

The Mathematics of Origami

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Overview

Axioms of Origami

General

- i. Brief History
- ii. Independence

Axioms of Origami

Seven Axioms of Origami

- i. Given two points p_1 and p_2 , we can fold a line connecting them.
- ii. Given two points p_1 and p_2 , we can fold p_1 onto p_2 .
- iii. Given two lines l_1 and l_2 , we can fold line l_1 onto l_2 .
- iv. Given a point p_1 and a line l_1 , we can make a fold perpendicular to l_1 passing through the point p_1 .

Axioms of Origami

Seven Axioms of Origami

- v. Given two points p_1 and p_2 and a line l_1 , we can make a fold that places p_1 onto l_1 and passes through the point p_2 .
- vi. Given two points p_1 and p_2 and two lines l_1 and l_2 , we can make a fold that places p_1 onto line l_1 and places p_2 onto line l_2 .
- vii. Given a point p_1 and two lines l_1 and l_2 , we can make a fold perpendicular to l_2 that places p_1 onto line l_1 .

Seven Axioms of Origami

- i. The first six axioms allow all quadratic and cubic equations with rational coefficients to be solved.
- ii. They also allow two of the three problems of antiquity, the trisection of an angle and the doubling of the cube, to be constructed.

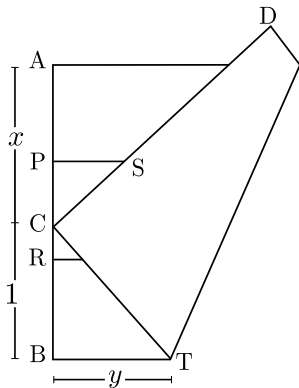
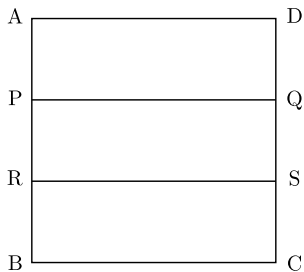
Problems of Antiquity

These were a trio of geometric problems whose solutions were attempted solely through the use of compass and straight-edge.

- i. Angle Trisection
- ii. Cube Duplication
- iii. Circle Squaring

INCLUDE PARTIALLY FOLDED DIAGRAMMS

Cube Duplication



- i. $\frac{\alpha}{\beta} = \sqrt[3]{2}$
- ii. Thus, a cube with a side length α will have twice the volume of a cube with side length β .

Squaring the Circle

- i. Impossible
- ii. π

Constructible Numbers

- i. Given two points p_0 and p_1 , construct a third point p'_1 a distance $|p_0p_1|$ from point p_0 such that $\overline{p_0p'_1}$ is perpendicular to $\overline{p_0p_1}$.
- ii. Given two points p_0 and p_1 , a third point p_2 can be constructed such that p_2 is collinear with p_0 and p_1 , thus $|p_0p_1| = |p_1p_2|$.

Constructible Numbers

- iii. Given two constructible numbers α and β , we can construct $\frac{\alpha}{\beta}$, their ratio.
- iv. Given two constructible numbers α and β , we can construct their sum $\alpha + \beta$ or their difference $\alpha - \beta$.
- v. Given two constructible numbers α and β , we can construct $\alpha\beta$, their product.

Given two constructible numbers α and β , we can construct $\frac{\alpha}{\beta}$, their ratio.

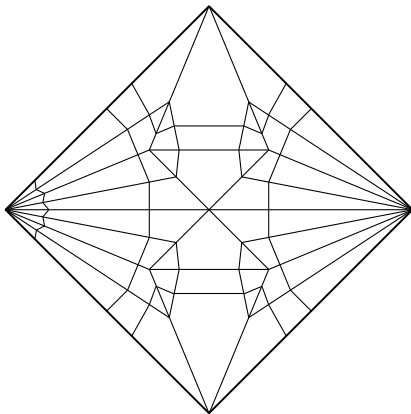
INCLUDE PARTIALLY FOLDED DIAGRAM AS WELL AS
FINAL FOLDED DIAGRAM

Constructible Numbers

- i. Since the set of constructible numbers is closed under addition, subtraction, multiplication, and division, it can be concluded that the set of constructible numbers form a field.
- ii. Ultimately, the field of origami constructible numbers are closed under taking both square roots and cube roots.
- iii. The construction of the square root of any constructible number implies the field of origami constructible numbers contains the field of compass and straightedge constructible numbers.

- i. Efficiency
- ii. Optimality

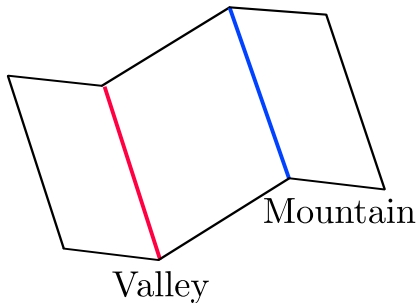
An example of a crease pattern:



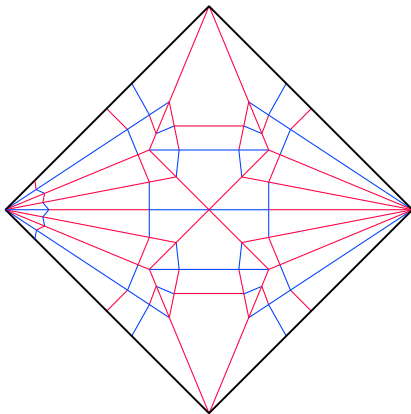
Foldability

Crease Assignments

A crease pattern doesn't contain all the information about a model, however.



A crease pattern with mountain-valley assignment:



Two questions

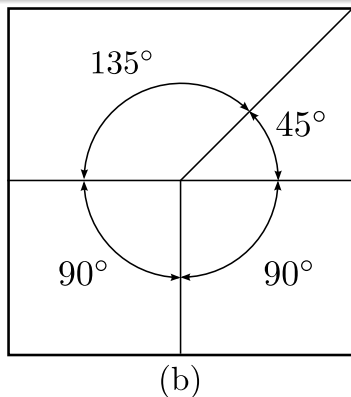
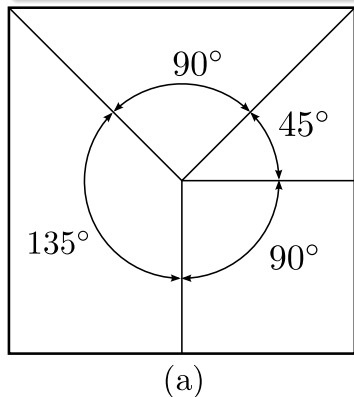
For a given crease pattern,

- i. is there a way to fold it flat?
- ii. is there a way to fold it at all?

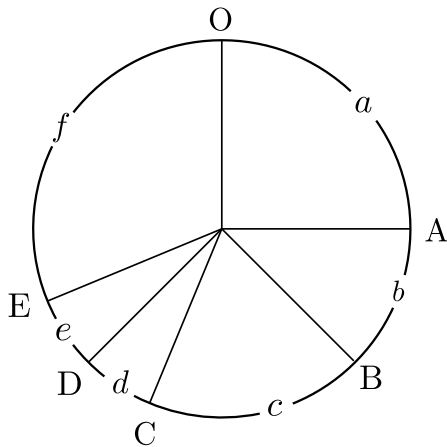
Flat-Foldability Conditions

Kawasaki's Theorem (1989)

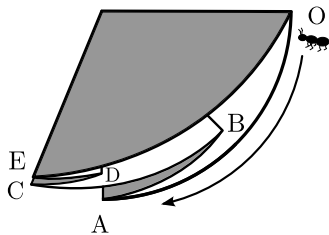
The alternate angles around a vertex must sum to 180° .



Kawasaki's Theorem



(a)

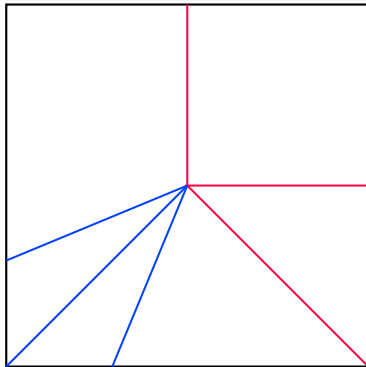
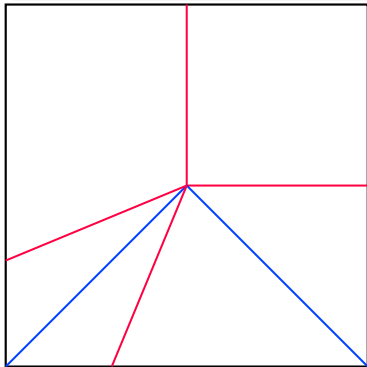


(b)

Flat-Foldability Conditions

Maekawa's Theorem (1986)

The sum of mountain + valley is ± 2 .

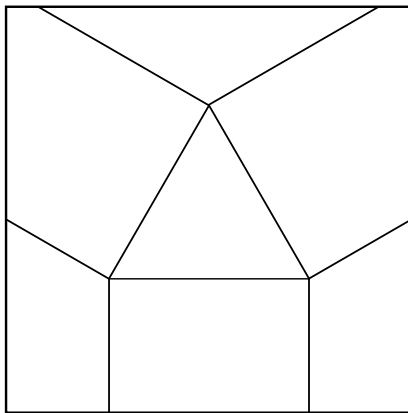


More Complicated Crease Patterns

How can we determine flat foldability for a more complicated pattern?

More Complicated Crease Patterns

Here's a crease pattern that can't fold flat!



How can we capture the notion of a folded paper?

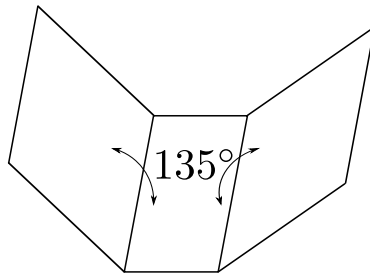
How can we capture the notion of a folded paper?

foldability_pix/unfolded.pdf

Foldings

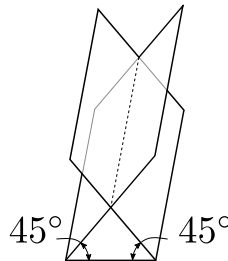
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An Invalid Folding

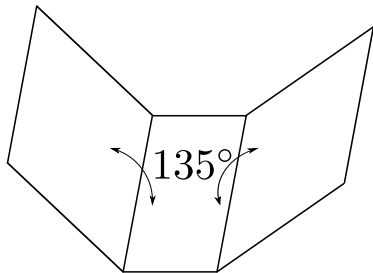
This folding has self-intersection.



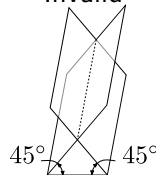
Valid Foldings

A valid folding is one that doesn't cause any self-intersection.

Valid



Invalid



Drawing pictures on Crease Patterns

Question

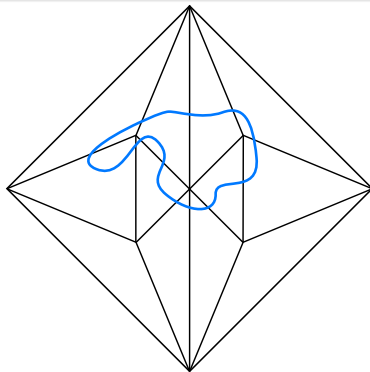
If we draw something on the flat piece of paper before folding it up, what can happen when we fold it?

Drawing pictures on Crease Patterns

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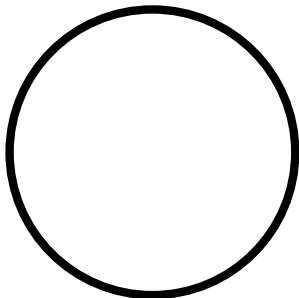
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We'll see what happens when we draw Jordan curves on the crease patterns, like the picture on the right.

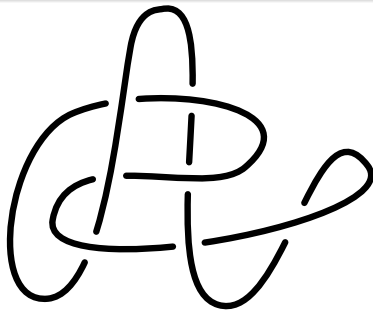
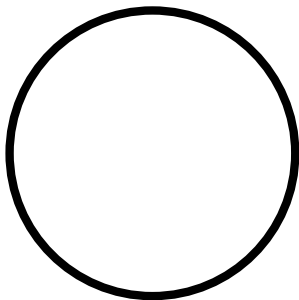


What is a knot?

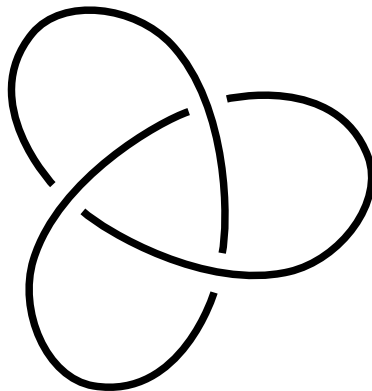
What is a knot?



What is a knot?



This is the *trefoil knot*. We can't “untangle” it, no matter how hard we try.



Connecting Topology and Origami

Theorem

A folding of a crease pattern is valid if and only if every Jordan curve embedded in the paper before folding is mapped to the unknot after folding.

Connecting Topology and Origami

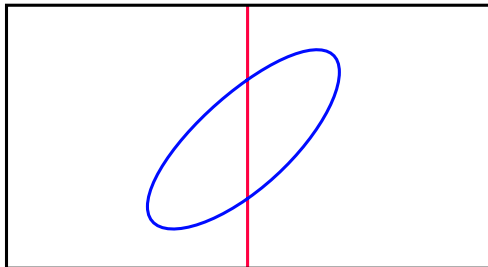


This direction is fairly intuitive.

Connecting Topology and Origami



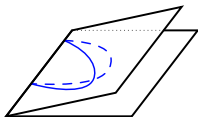
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Connecting Topology and Origami



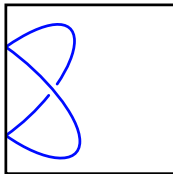
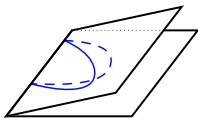
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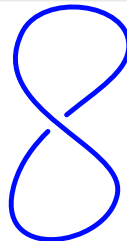
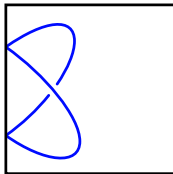
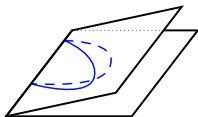
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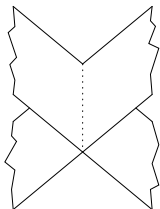
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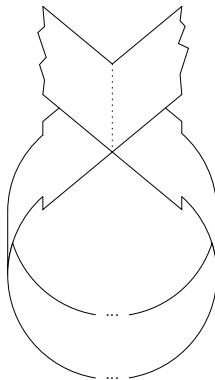
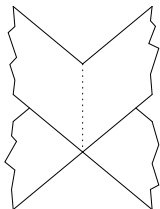
Invalid folding means intersection.



Connecting Topology and Origami



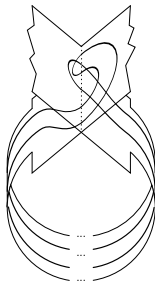
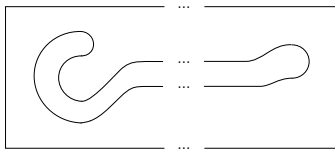
We can connect up the paper like this.



Connecting Topology and Origami



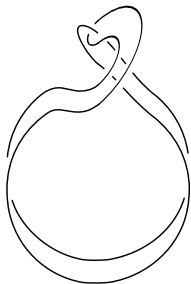
We can draw a curve like this.



Connecting Topology and Origami



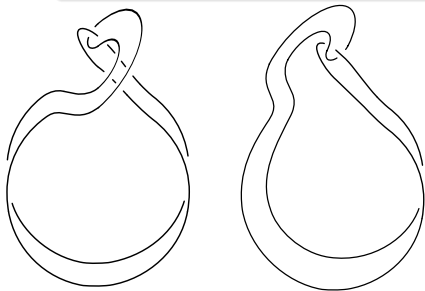
Is it a knot?



Connecting Topology and Origami



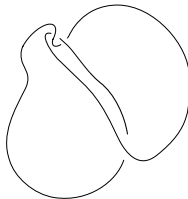
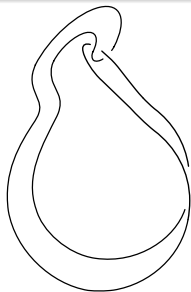
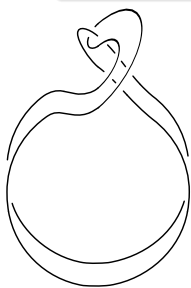
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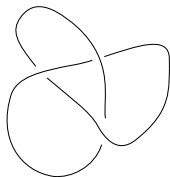
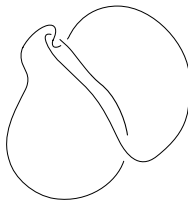
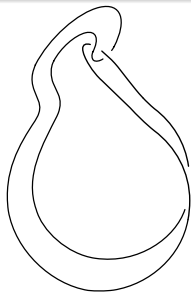
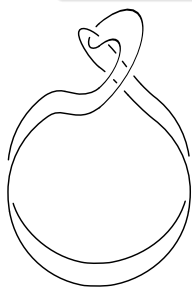
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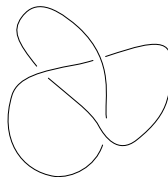
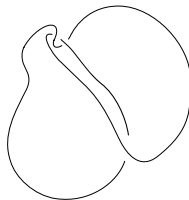
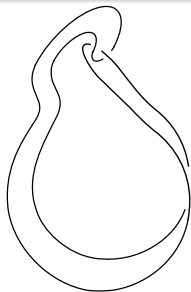
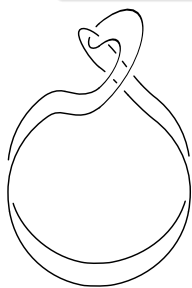
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Connecting Topology and Origami



Is it a knot?



Trefoil knot!

Connecting Topology and Origami

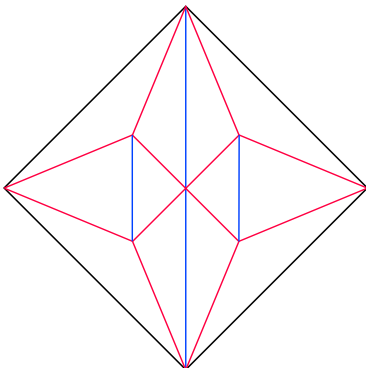
Now what?

- Maybe a characteristic for crease patterns
- Finding ways to describe the complexity of a crease pattern or a folding (e.g. the number of self-intersections)

Determining flat-foldability

Question:

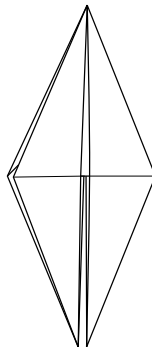
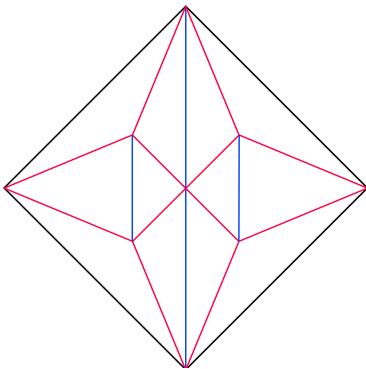
Is this crease pattern flat-foldable?



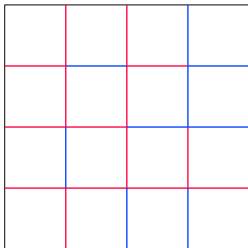
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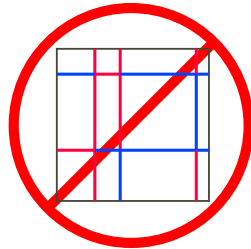
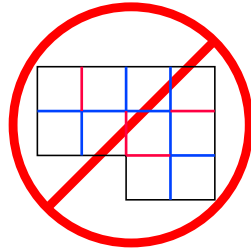
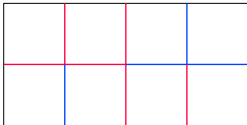
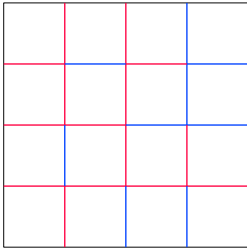
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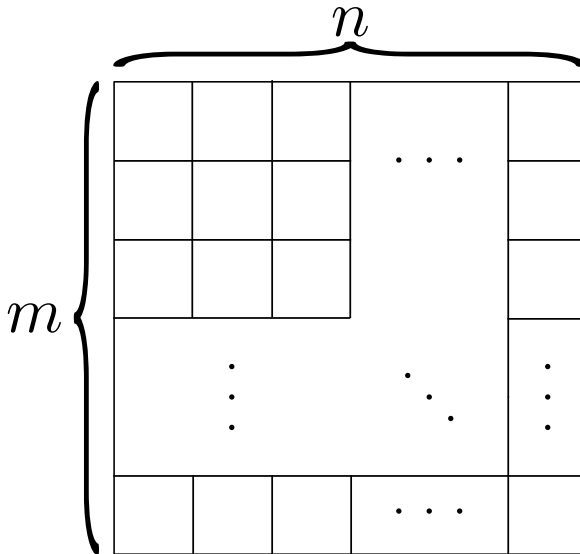
Introduction to Maps



Introduction to Maps



Introduction to Maps



Map Folding: An Open Problem

Open Problem:

How hard is it to determine whether or not a map is flat-foldable?

Map Folding: An Open Problem

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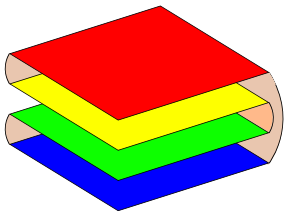
Easier Problem:

How hard is it to determine whether or not a map is *flat-folded*?

Linear Orderings

Question:

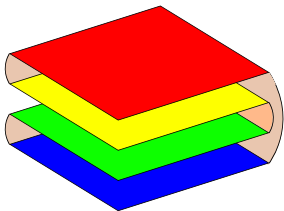
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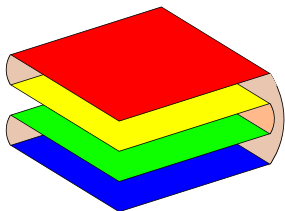


red \rightarrow yellow \rightarrow green \rightarrow blue

Linear Orderings

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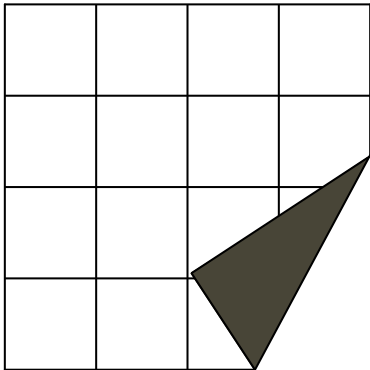
Note:

Each linear ordering corresponds with exactly one folded form.

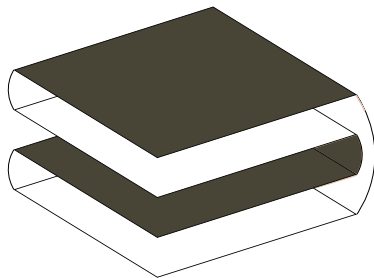
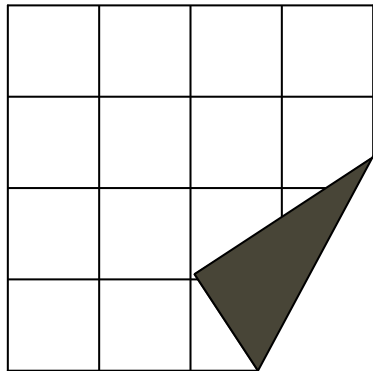
Question:

How can we tell if a linear ordering is realizable by a given crease pattern?

Checkerboard Pattern



Checkerboard Pattern



Checkerboard Pattern

Goal:

Label each face as being light-side or dark-side up in *any* folding.

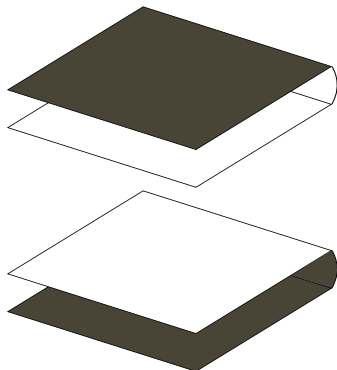
	?	?	?
?	?	?	?
?	?	?	?
?	?	?	?

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	?	?	?
?	?	?	?
?	?	?	?
?	?	?	?

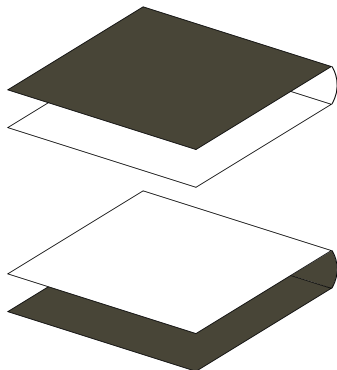


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		?	?
	?	?	?
?	?	?	?
?	?	?	?

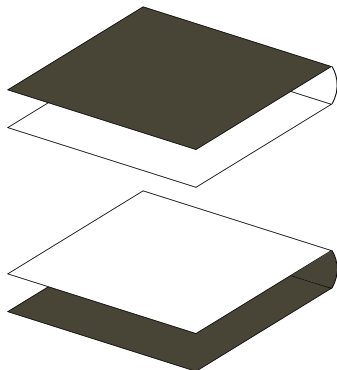


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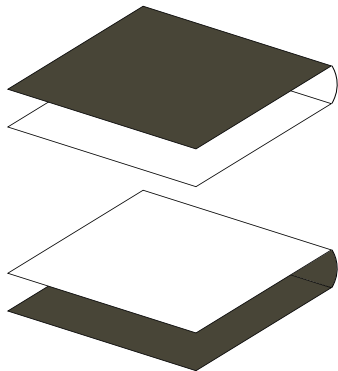
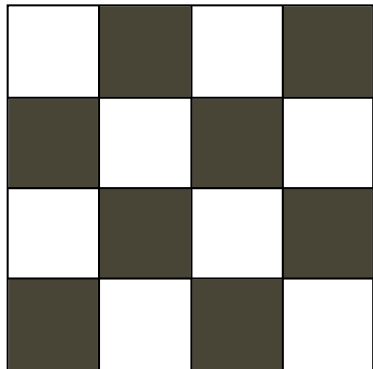
			?
		?	?
	?	?	?
?	?	?	?



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Partial Ordering

Definition:

A *partial ordering* P on a set S is a set of ordered pairs of elements of S that orders some of the elements of S .

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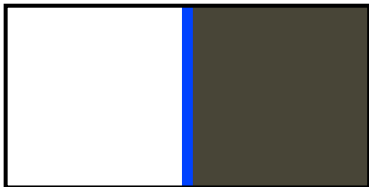
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...

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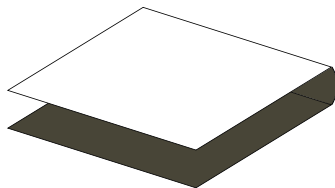
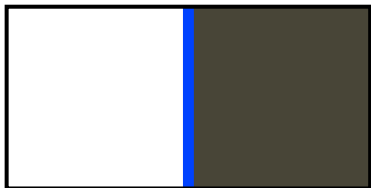
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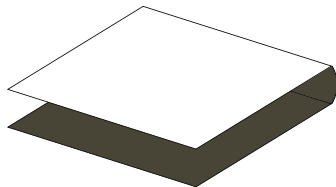
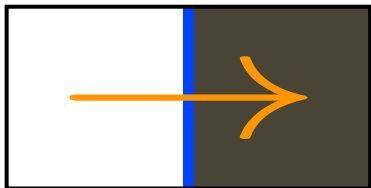
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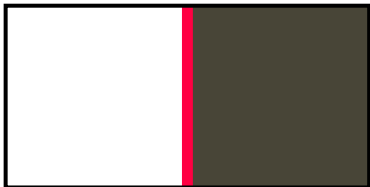
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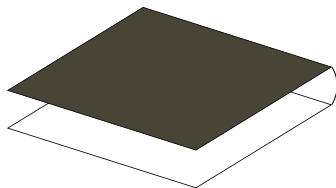
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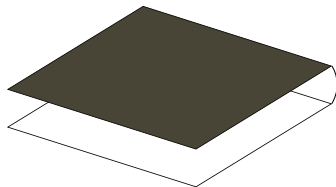
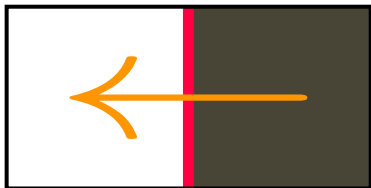
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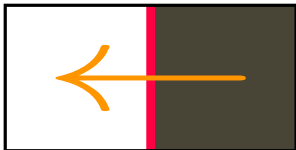
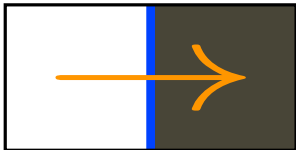
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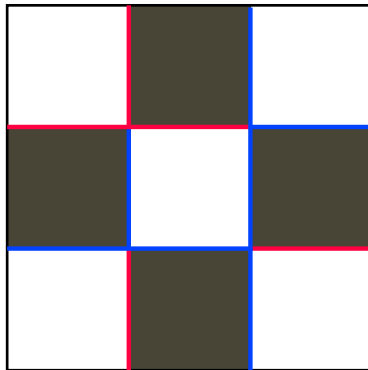
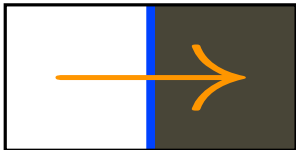
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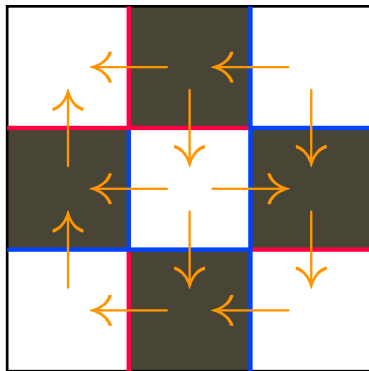
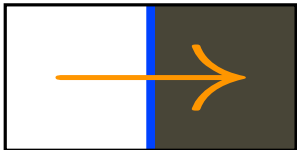
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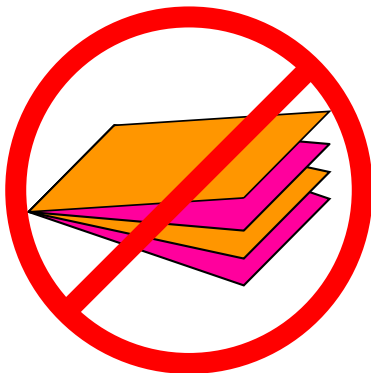
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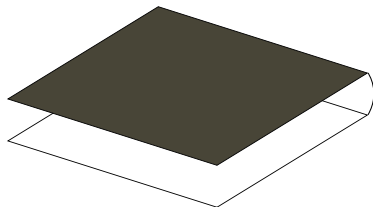
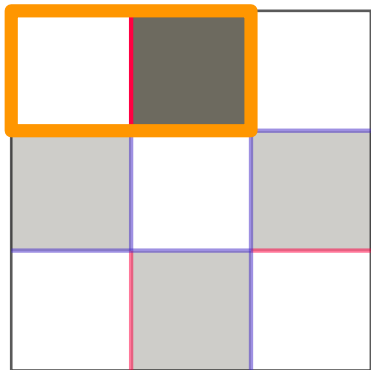
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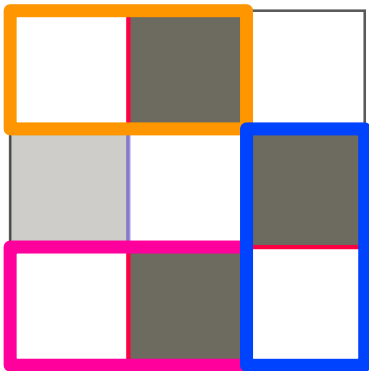
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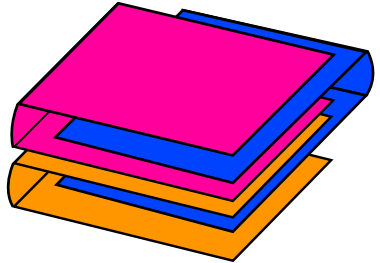
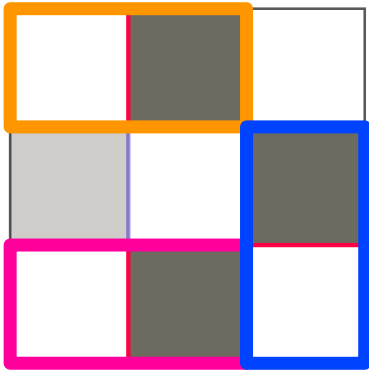
Butterflies



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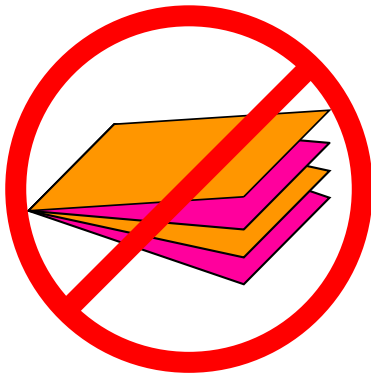
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Butterfly Condition

Goal:

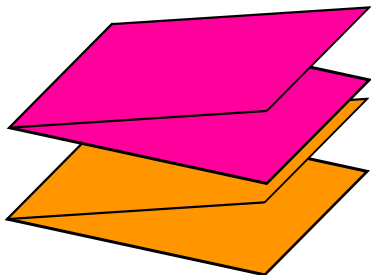
Enumerate the realizable configurations of twin butterfly pairs.



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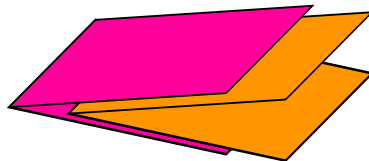
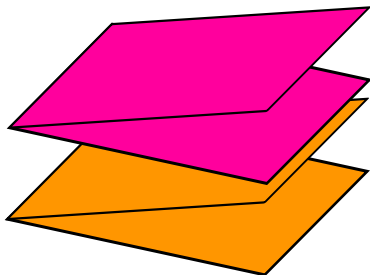
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Conditions for flat-foldability

Theorem: (Nishat and Whitesides 2013)

A linear ordering \mathcal{L} of faces is flat-foldable if and only if (i) \mathcal{L} satisfies the partial ordering given by the map and (ii) every pair of twin butterflies stacks or nests in \mathcal{L} .

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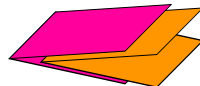
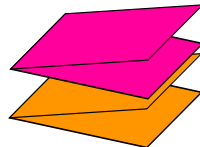
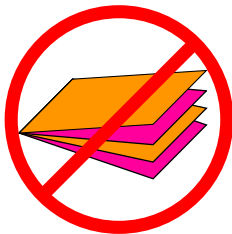


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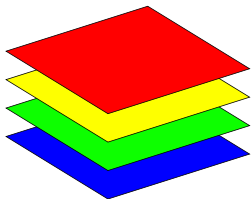
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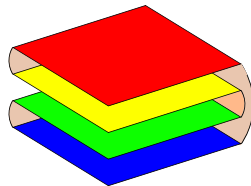
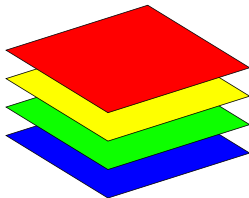


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Summary: If someone says “This map can be flat-folded, here is the folding,” we can quickly test whether or not they were correct.

Counting Origami

Counting Problems in Origami

- How many ways can we fold an $m \times n$ map?
- How many $m \times n$ map crease patterns have a valid folding?

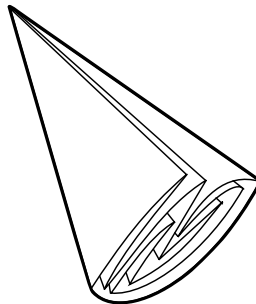
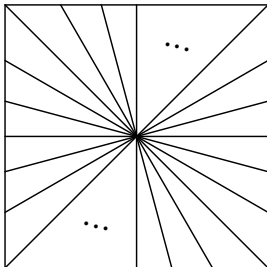
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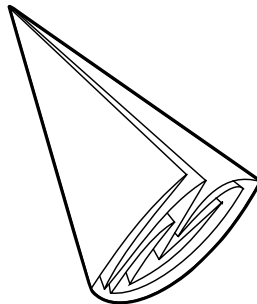
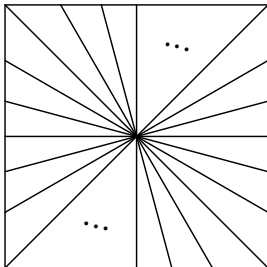
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Star Patterns

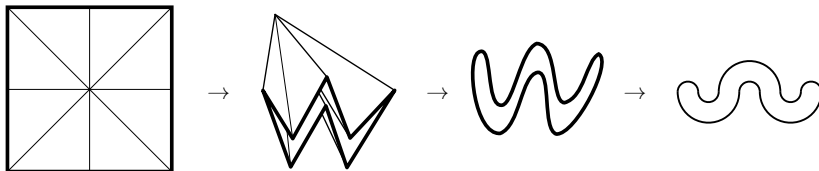


Star Patterns



Question: How many foldings are generated by star patterns with $2n$ creases?

Representing foldings of Star Patterns



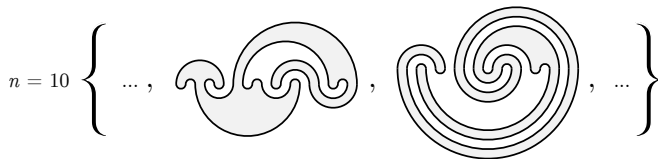
Examples of Meanders

$$n = 1 \quad \{ \bigcirc \}$$

$$n = 2 \quad \{ \text{U-shape}, \text{inverted U-shape} \}$$

$$n = 3 \quad \{ \text{C-shape}, \text{S-shape}, \text{Z-shape}, \text{W-shape}, \\ \text{M-shape}, \text{N-shape}, \text{O-shape}, \text{P-shape} \}$$

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Order n	# Meanders M_n
1	1
2	2
3	8
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Table: The sequence of Meandric Numbers

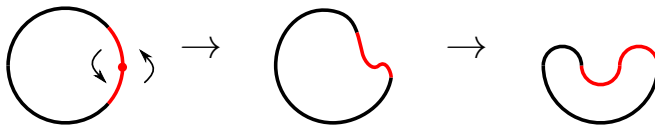
$$n = 1$$

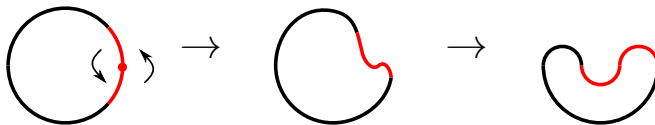


$$n = 2$$



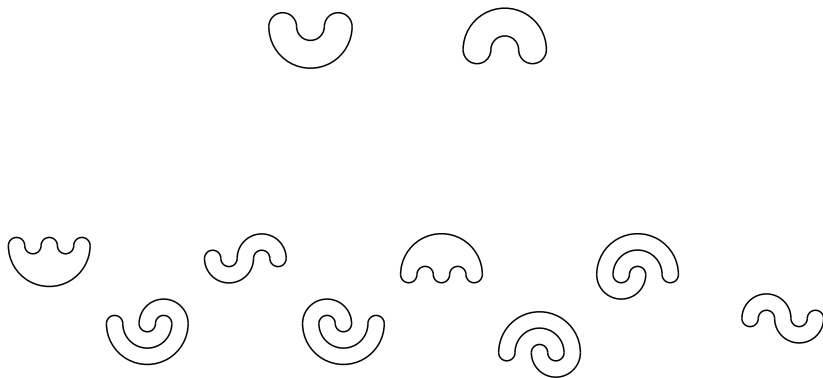
Game Plan



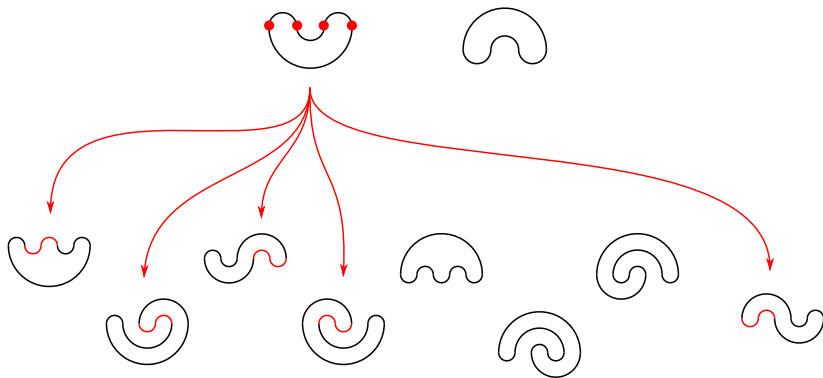


Idea: Produce larger meanders by adding “twists” to smaller meanders

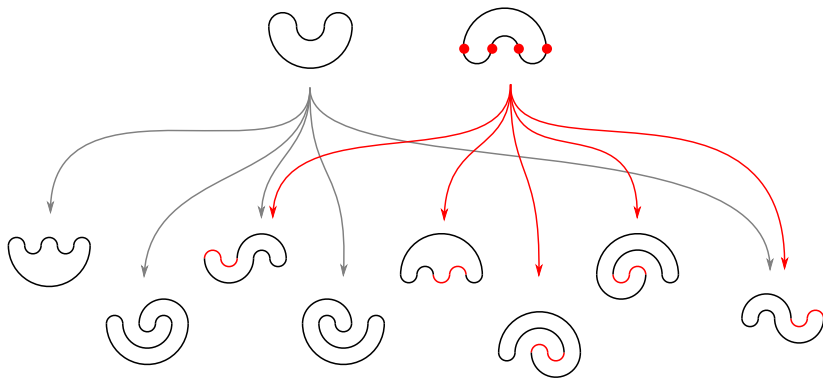
Will it work?



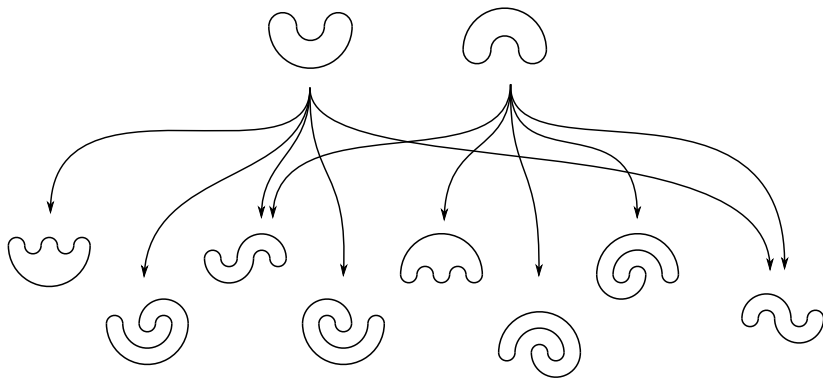
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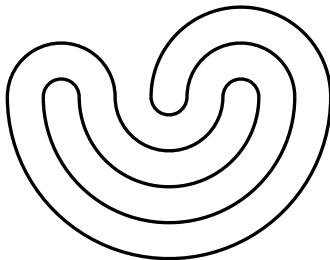


Question: Can we get all meanders by repeatedly doing this?

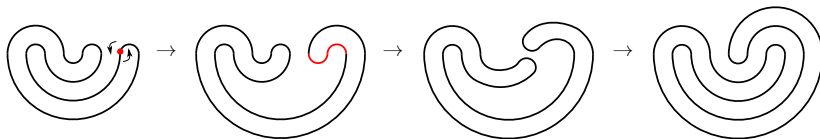
Will it work?

Question: Can we get all meanders by repeatedly doing this?

Answer: Sadly we cannot:



Will it work?



Theorem

All meanders may be built from $m_1 = \circ$ with at most one shuffle after each twist.

Definition (Simple Meander)

A *simple meander* is a meander of order n that can be constructed without shuffling.

Theorem (Recurrence Relation for Simple Meanders)

Let $\mathbb{P}(k, n) = \{(x_1, x_2, \dots, x_n) : \sum x_i = k \text{ and } x_i \geq 0 \forall i\}$. Then

$$r(n) = \sum_{i=1}^n \sum_{P \in \mathbb{P}(i, n+1)} \prod_{k=1}^{n+1} r(P_k)$$

$$H(n) = \sum_{i=1}^n r(i) \sum_{P \in \mathbb{P}(i, n)} \prod_{k=1}^n H(P_k)$$

$$M_n^S = 2H(n)$$

Make sure to stop by our
exhibit in the Gould Library!

Opening Spring Term 2016



Rahnuma Islam Nishat and Sue Whitesides (2013)

Map Folding

Canadian Conference on Computational Geometry (2013), p49-52