Geometric stacks in Synthetic Algebraic Geometry

Tim Lichtnau, Hugo Moeneclay

Oktober 2024

Abstract

This is meant as a short summary of the progress of the Master Thesis of Tim Lichtnau so far. We work in Synthetic Algebraic Geometry, i.e. Homotopy Type Theory + 3 Axioms. A type gets interpreted as a Zariski-sheaf on the site given by the opposite category of finitely presented algebras over a fixed ring [CCH23]. Affine types get interpreted as the representable sheaves.

Definition 0.1. A Grothendieck topology $\mathbb{T}$ is a subclass of affine schemes, such that

- $1 \in \mathbb{T}$
- $\mathbb{T}$ is $\sum$-stable, i.e. if $X : \mathbb{T}, B : X \to \mathbb{T}$, then $\sum_{x:X} Bx$ belongs to $\mathbb{T}$.

A map is a $\mathbb{T}$-cover iff its fibered in $\mathbb{T}$. Relating to the classical definition, we could call a family of maps of affines $\{U_i \to U\}_{i=1}^n$ covering iff

$$\sum_{i=1}^{n} U_i \to U$$

is a $\mathbb{T}$-cover.
We fix a topology $\mathbb{T}$, for which we want to define the notion of (geometric) stack.

Definition 0.2. A type X is a (higher) stack iff its $\|\operatorname{Spec} A\|$-local for any $\operatorname{Spec} A \in \mathbb{T}$. An n-stack is a stack that is an n-type.

This exactly captures the expected classical notion of a stack for a Grothendieck topology [Moe24b], e.g. a 0-type X is a stack, if for any $\mathbb{T}$-cover $A \to B$,

$$X^B \to X^A \rightrightarrows X^{A \times_B A}$$

is an equalizer diagram.

Example 0.3 ([Moe24a]).

- *The fppf topology is given by the faithfully flat affine schemes.*
- *The étale topology is given by formaly-étale + faithfully flat affine schemes.*
- *The smooth topology is given by smooth + faithfully flat affine schemes.*

We start by defining the relative setting (this corresponds to fibers of smooth morphism of geometric stacks in [Sim96]). There is a short inductive definition:

Definition 0.4. A stack X is covering, whenever inductively

- $X \in \mathbb{T}$ or
- X is equipped with a map $\mathbb{T} \ni \operatorname{Spec} A \to X$ fibered in covering stacks.

We call a map $X \to Y$ fibered in covering stacks a geometric cover (That corresponds to smooth morphisms in [Sim96])

Definition 0.5. A stack X is geometric iff it merely admits a geometric cover $\operatorname{Spec} A \to X$.

We have the following classical labels associated to geometric stacks depending on the topologies:

$\mathbb{T}$	Geometric stacks for $\mathbb{T}$
étale	(Higher) Deligne Mumford Stacks
smooth	(Higher) Artin stacks
fppf	Something similar to Artin stacks ?

Example 0.6. *Every stack that is a scheme is geometric.*

Theorem 0.7 (Stability Results)**.**

- *The class of covering / geometric stacks is $\sum$-stable.*
- *The class of covering / geometric stacks is closed under quotients: If $X \to Y$ is a geometric cover with X covering / geometric stack, then Y is covering / geometric*
- *Geometric stacks are closed under taking identity types.*
- *Every geometric stack is a geometric n-stack for some n*
- *Covering / Geometric stacks have descent: Both types* GeometricStack *and* CoveringStack *are a stack.*

It is maybe worth mentioning, that proving descent was surprisingly easy.

Under some very mild condition on the topology [1] (e.g. satisfied by étale or fppf), which is equivalent to saying that every geometric cover between affines is a $\mathbb{T}$-cover we have the following explicit description depending on the truncation level n:

Theorem 0.8. *An n-stack X is geometric if and only if*

$(n = 0)$*: there merely exists a map* $\operatorname{Spec} A \to X$ *whose fibers F merely admit a $\mathbb{T}$-cover $\mathbb{T} \ni$* $\operatorname{Spec} B \to F$. [2]

$(n \geq 1)$*: there merely exist a map* $\operatorname{Spec} A \to X$ *whose fibers are covering $(n-1)$-stacks. Additionally X is covering iff we can choose* $\operatorname{Spec} A$ *to lie in $\mathbb{T}$.*

For the étale topology we have the following notable results

Theorem 0.9.

- *Every Deligne-Mumford stack is a 1-gerbe, i.e. $X \to \|X\|_1^{\mathbb{T}}$ is a geometric cover, where the latter means the $\mathbb{T}$-sheafification of the 1-truncation of X.*
- *A Deligne-Mumford stack X is covering iff $\pi_0^{\mathbb{T}} X := \|X\|_0^{\mathbb{T}}$ and all higher homotopy groups*
$$\pi_i^{\mathbb{T}}(X, x) = \|\Omega^i(X, x)\|_0^{\mathbb{T}}, i \geq 1$$
are covering algebraic spaces for the étale topology.

[1] We can always enforce this condition without changing the notion of (covering / geometric) stack

[2] One can reformulate also as taking a quotient of $\operatorname{Spec} A$ by an equivalence relation satisfying a certain property.

1 Examples

We can reproduce examples from Stacks project: Let $\mu_\ell = \operatorname{Spec} R[T]/(T^\ell - 1)$ denote the group of ℓ.th roots of unity.

Example 1.1 (Non-example)**.** *If $2 \neq 0$, the sheaf quotient of $\mathbb{A}^1$ by the μ_2 action is not an algebraic space.*

Example 1.2 (Not locally-separated examples)**.** *Assume $\ell \neq 0$ prime. Let μ_ℓ act on $\operatorname{Spec} B$ in one of the following ways:*

1. *Let μ_ℓ act on $\operatorname{Spec} B = \mathbb{A}^1$.*
2. *Put $\ell = 2$. Let μ_2 act on the cross*

$$\operatorname{Spec} B \equiv \sum_{x,y:R} xy = 0$$

   via the swap.

Then $\operatorname{Spec} B/R_{\mu_\ell}$ is an algebraic space that is not a scheme.

There is a general way to produce examples of algebraic spaces:

Lemma 1.3. *Quotients of geometric stacks X by groups that are covering stacks are geometric stacks, if the isotropy stacks are covering. This happens for example if X has $\mathbb{T}$-flat identity types or if the group action is free.*

Theorem 1.4. *Schemes do not have descent, i.e. even if Schemes are stacks, the type of Schemes is not a stack!*

Proof. Given $a : R$, e.g. $a = 1$, the idea is to consider the twisted line with double origin:

$$\sum_{x:R} \left(\operatorname{Spec} R[T]/(T^2 + a)\right)^{x=0},$$

which is etale-merely a scheme, namely the line with double origin. Then one needs to show, that if it would be a scheme then $T^2 + a$ has a root. $\square$

Theorem 1.5. *geometric covers for the etale topology are formally étale*

Proof. Surprisingly involved. Here is some detail missing ala formally étale implies flat $\square$

Proposition 1.6. *For the etale or the smooth topology, geometric stacks are stable under taking tangent spaces.*

2 A notion of flat for any topology

Definition 2.1. Denote Top the topologies containing Bool, e.g. finer than the Zariski-topology. Let FLAT consists of all the classes of affines $\mathbb{P}$ containing $1, \perp$ stable under $\sum$. Given $\mathbb{P} : \mathsf{FLAT}, \mathbb{T} : \mathsf{Top}$ we say $\mathbb{P}$ flattens $\mathbb{T}$ iff ($\mathbb{T} \subset \mathbb{P}$ and)

$$\mathbb{T} = \{X : \mathbb{P} \mid \|X\|_{\mathbb{T}}\}$$

Theorem 2.2.

1. *There is at most one $\mathbb{P}$ that flattens a topology. Then we say, the topology is flatten.*
2. *A topology can be idempotently flattened without changing the stacks*

3. *For any* $\mathbb{P}$: FLAT *and any Lavwere Tierney Operator* j, $\{X : \mathbb{P} \mid \|X\|_j\}$ *is flattened by* $\mathbb{P}$.

Topology $\mathbb{T}$	$\mathbb{T}$-flat
fppf	flat affines
étale	formally étale + flat affines
Zariski	finite sum of principal opens

The notion of $\mathbb{T}$-flat turns out to be incredibly useful in the case of the étale -topology (because there $\mathbb{T}$-flat affines are stable under identity types), for example for showing that Deligne-Mumford stacks are 1-gerbes or that geometric covers are formally étale .
Question: When is an Artin stack Deligne Mumford?

References

[CCH23] Felix Cherubini, Thierry Coquand, and Matthias Hutzler. *A Foundation for Synthetic Algebraic Geometry*. 2023. arXiv: 2307.00073 [math.AG]. URL: https://arxiv.org/abs/2307.00073.

[Moe24a] Hugo Moeneclay. *Synthetic Differential Geometry*. HoTTUF 2024. 2024. URL: https://hott-uf.github.io/2024/abstracts/HoTTUF_2024_paper_11.pdf.

[Moe24b] Moeneclay, Hugo. "Sheaves". In: (2024). URL: www.felix-cherubini.de/sheaves.pdf.

[Sim96] Carlos Simpson. *Algebraic (geometric) n-stacks*. 1996. arXiv: alg-geom/9609014 [alg-geom]. URL: https://arxiv.org/abs/alg-geom/9609014.