Bioinformatics

Discrete Mathematics and Optimisation

Problem Sheet Graphs and Networks II

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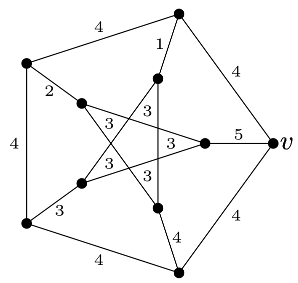
**1.** Consider the weighted graph in the figure




(a) Perform Kruskal's algorithm in the graph and display the consecutive subgraphs build by the algorithm. What is the weight of the resulting MST?

(b) Perform Prim's algorithm starting at vertex $v$ and display the consecutive subgraphs build by the algorithm. Is the resulting MST the same as in 2(a)?

**2.** Let $G$ be a weighted connected graph in which the weights are pairwise distinct. Prove that there exists a unique minimum weight spanning tree in $G$.

**3.** Let $G = (V, E)$ be a graph with $n$ vertices.

(a) Show that $G$ is connected if and only if, for every proper subset $X \subset V$ (i.e. $X \neq V$ and $\emptyset$), there is an edge with one endpoint in $X$ and one in $V \setminus X$.

(b) Deduce that if $G$ is connected then it has at least $n - 1$ edges.

(c) Deduce that $G$ is connected if and only if it contains a spanning tree as a subgraph.

**4.** A directed graph $G = (V, E)$ is *strongly connected* if for every pair of vertices $x, y \in V(G)$ there is a directed path from $x$ to $y$. An *arborescence* in a directed graph is a rooted spanning tree such that all edges are directed from the root to the leaves.

(a) Let $G$ be a a strongly connected directed graph and let $r \in V(G)$ be a vertex of $G$. Is it true that $G$ contains an arborescence rooted at $r$?

(b) Design an algorithm analogous to Prim's algorithm which, given a weighted strongly connected directed graph $G$ and a vertex $r \in V(G)$, outputs an arborescence rooted at $r$. Will this arborescence be of minimum weight?

**5.** The *travelling salesman problem* asks, given the weighted complete graph $K_n$, to find a cycle which visits every node exactly once (a Hamilton cycle) with minimum weight. Consider the following algorithm to build a Hamilton cycle in a weighted $K_n$:

- Build a minimum spanning tree with Prim's algorithm.
- Traverse the tree according to the Depth-first search algorithm.
- Return the list of vertices according to their first occurrence in the DFS algorithm.

A weighting of the complete graph satisfies the triangular inequality if, for every $x, y, z$, we have $w(xy) \leq w(xz) + w(zy)$. Show that if the weighting $w$ of $K_n$ satisfies the triangular inequality, the above algorithm returns a Hamilton cycle whose weight is not larger than twice the minimum weight of the optimal Hamilton cycle.

**6.** A *vertex cover* in a graph $G$ is a set $S \subset V(G)$ of vertices such that every edge is incident to some vertex in $S$. The MinCover problem asks the following: For a given input graph, output a vertex cover of minimum cardinality.

(a) Compute the minimum vertex cover of $K_n$, $K_{n,m}$ and $P_n$.

(b) Consider the following greedy algorithm to compute a vertex cover in a graph $G$.

- $G_0 = G$, $S = \emptyset$.
- While $|E(G_i)| > 0$, choose a vertex $x_i$ of maximum degree in $G_i$, set $S = S \cup \{x_i\}$ and $G_i = G_{i-1} - x_i$ (delete $x_i$ and all its incident edges from $G_{i-1}$.
- Output $S$.

Show that the algorithm outputs a vertex cover. Give an example where the obtained vertex cover is not a minimum cover.