The Axioms of the Greeks

The famous Greek mathematician Euclid (c. 300 BCE) produced a set of rules, known as axioms, that formalize what we intuitively think is true about doing geometry with a compass and straightedge.

Robert Simson
The Elements of Euclid
Philadelphia : Printed for Conrad and Co., 1810

Angle Trisection

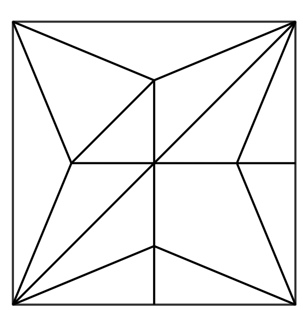
Mathematicians have struggled for millenia to perform certain geometric constructions using a compass and straightedge- for example, to divide any given angle into thirds. In the 19^{th} century it was proven that it is actually impossible to perfectly trisect an arbitrary angle with compass and straightedge. However...

The Axioms of Origami

There is a corresponding set of axioms for doing geometry with origami. These formalize what we can accomplish by folding lines on a piece of paper. There are seven axioms, four of which we include here.

A Better Answer

In fact, if our paper is thin enough, we can get an arbitrarily large perimeter! We can do this by tiling this crease pattern in a grid.



Angle Trisection with Origami

It is indeed possible to trisect any angle with folds! Here are the step by step instructions for doing so, making use of the axioms of origami pictured above. This technique was found in the 1970s by Hisashi Abe. What do you think makes origami more powerful than compass and straightedge?

Types of Foldability

Sometimes, a model can't be folded flat, but it can be folded in a 3D form. This flower is an example – you can see that if you tried to press it in a book, it would get crushed. Unfortunately, it's much more difficult to study 3D foldability. It isn't clear how to tell from the crease pattern that the flower won't fold flat.

Kawasaki's Theorem

A single vertex can be folded flat if and only if the sum of every other angle around the vertex is 180 degrees.

Maekawa's Theorem

A single vertex can be folded flat if and only if the difference between the number of mountain folds and the number of valley folds is plus or minus 2.

Some crease patterns are not foldable.

Here's an example of a crease pattern that can't be folded flat at all. In fact, it can't even be folded in a 3D form without curving the paper (unlike the flower above).

Notice that this crease pattern satisfies Maekawa's and Kawasaki's theorems--the fact that it still can't be folded flat demonstrates why these theorems are not sufficient.

How do we tell if a crease pattern can be folded?

Can we tell if a given crease pattern can be folded flat? Or more generally, if it can be folded at all? There are two important conditions that let us know when a crease pattern can be folded flat: Kawasaki's and Maekawa's theorems.

Note that these are necessary but not sufficient conditions for flat foldability; there are some crease patterns that satisfy these conditions but still cannot fold flat.

Map Foldability is a hard problem.

Unfortunately, studying maps is still difficult. It takes a computer a long time to figure out if a map crease pattern can be folded flat – in fact, the best way mathematicians know of is to simply try every possible way of folding the map and seeing if any way works.

Try it yourself!

Here are two map crease patterns. One of them can be folded flat, and the other can't. On the table to the left, you can try to find out which one can be folded. Remember that red lines are mountian folds and must point up; and blue lines are valley folds, and point down.

The Napkin Folding Problem

Can you fold a piece of paper (or a napkin) into a shape that is has a larger perimeter (distance around the outside) than the original square? Go ahead and to to do it at the table to the left. It seems impossible!

Answer

It might be surprising, but it turns out you can. Take a look at the model to the left—it has slightly greater perimeter than the starting square.

What is a Crease Pattern?

Crease patterns are the most important tool for studying origami. The lines specify where to fold the paper and the color indicates the direction of the fold, as shown to the right. We call the intersection of multiple creases at one point a *vertex*.