Bioinformatics

Discrete Mathematics and Optimisation

Problem Sheet Convexity and Newton Method

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1. Verify if the following functions are convex or not:
   (a) $f_1(x, y, z) = x^4 - 2xy + y^2 + z^2 + 2x^2$,
   (b) $f_2(x, y, z) = 5\,x^2 + 9\,y^4 + e^x + e^{-z}$,
   (c) $f_3(x, y) = -x^2 - y^2$,
   (d) $f_4(x, y) = x^2 - y^2$.

2. Let $f(x, y) = (x - y)^2 + (x + 2y + 1)^2 - 8xy$.
   (a) Express the function f in the form:
   $$f(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + c.$$
   (b) Show that $f(x, y)$ is convex in $\mathbb{R}^2$.
   (c) Find the global minimum of $f(x, y)$ in $\mathbb{R}^2$.

3. Let $f, g : C \subset \mathbb{R}^n \to \mathbb{R}$ be two convex functions on the convex set C.
   (a) Show that $h(x) = \max(f(x), g(x))$ is a convex function in C.
   (b) Show that the global minimum of $h(x)$ in C is either the minimum of one of the functions or a point at which $f(x) = g(x)$.

4. Let $g(x, y) = x^4 + 2x^2 - xy + y^2 + y^4$ and $f(x, y) = e^{g(x,y)}$.
   (a) Compute the gradient $\nabla g(x, y)$ and the Hessian $H_g(x, y)$ of $g(x, y)$.
   (b) Show that both $g(x, y)$ and $f(x, y)$ are strictly convex functions in $\mathbb{R}^2$.
   (c) Find the global minimum of $f(x, y)$ in $\mathbb{R}^2$.