1 Zolotarev Notes

The Zolotarev optimal rational function approximating the sign function $\mathcal{Z}(x) = 1$ over the range $x \in [r_1, r_2]$ is expressed in factored form by

$$\mathcal{Z}(x) = mx \frac{\prod_{j=1}^{j=N_n} x^2 - a_i}{\prod_{j=1}^{N_d} x^2 - d_i} \tag{1}$$

and in partial fraction form by

$$\mathcal{Z}(x) = mx[f + \sum_{j=1}^{N_d} \frac{c_i}{x^2 - d_i}] \tag{2}$$

Parameters r_1,r_2, and N are hard coded in to main.cpp in the function z.setZolo(r1,r2,N). The coefficients are output in zoloFactorCoeffs.dat and zolozoloPartFracCoeffs.dat respectively.

1.1 Overlap Operator

The overlap operator (with Wilson kernel) is then expressed directly as

$$D_{OL} = \frac{1+m}{2} + \frac{1-m}{2} m D_W (f + \sum_j \frac{c_j}{D_W^\dagger D_W - d_j}) \tag{3}$$

where $D_W \equiv D_W(M)$ is the usual Wilson Dirac operator with parameter $M \in (0, 2)$.

1.2 Domain Wall

The massless domain wall operator with Wilson kernel is illustrated by

$$D_{DW} = \begin{pmatrix} \omega_1 D^{\|} + I & (\omega_1 D^{\|} - I)P_- & 0 & 0 \\ (\omega_2 D^{\|} - I)P_+ & \omega_2 D^{\|} + I & (\omega_2 D^{\|} - I)P_- & 0 \\ 0 & (\omega_3 D^{\|} - I)P_+ & \omega_3 D^{\|} + I & (\omega_3 D^{\|} - I)P_- \\ 0 & 0 & (\omega_4 D^{\|} - I)P_+ & \omega_4 D^{\|} + I \end{pmatrix} \tag{4}$$

The coefficients ω_i are set to 1 to be equivalent to a hyperbolic tangent approximation of the sign function. To be equivalent to the Zolotarev approximation we set $\omega_i = 1/u_u$ where u_i are the roots of $\mathcal{Z}(x) - 1$. These roots are output in the file zroots.dat. This indirect form is more efficient in the construction of sea fermions in the RHMC algorithm for instance.

2 Hyperbolic Tangent

Partial fraction and factored coefficients are also output for the hyperbolic tangent approximation if required. This may be the preferable (and simpler) option when weakly coupled, with a small condition number, but is inefficient for stronger coupling with large condition number.