Thesis

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1 Saturated Topologies

Consider a topology $\mathbb{T}$ finer than the Zariski topology.

Definition 1.1. A smooth atlas of X is some $\hat{X} \in \mathbb{T}, \hat{X} \to X$ $\mathbb{T}$-cover

Definition 1.2. $\mathbb{T}$ is saturated if Beeing in the topology descents along $\mathbb{T}$-covers between affines, i.e. every affine schemes that has a smooth atlas lies itself in $\mathbb{T}$.
The saturated closure of a topology $\mathbb{T}$ is the topology $\mathbb{T}'$ defined by (todo finite sums of?)

$$X \in \mathbb{T}' \text{ iff } X \text{ is affine} \wedge \exists \text{ smooth atlas of } X$$

Lemma 1.3. *Using ZLC, this is the smallest saturated topology containing* $\mathbb{T}$*.*

Proof. Obviously $1 \in \mathbb{T}'$. Types which have a smooth atlas are stable by dependent sums by the proof of **??**. For the saturatedness consider some $\mathbb{T}'$-cover $\mathbb{T}' \ni X' \to X$. By replacing X' with some smooth atlas, we may assume that $X' \in \mathbb{T}$. As every fiber $X'_x \in \mathbb{T}'$, we merely find a smooth atlas $\tilde{X}'_x \to X'_x$. Then by Zariski local choice there exists a Zariski atlas $\hat{X} \to X$ and a commutative diagram

$$\begin{tikzcd}
Y \equiv \sum_{x:\hat{X}} \tilde{X}'_x \arrow[r] \arrow[d] & \sum_{x:X} X'_x = X' \arrow[d] \\
\hat{X} \arrow[r, "Zar"'] & X
\end{tikzcd}$$

As $X' \in \mathbb{T}$ and $Y \to X'$ is fibered in $\mathbb{T}$ (4.3) we have $Y \in \mathbb{T}$. But $Y \to \hat{X}$ is a $\mathbb{T}$-cover and $\hat{X} \to X$ is a $\mathbb{T}$-cover, $Y \to X$ is a $\mathbb{T}$-cover. Hence $X \in \mathbb{T}'$. $\square$

Lemma 1.4. *A type T is a sheaf wrt to $\mathbb{T}'$ iff it is a sheaf wrt to $\mathbb{T}$*

Proof. As $\mathbb{T} \subset \mathbb{T}'$ the $\to$ direction is clear. Now, let $X \in \mathbb{T}'$. We have to show that $T \to T^{\|X\|}$ is an equivalence. Choose $\mathbb{T} \ni Y \to X$. Then we have a commutative diagram

$$\begin{tikzcd}
T \arrow[r] \arrow[dr, "\simeq"'] & T^{\|X\|} \arrow[d] \\
 & T^{\|Y\|}
\end{tikzcd}$$

So $T \to T^{\|X\|}$ has a left-inverse. Thus it suffices to show that any $f : T^{\|X\|}$ has a preimage. Choose $t : T$, s.th. cnst_t^Y is the composite $\|Y\| \to \|X\| \xrightarrow{f} T$. We have $\|Y\| \to (\mathrm{cnst}_t^X = f)$. But as $Y \in \mathbb{T}$ and $\Delta_t = f$ is a sheaf (as an identitytype in the sheaf $T^{\|X\|}$) we are done. $\square$

Remark 1. We never used that we only talk about $\mathbb{T}$-covers.

Lemma 1.5. *Every saturated affine (i.e.* $\operatorname{Spec} A \in \mathbb{T}'$*) it* $\mathbb{T}$*-merely inhabited.*

Proof. We have $\|X\| \to \|\operatorname{Spec} A\|$ for some smooth atlas $\mathbb{T} \ni X \to \operatorname{Spec} A$. $\square$

Question 1. Does the converse hold, i.e. is every $\mathbb{T}$-merely inhabited affine saturated?

2 Lex Modalities

Lemma 2.1 (Stability resuls)**.** *Lex Modalities are stable under*

1. *Conjunction*

2. *Composition*

Lemma 2.2. *Let $\bigcirc$ be a lex-modality. Let X be $\bigcirc$-modal and $B : X \to \mathcal{U}_{\bigcirc}$ be a family of modal types. Then $\sum_{x:X} B_x$ is $\bigcirc$-modal*

Lemma 2.3. *Let $B : \bullet X \to \mathcal{U}$. Then $\bullet(\sum_{x:X} B(\eta x)) = \sum_{x:\bullet X} \bullet Bx$*

Proof. Observe that

$$\sum_{x:X} B\eta x \to \sum_{x:\bullet X} Bx$$

is a $\bullet$-equivalence, because for all modal types T, the type $Bx \to T$ is modal for any $x : \bullet X$. Then it follows by [ref?]. $\square$

3 Atlas

Definition 3.1. A $\mathbb{T}$-atlas of X is a $\mathbb{T}$-cover $\operatorname{Spec} A \to X$ out of an affine scheme.

Remark 2. Any good enough TODO scheme has a Zariski atlas. If $\mathbb{T}$ is finer than the Zariski-topology then in the definition we may replace affine scheme by good enough scheme, if its just about the question whether a type admits an atlas.

Example 3.2. *Let X be a (1-)type. X has a Zariski-atlas, iff there exists some $f : \operatorname{Spec} A \to X$ fibered in types of the form $\operatorname{Spec}(R_{f_1} \times \ldots \times R_{f_n})$ for $(f_1, \ldots, f_n) \in Um(R)$.*

Remark 3. If one applies ZLC to an affine scheme $\operatorname{Spec} A$ the resulting principal open cover $D(f_i), f_i \in A$ will induce indeed a zariski atlas $\bigsqcup D(f_i) \to \operatorname{Spec} A$, because the fiber over $x : \operatorname{Spec} A$ is $\bigsqcup D(f_i(x))$.

Question: Does every zariski atlas of $\operatorname{Spec} A$ have this form? Weird Zariski Atlasses

Example 3.3. *$\mathbb{P}^n$ has a zariski atlas given by the standart homogeneous principal opens $\sum_{i=0}^{n} D_+(x_i)$. The fiber over a point $[y_0 : \ldots . y_n]$ is $D(y_0) + \ldots D(y_n)$ where $(y_1, \ldots, y_n) \in Um(R)$.*

4 Local Choice

In this section let $\mathbb{T}$ denote a topology finer than the zariski topology.

Definition 4.1. Let *Cov* be a class of morphisms (which we think of n-atlasses of some n), containing $\mathbb{T}$-atlas, (stable under pullback NECESSARY TODO?) A type S has *local choice* wrt *Cov* if for any $\mathbb{T}$ -surjective map $X \to Y$ and any map $f : S \to Y$ there exists a map $p' : S' \to S$ in *Cov* and a commutative diagram

$$\begin{tikzcd} S' \arrow[r, dashed] \arrow[d] & X \arrow[d, "p"'] \\ S \arrow[r, "f"] & Y \end{tikzcd}$$

Proposition 4.2. *Assume that* Cov *is stable under composition and that Zariski-covers are in* Cov *.* S *has* $\mathbb{T}$*-local choice wrt* Cov *if it has a projective* Cov*er , i.e. there exists a projective (or, assuming ZLC, affine scheme resp.)* $\hat{S}$ *with a map* $g : \hat{S} \to S$ *in* Cov *.*

Proof. By stability under composition of *Cov* , We may assume that $g : \hat{S} \to S$ is the identity. As p is $\mathbb{T}$-surjective, for any $x : S$ there merely is a $\operatorname{Spec} B_x \in T$ and a map $\operatorname{Spec} B_x \to \| \operatorname{fib}_p(x)\|$. Claim: No matter on the assumptions (on $S = \hat{S}$), there exists a Zariski cover $S' \xrightarrow{p'} S$ with S' projective (affine resp.) and a term in

$$\prod_{x:S'} \sum_{\operatorname{Spec} B_x \in T} \operatorname{Spec} B_x \to \| \operatorname{fib}_p(fp'x)\|$$

Proof: In the case of projectivity, just use $p' = \mathrm{id}_S$ and in the case of having ZLC and S beeing affine, use ZLC (3). □(Claim)
By setting

$$(S'' := \sum_{x:S'} \operatorname{Spec} B_x) \xrightarrow{\pi} S'$$

the projection, we are now in the situation that for any $t : S''$ we merely have a point in $\operatorname{fib}_p((p''(t)))$ and $S'' \to S'$ is a $\mathbb{T}$-cover, thus it is in *Cov* . Moreover, S'' is a projective type (affine), as it is a dependent sum of projectives (affines). Hence again we now can find a lift $S'' \to X$. making

$$\begin{tikzcd} S'' \arrow[r] \arrow[d] & X \arrow[dd, "p"] \\ S' \arrow[d, "p'"] & \\ S \arrow[r, "\mathrm{id}"] & S \end{tikzcd}$$

commute. Now $S'' \to S' \to S$ as the composition of Zariski-covers and *Cov*er is a *Cov*er [...]as desired. □

The next lemma shows, that the class of types equipped with a $\mathbb{T}$-atlas is stable under dependent sums.

Lemma 4.3. *Let* $\mathcal{U}' \subset \mathcal{U}$ *be stable under dependent sums (e.g.* $\mathbb{T}$*-inhabited types) Let* X *be a type with a map* $p : U \to X$ *fibered in* $\mathcal{U}'$*. For any* $x : X$*, let* Y_x *be a type and moreover for any* $u : U$*, we are given a map* $q_u : V_u \to Y_{p(u)}$ *fibered in* $\mathcal{U}'$*. Then the induced map*

$$p : \sum_{u:U} V_u \to \sum_{x:X} Y_x$$

is fibered in $\mathcal{U}'$

Proof. The fiber of p over some $(x, y) \in \sum_{x:X} Y_x$ is given by

$$\sum_{u:\mathrm{fib}_p\, x} \mathrm{fib}_{q_u}(y')$$

where $y' : Y_{p(u)}$ (depending on u) is the transport of $y : Y_x$ along $x = p(u)$. As $\mathcal{U}'$ is stable under dependent sum those fibers are again in $\mathcal{U}'$. This shows the result. $\square$

5 Algebraic Space

We first need to define a notion of algebraic space and smooth algebraic space, which should be the smallest class of types that satisfies the following:

- Stability under finite limits 7.1
- has Descent
- (nice) Schemes are contained in it
- affines in $\mathbb{T}$ are smooth algebraic spaces. (there are probably more).
- stable under smooth quotients: If X is an algebraic space, Y modal 0-type and $X \to Y$ is $\mathbb{T}$-surjective and fibered in smooth algebraic spaces, then Y is an algebraic space. Additionally, if X is smooth, then Y is smooth.

Definition 5.1. An affine Scheme U is called flat, if

$$\|U\|_{\mathbb{T}} \to (U \in \mathbb{T})$$

Lemma 5.2. *The converse holds always*

Proof. because things in $\mathbb{T}$ are automatically $\mathbb{T}$-merely inhabited □

Recall the definition of $\mathbb{T}$-atlas 3.1

Definition 5.3. We call a modal proposition algebraic, if one of the equivalent conditions is satisfied:

1. its merely of the form $\|U\|_{\mathbb{T}}$ for some flat affine U.
2. There is a $\mathbb{T}$-surjective map out of a flat affine U.
3. It has a $\mathbb{T}$-atlas.

Proof.

$1 \Leftrightarrow 2$ Clear.

$1 \Rightarrow 3$ we show that $U \to \|U\|_{\mathbb{T}}$ is a $\mathbb{T}$-atlas. Every fiber is in $\mathbb{T}$, because U is flat.

$3 \Rightarrow 1$ Let $V \to P$ be a $\mathbb{T}$-atlas. have to show TFAE $\|V\|_{\mathbb{T}} \to P \to (V \in \mathbb{T}) \overset{5.2}{\to} \|V\|_{\mathbb{T}}$. Proof: $\|V\|_{\mathbb{T}} \to P$ as P is modal prop. Secondly, because $V \to P$ is a $\mathbb{T}$-cover.
Hence P is an algebraic proposition.

□

Lemma 5.4. *Algebraic propositions are algebraic spaces.*

Proof. We have $U \to \|U\|_{\mathbb{T}}$ where U is affine, hence an algebraic space and the fibers are in $\mathbb{T}$ by flatness of U, hence they are smooth algebraic spaces. By stability under quotients, our algebraic proposition is an algebraic space. □

Definition 5.5. An smooth equivalence relation on a set U is some equivalence relation $R : U^2 \to \mathrm{Prop}$, whose fibers are in $\mathbb{T}$

Lemma 5.6. *let U be an algebraic space (e.g. affine scheme) and $R : U^2 \to \mathrm{Prop}$ be a smooth equivalence relation Then U/R is an algebraic space*

Proof. The map $U \to U/R$ is fibered in $\mathbb{T}$, in particular fibered in smooth algebraic spaces.By stability under smooth quotients, U/R is an algebraic space. □

Corollary. *Let U be affine and R a smooth equivalence relation. The identity types of U/R, i.e. the propositions $R(x,y)$, are algebraic propositions.*

Proof. By 12.2, the class of types admitting a $\mathbb{T}$-atlas is closed under taking identity types. U/R is a type admitting a $\mathbb{T}$-atlas, hence its identity types admit them as well. $\square$

Definition 5.7. A modal set X is a classical algebraic space iff it is merely of the form U/R for some affine U and $R : U^2 \to \mathrm{Prop}$ a smooth equivalence relation. Equivalently there exists some $\mathbb{T}$-atlas $U \to X$ (i.e. out of an affine). We call X smooth if U can be choosen to be in $\mathbb{T}$.

Corollary (of 5.3)**.** *Classical Algebraic spaces that are propositions are algebraic propositions.*

Remark 4. Assume Saturatedness of the topology. smooth classical Algebraic spaces which are affine are in $\mathbb{T}$.

Question 2. Is the class of classical algebraic spaces stable under smooth quotients? If its not, how should we enlarge it?

Try: Assume R is fibered in smooth algebraic spaces. Choose $U \to T$ a $\mathbb{T}$-atlas. For any $x : U$ the fiber R_x merely has an atlas $\tilde{R}_x \to R_x$. As U has choice (its affine), we find some $\mathbb{T}$-cover $\tilde{U} = \sum_x \tilde{R}_x \to \sum_x R_x$. Goal: Find for all $t : U/R$ a $\mathbb{T}$-atlas $V_t \to \mathrm{fib}_{[]}(t)$. Then $\sum_t V_t$ will be affine, because its the total space of a $\mathbb{T}$-cover of an affine. Moreover, $\sum_t V_t \to \sum_t \mathrm{fib}_{[]}(t) \to U/R$ will be a $\mathbb{T}$-cover, as $V_t \in \mathbb{T}$. This is what we wanted to show.

6 n-stacks

Definition 6.1. Let $\mathbb{T}$ be a subcanonical topology finer than the Zariski topology. Let $n \geq -2$. A type X

- is a (smooth) -2-stack if it is contractible
- is A $(n+1)$-stack, if
  - X is a $\mathbb{T}$-sheaf
  - For any $x, y : X$ $x =_X y$ is a n-stack
  - There exists an n-atlas, i.e. a $\mathbb{T}$-surjective map $\operatorname{Spec} A \to X$ fibered in
    - $\mathbb{T}$, if $n \leq 0$
    - smooth n-stacks, if $n > 0$.
- X is a smooth $n+1$-stack if
  - X is a $(n+1)$-stack
  - There exists a n-atlas $\operatorname{Spec} A \to X$ with $\operatorname{Spec} A \in \mathbb{T}$

Lemma 6.2. *One could only alternatively talk about (smooth) n-stacks for $n \geq 1$, define them by induction as above. Then later define:*

- *A (smooth) -1-stack is a (smooth) 1- stack is a proposition.*

- *A (smooth) 0-stack is a (smooth) 1- that is a 0-type.*

Proof. □

Lemma 6.3. *A (smooth) n-stack is a (smooth) $n+1$-stack.*

Proof. Induction. Be aware of the induction start, where maybe no atlas is assumed! We need, that $\mathbb{T}$ is subcanonical to conclude that affines are $\mathbb{T}$-sheaves. □

Remark 5. If one changes the definition of atlas to be a map out of a scheme, then smooth -1 atlas will be scheme in $\mathbb{T}$. Otherwise propositional -1-stack are not 0-stacks.

7 Stability results

Theorem 7.1. *Let $n \geq -2$. Smooth / n-stacks are stable by dependent sums.*

Proof. Induction. For $n = -2$ its okay. Let $B : X \to \mathcal{U}$ be a family of $n+1$-stacks indexed over a $n+1$-stack X, then surely the total space $\sum_{x:X} Bx$ is a $\mathbb{T}$-sheaf as $\mathbb{T}$-sheaves are stable under dependend sum. The identity types in a $\sum$ type are $\sum$ of identity types
It remains to construct some n-atlas $\operatorname{Spec} A \to \sum_{x:X} B_x$ For any $x : X$ we merely have an n-atlas $V_x \to B_x$, i.e. with V_x affine. Claim: X has local choice for X wrt n-atlasses.
Proof: n-atlasses contain zariski-atlasses, because $\mathbb{T}$ is finer than the Zariski topology. n-stacks are stable under dependent sum by induction, thus n-atlasses are stable under composition. □(Claim)
By (4.2) for X, we merely find U affine, an n-atlas $p : U \to X$ with

$$\prod_{u:U} \sum_{V_{p(u)} \in T} (q : V_{p(u)} \to B_{p(u)}) \times (q \text{ fibered in smooth } n \text{ stacks })$$

Now the desired map is $\sum_{u:U} V_{pu} \to \sum_{x:X} B_x$, because it is an n-atlas by 4.3
If additionally, all the B_x and X are smooth n-stacks, just observe that we can choose the affine V_{pu} to lie in $\mathbb{T}$, Accordingly $\sum_{u:U} V_{pu} \in T$ as $\mathbb{T}$ is stable under Σ.

□

Corollary. *n-atlasses are stable under composition.*

Lemma 7.2. *$n+1$-stacks are closed under taking closed (open) subtypes.*

Proof. First we show:if X has an n-atlas and Y is a closed (open) subtype of X, then Y has an n-atlas. Choose an n-atlas $\operatorname{Spec} A \to X$. The pullback to Y has have the same fibers.
If Y is closed, and the total space is a closed subtype of $\operatorname{Spec} A$, hence it will be affine.
if Y is an open subtype of X, then the pullback is an open subtype of $\operatorname{Spec} A$, hence by zariski local choice merely of the form $\bigcup_{i=1}^{n} D(a_i) \subset A$. As n-atlasses are stable under composition 7, it suffices to show, that the map $f : \bigsqcup_i D(a_i) \to \bigcup_{i=1}^{n} D(a_i)$ is a Zariski-atlas, because then it will be an n-atlas as well. Let $x : \bigcup_{i=1}^{n} D(a_i)$, i.e. there merely exists an i, such that $a_i(x)$ is invertible. The fiber is exactly $D(a_1(x)) + \ldots + D(a_n(x))$. thus we are done. (MAYBE OUTSOURCE THIS and say open subschemes of affines have zariski atlas)

□

Corollary. *Let X be a quasi-projective scheme that is a sheaf. Then X is a 0-stack.*

Proof. It suffices to see that X has a zariski atlas. Use . □

Definition 7.3. A property of morphisms between n-stacks is local, if it is satisfied by identities, stable under composition and basechange/descent along *Cov* -maps, precomposition/right cancellability with *Cov* -maps.

Lemma 7.4. *Given a local property of types P. Then beeing fibered in P is a local property of morphisms.*

Lemma 7.5. *Given a local property P of morphisms of modal n-types, a morphism $f : X \to Y$ has P if there exists an n-atlas of f having P.*

The previous lemma tells us that we have the correct notion of smooth morphisms between n-stacks for $n = 0, 1$.

8 Descent

Theorem 8.1. *Let T be a modal n-type. The Proposition, that P is a (smooth) n-stack, is modal.*

9 Fundamental Theorem of algebraic spaces

9.1 For groupoids

Lemma 9.1. *If $R \twoheadrightarrow X \to X$ is a $\mathbb{T}$-htpy-coequalizer diagram of two $\mathbb{T}$-covers between affines, then X is a 1-stack.*

9.2 For sets

Lemma 9.2. *Denote $\mathbb{T}Set$ for the sets that are $\mathbb{T}$-sheaves. Assume given a $\mathbb{T}set$ X then the following maps are mutually inverse*

$$\sum_{R:X\to X\to \mathbb{T}\,\mathrm{Prop}} R \text{ equivalence relation} \simeq \sum_{Y:\mathbb{T}\mathsf{Set}} \sum_{p:X\to Y} p \text{ } \mathbb{T}\text{surjective}$$

$$R \mapsto (X/R, [\_])$$

$$\lambda x, y.(p(x) = p(y)) \leftarrow\!\shortmid (Y, p)$$

where X/R is defined by applying $L_T\|\_\|_0$ at the higher inductive type $X//R$.

Proof.

- Well-definedness: The map $[\_] : X \to \|X//R\|_0 \to L_T\|X//R\|_0$ is the composition of a surjective with a $\mathbb{T}$-surjective map [ref?], hence its $\mathbb{T}$-surjective.
Conversely given (Y, p) as Y is a sheaf, we have for all $x, y : X$ that $p(x) =_Y p(y)$ is a sheaf.

- If $x, y : X$ then we have a chain of equivalences

$$R(x, y) \simeq (\bar{x} =_{\|X//R\|_0} \bar{y}) \to ([x] =_{L_T\|X//R\|_0} [y])$$

where the first map is plain HoTT and the second map is ap, i.e. the unit of the modality [ref?], but as the $\bar{x} =_{\|X//R\|_0} \bar{y}$ is already a sheaf, it is an isomorphism as well.

- Let (Y, p) be in the RHS. Let $R(x, y) = (p(x) = p(y)) : \mathbb{T}\,\mathrm{Prop}$. By plain HoTT, There is a map $\eta : X//R \to Y$ (defined by the universal property of the set truncation and by induction on the higher inductive type $X//R$ on canonical terms through the map $p : X \to Y$). I claim η exhibits Y as the localization for $\mathbb{T}\,\mathsf{Set}$-modality of $X//R$. Let T be another $\mathbb{T}\,\mathsf{Set}$ equipped with a map $X//R \to T$. By precomposition we obtain a map $X \to T$. Claim: it factors uniquely through $p : X \to Y$.

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X//R \arrow[r] & T \\ & Y \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Proof:
Existence: We want to define a map $Y \to T$. Let $y : Y$. As p is $\mathbb{T}$-surjective and T is a sheaf, we may assume we merely have some element in the fiber of p over y. Now push this element through

$$\|\operatorname{fib}_p y\| \to \|X//R\|_0 \to T$$

where the first map is by Plain HoTT and the second one is induced from $X//R \to T$ by assumption and the fact that T is a set.. One can easily check this makes the diagram commute. Uniqueness follows from $X \to Y$ beeing $\mathbb{T}$-surjective and the following Fact: Two parellel maps $Y \rightrightarrows T$ into a $\mathbb{T}\,\mathsf{Set}$ T are already equal if the become equal after

precomposition with a $\mathbb{T}$-surjection $X \to Y$.
Proof of the fact : Let $y : Y$. The goal is an identity type of a $\mathbb{T}\,\mathsf{Set}$, hence a $\mathbb{T}\,\mathrm{Prop}$. Hence As the fiber over y in X is $\mathbb{T}$-merely inhabited, we may assume an actual term in the fiber. As $X \to Y$ equalizes the arrows, this term allows us to conclude. □(fact) □(Claim)
We apply the fact to the ($\mathbb{T}$-)surjectivity of $X \to X//R$ to get a unique factorization

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[dr] & X//R \arrow[r] \arrow[d] & T \\ & Y \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

making the right triangle commute. This is what we wanted to show.

□

Definition 9.3. An equivalence relation R on a type X is called:

- redundant if for all $x, y : X$ the proposition $R(x, y)$ is a -1-stack.
- smooth if its and for any $y : X$ its fibers:

$$R_y :\equiv \sum_{x:X} R(x, y)$$

  are affine in $\mathbb{T}$.

Lemma 9.4. *Assume that $\mathbb{T}$ satisfies descent for propositions and for sets 8.1, i.e. that a modal proposition being a (-1)-stack is a sheaf. Assume that a modal set beeing affine in $\mathbb{T}$ is a sheaf. Assume given a $\mathbb{T}$set X, then the following types are equivalent:*

- *The type of redundant smooth equivalence relations over X.*
- *The type of $\mathbb{T}$sets Y with identity types beeing stacks and an -1-atlas X to Y (in V2 a $\mathbb{T}$-cover).*

Proof. By the equivalence in 9.2, it is enough to check that:

- The identity types in X/R are (-1)-stacks if and only if the relation R is redundant . For any $x, y : X$ we know that:

$$R(x, y) \simeq [x] =_{X/R} [y]$$

  so the direct direction is immediate. For the converse we use the assumption that a modal proposition being a (-1)-stack is a sheaf and that the map $[\_] : X \to X/R$ is $\mathbb{T}$-surjective.

- The fibers of:

$$[\_] : X \to X/R$$

  are affine in $\mathbb{T}$ if and only if the relation R is smooth. For any $y : X$ we have that:

$$\sum_{x:X} R(x, y) \simeq \mathrm{fib}_{[\_]}([y])$$

  so the direct direction is immediate. Here as well the converse follows from $\mathbb{T}$-surjectivity of $[\_]$ and that the topology has descent.

□

Corollary. *Assume $\mathbb{T}$ satisfies descent for propositions and for sets. A type is a 0-stack iff its merely the $\mathbb{T}$-quotient of an affine scheme by a smooth equivalence relation.*

Theorem 9.5. *Assume $\mathbb{T}$ satisfies descent for propositions. The quotient of a* 0*-stack* $X \in \mathbb{T}\,\mathsf{Set}$ *by an* 0*-smooth equivalence relation* R *is a* 0*-stack. TODO*

Proof. The identity types in X/R are propositional 0-stacks, hence (-1)-$\mathbb{T}$runcations of -1-stacks by 11.2 as desired.
How to find an atlas: todo. How to proceed, if we could choose all atlasses we want at the same time?

□

Remark 6. This is equivalent to saying that 1-stacks that are 0-types are geomeric 0-stacks: One direction we prove later. If R is a 0-smooth equivalence relation on a 0-stack X, then X/R is a 1-stack by observing that any -1-atlas $X' \to X$ gives a 0-atlas $X' \to X \to X/R$. Moreover, X/R is a 0-type, hence by assumption a 0-stack.

Example 9.6. *There are open affine subschemes* U *of affine schemes* $\operatorname{Spec} A$*, which are not (disjoints unions of) principal open*

Proof. Consider $A = R[x, y, u, v]/(xy + ux^2 + vy^2), X = \operatorname{Spec} A$ and consider the open $U = D(x, y)$.
We cant expect U to be a disjoint union of principal opens (todo). However, $D(x, y)$ is affine: We have maps $U \to R$ given by $f = -v/x = (y + ux)/y^2, g = -u/y = (x + vy)/x^2$. Then $D(f) \cup D(g) = \operatorname{Spec} R^X$, as $yf + xg = 1$ in R^U. Taking preimages under the affinization map, $U_f \cup U_g = X$ and one checks this defines an open affine cover (for example : $U_f \simeq \operatorname{Spec} R[x, u, f^{\pm 1}, g]/(xy + ux^2 + uy^2)$ with $y := (1 - gx)/f$.) But on both of this open subsets the affinization map is an isomorphism hence the affinization of X is an isomorphism. compare (Hartshorne II.2.17) □

Lemma 9.7. *Let* $f : X \to Y$ *be surjective. There exists a Zariski Cover* $X' \to X$ *such that* $X' \to Y$ *is a Zariski cover iff there exists a Zariski Cover* $X' \to X$*, some* $n : \mathbb{N}$ *and an open affine embedding* $X' \hookrightarrow Y^n$ *over* Y*.*

10 Saturated Topologies revisited

Lemma 10.1 (1.1)**.** *We want that every $n-1$-atlas of a smooth n-atlas has the additional requirement in the definition of smooth n-atlas. It turns out, that for this topology needs to be saturated: The following are equivalent*

1. *Beeing in the topology descents along $\mathbb{T}$-covers between affines, i.e. $\mathbb{T}$ is saturated.*
2. *A smooth n -stack X that is an affine scheme lies in the Topology $\mathbb{T}$.*
3. *Let $n \geq 0$. If T is a smooth n-stack, then any $n-1$-atlas $U \to T$ satisfies $U \in \mathbb{T}$.*
4. *If $U \xrightarrow{f} V \xrightarrow{g} W$ are maps between affines and f and gf are $\mathbb{T}$ covers, then g is a $\mathbb{T}$ Cover*

Proof. $1 \Rightarrow 2$
Induction. This holds for $n = -1$. Assume it holds for $n-1$. Choose a $n-1$-atlas with T source, i.e. $T \ni \operatorname{Spec} A \to X$ fibered in smooth $n-1$-stacks. As it is affine, all the fibers of the atlas are affine smooth $n-1$-stacks, hence by induction they lie in $\mathbb{T}$, thus the atlas is a $\mathbb{T}$-cover between affines, hence $X \in \mathbb{T}$.
$2 \Rightarrow 3$
As $U \to T$ is fibered in smooth $n-1$ stacks, all the fibers are in particular smooth n-stacks by 6.3. By stability under dependent sum $U = \sum_{t:T} U_t$ is a smooth n-stack that is affine, hence by assumption (2) it lies in the topology.
$3 \Rightarrow 1$
Let $X \to Y$ be a $\mathbb{T}$-cover with X affine in $\mathbb{T}$ and Y affine. Then Y is a smooth 0-stack, But $Y \to Y$ is a -1-atlas, hence by assumption $Y \in T$.
$4 \Rightarrow 1$
Obvious
$1 \Rightarrow 4$
Check fiberwise ∎

If $n \geq$, replacing $\mathbb{T}$ by its saturation $\mathbb{T}'$ does change the notion of (smooth) n-stack, but we have the following statement, that tells us, that if we start with 0-$\mathbb{T}$-stacks then the notion of smoothness does not see the difference between $\mathbb{T}$ and its saturation.

Proposition 10.2. *Let X be a 0-stack that is a weak smooth 0-stack, i.e. there exists a $\mathbb{T}'$-atlas $\mathbb{T}' \ni X' \to X$ (i.e. fibered in $\mathbb{T}'$). Then X is a smooth 0-stack.*

Proof. Wlog $X' \in \mathbb{T}$. Choose a -1-atlas $\operatorname{Spec} A \to X$ (i.e. fibered in $\mathbb{T}$). As the fibers of $X' \to X$ merely have smooth atlasses $\tilde{X}'_x \to X'_x$, we can use Local choice to obtain a commutative diagram $Y = \sum_{x':X'} \tilde{X}'_x$

$$\begin{tikzcd} \tilde{X} \arrow[r, "\mathbb{T}"] \arrow[d, "\mathbb{T}"'] & \operatorname{Spec} A \arrow[d] \\ X' \arrow[r, "\mathbb{T}'"'] & X \end{tikzcd}$$

As $Y \to X'$ is a$\mathbb{T}$-cover and $X' \in \mathbb{T}$ we conclude $Y \in \mathbb{T}$. Hence we found a smooth $\mathbb{T}$-atlas of X. ∎

10.1 Zariski Topology is not saturated

Example 10.3 (Weird Zariski Atlasses)**.** *Assume those equivalent conditions on the Zariski topology. There exist Zariski atlasses of affines $\operatorname{Spec} A = X$ which are not of the form $D(a_1) + \ldots + D(a_n) \to \operatorname{Spec} A$ for $(a_1, \ldots, a_n) \in Um(A)$*

Proof. Indeed, using the first example, choose $U \subset \operatorname{Spec} A$ affine not principal open, then choosing a Zariski atlas $V \to U$ gives $V + X \to U + X \to X$ where $V + X \to X$ is a Zariski cover and $V + X \to U + X$ is a Zariski cover. From (4), we deduce that $U + X \to X$ is a Zariski cover, but U is not a disjoint union of principal opens in $\operatorname{Spec} A$. $\square$

Example 10.4. *Assume those equivalent conditions on the Zariski topology. Every affine open proposition U is principal open !*

Proof. Let $V \to U$ be a Zariski atlas. Then $V + 1 \to U + 1$ is a Zariski atlas with $V + 1 \in \mathbb{T}$ and $U + 1$ affine, hence by (1) $U + 1 \in \mathbb{T}$, hence U is a disjoint union of principal opens hence, as it is a proposition, its principal open. $\square$

11 beeing a stack is indepent of the truncation level

Lemma 11.1. *Let $n \geq 0$. A n-stack is an modal n-type.*

Proof. The n-$\mathbb{T}$runcationis an n-type. Now conclude by induction. $\square$

We want to show that the notion of stack makes sense, i.e. beeing a stack should not depend on the truncation level.

Lemma 11.2. *Assume $\mathbb{T}$ is saturated and satisfies descent for propositions. Let P be a modal proposition. Then TFAE*

1. *For some $m \geq 0$, P is a m stack*
2. *There exists some fp algebra A such that $\operatorname{Spec} A \to P$ and P is logically equivalent to $(\operatorname{Spec} A \in \mathbb{T})$.*
3. *P is equivalent to $\|\operatorname{Spec} A\|_{\mathbb{T}}$ for some fp A, i.e. P is a -1-stack.*

Proof.

1. $\Rightarrow$ 2. Let $\operatorname{Spec} A \to P$ be a $m - 1$ atlas. Assume $\operatorname{Spec} A \in \mathbb{T}$. Then $\|\operatorname{Spec} A\| \to P$ so as P is a sheaf, we have P. Conversely, if $x : P$, then the fiber over x is $\operatorname{Spec} A$ and a smooth $m - 1$ stack, hence belongs to the topology by 10.1.

2. $\Rightarrow$ 3. We have to show: There exists some flat algebra such that P is logically equivalent to $\|\operatorname{Spec} A\|_{\mathbb{T}}$. By assumption we have $\operatorname{Spec} A \to P \to (\operatorname{Spec} A \in \mathbb{T})$, so we deduce $\|\operatorname{Spec} A\|_{\mathbb{T}} \to P \to (\operatorname{Spec} A \in \mathbb{T})$, as P is a modal proposition. In particular A is flat. Conversely $P \to (\operatorname{Spec} A \in \mathbb{T}) \to \|\operatorname{Spec} A\|_{\mathbb{T}}$, where the first arrow is by assumption.

3. $\Rightarrow$ 1. 6.3

$\square$

Lemma 11.3. *A smooth -1-stack P is contractible.*

Proof. Choose a $\mathbb{T}$-cover $\mathbb{T} \ni \operatorname{Spec} A \to P$. As P is a proposition we have $\|\operatorname{Spec} A\| \to P$. As P is a sheaf we have P. $\square$

Example 11.4. *A 0-stack is a $\mathbb{T}$-sheaf whose identity types are (-1)-$\mathbb{T}$runcations of ((affine ?)) schemes and there exists a $\mathbb{T}$-atlas $\operatorname{Spec} A \to X$.*
Why are schemes 0-stacks? This holds in special case, for example if the scheme is quasi projective.

Theorem 11.5. *Let $\mathbb{T}$ be saturated. Assume the topology satisfies descent Let $m, n \geq -2$. Given an n-type T that is a (smooth) m-stack then T is a (smooth) n-stack.*

Proof. By 6.3 we may assume $m \geq n \geq -2$.
If $m \leq 1$ this is clear. Now assume $m \geq 2$. Induction. Inductionstart $m = 2$. Let us prove the case of $m = 2, n = 1$, the cases $-2 \leq n < 1$ are immediate from this.
Choose a 1-atlas $X' \to T$, i.e. its fibered in smooth 1-stacks. As T is a groupoid and X' is a set, the fibers are actually sets, i.e. smooth 0-stacks.
Now consider $R := X' \times_T X'$. As X' is in particular a 0-stack and 0-stacks are stable under dependent sums, R will be a 0-stack. Choose a a $\mathbb{T}$-cover $R' \to R$ with R' affine. Now $R' \to R \to X'$ is a map between affine schemes i.e it is fibered in smooth 0-stacks that are affine. As $\mathbb{T}$ is saturated, the fibers of $R' \to X'$ are in $\mathbb{T}$. As $X'//R'$ is a 1-stack by **??**, it suffices to show that $X'//R' \to X'//R$ is a $\mathbb{T}$-cover. Pick a term in $X'//R$. As the fiber beeing in $\mathbb{T}$ is sheaf If additionally T is assumed to be a smooth 2-stack, then we can assume X' to be in the topology. This will force R to be a smooth 0-stack, so we may choose R' Assume $m > 2$ and the statement is proven for all $(n', m') < (n, m)$ in lexicographical ordering. As the identity types of T are $n-1$-types and $m-1$ stacks by induction they are $n-1$ stacks. Let $X \to T$ be an $m-1$-atlas, i.e. fibered in smooth $m-1$-stacks with X affine. The fibers are in particular $n-1$-types, so by induction they are smooth $n-1$-stacks. Hence $X \to T$ is an $n-1$-atlas. If, additionally T is assumed to be a smooth m-stack, we can choose $X \in \mathbb{T}$, hence $X \to T$ witnesses that T is a smooth n-stack.

□

12 Stability under Quotients

Definition 12.1. A morphism between n-stacks is smooth if it is fibered in

- $\mathbb{T}$ if $n \leq 0$
- smooth n-stacks if $n > 0$.

Lemma 12.2. *Let C be a class stable under finite limits, i.e. containing 1, stable under dependent sums and identity types. The class* $\mathsf{HasAtlas}_C$ *of types Y which admit a map* $\operatorname{Spec} A \to Y$ *fibered in C is stable under finite limits*

Proof. Obviously 1 has an atlas, and the class of types admitting an atlas is stable by $\sum$ by some earlier result (using local choice arguments, in the presheaf model). It remains to show, that identity types in Y have an atlas provided that Y has an atlas.
By assumption we can choose a map $p : V \to Y$ out of an affine fibered in C. Let $y, y' : Y$. Then we have the map

$$(\operatorname{fib}_p y) \times_V (\operatorname{fib}_p y') \to y = y'$$
$$(v, q : y = pv), (v', q' : y' = pv'), (h : v = v') \mapsto q \cdot h \cdot q'^{-1}$$

The fiber over $j : y = y'$ looks like

$$\sum_v \underbrace{(\sum_{v'} (h : v = v'))}_{\mathsf{isContr}} \times (q : y = pv) \times (q' : y' = pv') \times (q \cdot h \cdot q'^{-1} = j) \simeq \sum_v (v = py) \simeq \operatorname{fib}_p y$$

Hence the map is fibered in C. It suffices to show, that $(\operatorname{fib}_p y) \times_V (\operatorname{fib}_p y')$ has an atlas, because then we can compose such an atlas with the above map to obtain an atlas of $y = y'$. By assumption the fibers of p have an atlas, so we can choose $q : W \to \operatorname{fib}_p y, q' : W' \to \operatorname{fib}_p y'$ atlasses. Then $W \times_V W' \to (\operatorname{fib}_p y) \times_V (\operatorname{fib}_p y')$ is an atlas: The domain is a fiber product of affines, hence affine. The fiber over (x, x') is equivalent to the product of fibers $(\operatorname{fib}_q x) \times (\operatorname{fib}_{q'} x')$ which is in C by stability under dependent sums (so in particular under finite products).

□

Theorem 12.3. *Let $f : X \to Y$ be a $\mathbb{T}$-surjective smooth morphism between modal n-types. If X is a (smooth) stack , then Y a (smooth) stack.*

(*) This can only hold if we define -1-stacks to be modal propositions with a -2-atlas $\operatorname{Spec} A \to P$, i.e. algebraic propositions 5.3

Proof. Induction. For $n = -2$ its clear. Let X be a n-stack. Lets first construct the $n-1$-atlas of Y. We merely find a $V \twoheadrightarrow X$ which is an $n-1$-atlas. Then $V \to X \to Y$ is an n-atlas because it is $\mathbb{T}$-surjective and is fibered in the correct $\sum$-stable class of types, i.e. $\mathbb{T}$ if $n \leq 1$ and smooth $n-1$-stacks for $n > 1$. Hence Y is an $n+1$-stack. As Y is an n-type, Y is an n-stack 11.5.
If additionally X is assumed to be smooth, then V can be assumed to lie in $\mathbb{T}$ which directly gives us that Y has a smooth atlas.
It remains to show that the identity types of Y are $n-1$-stacks. As Y has an $n-1$-atlas, by 12.2 we find some $n-1$-atlas $p : W \to y = y'$. The map is smooth. If $n = 0$, $y = y'$ is a -1-stack by (*). If $n > 0$, W is an $n-1$-stack and p is smooth, so by induction $y = y'$ is an $n-1$-stack.

□

Remark 7 (Using descent but not induction)**.** Hugo suggested an alternative argument proving that the identity types of Y are $n-1$-stacks, which presumable avoids 11.5 but uses descent for n-stacks: For $x : X, y : Y$ we have that

$$(f(x) = y) \simeq (1 \times_X \operatorname{fib}_f y)$$

is an n-stack by stability under $\sum$. Because it is an $n-1$-type, it is a $n-1$-stack by 11.5. Now conclude that every identity type of Y is an $n-1$-stack by using descent for $n-1$-stacks and $\mathbb{T}$-surjectivity of f.

13 Local properties

Definition 13.1. Let *Cov* be the property of morphisms of n-stacks defined by asking that the morphism is $\mathbb{T}$-surjective and fibered in smooth n-stacks. Its stable under basechange. A property of n-stacks is local if $P(1)$ holds, P is stable by dependent sums and given a *Cov*er $X \to Y$ we have PX iff PY.

Example 13.2. *beeing smooth n-stack is a local property of stacks.*

Proof. We have to show: If $f : X \to Y$ is a $\mathbb{T}$-surjective map fibered in smooth n-stacks between n-stacks, then X is a smooth n-stack iff Y is a smooth n-stack. The only if is clear by stability under dependent sums. The other direction is 12.3.

□

Definition 13.3. A property of morphisms between n-stacks is local, if it is satisfied by identities, stable under composition and basechange/descent along *Cov* -maps, precomposition/right cancellability with *Cov* -maps.

Lemma 13.4. *Given a local property of types P. Then beeing fibered in P is a local property of morphisms.*

Lemma 13.5 ([ref?])**.** *Given a local property P of morphisms of n-stacks, a morphism $f : X \to Y$ has P if there exists an n-atlas of f having P.*

Example 13.6. *A morphism of n-stacks is smooth iff there exists an n-atlas of f*

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r, "\tilde{f}"'] \arrow[d] & \operatorname{Spec} B \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

such that $\tilde{f}$ is a $\mathbb{T}$-cover.

The previous lemma tells us that we have the correct notion of smooth morphisms between n-stacks for $n = 0, 1$.