Thesis

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1 Preparation

Lemma 1.1. *Let C be a class of types stable under $\sum$. The class $\mathsf{HasAtlas}_C$ of types Y which admit a map $\operatorname{Spec} A \to Y$ fibered in C is stable under identity types.*

Proof. By assumption we can choose a map $p : V \to Y$ out of an affine fibered in C. Let $y, y' : Y$. Then we have the map

$$\begin{gathered}(\operatorname{fib}_p y) \times_V (\operatorname{fib}_p y') \to y = y' \\ (v, q : y = pv), (v', q' : y' = pv'), (h : v = v') \mapsto q \cdot h \cdot q'^{-1}\end{gathered}$$

The fiber over $j : y = y'$ looks like

$$\sum_v (\underbrace{\sum_{v'} (h : v = v')}_{\mathsf{isContr}}) \times (q : y = pv) \times (q' : y' = pv') \times (q \cdot h \cdot q'^{-1} = j) \simeq \sum_v (v = py) \simeq \operatorname{fib}_p y$$

Hence the map is fibered in C. It suffices to show, that $(\operatorname{fib}_p y) \times_V (\operatorname{fib}_p y')$ has an atlas, because then we can compose such an atlas with the above map to obtain an atlas of $y = y'$. By assumption the fibers of p have an atlas, so we can choose $q : W \to \operatorname{fib}_p y, q' : W' \to \operatorname{fib}_p y'$ atlasses. Then $W \times_V W' \to (\operatorname{fib}_p y) \times_V (\operatorname{fib}_p y')$ is an atlas: The domain is a fiber product of affines, hence affine. The fiber over (x, x') is equivalent to the product of fibers $(\operatorname{fib}_q x) \times (\operatorname{fib}_{q'} x')$ which is in C by stability under dependent sums (so in particular under finite products).

□

Lemma 1.2. *Let $\mathcal{U}' \subset \mathcal{U}$ be stable under dependent sums (e.g. $\mathbb{T}$-inhabited types) Let X be a type with a map $p : U \to X$ fibered in $\mathcal{U}'$. For any $x : X$, let Y_x be a type and moreover for any $u : U$, we are given a map $q_u : V_u \to Y_{p(u)}$ fibered in $\mathcal{U}'$. Then the induced map*

$$p : \sum_{u:U} V_u \to \sum_{x:X} Y_x$$

is fibered in $\mathcal{U}'$

Proof. The fiber of p over some $(x, y) \in \sum_{x:X} Y_x$ is given by

$$\sum_{u:\operatorname{fib}_p x} \operatorname{fib}_{q_u}(y')$$

where $y' : Y_{p(u)}$ (depending on u) is the transport of $y : Y_x$ along $x = p(u)$. As $\mathcal{U}'$ is stable under dependent sum those fibers are again in $\mathcal{U}'$. This shows the result. □

2 Lex Modalities

Lemma 2.1 (Stability resuls)**.** *Lex Modalities are stable under*

1. *Conjunction*
2. *Composition*

Lemma 2.2. *Let $\bigcirc$ be a lex-modality. Let X be $\bigcirc$-modal and $B : X \to \mathcal{U}_\bigcirc$ be a family of modal types. Then $\sum_{x:X} B_x$ is $\bigcirc$-modal*

Lemma 2.3. *Let $B : \bullet X \to \mathcal{U}$. Then $\bullet(\sum_{x:X} B(\eta x)) = \sum_{x:\bullet X} \bullet Bx$*

Proof. Observe that

$$\sum_{x:X} B\eta x \to \sum_{x:\bullet X} Bx$$

is a $\bullet$-equivalence, because for all modal types T, the type $Bx \to T$ is modal for any $x : \bullet X$. Then it follows by [ref?]. □

Lemma 2.4. *Let $\bullet$ be a lex modality. Let $x, y : X$. The map*

$$\bullet(x = y) \to \eta x =_{\bullet X} \eta y$$

induced by $ap_\eta : x = y \to \eta x =_{\bullet X} \eta y$ is an equivalence

Proof. By Modalities Theorem 3.1 [ix]. □

Definition 2.5. Let $\bullet$ be a lex modality. we call a type X $\bullet$-seperated if one of the following equivalent conditions hold

- the identity types of X are modal
- the unit $X \to \bullet X$ is an embedding

In this case

Proof. by 2.4 the vertical map in the commutative diagram

$$\begin{tikzcd} x =_X y \arrow[r, "\eta_{x=y}"] \arrow[dr, "ap_{\eta_X}"'] & L(x = y) \arrow[d, "\simeq"] \\ & \eta x =_{LX} \eta y \end{tikzcd}$$

is an equivalence. So $x = y$ is a sheaf if $\eta_{x=y}$ is an equivalence iff η_X is an embedding. □

3 Covering stacks

Fix $\mathbb{T}$ a topology, which we call the covering-affines.

Definition 3.1. Covering geometric stacks are the smallest class containing $\mathbb{T}$ such that: If Y is a sheaf and $\mathbb{T} \ni S \to Y$ is fibered in covering geometric stacks, then Y is a covering geometric stack.

We call such map $X \to Y$ whose fibers are covering stacks a geometric cover. If X is affine we call it a geometric atlas. If X is in $\mathbb{T}$ we call it a geometric catlas.

Proposition 3.2 (Recursion principle for covering stacks)**.** *Let $P : \mathcal{U}_{\mathbb{T}} \to \mathrm{Prop}$ be a property of sheaves. Assume*

- *Every covering affine has P*
- *If $\mathbb{T} \ni S \to Y$ is fibered in P then Y has P*

Then every covering geometric stack has P.

Proof. Replace P by $P \wedge \mathsf{is-covering-stack}$. Then usual induction

□

Definition 3.3. We call X a geometric stack if it merely has a geometric atlas, i.e some $\operatorname{Spec} A \to X$ fibered in covering geometric stacks.

Lemma 3.4. *The class of (covering) geometric stacks is $\sum$-stable.*

Proof. Define the predicate PX as ' the sum of every family B of (covering) geometric stacks is a (covering) geometric stack'. If X is a (covering) affine, by choice of X we can choose geometric (c)atlasses $S_x \to Bx$ for all $x : X$. Then $\sum_{x:X} S_x \to \sum_x Bx$ is a geometric catlas. If $f : S \to X$ is a map fibered in P with $S \in T$, then let $B : X \to \mathsf{CS}_{\mathcal{V}}$. By choice of S we can choose geometric catlasses $\tilde{B}s \to B(fs)$ for all $s : S$. Then consider $\sum_{s:S} \tilde{B}s \to \sum_{x:X} Bx$. Its domain is in $\mathbb{T}$. It remains to show, that the fiber over (x, t) is a covering stack. It is a dependent sum over $\mathrm{fib}_f\, x$, which by induction satisfies P that lets us conclude by definition of P. □

Lemma 3.5. *covers are stable under composition.*

Proof. covering stacks are stable under $\sum$. □

Proposition 3.6. *Every covering geometric stack X merely admits a geometric catlas.*

Proof.

- If X is covering affine, then $X \to X$ is a geometric catlas.
- If X is obtained as a quotient then it already is equipped with a catlas.

□

Proposition 3.7. *The class of (covering) geometric stacks is stable under quotients: If $X \to Y$ is fibered in covering stacks and X is a (covering) stack and Y is a sheaf then Y is a (covering) stack.*

Proof. Choose a geometric (c)atlas of X. Then the composition with the map $X \to Y$ is a cover by 3.5. As the domain is (covering) affine, its a geometric (c)atlas. □

Now we want to show that the clash of terminology regarding 'covering' is reasonable:

Proposition 3.8. *Let $\mathbb{T}$ be saturated. A covering stack X is affine iff its a covering affine.*

Proof. The converse is clear. The direct direction follows by the recursion principle. choosing a geometric catlas $S \to X$. As both S and X are affine the fibers are affine. By induction the fibers are covering affines. By saturatedness of the topology X is covering affine. □

Lemma 3.9. *Let $\mathbb{T}$ be saturated. Let X be a covering stack. Let $f : \operatorname{Spec} A \to X$ be a geometric atlas. Then* $\operatorname{Spec} A \in \mathbb{T}$

Proof. As $\operatorname{Spec} A \simeq \sum_{x:X} \operatorname{fib}_f x$ is a dependent sum of covering stacks, it is a covering stack again. We conclude by 3.8. □

3.1 Geometric stacks

Lemma 3.10. *geometric stacks are closed under* id*-types.*

Proof. This is 1.1, using that covering stacks are closed under $\sum$. □

warning. The previous lemma does not hold for covering stacks: Identity types of things in $\mathbb{T}$ could be empty.

Proposition 3.11 (Recursion principle for geometric stacks)**.** *Let $P : \mathsf{GS} \to \mathrm{Prop}$ be a property of geometric stacks. Assume*

- *Every affine has P*
- *If $S \to Y$ is fibered in covering stacks that have P then Y has P*

Then every geometric stack has P.

Proof. One could explain geometric stacks as the smallest class containing all affines and if $\operatorname{Spec} A \to X$ is fibered in geometric stacks that happens to be covering, then X is a geometric stack. □

3.2 About the smallest class in a subuniverse

Definition 3.12. Let $\mathcal{V} \supset \mathsf{Aff}$ be a superclass stable under $\sum$. covering geometric $\mathcal{V}$ stacks are the smallest intermediate class $\mathbb{T} \subset \mathsf{CS}_{\mathcal{V}} \subset \mathcal{V}$ such that: If $X : \mathbb{T}$ $Y : \mathcal{V}$ and $X \to Y$ is fibered in $\mathsf{CS}_{\mathcal{V}}$, then $Y \in \mathsf{CS}_{\mathcal{V}}$

Definition 3.13. We define the saturation of $\mathbb{T}$ as the class of covering Aff-stacks. We call a topology $\mathbb{T}$ saturated if it coincides with its saturation, or more concretely: Every affine schemes that has a catlas lies itself in $\mathbb{T}$.

In a further chapter we will develop this theory further.

Proposition 3.14. *Let $\mathcal{V}$ be stable under finite limits and containing (covering) affines. X is a (covering) $\mathcal{V}$-stack iff it is in $\mathcal{V}$ and a (covering) geometric stack.*

Proof. The direct direction is clear. For the converse we apply the recursion principle to the property '$X \in \mathcal{V}$ implies X is a (covering) $\mathcal{V}$-stack'. If $X \in \mathbb{T}$, its clear. Otherwise its equipped with a geometric (c)atlas. The fibers are in $\mathcal{V}$, as they can be written as a fiberproduct of $S, X, 1 \in \mathcal{V}$. By induction all fibers are covering $\mathcal{V}$-stacks. □

Proposition 3.15. *covering $\mathcal{V}$-stacks are stable under dependent sums. In particular the saturation of a topology defines a topology.*

Proof. Both the classes $\mathcal{V}$ and covering stacks are stable under dependent sums. Hence the intersection of them is $\sum$-stable as well.
The saturation is a class of affines, that in particular contains $1 \in \mathbb{T}$. We have argued its stable under $\sum$. □

Proposition 3.16. *A sheaf X merely admits some affine* $\operatorname{Spec} A \to X$ *fibered in covering $\mathcal{V}$-stacks, iff its a geometric stack whose identity types are geometric $\mathcal{V}$-stacks.*

Proof. The direct direction: By 1.1 the identity types are $\mathcal{V}$-stacks.
The converse direction: Choose a geometric atlas $f : S \to X$. As each fiber $\sum_{s:S} fs =_X x$ is in V by $\sum$-stability of $\mathcal{V}$ and is a covering stack, its a covering $\mathcal{V}$-stack by 3.14. □

Definition 3.17. Let $n \geq -2$. A (covering) geometric n-stack is a (covering) geometric stack that is an n-type.

Proposition 3.18. *Let X be a sheaf. For all $n \geq 0$, the following are equivalent:*

1. *X is a (covering) geometric $n+1$-stack*
2. *X merely admits some map $S \to X$ out of a (covering) affine fibered in covering n-stacks*
3. *X merely admits some (covering) geometric n-stack $Y \to X$ fibered in covering n-stacks.*

Proof.

1 . ⇔ 2. X is a (covering) geometric $n+1$ stack iff (3.10) its a (covering) geometric stack whose identity types are geometric n-stack iff (3.16) 2.

2 . ⇒ 3. S is a (covering) geometric n-stack

3. ⇒ 2 Y admits a map $S \to Y$ fibered in covering n-stacks with S (covering) affine, so the composition $S \to X$ will have the same property by 3.5.

□

3.3 Truncatedness

In this subsection we want to prove that every geometric stack is a geometric n-stack for some n.

Lemma 3.19. *Every covering $\mathcal{V}$-stack X is $\mathbb{T}$-merely inhabited.*

Proof.

- If X is in $\mathbb{T}$ then its clear.
- If X is obtained by a quotient, we have a map $\operatorname{Spec} A \to X$ with domain in $\mathbb{T}$. Now use that we get a map on $\mathbb{T}$-propositional-truncations and that Spec A is T-merely inhabited.

□

Lemma 3.20. *Let X be an $n+1$-type and Y a sheaf. If $X \to Y$ is a n-truncated $\mathbb{T}$-surjective map, then Y is an $n+1$-type.*

Proof. Use that $\mathsf{is-n-truncated}(y = y')$ is a sheaf for $y, y' : Y$. □

Theorem 3.21. *Every geometric stack is n-truncated for some $n : \mathbb{N}$.*

Proof. We apply the recursion principle for geometric stacks.

- If Y is affine its clear with $n = 0$.
- Assume Y is equipped with a $\mathcal{V}$-atlas $f : S \to Y$, such that every fiber in n-truncated for some n. f is $\mathbb{T}$-surjetive by 3.19. We apply 3.20. So it remains to find an n such that all fibers are n-truncated. For any $x : S$, By induction $\mathrm{fib}_f(fx)$ is n-truncated for some n. By projectivity of S, we find some n such that $\mathrm{fib}_f(fx)$ is n-truncated for all $x : S$. For general $y : Y$, using that is-n-truncated $\mathrm{fib}_f\, y$ is a sheaf , we can conclude by $\mathbb{T}$-surjectivity of f.

□

3.4 Descent

For this subsection lets assume $\mathcal{V}$ a subuniverse (stable under $\sum$), that satisfies:
If $Y \in \mathcal{V}$ is seperated, then $L_{\mathbb{T}}Y \in \mathcal{V}$. (*)
St a class of sheaves in $\mathcal{V}$, such that $\mathbb{T}$ is contained in it and for any $\mathbb{T}$-cover $X \to Y$ of sheaves in $\mathcal{V}$, $X \in \mathsf{St}$ iff $Y \in \mathsf{St}$. We call types in this class stacky.

Lemma 3.22. *Let $\mathbb{T}$ satisfy descent, i.e. beeing affine in the topology is a sheaf. If Y admits a $\mathbb{T}$-cover $f : X \to Y$ where $Y \in \mathcal{V}$ is seperated, then there is a $\mathbb{T}$-cover $X \to L_{\mathbb{T}}Y$.*

Proof. Consider $X \xrightarrow{f} Y \xrightarrow{\eta} L_{\mathbb{T}}Y$. As beeing affine in $\mathbb{T}$ is a sheaf, we may just show that for all $y : Y$, the fibers over $\eta y : L_{\mathbb{T}}Y$ are in $\mathbb{T}$. As η is a monomorphism by 2.5 , η restricts to an equivalence

$$\mathrm{fib}_f\, y \to \mathrm{fib}_{\eta f}(\eta y)$$

But the left hand side is in $\mathbb{T}$ by assumption. □

Lemma 3.23. *Assume $\mathbb{T}$ have descent. Let $X \in \mathsf{St}$ and $Y \in \mathcal{V}$. Let $f : X \twoheadrightarrow Y$ be fibered in $\mathbb{T}$ and surjective. Then $L_{\mathbb{T}}Y$ is stacky.*

Proof. As X is stacky, it suffices to show, that $L_{\mathbb{T}}Y$ admits a $\mathbb{T}$-cover. We want to apply 3.22. So it remains to show, that Y is seperated, because then we also know $L_{\mathbb{T}}Y \in \mathcal{V}$ by (*). By surjectivity of f we may only show that for any $x : X, y : Y$, the type $fx =_Y y$ is a sheaf. If we define U to be the fiber over y, it is in $\mathbb{T}$ by assumption. But then $fx =_Y y$ is the outer pullback

$$\begin{tikzcd}
fx = y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & U \in \mathbb{T} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, "y"'] \\
1 \arrow[r, "x"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

of stacky types, in particular sheaves. □(Claim)

□

Theorem 3.24. *Assume $\mathbb{T}$ have descent. Then St is a sheaf.*

Proof. St is seperated: This follows from the embedding St into the seperated (TODO) type of sheaves.
Let $U \in \mathbb{T}$ and $P : \|U\| \to \mathsf{St}$. We want to construct a filler

$$\begin{tikzcd}
\|U\| \arrow[r, "P"] \arrow[d] & \mathsf{St} \\
1 \arrow[ur, dashed] &
\end{tikzcd}$$

Claim: $L_{\mathbb{T}}(\sum_{x:\|U\|} Px)$ is stacky.

Proof. of the claim. We want to apply the previous lemma to the map

$$\sum_{x:U} P|x| \to \sum_{x:\|U\|} Px$$

The domain is in St by stability under $\sum$. The fibers are equivalent to $U \in \mathbb{T} \subset \mathsf{St}$. □

The claim provides the map $1 \to \mathsf{St}$. The diagram commutes: Assuming $x : \|\operatorname{Spec} A\|$ we wish to show $Px = \sum_{x:\|U\|} Px$. Using univalence, we may show that the maps

$$Px \to \sum_{x:\|U\|} Px \xrightarrow{\eta} L_{\mathbb{T}} \sum_{x:\|U\|} Px$$

are both equivalences. The first one is an equivalence as $\|U\|$ is contractible. Hence the middle term is a sheaf, thus the unit map is an equivalence as well.

$\square$

Corollary. *If $\mathbb{T}$ has descent, (covering) geometric stacks satisfy descent.*

Corollary. *If $\mathbb{T}$ has descent. For all $n : \mathbb{N}$, the class of (covering) (n-)stacks has descent.*

Proof. We set $\mathcal{V}$ as the n-truncated-type. We have to check the condition (*): If Y is a seperated n type , then $L_{\mathbb{T}}Y$ is an n-type. As a sheaf beeing n-truncated is a sheaf, we may just show that $\eta x = \eta y$ is $n-1$-truncated for all $x, y : Y$. hence, Apply 2.5 to the seperated Y, we know $\eta x =_{LX} \eta y \simeq (x = y)$ beeing an $n-1$-type.

$\square$

4 Saturated Topologies

Definition 4.1. A catlas of X is some $\hat{X} \in \mathbb{T}, \hat{X} \to X$ $\mathbb{T}$-cover

Proposition 4.2. *The saturation of a topology $\mathbb{T}$ is the class $\mathbb{T}'$ defined by*

$$X \in \mathbb{T}' \text{ iff } X \text{ is affine} \wedge \exists \text{ catlas of } X$$

Proof. As $\mathbb{T}'$ is definitely contained in the saturation, it suffices to show, that the class $\mathbb{T}'$ defined above is saturated. $\mathbb{T}'$ is $\sum$-stable by 6.3.
Consider some $\mathbb{T}'$-cover $\mathbb{T}' \ni X' \to X$. By replacing X' with some catlas (allowed as $\mathbb{T}'$-covers compose), we may assume that $X' \in \mathbb{T}$. As every fiber $X'_x \in \mathbb{T}'$ and X has choice, we can choose for all $x : X$ a catlas $\tilde{X}'_x \to X'_x$. We obtain commutative diagram

$$\begin{tikzcd} \tilde{X} \equiv \sum_{x:X} \tilde{X}'_x \arrow[r] \arrow[dr] & \sum_{x:X} X'_x \arrow[d] \\ & X \end{tikzcd}$$

As $X' \in \mathbb{T}$ and $\tilde{X} \to X'$ is fibered in $\mathbb{T}$ (1.2) we have $\tilde{X} \in \mathbb{T}$. And $X' \to X$ is a $\mathbb{T}$-cover hence $Y \to X$ is a $\mathbb{T}$-cover. Hence $X \in \mathbb{T}'$. □

Lemma 4.3. *A type T is a sheaf wrt to $\mathbb{T}'$ iff it is a sheaf wrt to $\mathbb{T}$*

Proof. As $\mathbb{T} \subset \mathbb{T}'$ the $\to$ direction is clear. Now, let $X \in \mathbb{T}'$. We have to show that $T \to T^{\|X\|}$ is an equivalence. Choose $\mathbb{T} \ni Y \to X$. Then we have a commutative diagram

$$\begin{tikzcd} T \arrow[r] \arrow[dr, "\simeq"'] & T^{\|X\|} \arrow[d] \\ & T^{\|Y\|} \end{tikzcd}$$

So $T \to T^{\|X\|}$ has a left-inverse. Thus it suffices to show that any $f : T^{\|X\|}$ has a preimage. Choose $t : T$, s.th. cnst_t^Y is the composite $\|Y\| \to \|X\| \xrightarrow{f} T$. We have $\|Y\| \to (\mathrm{cnst}_t^X = f)$. But as $Y \in \mathbb{T}$ and $\Delta_t = f$ is a sheaf (as an identitytype in the sheaf $T^{\|X\|}$) we are done. □

Remark 1. We never used that we only talk about $\mathbb{T}$-covers.

Lemma 4.4. *Every saturated affine (i.e.* $\operatorname{Spec} A \in \mathbb{T}'$*) it $\mathbb{T}$-merely inhabited.*

Proof. We have $\|X\| \to \|\operatorname{Spec} A\|$ for some catlas $\mathbb{T} \ni X \to \operatorname{Spec} A$. □

Question 1. Does the converse hold, i.e. is every $\mathbb{T}$-merely inhabited affine saturated?

5 Atlas

Definition 5.1. Given $\mathcal{V} \subset \mathcal{U}$ a subclass stable under $\sum$, a $\mathcal{V}$-cover is a map fibered in $\mathcal{V}$. A $\mathcal{V}$-atlas of X is a $\mathbb{T}$-cover $\operatorname{Spec} A \to X$ out of an affine scheme.
In the context of a topology $\mathbb{T}$, We call a $\mathcal{V}$-atlas $\operatorname{Spec} A \to X$ a $\mathcal{V}$-catlas, if the domain $\operatorname{Spec} A$ belongs to $\mathbb{T}$.

Example 5.2. *Let X be a (1-)type. X has a* Zar*-atlas, iff there exists some $f : \operatorname{Spec} A \to X$ fibered in types of the form $\operatorname{Spec}(R_{f_1} \times \ldots \times R_{f_n})$ for $(f_1, \ldots, f_n) \in Um(R)$.*

Remark 2. If one applies ZLC to an affine scheme $\operatorname{Spec} A$ the resulting principal open cover $D(f_i), f_i \in A$ will induce indeed a zariski atlas $\bigsqcup D(f_i) \to \operatorname{Spec} A$, because the fiber over $x : \operatorname{Spec} A$ is $\bigsqcup D(f_i(x))$.

Question: Does every zariski atlas of $\operatorname{Spec} A$ have this form? **??**

Example 5.3. *$\mathbb{P}^n$ has a zariski atlas given by the standart homogeneous principal opens $\sum_{i=0}^n D_+(x_i)$. The fiber over a point $[y_0 : \ldots . y_n]$ is $D(y_0) + \ldots D(y_n)$ where $(y_1, \ldots, y_n) \in Um(R)$.*

6 Local Choice

One of the goals of this chapter is to show descent for types admitting a $\mathbb{T}$-(c)atlas. In this section let $\mathbb{T}$ denote a topology finer than the zariski topology.

Definition 6.1. Let *Cov* be a class of morphisms (which we think of n-atlasses of some n), containing $\mathbb{T}$-atlas, (stable under pullback NECESSARY TODO?) A type S has *local choice* wrt *Cov* if for any $\mathbb{T}$ -surjective map $X \to Y$ and any map $f : S \to Y$ there exists a map $p' : S' \to S$ in *Cov* and a commutative diagram

$$\begin{tikzcd} S' \arrow[r, dashed] \arrow[d] & X \arrow[d, "p"] \\ S \arrow[r, "f"] & Y \end{tikzcd}$$

Proposition 6.2. *Assume that* Cov *is stable under composition.*

- *If $\hat{S} \to S$ is a* Cov*er and $\hat{S}$ has $\mathbb{T}$-local choice, then S has $\mathbb{T}$-local choice.*
- *Affine schemes have $\mathbb{T}$-local choice.*
- *Any type admitting a* Cov *- Atlas* $\operatorname{Spec} A \to S$ *has $\mathbb{T}$-local choice.*

Proof. The first point follows from stability under composition of *Cov* . the third point follows from the second. By the first point, we may assume that S is affine. As p is $\mathbb{T}$-surjective, for any $x : S$ there merely is a $\operatorname{Spec} B_x \in T$ and a map $\operatorname{Spec} B_x \to \|\operatorname{fib}_p(x)\|$. As S is projective, we have a term in

$$\prod_{x:S} \sum_{\operatorname{Spec} B_x \in T} \operatorname{Spec} B_x \to \|\operatorname{fib}_p(fx)\|$$

By setting

$$(S' := \sum_{x:S} \operatorname{Spec} B_x) \xrightarrow{\pi} S$$

the projection, we are now in the situation that for any $t : S'$ we merely have a point in $\operatorname{fib}_p((p'(t)))$ and $S' \to S$ is a $\mathbb{T}$-cover, thus it is in *Cov* . Moreover, S' is affine, as it is a dependent sum of affines. Hence again we now can find a lift $S' \to X$ making

$$\begin{tikzcd} S' \arrow[r] \arrow[d, "p'"'] & Y \arrow[d, "p"] \\ S \arrow[r, "f"] & X \end{tikzcd}$$

commute. ∎

The next lemma shows, that the class of types equipped with a $\mathbb{T}$-atlas is stable under dependent sums.

Theorem 6.3. *Let $\mathcal{U}'$ be a class stable under dependent sums. The class of types admitting a $\mathcal{U}'$-atlas is closed under dependent sums. If $\mathbb{T}$ is a topology, the same holds for $\mathcal{U}'$-catlasses.*

Proof. The stability under quotients is easy: Let us construct some atlas $\operatorname{Spec} A \to \sum_{x:X} B_x$ For any $x : X$ we merely have an atlas $V_x \to B_x$, i.e. with V_x affine. X has local choice wrt atlasses by (6.2) using $\mathcal{U}'$ is $\sum$-stable (we use the trivial topology).
If additionally, all the B_x and X are smooth n-stacks, just observe that we can choose the affine V_{pu} to lie in $\mathbb{T}$, Accordingly $\sum_{u:U} V_{pu} \in T$ as $\mathbb{T}$ is stable under Σ.
By Local choice for X, we merely find U affine, an atlas $p : U \to X$ with

$$\prod_{u:U} \sum_{V_{p(u)} \in T} (q : V_{p(u)} \to B_{p(u)}) \times (q \text{ fibered in smooth } n \text{ stacks })$$

Now the desired map is $\sum_{u:U} V_{pu} \to \sum_{x:X} B_x$, because it is an atlas by 1.2

□

Proposition 6.4. *Let $\mathcal{U}'$ be a class stable under dependent sums. The class of types admitting a $\mathcal{U}'$-(c)atlas is closed under $\mathcal{U}'$-covers: If $X \to Y$ is a $\mathcal{U}'$-cover, then X admits a $\mathcal{U}'$-(c)atlas iff Y admits a $\mathcal{U}'$-(c)atlas.*

Proof. One direction is the stability under dependent sums. For the other, if $S \to X$ is a $\mathcal{U}'$-atlas, then $S \to X \to Y$ is a $\mathcal{U}'$-atlas by $\sum$-stability of $\mathcal{U}'$. □

Corollary. *If $\mathbb{T}$ has descent, The class of sheaves merely admitting a $\mathbb{T}$-catlas has descent.*

Proof. We can set $\mathcal{V} = \mathcal{U}$, and we have to show, that if $X \to Y$ is a $\mathbb{T}$-cover than X admits a $\mathbb{T}$-catlas iff Y admits a $\mathbb{T}$-catlas. This follows from 6.4. □

7 Fundamental Theorem of algebraic spaces

Lemma 7.1. *Denote $\mathbb{T}Set$ for the sets that are $\mathbb{T}$-sheaves. Assume given a $\mathbb{T}set$ X then the following maps are mutually inverse*

$$\sum_{R:X\to X\to\mathbb{T}\,\mathrm{Prop}} R \text{ equivalence relation} \simeq \sum_{Y:\mathbb{T}\mathsf{Set}} \sum_{p:X\to Y} p \ \mathbb{T}\text{surjective}$$

$$R \mapsto (L_{\mathbb{T}}\|X//R\|_0, [\_])$$

$$\lambda x, y.(p(x) = p(y)) \leftarrow\!\shortmid (Y, p)$$

Proof.

- Well-definedness: The map $[\_] : X \to \|X//R\|_0 \to L_T\|X//R\|_0$ is the composition of a surjective with a $\mathbb{T}$-surjective map [ref?], hence its $\mathbb{T}$-surjective.
Conversely given (Y, p) as Y is a sheaf, we have for all $x, y : X$ that $p(x) =_Y p(y)$ is a sheaf.

- If $x, y : X$ then we have a chain of equivalences

$$R(x, y) \simeq (\bar{x} =_{\|X//R\|_0} \bar{y}) \overset{\mathsf{ap}_\eta}{\to} ([x] =_{L_T\|X//R\|_0} [y])$$

where the first map is plain HoTT, meaning that $\|X//R\|_0$ is seperated. The second map is equivalence by 2.5.

- Let (Y, p) be in the RHS. Let $R(x, y) = (p(x) = p(y)) : \mathbb{T}\,\mathrm{Prop}$. By plain HoTT, There is a map $\eta : X//R \to Y$ (defined by the universal property of the set truncation and by induction on the higher inductive type $X//R$ on canonical terms through the map $p : X \to Y$). I claim η exhibits Y as the localization for $\mathbb{T}\,\mathsf{Set}$-modality of $X//R$. Let T be another $\mathbb{T}\,\mathsf{Set}$ equipped with a map $X//R \to T$. By precomposition we obtain a map $X \to T$. Claim: it factors uniquely through $p : X \to Y$.

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X//R \arrow[r] & T \\ & Y \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Proof:
Existence: We want to define a map $Y \to T$. Let $y : Y$. As p is $\mathbb{T}$-surjective and T is a sheaf, we may assume we merely have some element in the fiber of p over y. Now push this element through

$$\|\operatorname{fib}_p y\| \to \|X//R\|_0 \to T$$

where the first map is by Plain HoTT and the second one is induced from $X//R \to T$ by assumption and the fact that T is a set.. One can easily check this makes the diagram commute. Uniqueness follows from $X \to Y$ beeing $\mathbb{T}$-surjective and the following Fact: Two parellel maps $Y \rightrightarrows T$ into a $\mathbb{T}\,\mathsf{Set}$ T are already equal if the become equal after precomposition with a $\mathbb{T}$-surjection $X \to Y$.
Proof of the fact : Let $y : Y$. The goal is an identity type of a $\mathbb{T}\,\mathsf{Set}$, hence a $\mathbb{T}\,\mathrm{Prop}$. Hence As the fiber over y in X is $\mathbb{T}$-merely inhabited, we may assume an actual term in the fiber. As $X \to Y$ equalizes the arrows, this term allows us to conclude. □(fact) □(Claim)
We apply the fact to the ($\mathbb{T}$-)surjectivity of $X \to X//R$ to get a unique factorization

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[dr] & X//R \arrow[r] \arrow[d] & T \\ & Y \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

making the right triangle commute. This is what we wanted to show.

□

Definition 7.2. A modal equivalence relation R on a type X is called covering, if for any $y : X$ the fibers

$$R_y :\equiv \sum_{x:X} R(x, y)$$

merely admits a $\mathbb{T}$-catlas.

Lemma 7.3. *Assume that the topology has descent. Given a $\mathbb{T}$set X, the following types are equivalent:*

- *The type of covering equivalence relations on X.*
- *The type of $\mathbb{T}$sets Y equipped with a map $X \to Y$ fibered in types admitting a $\mathbb{T}$-catlas.*

Proof. By the equivalence in 7.1 it is enough to check that The fibers of:

$$[\_] : X \to L_{\mathbb{T}}\|X//R\|_0$$

merely admit a $\mathbb{T}$-catlas if and only if the relation R is covering. For any $y : X$ we have that:

$$\sum_{x:X} R(x, y) \simeq \mathrm{fib}_{[\_]}([y])$$

so the direct direction is immediate. The converse follows from $\mathbb{T}$-surjectivity of $[\_]$ and from 6. $\square$

8 Algebraic Space

Recall the notion of (covering) 0-stacks. it is the smallest pair of classes that satisfies the following

- (covering) affines are (covering) algebraic spaces.
- stable under covering quotients: If X is an algebraic space, Y modal 0-type and $X \to Y$ is fibered in covering algebraic spaces, then Y is an algebraic space. Additionally, if X is covering, then Y is covering.

8.1 Geometric propositions

Definition 8.1. An affine Scheme U is called geometric, if

$$\|U\|_{\mathbb{T}} \to (U \in \mathbb{T})$$

Lemma 8.2. *The converse holds always*

Proof. because things in $\mathbb{T}$ are automatically $\mathbb{T}$-merely inhabited □

Recall the definition of $\mathbb{T}$-atlas 5.1

Definition 8.3. We call a modal proposition geometric, if one of the equivalent conditions is satisfied:

1. its merely of the form $\|U\|_{\mathbb{T}}$ for some geometric affine U.
2. There is a $\mathbb{T}$-surjective map out of a geometric affine U.
3. It has a $\mathbb{T}$-atlas.

Proof.

$1 \Leftrightarrow 2$ Clear.

$1 \Rightarrow 3$ we show that $U \to \|U\|_{\mathbb{T}}$ is a $\mathbb{T}$-atlas. Every fiber is in $\mathbb{T}$, because U is geometric.

$3 \Rightarrow 1$ Let $V \to P$ be a $\mathbb{T}$-atlas. have to show TFAE $\|V\|_{\mathbb{T}} \to P \to (V \in \mathbb{T}) \overset{8.2}{\to} \|V\|_{\mathbb{T}}$. Proof: $\|V\|_{\mathbb{T}} \to P$ as P is modal prop. Secondly, because $V \to P$ is a $\mathbb{T}$-cover.
Hence P is a geometric proposition.

□

8.2 Algebraic spaces

Lemma 8.4. *Consider a modal equivalence relation $R : S^2 \to \mathbb{T}\,\mathrm{Prop}$ on an affine S. TFAE*

- *R is covering, i.e. every fiber $R_s \equiv \sum_{t:S} Rst$ admits a $\mathbb{T}$-catlas.*
- *every fiber $R_s \equiv \sum_{t:S} Rst$ is a covering 0-stack.*

Proof. Every type admitting a $\mathbb{T}$-catlas is a covering 0-stack. Conversely: if the fibers are covering 0-stacks. Let us first observe, that for all $s, t : S$, Rst is a geometric proposition: Rst is the fiber of the projection $\sum_{t:S} Rst \to S$ between geometric stacks, which are stable under finite limits.

For all $t : S$ we can choose a geometric atlas $\mathrm{Spec}\, A_t \to Rst$ by 8.3. Then

$$\sum_{t:S} \mathrm{Spec}\, A_t \to \sum_{t:S} Rst$$

is a $\mathbb{T}$-atlas. As $\sum_{t:S} Rst$ is a covering 0-stack by assumption, the map has to be a $\mathbb{T}$-catlas by 3.9. □

Theorem 8.5. *Let X be a modal set. The following are equivalent:*

1. *X is a (covering) geometric 0-stack*
2. *X is merely of the form $L_{\mathbb{T}}(U/R)$ for some (covering) affine U and $R : U^2 \to \mathrm{Prop}$ a covering equivalence relation.*
3. *there exists some map $S \to X$ with S (covering) affine whose fibers merely have $\mathbb{T}$-catlasses.*

We call this class (covering) algebraic spaces.

Proof.

$2 \leftrightarrow 3$ 7.3

$2 \to 1$ Choose a presentation $R : U^2 \to \mathrm{Prop}$. It suffices to show, that the map $f : U \to L_{\mathbb{T}}(U/R)$ is a geometric (c)atlas. The map f is $\mathbb{T}$-surjective by the well-definedness of the bijection 7.1. By descent we may just show, that the fibers $\mathrm{fib}_f(f(s))$ for $s : U$ are covering 0-stacks. But by the bijection in 7.1 those are equivalent to the fibers R_s, which are covering 0-stacks as the equivalence relation is covering.

$1 \to 2$ This can be reformulated in the following way, using the recursion principle for (covering) geometric 0-stacks: Let X be a sheaf of sets. Let S be (covering-) affine and $f : S \to X$ be fibered in covering algebraic spaces. Then X is a (covering) algebraic space. This follows from the observation, that the equivalence relation determined by f is covering 8.4 , because the fibers of f are covering 0-stacks.

□

8.3 Schemes are algebraic Spaces for the Zariski Topology

Definition 8.6. A proposition U is open iff its merely of the form $f_1\ inv \vee \ldots f_n inv$ for some $f_i : R$.

Definition 8.7. A Zariski sheaf X is a scheme if there merely exists some affine S map $S \to X$ whose fibers are Zariski-merely inhabited finite sums of open propositions

Lemma 8.8. *Given $f_1, \ldots, f_n : R$ such that $\|D(f_1)+\ldots+D(f_n)\|_{\mathsf{Zar}}$ then $\sum_{i=1}^n D(f_i) \in \mathsf{Zar}$.*

Proof. We have to show that $(f_1, \ldots, f_n) = 1$. Claim: $(f_1, \ldots, f_n) = 1$ is a sheaf. Proof: Recall that the Zariski topology is subcanonical ([ref?]). Now the type is equivalent to the statement that the map between the two sheaves $\mathrm{Spec}\, 0 \to \mathrm{Spec}\, R/(f_1, \ldots, f_n)$ is an equivalence. This is a sheaf [ref?]. □

Proposition 8.9. *Every Zariski-merely-inhabited type that is merely of the form $U_1 + \ldots + U_n$ for open propositions U_i admits a Zar-catlas.*

Proof. By definition of openness, We can choose a surjection $\coprod_{j=1}^{n_i} D(f_{ij}) \twoheadrightarrow U_i$ for any i. We want to show, that the map

$$\coprod_{i,j} D(f_{ij}) \twoheadrightarrow U_1 + \ldots U_n$$

is a Zar-catlas.

- Let us first show that the fibers are in Zar. Assume U_i holds. So we find a term in $\coprod_j D(f_{ij})$. In particular we have $\| \coprod_j D(f_{ij})\|_{\mathsf{Zar}}$. By the lemma we conclude, that the fiber $\sum_j D(f_{ij})$ belongs to Zar.

- The total space is in Zar: This follows as the surjection after $\mathbb{T}$-truncation becomes an equivalence. As we have $\|U_1 + \ldots + U_n\|_{\mathbb{T}}$, we can conclude by the lemma.

□

warning. The converse does not hold! Apply 3.22 to the map

$$\mathsf{Zar} \ni 1 + 1 \to \sum D(f)$$

$\sum D(f)$ is seperated as $D(f)$ is a sheaf. All the fibers are equivalent to $1 + X$, hence they are in the Zariski topology. Use that beeing in the Zariski topology has Zariski-descent.

Corollary. *Every scheme is an algebraic space for the Zariski topology.*

Question 2. Is every algebraic space for the zariski topology a scheme?

Lemma 8.10. *Every* Zar*-sheaf that admits a* Zar*-atlas is a scheme. Hence, every geometric proposition is a scheme*

Proof. Obvious. □

9 Local properties

Lemma 9.1. *Given a commutative triangle*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

with $X \to Y$ a geometric cover. Then h is a geometric cover iff g is a geometric cover.

Proof. Reduce to the case of $Z = 1$. If $X \to Y$ is a geometric cover, then X is a covering stack iff Y is a covering stack by stability under quotients and under sums. If both are coverings stacks, then the fibers ☐

Lemma 9.2. *A morphism between geometric stacks $f : X \to Y$ is a geometric cover iff there exist atlasses and a $\mathbb{T}$-cover on affines*

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, dashed, "\hat{f}"] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Proof. The converse follows by the previous lemma. The direct direction follows by choosing a geometric atlas $\operatorname{Spec} B \to Y$ and taking the pullback

$$\begin{tikzcd} X \times_Y \operatorname{Spec} A \arrow[r, "f'"] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

f' has the same fibers as f, hence it will be geometric cover. Now $X \times_Y \operatorname{Spec} A$ is a geometric stack, hence we can choose a geometric atlas $\operatorname{Spec} B \to X \times_Y \operatorname{Spec} A$. The composition will be a geometric cover between affines, hence a $\mathbb{T}$-cover. ☐