Block SVD Power Method

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1 SVD Power Method Proof

From the definition of SVD the matrix A can be represented like this

$$A \in R^{n \times m} = \sum_{j=1}^{r} \sigma_j u_j v_j^t$$

An iteration of the SVD Power Method looks like this:

$$v^{(0)} \in R^m \Rightarrow v^{(0)} = \sum_{j=1}^{r} y_j v_j$$

$$w^{(1)} = Av^{(0)} = \sum_{j=1}^{r} \sigma_j u_j v_j^t \sum_{i=1}^{r} y_i v_i = \sum_{j=1}^{r} \sigma_j y_j u_j$$

The v terms disappear because v_j is orthogonal to all v_i, $i \in [1, n]$, $i \neq j$

And $v_j^t v_j = 1$

$$u^{(0)} = \frac{w^{(1)}}{\left\|w^{(1)}\right\|} = \alpha_1 w^{(1)} = \alpha_1 \sum_{j=1}^{r} \sigma_j y_j u_j$$

$$z^{(1)} = A^t u^{(0)} = \alpha_1 \sum_{j=1}^{r} v_j u_j^t \sigma_j \sum_{j=1}^{r} \sigma_j y_j u_j = \alpha_1 \sum_{j=1}^{r} \sigma_j^2 y_j v_j$$

$$v^{(1)} = \frac{z^{(1)}}{\left\|z^{(1)}\right\|} = \beta_1 z^{(1)} = \alpha_1 \beta_1 \sum_{j=1}^{r} \sigma_j^2 y_j v_j$$

As the iteration count increases we will have this:

$$u^{(1)} = \alpha_1 \beta_1 \alpha_2 \sum_{j=1}^{r} \sigma_j^3 y_j u_j$$

$$v^{(2)} = \alpha_1 \beta_1 \alpha_2 \beta_2 \sum_{j=1}^{r} \sigma_j^4 y_j v_j$$

...

$$v^{(k)} = \delta_{2k} \sum_{j=1}^{r} \sigma_j^{2k} y_j v_j$$

$$u^{(k)} = \delta_{2k-1} \sum_{j=1}^{r} \sigma_j^{2k-1} y_j u_j$$

$$A\vec{v} = \sigma\vec{u} \quad \wedge \quad A^t\vec{u} = \sigma\vec{v}$$

$$v = v^{(k)} \quad \wedge \quad u = u^{(k)}$$

$$\Rightarrow Av^k = (\sum_{j=1}^{r} u_j v_j^t)\delta_{2k} \sum_{i=1}^{r} \sigma_i^{2k} y_i v_i = \delta_{2k} \sum_{j=1}^{r} \sigma_j^{2k+1} y_j u_j$$

$$\sigma u^k = \sigma\delta_{2k-1} \sum_{j=1}^{r} \sigma_j^{2k+1} y_j u_j$$

If we can prove that $\lim_{k\to\infty} \sigma\delta_{2k+1} = \delta_{2k}$ then the proof is done

$$\left\|u^{(k)}\right\|^2 = 1 = \delta_{2k-1}^2 \sum_{j=1}^{r} \sigma_j^{4k-2} y_j^2$$

u_j disappears because it is a unit vector

$$\left\|v^{(k)}\right\|^2 = 1 = \delta_{2k}^2 \sum_{j=1}^{r} \sigma_j^{4k} y_j^2$$

Let $\sigma = \sigma_1 = \sigma_{max}$

$$\sigma^{4k+2}\delta_{2k+1}^2(y_1^2 + \sum_{i=2}^{r}(\frac{\sigma_j}{\sigma})^{4k+2} y_j^2) = 1$$

$$\sigma^{4k}\delta_{2k}^2(y_1^2 + \sum_{i=2}^{r}(\frac{\sigma_j}{\sigma})^{4k} y_j^2) = 1$$

$$\lim_{k\to\infty}(\frac{\sigma_j}{\sigma})^{4k} = 0$$

$$\Rightarrow \lim_{k\to\infty} \sigma^2 \frac{\delta_{2k+1}^2}{\delta_{2k}^2} = 1$$

σ is positive because it is a singular value and the δs are positive by definition

$$\Rightarrow \lim_{k\to\infty} \sigma\delta_{2k+1} = \delta_{2k} \quad \square$$