R[X_1, \ldots, X_n]/(\tilde{P}_1, \ldots, \tilde{P}_n)\right)_{\tilde{G}}$$

is a standart étale R-algebra such that $\hat{T} \otimes_R R/I = T$.
For this we only need to see flatness. For this define $\mathsf{StdEt'}_R$ just as StdEt_R but without the flatness condition. The proof that was given shows that $\mathsf{StdEt'}_R$ is a formally étale subuniverse. Thus by 17.10, The unique filler $\operatorname{Spec} \hat{T}$ is given by $\prod_p \operatorname{Spec} T_p$, which is not $\sum_p \operatorname{Spec} T_p = \operatorname{Spec} \prod_p T_p$ (you can set $\operatorname{Spec} T_p = 1$).

which is flat by 17.17 TODO

□

Remark 4. $P = \operatorname{Spec} A$. Let $P \to \mathsf{StdEt}_R$ correspond to some $B : \mathsf{StdEt}_A$. The $\sum$ corresponds to $\operatorname{Spec}_R B$ where we restricted scalars.

Proof. Indeed

$$\left(\sum_{\mathfrak{p}:P} \operatorname{Spec} B_{\mathfrak{p}}\right) \to R = \prod_{\mathfrak{p}:P} B_{\mathfrak{p}}$$

as an R-algebra. □

17.4 étale -flat stacks form a formally étale subuniverse

Warning. $\mathsf{EF} \hookrightarrow \mathsf{GS}$ is probably not formally étale .

Lemma 17.20. *An étale -flat geometric stack is formally étale , if it admits a geometric $\mathbb{F}$-cover with formally étale domain for $\mathbb{F}$ some formally étale subuniverse contained in* EF*.*

Proof. Choose $f : W \to X$ a geometric $\mathbb{F}$-cover with W a formally étale geometric stack. By 17.3, we may show that $W \to X^P$ is an $\mathbb{F}$-cover (thus an EF-cover). The fiber over any $x : X^P$ is $\prod_p \mathrm{fib}_{f_p} x_p$, a dependent product of things in $\mathbb{F}$, thus in $\mathbb{F}$ by 17.10. □

Lemma 17.21. *Let $\mathbb{F} \subset \mathsf{EF}$ be a formally étale subuniverse. Then $\mathsf{CS} \cap \mathbb{F}$ is a formally étale subuniverse.*

Proof. As $\mathsf{CS} \cap \mathbb{F} \subset \mathbb{F} \subset FET$, we only need to show, that for any $X : \mathbb{F}$, $X \in \mathsf{CS}$ is formally smooth. But for some $X : \mathsf{EF}$ beeing $\neg\neg$ inhabited is formally smooth, so conclude by 17.1. □

Lemma 17.22. *Every affine in the étale topology merely admits a Zariski cover with domain in* StdEt

Proof. [ref?] □

Theorem 17.23. *The type of* EF*-stacks is a formally étale subuniverse.*

Proof. By truncatedness of EF-stacks we have $\mathsf{EF} = \bigcup_n \mathsf{EF}_n$, so by 5.2 we may just show, that EF_n form a formally étale subuniverse for each n.

- $n = 0$:

  An algebraic space is called

  - 0-étale -flat, if its Zariski-flat, i.e. merely a finite sum of open propositions:
  - $n+1$-étale -flat, if it is merely the quotient of some stdétale by an EtProp-valued equivalence relation fibered in n-étale -flat covering algebraic spaces.

  We have that k-étale -flat algebraic spaces are étale -flat geometric stacks by induction, using that StdEt is a subtype of formally étale + flat affines. Observe that, the respective classes contain in particular

  1. -étale -flat algebraic spaces contain the étale -topology 17.22.
  2. -étale -flat algebraic spaces contains sheaves that merely admit an étale -catlas $S' \to F$, i.e. whose fibers as well as S' belongs to the étale topology. This is because by 17.22 we can choose a Zariski cover $\mathsf{StdEt} \ni \hat{S} \to S'$ and then $\hat{S} \to S' \to F$ is still an étale -cover.
  3. -étale -flat algebraic spaces consists already of all EF-algebraic spaces: Let $f : \operatorname{Spec} A \to X$ be a geometric atlas with $\operatorname{Spec} A$ beeing formally étale + flat. We may assume that $\operatorname{Spec} A$ is standart étale by 17.22. The equivalence relation R on $\operatorname{Spec} A$ induced by f is covering, thus its fibers merely admit an étale -catlas by 11.4. Hence R is fibered in covering 2-étale -flat algebraic spaces.

  We want to prove by induction, that k-étale -flat algebraic spaces form a formally étale subuniverse.

  - Zariski-Flat types form a formally étale subuniverse.
    - Every finite sum of opens is formally étale
    - The type $\mathcal{P}_{\mathsf{Zar}}$ is formally smooth: Apply 5.3 to the map into a magma $\mathsf{Open} \to (\mathcal{P}_{\mathsf{Zar}}, +)$.

- For the induction step $n \mapsto n+1$:
  * First we show that the type of $n+1$-étale -flat algebraic spaces is formally smooth. Denote $\mathbb{F}$ the covering n-étale -flat algebraic spaces, which are a formally étale subuniverse by the induction hypothesis and 17.21. Then the type of $n+1$-étale -flat algebraic spaces admits a surjection out of

$$\sum_{X:\mathsf{StdEt}} (R : \mathrm{EqRel}(X, \mathsf{EtProp})) \times (\prod_{x:X} R_x \in \mathbb{F})$$

    So it suffices to see, that this is formally étale , which is a modality, thus its enough to see
    1. StdEt is a formally étale subuniverse by 17.19
    2. EtProp is formally étale subuniverse by 17.4
    3. For any $x : X$, $R_x \in \mathbb{F}$ is formally étale : $R_x \equiv \sum_{y:X} Rxy$ is formally étale as X is formally étale by 1. and Rxy is by 2. So $R_x \in \mathbb{F}$ is the fiber over R_x of $\mathbb{F} \to FET$, which is a map between formally étale types.
  * Every $n+1$-étale -flat algebraic space admits a geometric cover fibered in n-étale -flat algebraic spaces with formally étale domain, thus it is formally étale by the induction hypothesis and 17.20.

- $n \mapsto n+1$. By definition we need to show:
  - Any étale -flat geometric $n+1$-stack is formally étale by 17.20 using that étale -flat geometric n-stacks are a formally étale subuniverse by the induction hypothesis.
  - To show EF_{n+1} beeing formally smooth, we may show [ref?]that the domain of the surjection

$$\textstyle\sum_{X:\mathsf{St}\cap FET}(X \xrightarrow{F} \mathsf{CS}_n) \times (\sum_X F \in \mathsf{EF}_n) \twoheadrightarrow \sum_{X:\mathsf{St}\cap FET} X \in \mathsf{EF}_{n+1} = \mathsf{EF}_{n+1}$$

    is formally étale , where the right equality uses the previous paragraph.
    As beeing formaly étale is a modality, we may only show that the following types are formally étale
    * $\mathsf{St} \cap FET$ by 17.12
    * CS_n. It embeds into the formally étale EF_n so conclude by 17.21.
    * $\sum_X F \in \mathsf{EF}_n$. Here just use that the map $\mathsf{EF}_n \hookrightarrow \mathsf{St}\cap FET \ni \sum_X F$ between formally étale types has formally étale fibers.

□

18 Tangent Spaces

Definition 18.1. A pointed type $(D, 0)$ is tiny if

- it has choice
- for any $W : D \to \mathsf{Aff}$, $\prod_d Wd$ is affine
- D is flat affine

Remark 5. Closed dense propositions are probably not tiny: Put $Wd = \mathbb{A}^1$, then if $\prod_{d:D} Wd = R/\varepsilon$ is affine, it would be an affine finitely copresented module, hence maybe free?

Fix a topology $\mathbb{T}$ which is stable under tiny exponentials, i.e. such that for any D tiny, and any $W : D \to \mathbb{T}$, the affine $\prod_{d:D} W_d$ belongs to $\mathbb{T}$.

Lemma 18.2 (TODO)**.** *The following topologies are stable under tiny exponentials.*

- *The étale topology*
- *The smooth topology*

Proof.
- TODO
- TODO

□

Warning. The fppf topology is not stable under tiny exponentials! 18.1

Theorem 18.3. *Covering / Geometric stacks are stable under exponentials over tiny types.*

Proof. Let $P : D \to \mathsf{GS}$. We first prove the covering case by the W induction principle of covering stacks: By choice of D We may assume that $P : D \to W_n$. If $n = 0$ its fine by assumption on $\mathbb{T}$. by choice of D we can choose W_{n-1} atlasses $pd : Xd \to Pd$ for $d : D$. Claim : $\prod_{d:D} X(d) \to \prod_{d:D} Pd$ is a geometric atlas. Proof : Indeed the fiber over f is $\prod_{d:D} \mathrm{fib}_{pd}(fd)$ which is a dependent product over W_{n-1} types, hence covering by induction. Hence $\prod_d Pd$ is geometric. If, additionally, all the Pd are covering then we may choose the Xd to be covering affine. By assumption on $\mathbb{T}$, $\prod_{d:D} Xd$ belongs to $\mathbb{T}$, hence $\prod_{d:D} Pd$ is a covering stack. □

Corollary 18.4. *For any D tiny and $W_d \to X_d$ a family of geometric atlasses the map $\prod_{d:D} W_d \to \prod_{d:D} X_d$ is a geometric atlas.*

Corollary 18.5. *Geometric stacks are stable under taking tangent spaces.*

Lemma 18.6. *For any $B : D \to \mathcal{U}$ a type family we have*

$$\forall d : D \|Bd\|_{\mathbb{T}} \to \|\prod_d Bd\|_{\mathbb{T}}$$

if D has choice such that $\mathbb{T}$ is $\prod$-stable over D.

Proof.
1. By choice of D we find $S_d \in \mathbb{T}$ and $S_d \to \|Bd\|$. by $\prod$-stability over D on $\mathbb{T}$ we have $\prod_d S_d \in \mathbb{T}$,s in particular $\|\prod_d S_d\|_{\mathbb{T}}$. Hence we $\mathbb{T}$-merely have $\prod_d \|Bd\|$. By choice of D we $\mathbb{T}$-merely get $\prod_d Bd$.
2. If D is a proposition just observe, that

$$\|\prod_d Bd\|_{\mathbb{T}} = \|D \to \sum_d Bd\|_{\mathbb{T}} = D \to \|\sum_d Bd\|_{\mathbb{T}}.$$

□

Lemma 18.7. *Let D be a finite wedge of infinitesimal varieties. Consider a family of smooth maps $f_d : W_d \to X_d$ and an element $w : W_0$. Consider $g : \prod_d X_d$ such that $p : g_0 = f_0 w$. Then we merely find some $h : \prod_{d:D} W_d$ such that $h_0 = w$ with $q_d : g_d = f_d(h_d)$.*

Proof. Let us first treat the special case where D is tiny. For any $d : D$ by smoothness of $W_d \to X_d$ we merely have a lift

$$\begin{tikzcd} d = 0 \arrow[r] \arrow[d] & \mathrm{fib}_{f_d}\, g_d \\ 1 \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

where the above map is given by transport of $(w, p) : \mathrm{fib}_{f_0}(g_0)$. By choice of D we can produce a term in

$$\prod_d (h_d : \mathrm{fib}_{f_d}\, g_d) \times ((r : d = 0) \to h_d = \mathrm{tp}_r(w, p)) \simeq \left(\prod_d (h_d : \mathrm{fib}_{f_d}\, g_d) \right) \times h_0 = w$$

Which is the datum of a filler. This concludes the special case of D beeing tiny.
If $D = \bigvee_{i=1}^n D^i$ we can produce by the special case sections $h^i : \prod_{d:D_i} W_d$ such that $h^i_0 = w$ with $q_d : g_d = f_d(h^i_d)$. As the h_i agree on the basepoint, we get a dependent section $h : \prod_{d:D} W_d$ with $h_0 = w$and $g_d = f_d(h_d)$. ☐

Lemma 18.8. *Let D be a finite wedge of infinitesimal varieties. Given a family of $\mathbb{T}$-surjective smooth maps $W_d \to X_d$, the map $\prod_{d:D} W_d \to \prod_{d:D} X_d$ is $\mathbb{T}$-surjective*

Proof. To apply the previous lemma, we just use that $W_0 \to X_0$ is $\mathbb{T}$-surjective. ☐

Proposition 18.9. *Let $j : D \to D'$ be a map between a finite wedge of infinitesimal varieties D and a tiny type D', that is local wrt to all affine schemes, i.e. $X^{D'} \to X^D$ is an equivalence for any affine X. Then its local wrt to all geometric stacks. In particular, geometric stacks are infinitesimally linear, i.e. a geometric stack X is local wrt $j : \mathbb{D}(n_1) \vee \ldots \vee \mathbb{D}(n_k) \to \mathbb{D}(n_1 + \ldots + n_k)$ for any $n_1, \ldots, n_k : \mathbb{N}$.*

Proof. We prove more generally, that for any family $X : D' \to \mathsf{GS}$, the map

$$\prod_{d:D'} Xd \to \prod_{d:D} X(jd) \tag{$\star$}$$

is an equivalence. Lets first check the special case where all the Xd are affine: We have a pullback

$$\begin{tikzcd} \prod_{d:D'} Xd \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_{d:D} X(jd) \arrow[d] \\ \left(\sum_{d:D'} Xd\right)^{D'} \arrow[r] & \left(\sum_{d:D} Xjd\right)^{D} \end{tikzcd}$$

The lower map is an equivalence, as $\sum_{d:D'} Xd$ is affine scheme, hence infinitesimally linear. So the above map is an equivalence as well.
We split the prove of equivalence up into $\mathbb{T}$-surjectivity and beeing an embedding.

- By choice of D' we find geometric atlasses $Wd \to Xd$ for $d : D'$. Then by the special case and 18.4 we can the following commutative diagram

$$\begin{tikzcd} \prod_{d:D'} Wd \arrow[r, "\sim"] \arrow[d] & \prod_{d:D} W(jd) \arrow[d, "\mathbb{T}-surj"] \\ \prod_{d:D'} Xd \arrow[r] & \prod_{d:D} X(jd) \end{tikzcd}$$

- Now we need to show that the map is an embedding. Induction over the truncatedness of X. For $n = -2$ its fine. For the induction step $n \mapsto n+1$, use function extensionality and observe that the identity types of X are geometric n-stacks, so the map ($\star$) where we replace Xd by its appropriate identity type, is an equivalence by induction.

$\square$

Lemma 18.10. *Let $A = R[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$ be a finitely presented algebra. Let $\mathfrak{p} : \operatorname{Spec} A$ be a point. The tangent space of $\operatorname{Spec} A$ at $\mathfrak{p} : \operatorname{Spec} A \subset \mathbb{A}^n$ is affine whose algebra is cut out by the polynomials $g_i = \sum_k \frac{\partial f_i}{\partial x_k}(\mathfrak{p}) x_k$:*

Proof. I give two proofs

- Write V to the A-module R obtained by $\mathfrak{p}$. We have a bijection

$$\begin{tikzcd} \operatorname{Der}(R[X_1, \ldots, X_n], V) \arrow[r, leftrightarrow, "\cong"] & \operatorname{Spec} R[Y_1, \ldots, Y_n] \\ d \arrow[r, mapsto] & (dx_1, \ldots, dx_n) \\ (h \mapsto \sum_k \frac{\partial h}{\partial x_k}(p) \cdot v_k) & v \arrow[l, mapsto] \end{tikzcd}$$

  This restricts to a bijection $\operatorname{Der}(A, V) \cong V(g_1, \ldots, g_n)$ by construction of the g_i.

- We can write $\operatorname{Spec} A$ as a pullback

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & R^n \arrow[d, "F"] \\ 1 \arrow[r, "0"] & R^m \end{tikzcd}$$

  Now one can apply Tangentspaces to get a pullback again

$$\begin{tikzcd} T_{\mathfrak{p}} \operatorname{Spec} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & T_{\mathfrak{p}} R^n = R^n \arrow[d, "DF_{\mathfrak{p}}"'] \\ 1 \arrow[r, "0"] & T_0 R^m = R^m \end{tikzcd}$$

$\square$

Remark 6. Given a pointed geometric stack (X, x), we can look at $Y = T_x X$ and the map $Y \to \|Y\|_0$, the fibers are deloopings of ΩY. Now there exists a ΩY-equivariant isomorphism $Y \cong \|Y\|_0 \times B\Omega Y$ over $\|Y\|_0$ iff the map $Y \to \|Y\|_0^{\mathbb{T}}$ has a section.

Tangent spaces of Deligne Mumford stacks are somewhat uninteresting.

Lemma 18.11. *The tangent spaces of Deligne Mumford stacks are 0-types. Maybe finitely copresented?*

Proof. Let X be a Deligne Mumford Stack. Choose a geometric atlas $f : \operatorname{Spec} A \to X$. Let $x : X$. As the tangent space $T_x X$ is a stack, it beeing a set is a sheaf, so by $\mathbb{T}$-surjectivity of f we may assume that $x = f(y)$ for some $y : \operatorname{Spec} A$. But then $D_f y : T_y(\operatorname{Spec} A) \to T_x X$ is a bijection, as f is formally étale [ref?]. The next question answers what happens for finitely copresented. $\square$

Question 9. Is for a sheaf R-module to be finitely copresented a sheaf?

18.1 Flatness examples

Lemma 18.12. *$I^2 = I$ implies $I = 0$ or $I = R$.*

Proof. By Nakayama we find r such that $r \in I$ but $(1-r)I = 0$. Then $(r) = I$ because if $x \in I$ then $(1-r)x = 0$, hence $r|x$. As R is local we can decide either $r = 0$ or $r = 1$. $\square$

Lemma 18.13. *A closed proposition is decidable in each of the following cases*

- *It is $\neg\neg$-stable.*
- *It is flat.*

Proof. We can write the closed proposition as $I = 0$ for some finitely generated ideal I. Let us show, that in each case $I^2 = I$.

- If $I = 0$ is $\neg\neg$-stable, this means that $I = I^2$. TODO
- If R/I is a flat R-module. The map $I \otimes R/I \to R/I$ is injective by flatness. But its the zero map. Hence $I \otimes R/I = I/I^2 = 0$.

$\square$

Example 18.14. *Let ε be nilpotent. R/ε is not$_\varepsilon$ flat.*

From now on we try to only argue geometrically instead of algebraically. For example $\operatorname{Spec} R[Z,T]/TZ \to \operatorname{Spec} R[T]$ is not flat by the following lemma.

Lemma 18.15. *If ε is nilpotent, then $R[z]/\varepsilon z$ is not$_\varepsilon$ flat .*

Proof. We have an R-linear isomorphism

$$\begin{aligned} R \oplus R/\varepsilon[Y] &\to R[Y]/(\varepsilon Y) \\ (r, f) &\mapsto r + Yf \end{aligned}$$

As the RHS, the second factor of the LHS is flat over R. As $R/\varepsilon[Y]$ is a faithfully flat algebra over R/ε, we deduce that R/ε is flat over R. By the lemma We conclude that $\varepsilon = 0$ as desired. $\square$

Example 18.16. $\operatorname{Spec} R[X,Y,T]/(T - XY) \to \operatorname{Spec} R[T]$ *is flat, but* $\operatorname{Spec} R[X,Y,T]/(T - TXY) \to \operatorname{Spec} R[T]$ *is not flat.*

Proof. The first example is flat todo, but in the second example the fiber $SpecR[X,Y]/(\varepsilon(XY-1))$ is not$_\varepsilon$ flat:

$$\begin{tikzcd}
\operatorname{Spec} R[X,Y]/(\varepsilon(XY-1)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{<}{\underset{fppf}{\Longleftarrow}}"] & \operatorname{Spec} R[X,Y] \arrow[d] \\
\operatorname{Spec} R[Z]/\varepsilon Z \arrow[r] & \operatorname{Spec} R[Z]
\end{tikzcd}$$

$\operatorname{Spec} R[X,Y] \to \operatorname{Spec} R[Z]$ on algebras sending $Z \mapsto XY - 1$, which is fppf by the first part of the example. $\square$

Example 18.17 (TODO)**.** *The affine veronese map (not an embedding)*

$$\begin{aligned} v_2 : \operatorname{Spec} R[X,Y] &\to \operatorname{Spec}[X_1, X_2, X_3]/(X_1X_3 - X_2^2) \\ (x, y) &\mapsto (x^2, xy, y^2) \end{aligned}$$

is not flat.

Proof. Let us show that the fiber over (x_1, ε, x_3) is not$_\varepsilon$ flat.
We have an embedding into a flat scheme

$$\mathrm{fib}_{v_2}(x_1, \varepsilon, x_3) \hookrightarrow \operatorname{Spec} R[X]/(X^2 - X_1) \times \operatorname{Spec} R[Y]/(Y^2 - X_3)$$

□

Warning. Tangent spaces of faithfully flat affines are not flat in general. Let $p \neq 0$ be prime. $R[X]/X^p$ is a faitfully flat algebra as X^p is a monic polynomial [ref?]. Then it is not the case that all tangent spaces are flat.

Proof.

$$T_\varepsilon \operatorname{Spec} R[X]/(X^p) = \operatorname{Spec} R[Y]/(p\varepsilon^{p-1}Y) = \operatorname{Spec} R[Y]/(\varepsilon^{p-1}Y)$$

By the lemma For any ε nilpotent, $T_\varepsilon \operatorname{Spec} R[X]/(X^p)$ is not$_{\varepsilon^{p-1}}$ flat . Thats enough by duality because the composite $\operatorname{Spec} R[Y]/(Y^{p-1}) \hookrightarrow \operatorname{Spec} R[Y]/(Y^p) \hookrightarrow \mathcal{N}_\infty(0)$ is not an equivalence

□

19 Questions // TODO

Theorem 19.1 (TODO)**.** *An Artin stack X is Deligne Mumford iff one of the following conditions is satisfied:*

1. *There exists a geometric atlas $W \to X$*
2. *The identity types of X are $\mathbb{P}$-seperated*

Proof. 1. $\Rightarrow$2. **??**

2. $\Rightarrow$ 1 Residual ??? [06MF]

$\square$

Prove 17.19!!!

Question 10. if $\mathbb{T} \subset \mathbb{T}'$ do we have that for each $X : \mathsf{GS}_{\mathbb{T}}$ $L_{\mathbb{T}'}X \in \mathsf{GS}_{\mathbb{T}'}$?

Theorem 19.2 (TODO)**.** *The class of flat affines is stable under $\sum$. Moreover flatness can be defined fiberwise.*

20 Not clear where to put that

Lemma 20.1. *Let $\rho \neq 0$.* $\operatorname{Spec} R[T]/(T^2+1)$ *is compact.*

Proof. Let $U \subset \operatorname{Spec} C$ be open. Then we find $f_1, \dots, f_n : C$, such that $U = D(f_1, \dots, f_n)$. Choose representatives $f_i = a_i + b_i T \mod T^2 + \rho$. Then consider the following numbers

$$r_{ij} = \begin{cases} a_i b_j - a_j b_i & , i \neq j \\ a_i^2 + \rho b_j^2 & , i = j \end{cases}$$

We will show that $D((r_{ij})_{i,j}) \leftrightarrow (\operatorname{Spec} C \subset U)$.
'$\rightarrow$'
Assume $r_{ij} \neq 0$. If $i = j$, then $\operatorname{Spec} C \subset D(f_i) \subset U$.
If $i \neq j$, then $\operatorname{Spec} C \subset D(f_i, f_j) \subset U$.
'$\leftarrow$'
Because this statement is a propositional sheaf, we may assume a term $x : \operatorname{Spec} C$. Choose i, j, s.th. $x \in D(f_i), -x \in D(f_j)$. In both cases $i = j$ and $i \neq j$, then $r_{ij} \neq 0$. $\square$

Lemma 20.2 (Not needed)**.** *Open subtypes of $\mathbb{A}^1$ are $\neg\neg$ principal open.*

Proof.

- An open affine subscheme of $\mathbb{A}^1$ is $\neg\neg$ principal open: Let $D(f_1, \dots, f_n) \subset \mathbb{A}^1$ be an arbitrary open subset. We may assume that each $f_i : R[X]$ is non constant (in particular non zero). By [ref?], $\neg\neg$-merely each $D(f_i) \subset R$ is cofinite. Thus $\neg\neg$-merely, the finite union $\bigcup_{i=1}^n D(f_i) \subset R$ is cofinite as well, hence principal open.

$\square$

Proposition 20.3. *Assume covering stacks are Ω-stable. A truncated stack (e.g. geometric stack) is covering iff $\pi_0^{\mathbb{T}} X := \|X\|_0^{\mathbb{T}}$ and all higher homotopy groups*

$$\pi_i^{\mathbb{T}}(X, x) = \|\Omega^i(X, x)\|_0^{\mathbb{T}}, i \geq 1$$

are covering algebraic spaces.

Proof. Let X be an n-stack. If X is covering, then by Ω-stability all the $\pi_i^{\mathbb{T}}$ are covering 15.6 Now the converse. Consider the postnikov tower

$$X = \|X\|_n^{\mathbb{T}} \to \|X\|_{n-1}^{\mathbb{T}} \to \ldots \to \|X\|_1^{\mathbb{T}} \to \|X\|_0^{\mathbb{T}}$$

As $\|X\|_0^{\mathbb{T}}$ is covering, by quotient stability of covering stacks we may show that all the maps are geometric covers. Let $1 \le k \le n$ and consider the map $f_k^X : \|X\|_k^{\mathbb{T}} \to \|X\|_{k-1}^{\mathbb{T}}$. By descent for covering stacks, we may only consider the fiber over $|x|$, as the $\eta_{k-1}^{\mathbb{T}}$ is $\mathbb{T}$-surjective. It suffices to show, that the fiber is given by $B_{\mathbb{T}}^k \pi_k^{\mathbb{T}}(X, x)$ as deloopings of covering stacks are covering 13.4.
We apply 13.3. First observe that $\Omega^k(\text{fib}(f_k^X)|x| = \text{fib}(\Omega^k(f_k^X, x))$ is equivalent to the basefiber of

$$\pi_k^{\mathbb{T}}(X, x) \equiv \|\Omega^k X\|_0^{\mathbb{T}} \simeq \Omega^k(\|X\|_k^{\mathbb{T}}) \to \Omega^k \|X\|_{k-1}^{\mathbb{T}} \simeq 1$$

So it suffices to show by induction over k, that for all pointed stacks (X, x), $\text{fib}(f_k^X)|x|$ is $\mathbb{T}$-k-connected.
This is definitely $\mathbb{T}$-connected by using that any term $(y, p) : \text{fib}(f_k^X)|x| = \sum_{y:\|X\|_n^{\mathbb{T}}} \|x = y\|_{n-1}^{\mathbb{T}}$ yields a witness of $\|x - y\|^{\mathbb{T}}$. Then $\Omega(\text{fib}(f_k^X)|x| = \text{fib}(\Omega(f_k^X, x)) = \text{fib}(f_{k-1}^{\Omega(X,x)})$ which is $\mathbb{T}$-$k-1$-connected by induction. □

20.1 Remarks about weakly flat affines

Lemma 20.4. *The proposition $\|X\|_{\mathbb{T}}$ is geometric iff there exists a map from a weakly flat affine $\operatorname{Spec} B \to X$ such that $\|\operatorname{Spec} B\|_{\mathbb{T}} \to \|X\|_{\mathbb{T}}$ is an equivalence.*

Proof. '$\leftarrow$' is clear.
'$\to$'. Choose $\operatorname{Spec} B'$ weakly flat such that $\|X\|_{\mathbb{T}} = \|\operatorname{Spec} B'\|_{\mathbb{T}}$. As the map $X \to \|X\|_{\mathbb{T}}$ is $\mathbb{T}$-surjective, by $\mathbb{T}$-local choice we find a $\mathbb{T}$-cover $\operatorname{Spec} B \to \operatorname{Spec} B'$ and a commutative diagram

$$\begin{tikzcd} \exists \operatorname{Spec} B \arrow[r, dashed] \arrow[d, dashed] & X \arrow[d] \\ \operatorname{Spec} B' \arrow[r] & \|X\|_{\mathbb{T}} \end{tikzcd}$$

As $\operatorname{Spec} B'$ was weakly flat and the left vertical map is a $\mathbb{T}$-cover, $\operatorname{Spec} B$ is weakly flat. □

Lemma 20.5 (DM)**.** *If $\operatorname{Spec} A + \operatorname{Spec} B$ is weakly flat affine, then $\operatorname{Spec} A$ is weakly flat.*

Proof. Indeed

$$\|X\|_{\mathbb{T}} \to \|X + Y\|_{\mathbb{T}} \to X + Y \in \mathbb{T} \to X \in \mathbb{P}$$

but $\|X\|_{\mathbb{T}} \wedge X \in \mathbb{P} \to X \in \mathbb{T}$.

□

Lemma 20.6. *if the topology is saturated Beeing weakly-flat descends along $\mathbb{T}$-covers.*

Lemma 20.7 (DM)**.** *If $\|P + Q\|_{\mathbb{T}}$ is a geometric prop, then TODO*

Proof. By the previous two lemma and we find a map out of a weakly flat affine $\operatorname{Spec} B \to P + Q$ that induces an equivalence on $\mathbb{T}$-truncations, but it splits into two map out of a weakly affine $\operatorname{Spec} B_1 \to P, \operatorname{Spec} B_2 \to Q$. □

Notation. For $P : (\varepsilon : \mathcal{N}_\infty(0)) \to X \to \text{Prop}$, let $\varepsilon : \mathcal{N}_\infty(0) \vdash x : X$. We say x is not$_\varepsilon$ P, if $\forall \varepsilon$, $P_\varepsilon x \to \varepsilon = 0$. Observe, if x is not$_\varepsilon$ P for any $\varepsilon^2 = 0$, then x is not P.

Remark 7. If $2 \neq 0$. Let $\varepsilon, \varepsilon' : \mathcal{N}_\infty(0)$. $\varepsilon = \varepsilon' + \varepsilon = -\varepsilon'$ is not$_\varepsilon$ weakly-flat

Proof. We prove that once its $\mathbb{T}$-merely inhabited, then its not$_\varepsilon$ covering, which is enough as $\neg\neg(\varepsilon = \varepsilon' + \varepsilon = -\varepsilon')$.As the goal is a stack we may assume $\varepsilon = \varepsilon' + \varepsilon = -\varepsilon'$. wlog the first case. Then assume $1 + (\varepsilon = -\varepsilon) \simeq 1 + \varepsilon = 0$ is covering. Then $\varepsilon = 0$ is formally étale , thus inhabited as a formally étale closed dense proposition. □

Example 20.8 (Obsolete)**.** *The map* $q : \mathbb{A}^1 \to \mathbb{A}^1/\mu_\ell$ *is not a geometric cover.*

Proof. The map factors through the geometric cover $\mathbb{A}^1 \to \mathbb{A}^1//\mu_\ell$. Thus its enough to show that $\mathbb{A}^1//\mu_\ell$ is not a 0-gerbe, or that not every loop space is covering. Let us show that, $\Omega(\mathbb{A}^1//\mu_\ell, \varepsilon)$ is not$_\varepsilon$ covering. Assume it is covering for some $\varepsilon \in \mathcal{N}_\infty(0)$. As μ_ℓ has decidable equality,

$$\begin{aligned}
\Omega(\mathbb{A}^1//\mu_\ell, \varepsilon) &= \left(\sum_{g:\mu_\ell} g\varepsilon = \varepsilon\right) \\
&= (\varepsilon = \varepsilon) + \sum_{g:\mu_\ell \setminus \{1\}} (g-1)\varepsilon = 0 \\
&= 1 + \mu_\ell \setminus \{1\} \times (\varepsilon = 0)
\end{aligned}$$

Thus $(\varepsilon = 0) \times (\mu_\ell \setminus \{1\})$ is an étale -flat geometric stack. Moreover $(\mu_\ell \setminus \{1\})$ is a covering stack by 12.12. Thus $\varepsilon = 0$ is an affine étale -flat geometric stack, thus formally étale + flat affine by saturatedness of the étale topology 17.8. So as a formally étale + closed dense proposition, $\varepsilon = 0$ holds as desired. □

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