The type of Schemes is not modal

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Consider a modality L, such that

L1 Any $r : R$ invertible L-merely has a squareroot.

L2 $\bot$ is modal

The key ingredient to prove that Sch is not modal, is the following:

Proposition 0.1. *Let $2 \neq 0$. Let $r : R^\times$. Denote*

$$\begin{aligned} C_r &= R[X]/(X^2 - r) \\ X_r &= \sum_{x:R} (\operatorname{Spec} C_r)^{x=0} \end{aligned}$$

The following types (referred as $\mathcal{A}$ $\mathcal{B}$ $\mathcal{C}$)are logically equivalent, i.e. we find functions

$$\begin{tikzcd} \| \operatorname{Spec} C_r \| \arrow[d] \\ \operatorname{isScheme}(X_r) \arrow[d] \\ \sum_{Y:R\to\mathsf{Sch}} Y_0 =_{\mathsf{Sch}} \operatorname{Spec} C_r \times (\nexists \text{ an open affine neigbhorhood of } Y_0 \text{ in } \sum_{x:R} Y_x) \arrow[uu, bend left] \end{tikzcd}$$

Remark 1. If Sch is a modal type, the advantage of $\mathcal{C}$ is that it is modal, even if schemes are not assumed to be modal.

Corollary 0.2. *The type of Schemes* Sch *is not modal*

Proof. Assume Sch is modal. Lets call $C_r = R[X]/(X^2 - r)$. By [1] A . 0.3. its enough to show $\| \operatorname{Spec} C_r \|$ for all $r : R^\times$.
Let $r : R^\times$. First I give a conceptual proof in the case where every scheme is modal only needing $\mathcal{A} \leftrightarrow \mathcal{B}$:

$$\begin{aligned} L1 &\to L\| \operatorname{Spec} C_r \| \\ &\overset{\mathcal{A}\to\mathcal{B}}{\to} L(X_r \in \mathsf{Sch}) \\ &\overset{*}{\leftrightarrow} (X_r \in \mathsf{Sch}) \\ &\overset{\mathcal{B}\to\mathcal{A}}{\to} \| \operatorname{Spec} C_r \| \end{aligned}$$

where at $(*)$ we used that $(X_r \in \mathsf{Sch}) \simeq \sum_{X:\mathsf{Sch}} (X = X_r)$ is modal: because both X and X_r are modal, the type of equivalences $X \simeq X_r$ is modal as well, so conclude by univalence

Now we give a proof for the general case, where morally we replace $\mathcal{B}$ in the previous proof by the modal type $\mathcal{C}$. If Sch is a modal type, then the type $\mathcal{C}$ is modal:

- modal types are stable under $\sum$
- function types into modal types are modal,
- identity types in Sch are modal,
- $\bot$ is modal (L2).

Then conclude by

$$(L1) \to L\|\operatorname{Spec} C_r\| \overset{\mathcal{A}\to\mathcal{C}}{\to} L\mathcal{C} \simeq \mathcal{C} \to \mathcal{A}$$

□

1 Line with two origins

Lemma 1.1. *Closed propositions are connected.*

Proof. Let $P = (I = 0)$ be a closed proposition, for $I \subset R$ an ideal. We wish to show, that the map $\mathsf{Bool} \to \mathsf{Bool}^P$ is surjective. Maps $\operatorname{Spec} A \to \mathsf{Bool} = \operatorname{Spec}(R \times R)$ biject with idempotents of $\operatorname{Spec} A$ (Indeed the image of $R \times R$ classifies idempotents by the image of $(1, 0)$ in A). In our case $A = R/I$ has only the two trivial idempotents 0 and 1: Indeed: any idempotent of A can be lifted to an idempotent of A, using locality of R. But then, using locality of R again, we conclude.

□

Lemma 1.2. *For any $x : R$: the map*

$$\begin{aligned} f_x : \mathsf{Susp}(x \neq 0) &\to \left(\mathsf{Bool}^{x=0}\right) \\ N &\mapsto \lambda\_.true \\ S &\mapsto \lambda\_.false \end{aligned}$$

is an equivalence.
In particular we have two equivalent models of the line with two origins

$$\sum_{x:R} \mathsf{Bool}^(x = 0) \simeq \sum_{x:R} \mathsf{Susp}(x \neq 0)$$

Proof. It is well-defined: We have to check that $f_x(N) = f_x(S)$ if $x \neq 0$. But in this case the function type $\mathsf{Bool}^{x=0}$ is contractible.
Is is surjective as closed propositions are connected. It is injective: As $\mathsf{Bool} \to \mathsf{Susp}(x \neq 0)$ is surjective, we may only study the points N and S. By case-analysis we only need to show, that if $f_x(N) = f_x(S)$, then $x \neq 0$. But if $x = 0$, then $true =_{\mathsf{Bool}} false$, a contradiction. □

Lemma 1.3 (0 is a regular point of R)**.** *If $0 \in U \subset R$ is an open neighbhorhood, then the restriction map $R^U \to R^{U\setminus\{0\}}$ is injective.*

Proof. We may assume that $U = D(f)$ is a principal open neigbhorhood. Then, the element $X : R[X]_f$ is regular, as $X : R[X]$ is regular: Indeed if $XPf^n =_{R[X]} 0$ for some $P : R[X]$, then $Pf^n = 0$, thus $P =_{R[X]_f} 0$.
In other words, the map $R[X]_f \to R[X]_{fX}$ is injective, which is a reformulation of the goal. □

Remark 2. One can define regularity of a point 0 in a scheme X generally by asking that it admits a open affine neigbhorhood $0 \in \operatorname{Spec} A \subset X$ such that $\operatorname{Spec} A \setminus \{0\} = D(g_1, \ldots, g_n)$ for $A \to \prod_{i=1}^n A_{g_i}$ injective. This yields a well-behaved notion. [2]

Lemma 1.4. *There is no open affine subset of the line with two origins L containing both points.*

Proof. Let us write $p : L \to R$ for the first projection. Assume there is an open affine subset of the line with two origins such that $\mathrm{fib}_p(0) \subset U \subset L$. Then $p(U) \subset R$ is an open neigbhorhood of 0, as

$$x \in p(U) \leftrightarrow (x, true) \in U \vee (x, false) \in U$$

Claim: the map $R^{p(U)} \to R^U$ is an equivalence. If we have shown that: As U is affine we conclude that the map

$$\begin{aligned} U &\to \mathrm{Spec}(R^{p(U)}) \\ x &\mapsto \phi \mapsto (\phi(px)) \end{aligned}$$

is an equivalence, which is a contradiction to the assumption, that U contains both (distinct!) origins.
Proof of claim: First the Proof idea: As U is a subset of a quotient of $R+R$, the function $U \to R$ determines two (partially defined on open domain) functions to R that coincide away from the origin, which is a regular point. Thus by 1.3 they coincide everywhere. More precisely:
Injectivity: If two maps $f, g : p(U) \to R$ coincide after precomposing with $U \to p(U)$, then they coincide away from 0 so conclude by 1.3.
Surjectivity: Given a map $L \supset U \to R$, by pulling back along $p : R + R \to L$ we can view it as a map $R + R \supset p^{-1}(U) \to R$ defined at both origins, so in particular as a pair of maps to R defined on some open neigbhorhood of 0 in R. They coincide away from 0 so by 1.3 they are equal. □

2 Twisted line with double origin

Lemma 2.1.
- *Any embedding* $\mathsf{Bool} \hookrightarrow \mathrm{Spec}\, C_r$ *is already an equivalence*
- *If* $2 \neq 0$, $\|\mathrm{Spec}\, C_r\| \leftrightarrow \|\mathsf{Bool} \simeq \mathrm{Spec}\, C_r\|$

Proof.
- Any embedding

$$\begin{aligned} \mathsf{Bool} &\to \mathrm{Spec}\, C_r \\ true &\mapsto i \\ false &\mapsto i' \end{aligned}$$

  is already an equivalence, as for any $x : R$ if $(x + i')(x + i) = 0$ we know that one of the factors is invertible by locality ($i - i' \neq 0$) and the other is zero.

- '$\leftarrow$' Obvious
  '$\rightarrow$' Because $i \neq -i$, this determines an embedding.

□

Lemma 2.2. *Consinder an open subset $U \subset \mathrm{Spec}\, C_r$, such that $\neg(U = \mathrm{Spec}\, C_r)$. Then U is an open proposition.*

Proof. Note, that U is a proposition: If $x, x' : U$, then $x = x' \simeq \neg\neg(x = x')$ by decidable equality of U, using that $\mathrm{Spec}\, C_r$ is a formally étale affine. But if $x \neq x'$, then $\{x, x'\} \hookrightarrow \mathrm{Spec}\, C_r$ is an embedding, so by 2.1 an equivalence, but then $U = \mathrm{Spec}\, C_r$, contradiction.

We first reduce to the case where U is a principal open of $\operatorname{Spec} C$. By [1] we find $f_1, \ldots, f_n : C_r$ such that $U = \bigcup_{i=1}^n D(f_i)$. As the left hand side is a proposition we have

$$U \leftrightarrow \bigvee_{i=1}^n D(f_i)$$

so we may show, that each $D(f_i) \subset \operatorname{Spec} C_r$ is an open proposition.
Let $f : C_r$ such that $D(f)$ is a proposition. Choose a representative $a + bT : R[T]$.
Let us show $(2a \neq 0) \leftrightarrow D(f)$, which is a modal proposition, as open propositions are $\neg\neg$-stable, thus modal. By (L1) we may assume $x : \operatorname{Spec} C_r$. Using that $D(f) \subset \operatorname{Spec} C_r = \{x, -x\}$ and that $D(f)$ is a proposition we have

$$D(f) = (a + bx \neq 0) + (a - bx \neq 0) \xrightarrow{\sim} (a + bx \neq 0) \vee (a - bx \neq 0) \tag{1}$$

We may show both implications $2a \neq 0 \leftrightarrow (a + bx \neq 0) \vee (a - bx \neq 0)$.
$'\rightarrow'$ $(a + bx) + (a - bx)$ is invertible, so by locality one of the summands is invertible.
$'\leftarrow'$ by symmetry wlog $a + bx \neq 0$. Then by the first equation of (1) and the fact that $D(f)$ is a proposition, $\neg\neg(a - bx = 0)$. Thus $\neg\neg(a + a = a + bx \neq 0)$, hence $2a \neq 0$. □

The rest of this section is devoted for the proof of 0.1.

$\mathcal{B} \to \mathcal{C}$ we have $p : X_r \to R$ the first projection so we may use $Y_x :\equiv \operatorname{fib}_p x$ and the evident equivalence $\operatorname{fib}_p(0) \simeq \operatorname{Spec} C_r$. There is no open affine subset of X_r containing $\operatorname{fib}_p(0)$: Indeed as the goal is $\neg\neg$-stable, it is modal by L2. So we may assume X_r beeing the line with two origins, using (L1). So we can conclude by 1.4.

$\mathcal{A} \to \mathcal{B}$ if $\| \operatorname{Spec} C_r \|$, then X_r is a scheme:
$\sum_{x:R} \mathsf{Bool}^{x=0}$ is the line with two origins by 1.2, which is known to be a scheme. So by 2.1, $X_r \equiv \sum_{x:R} (\operatorname{Spec} C_r)^{x=0}$ is a scheme as well.

$\mathcal{C} \to \mathcal{A}$ Let $p : X \to R$ be a map out of a scheme that comes with an equivalence $\operatorname{fib}_p(0) \simeq \operatorname{Spec} C_r$, such that X does not admit an open affine neigbhorhood of $\operatorname{fib}_p(0)$. We wish to show $\| \operatorname{Spec} C \|$. Any finite open affine cover of X can be restricted to a finite open affine cover $\operatorname{Spec} C_r = \bigcup_{j=0}^n U_j$ of the basefiber $\operatorname{Spec} C$ consisting of strictly smaller open subsets, using the assumption that $\operatorname{fib}_p(0)$ does not have an open affine neighborhood. Then the goal is

$$\| \operatorname{Spec} C_r \| = \| \bigcup_{j=0}^n U_j \| \leftrightarrow \| \sum_{j=0}^n U_j \| = \bigvee_j U_j$$

an open propoosition by 2.2, thus $\neg\neg$-stable, hence modal by (L2). So it is inhabited, as $L\| \operatorname{Spec} C_r \|$ is inhabited (L1).

References

[1] Felix Cherubini, Thierry Coquand, and Matthias Hutzler. *A Foundation for Synthetic Algebraic Geometry*. 2023. arXiv: 2307.00073 [math.AG]. URL: https://arxiv.org/abs/2307.00073.

[2] Tim Lichtnau. "Higher Geometric Stacks in SAG". In: (2024). URL: https://raw.githubusercontent.com/timlichtnau/MasterThesis/Main/main.pdf.