Problem 1 (Enumeration, 1.5 pt)**.** Obtain the exact number in each case:

a) (0.5 pt) The number of words that one can create by permuting the letters of 'ABRACADABRA';

b) (0.5 pt) We have 5 boys and 5 girls. The number of groups with 3 people and at least one girl;

c) (0.5 pt) The number words of length n that can be made with an alphabet of k symbols with exactly one pair of repeated consecutive letters;

Solution.

a) We have as repeated letters 2 B's, 2 R's and 5 A's (and a total of 11 letters). This makes the counting as follows:

$$\frac{11!}{2!2!5!1!1!} = 83160.$$

b) We can count it by taking all possible groups ($\binom{10}{3}$) and delete those ones with only boys ($\binom{5}{3}$). This is equal to

$$\binom{10}{3} - \binom{5}{3} = 110.$$

c) We can form $k(k-1)^{n-2}$ different words of length $n-1$ without consecutive letters. If we just want a single pair of consecutive repeated letters we need to choose which letter we want to do so, ($n-1$ possibilities) so we get

$$(n-1)k(k-1)^{n-2}.$$

Problem 2 (Recurrences, 2 pt)**.** Let $\{A, B, C\}$ be an alphabet of 3 letters. Let a_n be the number of words of length n where chains of A's (namely, blocks of consecutive A's) are of even length.

a) (1 pt) Find a_1, a_2 and show that a_n is equal to $2a_{n-1} + a_{n-2}$ when $n \geq 3$.

b) (1 pt) Get an explicit expression for a_n.

Solution.

a) It is obvious that $a_1 = 2$ (we cannot have the word 'A') and $a_2 = 5$: we have to discard words with just one A, which are 4 (AB, AC, BA, CA). Let us go now to the general case. We do it by looking at the last letter:

- If finishing with either B or C, the removal of the last letter (giving rise to a word of length $n-1$) gives rise to a word of length $n-1$ where each chain of A's have even length. This gives the term $2a_{n-1}$.

- If finishing with A, then we know that the letter $n-1$ must be an A as well (otherwise we would have a chain of A's of odd length). So by removing the last two A's we get a word of length $n-2$ where each chain of A's have even length. This case contributes with a a_{n-2}.

Summing up all contributions we get the desired equality $a_n = 2a_{n-1} + a_{n-2}$

b) From the recurrence relation in a) we get that the characteristic polynomial is $x^2 - 2x - 1$, whose roots are

$$\frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}.$$

We call $\lambda_1 = 1 + \sqrt{2}$ and $\lambda_2 = 1 - \sqrt{2}$. Hence, a_n has the general form $a_n = c_1\lambda_1^n + c_2\lambda_2^n$, for some constants c_1 and c_2 that we will find right now.

From the fact that $a_1 = 2$ and $a_2 = 5$ we get using the recurrence relation that a_0 must be 1. Hence,

$$\begin{aligned} a_0 = 1 = &\quad c_1 + c_2 \\ a_1 = 2 = &\quad c_1\lambda_1 + c_2\lambda_2 \end{aligned}$$

So this can be written as

$$\begin{aligned} \lambda_1 = &\quad c_1\lambda_1 + c_2\lambda_1 \\ 2 = &\quad c_1\lambda_1 + c_2\lambda_2 \end{aligned}$$

So $2 - \lambda_1 = c_2(\lambda_2 - \lambda_1)$. So, we get that $1 - \sqrt{2} = c_2(-2\sqrt{2})$ and hence $c_2 = \frac{2-\sqrt{2}}{4}$. By using the first equality we get that $c_1 = \frac{2+\sqrt{2}}{4}$ and so

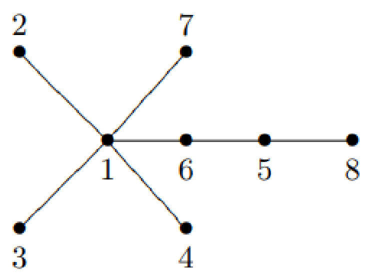
$$a_n = \frac{2+\sqrt{2}}{4}\left(1+\sqrt{2}\right)^n + \frac{2-\sqrt{2}}{4}\left(1-\sqrt{2}\right)^n.$$

Problem 3 (Graphs, 3.5 pt)**.** Answer the following questions. They can be answered independently.

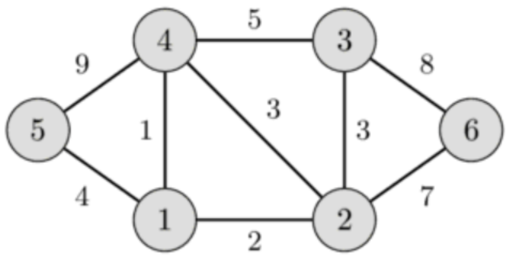
a) 0.75 pt Does there exist a graph on 5 vertices with vertex degrees 2,3,3,3,4?

b) 0.75 pt Find a graph on $2n$ vertices with exactly n spanning trees.

c) 1 pt Find the Prüfer code associated to the following tree



d) 1 pt Apply Prim's algorithm to the following graph, starting at vertex 1 as root:



Solution.

a) Not possible: by the Handshaking lemma we know that the number of vertices of odd degree is even, and this condition is not satisfied here.

b) There are many ways to do that. Pick a cycle of length n and attach to each vertex a new one, in such a way that the total number of vertices is equal to $2n$. Any spanning tree of the resulting graph must contain the pendant vertices, as well as $n-1$ edges of the cycle. In order to make this choice we just need to choose the missing edge from the cycle, and we have a total of n edges. This shows that the total number of spanning trees is equal to n, as we wanted to show.

c) In order to define the code we need, on each step of the algorithm, 1) identify the leaf with smallest label 2) identify its neighbour (and write it in the Prüfer code) 3) delete the corresponding leaf and repeat. We will need to run the procedure for 6 steps.

So, we have:

- Step 1: the smallest leave is 2, which is connected to 1 $\to (1, ...)$. We delete vertex 2.
- Step 2: the smallest leave now is 3, which is connected to 1 $\to (1, 1, ...)$. We delete vertex 3.
- Step 3: the smallest leave now is 4, which is connected to 1 $\to (1, 1, 1, ...)$. We delete vertex 4.
- Step 4: the smallest leave now is 7, which is connected to 1 $\to (1, 1, 1, 1, ...)$. We delete vertex 7.
- Step 5: the smallest leave now is 1, which is connected to 6 $\to (1, 1, 1, 1, 6...)$. We delete vertex 1.
- Step 6: the smallest leave now is 6, which is connected to 5 $\to (1, 1, 1, 1, 6, 5)$. We delete vertex 6 and we are done.

Hence, the Prüfer code is $(1, 1, 1, 1, 6, 5)$.

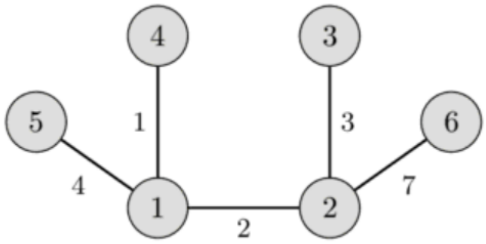
d) To apply Prim's algorithm we need, at every step, to add a new edge (not creating a cycle) with an end vertex not explored yet.

As the presented graph has 6 vertices, we will have to use 5 steps:

- Step 1: The smallest edge incident with 1 is $\{1, 4\}$. So we update with $V = \{1, 4\}$ and $E = \{\{1, 4\}\}$.
- Step 2: The smallest edge incident with 1 or 4 not creating a cycle is $\{1, 2\}$. So we update with $V = \{1, 2, 4\}$ and $E = \{\{1, 4\}, \{1, 2\}\}$.

- Step 3: The smallest edge incident with 1 or 4 or 2 not creating a cycle is $\{2,3\}$. So we update with $V = \{1,2,3,4\}$ and $E = \{\{1,4\},\{1,2\},\{2,3\}\}$.
- Step 4: The smallest edge incident with 1 or 4 or 2 or 3 not creating a cycle is $\{1,5\}$. So we update with $V = \{1,2,3,4,5\}$ and $E = \{\{1,4\},\{1,2\},\{2,3\},\{1,5\}\}$.
- Step 5: finally, the smallest edge incident with 1 or 4 or 2 or 3 or 5 not creating a cycle is $\{2,6\}$. So we update with $V = \{1,2,3,4,5,6\}$ and $E = \{\{1,4\},\{1,2\},\{2,3\},\{1,5\},\{2,6\}\}$.

The resulting spanning tree of minimum weight is then



Problem 4 (Linear Programming, 3pt)**.** Consider the following Linear Program.

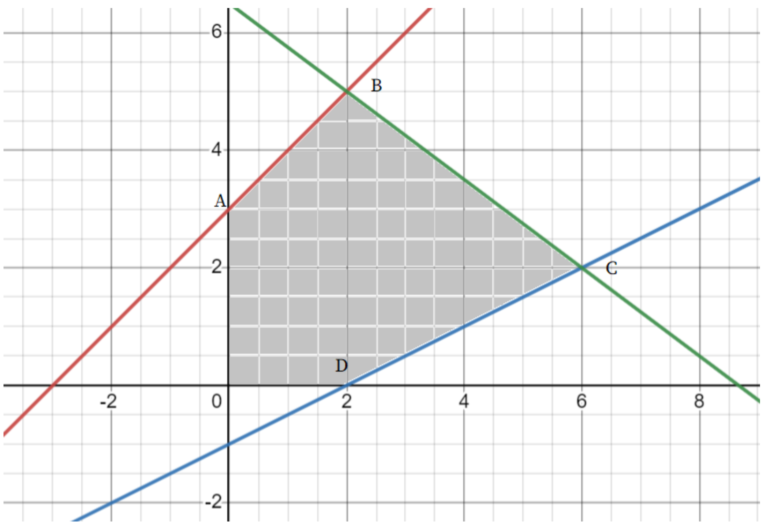
$$\begin{array}{lr} \text{maximize} & 2x+3y \\ \text{subject to} & -x+y \leq 3 \\ & x-2y \leq 2 \\ & 3x+4y \leq 26 \\ & x,y \geq 0. \end{array}$$

a) (1.5 pt) Identify the region of feasible solutions and give a geometric interpretation for the solution of the problem.

b) (1.5 pt) Rewrite the linear program in an appropriate form and apply the Simplex algorithm to solve it, starting the algorithm from the feasible solution $(x,y) = (0,0)$.

Solution.

a) The region of feasible solutions is contained in the region $A = \{(x,y) \in \mathbb{R}^2 : x,y \geq 0\}$, which can be obtained by intersection of 3 lines with A. By drawing these lines we get:



The points of intersection are the ones obtained by intersecting each pair of lines, obtaining $A = (0,3)$, $B = (2,5)$, $C = (6,2)$, $D = (2,0)$.

Now, let us argue on the point that maximizes the objective function. The line we need to consider is $2x + 3y = C$, and we need to know which is the largest value of C which makes this line intersect the region of feasible solutions. We know that such point must be on the boundary, so evaluating $2x + 3y$ on the vertices of the region and taking the largest one would give what we need. In this case, by evaluating at B we get $4 + 15 = 19$ which is the maximum value.

Hence the solution is given when taking $(x, y) = (2, 5)$.

Another way of seeing that is moving a line parallel to $2x + 3y = C$ from left to right (namely, increasing the value of C). The last point the line will intersect from the region will be precisely the point $(2, 5)$.

b) We start introducing new variables to make inequalities to be equalities. We use letters s_1, s_2 and s_3:

$$\begin{array}{lcccccc}
\text{maximize} & 2x & +3y & & & & \\
\text{subject to} & -x & +y & +s_1 & & & = 3 \\
 & x & -2y & & +s_2 & & = 2 \\
 & 3x & +4y & & & +s_3 & = 26 \\
 & x, y \geq 0. & & & & &
\end{array}$$

We know have a solution to the system $(x, y, s_1, s_2, s_3) = (0, 0, 3, 2, 26)$, which is our base initial solution. From here we will apply the simplex algorithm. We initialize it as:

$$\begin{array}{cllll|l}
s_1 & = 3 & +x & -y & & \text{eq.1} \\
s_2 & = 2 & -x & +2y & & \text{eq.2} \\
s_3 & = 26 & -3x & -4y & & \text{eq.3} \\
\hline
z & = 2x & +3y & & & z = 0
\end{array} \quad (0,0,3,2,26) \text{ basic feasible solution.}$$

1st iteration: we choose, for instance, y to enter, which makes s_1 to leave (is the most restrictive condition for y). This makes

$$y = 3 + x - s_1$$

and after the first iteration the system we get is

$$\begin{array}{ccccc|l}
y & = & 3 & +x & -s_1 & \text{eq.1} \\
s_2 & = & 8 & +x & -2s_1 & \text{eq.2} \\
s_3 & = & 14 & -7x & +4s_1 & \text{eq.3} \\
\hline
z & = 9 & +5x & -3s_1 & & z = 9
\end{array} \quad (0,3,0,8,14) \text{ basic feasible solution.}$$

2nd iteration: we choose x to leave. Looking at the equations, we need to pick s_3 to enter, as it is the most restrictive condition. Hence, we have

$$x = 2 + \frac{4}{7}s_1 - \frac{1}{7}s_3.$$

By substituting this relation in eq.1 and eq.2 we get

$$
\begin{array}{llllll|l}
y & = & 5 & +\frac{11}{7}s_1 & -\frac{1}{7}s_3 & & \text{eq.1} \\
s_2 & = & 10 & -\frac{10}{7}s_1 & -\frac{1}{7}s_3 & & \text{eq.2} \\
x & = & 2 & +\frac{4}{7}s_1 & -\frac{1}{7}s_3 & & \text{eq.3} \\
\hline
z & = 19 & -\frac{1}{7}s_1 & -\frac{1}{7}s_3 & & & z = 19
\end{array}
$$

$(2, 5, 0, 10, 0)$ basic feasible solution.

Which fits with the solution obtained geometrically in a).

If starting pivoting with x instead of y would need of 3 iterations instead of 2.