

# The Mathematics of Origami

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# Overview

- 1 Origami Constructions
  - Axioms, Euclid &c.
- 2 The Basics of Foldability
  - Local Flat Foldability
  - General Foldability
  - Knots
- 3 Map Folding: An Open Problem
  - Overview
  - Linear Orderings
  - Formal Proof
- 4 Meanders
  - Our Method

# Axioms of Origami

## General

- i. Brief History
- ii. Independence

# Axioms of Origami

## Seven Axioms of Origami

- i. Given two points  $p_1$  and  $p_2$ , we can fold a line connecting them.
- ii. Given two points  $p_1$  and  $p_2$ , we can fold  $p_1$  onto  $p_2$ .
- iii. Given two lines  $l_1$  and  $l_2$ , we can fold line  $l_1$  onto  $l_2$ .
- iv. Given a point  $p_1$  and a line  $l_1$ , we can make a fold perpendicular to  $l_1$  passing through the point  $p_1$ .

# Axioms of Origami

## Seven Axioms of Origami

- v. Given two points  $p_1$  and  $p_2$  and a line  $l_1$ , we can make a fold that places  $p_1$  onto  $l_1$  and passes through the point  $p_2$ .
- vi. Given two points  $p_1$  and  $p_2$  and two lines  $l_1$  and  $l_2$ , we can make a fold that places  $p_1$  onto line  $l_1$  and places  $p_2$  onto line  $l_2$ .
- vii. Given a point  $p_1$  and two lines  $l_1$  and  $l_2$ , we can make a fold perpendicular to  $l_2$  that places  $p_1$  onto line  $l_1$ .

# Axioms of Origami

## Seven Axioms of Origami

- i. The first six axioms allow all quadratic and cubic equations with rational coefficients to be solved.
- ii. They also allow two of the three problems of antiquity, the trisection of an angle and the doubling of the cube, to be constructed.

# Greek Problems of Antiquity

## Problems of Antiquity

These were a trio of geometric problems whose solutions were attempted solely through the use of compass and straight-edge.

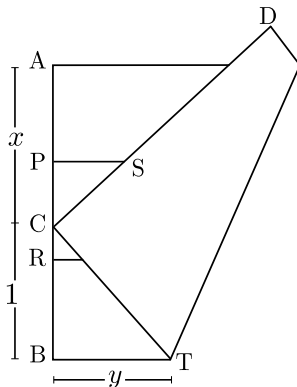
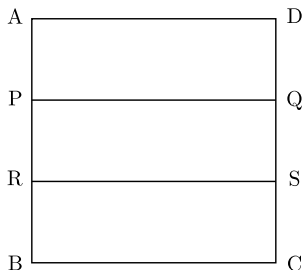
- i. Angle Trisection
- ii. Cube Duplication
- iii. Circle Squaring

# Angle Trisection

INCLUDE PARTIALLY FOLDED DIAGRAMS



# Cube Duplication



- i.  $\frac{\alpha}{\beta} = \sqrt[3]{2}$
- ii. Thus, a cube with a side length  $\alpha$  will have twice the volume of a cube with side length  $\beta$ .

# Squaring the Circle

- i. Impossible
- ii.  $\pi$

# Constructability

## Constructible Numbers

- i. Given two points  $p_0$  and  $p_1$ , construct a third point  $p'_1$  a distance  $|p_0p_1|$  from point  $p_0$  such that  $\overline{p_0p'_1}$  is perpendicular to  $\overline{p_0p_1}$ .
- ii. Given two points  $p_0$  and  $p_1$ , a third point  $p_2$  can be constructed such that  $p_2$  is collinear with  $p_0$  and  $p_1$ , thus  $|p_0p_1| = |p_1p_2|$ .

# Constructability

## Constructible Numbers

- iii. Given two constructible numbers  $\alpha$  and  $\beta$ , we can construct  $\frac{\alpha}{\beta}$ , their ratio.
- iv. Given two constructible numbers  $\alpha$  and  $\beta$ , we can construct their sum  $\alpha + \beta$  or their difference  $\alpha - \beta$ .
- v. Given two constructible numbers  $\alpha$  and  $\beta$ , we can construct  $\alpha\beta$ , their product.

# Constructability

Given two constructible numbers  $\alpha$  and  $\beta$ , we can construct  $\frac{\alpha}{\beta}$ , their ratio.

INCLUDE PARTIALLY FOLDED DIAGRAMS AS WELL AS  
FINAL FOLDED DIAGRAM

# Constructability

## Constructible Numbers

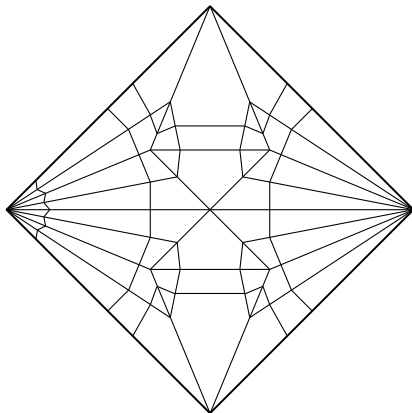
- i. Since the set of constructible numbers is closed under addition, subtraction, multiplication, and division, it can be concluded that the set of constructible numbers form a field.
- ii. Ultimately, the field of origami constructible numbers are closed under taking both square roots and cube roots.
- iii. The construction of the square root of any constructible number implies the field of origami constructible numbers contains the field of compass and straightedge constructible numbers.

# Future Work

- i. Efficiency
- ii. Optimality

# Foldability

An example of a crease pattern:

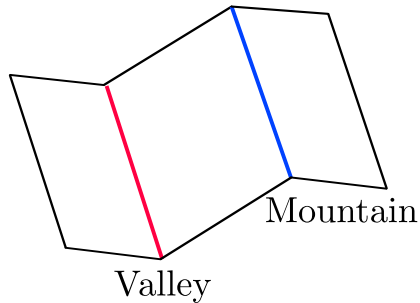




# Foldability

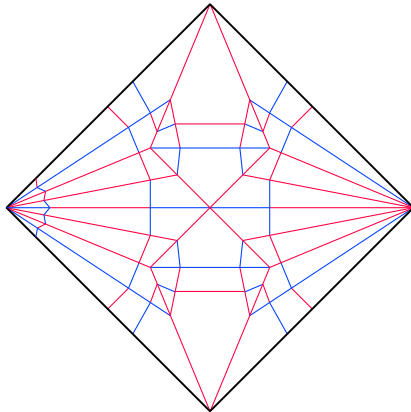
## Crease Assignments

A crease pattern doesn't contain all the information about a model, however.



# Foldability

A crease pattern with mountain-valley assignment:



# Foldability

## Two questions

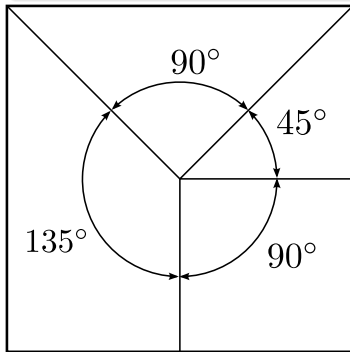
For a given crease pattern,

- i. is there a way to fold it flat?
- ii. is there a way to fold it at all?

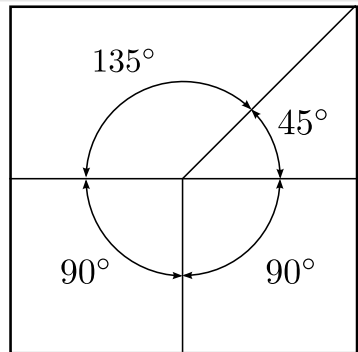
# Flat-Foldability Conditions

## Kawasaki's Theorem (1989)

The alternate angles around a vertex must sum to  $180^\circ$ .

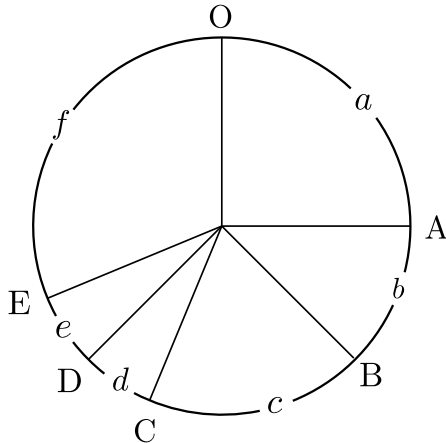


(a)

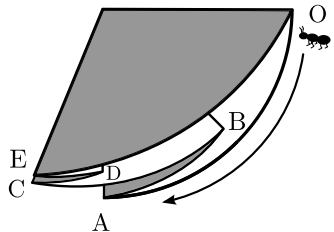


(b)

# Kawasaki's Theorem



(a)

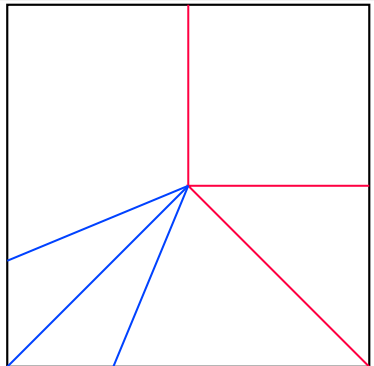
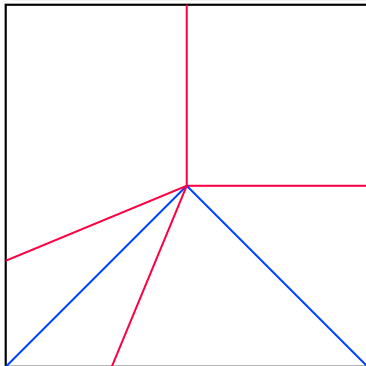


(b)

# Flat-Foldability Conditions

## Maekawa's Theorem (1986)

The sum of mountain + valley is  $\pm 2$ .

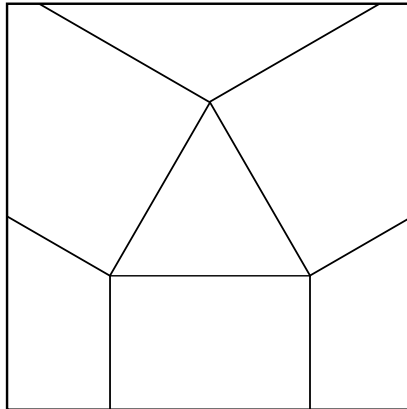


# More Complicated Crease Patterns

How can we determine flat foldability for a more complicated pattern?

# More Complicated Crease Patterns

Here's a crease pattern that can't fold flat!



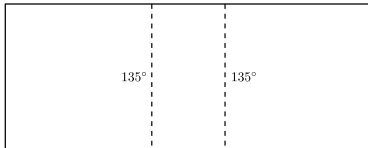


# Foldings

How can we capture the notion of a folded paper?

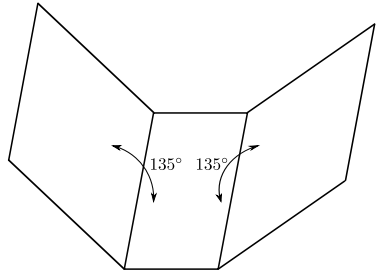
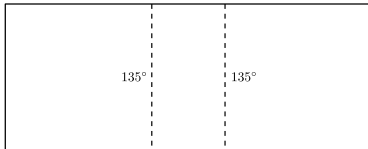
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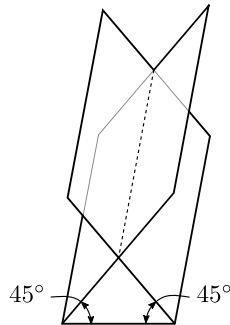
# Foldings

How can we capture the notion of a folded paper?



# An Invalid Folding

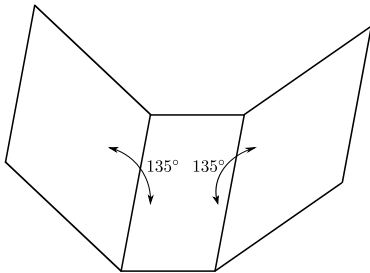
This folding has  
self-intersection.



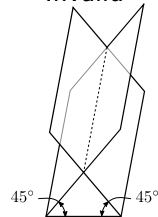
# Valid Foldings

A valid folding is one that doesn't cause any self-intersection.

Valid



Invalid



# Drawing pictures on Crease Patterns

## Question

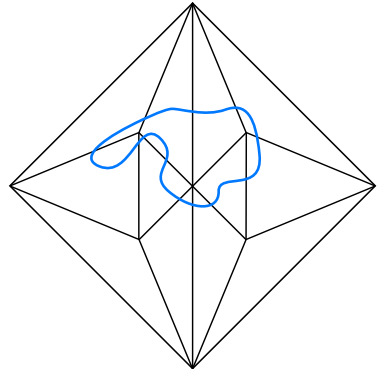
If we draw something on the flat piece of paper before folding it up, what can happen when we fold it?

# Drawing pictures on Crease Patterns

## Question

If we draw something on the flat piece of paper before folding it up, what can happen when we fold it?

We'll see what happens when we draw Jordan curves on the crease patterns, like the picture on the right.



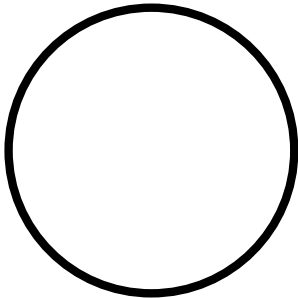
# Knot Theory 101

What is a knot?



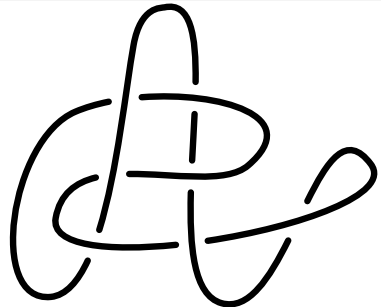
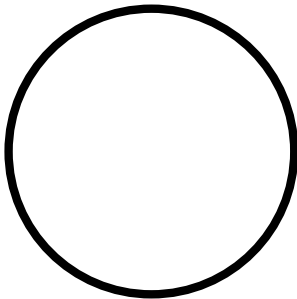
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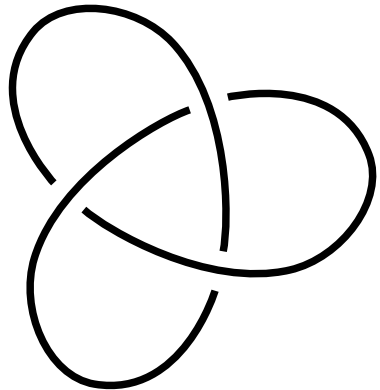
# Knot Theory 101

What is a knot?



# Knot Theory 101

This is the *trefoil knot*. We can't “untangle” it, no matter how hard we try.



# Connecting Topology and Origami

## Theorem

*A folding of a crease pattern is valid if and only if every Jordan curve embedded in the paper before folding is mapped to the unknot after folding.*

# Connecting Topology and Origami

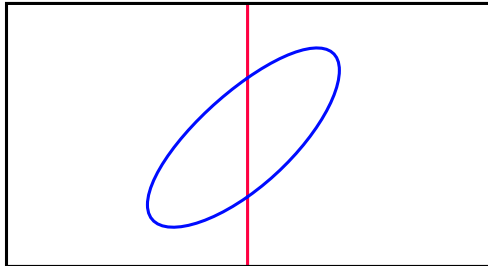


This direction is fairly intuitive.

# Connecting Topology and Origami



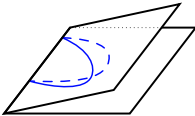
This direction is fairly intuitive.



# Connecting Topology and Origami



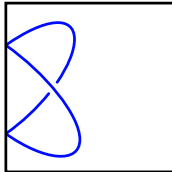
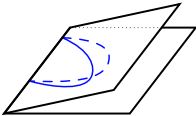
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# Connecting Topology and Origami



This direction is fairly intuitive.

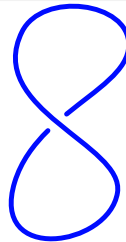
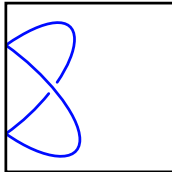
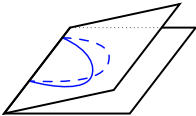




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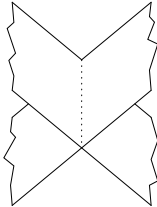
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# Connecting Topology and Origami



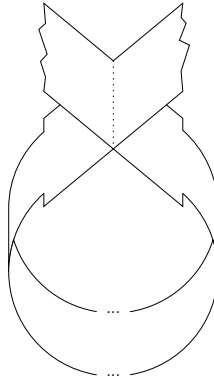
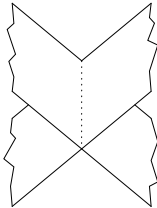
Invalid folding means intersection.



# Connecting Topology and Origami



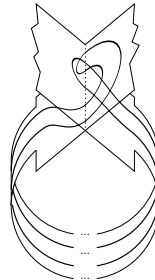
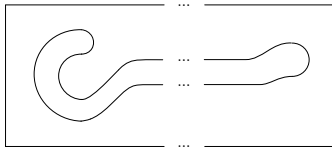
We can connect up the paper like this.



# Connecting Topology and Origami



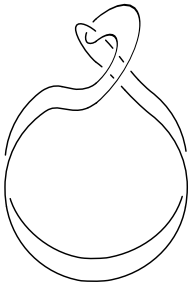
We can draw a curve like this.



# Connecting Topology and Origami



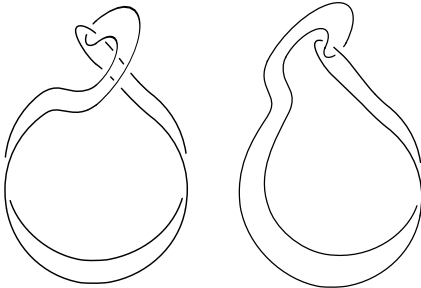
Is it a knot?



# Connecting Topology and Origami



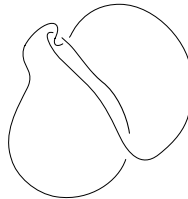
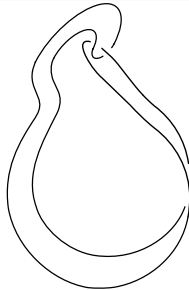
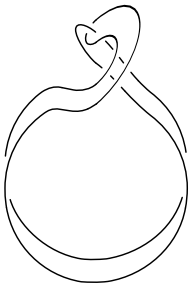
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# Connecting Topology and Origami



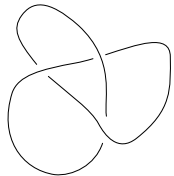
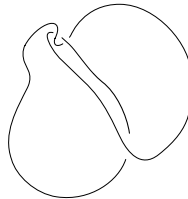
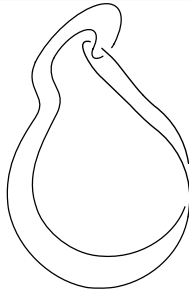
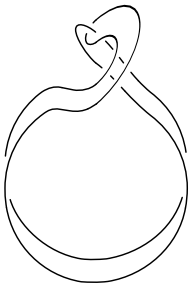
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# Connecting Topology and Origami



Is it a knot?

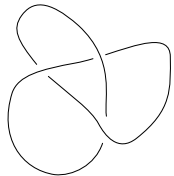
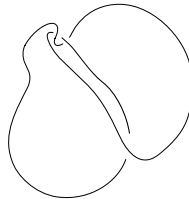
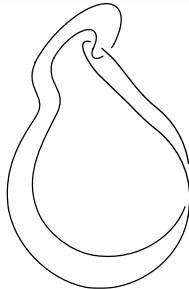
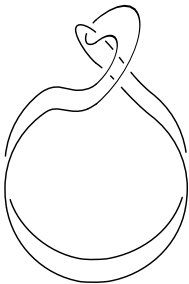




# Connecting Topology and Origami



Is it a knot?



Trefoil knot!

# Connecting Topology and Origami

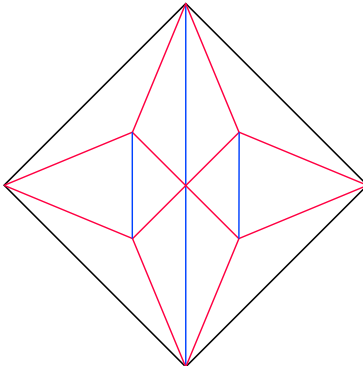
## Now what?

- Maybe a characteristic for crease patterns
- Finding ways to describe the complexity of a crease pattern or a folding (e.g. the number of self-intersections)

# Determining flat-foldability

Question:

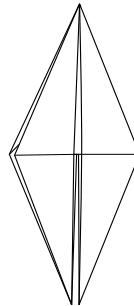
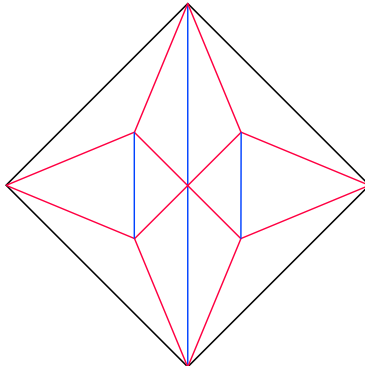
Is this crease pattern flat-foldable?



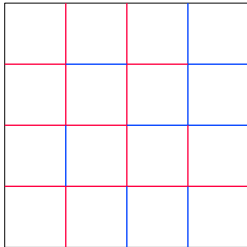
# Determining flat-foldability

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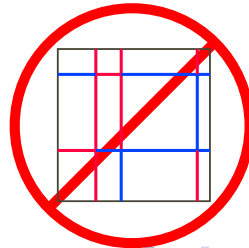
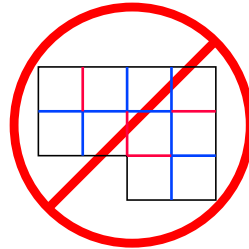
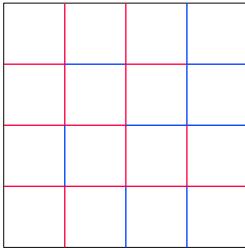
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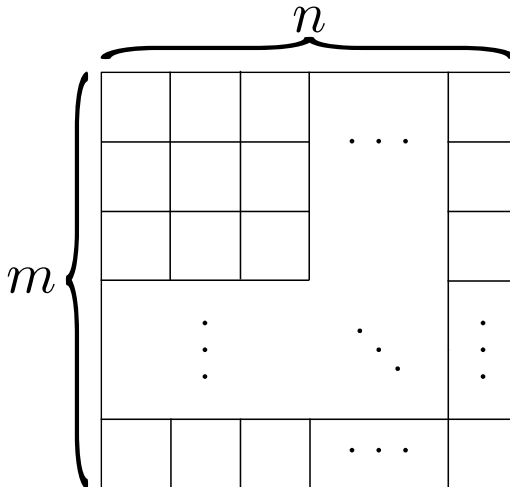
# Introduction to Maps



# Introduction to Maps



# Introduction to Maps



# Map Folding: An Open Problem

## Open Problem:

How hard is it to determine whether or not a map is flat-foldable?



# Map Folding: An Open Problem

## Open Problem:

How hard is it to determine whether or not a map is flat-foldable?

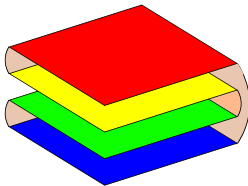
## Easier Problem:

How hard is it to determine whether or not a map is *flat-folded*?

# Linear Orderings

Question:

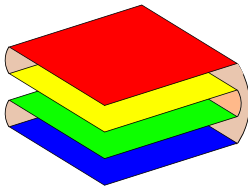
How can we represent a folded form?



# Linear Orderings

Question:

How can we represent a folded form?

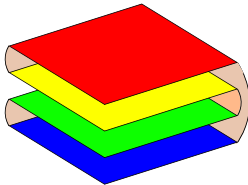


red  $\rightarrow$  yellow  $\rightarrow$  green  $\rightarrow$  blue

# Linear Orderings

Question:

How can we represent a folded form?



red  $\rightarrow$  yellow  $\rightarrow$  green  $\rightarrow$  blue

Note:

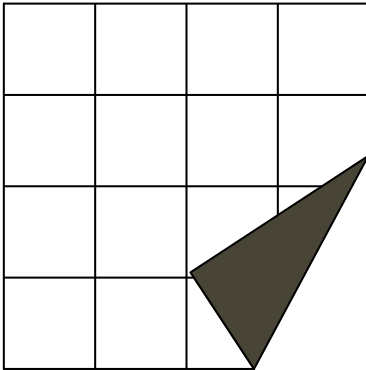
Each linear ordering corresponds with exactly one folded form.

# Linear Orderings

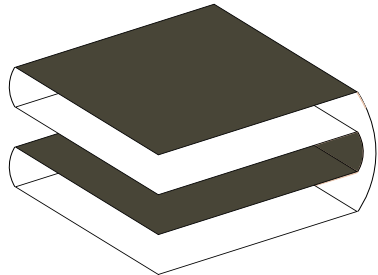
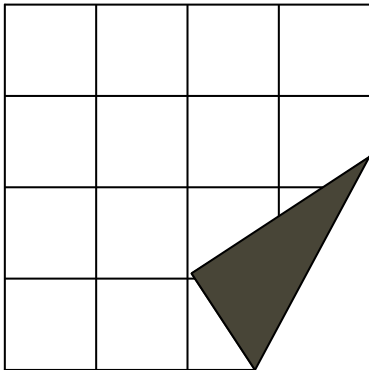
## Question:

How can we tell if a linear ordering is realizable by a given crease pattern?

# Checkerboard Pattern



# Checkerboard Pattern



# Checkerboard Pattern

## Goal:

Label each face as being light-side or dark-side up in *any* folding.

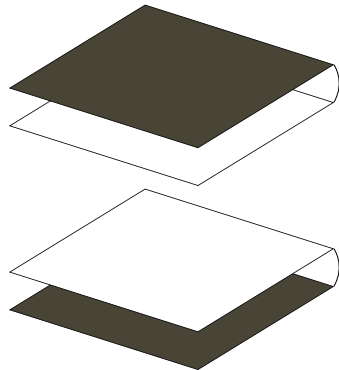
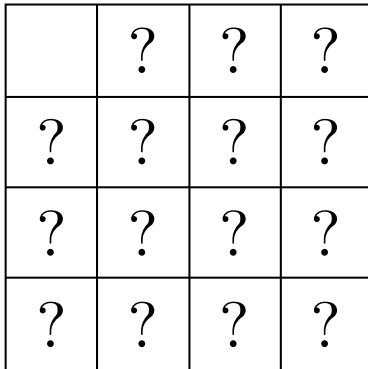
	?	?	?
?	?	?	?
?	?	?	?
?	?	?	?



# Checkerboard Pattern

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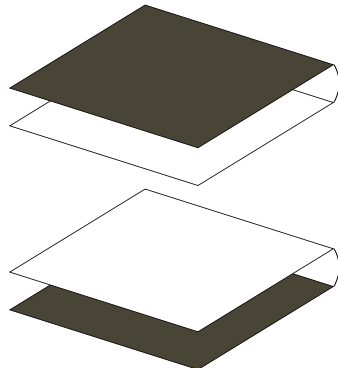


# Checkerboard Pattern

## Goal:

Label each face as being light-side or dark-side up in *any* folding.

		?	?
	?	?	?
?	?	?	?
?	?	?	?

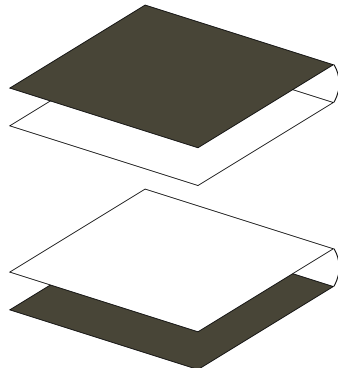


# Checkerboard Pattern

## Goal:

Label each face as being light-side or dark-side up in *any* folding.

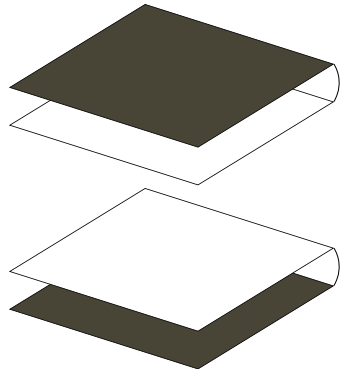
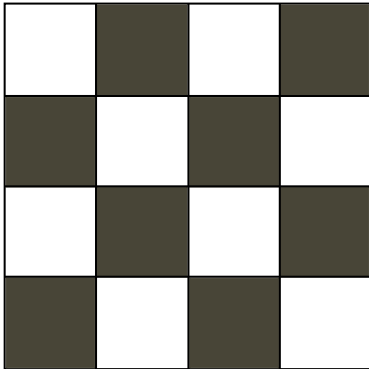
			?
		?	?
	?	?	?
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## Definition:

A *partial ordering*  $P$  on a set  $S$  is a set of ordered pairs of elements of  $S$  that orders some of the elements of  $S$ .

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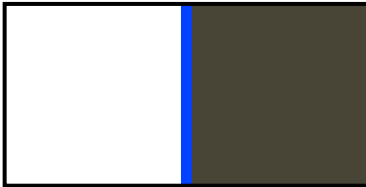
$$c \rightarrow d \rightarrow a \rightarrow b$$

...

# Partial Ordering

## Goal:

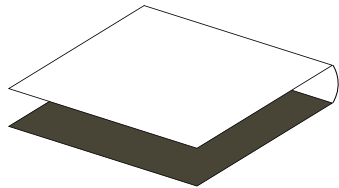
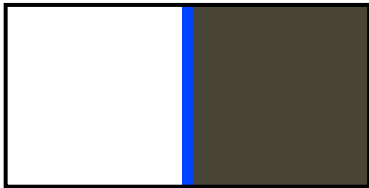
Use checkerboard pattern and crease assignments to tell, for each pair of adjacent faces, which face comes first in *all* linear orderings.



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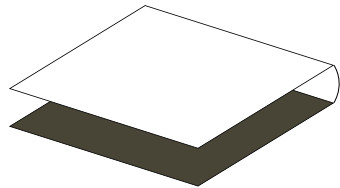
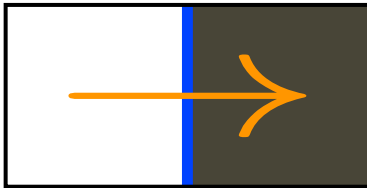
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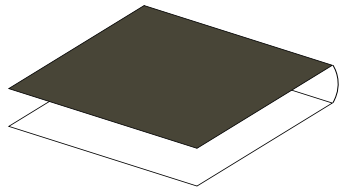
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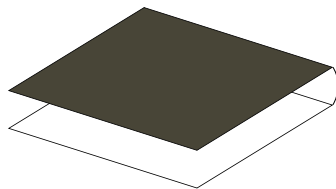
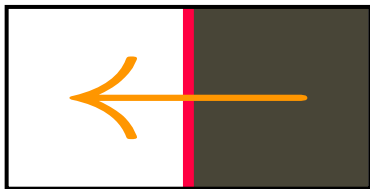




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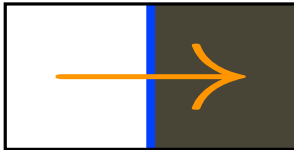
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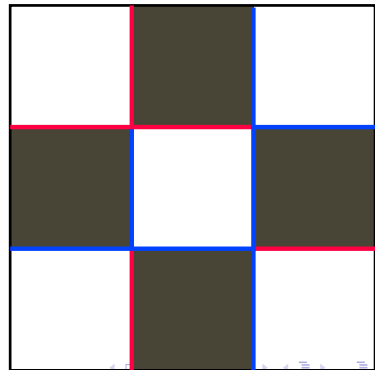
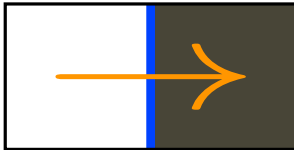
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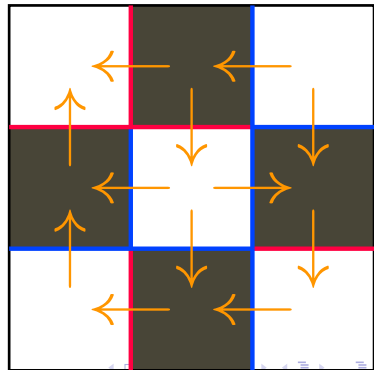
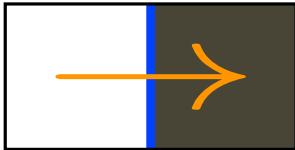
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# Linear Orderings

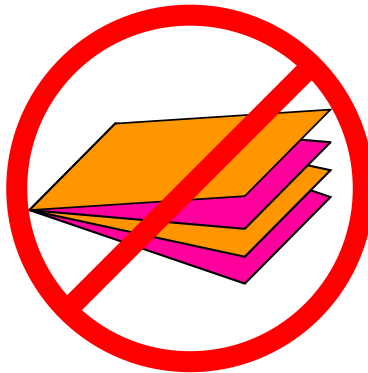
Is satisfying this partial ordering enough to ensure foldability?

# Linear Orderings

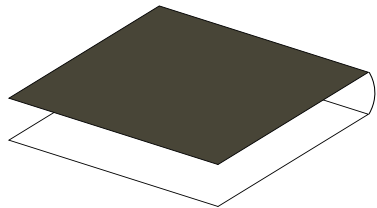
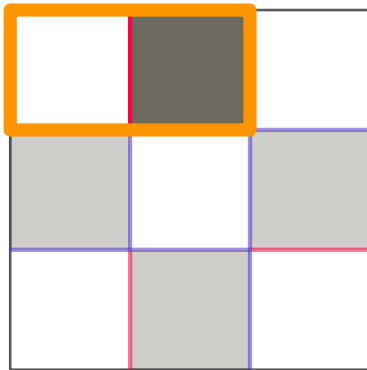
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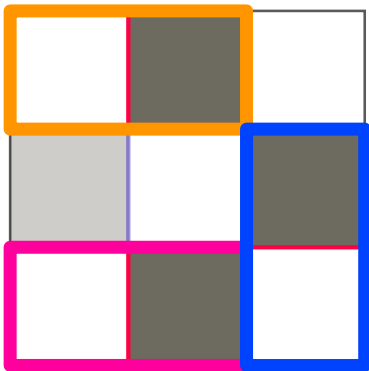


# Butterflies

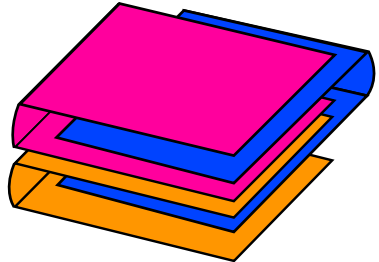
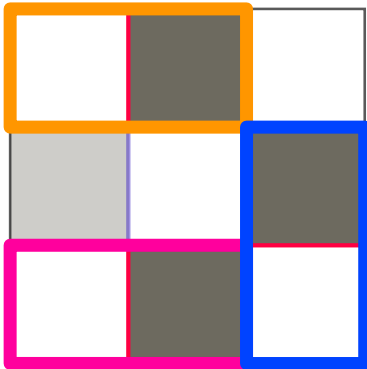




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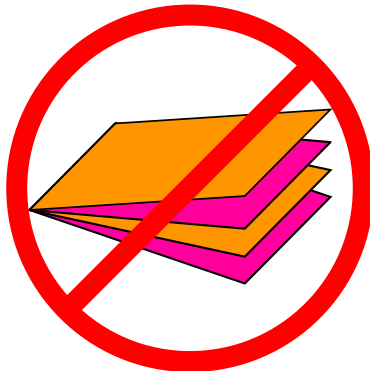
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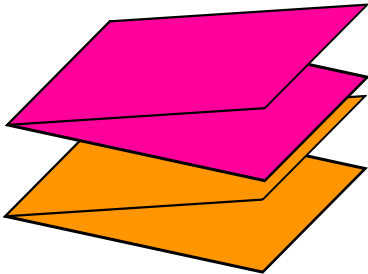
Enumerate the realizable configurations of twin butterfly pairs.



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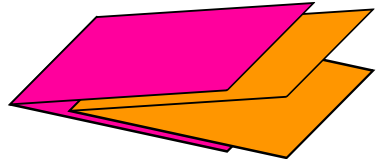
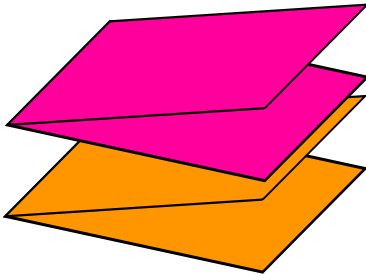
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# Conditions for flat-foldability

Theorem: (Nishat and Whitesides 2013)

A linear ordering  $\mathcal{L}$  of faces is flat-foldable if and only if (i)  $\mathcal{L}$  satisfies the partial ordering given by the map and (ii) every pair of twin butterflies stacks or nests in  $\mathcal{L}$ .

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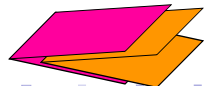
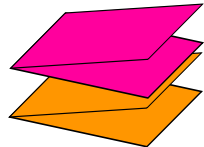
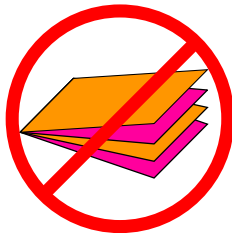


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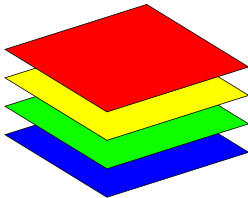
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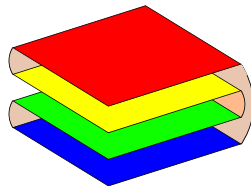
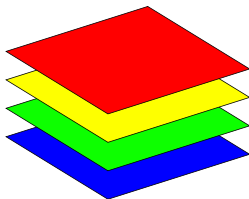


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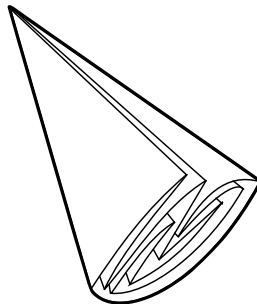
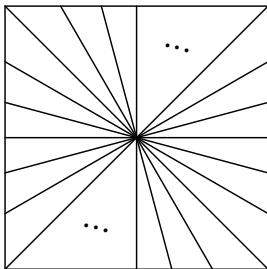
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**Summary:** If someone says “This map can be flat-folded, here is the folding,” we can quickly test whether or not they were correct.

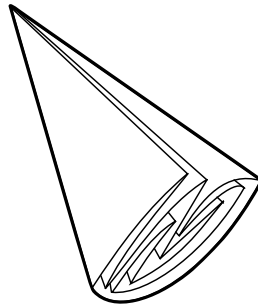
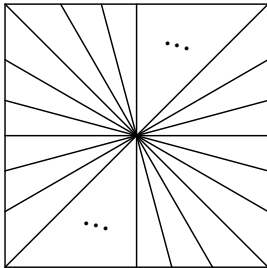
# Counting Problems in Origami

- How many ways can we fold an  $n \times m$  map?
  - How many ways can we fold a  $1 \times m$  map?
- How many  $n \times m$  map crease patterns can be folded, period?
  - How many ways can we fold a  $1 \times m$  map?

# Star Patterns

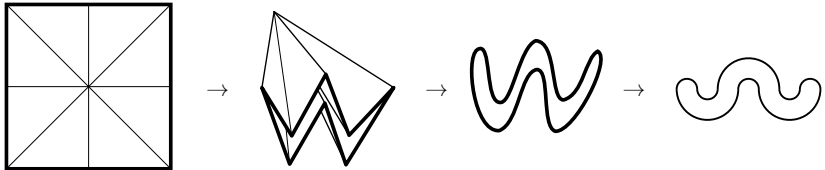


# Star Patterns



**Question:** How many foldings are generated by star patterns with  $2n$  creases?

# Representing foldings of Star Patterns



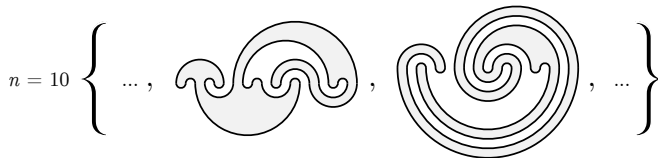
# Examples of Meanders

$$n = 1 \quad \{ \bigcirc \}$$

$$n = 2 \quad \{ \text{U-shape}, \text{inverted U-shape} \}$$

$$n = 3 \quad \{ \text{3-lobed top}, \text{3-lobed bottom}, \text{3-lobed left}, \text{3-lobed right}, \text{3-lobed top}, \text{3-lobed bottom}, \text{3-lobed left}, \text{3-lobed right} \}$$

# Examples of Meanders



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Order $n$	# Meanders $M_n$
1	1
2	2
3	8
4	42
5	262
$\vdots$	$\vdots$



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3	8
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$\vdots$	$\vdots$

Table : The sequence of Meandric Numbers

# Game Plan

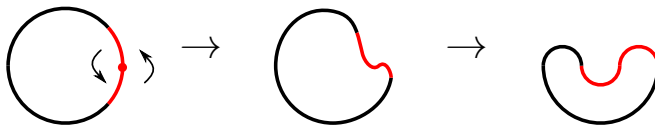
$$n = 1$$



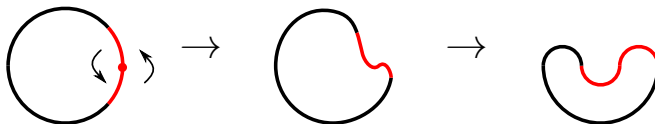
$$n = 2$$



# Game Plan



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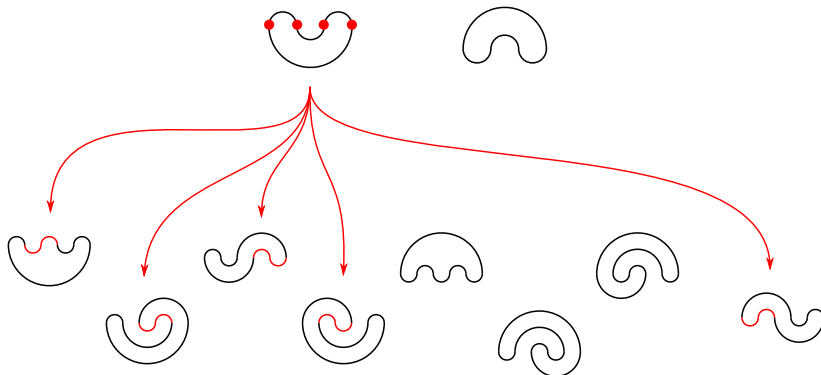


**Idea:** Produce larger meanders by adding “twists” to smaller meanders

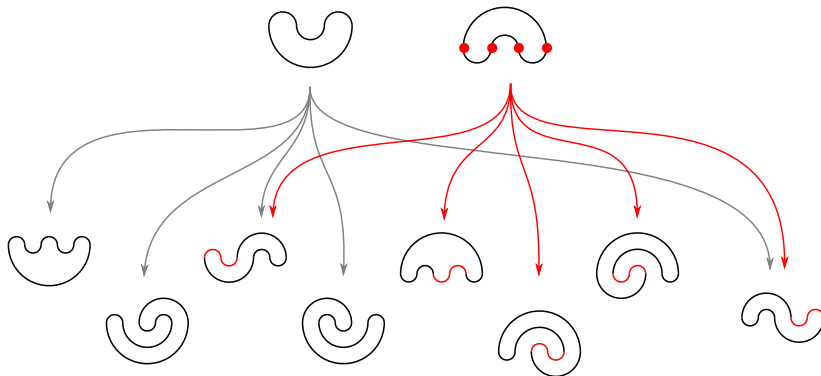
# Will it work?



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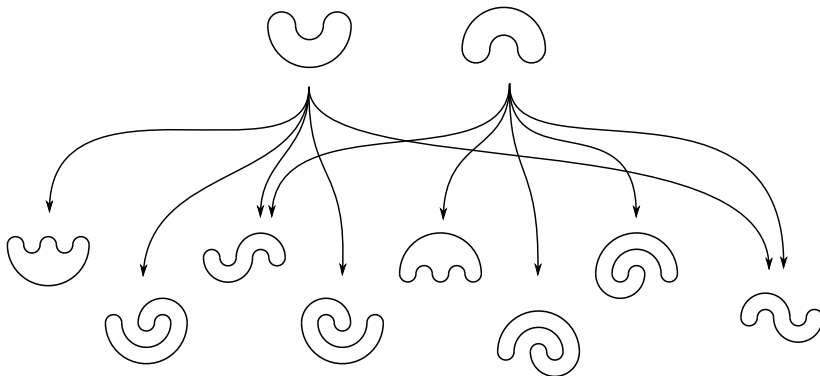


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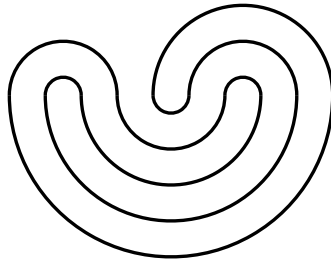
# Will it work?

**Question:** Can we get all meanders by repeatedly doing this?

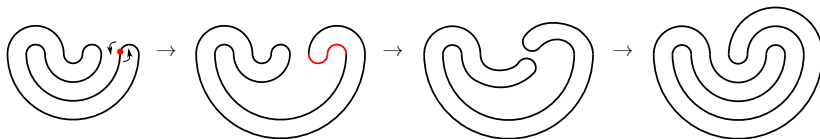
# Will it work?

**Question:** Can we get all meanders by repeatedly doing this?

**Answer:** Sadly we cannot:



# Will it work?



# Meanders

## Theorem

*All meanders may be built from  $m_1 = \circ$  with at most one shuffle after each twist.*

# Simple Meanders

## Definition (Simple Meander)

A *simple meander* is a meander of order  $n$  that can be constructed without shuffling.

# Simple Meanders

## Theorem (Recurrence Relation for Simple Meanders)

Let  $\mathbb{P}(k, n) = \{(x_1, x_2, \dots, x_n) : \sum x_i = k \text{ and } x_i \geq 0 \forall i\}$ . Then

$$r(n) = \sum_{i=1}^n \sum_{P \in \mathbb{P}(i, n+1)} \prod_{k=1}^{n+1} r(P_k)$$

$$H(n) = \sum_{i=1}^n r(i) \sum_{P \in \mathbb{P}(i, n)} \prod_{k=1}^n H(P_k)$$

$$M_n^S = 2H(n)$$

# Make sure to stop by our exhibit in the Gould Library!

*Opening Spring Term 2016*



# References



Rahnuma Islam Nishat and Sue Whitesides (2013)

Map Folding

*Canadian Conference on Computational Geomery (2013)*, p49-52