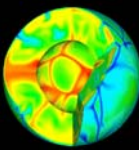- Ray theory: basic principles

  Wavefronts, Huygens principle, Fermat's principle, Snell's Law
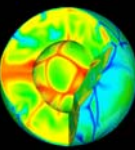
- Rays in layered media

  Travel times in a layered Earth, continuous depth models,
  Travel time diagrams, shadow zones, Abel's Problem, Wiechert-Herglotz
  Problem

- Travel times in a spherical Earth

  Seismic phases in the Earth, nomenclature, travel-time curves for
  teleseismic phases

- Ray definition

  Rays are defined as the normals to the wavefront and thus point in the direction of propagation.

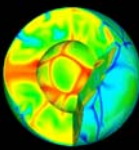- Rays in smoothly varying or not too complex media

  Rays corresponding to P or S waves behave much as light does in materials with varying index of refraction: rays bend, focus, defocus, get diffracted, birefringence et.

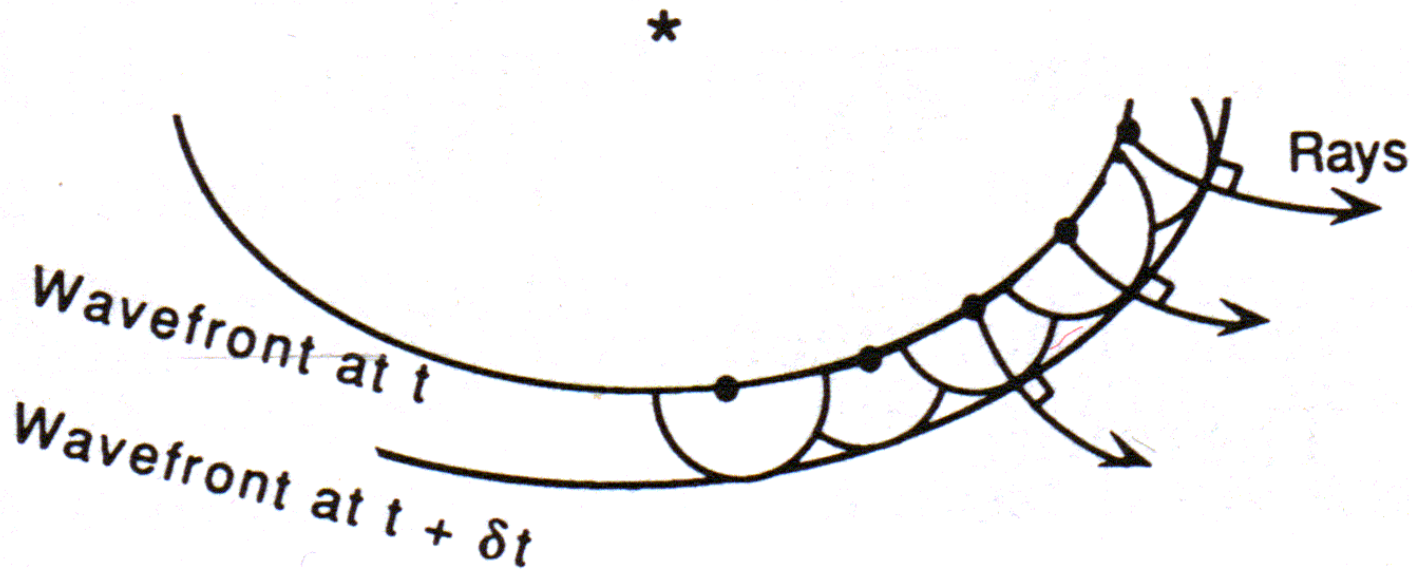- Ray theory is a high-frequency approximation

  This statement is the same as saying that the medium (apart from sharp discontinuities, which can be handled) must vary smoothly compared to the wavelength.
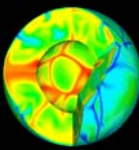
Huygens principle states that each point on the wavefront serves as a secondary source. The tangent surface of the expanding waves gives the wavefront at later times.
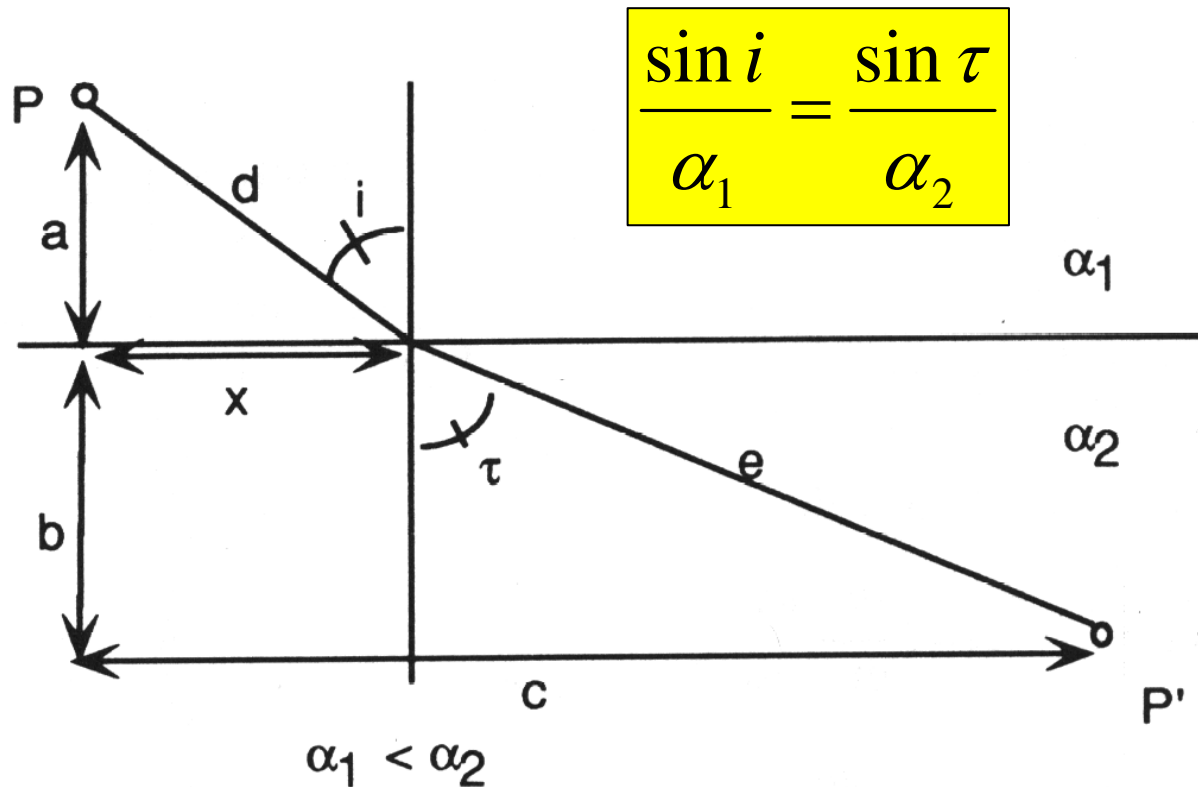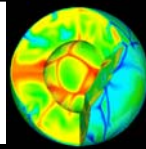
Fermat's principle governs the geometry of the raypath. The ray will follow *a minimum-time* path. From Fermat's principle follows directly Snell's Law
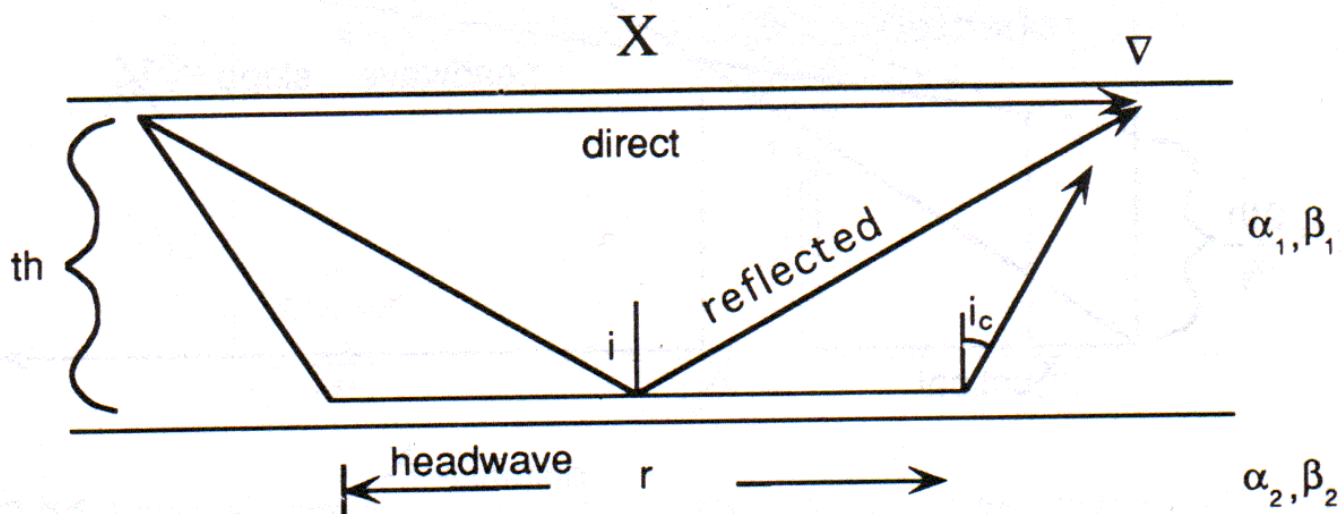
$$\frac{\sin i}{\alpha_1} = \frac{\sin \tau}{\alpha_2}$$
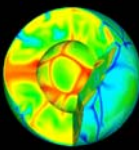

$\alpha_1 < \alpha_2$

Much information can be learned by analysing recorded seismic signals in terms of layered structured (e.g. crust and Moho). We need to be able to predict the arrival times of reflected and refracted signals ...

... the rest is geometry ...

Let us calculate the arrival times for reflected and refracted waves as a function of layer depth d and velocities α_i i denoting the i-th layer:
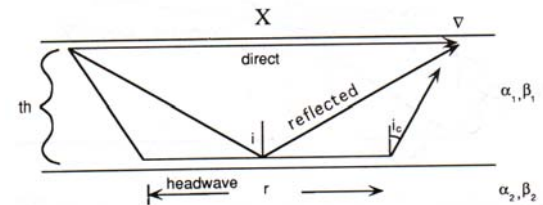
We find that the travel time for the reflection is

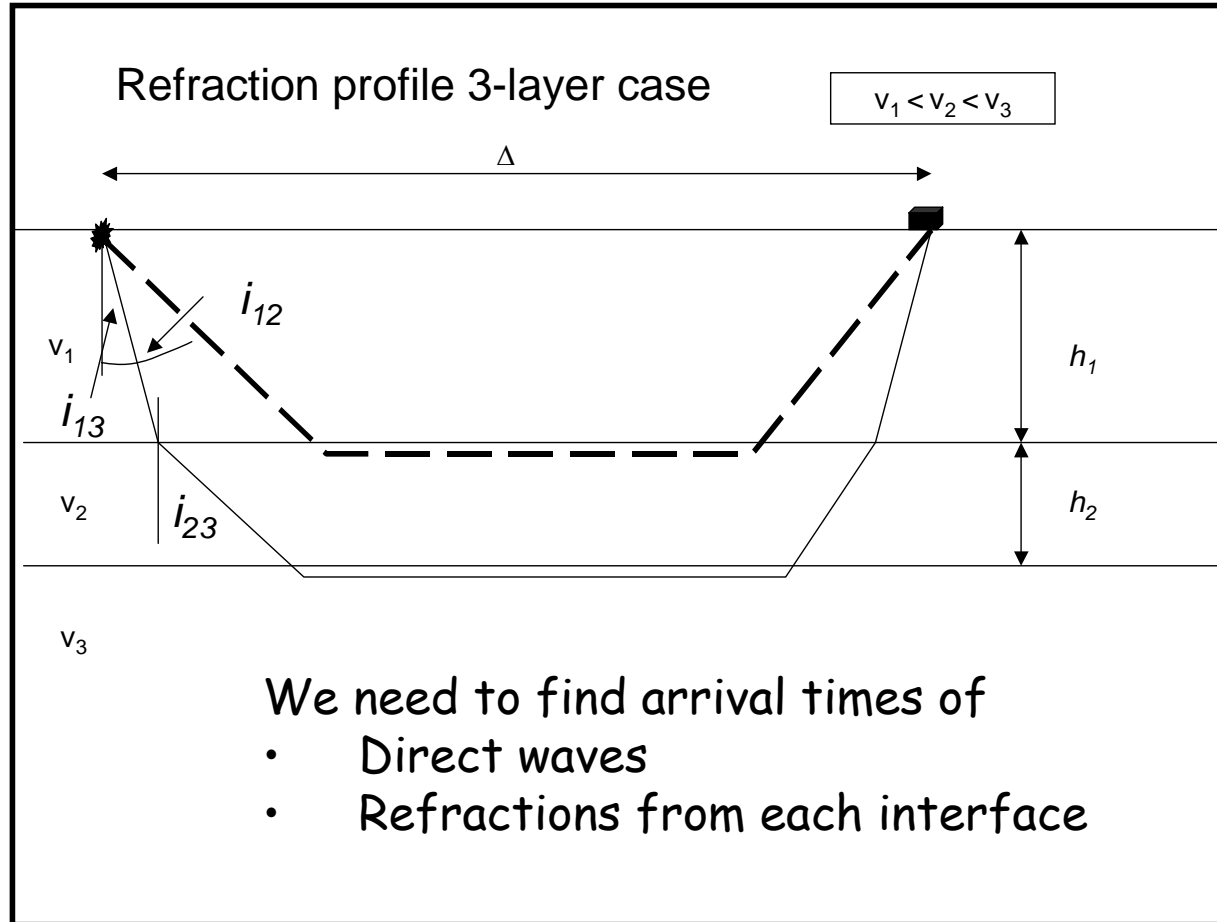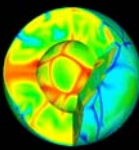$$T_{refl} = \frac{2d}{\alpha_1 \cos i}$$

And the refraction

$$T_{refr} = \frac{2d}{\alpha_1 \cos i} + \frac{r}{\alpha_2}$$
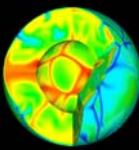
$$r = X - 2d \tan i_c$$

Refraction profile 3-layer case

$v_1 < v_2 < v_3$

Δ

i_{12}

v_1

h_1

i_{13}

v_2

i_{23}

h_2

v_3

We need to find arrival times of
- Direct waves
- Refractions from each interface

Direct wave

$$t_1 = \Delta / v_1$$

Refraction Layer 2

$$t_2 = \frac{2h_1 \cos i_{12}}{v_1} + \frac{\Delta}{v_2} = t^{i2} + \frac{\Delta}{v_2}$$

Refraction Layer 3

$$t_3 = \underbrace{\frac{2h_1 \cos i_{13}}{v_1} + \frac{2h_2 \cos i_{23}}{v_2}}_{t^{i3}} + \frac{\Delta}{v_3} = t^{i3} + \frac{\Delta}{v_3}$$
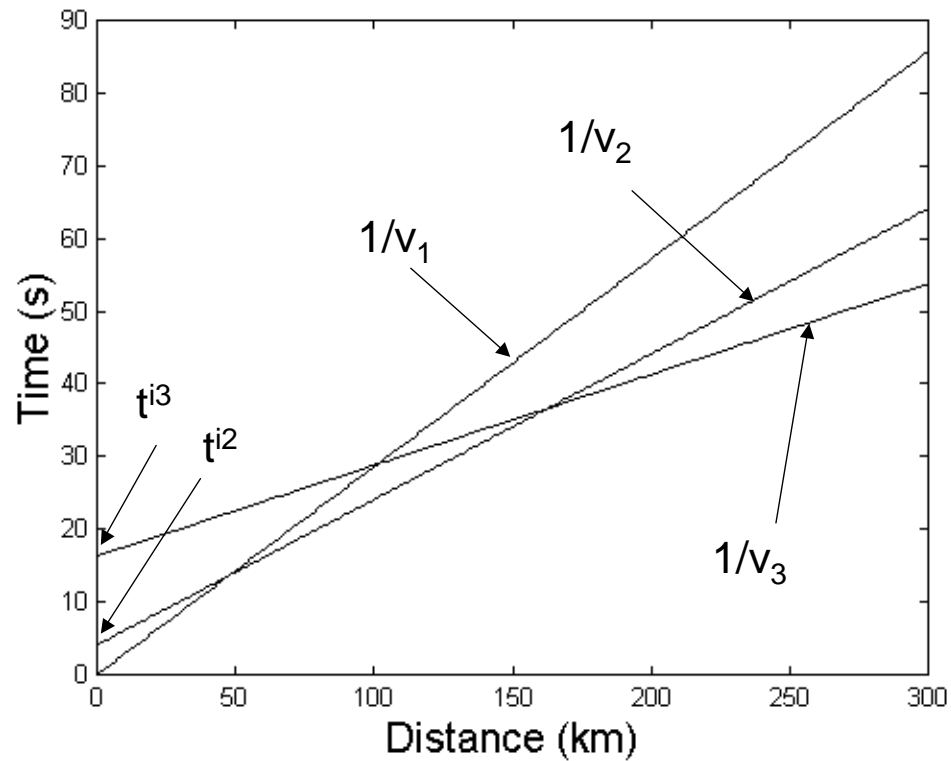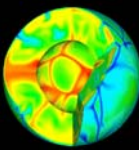
using ...

$$\frac{\sin i_{13}}{v_1} = \frac{\sin i_{23}}{v_2} = \frac{\sin i_{33}}{v_3} = \frac{1}{v_3}$$
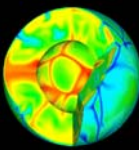
Travel-time diagram for the 3-layer case

Thus the refracted wave arrival is

$$T_{refr} = \frac{2d}{\alpha_1 \cos i_c} + \frac{1}{\alpha_2}\left(X - \frac{2d\alpha_1}{\alpha_2 \cos i_c} \right)$$
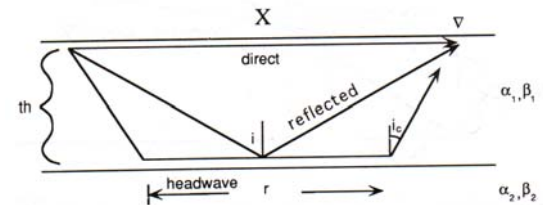
where we have made use of Snell's Law.

We can rewrite this using

$$1/\alpha_2 = p \qquad \cos i_c = (1 - \sin^2 i_c)^{1/2} = (1 - p^2\alpha_1^2)^{1/2} = \eta_1$$
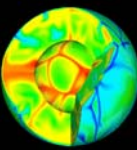
to obtain

$$T_{refr} = Xp + 2d\eta_1$$

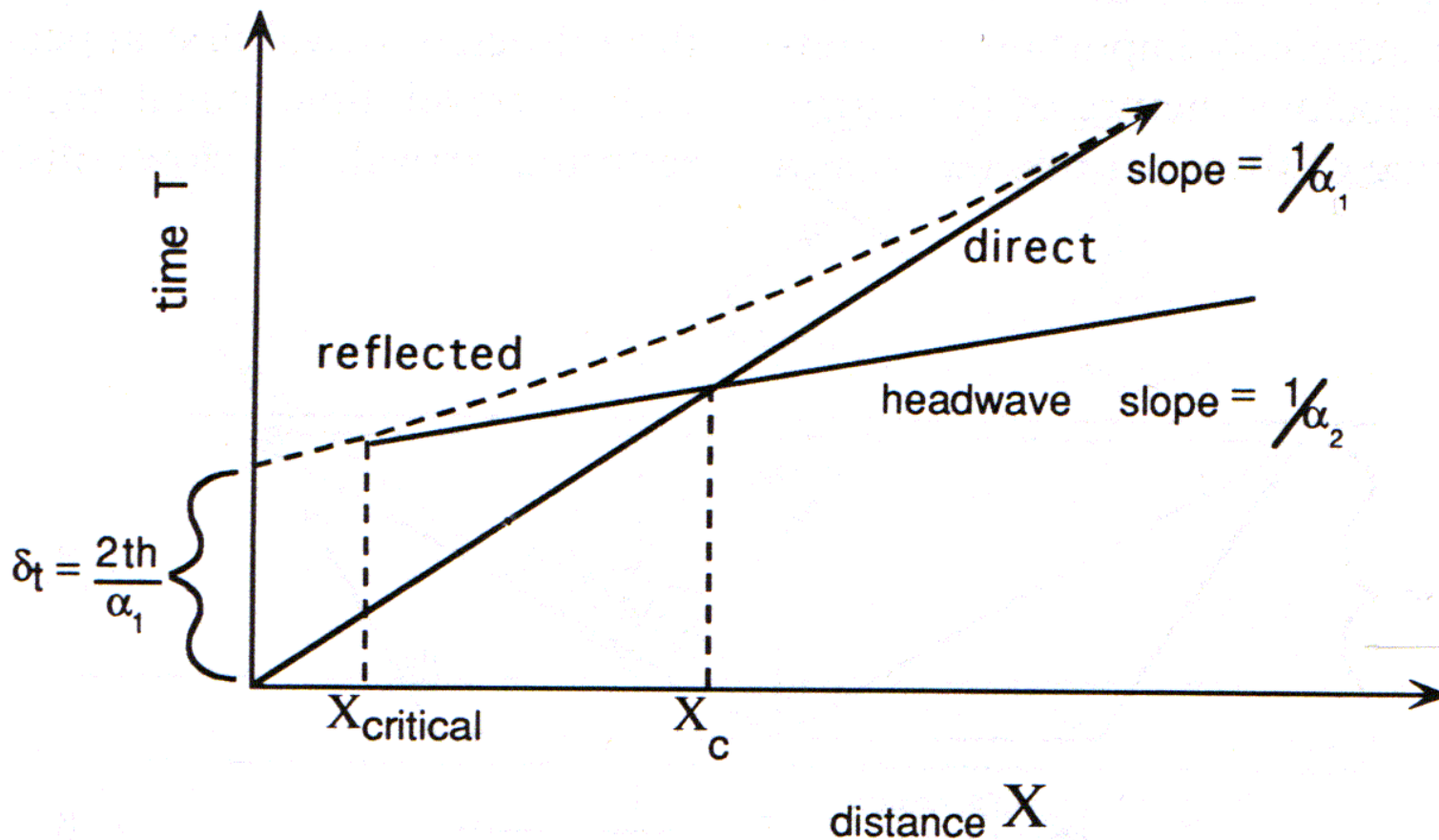Which is very useful as we have separated the result into a vertical and horizontal term.
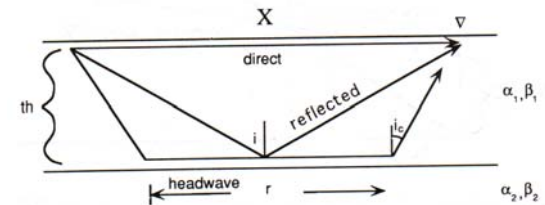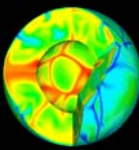
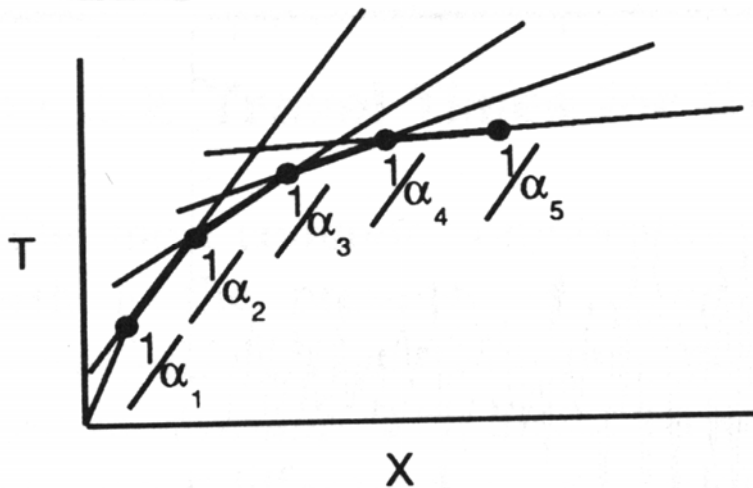What can we determine if we have recorded the following travel time curves?

Generalization to many layers

The previous relation for the travel times easily generalizes to many layers:

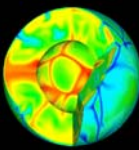$$T_{refr} = Xp + \sum_{i=1}^{n} 2d_i \eta_i$$

Travel time curve for a finely layered Earth. The first arrival is comprised of short segments of the head wave curves for each layer.

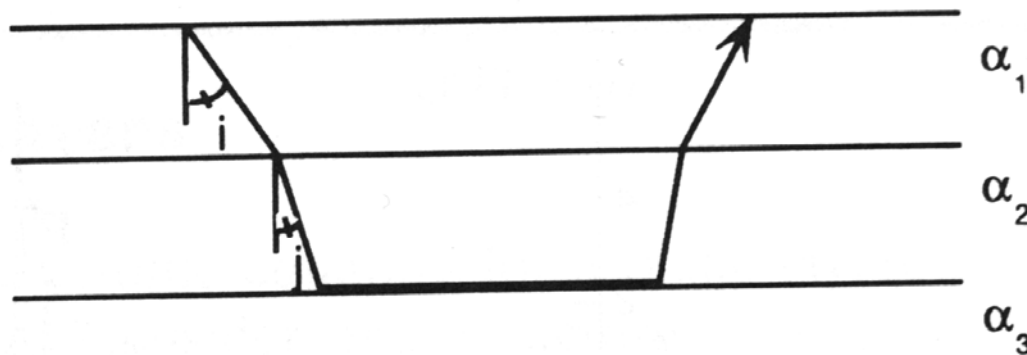This naturally generalizes to infinite layers i.e. to a continuous depth model.

What happens if we have a low-velocity zone?

Then no head wave exists on the interface between the first and second layer.

$$\alpha_2 < \alpha_1 < \alpha_3, \quad i_c = \sin^{-1}\left(\alpha_1/\alpha_3\right)$$
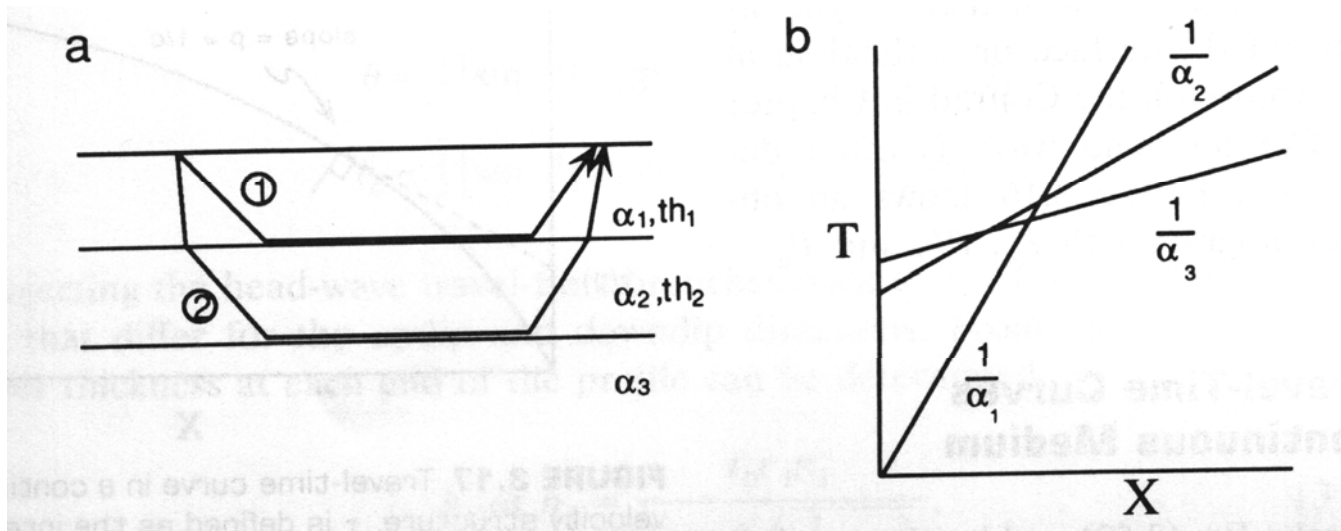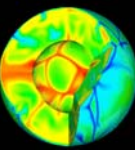
In this case only a refracted wave from the lower half space is observed. This could be misinterpreted as a two layer model. In such cases this leads to an overestimation of the depth of layer 3.
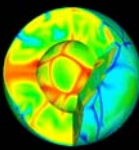
The situation may arise that a layer is so thin that its head wave is never a first arrival.

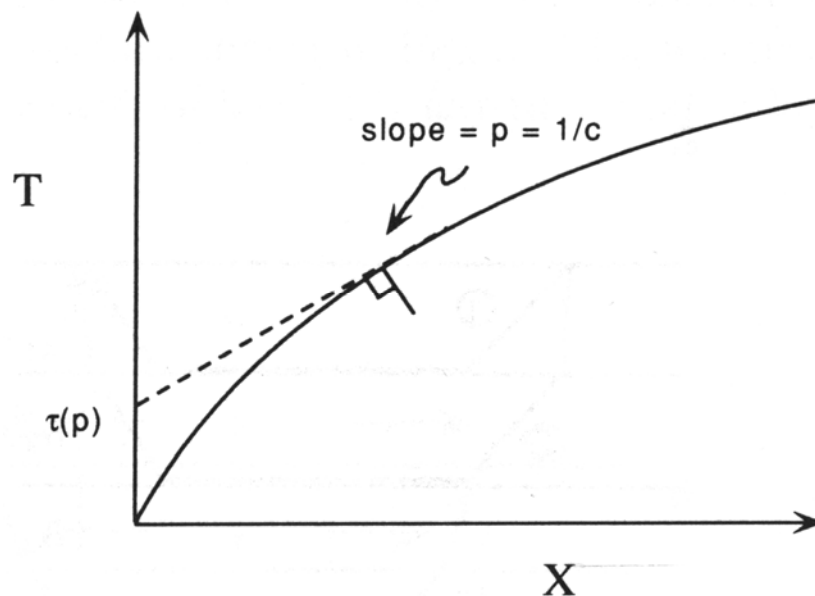From this we learn that the observability of a first arrival depends on the layer thickness **and** the velocity contrast.
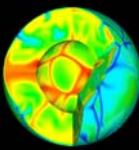
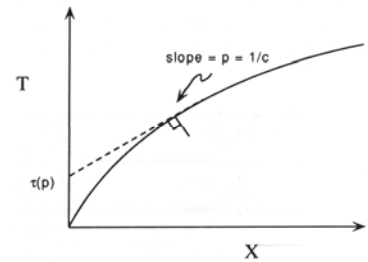We now let the number of layers go to infinity and the thickness to zero. Then the summation is replaced by integration.

Now we have to introduce the concept of intercept time τ of the tangent to the travel time curve and the slope p.
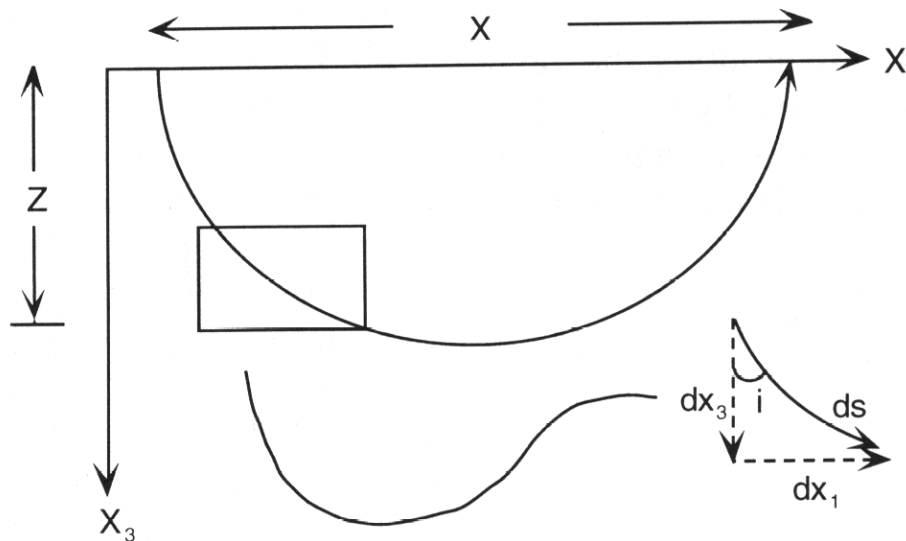
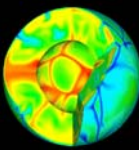Let us assume we know (observe) the travel time as a function of distance X. We then can calculate the slope $dT/dX=p=1/c$.

Let us first derive the equations for the travel time in a flat Earth. We have the following geometry (assuming increasing velocities):
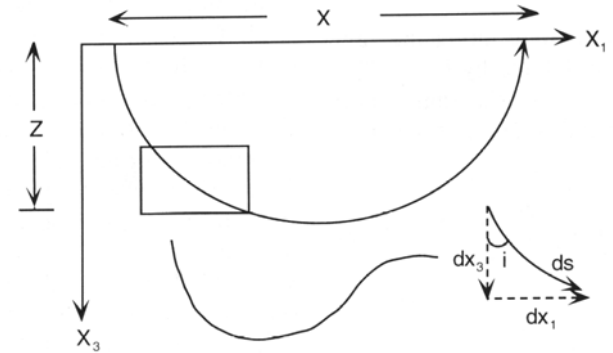
At each point along the ray we have

$$\sin i = \frac{dx}{ds} = cp$$

Remember that the ray parameter *p* is constant. In this case *c* is the *local* velocity at depth. We also make use of
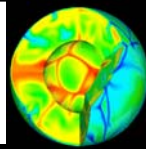
$$\cos i = \frac{dz}{ds} = \sqrt{1 - \sin^2 i} = \sqrt{1 - c^2 p^2}$$

$$\Rightarrow dx = ds \sin i = \frac{dz}{\cos i} cp$$

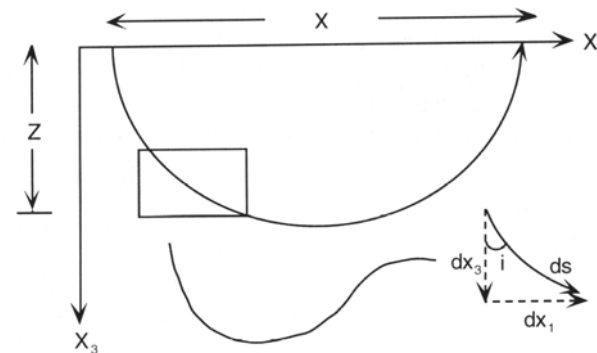$$\Rightarrow dx = \frac{cp}{\sqrt{1 - c^2 p^2}} dz$$

Now we can integrate over depth

$$X = 2\int_0^z \frac{cp}{\sqrt{1-c^2 p^2}}\, dz$$

This equation allows us to predict the distance a ray will emerge for a given p (or emergence angle) and velocity structure, but how long does the ray travel?
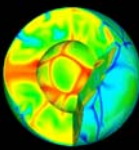
Similarly

$$dT = \frac{ds}{c} \Rightarrow T = \int_{path} \frac{ds}{c(s)} = 2\int_0^z \frac{dz}{c(z)\cos i}$$

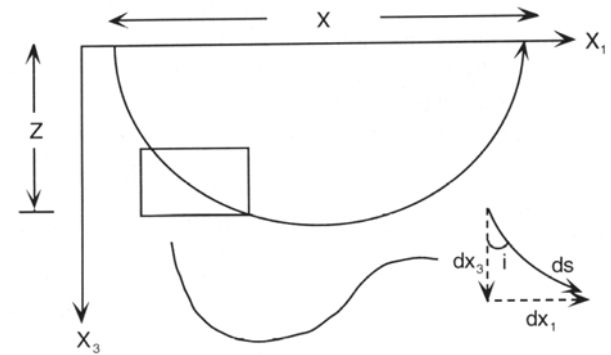$$T = 2\int_0^z \frac{dz}{c^2 \sqrt{1/c^2 - p^2}}$$

This can be rewritten to

$$T = pX + 2\int_0^z \sqrt{1/c^2(z) - p^2}\, dz$$

Remember this is in the same form as what we obtained for a stack of layers.
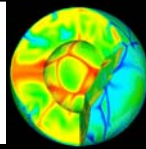
$$T_{refr} = pX + 2\sum_{i=1}^n d_i \eta_i$$

Let us now get back to our travel time curve we have

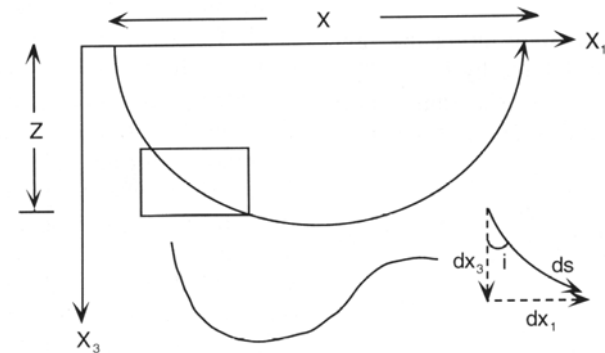$$\tau(p) = T - pX = 2\int_0^z \sqrt{1/c^2(z) - p^2}\, dz$$
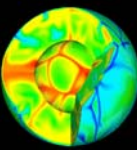
The intercept time is defined at X=0, thus

$$\frac{d\tau}{dp} = \frac{d}{dp}\left(2\int_0^z \sqrt{1/c^2(z) - p^2}\, dz \right)$$

$$= 2\int_0^z \frac{-p}{\sqrt{1/c^2(z) - p^2}}\, dz$$

$$= -X$$

As p increases (the emergence angle gets smaller) X decreases and τ will decrease. Note that $\tau(p)$ is a single valued function, which makes it easier to analyze than the often multi-valued travel times.
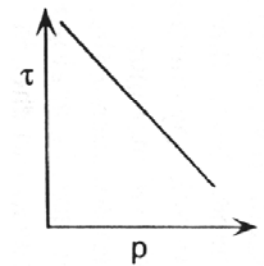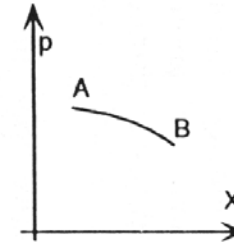
Velocity Model — Ray Paths — Travel Time — p vs X — τ vs p
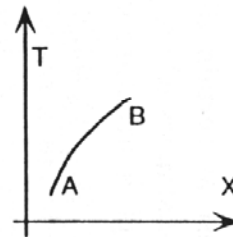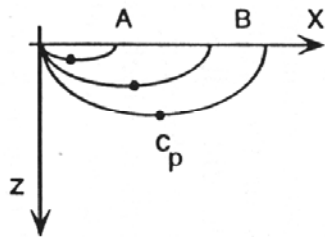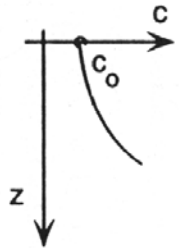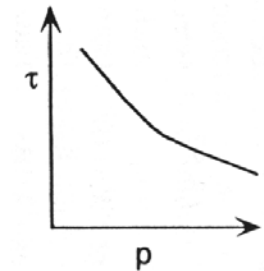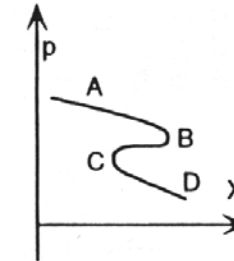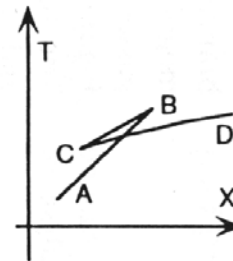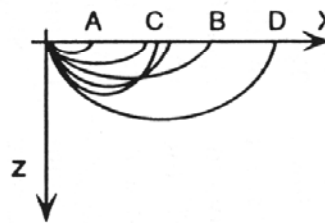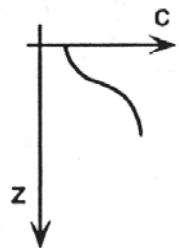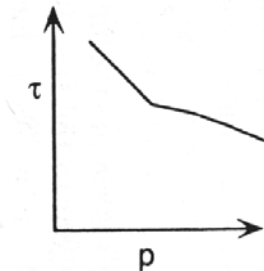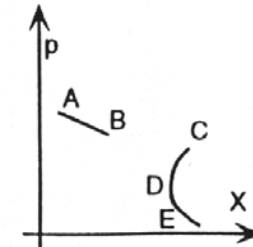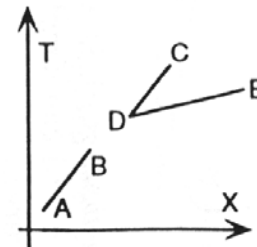
It seems that now we have the means to predict arrival times and the travel distance of a ray for a given emergence angle (ray parameter) and given structure. This is also termed a *forward problem.*

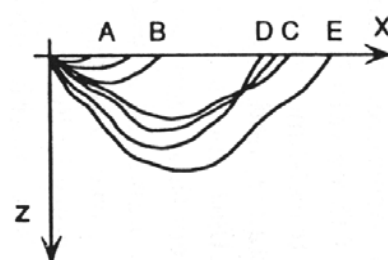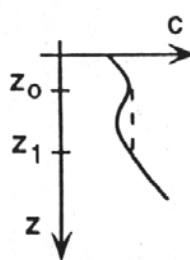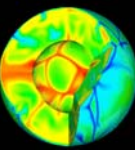But what we really want is to solve the *inverse problem.* We have recorded a set of travel times and we want to determine the structure of the Earth.
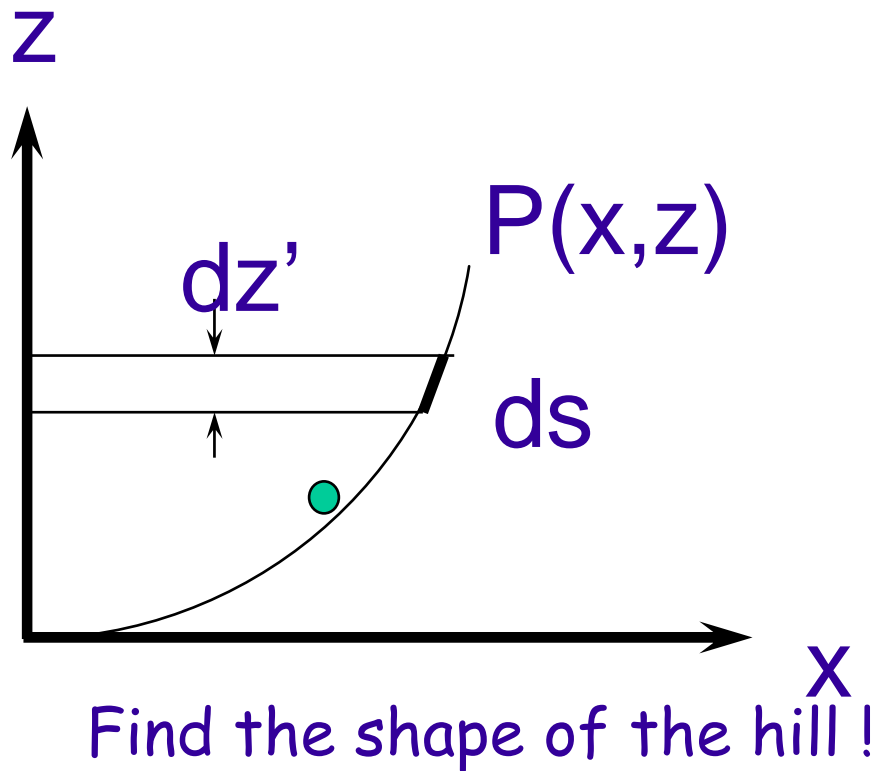
In a very general sense we are looking for an Earth model that minimizes the difference between a theoretical prediction and the observed data:

$$\sum_{traveltimes} T_{obs} - T_{theory}(m) = Min!$$

where m is an Earth model. For the problem of travel times there is an interesting analogy: Abel's Problem

z

P(x,z)

dz'

ds

Find the shape of the hill !

For a given initial velocity and measured time of
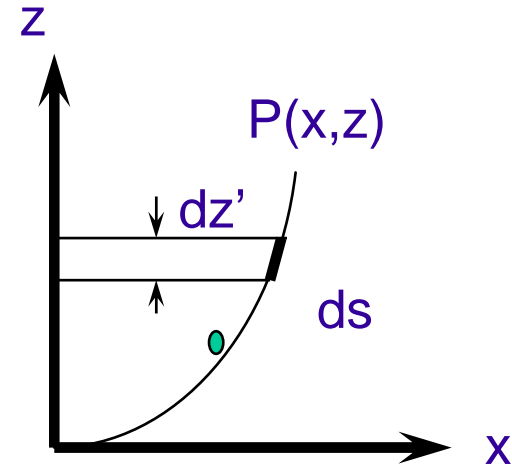the ball to come back to the origin.

X

z

P(x,z)

dz'

ds

x
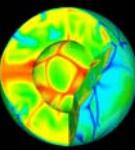
At any point:

$$gz = \frac{1}{2}v_0^2$$

At z-z':

$$-mg(z-z') = \frac{1}{2}m(ds/dt)^2$$

After integration:

$$t(z) = \int_0^z \frac{ds/dz'}{\sqrt{2g(z-z')}}dz'$$

$$t(z) = \int_0^z \frac{ds/dz'}{\sqrt{2g(z-z')}} dz'$$

z

$P(x,z)$

dz'

ds

x
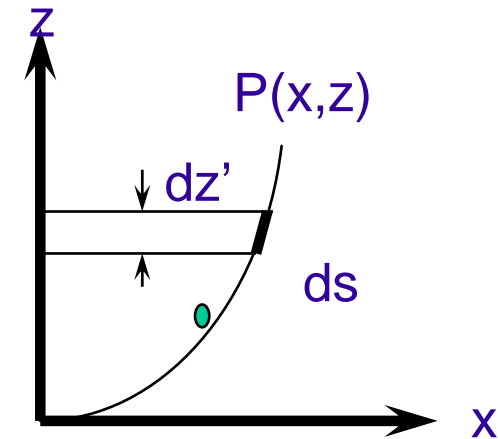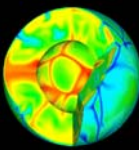
After change of variable and integration, and...
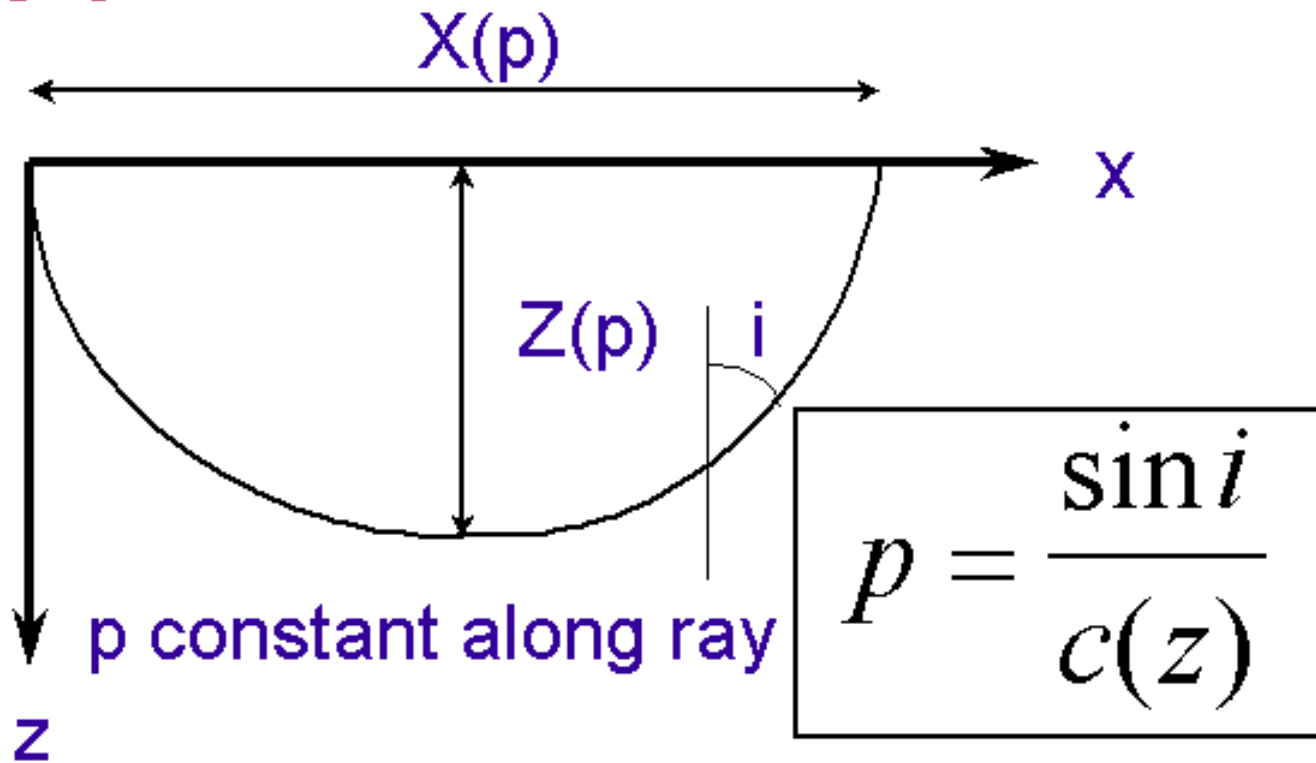
$$f(z') = -\frac{1}{\pi} \frac{d}{dz'} \int_{z'}^a \frac{t(z)dz}{\sqrt{z-z'}}$$

Ray path in a flat Earth model

X(p)

Z(p) i

$$p = \frac{\sin i}{c(z)}$$

p constant along ray
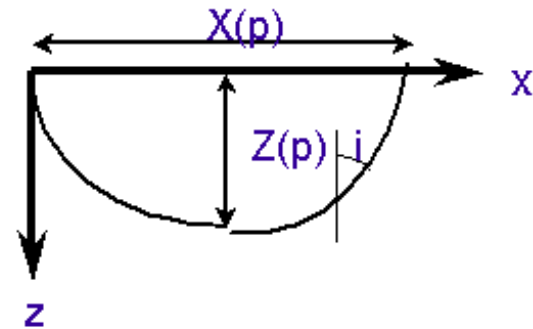
z

Find seismic velocities c(z) for observed X(p)

Wiechert-Herglotz

$$X(p) = 2 \int_0^{Z(p)} \tan i \, dz$$

using p=sin$i$/c(z) ...

$$X(p) = 2 \int_0^{Z(p)} \frac{p \, dz}{\sqrt{c(z)^{-2} - p^2}}$$

Wiechert-Herglotz
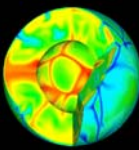
$$X(p) = 2 \int_0^{Z(p)} \frac{p\,dz}{\sqrt{c(z)^{-2} - p^2}}$$

... or in terms of travel times

$$T(p) = 2 \int_0^{Z(p)} \frac{c(z)^{-2}\,dz}{\sqrt{c(z)^{-2} - p^2}}$$

Wiechert-Herglotz

$X(p)$

$Z(p)$
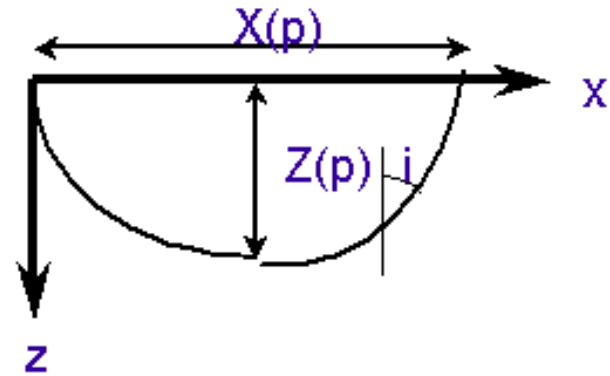
$$X(p) = 2 \int_0^{Z(p)} \frac{p\,dz}{\sqrt{c(z)^{-2} - p^2}}$$

… after some manipulation …

$$z(c) = -\frac{1}{\pi} \int_{c_0^{-1}}^{c^{-1}} \frac{X(p)}{\sqrt{p^2 - c^{-2}}}\,dp$$

Wiechert-Herglotz

$X(p)$

$Z(p) \!-\! i$

x

z
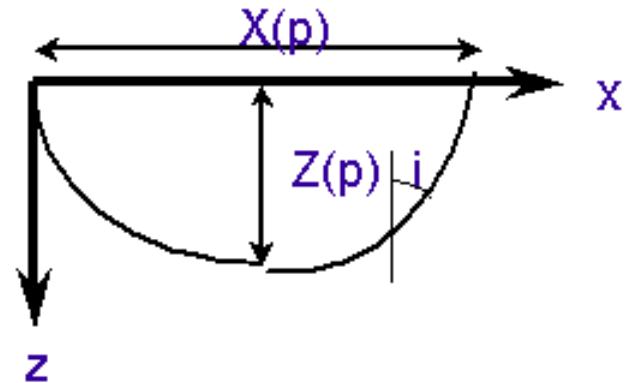
...conditions for $X(p)$ and $c(z)$

• derivative of $X(p)$ may be discontinuous
• $X(p)$ must be continuous
• no low velocity channels
• rapid velocity increase is allowed

How can we generalize these results to a spherical Earth which should allow us to *invert* observed travel times and find the internal velocity structure of the Earth?
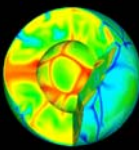
Snell's Law applies in the same way:

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_{1'}}{v_2}$$

From the figure it follows

$$\frac{r_1 \sin\theta_1}{v_1} = \frac{r_2 \sin\theta_2}{v_2}$$

which is a general equation along the raypath (i.e. it is constant)

... thus the ray parameter in a spherical Earth is defined as :
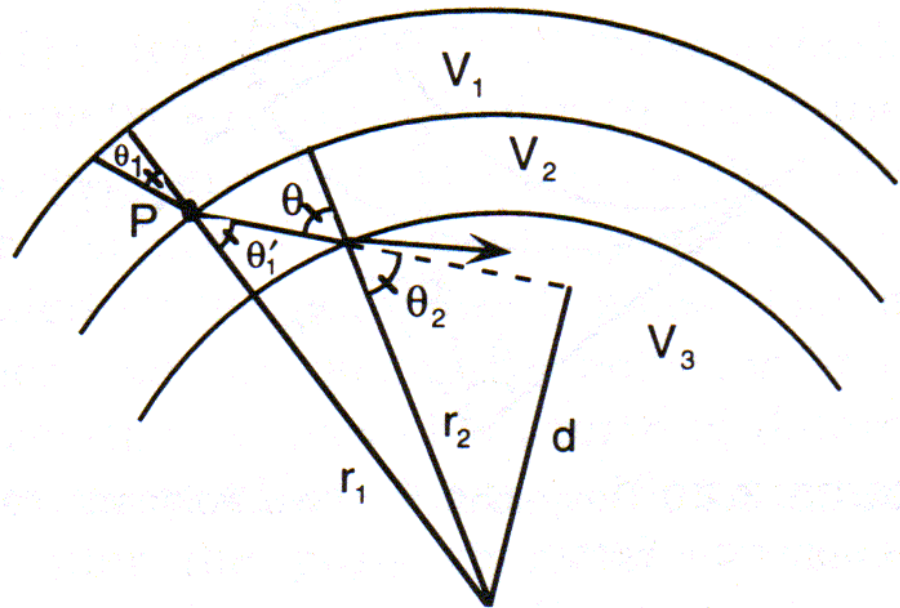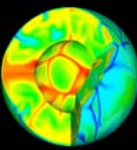
$$\frac{r \sin \theta}{v} = p$$

Note that the units (s/rad or s/deg) are different than the corresponding ray parameter for a flat Earth model.

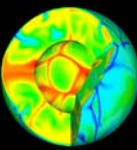The meaning of p is the same as for a flat Earth: it is the slope of the travel time curve.

$$p = \frac{dT}{d\Delta}$$

The equations for the travel distance and travel time have very similar forms than for the flat Earth case!

Flat

$$X = 2\int_0^z \frac{cp}{\sqrt{1 - c^2 p^2}}\, dz$$

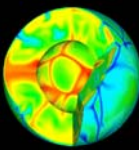$$T = 2\int_0^z \frac{dz}{c^2\sqrt{1/c^2 - p^2}}$$

Spherical

$$\Delta = 2\int_{r_1}^{r_0} \frac{cp}{r\sqrt{r^2 - c^2 p^2}}\, dr$$

$$T = 2\int_{r_1}^{r_0} \frac{r^2 dr}{c^2 r\sqrt{r^2/c^2 - p^2}}$$

Analogous to the flat case the equations for the travel time can be seperated into the following form:

Flat

Spherical

$$T = pX + 2\int_{0}^{z} \sqrt{1/c^2(z) - p^2}\, dz$$

$$T = p\Delta + 2\int_{r_0}^{r_1} \frac{\sqrt{r^2/c^2(z) - p^2}}{r^2}\, dr$$

The first term depends only on the horizontal distance and the second term and the second term only depends on r (z), the vertical dimension.

These results imply that what we have learned from the flat case can directly be applied to the spherical case!