Problem 1 (Counting and Graphs)**.** Let a_n be the number of words of length n on the alphabet $\{0,1\}$ which contain no two consecutive 0's.

(a) Write a recurrence relation for a_n. Solve the recurrence and give a formula for a_n.

(b) Consider the graph $G=(V,E)$ whose vertices are the words of length four on the alphabet $\{0,1\}$ which contain no two consecutive 0's, and two words are adjacent if the number of zeros differ by one. For example, 1111 and 0111 are adjacent but 1111 and 0110 are not. How many edges has the graph? Is it bipartite?

(c) Give a weight to every edge of G by the number of digits that their end vertices differ. For instance, the weight of the edge $\{1111, 0111\}$ is one and $\{1010, 1101\}$ has weight three. Run the Kruskal algorithm on this weighted graph and give a minimum spanning tree.

(d) Number the vertices of G from 1 to a_4 in some way and give the Prüfer code of the tree you obtained in (c) according to this numbering.

Solution. (a) A sequence of length n can be obtained from a sequence of length $n-1$ by adding either 0 or 1 if the last digit is 1, or adding only 0 if the last digit is 0.

Let b_{n-1} be the number of sequences of length $n-1$ ending in 1. Then,

$$a_n = 2b_{n-1} + (a_{n-1} - b_{n-1}) = b_{n-1} + a_{n-1}.$$

The number of sequences of length $n-1$ ending in 1 is precisely a_{n-2} (adding 1 to every sequence of length $n-2$). The recurrence is

$$a_n = a_{n-1} + a_{n-2},\ n \geq 3, a_1 = 2, a_2 = 3.$$

The first values are $2, 3, 5, 8, \ldots$. The characteristic polynomial of the recurrence is $x^2 - x - 1$ with roots $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Accordingly, the general solution is

$$a_n = a\alpha^n + b\beta^n, n \geq 3,$$

where the values of the initial values $a_1 = 2, a_2 = 3$ allows for the computation of the constants a, b by solving the linear system

$$\begin{aligned} a\alpha + b\beta &= 2 \\ a\alpha^2 + b\beta^2 &= 5 \end{aligned}$$

which gives $a = (3+\sqrt{5})/2\sqrt{5}, b = (\sqrt{5}-3)/2\sqrt{5}$.

(b) The graph has $a_4 = 8$ vertices and every vertex has four neighbours, the number of edges is 16. The graph is depicted in Figure 2 which shows a bipartition of G.

(c) The edges have either weight 1 or weight 3. The weights and a possible running of the Kruskal algorithm are depicted in Figure **??**

A minimum spanning tree has weight 7 (as edges of weight one form a connected subgraph)

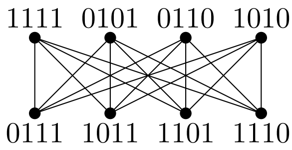

Figure 1: The graph G

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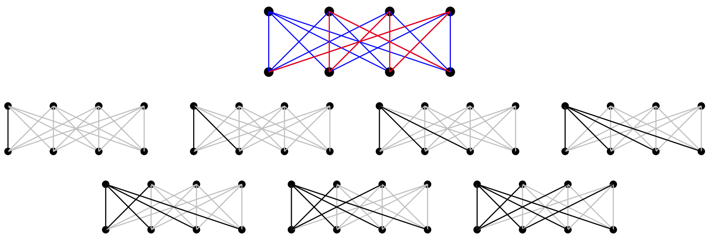


Figure 2: Blue edges have weight one and red edges weight three. The last picture shows a possible output of Kruskal algorithm.

(d) By labeling the vertices as in Figure 3 the Prüfer code of the tree obtained in (c) by Kruskal algorithm is

$$(5, 5, 6, 1, 1, 1)$$

Problem 2 (Linear Programing)**.** An algorithm A must be executed and two computers P_1 and P_2 are available with the following features per execution

	execution time	energy consumption
P_1	$5\mu s$	$4mW$
P_2	$7\mu s$	$3mW$

We want to maximize the number of executions of the algorithm with the constraints that the maximum amount of energy is $76mW$ and the maximum computing time is $90\mu s$. Moreover there must be a balance between the number of executions in the two machines, so that three times the number of executions on P_2 can not exceed the number of executions on P_1 plus 26, and three times the number of executions on P_1 can not exceed twice the number of executions on P_2 plus 23.

(a) Write a Linear Program to solve the problem.

(b) Draw the feasible region and identify graphically the solution.

(c) Write the Linear Program in equational form and run the simplex algorithm to find the solution.

Solution. (a) Let x and y be the number of executions of P_1 and P_2 respectively. The linear program which solves the problem is:

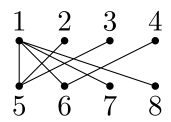


Figure 3: A numbering of the vertices of the tree.

$$\begin{array}{ll} \text{Maximize} & x+y \\ \text{Subject to} & 5x+7y \leq 90 \\ & 4x+3y \leq 76 \\ & -x+3y \leq 26 \\ & 3x-2y \leq 23 \\ & x,y \geq 0. \end{array}$$

(b) The feasible region is indicated in the Figure 4. The red line is the optimizing function and the optimal point is $(11,5)$ which correspond to 16 executions under the constraints.

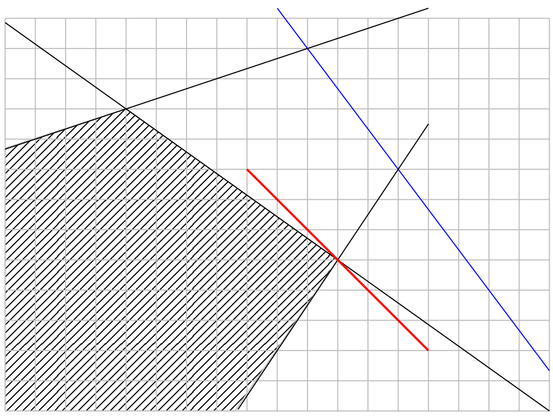


Figure 4: A graphical solution of (b). The red line is the optimizing function.

(c) We observe that the constraint $4x+3y \leq 76$ plays no role as, for $x,y \geq 0$, it is implied by the inequality $5x+7y \leq 90$ (this can be observed in the drawing in part (b)).

Introduce the slack variables z,u,v to get

$$\begin{array}{ll} \text{Maximize} & x+y \\ \text{Subject to} & 5x+7y+z = 90 \\ & -x+3y+u = 26 \\ & 3x-2y+v = 23 \\ & x,y,z,u,v \geq 0. \end{array}$$

A basic feasible solution is $(0,0,90,26,23)$. The initial tableau is

$$\begin{array}{l|l} z = 90-5x-7y & \\ u = 26+x-3y & \\ v = 23-3x+2y & \\ \hline x+y & 0 \end{array}$$

We update the tableau by letting x enter and v leave (the one restricting most the value of y). The new basic feasible solution is $(23/3, 0, 155/3, 101/3, 0)$ and the new tableau

$$\begin{array}{l|l} z = (155/3) - (31/3)y + (5/3)v & \\ u = (101/3) - (7/3)y - (1/3)v & \\ x = (23/3) + (2/3)y - (1/3)v & \\ \hline (23/3) + (5/3)y - (1/3)v & 23/3 \end{array}$$

Entering y (the only variable with positive coefficient in the cost function) and leaving z (the one restricting most the value of y) we get the the new tableau

$$\begin{array}{l|l} y = 5 - (3/31)z + (5/31)v & \\ u = 11 + (7/31)z - (22/31) & \\ x = 11 - (2/31)z - (37/93)v & \\ \hline \text{16-(5/31)z-(2/31)v} & 16 \end{array}$$

and the basic feasible solution $(5, 11, 0, 0, 0)$. Since the coefficients of the cost function are all negative, the algorithm stops giving the claimed solution.

Problem 3 (Extrema of functions)**.** Let $f(x, y) = 2x^2 + y^2 - xy + 2x + y + 4$.

(a) Write f as a quadratic function $f(x, y) = (x, y)\, Q\, \binom{x}{y} + (b_1, b_2)\binom{x}{y} + c$.

(b) Explain why f has a global minimum and find it by the Newton Method with initial point $(-1, -1)$.

(c) Find the first iteration of the Steepest Descent Method with initial point $(-1, -1)$.

Solution. (a) We can write

$$f(x, y) = (x, y) \begin{pmatrix} 2 & -1/2 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + (2, 1) \begin{pmatrix} x \\ y \end{pmatrix} + 4.$$

(b) The Hessian of f at any point is $2Q = \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$ which is positive definite (the principal determinants are 4 and 7, both positive). Therefore f is convex in the whole $\mathbb{R}^2$ (a convex set) and it has a global minimum.

By the Newton method with initial point $(x_0, y_0) = (-1, -1)$ we obtain with the first iteration

$$\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix} \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right) = - \begin{pmatrix} -3 \\ -1 \end{pmatrix}.$$

By solving the linear system we obtain the solution $(x_1, y_1) = (0, 0)$. Since the function is quadratic this is already the point where f attains its minimum $f(0, 0) = 4$.

(c) Starting at $(x_0, y_0) = (-1, -1)$ the next point in the Steepest Descent Method is

$$(x_1, y_1) = (x_0, y_0) - t\nabla f(-1, 1),$$

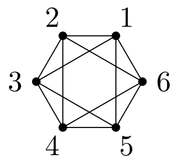
where t is chosen as to minimize the function

$$\phi(t) =$$

.

Problem 4 (Questions). (a) A connected graph G has 10 vertices. What is the minimum number of edges it can have? If G has the minimum possible number of edges, what is the minimum degree of a vertex in G.

(b) Give a tree which can be the output of the Breadth First Search algorithm on the graph of the figure

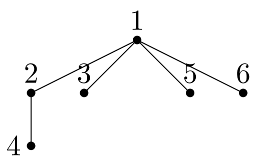


(c) We want to *minimize* the linear function $f(x, y, z) = 2x - 3y + 4z$ under the linear constraints $x + 6y + z \leq 1$, $2x - 3y + 5z \leq 12$, $-x + y - z \leq 20$, $x, y, z \geq 0$. Formulate it as a Linear Programming in equational form.

(d) We want to find the minimum of the function $f(x, y) = x^2 + y^2$ by the Gradient method starting at $(0, 1)$. Will the method provide the exact solution? If yes, in how many steps? (explain, you need no computations to answer).

Solution. (a) A connected graph with the minimum number of edges is a tree. If it has n vertices, the number of edge sis $n - 1$. Every tree has at elast two vertices of degree 1

(b) By starting at 1 the BFS algorithm returns the tree



(c) We must change 'minimize $2x - 3y + 4z$' by 'maximize $-2x + 3y - 4z$'.

(d) The level curves are concentric circles centerd at the origin. Therefore, starting at $(1, 0)$ the gradient points to the origin and the first step of the method reaches it, which is the global minimum, as the length of the step is as much as the first time where the function is no longer decreasing in its direction.