# The Mathematics of Origami

Thomas Bertschinger, Joseph Slote, Claire Spencer,
& Samuel Vinitsky

Carleton College

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# Origami? Math?

Image of traditional crane; image of fortune teller

# Origami? Math?

Image of solar panel deployment; image of chemistry microstructure stuff

# Overview

1. Origami Constructions
   - Axioms, Euclid &c.
2. The Basics of Foldability
   - Local Flat Foldability
   - General Foldability
   - Knots
3. Map Folding: An Open Problem
   - Overview
   - Linear Orderings
   - Formal Proof
4. Combinatorics of Origami
   - Finding a good problem
   - Meanders
   - Our Method
   - Partial Solution

# Axioms of Origami

## General

i. Brief History

ii. Independence

# Axioms of Origami

## Seven Axioms of Origami

i. Given two points $p_1$ and $p_2$, we can fold a line connecting them.

ii. Given two points $p_1$ and $p_2$, we can fold $p_1$ onto $p_2$.

iii. Given two lines $l_1$ and $l_2$, we can fold line $l_1$ onto $l_2$.

iv. Given a point $p_1$ and a line $l_1$, we can make a fold perpendicular to $l_1$ passing through the point $p_1$.

# Axioms of Origami

## Seven Axioms of Origami

v. Given two points $p_1$ and $p_2$ and a line $l_1$, we can make a fold that places $p_1$ onto $l_1$ and passes through the point $p_2$.

vi. Given two points $p_1$ and $p_2$ and two lines $l_1$ and $l_2$, we can make a fold that places p1 onto line $l_1$ and places $p_2$ onto line $l_2$.

vii. Given a point $p_1$ and two lines $l_1$ and $l_2$, we can make a fold perpendicular to $l_2$ that places $p_1$ onto line $l_1$.

# Axioms of Origami

## Seven Axioms of Origami

i. The first six axioms allow all quadratic and cubic equations with rational coefficients to be solved.

ii. They also allow two of the three problems of antiquity, the trisection of an angle and the doubling of the cube, to be constructed.

# Greek Problems of Antiquity

## Problems of Antiquity

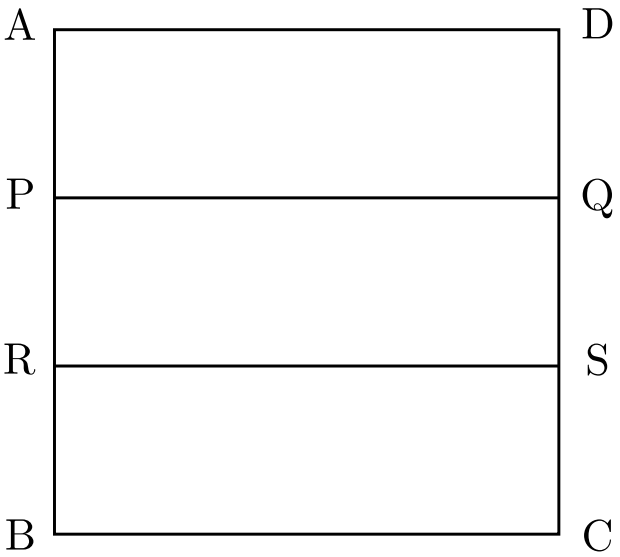
These were a trio of geometric problems whose solutions were attempted solely through the use of compass and straight-edge.

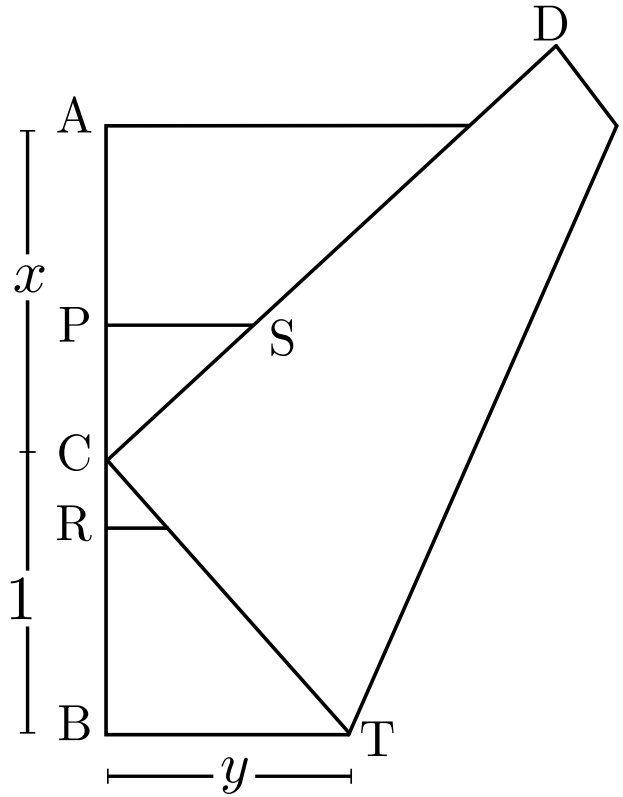
i. Angle Trisection
ii. Cube Duplication
iii. Circle Squaring

# Angle Trisection

INCLUDE PARTIALLY FOLDED DIAGRAMS

# Cube Duplication




i. $\frac{\alpha}{\beta} = \sqrt[3]{2}$

ii. Thus, a cube with a side length $\alpha$ will have twice the volume of a cube with side length $\beta$.

# Squaring the Circle

i. Impossible

ii. $\pi$

# Constructability

## Constructible Numbers

i. Given two points $p_0$ and $p_1$, construct a third point $p_1'$ a distance $|p_0p_1|$ from point $p_0$ such that $\overline{p_0p_1'}$ is perpendicular to $\overline{p_0p_1}$.

ii. Given two points $p_0$ and $p_1$, a third point $p_2$ can be constructed such that $p_2$ is collinear with $p_0$ and $p_1$, thus $|p_0p_1| = |p_1p_2|$.

# Constructability

## Constructible Numbers

iii. Given two constructible numbers $\alpha$ and $\beta$, we can construct $\frac{\alpha}{\beta}$, their ratio.

iv. Given two constructible numbers $\alpha$ and $\beta$, we can construct their sum $\alpha + \beta$ or their difference $\alpha - \beta$.

v. Given two constructible numbers $\alpha$ and $\beta$, we can construct $\alpha\beta$, their product.

# Constructability

Given two constructible numbers $\alpha$ and $\beta$, we can construct $\frac{\alpha}{\beta}$, their ratio.
INCLUDE PARTIALLY FOLDED DIAGRAMS AS WELL AS FINAL FOLDED DIAGRAM

# Constructability

## Constructible Numbers

i. Since the set of constructible numbers is closed under addition, subtraction, multiplication, and division, it can be concluded that the set of constructible numbers form a field.

ii. Ultimately, the field of origami constructible numbers are closed under taking both square roots and cube roots.

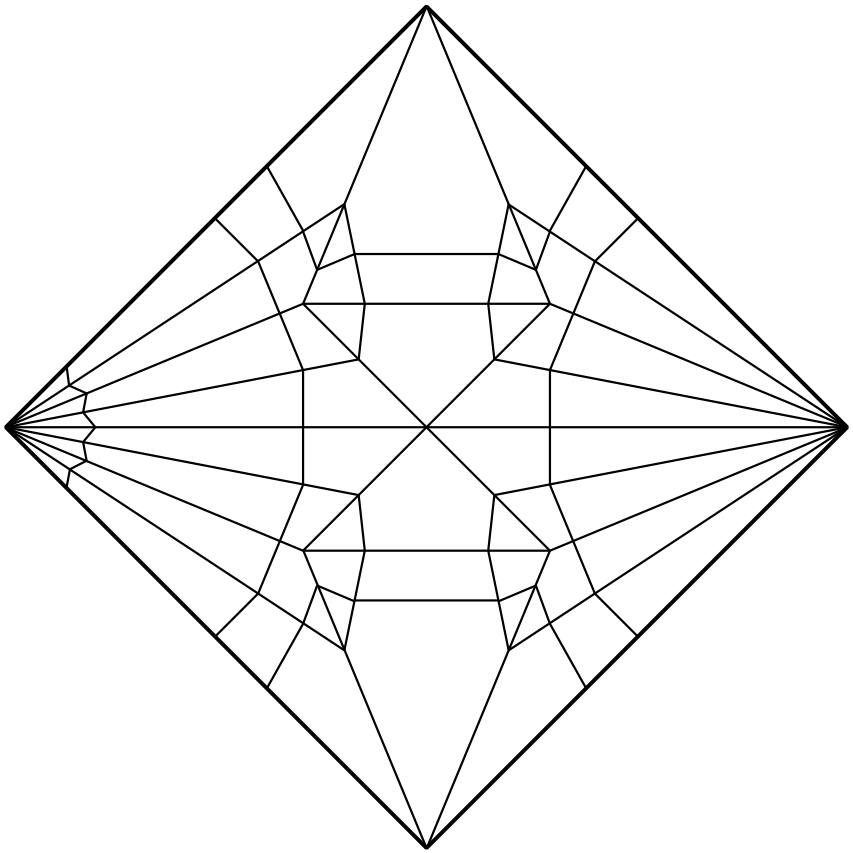
iii. The construction of the square root of any constructible number implies the field of origami constructible numbers contains the field of compass and straightedge constructible numbers.

# Future Work

i. Efficiency
ii. Optimality

# Foldability

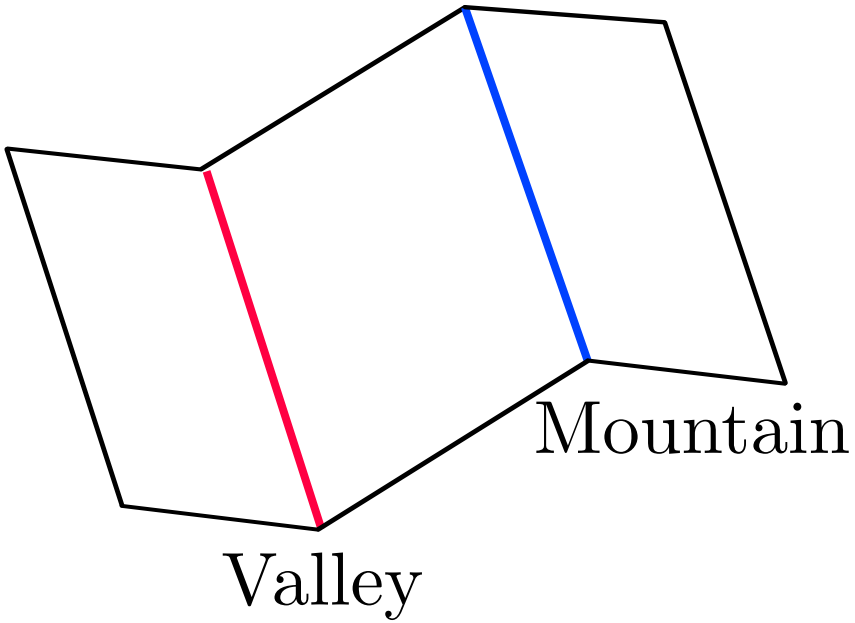
An example of a crease pattern:



# Foldability

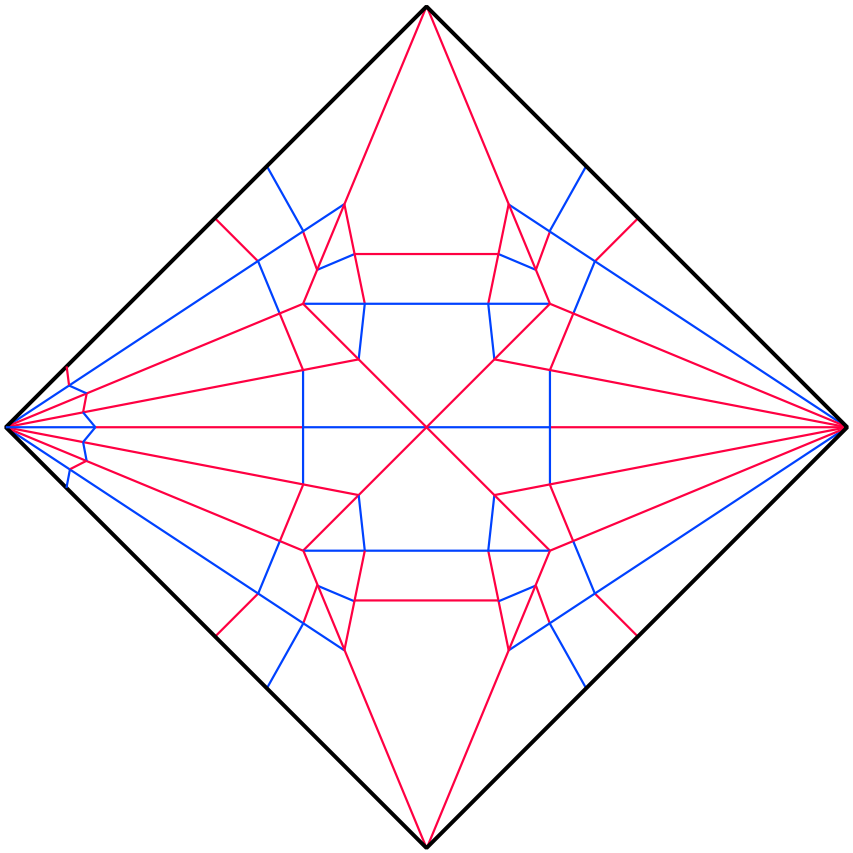
## Crease Assignments

A crease pattern doesn't contain all the information about a model, however.



# Foldability

A crease pattern with mountain-valley assignment:



# Foldability

## Two questions

For a given crease pattern,

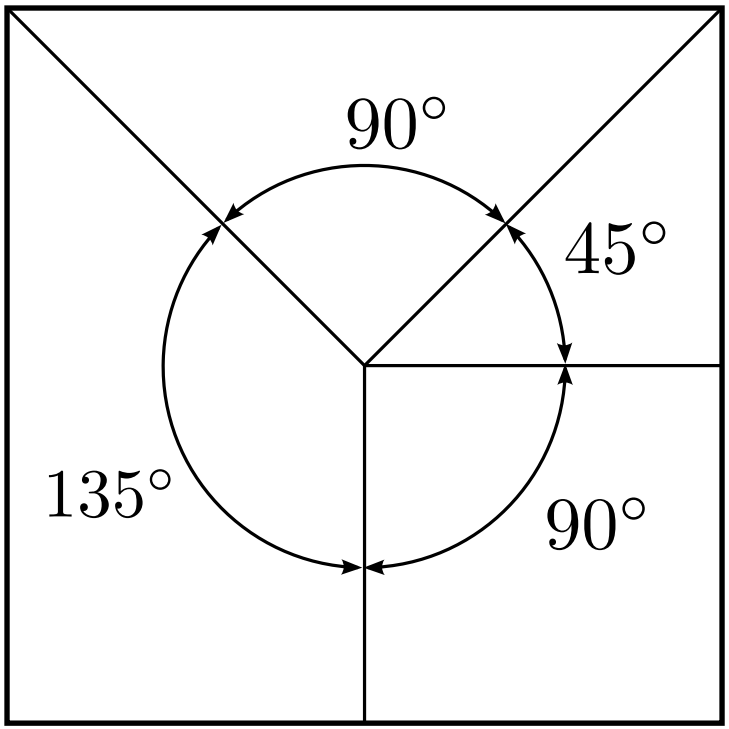
i. is there a way to fold it flat?

ii. is there a way to fold it at all?

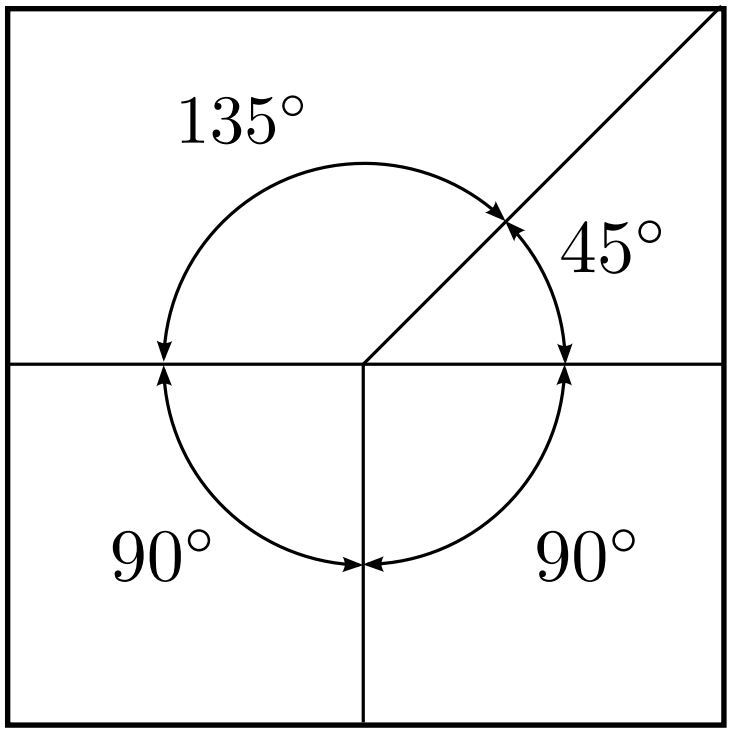
# Flat-Foldability Conditions

## Kawasaki's Theorem (1989)

The alternate angles around a vertex must sum to 180°.

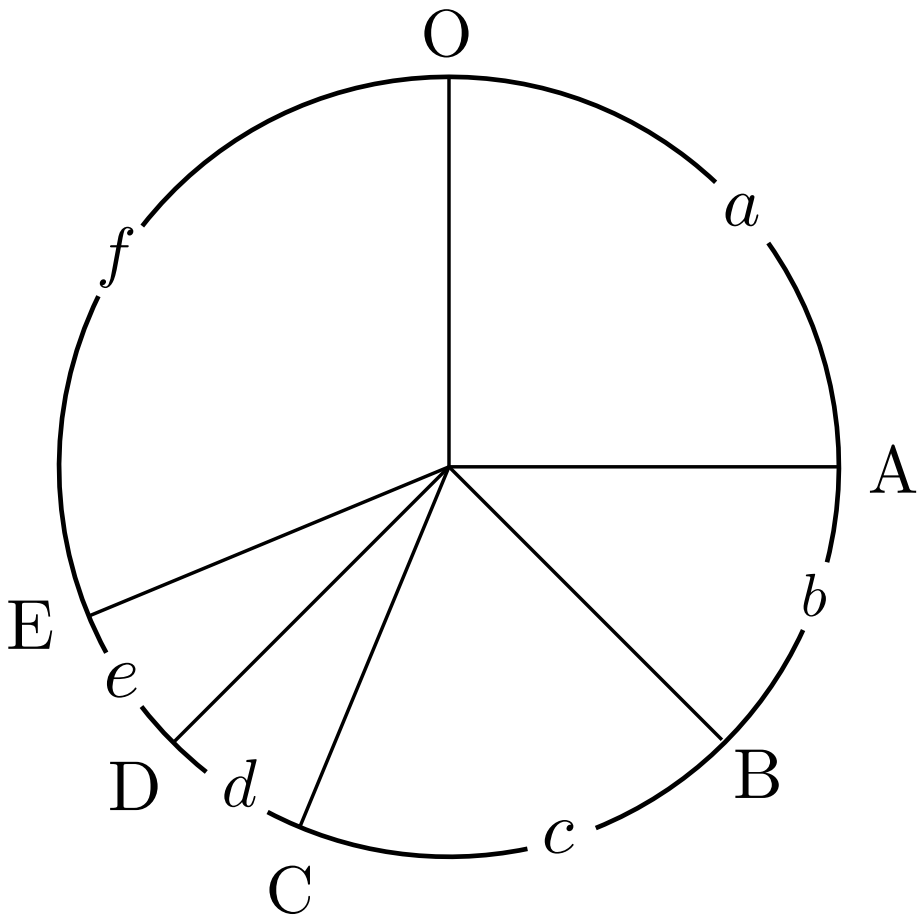



(a)




(b)

# Kawasaki's Theorem

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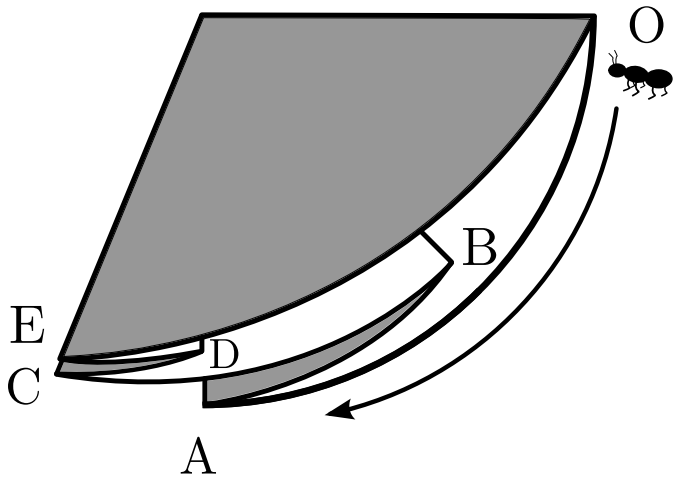
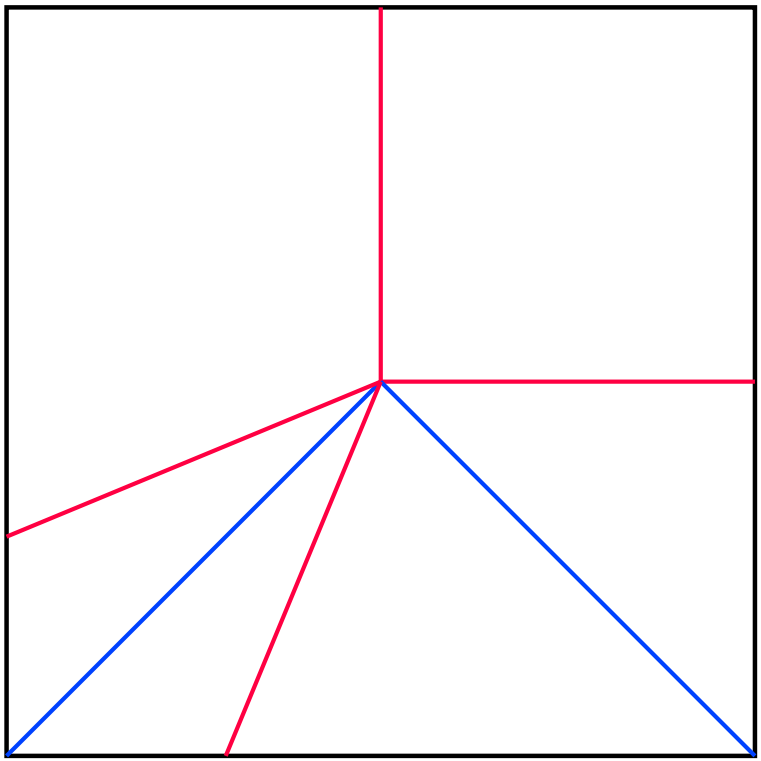



(b)

# Flat-Foldability Conditions

## Maekawa's Theorem (1986)

The sum of mountain + valley is $\pm 2$.



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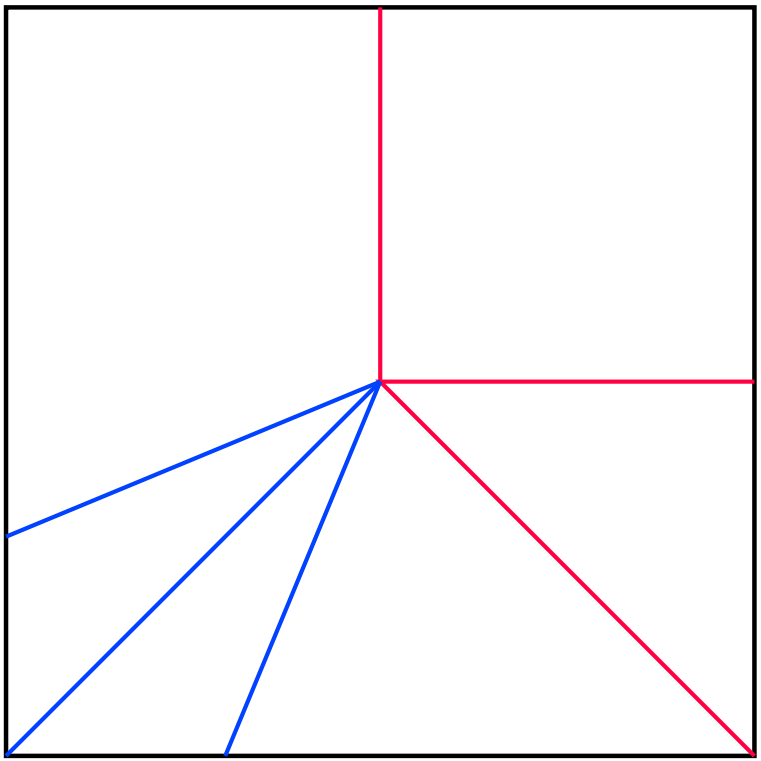


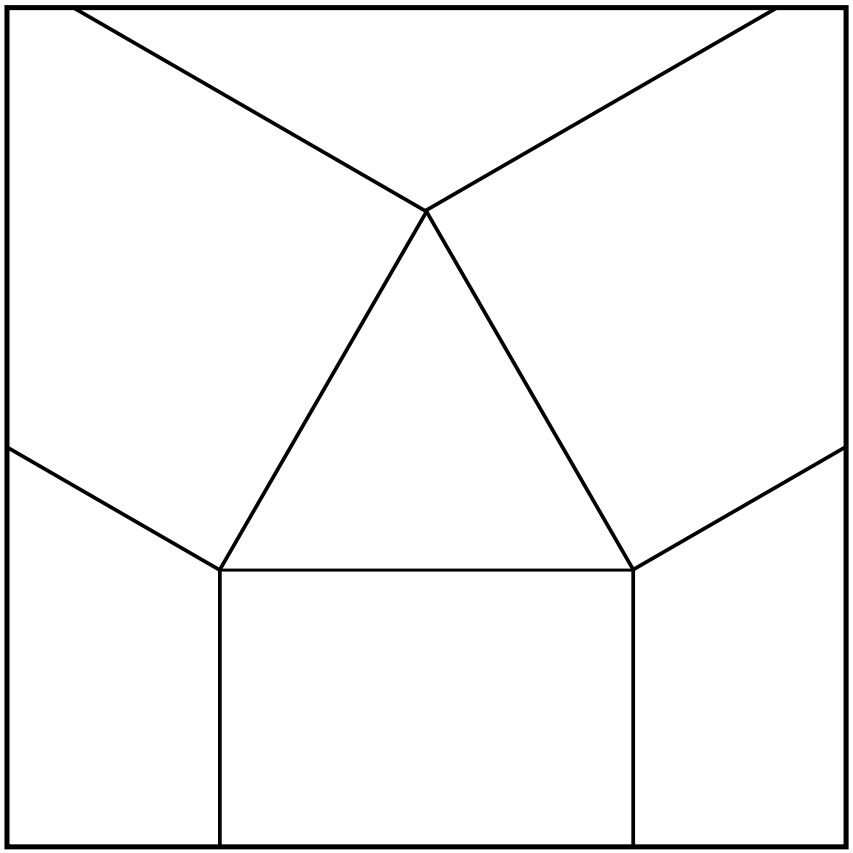
(b)

# More Complicated Crease Patterns

How can we determine flat foldability for a more complicated pattern?

# More Complicated Crease Patterns

Here's a crease pattern that can't fold flat!



# Foldings

How can we capture the notion of a folded paper?

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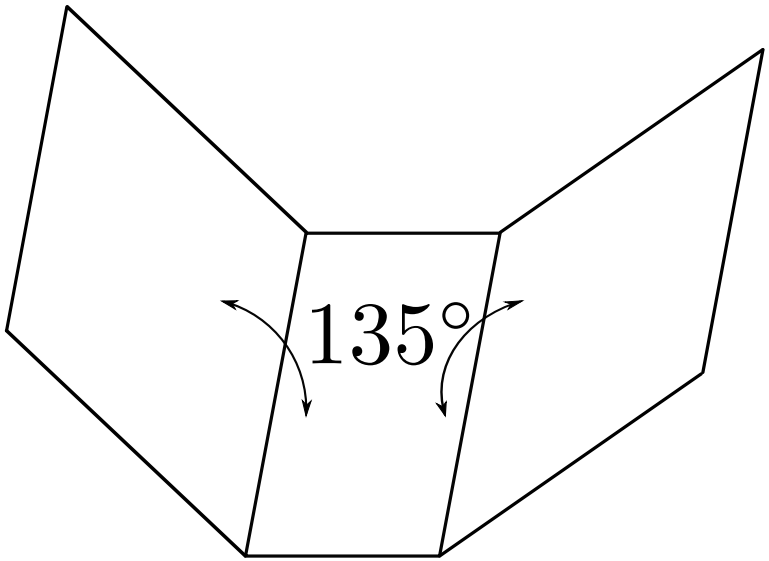
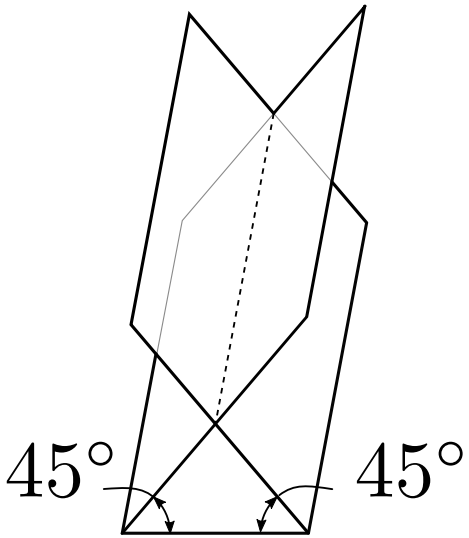


# An Invalid Folding

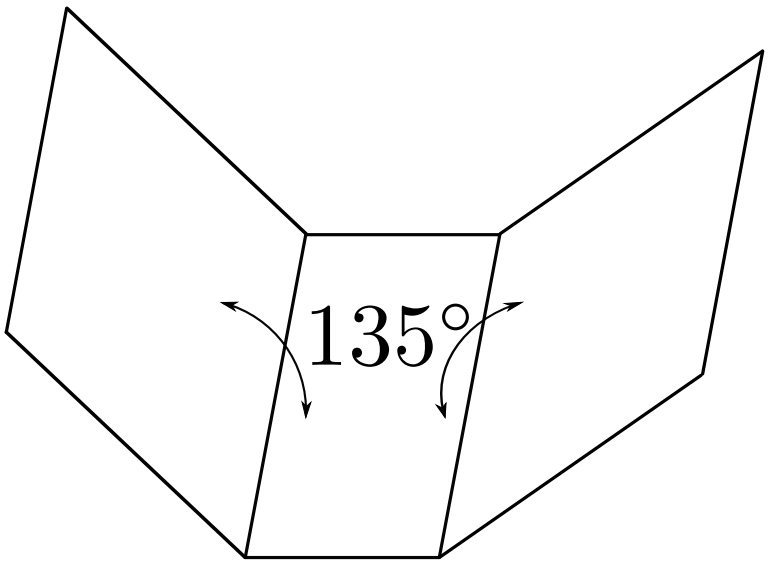
This folding has self-intersection.



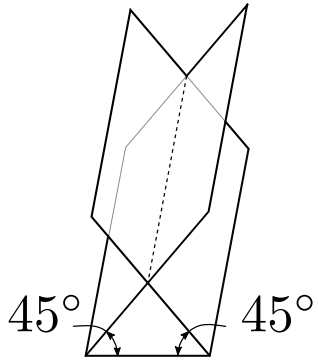
# Valid Foldings

A valid folding is one that doesn't cause any self-intersecton.

Valid




Invalid



# Drawing pictures on Crease Patterns

## Question

If we draw something on the flat piece of paper before folding it up, what can happen when we fold it?

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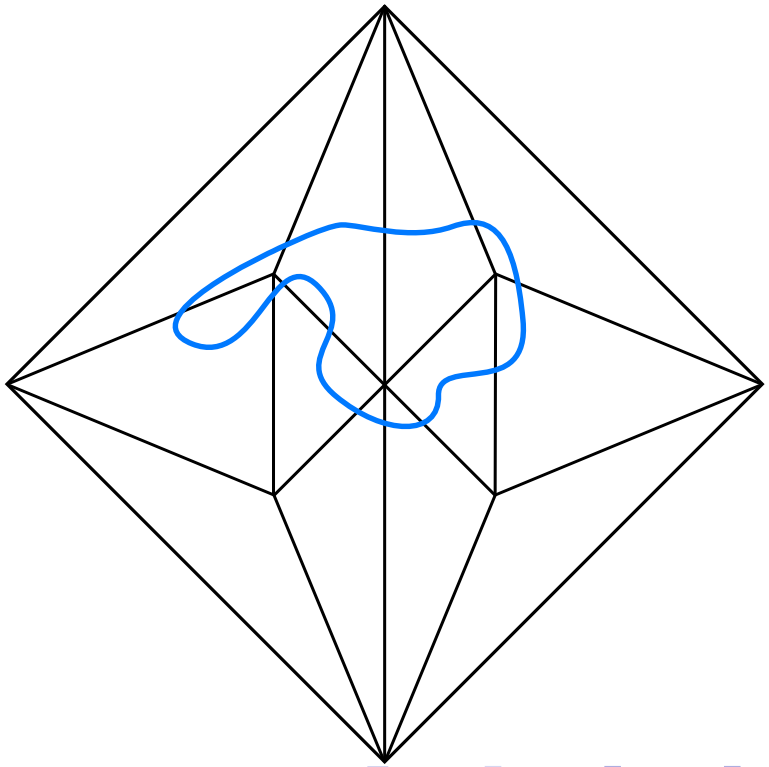
We'll see what happens when we draw Jordan curves on the crease patterns, like the picture on the right.



# Knot Theory 101

What is a knot?

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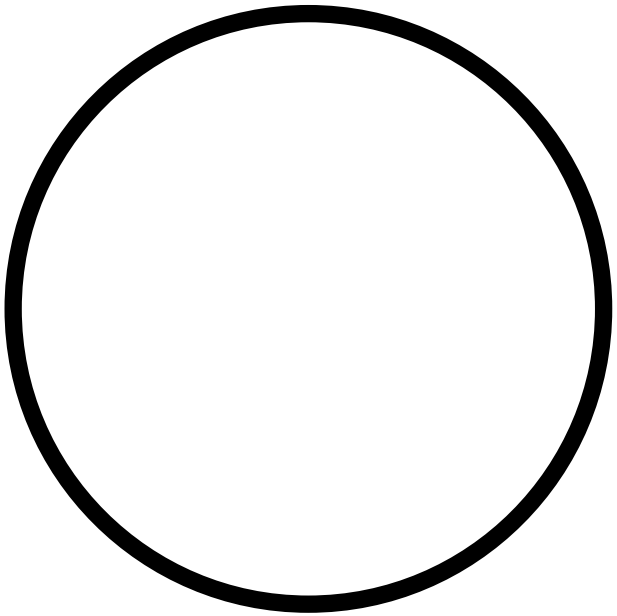

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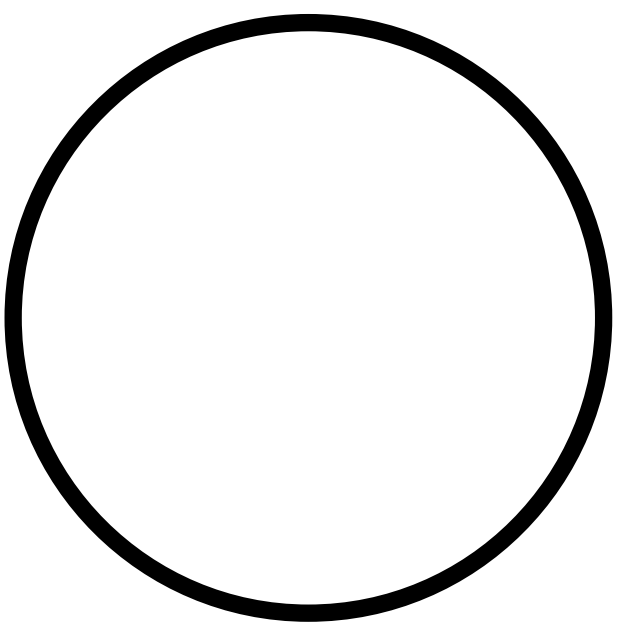
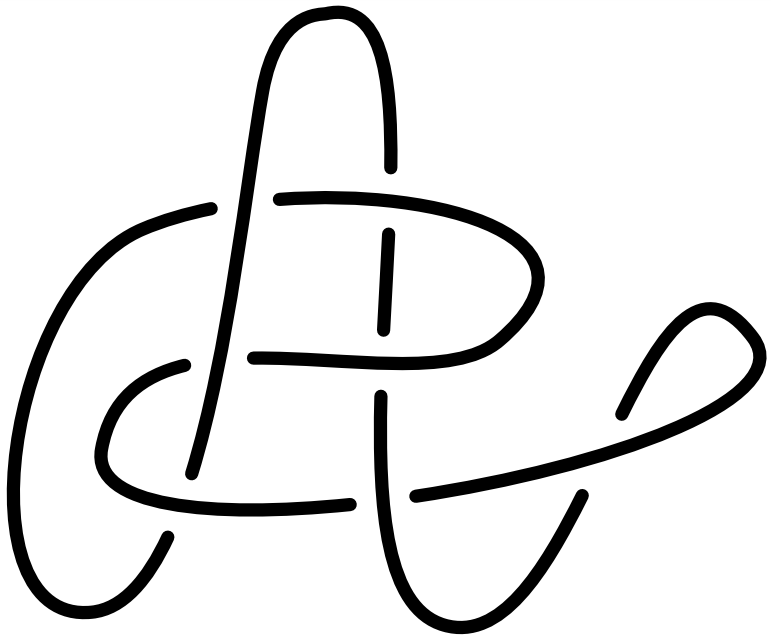
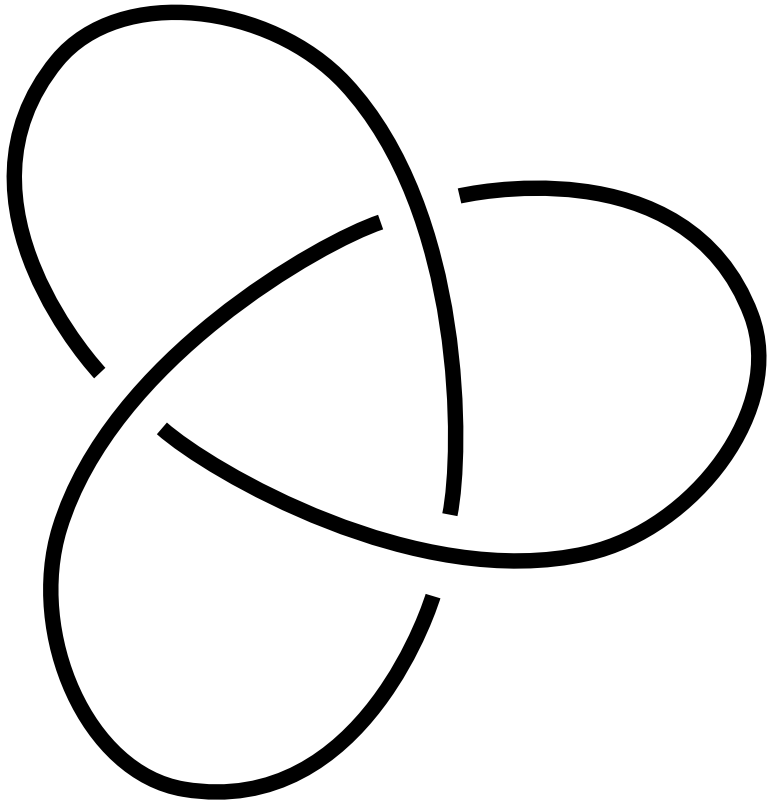




# Knot Theory 101

This is the *trefoil knot*. We can't "untangle" it, no matter how hard we try.



# Connecting Topology and Origami

**Theorem**

*A folding of a crease pattern is valid if and only if every Jordan curve embedded in the paper before folding is mapped to the unknot after folding.*

# Connecting Topology and Origami

$\Longrightarrow$

This direction is fairly intuitive.

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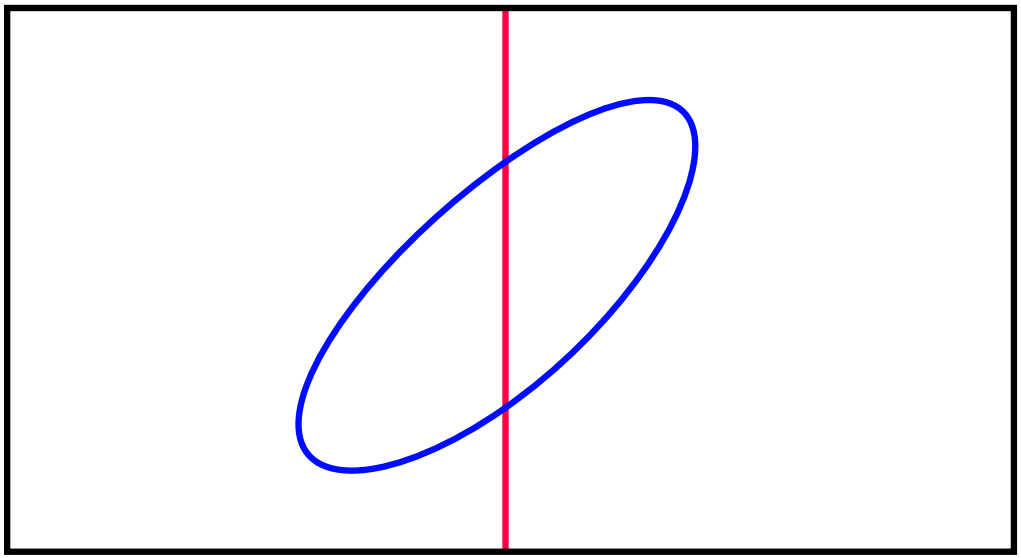

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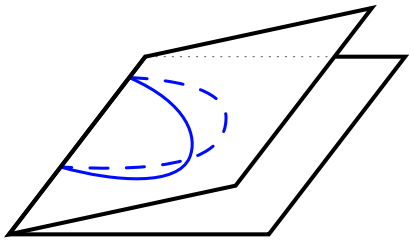

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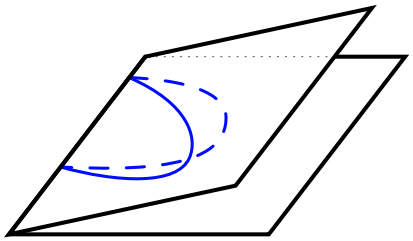

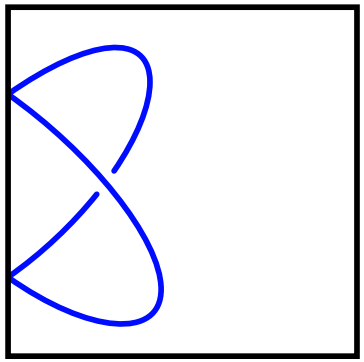

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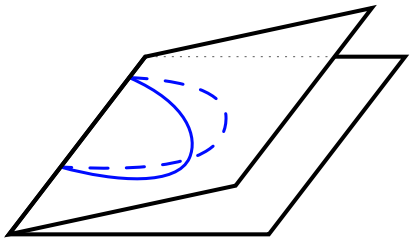
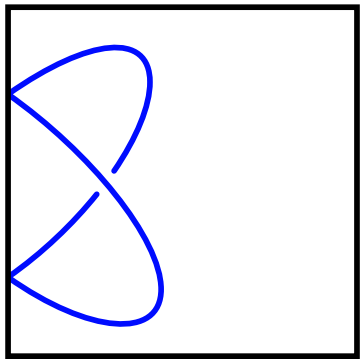
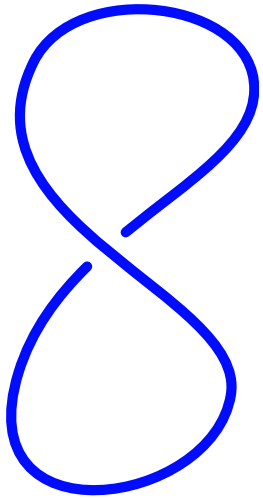


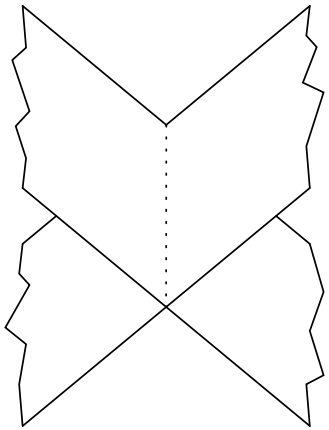




# Connecting Topology and Origami

$\Longleftarrow$

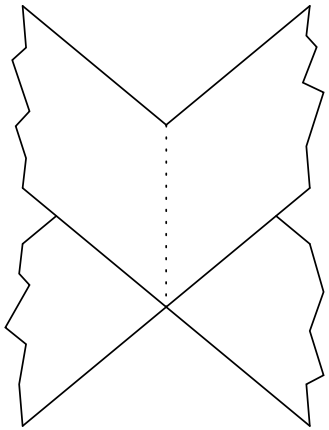
Invalid folding means intersection.

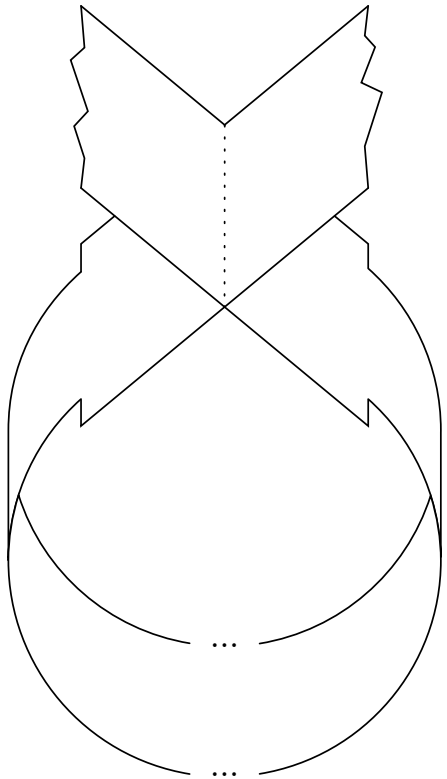


# Connecting Topology and Origami

$\Longleftarrow$

We can connect up the paper like this.

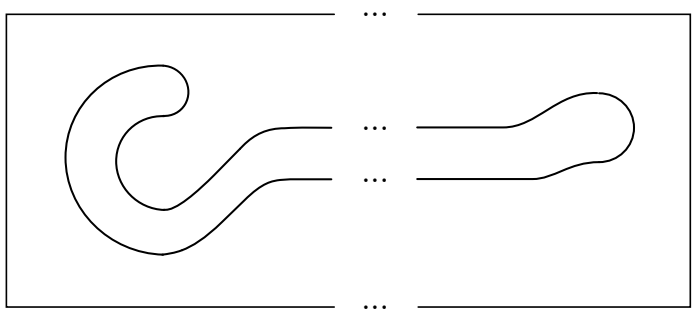


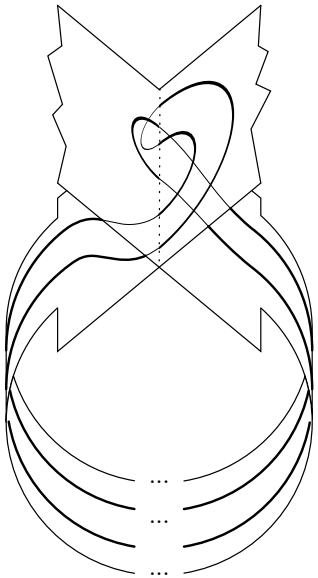


# Connecting Topology and Origami

$\Longleftarrow$

We can draw a curve like this.

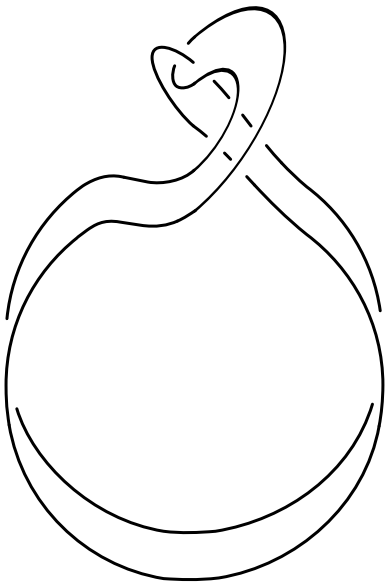




# Connecting Topology and Origami

$\Longleftarrow$

Is it a knot?



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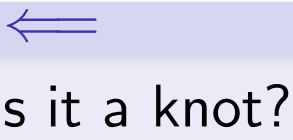

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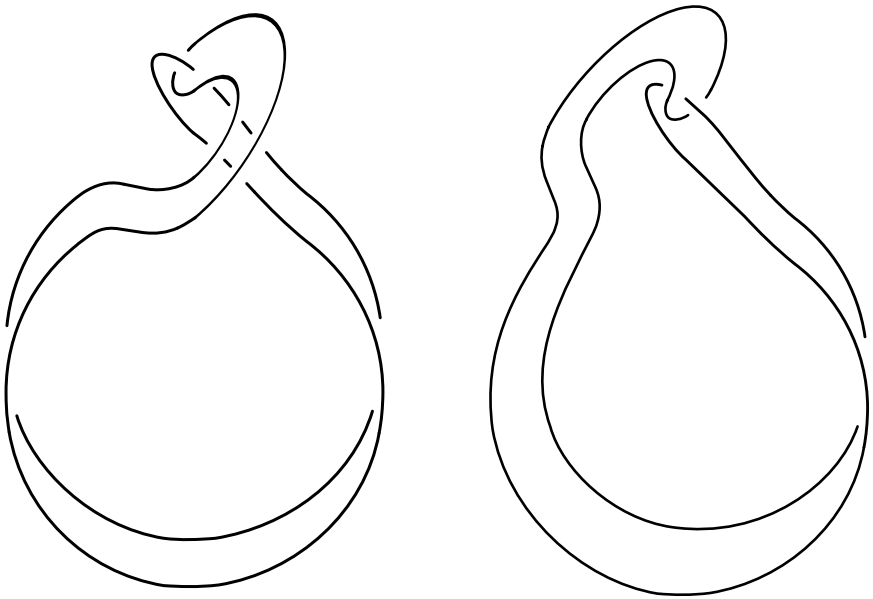

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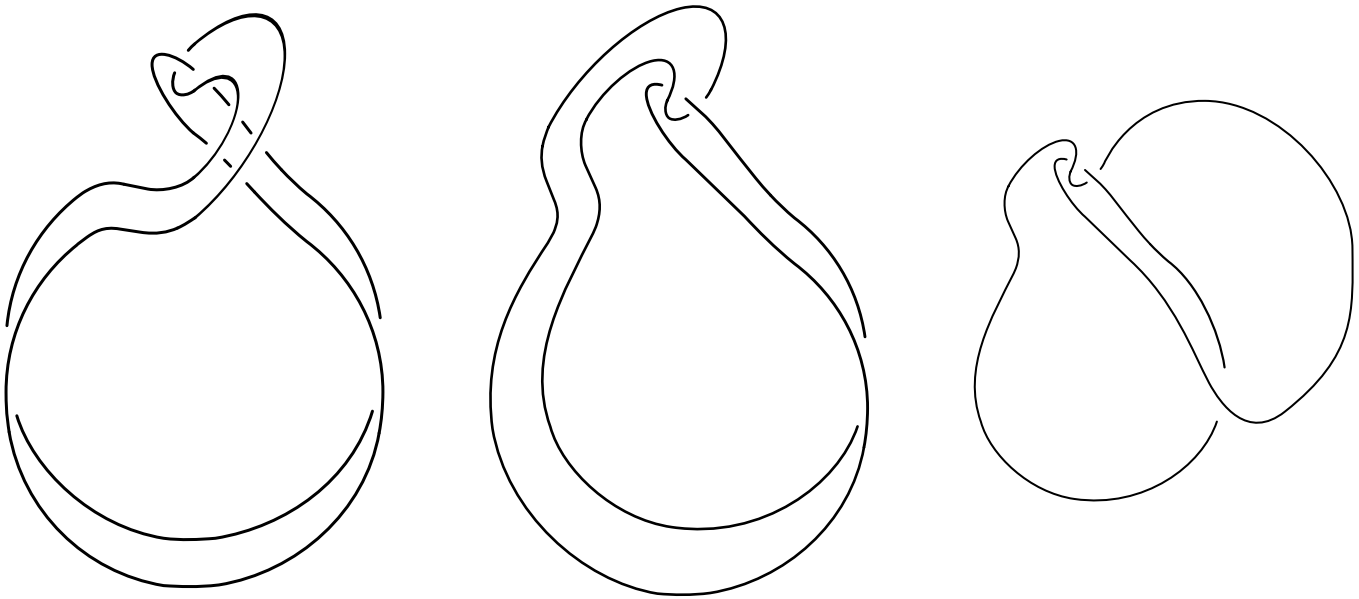

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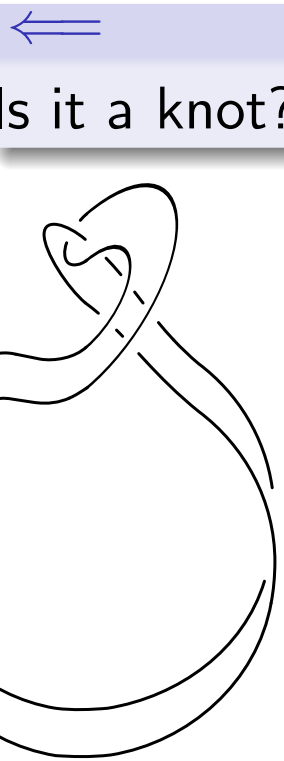

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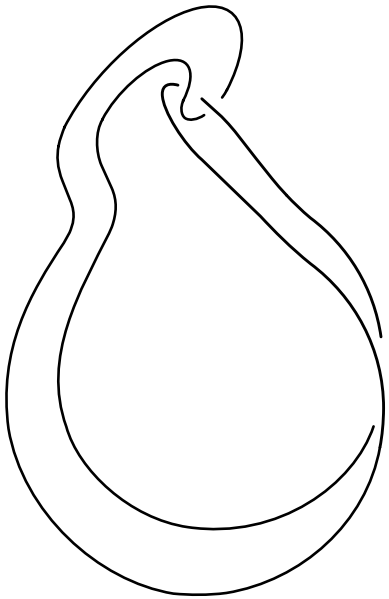

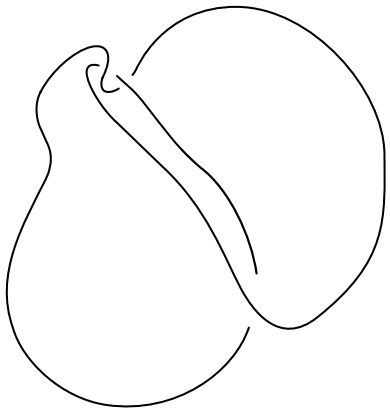

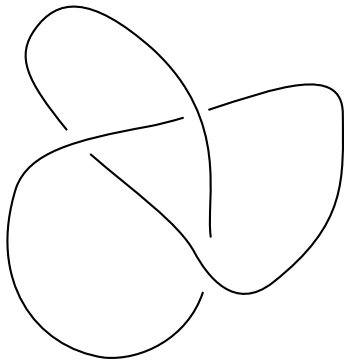

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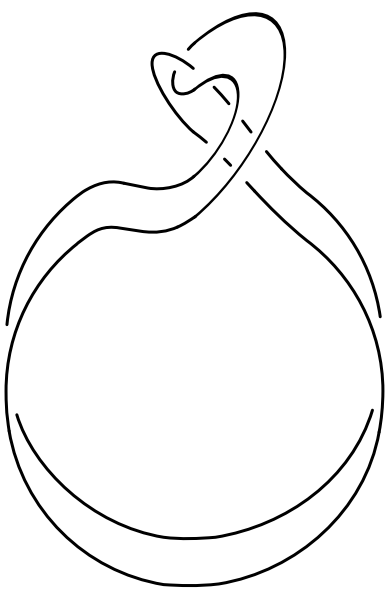
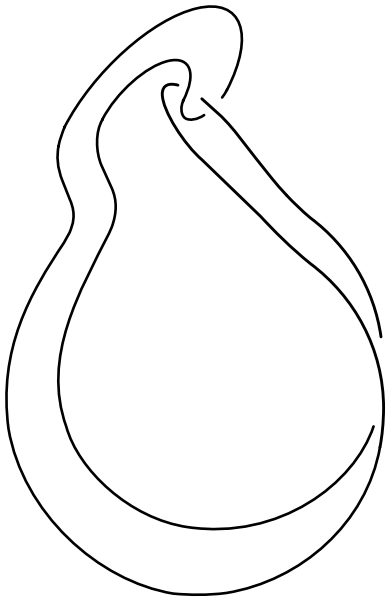
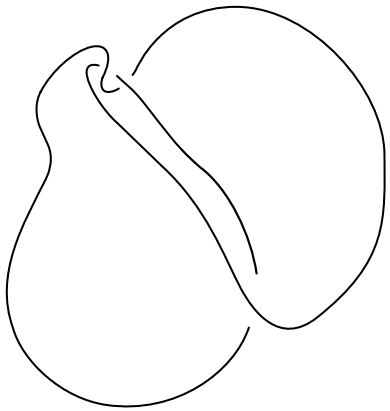
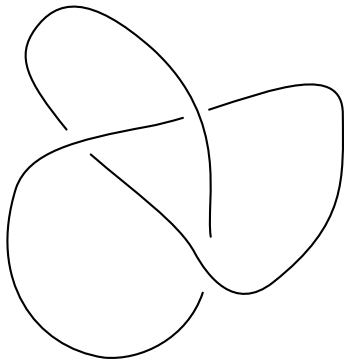








Trefoil knot!

# Connecting Topology and Origami

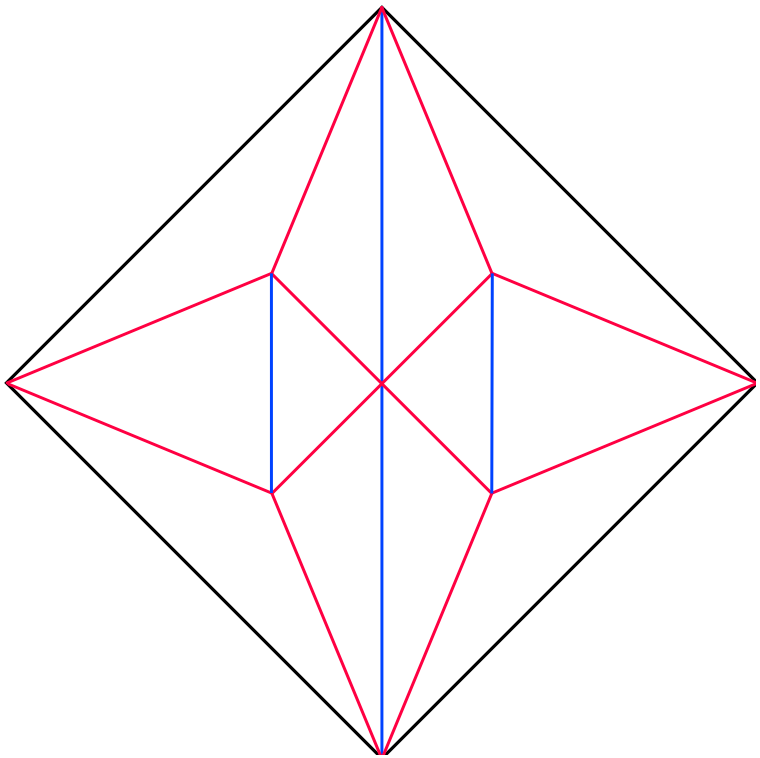
## Now what?

- Maybe a characteristic for crease patterns
- Finding ways to describe the complexity of a crease pattern or a folding (e.g. the number of self-intersections)

# Determining flat-foldability

**Goal:**

Given a crease pattern, determine if it is flat-foldable.



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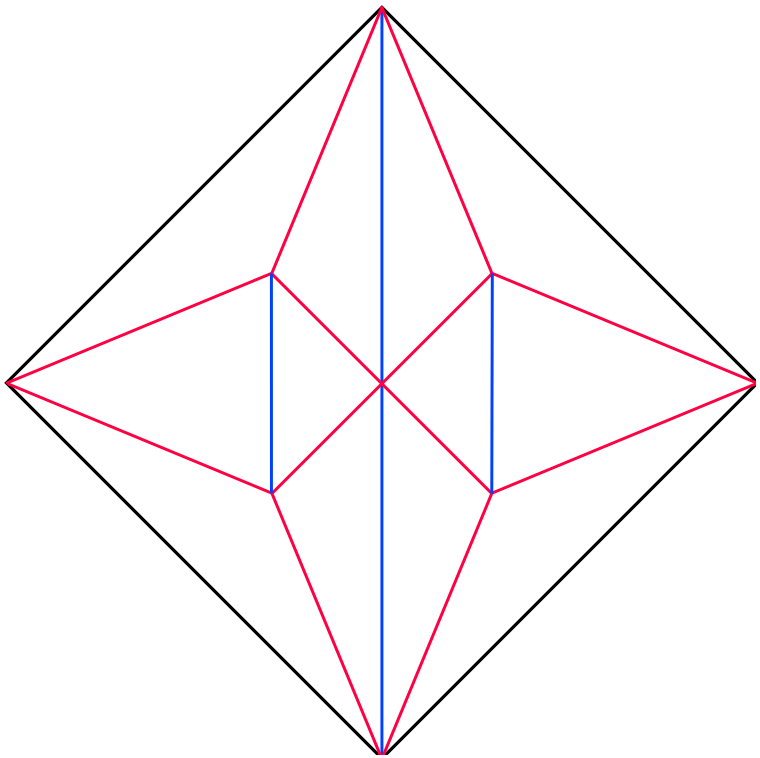
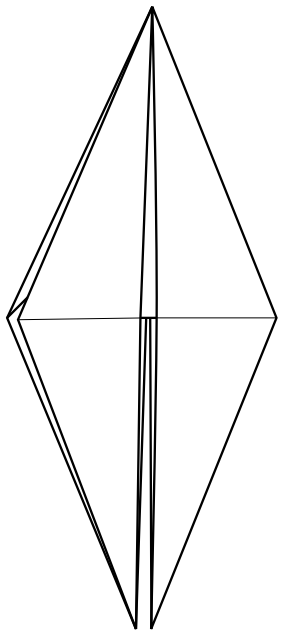
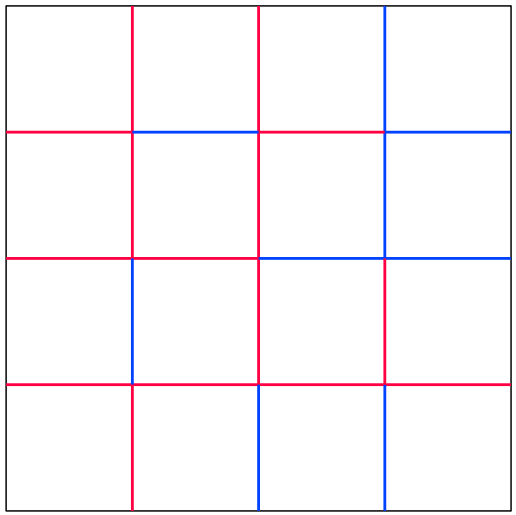
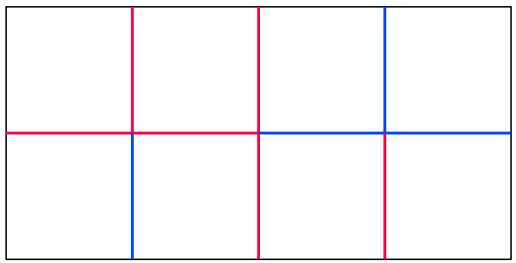




# Introduction to Maps





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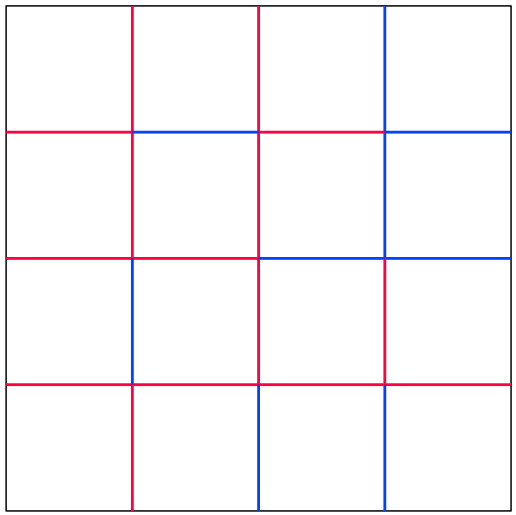

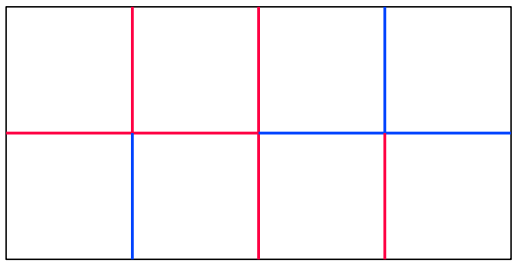

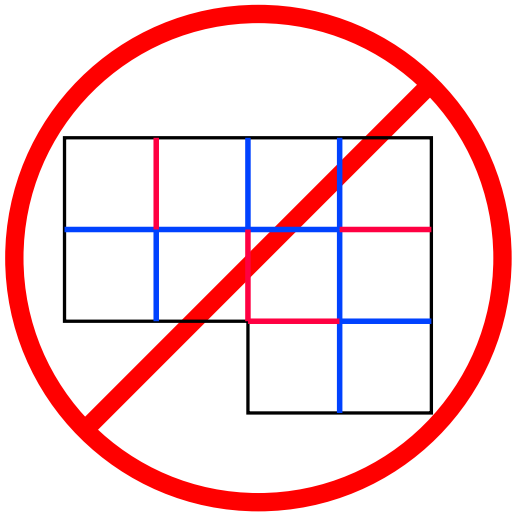

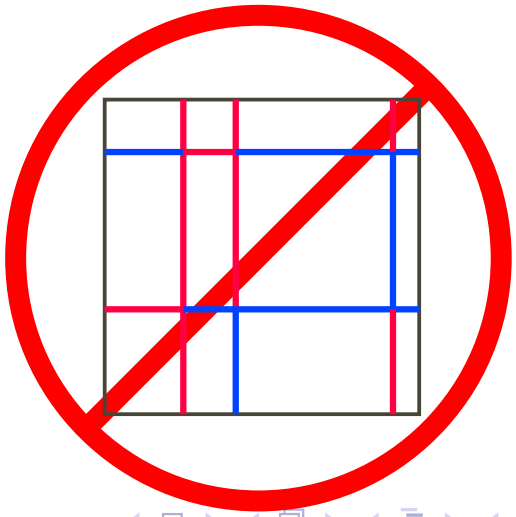

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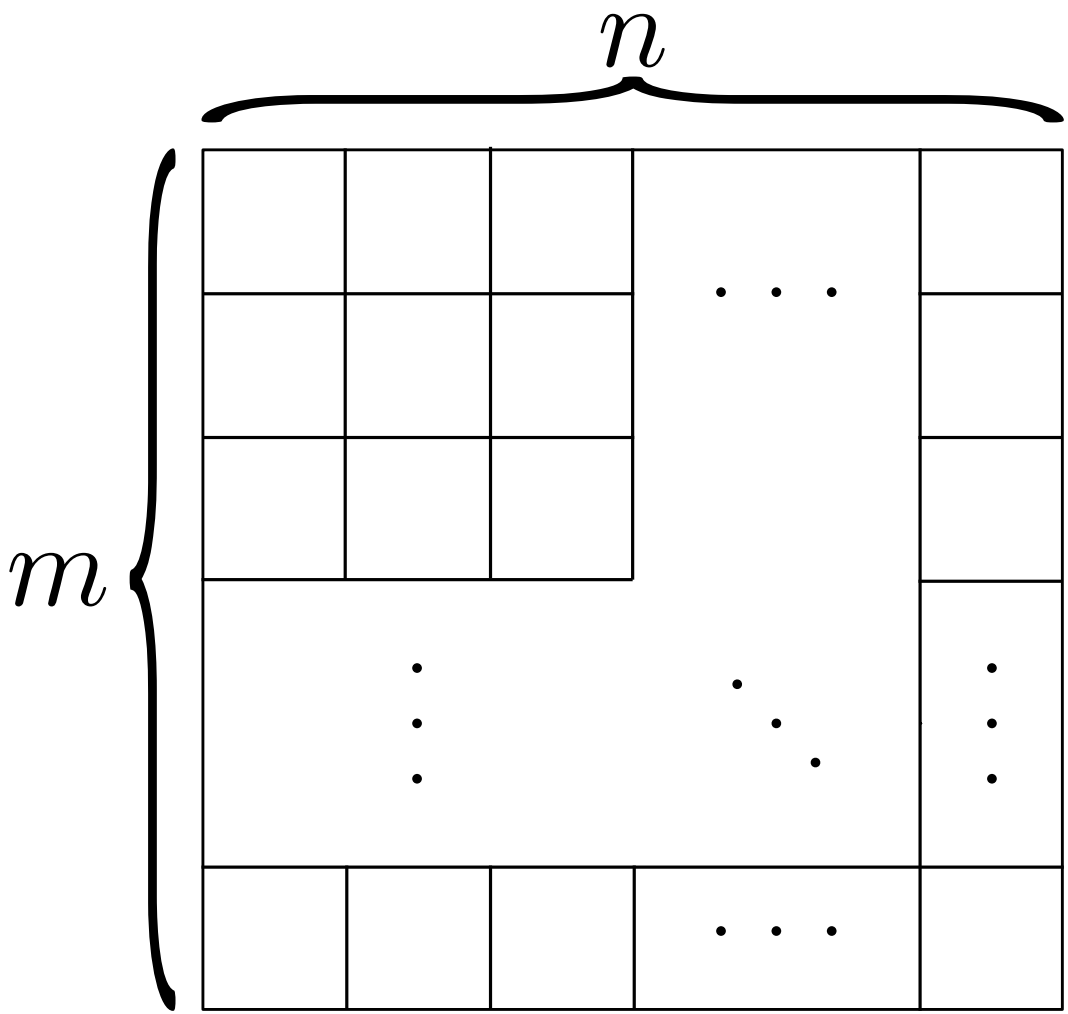

# Map Folding: An Open Problem

**Open Problem:**

How hard is it to determine whether or not a map is flat-foldable?

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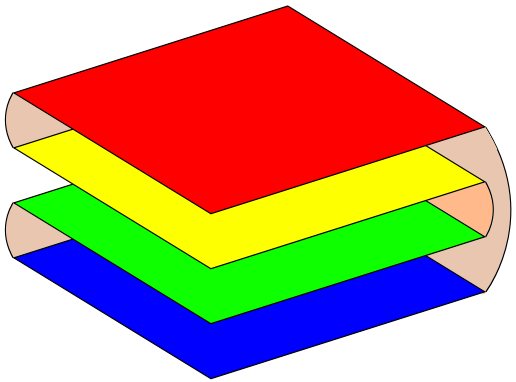
**Easier Problem:**

How hard is it to determine whether or not a map is *flat-folded*?

# Linear Orderings

**Question:**

How can we represent a folded form?



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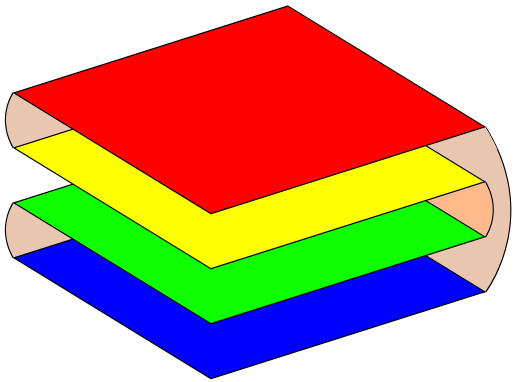

red $\rightarrow$ yellow $\rightarrow$ green $\rightarrow$ blue

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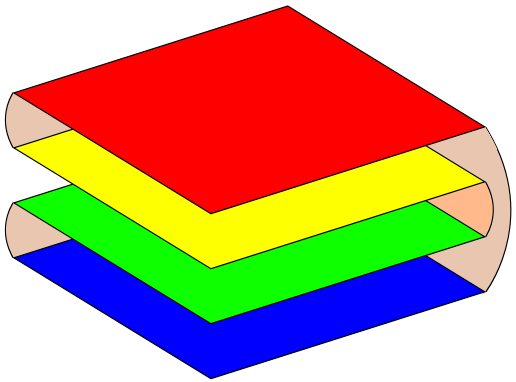

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**Note:**

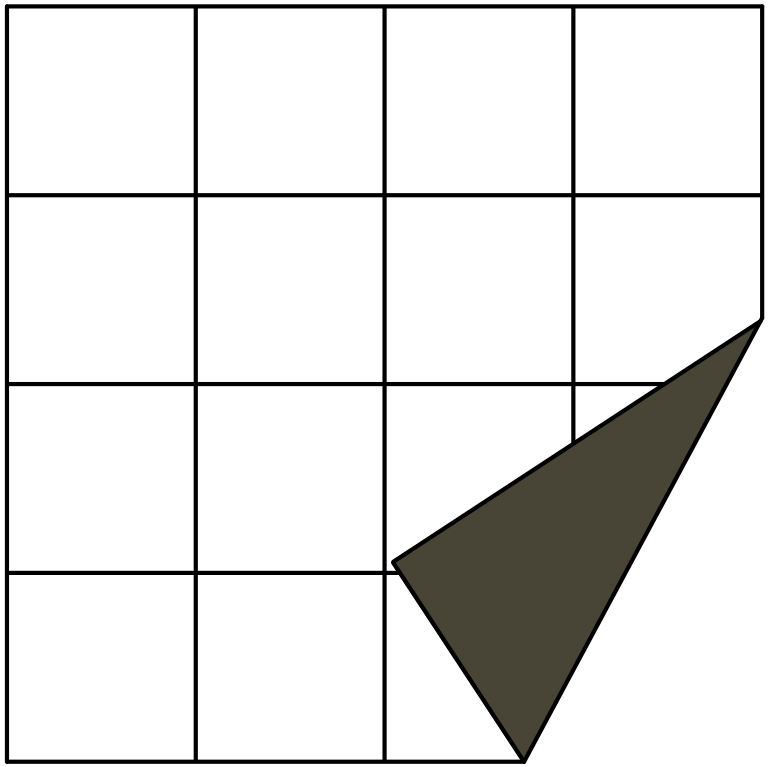
Each linear ordering corresponds with exactly one folded form.

# Linear Orderings

**Question:**

How can we tell if a linear ordering is realizable by a given crease pattern?

# Checkerboard Pattern



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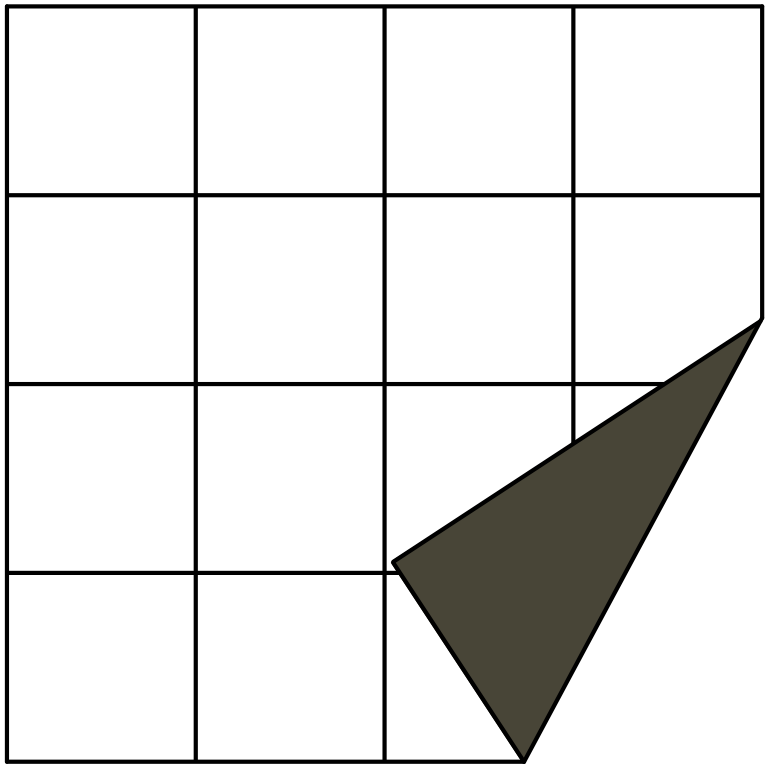

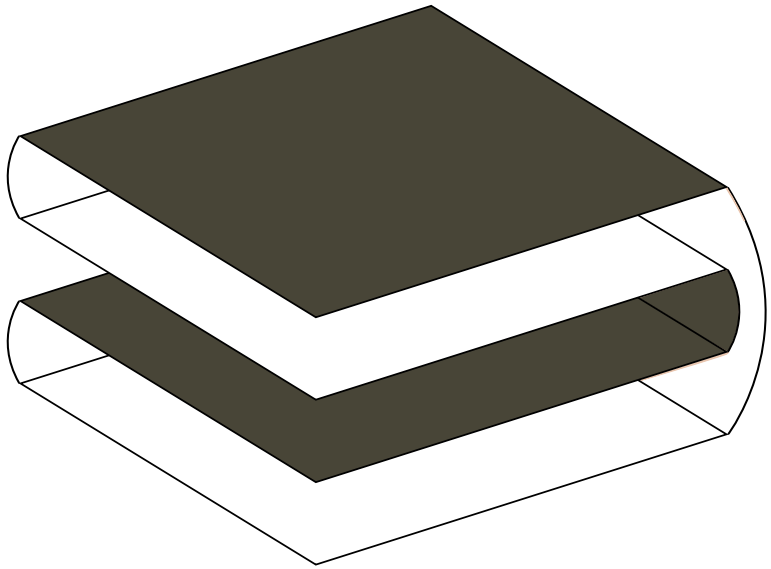

# Checkerboard Pattern

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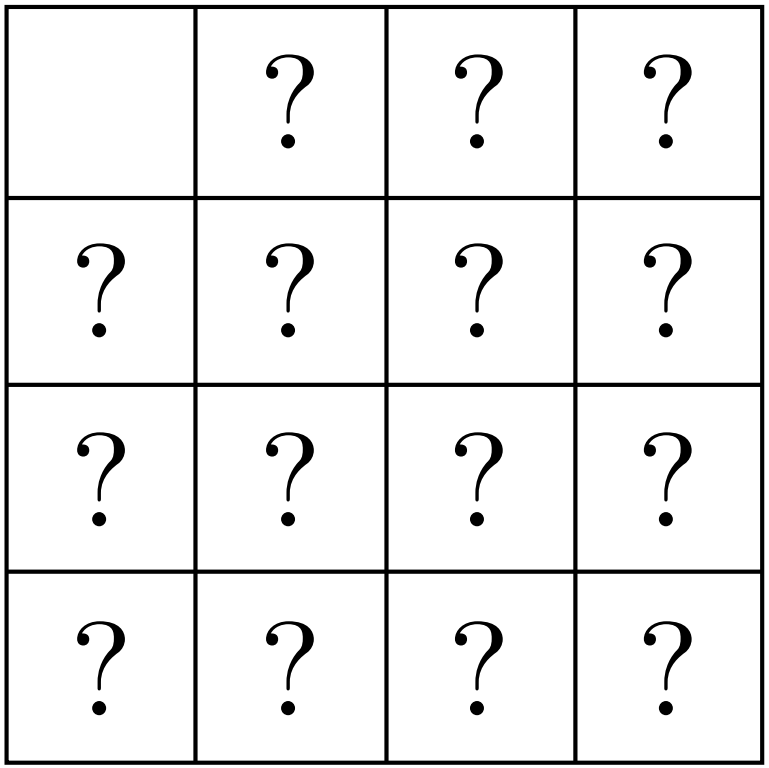
Label each face as being light-side or dark-side up in *any* folding.



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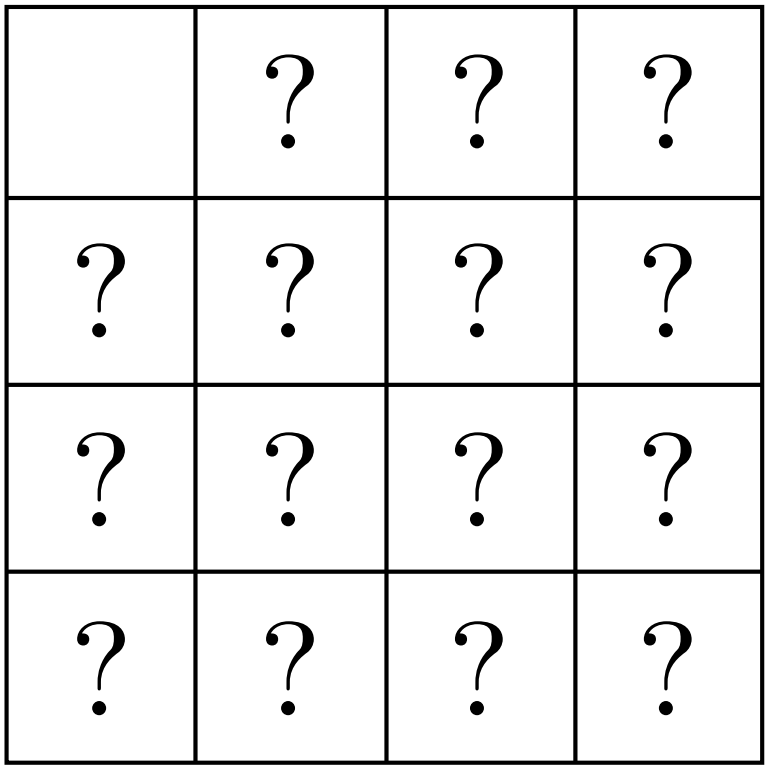


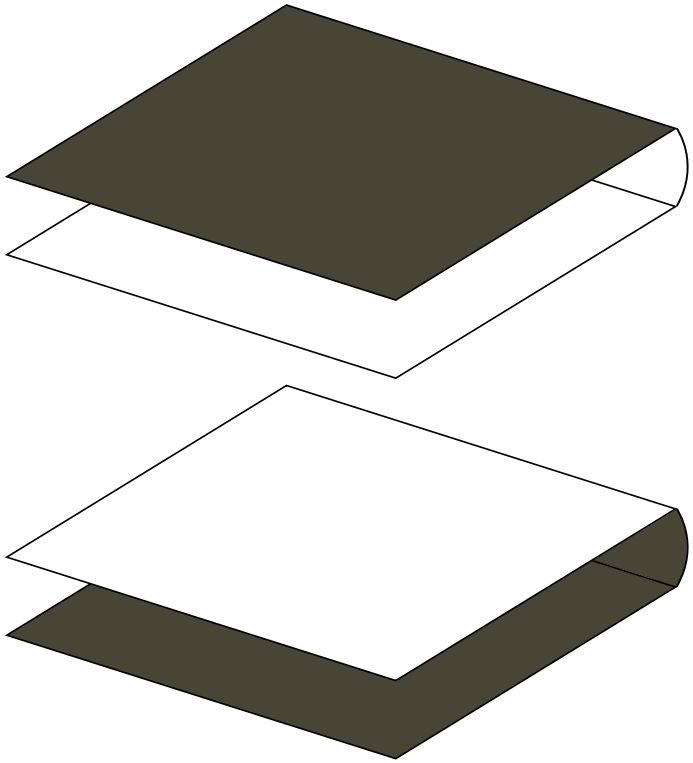

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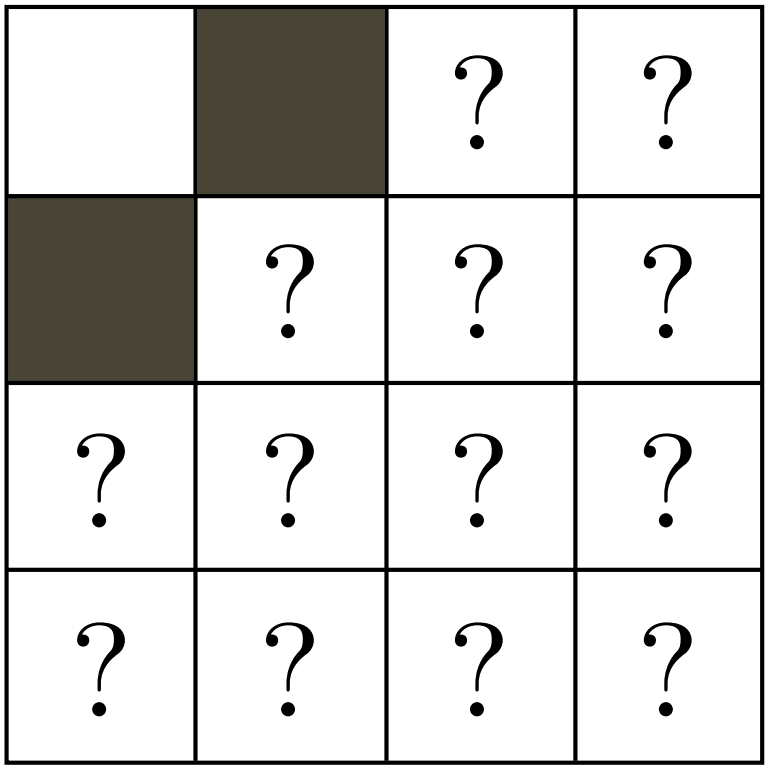


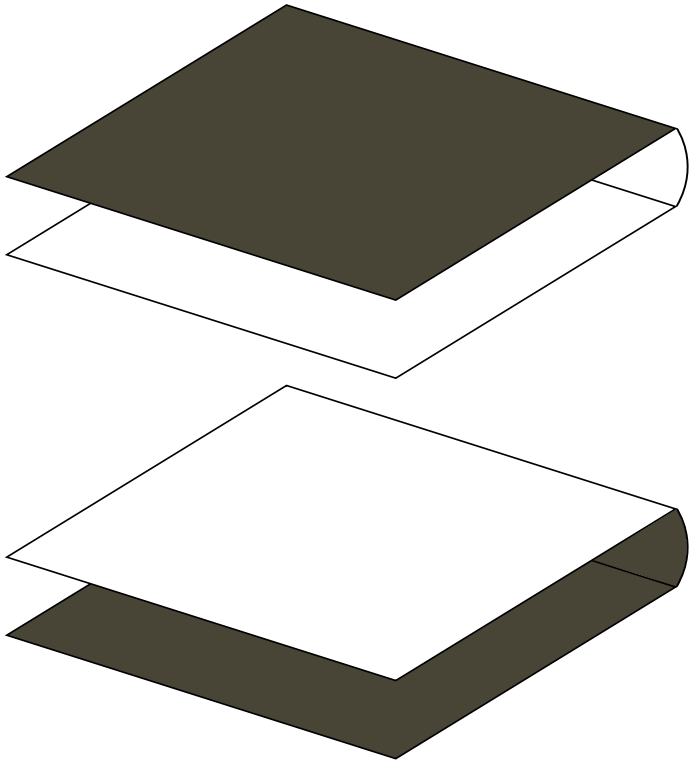

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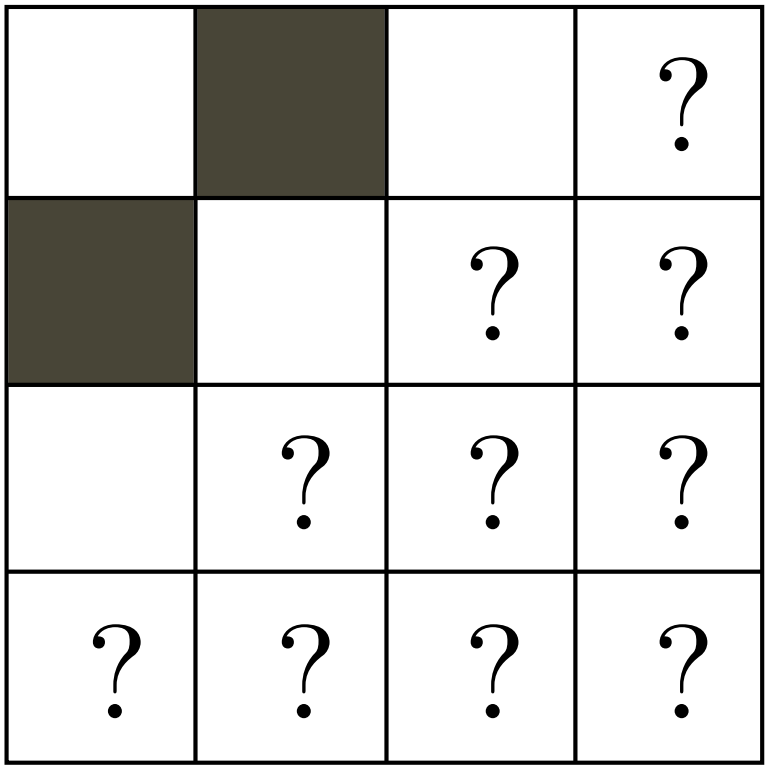


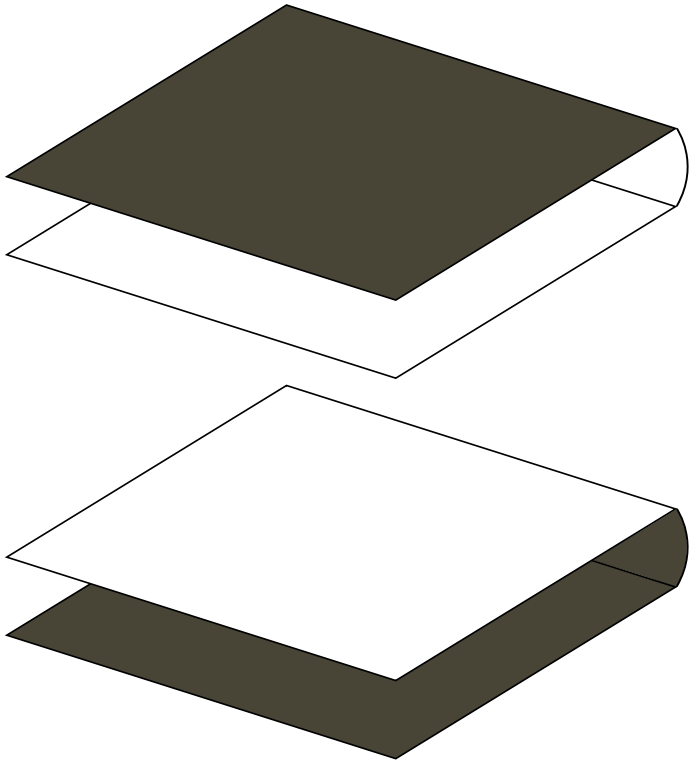

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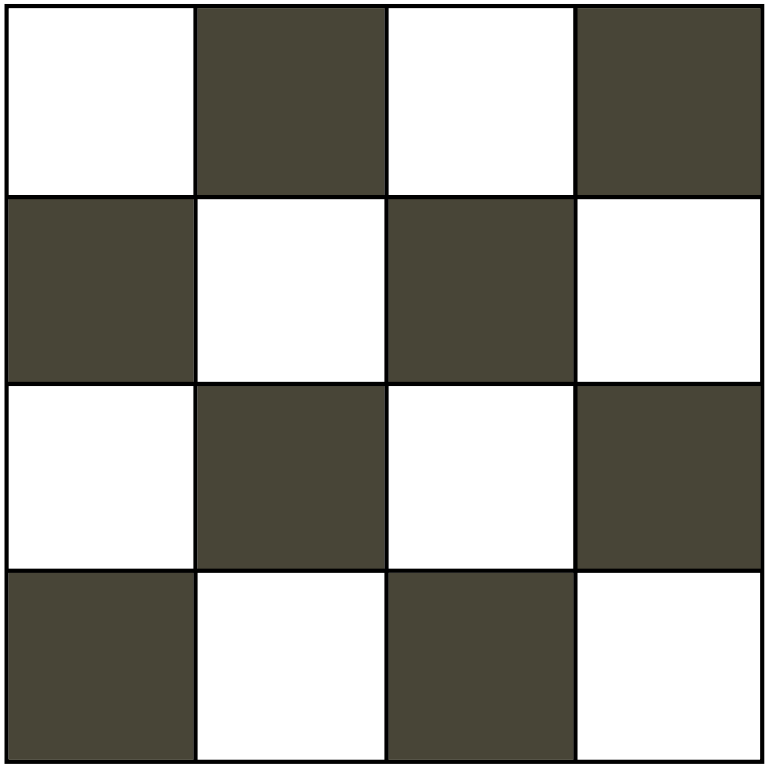

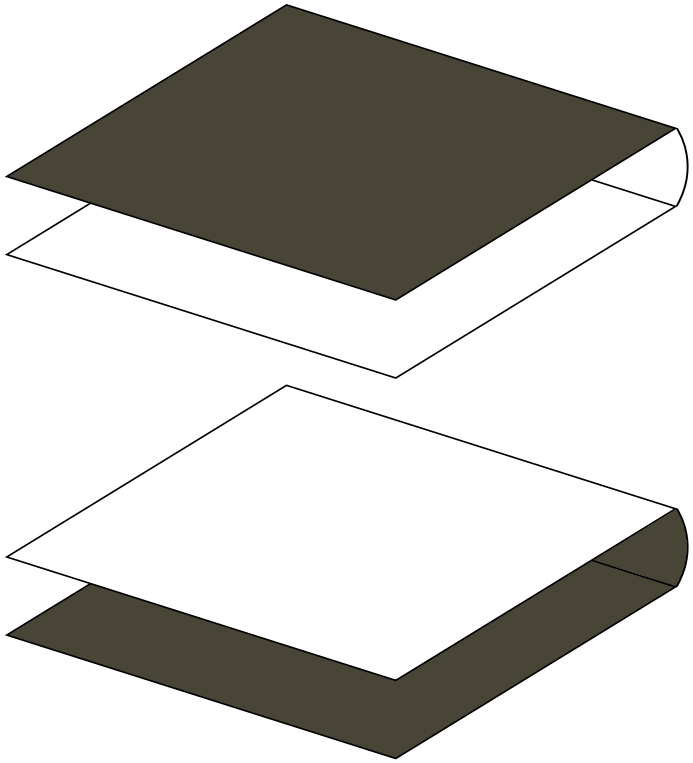

# Partial Ordering

**Definition:**

A *partial ordering* $P$ on a set $S$ is a set of ordered pairs of elements of $S$ that orders some of the elements of $S$.

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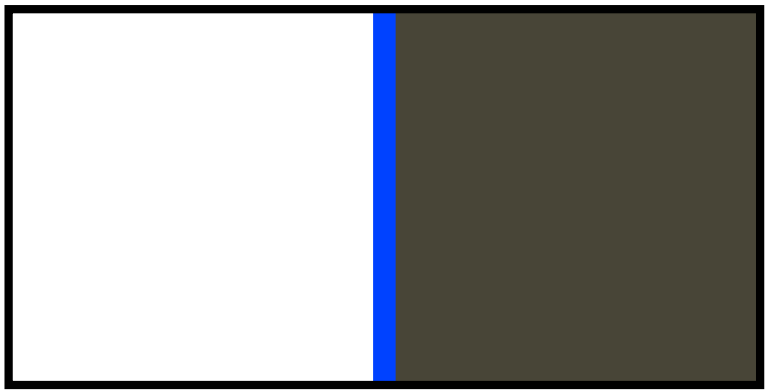
We will denote this an ordered pair as $a \to b$ ("$a$ covers $b$")

# Partial Ordering

**Goal:**

Use checkerboard pattern and crease assignments to tell, for each pair of adjacent faces, which face comes first in *all* linear orderings.



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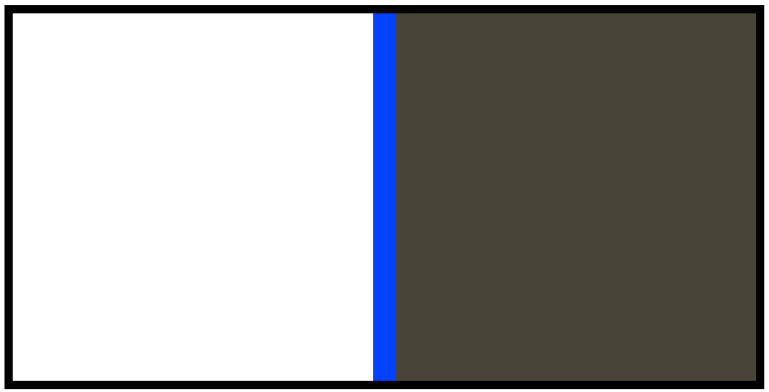

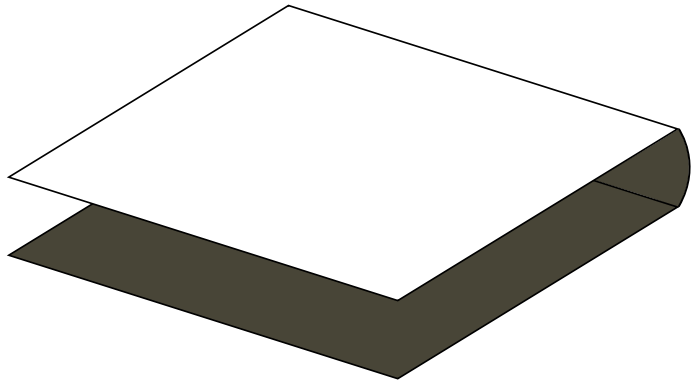

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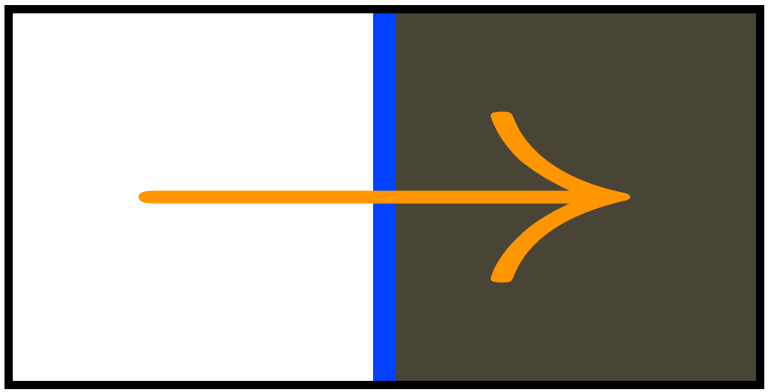

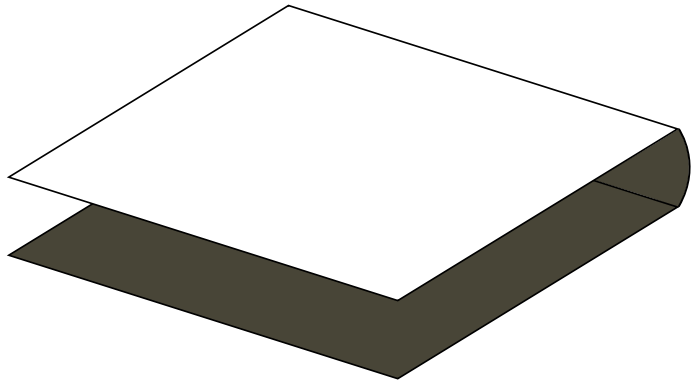

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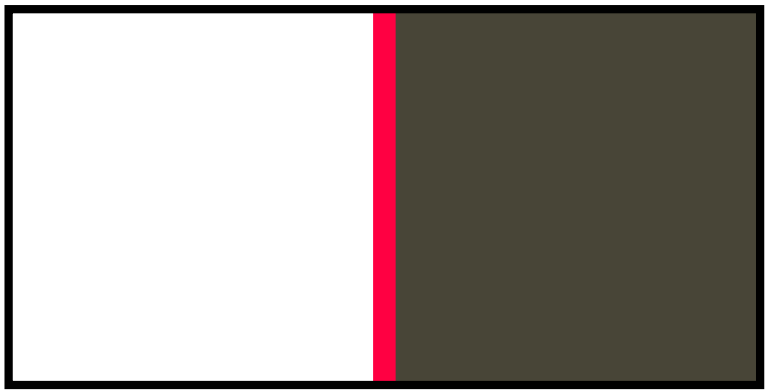

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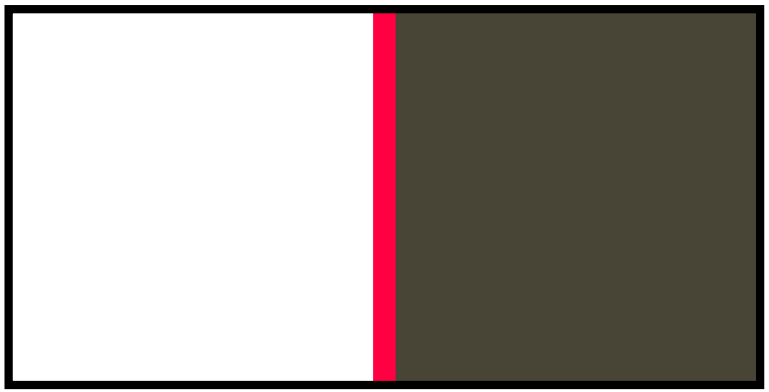

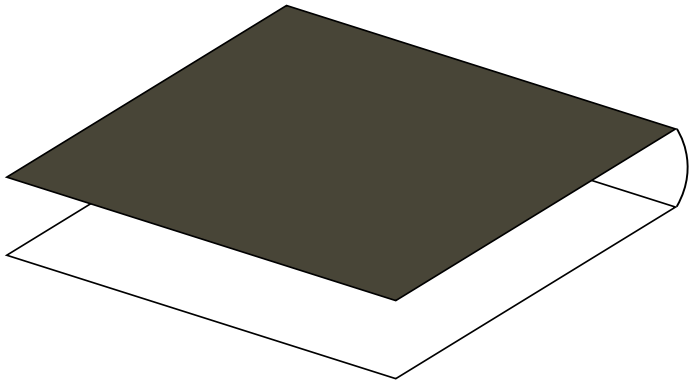

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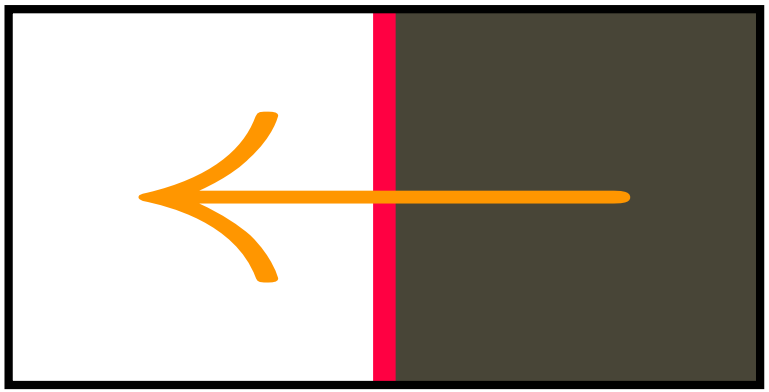

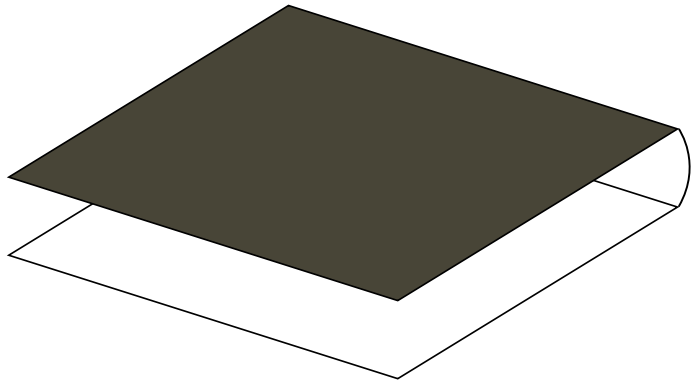

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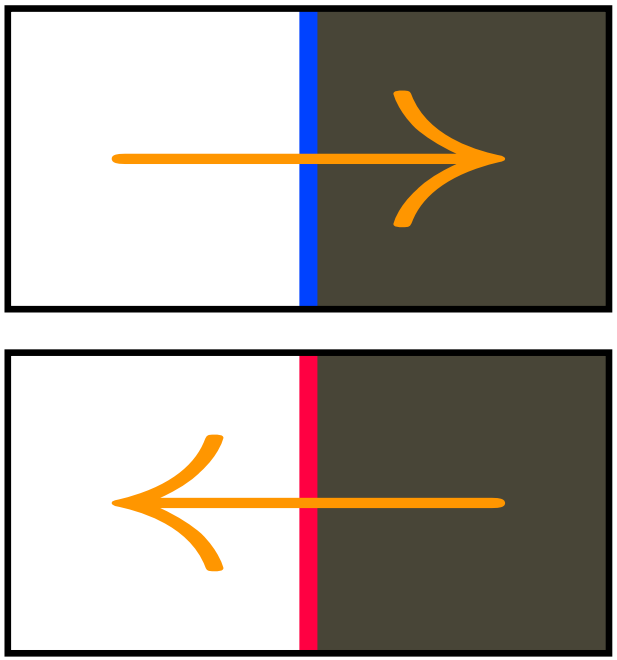

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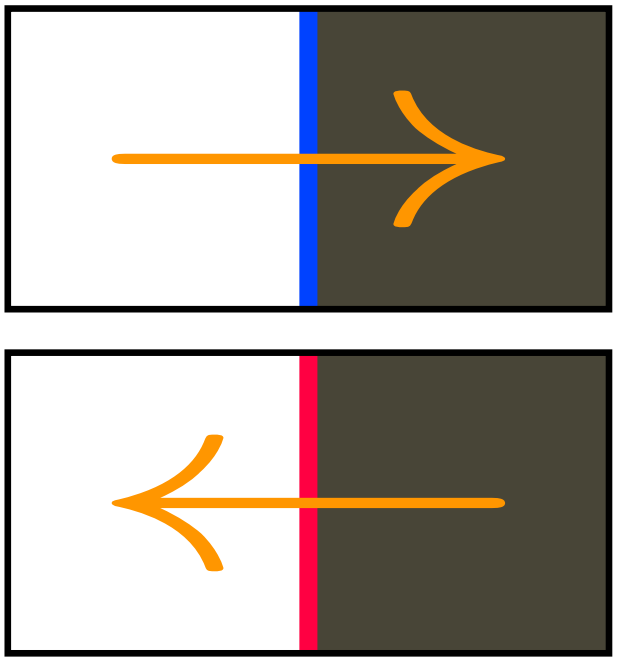

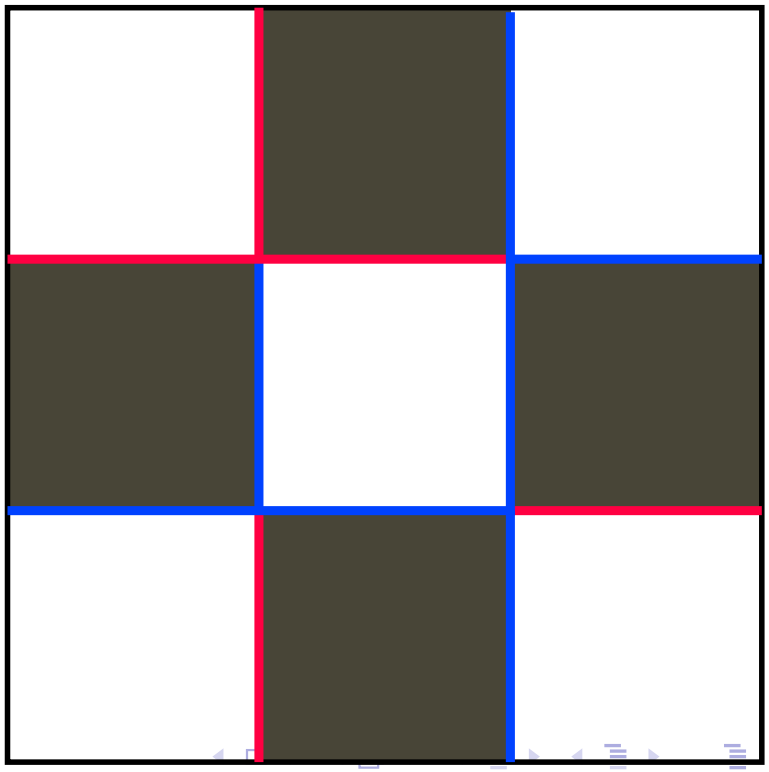

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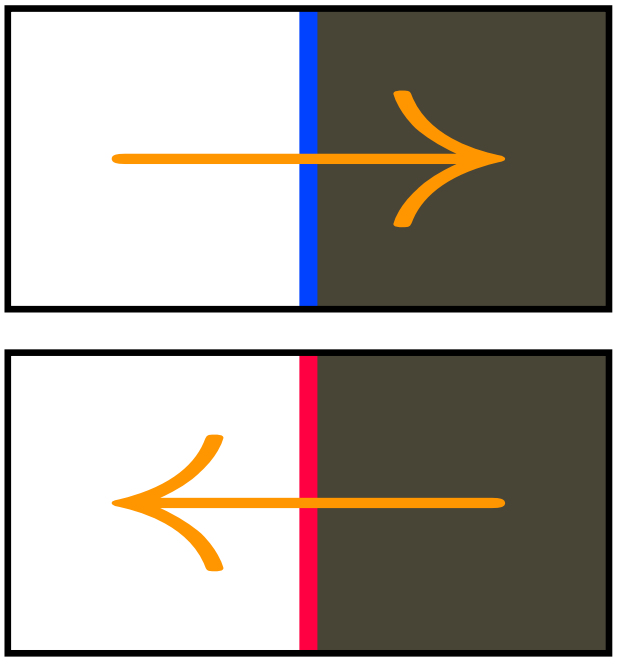

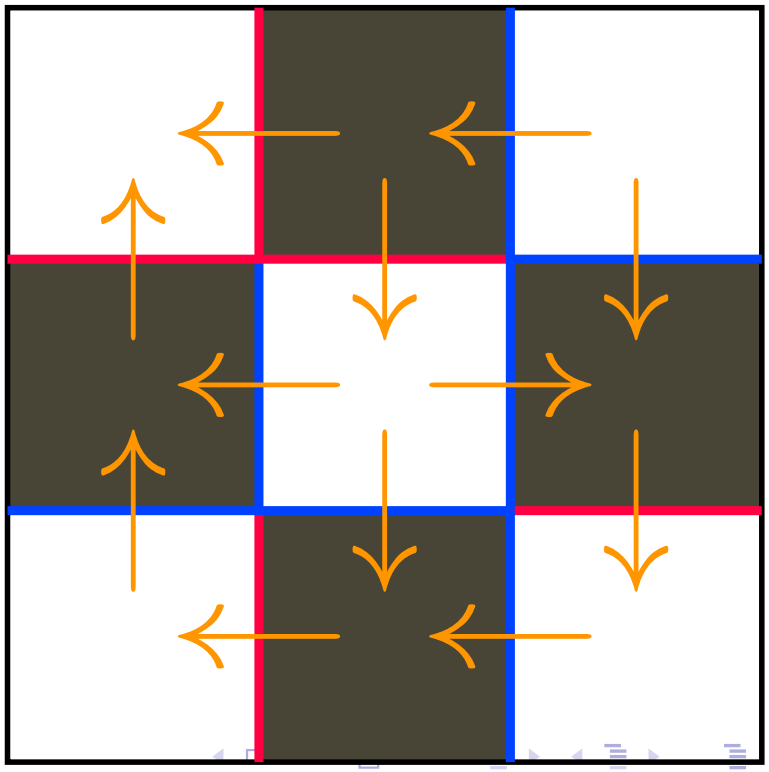

# Linear Orderings

Is satisfying this partial ordering enough to ensure foldability?

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Is satisfying this partial ordering enough to ensure foldability? **No!**

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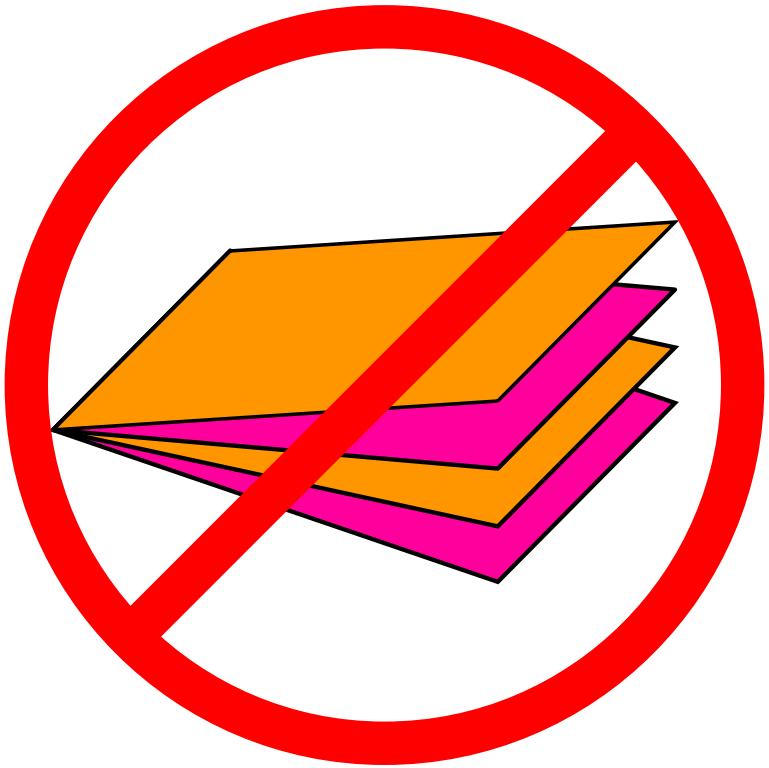
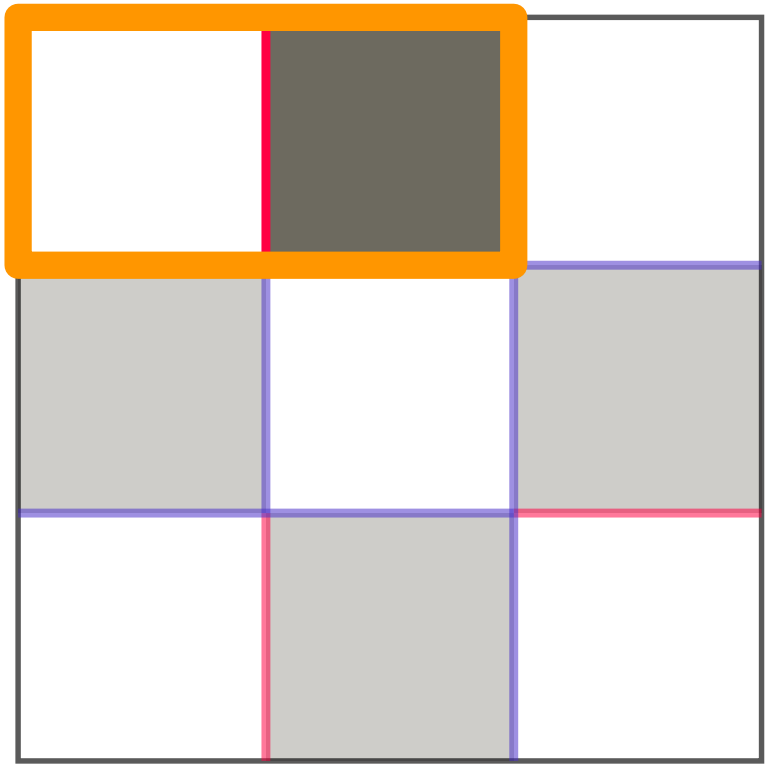
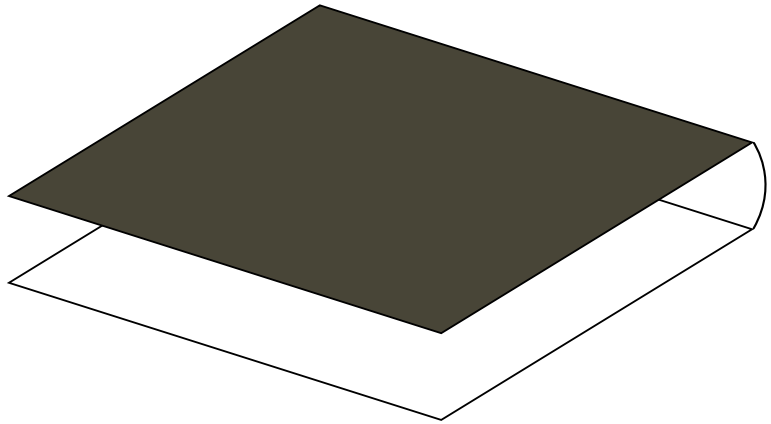


# Butterflies





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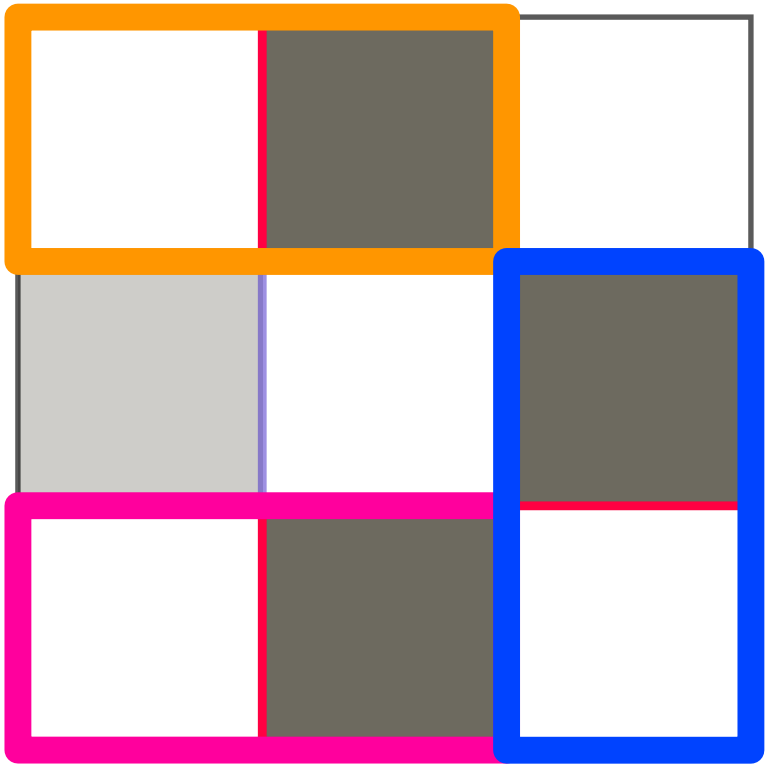

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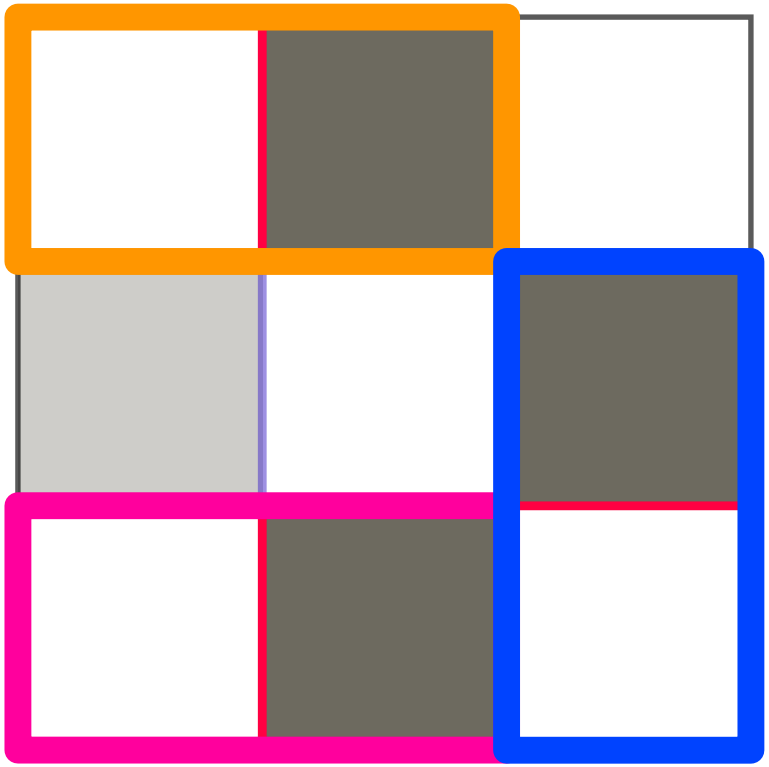

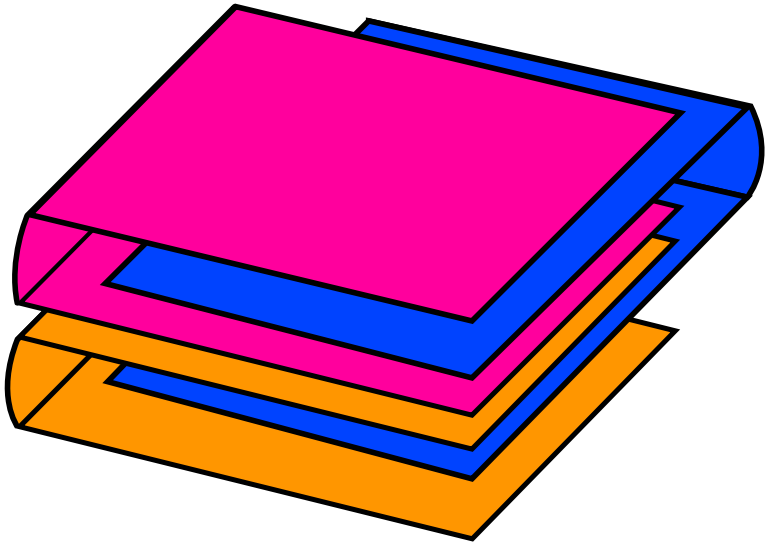

# Butterfly Condition

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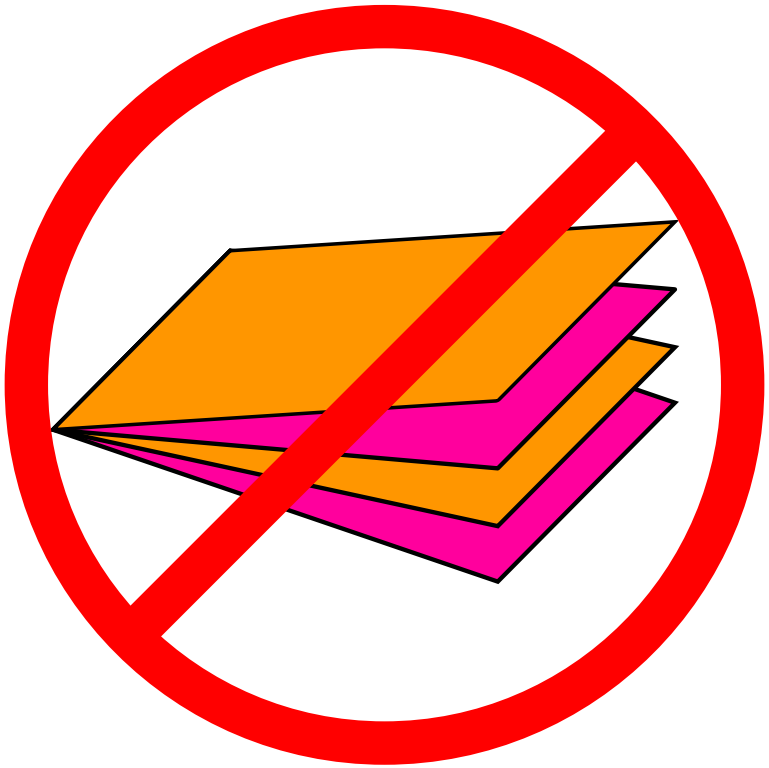
Enumerate the realizable configurations of twin butterfly pairs.



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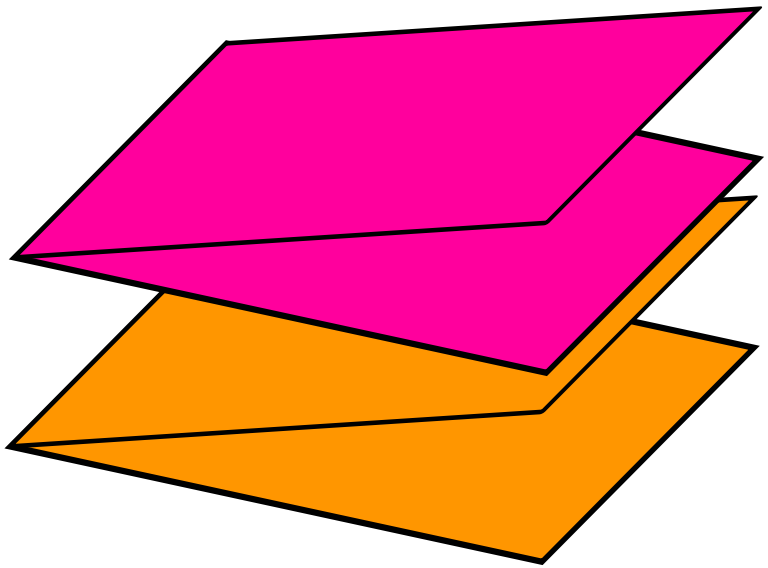

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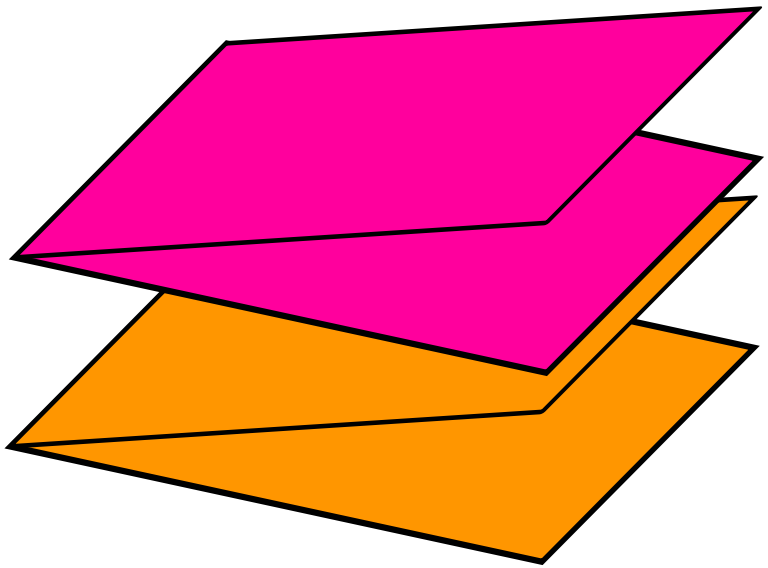

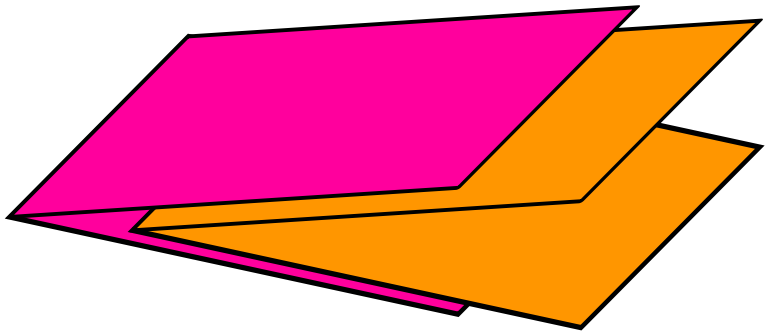

# Linear Orderings: Conditions for Validity

**Theorem: (Nishat and Whitesides 2013)**

A linear ordering $\mathcal{L}$ of faces is flat-foldable if and only if $(i)$ $\mathcal{L}$ satisfies the partial ordering given by the map and $(ii)$ every pair of twin butterflies stacks or nests in $\mathcal{L}$.

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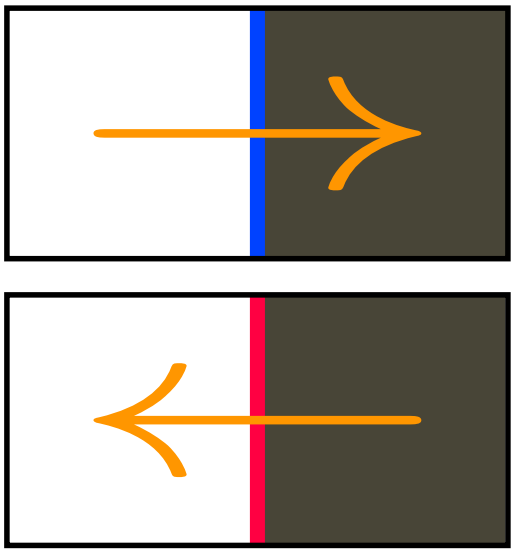

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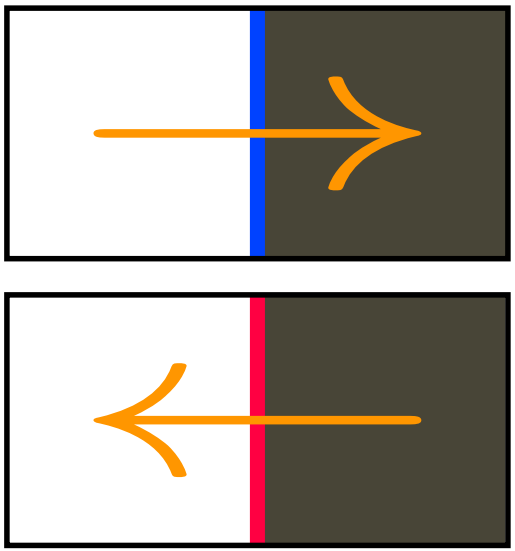

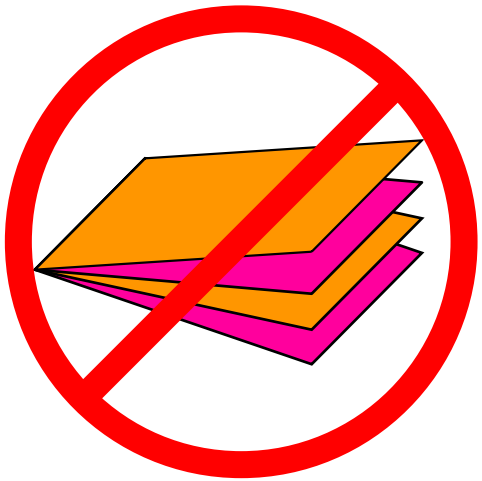

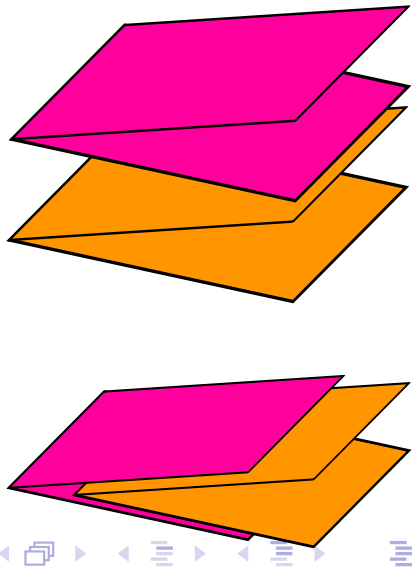

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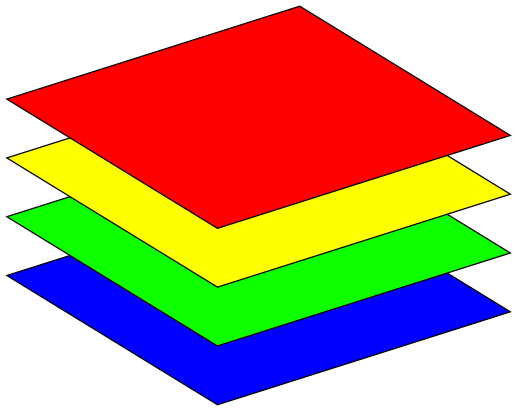

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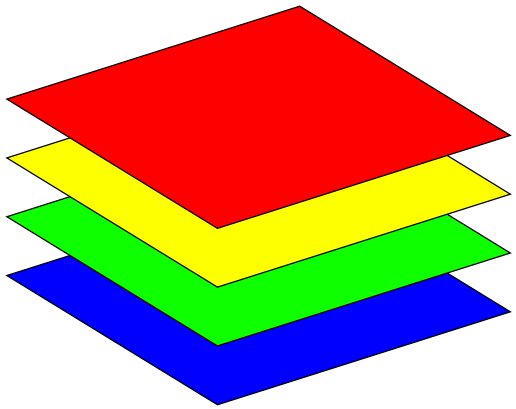

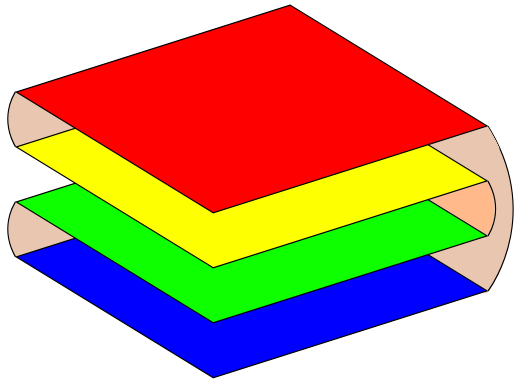

# Computational Complexity of Validity Testing

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**Summary:** If someone says "This map can be flat-folded, here is the folding," we can quickly test whether or not they were correct.

**Goal:**

Given a crease pattern, determine if it is flat-foldable.

**Result:**

Given a linear order, we can determine if it is flat-folded really fast.

**Leveraging this into a solution:**

Given a crease pattern, check every linear ordering.

If any of them are valid, then it is flat-foldable.

If none of them are valid, then it is not flat-foldable.

**Theorem: (Vinitsky 2016)**

Map Flat Foldability $\in$ NP

# Counting Problems

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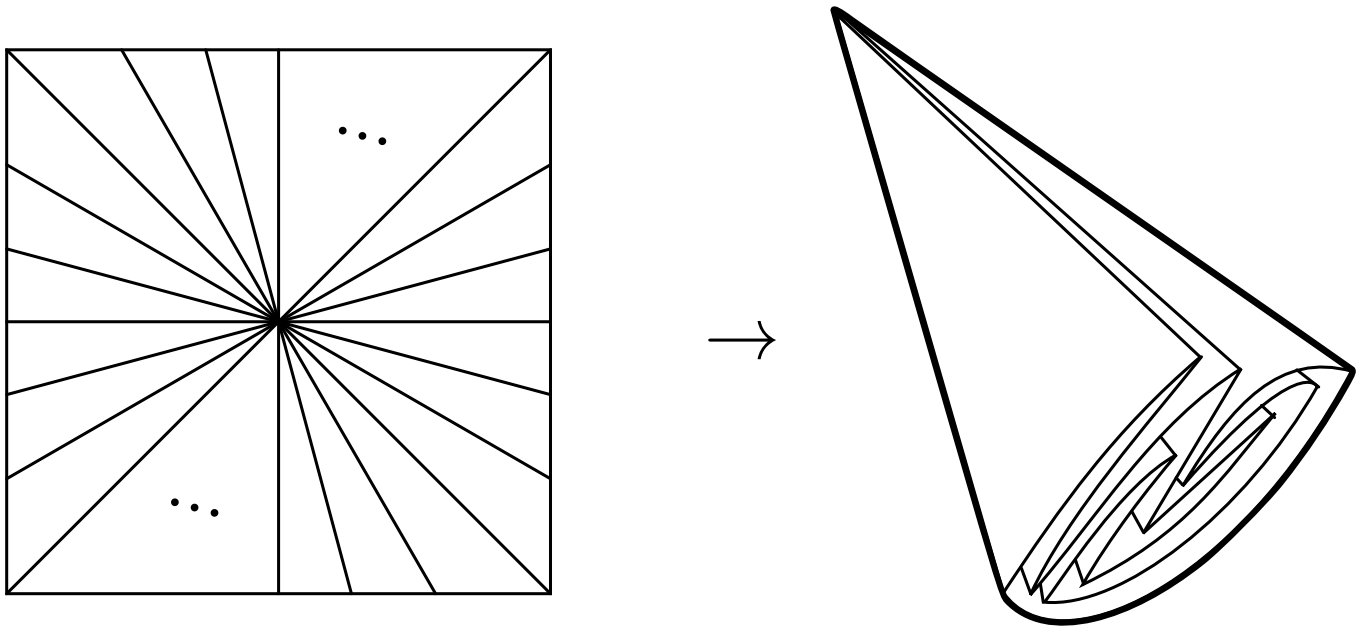

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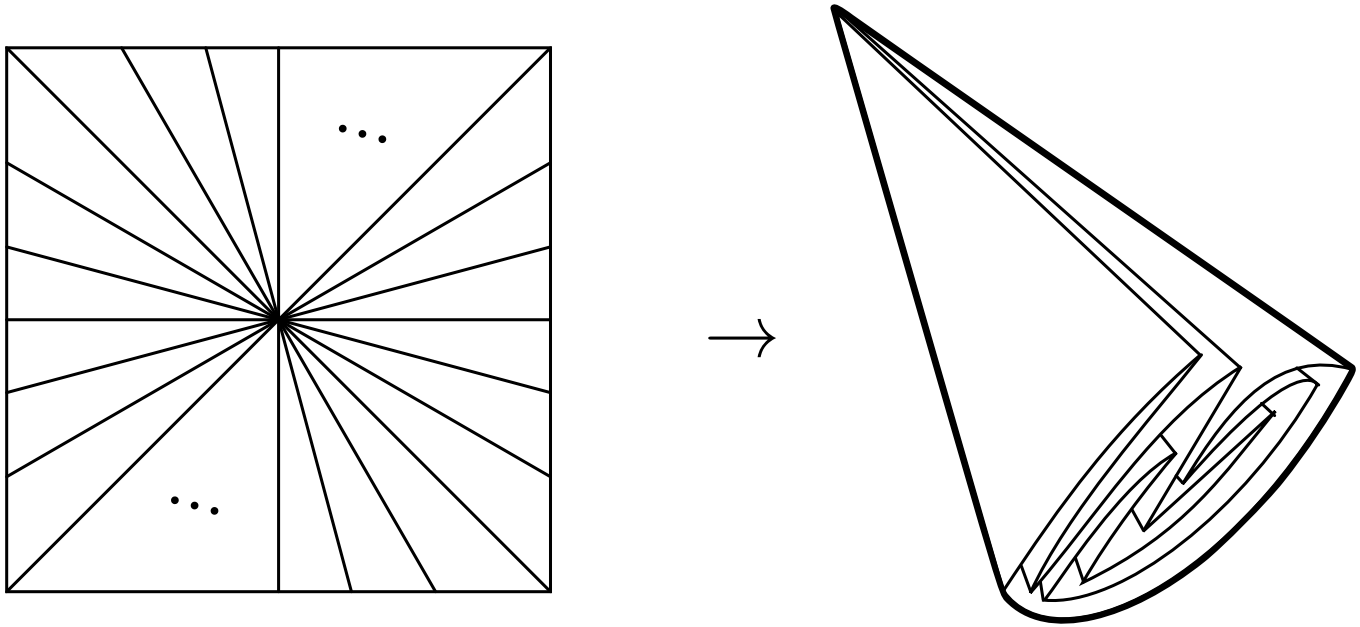

**Question:** How many foldings are generated by star patterns with $2n$ creases?

# Representing foldings of Star Patterns

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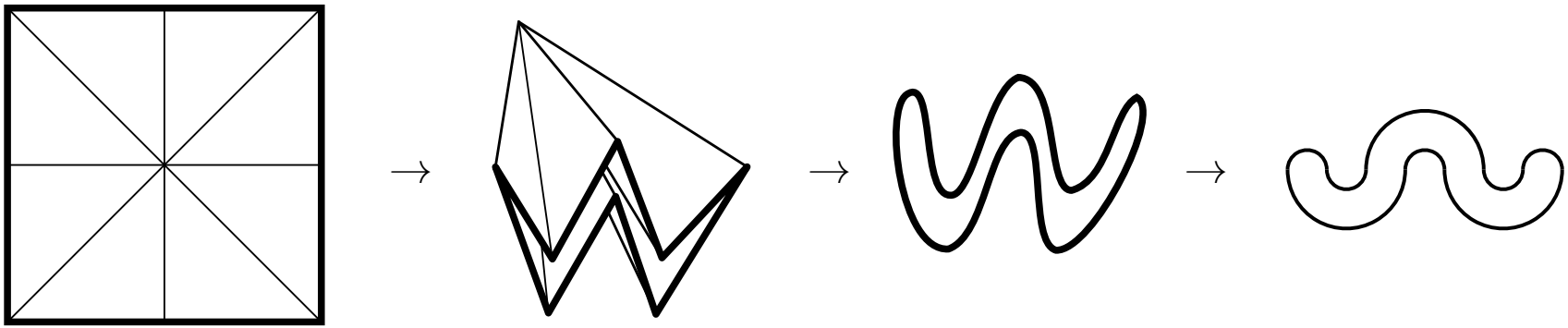
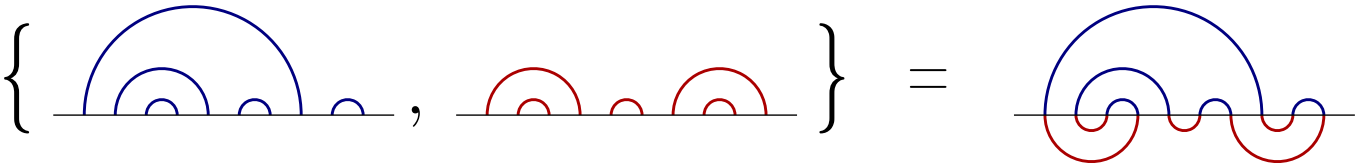


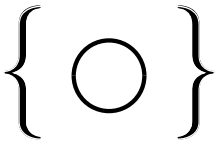
# Meanders

**Definition (Closed Meanders)**

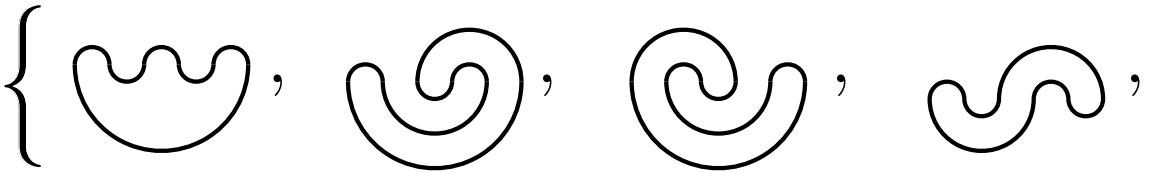
An *closed meander* of order $n$ has two collections of $n$ arches such that when they are placed opposite each other on a line, they form a Jordan curve.

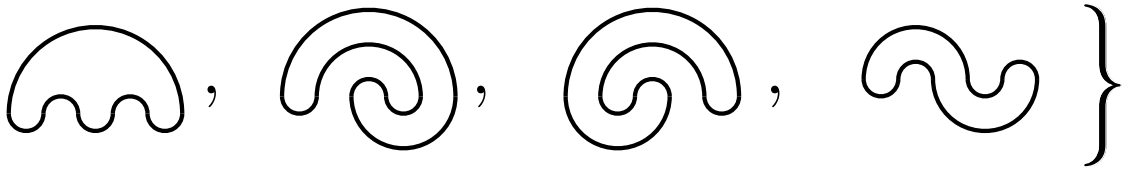


# Examples of Meanders

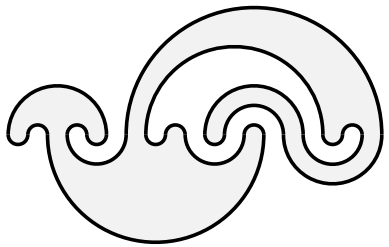
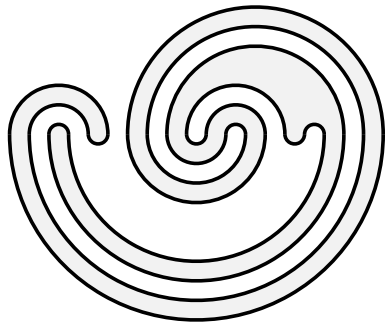
$n = 1$ 

$n = 2$

$n = 3$ 



# Examples of Meanders

$n = 10$ { ... ,  ,  , ... }

# Examples of Meanders

| **Order** $n$ | **#** **Meanders** $M_n$ |
|---|---|
| 1 | 1 |
| 2 | 2 |
| 3 | 8 |
| 4 | 42 |
| 5 | 262 |
| ⋮ | ⋮ |

Table: The sequence of Meandric Numbers

# Examples of Meanders

| **Order** $n$ | **#** **Meanders** $M_n$ |
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| 1 | 1 |
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| 10 | |
| ⋮ | ⋮ |

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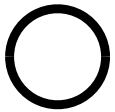
| 4 | 42 |
| 5 | 262 |
| ⋮ | ⋮ |
| 10 | 8,152,860 |
| ⋮ | ⋮ |

Table: The sequence of Meandric Numbers

# Our Technique

# Game Plan

$n = 1$



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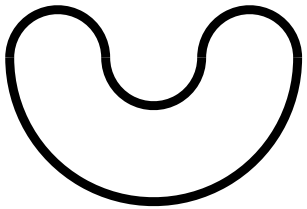

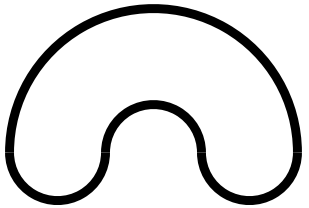

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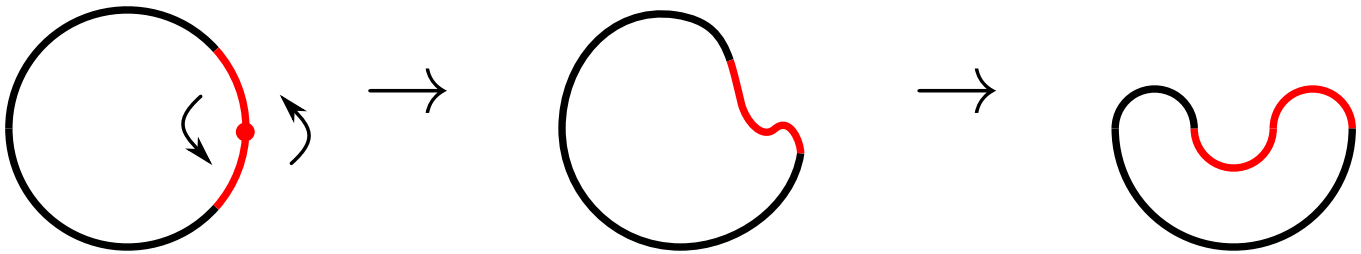

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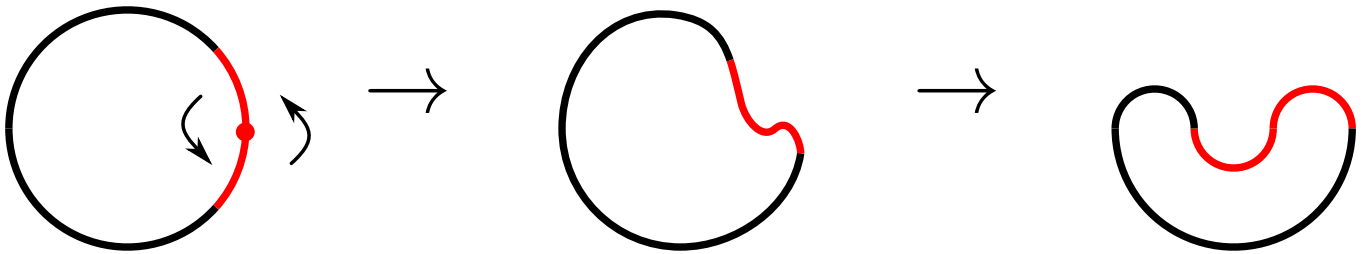
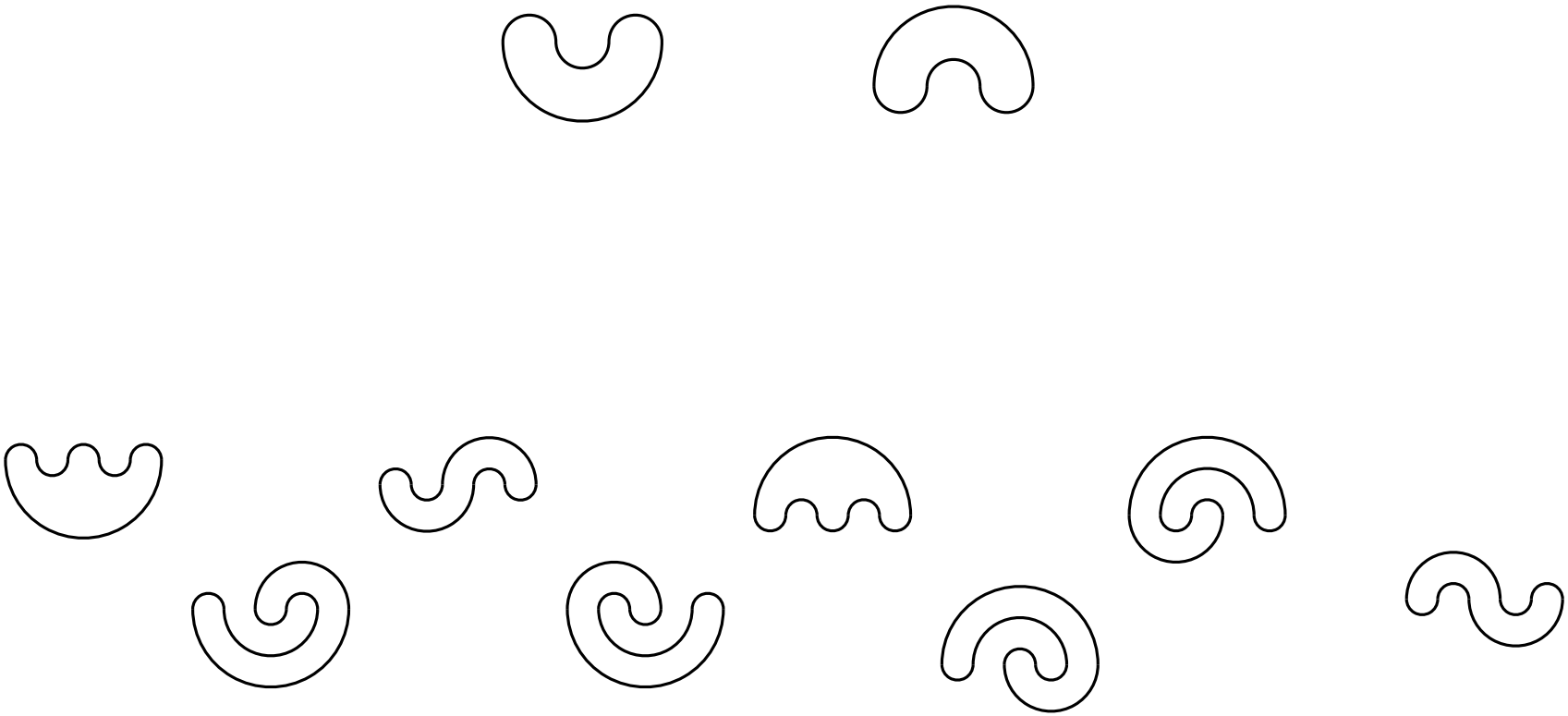


**Idea:** Produce larger meanders by adding "twists" to smaller meanders

# Will it work?



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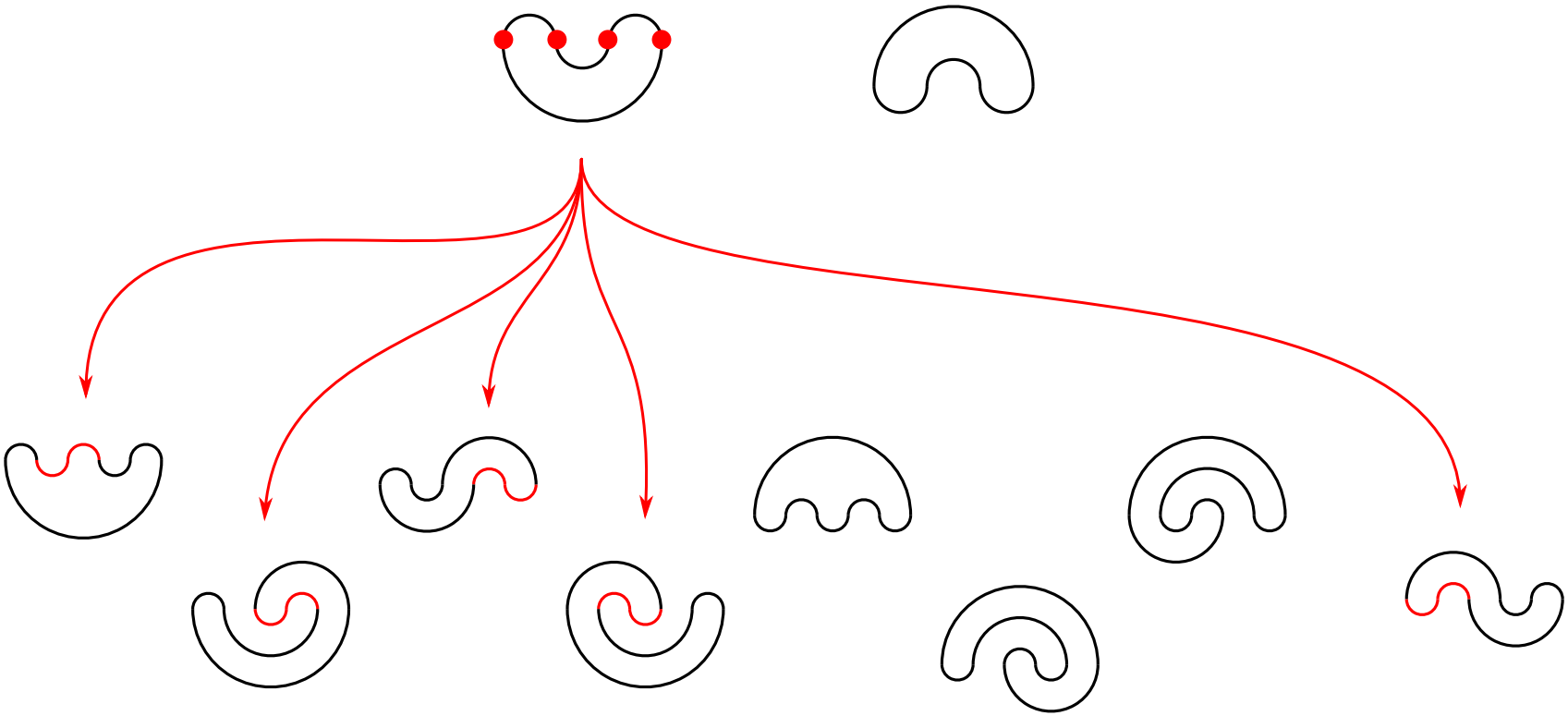

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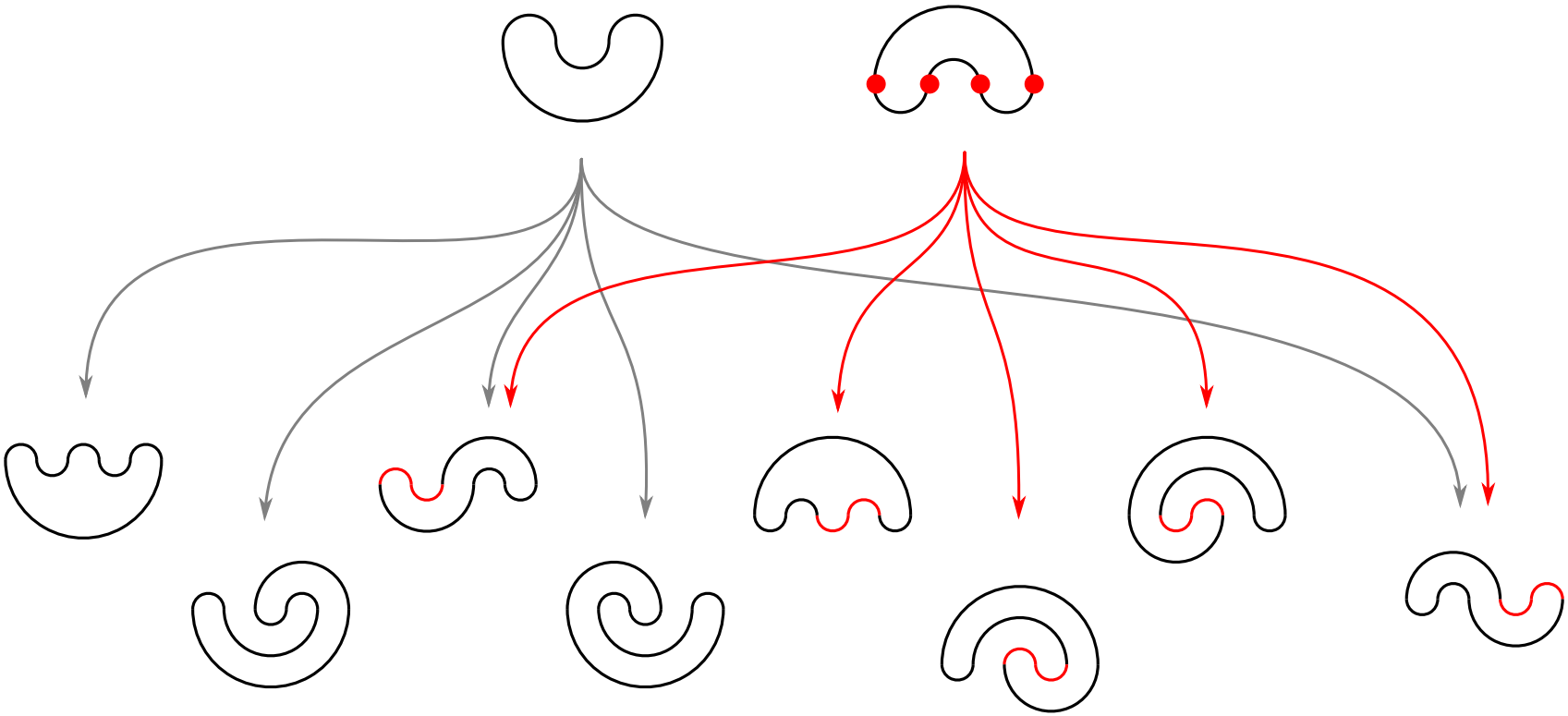

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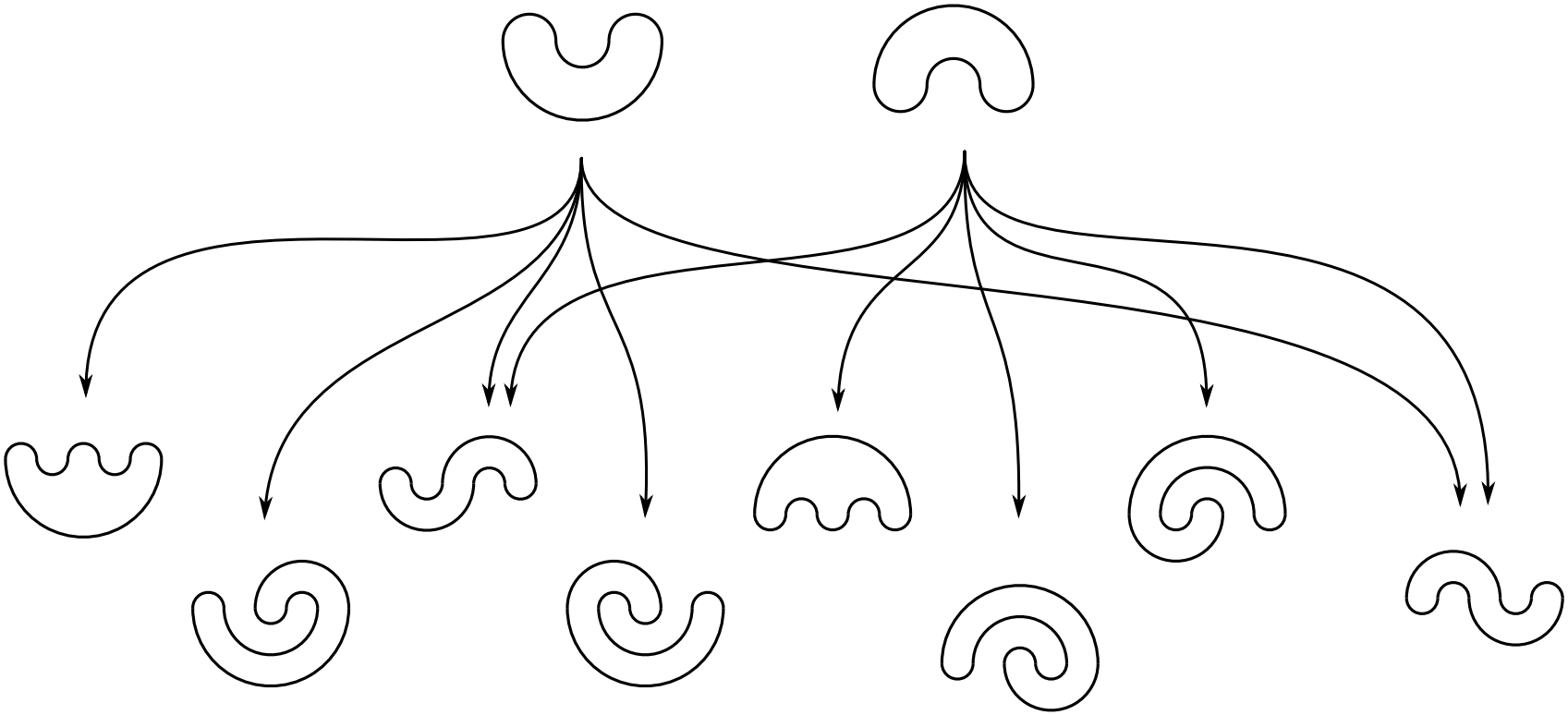

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**Answer:**

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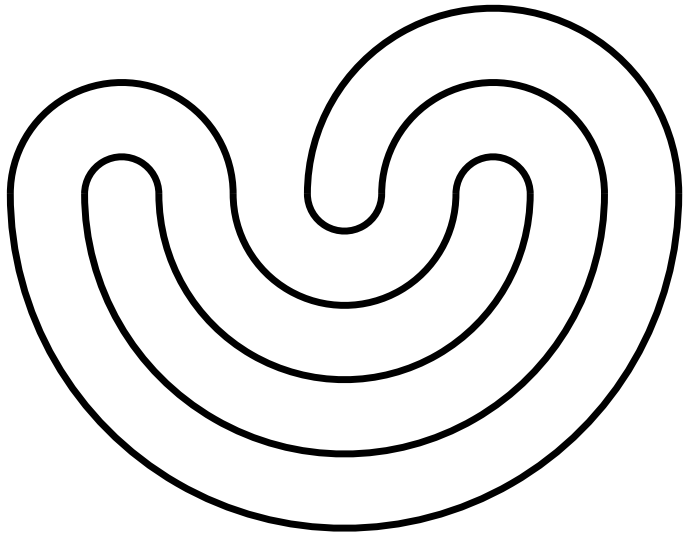
**Answer:**Sadly we cannot:



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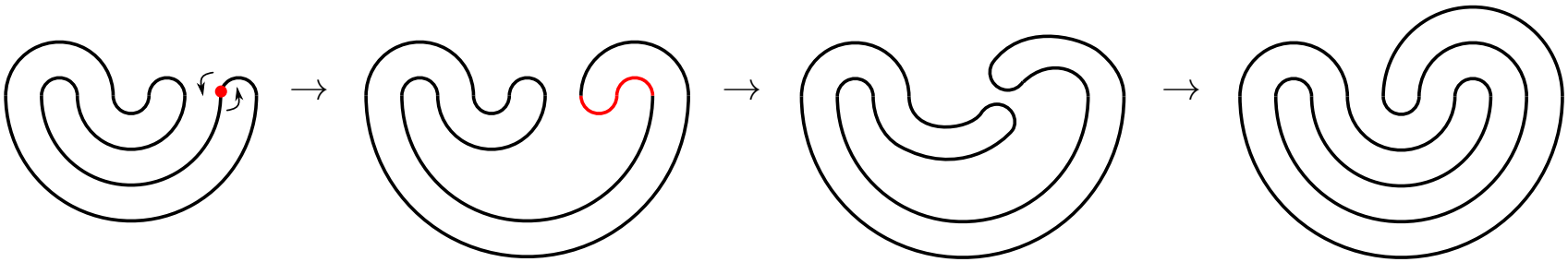

# Meanders

**Theorem**

*For every meander of order n can be produced from some meander of order $n-1$ with a single twist and at most one shuffle.*

Two issues to address:

1. Different twists yield the same meander (double counting)
2. Shuffling is necessary but greatly decreases double counting

# Simple Meanders

**Definition (Simple Meander)**

A *simple meander* is a meander of order $n$ that can be constructed without shuffling.

# Simple Meanders

## Theorem (Recurrence Relation for Simple Meanders)

*Let* $\mathbb{P}(k,n) = \{(x_1, x_2, \ldots, x_n) : \sum x_i = k \text{ and } x_i \geq 0 \forall i\}$. *Then*

$$r(n) = \sum_{i=1}^{n} \sum_{P \in \mathbb{P}(i,n+1)} \prod_{k=1}^{n+1} r(P_k)$$

$$H(n) = \sum_{i=1}^{n} r(i) \sum_{P \in \mathbb{P}(i,n)} \prod_{k=1}^{n} H(P_k)$$

$$M_n^s = 2H(n)$$

# Make sure to stop by our exhibit in the Gould Library!

*Opening Spring Term 2016*

# References

Rahnuma Islam Nishat and Sue Whitesides (2013)

Map Folding

*Canadian Conference on Computational Geomery (2013)*, p49-52