The Mathematics of Origami

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Overview

Axioms of Origami

General

i. Brief History
ii. Independence

Axioms of Origami

Seven Axioms of Origami

i. Given two points p_1 and p_2, we can fold a line connecting them.
ii. Given two points p_1 and p_2, we can fold p_1 onto p_2.
iii. Given two lines l_1 and l_2, we can fold line l_1 onto l_2.
iv. Given a point p_1 and a line l_1, we can make a fold perpendicular to l_1 passing through the point p_1.

Axioms of Origami

Seven Axioms of Origami

v. Given two points p_1 and p_2 and a line l_1, we can make a fold that places p_1 onto l_1 and passes through the point p_2.

vi. Given two points p_1 and p_2 and two lines l_1 and l_2, we can make a fold that places p1 onto line l_1 and places p_2 onto line l_2.

vii. Given a point p_1 and two lines l_1 and l_2, we can make a fold perpendicular to l_2 that places p_1 onto line l_1.

Axioms of Origami

Seven Axioms of Origami

i. The first six axioms allow all quadratic and cubic equations with rational coefficients to be solved.

ii. They also allow two of the three problems of antiquity, the trisection of an angle and the doubling of the cube, to be constructed.

Greek Problems of Antiquity

Problems of Antiquity

These were a trio of geometric problems whose solutions were attempted solely through the use of compass and straight-edge.

i. Angle Trisection
ii. Cube Duplication
iii. Circle Squaring

Angle Trisection

INCLUDE PARTIALLY FOLDED DIAGRAMS

Cube Duplication

INCLUDE PARTIALLY FOLDED DIAGRAMS

i. $\frac{\alpha}{\beta} = \sqrt[3]{2}$

ii. Thus, a cube with a side length α will have twice the volume of a cube with side length β.

Squaring the Circle

i. Impossible

ii. π

Constructability

Constructible Numbers

i. Given two points p_0 and p_1, construct a third point p_1' a distance $|p_0 p_1|$ from point p_0 such that $\overline{p_0 p_1'}$ is perpendicular to $\overline{p_0 p_1}$.

ii. Given two points p_0 and p_1, a third point p_2 can be constructed such that p_2 is collinear with p_0 and p_1, thus $|p_0 p_1| = |p_1 p_2|$.

Constructability

Constructible Numbers

iii. Given two constructible numbers α and β, we can construct $\frac{\alpha}{\beta}$, their ratio.

iv. Given two constructible numbers α and β, we can construct their sum $\alpha + \beta$ or their difference $\alpha - \beta$.

v. Given two constructible numbers α and β, we can construct $\alpha\beta$, their product.

Constructability

Given two constructible numbers α and β, we can construct $\frac{\alpha}{\beta}$, their ratio.
INCLUDE PARTIALLY FOLDED DIAGRAMS AS WELL AS FINAL FOLDED DIAGRAM

Constructability

Constructible Numbers

i. Since the set of constructible numbers is closed under addition, subtraction, multiplication, and division, it can be concluded that the set of constructible numbers form a field.

ii. Ultimately, the field of origami constructible numbers are closed under taking both square roots and cube roots.

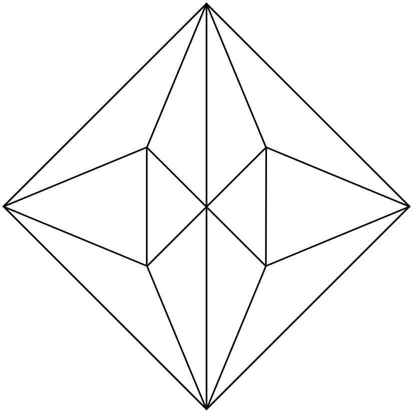
iii. The construction of the square root of any constructible number implies the field of origami constructible numbers contains the field of compass and straightedge constructible numbers.

Future Work

i. Efficiency
ii. Optimality

Foldability

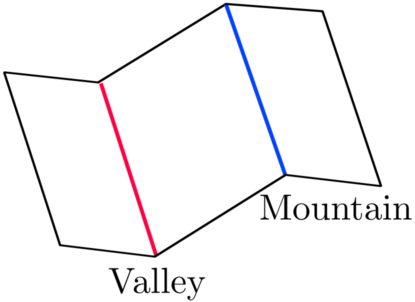
An example of a crease pattern:



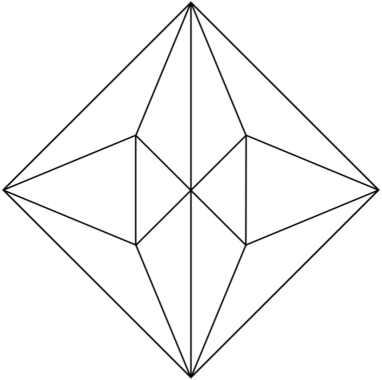
Foldability

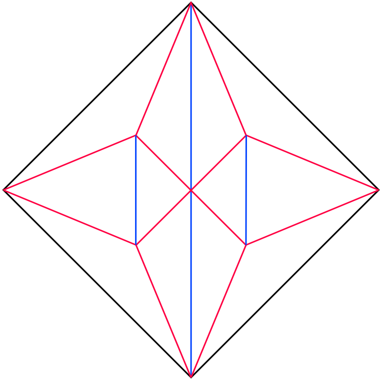
Crease Assignments

A crease pattern doesn't contain all the information about a model, however.



Foldability





Foldability

Two questions

For a given crease pattern,

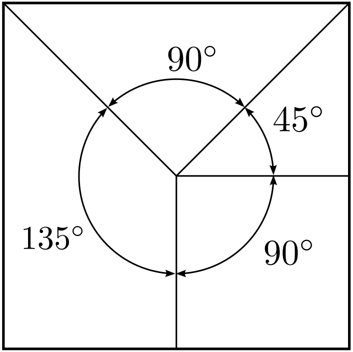
i. is there a way to fold it flat?

ii. is there a way to fold it at all?

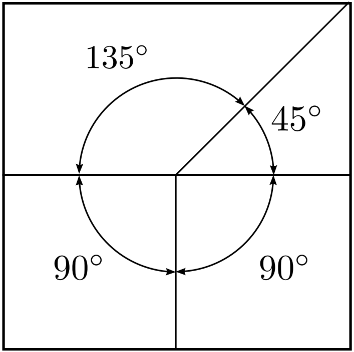
Kawasaki's Theorem

Theorem

The alternate angles around a vertex must sum to 180°.

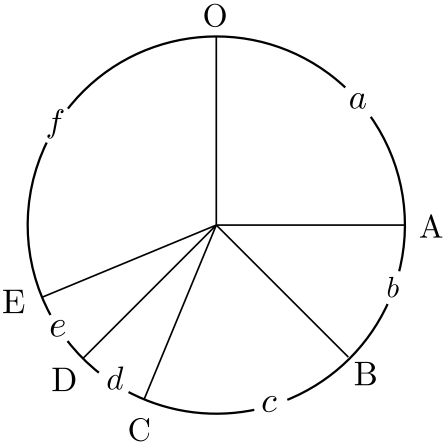


(a)



(b)

Kawasaki's Theorem



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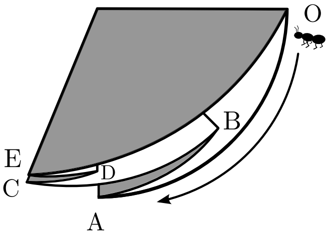

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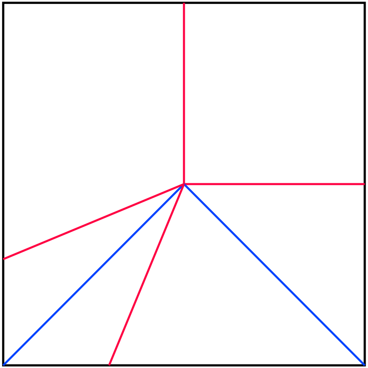
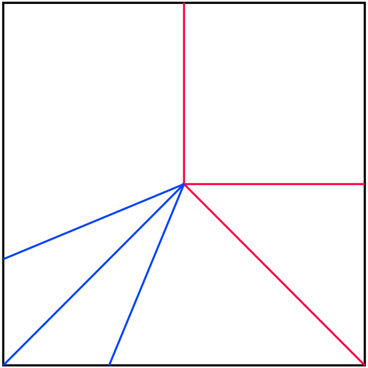
Maekawa's Theorem

Theorem

The sum of mountain + valley is ± 2.



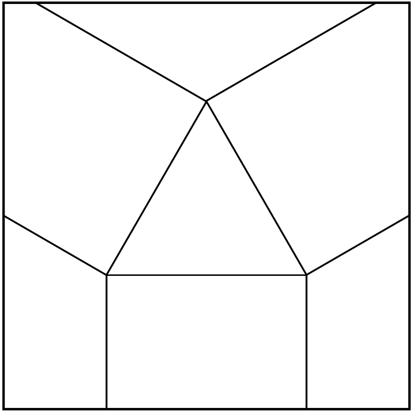


More Complicated Crease Patterns

How can we determine flat foldability for a more complicated pattern?
(a picture of a complicated crease pattern)

More Complicated Crease Patterns

Here's a crease pattern that can't fold flat!



Foldings

How can we capture the notion of a folded paper?

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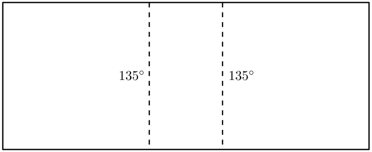

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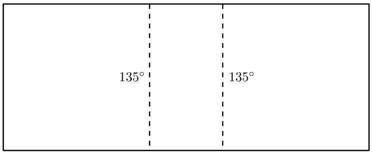
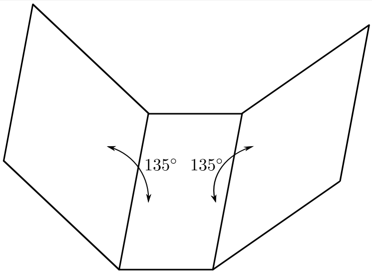
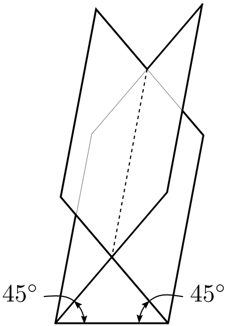




An Invalid Folding

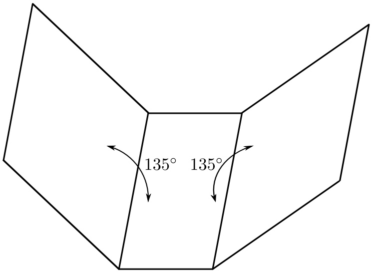
This folding has self-intersection.



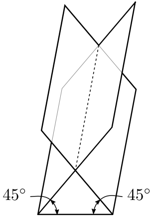
Valid Foldings

A valid folding is one that doesn't cause any self-intersecton.

Valid



Invalid



Drawing pictures on Crease Patterns

Question

If we draw something on the flat piece of paper before folding it up, what can happen when we fold it?

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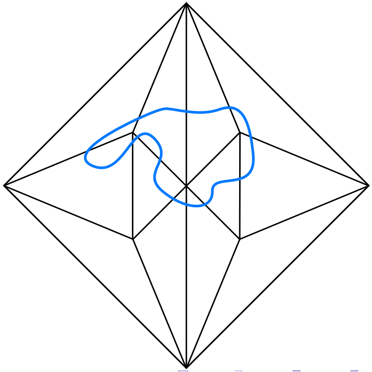
We'll see what happens when we draw Jordan curves on the crease patterns, like the picture on the right.

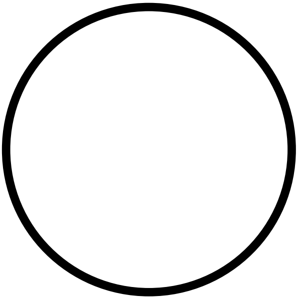


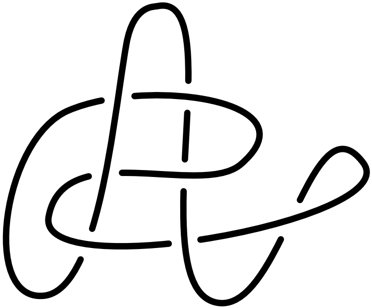
Knot Theory 101

What is a knot?

Here's how we think of a knot mathematically.

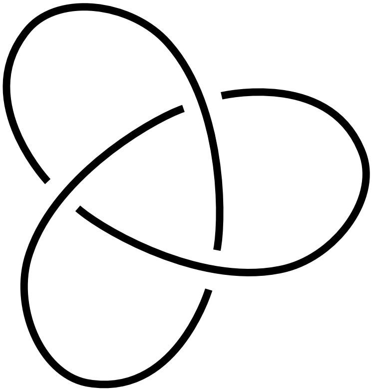
Knot Theory 101





Knot Theory 101

This is the *trefoil knot*. We can't "untangle" it, no matter how hard we try.



Connecting Topology and Origami

Theorem

A folding of a crease pattern C is valid if and only if every Jordan curve embedded in the paper before folding is mapped to the unknot after folding.

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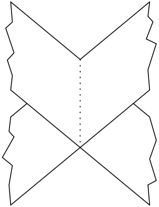
It's obvious that the first direction is true (we can't cause knot without invalid folding)

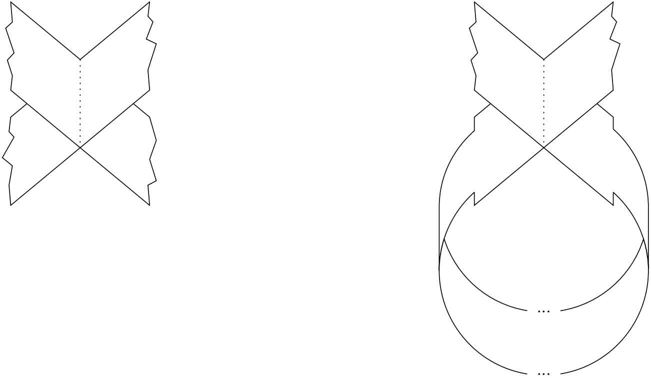
Connecting Topology and Origami

Invalid folding means intersection.



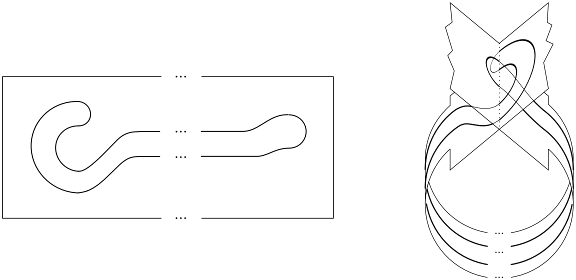
Connecting Topology and Origami

We can connect up the paper like this.



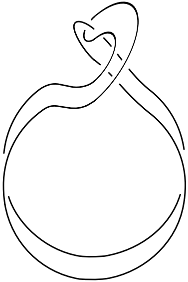
Connecting Topology and Origami

We can draw a curve like this.



Connecting Topology and Origami

Is it a knot?



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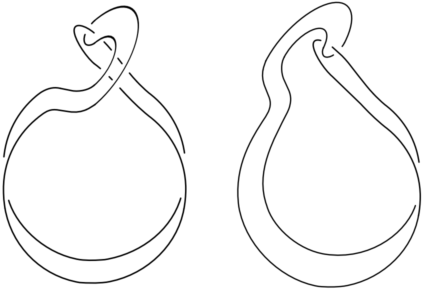

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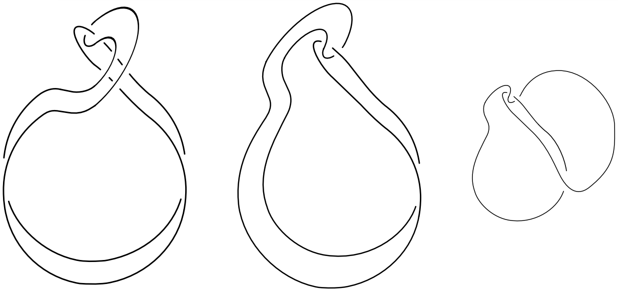

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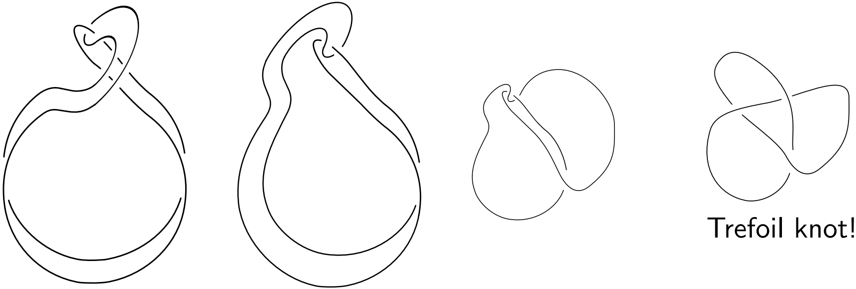


Trefoil knot!

Connecting Topology and Origami

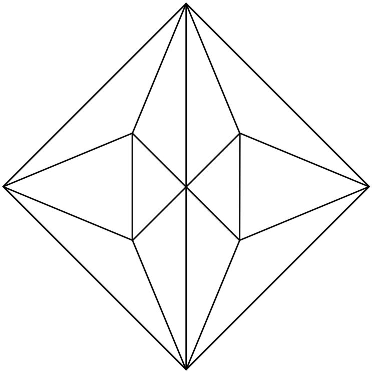
Now what?

- Maybe a characteristic for crease patterns
- =)

Determining flat-foldability

Question:

Is this crease pattern flat-foldable?



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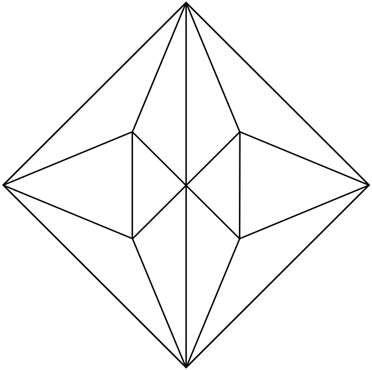
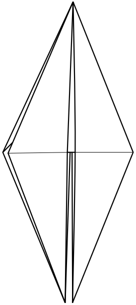
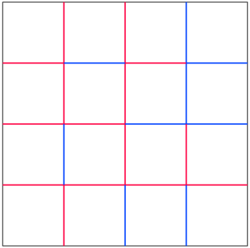
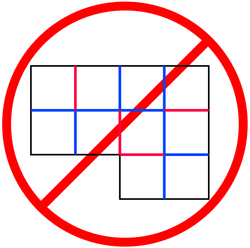
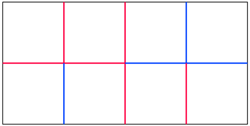


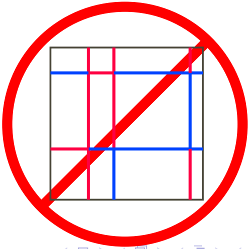


Introduction to Maps









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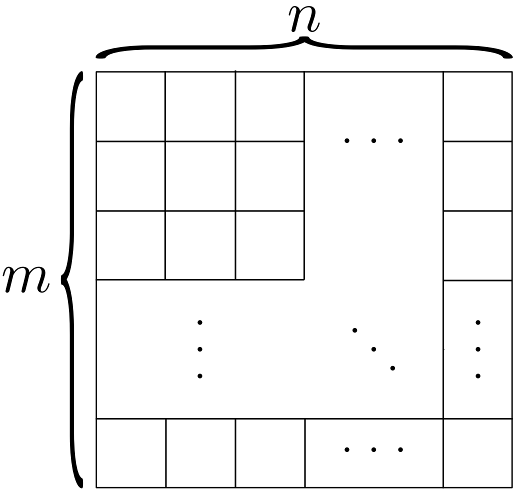

Map Folding: An Open Problem

Open Problem:

How hard is it to determine whether or not a map is flat-foldable?

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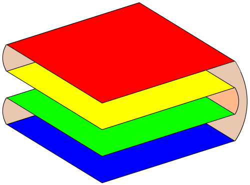
Easier Problem:

How hard is it to determine whether or not a map is *flat-folded*?

Linear Orderings

Question:

How can we represent a folded form?



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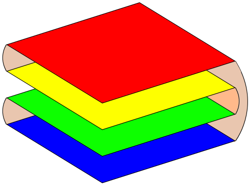


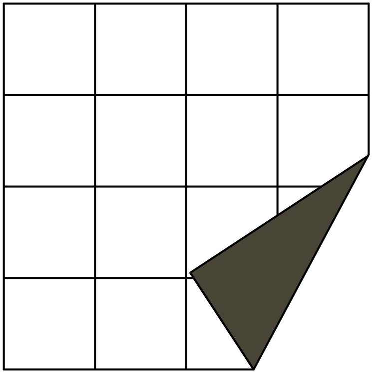
red $\rightarrow$ yellow $\rightarrow$ green $\rightarrow$ blue

Linear Orderings

Question:

How can we tell if a linear ordering is realizable by a given crease pattern?

Checkerboard Pattern



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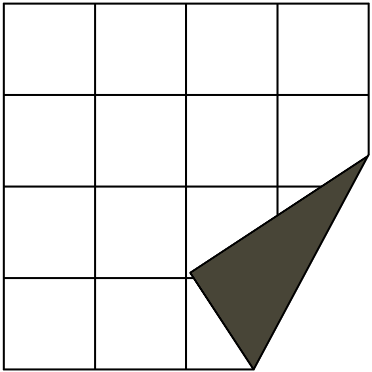

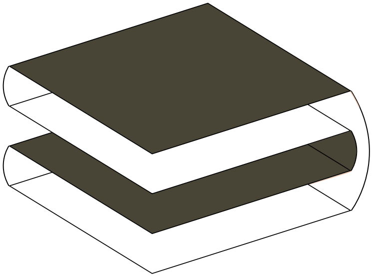

Checkerboard Pattern

Goal:

Label each face as being light-side or dark-side up in *any* folding.

	?	?	?
?	?	?	?
?	?	?	?
?	?	?	?

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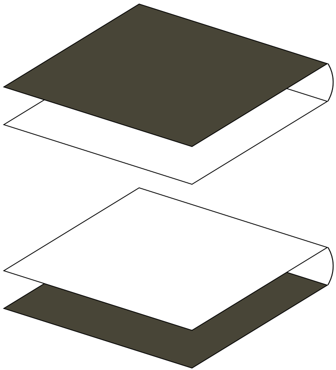

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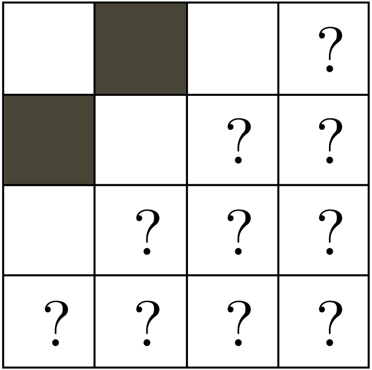

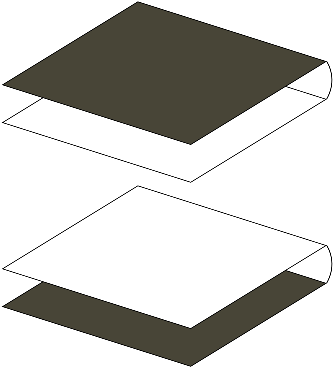

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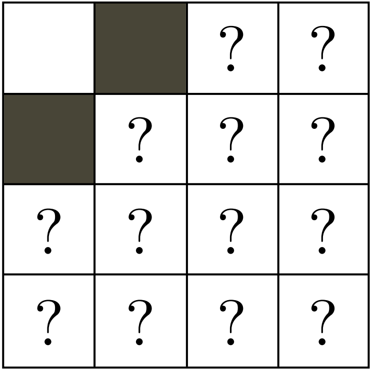

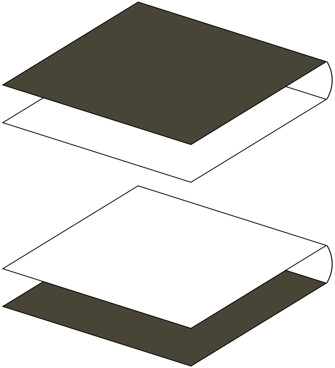

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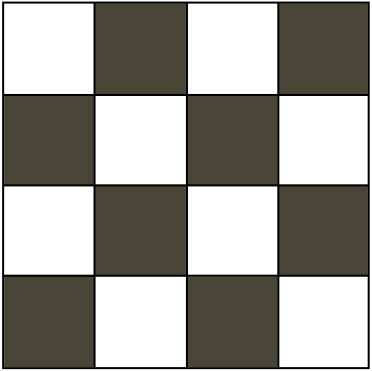

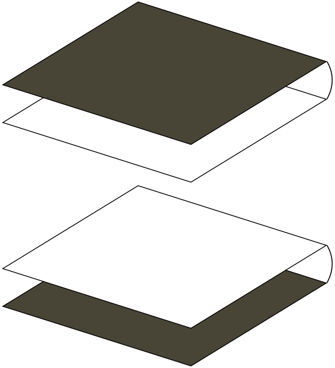

Partial Ordering

Definition:

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$a \to c \to b \to d$

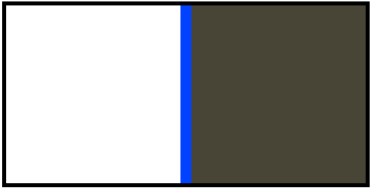
$c \to d \to a \to b$

...

Partial Ordering

Goal:

Use checkerboard pattern and crease assignments to tell, for each pair of adjacent faces, which face comes first in *all* linear orderings.



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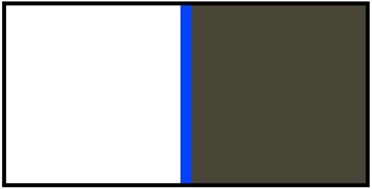

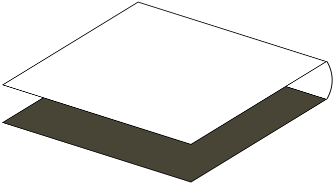

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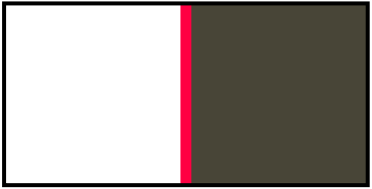

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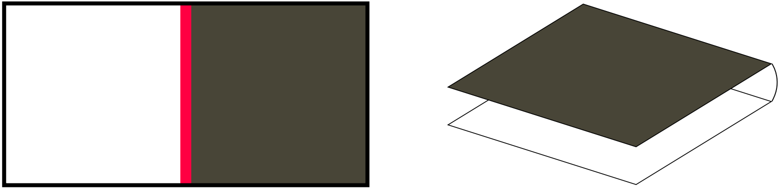

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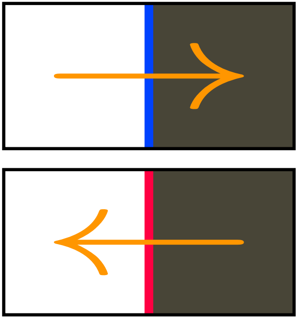

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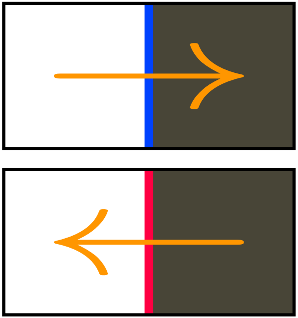

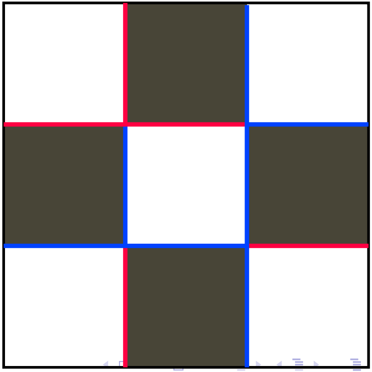

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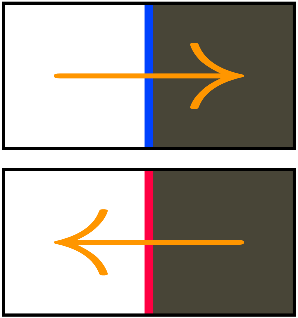

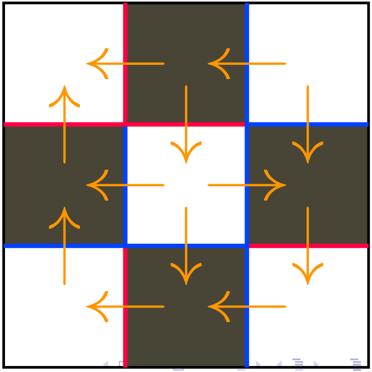

Linear Orderings

Is satisfying this partial ordering enough to ensure foldability?

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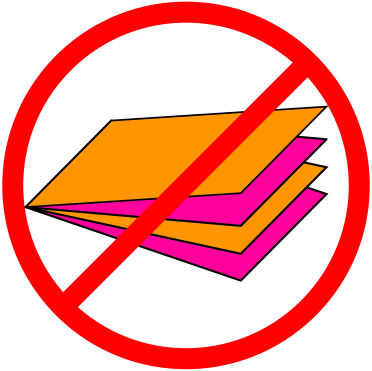
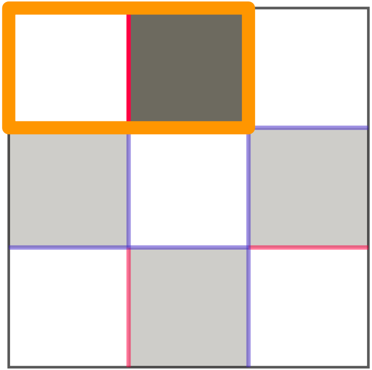
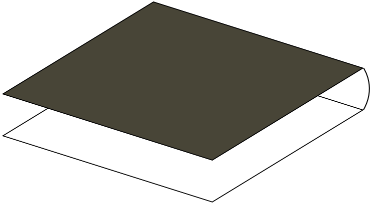


Butterflies





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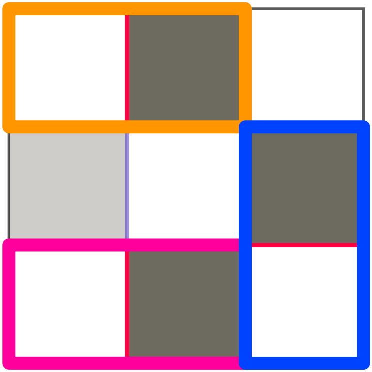

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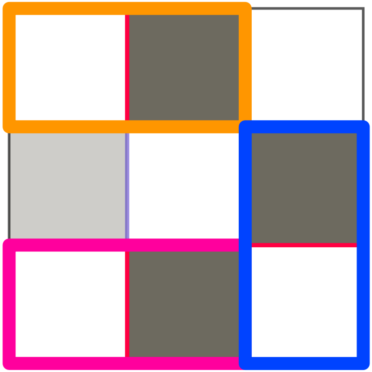
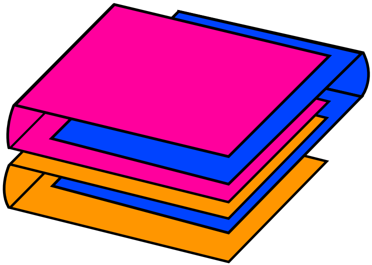
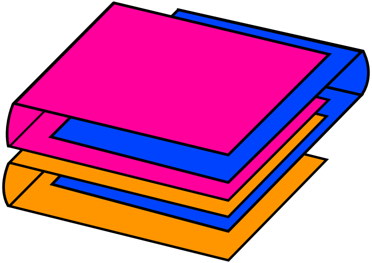




Butterfly Condition

Goal:

Enumerate the realizable configurations of *twin butterfly* pairs.



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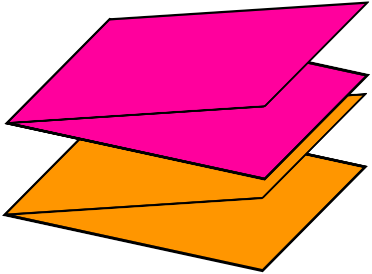

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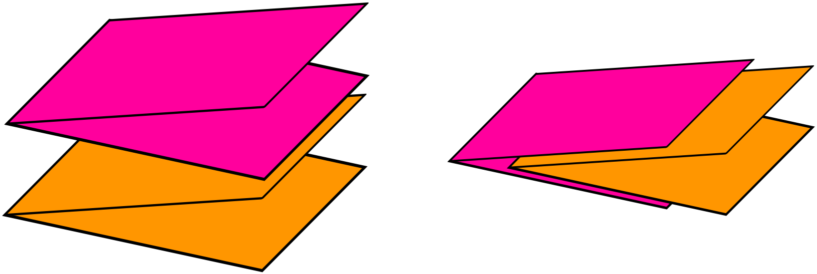

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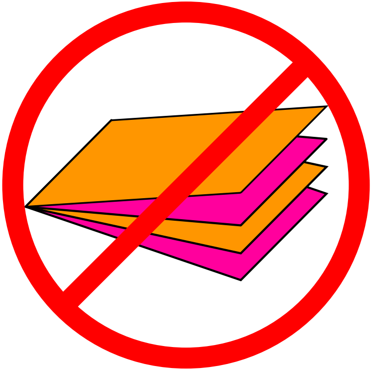

Conditions for flat-foldability

Theorem:

A linear ordering $\mathcal{L}$ of faces is flat-foldable if and only if (i) $\mathcal{L}$ satisfies the partial ordering given by the map and (ii) every pair of twin butterflies stacks or nests in $\mathcal{L}$.

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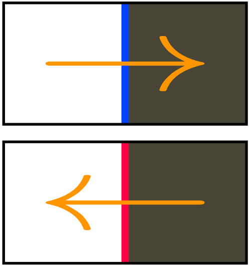

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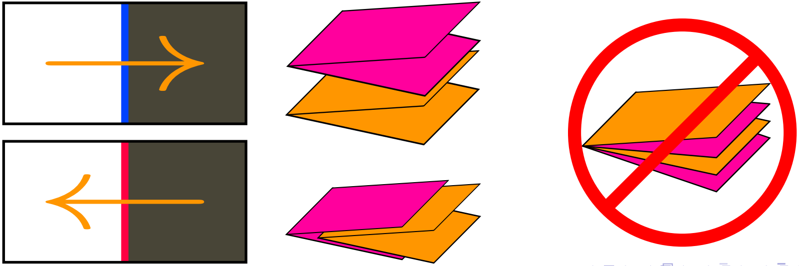

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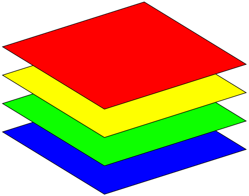

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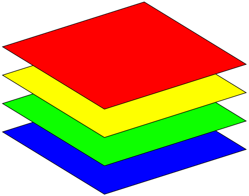

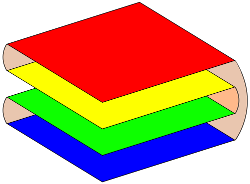

Computational Complexity of Validity Testing

Algorithm for determining whether or not a linear ordering is valid:

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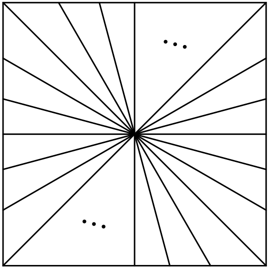
Summary: If someone says "This map can be flat-folded, here is the folding," we can quickly test whether or not they were correct.

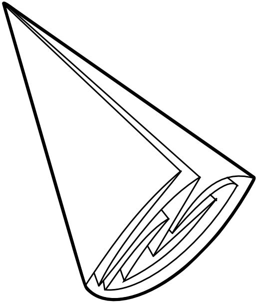
Counting Problems in Origami

- How many ways can we fold an $n \times m$ map?
    - How many ways can we fold a $1 \times m$ map?
- How many $n \times m$ map crease patterns can be folded, period?
    - How many ways can we fold a $1 \times m$ map?

Star Patterns



$\rightarrow$



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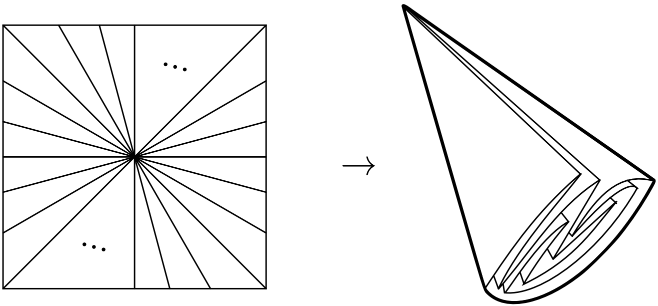
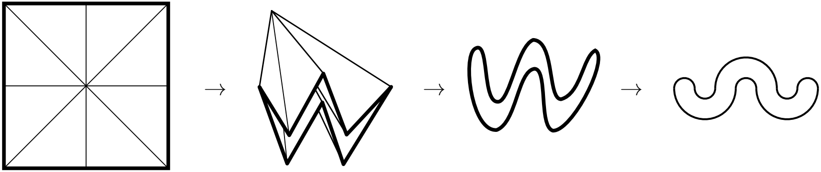


Question: How many foldings are generated by star patterns with $2n$ creases?

Representing foldings of Star Patterns

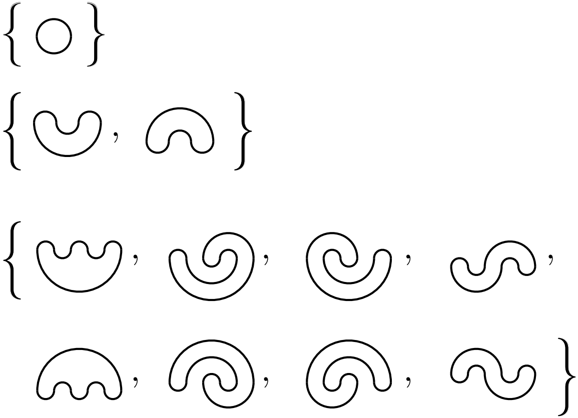


Examples of Meanders

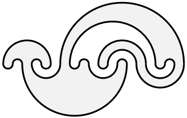
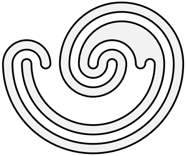
$n = 1$

$n = 2$

$n = 3$



Examples of Meanders

$n = 10 \; \{ \; \ldots,$  ,  $, \; \ldots \; \}$

Examples of Meanders

Order n	**#** **Meanders** M_n
1	1
2	2
3	8
4	42
5	262
$\vdots$	$\vdots$

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Order n	**# Meanders** M_n
1	1
2	2
3	8
4	42
5	262
$\vdots$	$\vdots$
10	

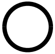
Examples of Meanders

Order n	**# Meanders** M_n
1	1
2	2
3	8
4	42
5	262
$\vdots$	$\vdots$
10	8,152,860
$\vdots$	$\vdots$

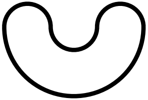
Table : The sequence of Meandric Numbers

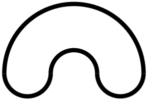
Game Plan

$$n = 1$$



$$n = 2$$





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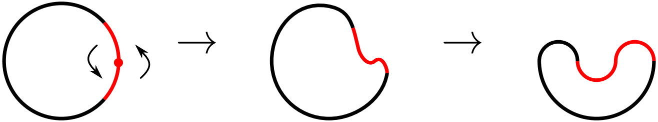

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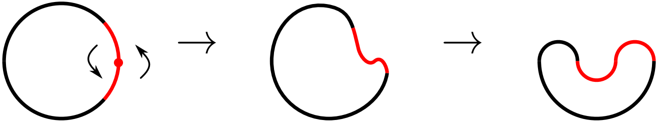
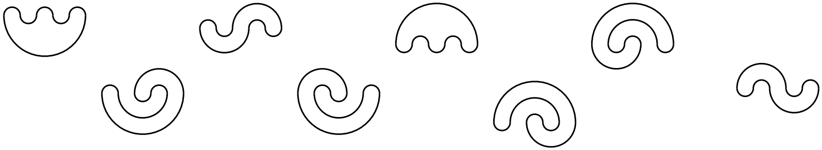


Idea: Produce larger meanders by adding “twists” to smaller meanders

Will it work?



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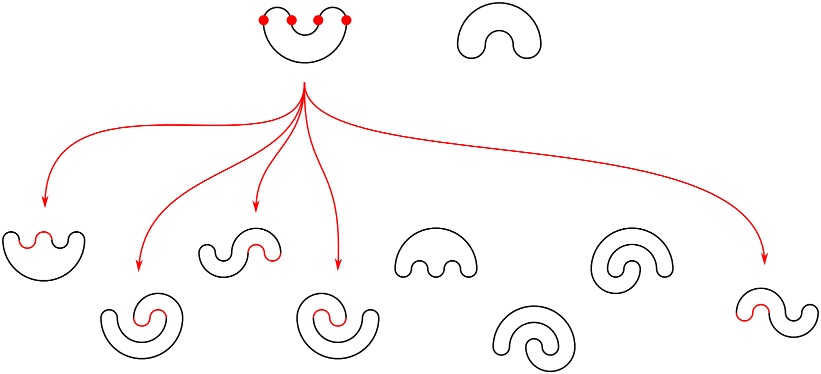

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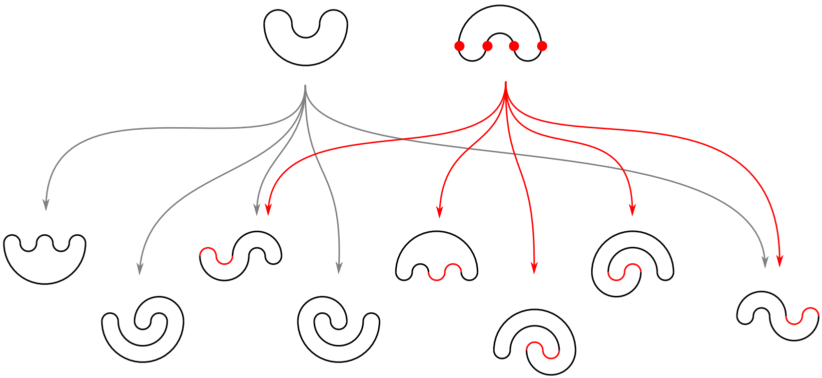

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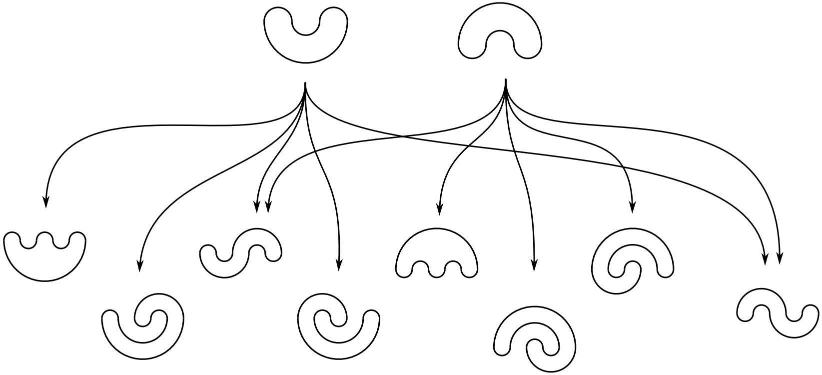

Will it work?

Question: Can we get all meanders by repeatedly doing this?

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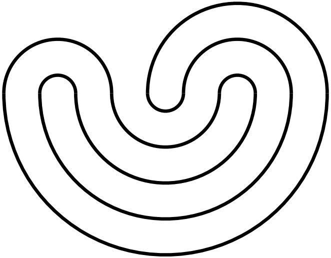
Answer: Sadly we cannot:



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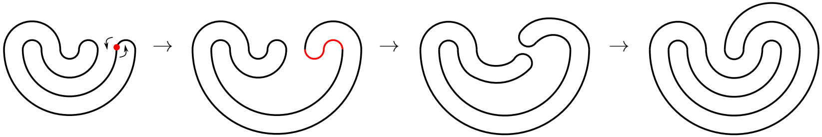

Meanders

Theorem

All meanders may be built from $m_1 = \circ$ with at most one shuffle after each twist.

Simple Meanders

Definition (Simple Meander)

A *simple meander* is a meander of order n that can be constructed without shuffling.

Simple Meanders

Theorem (Recurrence Relation for Simple Meanders)

Let $\mathbb{P}(k,n) = \{(x_1, x_2, \ldots, x_n) : \sum x_i = k \text{ and } x_i \geq 0 \forall i\}$. *Then*

$$r(n) = \sum_{i=1}^{n} \sum_{P \in \mathbb{P}(i,n+1)} \prod_{k=1}^{n+1} r(P_k)$$

$$H(n) = \sum_{i=1}^{n} r(i) \sum_{P \in \mathbb{P}(i,n)} \prod_{k=1}^{n} H(P_k)$$

$$M_n^s = 2H(n)$$

References

John Smith (2012)
Title of the publication
Journal Name 12(3), 45 – 678.

Q.E.D.