definition

Thesis

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1 Does not fit yet

Algebraic space = Classical algebraic space. Let $\mathbb{T}$ be a saturated topology. Let U be an affine in $\mathbb{T}$, $R : U^2 \to \mathrm{Prop}$ be a covering equivalence relation, meaning that the fibers R_x are covering algebraic spaces for $x : U$. I have shown previously that the identity types then are algebraic spaces. Let U/R denote the sheafifaction of the set truncation of the homotopy quotient. We want to show, that As U is projective we can choose $\tilde{R} : U^2 \to \mathbb{T}$ such that $\|\tilde{R}xy\|_{\mathbb{T}} = Rxy$. Consider the sheafification of the homotopy quotient $U//\tilde{R}$, this will be a 1-stack whose identity types are in $\mathbb{T}$. Hence it suffices to show that the map $f : U//\tilde{R} \to U/R$ is fibered in covering 0-stacks. Consider a term in U/R. By descent we may assume its of the form $[x]$ for some $x : U$. I claim that the map

$$\tilde{R}_x \to \mathrm{fib}_f[x] = \sum_{t:U//\tilde{R}} ft =_{U/R} [x]$$

is an equivalence, this is en

2 The inductive approach

3 A minimal approach

Fix $\mathbb{T}$ a topology, which we call the covering-affines.

Definition 3.1. Let $\mathcal{V} \supset \mathbb{T}$ be a superclass stable under $\sum$. covering stacks are the smallest intermediate class $\mathbb{T} \subset \tilde{\mathbb{T}} \subset \mathcal{V}$ that is closed under covers between types in $\mathcal{V}$: If $X, Y : \mathcal{V}$ and $X \to Y$ is fibered in $\mathsf{CS}_{\mathcal{V}}$, then $X \in \tilde{\mathbb{T}}$ iff $Y \in \tilde{\mathbb{T}}$.

We call such map $X \to Y$ whose fibers are covering $\mathcal{V}$-stacks a $\mathcal{V}$-cover. If X is affine we call it an $\mathcal{V}$-atlas. If X is in $\mathbb{T}$ we call it a $\mathcal{V}$-catlas. In Case of $\mathcal{V} = \mathcal{U}_{\mathbb{T}}$ the sheaves we call it a geometric cover / geometric atlas / geometric catlas.

Proposition 3.2 (Recursion principle)**.** *Let $P : \mathcal{V} \to \mathrm{Prop}$ be a property of types in $\mathcal{V}$. Assume*

- *Every covering affine has P*
- *If $X \to Y$ is fibered in P then X has P iff Y has P*

Then every covering $\mathcal{V}$-stack has P.

Proof. Replace P by $P \wedge \mathsf{is-covering-stack}$. Then usual induction

$\square$

Lemma 3.3. *$\mathcal{V}$-covers are stable under composition.*

Proof. covering $\mathcal{V}$-stacks are stable under $\sum$. $\square$

Proposition 3.4. *Every covering $\mathcal{V}$-stack X merely admits a $\mathcal{V}$-catlas, i.e. a $\mathcal{V}$-cover $Y \to X$ with $Y \in \mathbb{T}$.*

Proof. We apply the recursion principle of covering stacks

- If X is covering affine, then $X \to X$ is a $\mathcal{V}$-catlas with covering domain.

- If X is obtained as a quotient, i.e. if its equipped with a cover $Y \to X$ with Y a covering stack, then by induction Y admits a $\mathcal{V}$-catlas $S \to Y$. Then $S \to Y \to X$ is a $\mathcal{V}$-catlas by 3.3.

- If X is obtained as a sum, i.e. we have a $\mathcal{V}$-cover $f : X \to Y$, then by induction Y admits an $\mathcal{V}$-catlas $g : S \to Y$ and the fibers merely have $\mathcal{V}$-catlasses $S_y \to \mathrm{fib}_f\, y$ s. By choice of S, we can choose such catlasses $S_{gs} \to \mathrm{fib}_f(gs)$ for all $s : S$. By 7.3 the map
$$\sum_{s:S} S_{gs} \to (\sum_{y:Y} \mathrm{fib}_f\, y) \simeq X$$
is a $\mathcal{V}$-cover. Its domain is a covering affine as $\mathbb{T}$ is $\sum$-stable. Hence X admits a $\mathcal{V}$-catlas .

$\square$

Now we want to show that the clash of terminology regarding 'covering' is reasonable: For this we need a stronger recursion principle

Lemma 3.5. *Let $P : \mathcal{V} \to \mathrm{Prop}$ be a property of $\mathcal{V}$ types. Assume*

- *stable under $\sum$*
- *Every covering affine belongs to P*
- *If $S \to X$ is fibered in P with $S \in \mathbb{T}$, then PX.*
- *If PX, then X admits a map $\mathbb{T} \ni S \to X$ with fibers satisfying P.*

Then P is satisfied by all covering $\mathcal{V}$-stacks.

Proof. By the recursion principle, we only have to show quotients as covering affines have P and P is stable under $\sum$ by assumption.
Let $Y \to X$ be fibered in P and assume PY. by the third point Y admits a map $\mathbb{T} \ni S \to Y$ with fibers satisfying P. Then the composition $S \to Y \to X$ is fibered in P as well, because P is $\sum$-stable. Hence by the second point, X has P. $\square$

Proposition 3.6. *Let $\mathbb{T}$ be saturated. A covering stack X is affine iff its a covering affine.*

Proof. The converse is clear. The direct direction follows by the stronger recursion principle. choosing a $\mathcal{V}$-catlas $S \to X$. As both S and X are affine the fibers are affine. By induction the fibers are covering affines. By saturatedness of the topology X is covering affine. $\square$

Lemma 3.7. *Let X be a covering $\mathcal{V}$-stack. Let f : $\mathrm{Spec}\, A \to X$ be a $\mathcal{V}$-atlas. Then* $\mathrm{Spec}\, A \in \mathbb{T}$

Proof. As $\mathrm{Spec}\, A \simeq \sum_{x:X} \mathrm{fib}_f\, x$ is a dependent sum of covering $\mathcal{V}$-stacks, it is a covering $\mathcal{V}$-stack again. We conclude by 3.6. $\square$

3.1 Truncatedness

In this subsection we want to prove

Theorem 3.8. *Every covering geometric stack is n-truncated for some $n : \mathbb{N}$.*

Lemma 3.9. *Every covering $\mathcal{V}$-stack X is $\mathbb{T}$-merely inhabited.*

Proof.

- If X is in $\mathbb{T}$ then its clear.
- If X is obtained by a quotient, we have a map $\operatorname{Spec} A \to X$ with domain in $\mathbb{T}$. Now use that we get a map on $\mathbb{T}$-propositional-truncations and that Spec A is T-merely inhabited.
- if X is obtained by $X = \sum_{y:Y} By$ for Y a covering $\mathcal{V}$-stack and By covering $\mathcal{V}$-stacks, by induction all the By are $\mathbb{T}$-merely inhabited. Hence, for all $y : Y$, we can conclude $\|X\|_{\mathbb{T}}$. As Y is $\mathbb{T}$-merely inhabited by induction and the goal is a sheaf, we can conclude.

□

Lemma 3.10. *Let X be an $n{+}1$-type and Y a sheaf. If $X \to Y$ is a n-truncated $\mathbb{T}$-surjective map, then Y is an $n+1$-type.*

Proof. Use that $\mathsf{is} - \mathsf{n} - \mathsf{truncated}(y = y')$ is a sheaf for $y, y' : Y$. □

Proof. of the theorem. We apply the stronger recursion principle from above

- If Y is in the topology its clear with $n = 0$.
- Assume Y is equipped with a $\mathcal{V}$-catlas $f : S \to Y$, such that every fiber in n-truncated for some n. f is $\mathbb{T}$-surjetive by 3.9. We apply 3.10. So it remains to find an n such that all fibers are n-truncated. For any $x : S$, By induction $\mathrm{fib}_f(fx)$ is n-truncated for some n. By projectivity of S, we find some n such that $\mathrm{fib}_f(fx)$ is n-truncated for all $x : S$. For general $y : Y$, using that is-n-truncated $\mathrm{fib}_f\, y$ is a sheaf , we can conclude by $\mathbb{T}$-surjectivity of f.
- Let X be an n-truncated covering geometric stack. By 3.4 we find a geometric catlas $S \to X$. All the fibers are (at most) n-truncated.

□

3.2 Descent

For this subsection lets assume St a class of sheaves, such that $\mathbb{T}$ is contained in it and for any map $X \to Y$ fibered in $\mathbb{T}$, $X \in \mathsf{St}$ iff $Y \in \mathsf{St}$. We call types in this class stacky.

Lemma 3.11. *Let $\mathbb{T}$ satisfy descent, i.e. beeing affine in the topology has descent. Let $X \in \mathsf{St}$ and Y a type. Let $f : X \twoheadrightarrow Y$ be fibered in $\mathbb{T}$ and surjective. Then $\bullet Y$ is stacky.*

Proof. Claim: Y is seperated. Proof: By surjectivity of f we may only show that for any $x : X, y : Y$, the type $fx =_Y y$ is a sheaf. If we define U to be the fiber over y, it is in $\mathbb{T}$ by assumption. But then $fx =_Y y$ is the outer pullback

$$\begin{tikzcd}
fx = y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & U \in \mathbb{T} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d, "y"'] \\
1 \arrow[r, "x"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

of stacky types, in particular sheaves. □(Claim)

Consider $X \xrightarrow{f} Y \xrightarrow{\eta} \bullet Y$. As X is stacky, it suffices to show, that the fibers are in $\mathbb{T}$. As beeing affine in $\mathbb{T}$ is a sheaf, we may just show that for all $y : Y$, the fibers over $\eta y : \bullet Y$ are in $\mathbb{T}$. As η is a monomorphism by 5.4 , η restricts to an equivalence

$$\mathrm{fib}_f\, y \to \mathrm{fib}_{\eta f}(\eta y)$$

But the left hand side is in $\mathbb{T}$ by assumption. $\square$

Theorem 3.12. *Assume $\mathbb{T}$ have descent. Then* St *is a sheaf.*

Proof. St is seperated: This follows from the embedding St into the seperated (TODO) type of sheaves.
Let $U \in \mathbb{T}$ and $P : \|U\| \to \mathsf{St}$. We want to construct a filler

$$\begin{tikzcd} \|U\| \arrow[r, "P"] \arrow[d] & \mathsf{St} \\ 1 \arrow[ur, dashed] & \end{tikzcd}$$

Claim: $\bullet(\sum_{x:\|U\|} Px)$ is stacky.

Proof. of the claim. We want to apply the previous lemma to the map

$$\sum_{x:U} P|x| \to \sum_{x:\|U\|} Px$$

The domain is in St by stability under $\sum$. The fibers are equivalent to $U \in \mathbb{T} \subset \mathsf{St}$. $\square$

The claim provides the map $1 \to \mathsf{St}$. The diagram commutes: Assuming $x : \|\operatorname{Spec} A\|$ we wish to show $Px = \sum_{x:\|U\|} Px$. Using univalence, we may show that the maps

$$Px \to \sum_{x:\|U\|} Px \xrightarrow{\eta} \bullet \sum_{x:\|U\|} Px$$

are both equivalences. The first one is an equivalence as $\|U\|$ is contractible. Hence the middle term is a sheaf, thus the unit map is an equivalence as well.

$\square$

Corollary. *For all $n : \mathbb{N} \cup \{\infty\}$, the class of (covering) (n-)stacks satisfy descent.*

4 Saturated Topologies

Consider a topology $\mathbb{T}$ finer than the Zariski topology.

Definition 4.1. A covering atlas of X is some $\hat{X} \in \mathbb{T}, \hat{X} \to X$ $\mathbb{T}$-cover

Definition 4.2. $\mathbb{T}$ is saturated if Beeing in the topology descents along $\mathbb{T}$-covers between affines, i.e. every affine schemes that has a covering atlas lies itself in $\mathbb{T}$.
The saturated closure of a topology $\mathbb{T}$ is the topology $\mathbb{T}'$ defined by (todo finite sums of?)

$$X \in \mathbb{T}' \text{ iff } X \text{ is affine} \wedge \exists \text{ covering atlas of } X$$

Lemma 4.3. *Using ZLC, this is the smallest saturated topology containing* $\mathbb{T}$.

Proof. Obviously $1 \in \mathbb{T}'$. Types which have a covering atlas are stable by dependent sums by the proof of **??**. For the saturatedness consider some $\mathbb{T}'$-cover $\mathbb{T}' \ni X' \to X$. By replacing X' with some covering atlas, we may assume that $X' \in \mathbb{T}$. As every fiber $X'_x \in \mathbb{T}'$, we merely find a covering atlas $\tilde{X}'_x \to X'_x$. Then by Zariski local choice there exists a Zariski atlas $\hat{X} \to X$ and a commutative diagram

$$\begin{tikzcd} Y \equiv \sum_{x:\hat{X}} \tilde{X}'_x \arrow[r] \arrow[d] & \sum_{x:X} X'_x = X' \arrow[d] \\ \hat{X} \arrow[r, "Zar"'] & X \end{tikzcd}$$

As $X' \in \mathbb{T}$ and $Y \to X'$ is fibered in $\mathbb{T}$ (7.3) we have $Y \in \mathbb{T}$. But $Y \to \hat{X}$ is a $\mathbb{T}$-cover and $\hat{X} \to X$ is a $\mathbb{T}$-cover, $Y \to X$ is a $\mathbb{T}$-cover. Hence $X \in \mathbb{T}'$. $\square$

Lemma 4.4. *A type T is a sheaf wrt to $\mathbb{T}'$ iff it is a sheaf wrt to $\mathbb{T}$*

Proof. As $\mathbb{T} \subset \mathbb{T}'$ the $\to$ direction is clear. Now, let $X \in \mathbb{T}'$. We have to show that $T \to T^{\|X\|}$ is an equivalence. Choose $\mathbb{T} \ni Y \to X$. Then we have a commutative diagram

$$\begin{tikzcd} T \arrow[r] \arrow[dr, "\simeq"'] & T^{\|X\|} \arrow[d] \\ & T^{\|Y\|} \end{tikzcd}$$

So $T \to T^{\|X\|}$ has a left-inverse. Thus it suffices to show that any $f : T^{\|X\|}$ has a preimage. Choose $t : T$, s.th. cnst_t^Y is the composite $\|Y\| \to \|X\| \xrightarrow{f} T$. We have $\|Y\| \to (\mathrm{cnst}_t^X = f)$. But as $Y \in \mathbb{T}$ and $\Delta_t = f$ is a sheaf (as an identitytype in the sheaf $T^{\|X\|}$) we are done. $\square$

Remark 1. We never used that we only talk about $\mathbb{T}$-covers.

Lemma 4.5. *Every saturated affine (i.e.* $\operatorname{Spec} A \in \mathbb{T}'$*) it $\mathbb{T}$-merely inhabited.*

Proof. We have $\|X\| \to \|\operatorname{Spec} A\|$ for some covering atlas $\mathbb{T} \ni X \to \operatorname{Spec} A$. $\square$

Question 1. Does the converse hold, i.e. is every $\mathbb{T}$-merely inhabited affine saturated?

5 Lex Modalities

Lemma 5.1 (Stability resuls)**.** *Lex Modalities are stable under*

1. *Conjunction*
2. *Composition*

Lemma 5.2. *Let $\bigcirc$ be a lex-modality. Let X be $\bigcirc$-modal and $B : X \to \mathcal{U}_\bigcirc$ be a family of modal types. Then $\sum_{x:X} B_x$ is $\bigcirc$-modal*

Lemma 5.3. *Let $B : \bullet X \to \mathcal{U}$. Then $\bullet(\sum_{x:X} B(\eta x)) = \sum_{x:\bullet X} \bullet Bx$*

Proof. Observe that

$$\sum_{x:X} B\eta x \to \sum_{x:\bullet X} Bx$$

is a $\bullet$-equivalence, because for all modal types T, the type $Bx \to T$ is modal for any $x : \bullet X$. Then it follows by [ref?]. ☐

Lemma 5.4. *For a type X the following are equivalent:*

- *the identity types of X are sheaves*
- *the unit $X \to \bullet X$ is a monomorphism*

In this case we call X seperated

6 Atlas

Definition 6.1. A $\mathbb{T}$-atlas of X is a $\mathbb{T}$-cover $\operatorname{Spec} A \to X$ out of an affine scheme.

Remark 2. Any good enough TODO scheme has a Zariski atlas. If $\mathbb{T}$ is finer than the Zariski-topology then in the definition we may replace affine scheme by good enough scheme, if its just about the question whether a type admits an atlas.

Example 6.2. *Let X be a (1-)type. X has a Zariski-atlas, iff there exists some $f : \operatorname{Spec} A \to X$ fibered in types of the form $\operatorname{Spec}(R_{f_1} \times \ldots \times R_{f_n})$ for $(f_1, \ldots, f_n) \in Um(R)$.*

Remark 3. If one applies ZLC to an affine scheme $\operatorname{Spec} A$ the resulting principal open cover $D(f_i), f_i \in A$ will induce indeed a zariski atlas $\bigsqcup D(f_i) \to \operatorname{Spec} A$, because the fiber over $x : \operatorname{Spec} A$ is $\bigsqcup D(f_i(x))$.

Question: Does every zariski atlas of $\operatorname{Spec} A$ have this form? Weird Zariski Atlasses

Example 6.3. *$\mathbb{P}^n$ has a zariski atlas given by the standart homogeneous principal opens $\sum_{i=0}^{n} D_+(x_i)$. The fiber over a point $[y_0 : \ldots . y_n]$ is $D(y_0) + \ldots D(y_n)$ where $(y_1, \ldots, y_n) \in Um(R)$.*

7 Local Choice

In this section let $\mathbb{T}$ denote a topology finer than the zariski topology.

Definition 7.1. Let *Cov* be a class of morphisms (which we think of n-atlasses of some n), containing $\mathbb{T}$-atlas, (stable under pullback NECESSARY TODO?) A type S has *local choice* wrt *Cov* if for any $\mathbb{T}$ -surjective map $X \to Y$ and any map $f : S \to Y$ there exists a map $p' : S' \to S$ in *Cov* and a commutative diagram

$$\begin{tikzcd} S' \arrow[r, dashed] \arrow[d] & X \arrow[d, "p"] \\ S \arrow[r, "f"] & Y \end{tikzcd}$$

Proposition 7.2. *Assume that* Cov *is stable under composition.*

- *If $\hat{S} \to S$ is a* Cov*er and $\hat{S}$ has $\mathbb{T}$-local choice, then S has $\mathbb{T}$-local choice.*
- *Affine schemes have $\mathbb{T}$-local choice.*
- *Any type admitting a* Cov *- Atlas $\operatorname{Spec} A \to S$ has $\mathbb{T}$-local choice.*

Proof. The first point follows from stability under composition of *Cov* . the third point follows from the second. By the first point, we may assume that S is affine. As p is $\mathbb{T}$-surjective, for any $x : S$ there merely is a $\operatorname{Spec} B_x \in T$ and a map $\operatorname{Spec} B_x \to \| \operatorname{fib}_p(x) \|$. As S is projective, we have a term in

$$\prod_{x:S} \sum_{\operatorname{Spec} B_x \in T} \operatorname{Spec} B_x \to \| \operatorname{fib}_p(fx) \|$$

By setting

$$(S' := \sum_{x:S} \operatorname{Spec} B_x) \xrightarrow{\pi} S$$

the projection, we are now in the situation that for any $t : S'$ we merely have a point in $\operatorname{fib}_p((p'(t)))$ and $S' \to S$ is a $\mathbb{T}$-cover, thus it is in *Cov* . Moreover, S' is affine, as it is a dependent sum of affines. Hence again we now can find a lift $S' \to X$ making

$$\begin{tikzcd} S' \arrow[r] \arrow[d, "p'"'] & Y \arrow[d, "p"] \\ S \arrow[r, "f"] & X \end{tikzcd}$$

commute. $\square$

The next lemma shows, that the class of types equipped with a $\mathbb{T}$-atlas is stable under dependent sums.

Lemma 7.3. *Let $\mathcal{U}' \subset \mathcal{U}$ be stable under dependent sums (e.g. $\mathbb{T}$-inhabited types) Let X be a type with a map $p : U \to X$ fibered in $\mathcal{U}'$. For any $x : X$, let Y_x be a type and moreover for any $u : U$, we are given a map $q_u : V_u \to Y_{p(u)}$ fibered in $\mathcal{U}'$. Then the induced map*

$$p : \sum_{u:U} V_u \to \sum_{x:X} Y_x$$

is fibered in $\mathcal{U}'$

Proof. The fiber of p over some $(x, y) \in \sum_{x:X} Y_x$ is given by

$$\sum_{u:\mathrm{fib}_p\, x} \mathrm{fib}_{q_u}(y')$$

where $y' : Y_{p(u)}$ (depending on u) is the transport of $y : Y_x$ along $x = p(u)$. As $\mathcal{U}'$ is stable under dependent sum those fibers are again in $\mathcal{U}'$. This shows the result. $\square$

Theorem 7.4. *Let $\mathcal{U}'$ be a class stable under dependent sums. The class of types admitting a $\mathcal{U}'$-atlas is closed under dependent sums. If $\mathbb{T}$ is a topology, the same holds for $\mathcal{U}'$-atlasses with domain in $\mathbb{T}$.*

Proof. Let us construct some atlas $\mathrm{Spec}\, A \to \sum_{x:X} B_x$ For any $x : X$ we merely have an atlas $V_x \to B_x$, i.e. with V_x affine. X has local choice wrt atlasses by (7.2) using $\mathcal{U}'$ is $\sum$-stable (we use the trivial topology).
If additionally, all the B_x and X are smooth n-stacks, just observe that we can choose the affine V_{pu} to lie in $\mathbb{T}$, Accordingly $\sum_{u:U} V_{pu} \in T$ as $\mathbb{T}$ is stable under Σ.
By Local choice for X, we merely find U affine, an atlas $p : U \to X$ with

$$\prod_{u:U} \sum_{V_{p(u)} \in T} (q : V_{p(u)} \to B_{p(u)}) \times (q \text{ fibered in smooth } n \text{ stacks })$$

Now the desired map is $\sum_{u:U} V_{pu} \to \sum_{x:X} B_x$, because it is an atlas by 7.3

$\square$

8 Fundamental Theorem of algebraic spaces

8.1 For groupoids

Lemma 8.1. *If $R \twoheadrightarrow X \to X$ is a $\mathbb{T}$-htpy-coequalizer diagram of two $\mathbb{T}$-covers between affines, then X is a 1-stack.*

8.2 For sets

Lemma 8.2. *Denote $\mathbb{T}Set$ for the sets that are $\mathbb{T}$-sheaves. Assume given a $\mathbb{T}set$ X then the following maps are mutually inverse*

$$\sum_{R:X\to X\to \mathbb{T}\,\mathrm{Prop}} R \text{ equivalence relation} \simeq \sum_{Y:\mathbb{T}\mathsf{Set}} \sum_{p:X\to Y} p \ \mathbb{T}\text{surjective}$$

$$R \mapsto (X/R, [\_])$$

$$\lambda x, y.(p(x) = p(y)) \leftarrow\!\shortmid (Y, p)$$

where X/R is defined by applying $L_T\|\_\|_0$ at the higher inductive type $X//R$.

Proof.

- Well-definedness: The map $[\_] : X \to \|X//R\|_0 \to L_T\|X//R\|_0$ is the composition of a surjective with a $\mathbb{T}$-surjective map [ref?], hence its $\mathbb{T}$-surjective.
  Conversely given (Y, p) as Y is a sheaf, we have for all $x, y : X$ that $p(x) =_Y p(y)$ is a sheaf.

- If $x, y : X$ then we have a chain of equivalences

  $$R(x, y) \simeq (\bar{x} =_{\|X//R\|_0} \bar{y}) \to ([x] =_{L_T\|X//R\|_0} [y])$$

  where the first map is plain HoTT and the second map is ap, i.e. the unit of the modality [ref?], but as the $\bar{x} =_{\|X//R\|_0} \bar{y}$ is already a sheaf, it is an isomorphism as well.

- Let (Y, p) be in the RHS. Let $R(x, y) = (p(x) = p(y)) : \mathbb{T}\,\mathrm{Prop}$. By plain HoTT, There is a map $\eta : X//R \to Y$ (defined by the universal property of the set truncation and by induction on the higher inductive type $X//R$ on canonical terms through the map $p : X \to Y$). I claim η exhibits Y as the localization for $\mathbb{T}\,\mathsf{Set}$-modality of $X//R$. Let T be another $\mathbb{T}\,\mathsf{Set}$ equipped with a map $X//R \to T$. By precomposition we obtain a map $X \to T$. Claim: it factors uniquely through $p : X \to Y$.

  $$\begin{tikzcd} X \arrow[r] \arrow[dr] & X//R \arrow[r] & T \\ & Y \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

  Proof:
  Existence: We want to define a map $Y \to T$. Let $y : Y$. As p is $\mathbb{T}$-surjective and T is a sheaf, we may assume we merely have some element in the fiber of p over y. Now push this element through

  $$\|\operatorname{fib}_p y\| \to \|X//R\|_0 \to T$$

  where the first map is by Plain HoTT and the second one is induced from $X//R \to T$ by assumption and the fact that T is a set.. One can easily check this makes the diagram commute. Uniqueness follows from $X \to Y$ beeing $\mathbb{T}$-surjective and the following Fact: Two parellel maps $Y \rightrightarrows T$ into a $\mathbb{T}\,\mathsf{Set}$ T are already equal if the become equal after

precomposition with a $\mathbb{T}$-surjection $X \to Y$.
Proof of the fact : Let $y : Y$. The goal is an identity type of a $\mathbb{T}\,\mathsf{Set}$, hence a $\mathbb{T}\,\mathrm{Prop}$. Hence As the fiber over y in X is $\mathbb{T}$-merely inhabited, we may assume an actual term in the fiber. As $X \to Y$ equalizes the arrows, this term allows us to conclude. $\square$(fact) $\square$(Claim)
We apply the fact to the ($\mathbb{T}$-)surjectivity of $X \to X//R$ to get a unique factorization

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[dr] & X//R \arrow[r] \arrow[d] & T \\ & Y \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

making the right triangle commute. This is what we wanted to show.

$\square$

Definition 8.3. An equivalence relation R on a type X is called:

- redundant if for all $x, y : X$ the proposition $R(x, y)$ is a -1-stack.
- covering if its and for any $y : X$ its fibers:

$$R_y :\equiv \sum_{x:X} R(x, y)$$

  are affine in $\mathbb{T}$.

Lemma 8.4. *Assume that $\mathbb{T}$ satisfies descent for propositions and for sets* **??**, *i.e. that a modal proposition being a (-1)-stack is a sheaf. Assume that a modal set beeing affine in $\mathbb{T}$ is a sheaf. Assume given a $\mathbb{T}$set X, then the following types are equivalent:*

- *The type of redundant covering equivalence relations over X.*
- *The type of $\mathbb{T}$sets Y with identity types beeing stacks and an -1-atlas X to Y (in V2 a $\mathbb{T}$-cover).*

Proof. By the equivalence in 8.2, it is enough to check that:

- The identity types in X/R are (-1)-stacks if and only if the relation R is redundant . For any $x, y : X$ we know that:

$$R(x, y) \simeq [x] =_{X/R} [y]$$

  so the direct direction is immediate. For the converse we use the assumption that a modal proposition being a (-1)-stack is a sheaf and that the map $[\_] : X \to X/R$ is $\mathbb{T}$-surjective.
- The fibers of:

$$[\_] : X \to X/R$$

  are affine in $\mathbb{T}$ if and only if the relation R is covering. For any $y : X$ we have that:

$$\sum_{x:X} R(x, y) \simeq \mathrm{fib}_{[\_]}([y])$$

  so the direct direction is immediate. Here as well the converse follows from $\mathbb{T}$-surjectivity of $[\_]$ and that the topology has descent.

$\square$

Corollary. *Assume $\mathbb{T}$ satisfies descent for propositions and for sets. A type is a 0-stack iff its merely the $\mathbb{T}$-quotient of an affine scheme by a covering equivalence relation.*

Theorem 8.5. *Assume $\mathbb{T}$ satisfies descent for propositions. The quotient of a 0-stack $X \in \mathbb{T}\,\mathsf{Set}$ by an 0-covering equivalence relation R is a 0-stack. TODO*

Proof. The identity types in X/R are propositional 0-stacks, hence (-1)-$\mathbb{T}$runcations of -1-stacks by 13.2 as desired.
How to find an atlas: todo. How to proceed, if we could choose all atlasses we want at the same time?

□

Remark 4. This is equivalent to saying that 1-stacks that are 0-types are geomeric 0-stacks: One direction we prove later. If R is a 0-covering equivalence relation on a 0-stack X, then X/R is a 1-stack by observing that any -1-atlas $X' \to X$ gives a 0-atlas $X' \to X \to X/R$. Moreover, X/R is a 0-type, hence by assumption a 0-stack.

Example 8.6. *There are open affine subschemes U of affine schemes $\operatorname{Spec} A$, which are not (disjoints unions of) principal open*

Proof. Consider $A = R[x,y,u,v]/(xy + ux^2 + vy^2), X = \operatorname{Spec} A$ and consider the open $U = D(x,y)$.
We cant expect U to be a disjoint union of principal opens (todo). However, $D(x,y)$ is affine: We have maps $U \to R$ given by $f = -v/x = (y+ux)/y^2, g = -u/y = (x+vy)/x^2$. Then $D(f) \cup D(g) = \operatorname{Spec} R^X$, as $yf + xg = 1$ in R^U. Taking preimages under the affinization map, $U_f \cup U_g = X$ and one checks this defines an open affine cover (for example : $U_f \simeq \operatorname{Spec} R[x,u,f^{\pm 1}, g]/(xy+ux^2+uy^2)$ with $y := (1-gx)/f$.) But on both of this open subsets the affinization map is an isomorphism hence the affinization of X is an isomorphism. compare (Hartshorne II.2.17) □

Lemma 8.7. *Let $f : X \to Y$ be surjective. There exists a Zariski Cover $X' \to X$ such that $X' \to Y$ is a Zariski cover iff there exists a Zariski Cover $X' \to X$, some $n : \mathbb{N}$ and an open affine embedding $X' \hookrightarrow Y^n$ over Y.*

9 Algebraic Space

Recall the notion of (covering) 0-stacks. it is the smallest pair of classes that satisfies the following

- Stability under $\sum$ 11.1
- (covering) affines are (covering) algebraic spaces.
- stable under covering quotients: If X is an algebraic space, Y modal 0-type and $X \to Y$ is fibered in covering algebraic spaces, then Y is an algebraic space. Additionally, if X is covering, then Y is covering.

9.1 Geometric propositions

Definition 9.1. An affine Scheme U is called geometric, if

$$\|U\|_{\mathbb{T}} \to (U \in \mathbb{T})$$

Lemma 9.2. *The converse holds always*

Proof. because things in $\mathbb{T}$ are automatically $\mathbb{T}$-merely inhabited $\square$

Recall the definition of $\mathbb{T}$-atlas 6.1

Definition 9.3. We call a modal proposition geometric, if one of the equivalent conditions is satisfied:

1. its merely of the form $\|U\|_{\mathbb{T}}$ for some geometric affine U.
2. There is a $\mathbb{T}$-surjective map out of a geometric affine U.
3. It has a $\mathbb{T}$-atlas.

Proof.

$1 \Leftrightarrow 2$ Clear.

$1 \Rightarrow 3$ we show that $U \to \|U\|_{\mathbb{T}}$ is a $\mathbb{T}$-atlas. Every fiber is in $\mathbb{T}$, because U is geometric.

$3 \Rightarrow 1$ Let $V \to P$ be a $\mathbb{T}$-atlas. have to show TFAE $\|V\|_{\mathbb{T}} \to P \to (V \in \mathbb{T}) \overset{9.2}{\to} \|V\|_{\mathbb{T}}$. Proof: $\|V\|_{\mathbb{T}} \to P$ as P is modal prop. Secondly, because $V \to P$ is a $\mathbb{T}$-cover.
Hence P is a geometric proposition.

$\square$

Lemma 9.4. *geometric propositions are algebraic spaces.*

Proof. We have $U \to \|U\|_{\mathbb{T}}$ where U is affine, hence an algebraic space and the fibers are in $\mathbb{T}$ by geometricness of U, hence they are covering algebraic spaces. By stability under quotients, our geometric proposition is an algebraic space. $\square$

9.2 Algebraic spaces

Definition 9.5. Consider a modal equivalence relation $R : U^2 \to \mathsf{GeomProp}$ on an affine U. We call it covering if every fiber $R_s \equiv \sum_{t:S} Rst$ satisfy one of the following equivalent conditions

- admit a $\mathbb{T}$-catlas.
- is a covering 0-stack.

Proof. Every type admitting a $\mathbb{T}$-catlas is a covering 0-stack. Conversely: if the fibers are covering 0-stacks. For all $t : S$ we can choose a geometric atlas $\operatorname{Spec} A_t \to Rst$ by 9.3. Then

$$\sum_{t:S} \operatorname{Spec} A_t \to \sum_{t:S} Rst$$

is a $\mathbb{T}$-atlas. As $\sum_{t:S} Rst$ is a covering 0-stack by assumption, the map has to be a $\mathbb{T}$-catlas by 3.7. $\square$

Definition 9.6. A modal set X is an algebraic space iff it is merely of the form $L_{\mathbb{T}}(U/R)$ for some affine U and $R : U^2 \to \mathrm{Prop}$ a covering equivalence relation. Equivalently there exists some map $U \to X$ whose fibers merely have $\mathbb{T}$-catlasses. We call X covering if U can be choosen to be in $\mathbb{T}$.

Lemma 9.7. *Every (covering) algebraic space is a (covering) geometric 0-stack.*

Proof. Choose a presentation $R : U^2 \to \mathrm{Prop}$. It suffices to show, that the map $f : U \to L_{\mathbb{T}}(U/R)$ is a geometric (c)atlas. The map f is $\mathbb{T}$-surjective by the well-definedness of the bijection 8.2. By descent we may just show, that the fibers $\mathrm{fib}_f(f(s))$ for $s : U$ are covering 0-stacks. But by the bijection in 8.2 those are equivalent to the fibers R_s, which are covering 0-stacks as the equivalence relation is covering. $\square$

Corollary. *The identity types of algebraic spaces are geometric propositions.*

Proof. By the previous lemma and 9.17 $\square$

Lemma 9.8. *Let P be a sheaf and a proposition that admits a map* $\operatorname{Spec} A \to P$ *fibered in covering algebraic spaces. Then P is a geometric proposition.*

Proof. The fibers are covering algebraic spaces and affine, hence covering affine. By 9.3 we conclude. $\square$

Theorem 9.9. *Let X be a sheaf of sets. Let S be (covering-) affine and $f : S \to X$ be fibered in covering algebraic spaces. Then X is a (covering) algebraic space.*

Proof. The identity types of X admit a map fibered in covering algebraic spaces (todo check stability under $\sum$) out of an affine by 14.2. by 9.8 they are geometric propositions. The equivalence relation determined by f is covering 9.5 , because the fibers of f are covering 0-stacks. $\square$

9.3 Stability under covers TODO

In this subsection we want to prove the following:

Theorem 9.10 (TODO)**.** *The class of covering $\mathcal{V}$-stacks is the smallest intermediate class $\mathbb{T} \subset \tilde{T} \subset \mathcal{V}$ such that whenever $X \in \tilde{\mathbb{T}}, Y \in \mathcal{V}$ and $X \to Y$ is fibered in $\tilde{\mathbb{T}}$, then $Y \in \tilde{\mathbb{T}}$.*

Lemma 9.11. *Covering stacks are stable by dependent sums: If $X \in \mathsf{CS}_{\mathcal{V}}$, $Y : X \to \mathsf{CS}_{\mathcal{V}}$, then $\sum_{x:X} Yx \in CS$.*

Proof. Lets first prove the special case where $X \in \mathbb{T}$. By choice of X we can choose a C-atlas $Qx \to Yx$ for every x. Now $\sum_{x:X} Qx \to \sum_{x:X} Yx$ is fibered in C by 7.3 and the domain is in $\mathbb{T}$ by $\sum$-stability of $\mathbb{T}$.
For the general case, choose a C-atlas $p : T \to X$ with $T \in \mathbb{T}$. Then we have a map

$$\sum_{t:T} Y(pt) \to \sum_{x:X} Yx$$

where every fiber is equivalent to a fiber of p, i.e. its a covering C-stack. As its domain is a covering C-stack by the previous case, we can choose an atlas . $\square$

Proof. The first class is definitely contained in the second class. To show that they coincide we need to show, that the first class is stable under $\sum$ and under quotients. For the first point we use choice of affines. The second point $\square$

The first point is the minimal definition which is good mapping out of the class of coverings stacks and the second one is useful to keep in mind the stability results. The closedness under covers assumption is the conjunction of closed under $\sum$ (as C $\sum$-stable) and closed under quotients.

Lemma 9.12. *covering C-stacks contain* 1 *and are closed under* $\sum$.

Example 9.13. *covering Aff-stacks = saturation of* $\mathbb{T}$. *Indeed: By definition, the saturation of* $\mathbb{T}$ *is is obtained by quotients of* $\mathbb{T}$ *by* $\mathbb{T}$*-covers. We have shown, that its closed under covers between affines.*

Definition 9.14. We call X a C'-stack, iff there merely exists some affine $\operatorname{Spec} A \to X$ fibered in covering C-stacks.
We call X a C-stack, iff its a C'-stack and $X \in C$.

Definition 9.15. The (covering)∞-stacks are the (covering) $\mathcal{U}$-stacks.

Lemma 9.16. *X is a n' stack iff its an $n+1$-stack*

Proof. If its $\square$

Lemma 9.17. *C-stacks are closed under* id*-types.*

Proof. This is similar to 14.2. $\square$

Warning. The previous lemma does not hold for covering stacks: Identity types of things in $\mathbb{T}$ could be empty.

THIS IS UNUSUAL, but surprisingly useful. Let $n \geq 0$.

Example 9.18. *Affine covering* 0*-stacks are the saturation of* $\mathbb{T}$.

Definition 9.19. X is a (covering) 0-stack, if its a (covering) 0-type-stack.

Theorem 9.20 (TODO)**.** *Let X be a type. TFAE for all n :*

1. *X is a covering n-type-stack.*
2. *Inductively, There merely exists some $U \in \mathbb{T}$ with a map $U \to X$ fibered in covering $n-1$-stacks.*

2' *Inductively, as the previous one but additionally the* id*-types of X are $n-1$-stacks.*

3. *Inductively, There merely exists some covering $n-1$-stack U with a map $U \to X$ fibered in covering $n-1$-stacks.*

3' Inductively, as the previous one but additionally the id*-types of X are $n-1$-stacks.*

If one of the conditions is satisfied we call X a covering n-stack.

Proof. Induction $n-1 \mapsto n$, $n \geq 1$.

1. $\Rightarrow$ 2 We have to show, that the class in 2. is closed under $\sum$ and closed under quotients between n-types. This was already done.

2. $\Rightarrow$ 3 Clear

3. $\Rightarrow$ 3' By 9.17 and independence of the truncation level (TODO).

3'. $\Rightarrow$ 3 , 2' $\Rightarrow$ 2 Clear

3'. $\Rightarrow$ 1. by induction, covering $n-1$-stacks = covering $n-1$-type-stacks $\subset$ covering n-type-stacks. Now use stability under covers between n-types.

3' $\Rightarrow$ 2' Use 3 $\Rightarrow$ 1 $\Rightarrow$ 2.

$\square$

10 n-stacks

Definition 10.1. Let $\mathbb{T}$ be a subcanonical topology finer than the Zariski topology. Let $n \geq -2$. A type X

- is a (covering) -2-stack if it is contractible
- is A $(n+1)$-stack, if
  - X is a $\mathbb{T}$-sheaf
  - For any $x, y : X$ $x =_X y$ is a n-stack
  - There exists an n-atlas, i.e. a $\mathbb{T}$-surjective map $\operatorname{Spec} A \to X$ fibered in
    - $\mathbb{T}$, if $n \leq 0$
    - covering n-stacks, if $n > 0$.
- X is a covering $n+1$-stack if
  - X is a $(n+1)$-stack
  - There exists a n-atlas $\operatorname{Spec} A \to X$ with $\operatorname{Spec} A \in \mathbb{T}$

Lemma 10.2. *One could only alternatively talk about (covering) n-stacks for $n \geq 1$, define them by induction as above. Then later define:*

- *A (covering) -1-stack is a (covering) 1- stack is a proposition.*

- *A (covering) 0-stack is a (covering) 1- that is a 0-type.*

Proof. □

Lemma 10.3. *A (covering) n-stack is a (covering) $n+1$-stack.*

Proof. Induction. Be aware of the induction start, where maybe no atlas is assumed! We need, that $\mathbb{T}$ is subcanonical to conclude that affines are $\mathbb{T}$-sheaves. □

Remark 5. If one changes the definition of atlas to be a map out of a scheme, then covering -1 atlas will be scheme in $\mathbb{T}$. Otherwise propositional -1-stack are not 0-stacks.

11 Stability results

Theorem 11.1. *Let $n \geq -2$. covering / n-stacks are stable by dependent sums.*

Proof. Induction. For $n = -2$ its okay. Let $B : X \to \mathcal{U}$ be a family of $n+1$-stacks indexed over a $n+1$-stack X, then surely the total space $\sum_{x:X} Bx$ is a $\mathbb{T}$-sheaf as $\mathbb{T}$-sheaves are stable under depentend sum. The identity types in a $\sum$ type are $\sum$ of identity types.
Admitting an n-atlas is stable under dependent sum: We apply 7.4 to the class of (covering) n-atlasses, which is stable under depent sum by induction.

□

Corollary. *n-atlasses are stable under composition.*

Lemma 11.2. *$n+1$-stacks are closed under taking closed (open) subtypes.*

Proof. First we show:if X has an n-atlas and Y is a closed (open) subtype of X, then Y has an n-atlas. Choose an n-atlas $\operatorname{Spec} A \to X$. The pullback to Y has have the same fibers.
If Y is closed, and the total space is a closed subtype of $\operatorname{Spec} A$, hence it will be affine.
if Y is an open subtype of X, then the pullback is an open subtype of $\operatorname{Spec} A$, hence by zariski local choice merely of the form $\bigcup_{i=1}^n D(a_i) \subset A$. As n-atlasses are stable under composition 11, it suffices to show, that the map $f : \bigsqcup_i D(a_i) \to \bigcup_{i=1}^n D(a_i)$ is a Zariski-atlas, because then it will be an n-atlas as well. Let $x : \bigcup_{i=1}^n D(a_i)$, i.e. there merely exists an i, such that $a_i(x)$ is invertible. The fiber is exactly $D(a_1(x)) + \ldots + D(a_n(x))$. thus we are done. (MAYBE OUTSOURCE THIS and say open subschemes of affines have zariski atlas) □

Corollary. *Let X be a quasi-projective scheme that is a sheaf. Then X is a 0-stack.*

Proof. It suffices to see that X has a zariski atlas. Use . □

Definition 11.3. A property of morphisms between n-stacks is local, if it is satisfied by identities, stable under composition and basechange/descent along *Cov* -maps, precomposition/right cancellability with *Cov* -maps.

Lemma 11.4. *Given a local property of types P. Then beeing fibered in P is a local property of morphisms.*

Lemma 11.5. *Given a local property P of morphisms of modal n-types, a morphism $f : X \to Y$ has P if there exists an n-atlas of f having P.*

The previous lemma tells us that we have the correct notion of covering morphisms between n-stacks for $n = 0, 1$.

12 Saturated Topologies revisited

Lemma 12.1 (1.1)**.** *We want that every $n-1$-atlas of a covering n-atlas has the additional requirement in the definition of covering n-atlas. It turns out, that for this topology needs to be saturated: The following are equivalent*

1. *Beeing in the topology descents along $\mathbb{T}$-covers between affines, i.e. $\mathbb{T}$ is saturated.*
2. *A covering n -stack X that is an affine scheme lies in the Topology $\mathbb{T}$.*
3. *Let $n \geq 0$. If T is a covering n-stack, then any $n-1$-atlas $U \to T$ satisfies $U \in \mathbb{T}$.*
4. *If $U \xrightarrow{f} V \xrightarrow{g} W$ are maps between affines and f and gf are $\mathbb{T}$ covers, then g is a $\mathbb{T}$ Cover*

Proof. $1 \Rightarrow 2$
Induction. This holds for $n = -1$. Assume it holds for $n-1$. Choose a $n-1$-atlas with T source, i.e. $T \ni \operatorname{Spec} A \to X$ fibered in covering $n-1$-stacks. As it is affine, all the fibers of the atlas are affine covering $n-1$-stacks, hence by induction they lie in $\mathbb{T}$, thus the atlas is a $\mathbb{T}$-cover between affines, hence $X \in \mathbb{T}$.
$2 \Rightarrow 3$
As $U \to T$ is fibered in covering $n-1$ stacks, all the fibers are in particular covering n-stacks by 10.3. By stability under dependent sum $U = \sum_{t:T} U_t$ is a covering n-stack that is affine, hence by assumption (2) it lies in the topology.
$3 \Rightarrow 1$
Let $X \to Y$ be a $\mathbb{T}$-cover with X affine in $\mathbb{T}$ and Y affine. Then Y is a covering 0-stack, But $Y \to Y$ is a -1-atlas, hence by assumption $Y \in T$.
$4 \Rightarrow 1$
Obvious
$1 \Rightarrow 4$
Check fiberwise □

If $n \geq$, replacing $\mathbb{T}$ by its saturation $\mathbb{T}'$ does change the notion of (covering) n-stack, but we have the following statement, that tells us, that if we start with 0-$\mathbb{T}$-stacks then the notion of coveringness does not see the difference between $\mathbb{T}$ and its saturation.

Proposition 12.2. *Let X be a 0-stack that is a weak covering 0-stack, i.e. there exists a $\mathbb{T}'$-atlas $\mathbb{T}' \ni X' \to X$ (i.e. fibered in $\mathbb{T}'$). Then X is a covering 0-stack.*

Proof. Wlog $X' \in \mathbb{T}$. Choose a -1-atlas $\operatorname{Spec} A \to X$ (i.e. fibered in $\mathbb{T}$). As the fibers of $X' \to X$ merely have covering atlasses $\tilde{X}'_x \to X'_x$, we can use Local choice to obtain a commutative diagram $Y = \sum_{x':X'} \tilde{X}'_x$

$$\begin{tikzcd} \tilde{X} \arrow[r, "\mathbb{T}"] \arrow[d, "\mathbb{T}"'] & \operatorname{Spec} A \arrow[d] \\ X' \arrow[r, "\mathbb{T}'"'] & X \end{tikzcd}$$

As $Y \to X'$ is a$\mathbb{T}$-cover and $X' \in \mathbb{T}$ we conclude $Y \in \mathbb{T}$. Hence we found a covering $\mathbb{T}$-atlas of X. □

12.1 Zariski Topology is not saturated

Example 12.3 (Weird Zariski Atlasses)**.** *Assume those equivalent conditions on the Zariski topology. There exist Zariski atlasses of affines* $\operatorname{Spec} A = X$ *which are not of the form* $D(a_1) + \ldots + D(a_n) \to \operatorname{Spec} A$ *for* $(a_1, \ldots, a_n) \in Um(A)$

Proof. Indeed, using the first example, choose $U \subset \operatorname{Spec} A$ affine not principal open, then choosing a Zariski atlas $V \to U$ gives $V + X \to U + X \to X$ where $V + X \to X$ is a Zariski cover and $V + X \to U + X$ is a Zariski cover. From (4), we deduce that $U + X \to X$ is a Zariski cover, but U is not a disjoint union of principal opens in $\operatorname{Spec} A$. □

Example 12.4. *Assume those equivalent conditions on the Zariski topology. Every affine open proposition U is principal open !*

Proof. Let $V \to U$ be a Zariski atlas. Then $V + 1 \to U + 1$ is a Zariski atlas with $V + 1 \in \mathbb{T}$ and $U + 1$ affine, hence by (1) $U + 1 \in \mathbb{T}$, hence U is a disjoint union of principal opens hence, as it is a proposition, its principal open. □

13 beeing a stack is indepent of the truncation level

Lemma 13.1. *Let $n \geq 0$. A n-stack is an modal n-type.*

Proof. The n-$\mathbb{T}$runcationis an n-type. Now conclude by induction. □

We want to show that the notion of stack makes sense, i.e. beeing a stack should not depend on the truncation level.

Lemma 13.2. *Assume $\mathbb{T}$ is saturated and satisfies descent for propositions. Let P be a modal proposition. Then TFAE*

1. *For some $m \geq 0$, P is a m stack*
2. *There exists some fp algebra A such that* $\operatorname{Spec} A \to P$ *and P is logically equivalent to* $(\operatorname{Spec} A \in \mathbb{T})$.
3. *P is equivalent to* $\|\operatorname{Spec} A\|_{\mathbb{T}}$ *for some fp A, i.e. P is a -1-stack.*

Proof.

1. $\Rightarrow$ 2. Let $\operatorname{Spec} A \to P$ be a $m-1$ atlas. Assume $\operatorname{Spec} A \in \mathbb{T}$. Then $\|\operatorname{Spec} A\| \to P$ so as P is a sheaf, we have P. Conversely, if $x : P$, then the fiber over x is $\operatorname{Spec} A$ and a covering $m-1$ stack, hence belongs to the topology by 12.1.

2. $\Rightarrow$ 3. We have to show: There exists some flat algebra such that P is logically equivalent to $\|\operatorname{Spec} A\|_{\mathbb{T}}$. By assumption we have $\operatorname{Spec} A \to P \to (\operatorname{Spec} A \in \mathbb{T})$, so we deduce $\|\operatorname{Spec} A\|_{\mathbb{T}} \to P \to (\operatorname{Spec} A \in \mathbb{T})$, as P is a modal proposition. In particular A is flat. Conversely $P \to (\operatorname{Spec} A \in \mathbb{T}) \to \|\operatorname{Spec} A\|_{\mathbb{T}}$, where the first arrow is by assumption.

3. $\Rightarrow$ 1. 10.3

□

Lemma 13.3. *A covering -1-stack P is contractible.*

Proof. Choose a $\mathbb{T}$-cover $\mathbb{T} \ni \operatorname{Spec} A \to P$. As P is a proposition we have $\|\operatorname{Spec} A\| \to P$. As P is a sheaf we have P. □

Example 13.4. *A 0-stack is a $\mathbb{T}$-sheaf whose identity types are (-1)-$\mathbb{T}$runcations of ((affine ?)) schemes and there exists a $\mathbb{T}$-atlas $\operatorname{Spec} A \to X$.*
Why are schemes 0-stacks? This holds in special case, for example if the scheme is quasi projective.

Theorem 13.5. *Let $\mathbb{T}$ be saturated. Assume the topology satisfies descent Let $m, n \geq -2$. Given an n-type T that is a (covering) m-stack then T is a (covering) n-stack.*

Proof. By 10.3 we may assume $m \geq n \geq -2$.
If $m \leq 1$ this is clear. Now assume $m \geq 2$. Induction. Inductionstart $m = 2$. Let us prove the case of $m = 2, n = 1$, the cases $-2 \leq n < 1$ are immediate from this.
Choose a 1-atlas $X' \to T$, i.e. its fibered in covering 1-stacks. As T is a groupoid and X' is a set, the fibers are actually sets, i.e. covering 0-stacks.
Now consider $R := X' \times_T X'$. As X' is in particular a 0-stack and 0-stacks are stable under dependent sums, R will be a 0-stack. Choose a a $\mathbb{T}$-cover $R' \to R$ with R' affine. Now $R' \to R \to X'$ is a map between affine schemes i.e it is fibered in covering 0-stacks that are affine. As $\mathbb{T}$ is saturated, the fibers of $R' \to X'$ are in $\mathbb{T}$. As $X'//R'$ is a 1-stack by **??**, it suffices to show that $X'//R' \to X'//R$ is a $\mathbb{T}$-cover. Pick a term in $X'//R$. As the fiber beeing in $\mathbb{T}$ is sheaf If additionally T is assumed to be a covering 2-stack, then we can assume X' to be in the topology. This will force R to be a covering 0-stack, so we may choose R' Assume $m > 2$ and the statement is proven for all $(n', m') < (n, m)$ in lexicographical ordering. As the identity types of T are $n-1$-types and $m-1$ stacks by induction they are $n-1$ stacks. Let $X \to T$ be an $m-1$-atlas, i.e. fibered in covering $m-1$-stacks with X affine. The fibers are in particular $n-1$-types, so by induction they are covering $n-1$-stacks. Hence $X \to T$ is an $n-1$-atlas. If, additionally T is assumed to be a covering m-stack, we can choose $X \in \mathbb{T}$, hence $X \to T$ witnesses that T is a covering n-stack.

□

14 Stability under Quotients

Definition 14.1. A morphism between n-stacks is covering if it is fibered in

- $\mathbb{T}$ if $n \leq 0$
- covering n-stacks if $n > 0$.

Lemma 14.2. *Let C be a class of types stable under $\sum$. The class $\mathsf{HasAtlas}_C$ of types Y which admit a map $\operatorname{Spec} A \to Y$ fibered in C is stable under finite limits, i*

Proof. Obviously 1 has an atlas, and the class of types admitting an atlas is stable by $\sum$ by 7.4. It remains to show, that identity types in Y have an atlas provided that Y has an atlas.
By assumption we can choose a map $p : V \to Y$ out of an affine fibered in C. Let $y, y' : Y$. Then we have the map

$$(\mathrm{fib}_p\, y) \times_V (\mathrm{fib}_p\, y') \to y = y'$$
$$(v, q : y = pv), (v', q' : y' = pv'), (h : v = v') \mapsto q \cdot h \cdot q'^{-1}$$

The fiber over $j : y = y'$ looks like

$$\sum_v (\underbrace{\sum_{v'} (h : v = v')}_{\mathsf{isContr}}) \times (q : y = pv) \times (q' : y' = pv') \times (q \cdot h \cdot q'^{-1} = j) \simeq \sum_v (v = py) \simeq \mathrm{fib}_p\, y$$

Hence the map is fibered in C. It suffices to show, that $(\mathrm{fib}_p\, y) \times_V (\mathrm{fib}_p\, y')$ has an atlas, because then we can compose such an atlas with the above map to obtain an atlas of $y = y'$. By assumption the fibers of p have an atlas, so we can choose $q : W \to \mathrm{fib}_p\, y, q' : W' \to \mathrm{fib}_p\, y'$ atlasses. Then $W \times_V W' \to (\mathrm{fib}_p\, y) \times_V (\mathrm{fib}_p\, y')$ is an atlas: The domain is a fiber product of affines, hence affine. The fiber over (x, x') is equivalent to the product of fibers $(\mathrm{fib}_q\, x) \times (\mathrm{fib}_{q'}\, x')$ which is in C by stability under dependent sums (so in particular under finite products).

□

Theorem 14.3. *Let $f : X \to Y$ be a $\mathbb{T}$-surjective covering morphism between modal n-types. If X is a (covering) stack , then Y a (covering) stack.*

(*) This can only hold if we define -1-stacks to be modal propositions with a -2-atlas $\mathrm{Spec}\, A \to P$, i.e. algebraic propositions 9.3

Proof. Induction. For $n = -2$ its clear. Let X be a n-stack. Lets first construct the $n-1$-atlas of Y. We merely find a $V \twoheadrightarrow X$ which is an $n-1$-atlas. Then $V \to X \to Y$ is an n-atlas because it is $\mathbb{T}$-surjective and is fibered in the correct $\sum$-stable class of types, i.e. $\mathbb{T}$ if $n \leq 1$ and covering $n-1$-stacks for $n > 1$. Hence Y is an $n+1$-stack. As Y is an n-type, Y is an n-stack 13.5.
If additionally X is assumed to be covering, then V can be assumed to lie in $\mathbb{T}$ which directly gives us that Y has a covering atlas.
It remains to show that the identity types of Y are $n-1$-stacks. As Y has an $n-1$-atlas, by 14.2 we find some $n-1$-atlas $p : W \to y = y'$. The map is covering. If $n = 0$, $y = y'$ is a -1-stack by (*). If $n > 0$, W is an $n-1$-stack and p is covering, so by induction $y = y'$ is an $n-1$-stack.

□

Remark 6 (Using descent but not induction)**.** Hugo suggested an alternative argument proving that the identity types of Y are $n-1$-stacks, which presumable avoids 13.5 but uses descent for n-stacks: For $x : X, y : Y$ we have that

$$(f(x) = y) \simeq (1 \times_X \mathrm{fib}_f\, y)$$

is an n-stack by stability under $\sum$. Because it is an $n-1$-type, it is a $n-1$-stack by 13.5. Now conclude that every identity type of Y is an $n-1$-stack by using descent for $n-1$-stacks and $\mathbb{T}$-surjectivity of f.

15 Local properties

Definition 15.1. Let *Cov* be the property of morphisms of n-stacks defined by asking that the morphism is $\mathbb{T}$-surjective and fibered in covering n-stacks. Its stable under basechange. A property of n-stacks is local if $P(1)$ holds, P is stable by dependent sums and given a *Cov*er $X \to Y$ we have PX iff PY.

Example 15.2. *beeing covering n-stack is a local property of stacks.*

Proof. We have to show: If $f : X \to Y$ is a $\mathbb{T}$-surjective map fibered in covering n-stacks between n-stacks, then X is a covering n-stack iff Y is a covering n-stack. The only if is clear by stability under dependent sums. The other direction is 14.3.

□

Definition 15.3. A property of morphisms between n-stacks is local, if it is satisfied by identities, stable under composition and basechange/descent along *Cov* -maps, precomposition/right cancellability with *Cov* -maps.

Lemma 15.4. *Given a local property of types P. Then beeing fibered in P is a local property of morphisms.*

Lemma 15.5 ([ref?])**.** *Given a local property P of morphisms of n-stacks, a morphism $f : X \to Y$ has P if there exists an n-atlas of f having P.*

Example 15.6. *A morphism of n-stacks is covering iff there exists an n-atlas of f*

$$\begin{array}{ccc} \operatorname{Spec} A & \xrightarrow[\tilde{f}]{} & \operatorname{Spec} B \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

such that $\tilde{f}$ is a $\mathbb{T}$-cover.

The previous lemma tells us that we have the correct notion of covering morphisms between n-stacks for $n = 0, 1$.