

Extreme values: Convexity.

Discrete Mathematics and Optimization
Bioinformatics

1. Convex sets

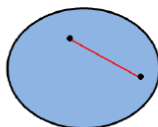
When local extrema can be ensured to be global?

Convexity is an important notion

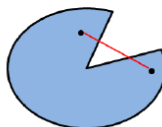
Definition

A subset $C \subset \mathbb{R}^n$ is **convex** if, for every $\mathbf{x}, \mathbf{y} \in C$, C contains all points in the segment joining \mathbf{x} and \mathbf{y} .

Convex



Non-convex



For every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ the segment joining \mathbf{x} and \mathbf{y} consists of the points

$$\lambda \mathbf{x} + (1 - \lambda) \mathbf{y}, \quad 0 \leq \lambda \leq 1.$$

1. Convex sets

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Examples of convex sets

- Balls $B(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{y}\| \leq r\}$ are convex.
- Translates of vector subspaces of \mathbb{R}^n are convex (lines, planes,...)
- If $C_1, C_2 \subset \mathbb{R}^n$ are convex then $C_1 \cap C_2$ is convex.
- Polytopes (constraints in Linear Programming) are convex.
- If $S = \{\mathbf{x}_1, \dots, \mathbf{x}_k\} \subset \mathbb{R}^n$, the **convex hull** of S

$$\text{Conv}(S) = \{\lambda_1 \mathbf{x}_1 + \dots + \lambda_k \mathbf{x}_k : \lambda_i \geq 0, \lambda_1 + \dots + \lambda_k = 1\},$$

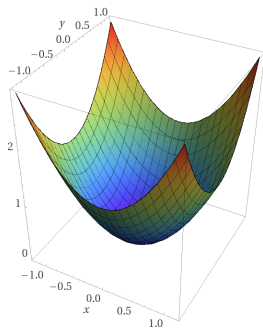
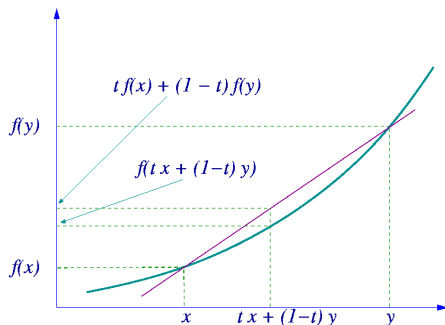
is convex.

2. Convex functions

Definition

A function $f : C \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is **convex** on the convex set C if, for every $\mathbf{x}, \mathbf{y} \in C$ and all $0 \leq \lambda \leq 1$,

$$f(\lambda \mathbf{x} + (1 - \lambda) \mathbf{y}) \leq \lambda f(\mathbf{x}) + (1 - \lambda) f(\mathbf{y}).$$



Computed by WolframAlpha

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- **Strict convex** if the inequality is strict.
- **Concave** if the opposite inequality holds.
- $f(\mathbf{x})$ is convex if and only if $-f(\mathbf{x})$ is concave.
- If f is convex on C and $\lambda_1 + \dots + \lambda_k = 1$, $\lambda_i \geq 0$ then, for every $\mathbf{x}_1, \dots, \mathbf{x}_k \in C$,

$$f(\lambda_1 \mathbf{x}_1 + \dots + \lambda_k \mathbf{x}_k) \leq \lambda_1 f(\mathbf{x}_1) + \dots + \lambda_k f(\mathbf{x}_k).$$

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Why convexity is important in optimization?

Theorem

Let $f : C \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be a convex function on the convex set C .
Then a (strict) local minimum of f is a (strict) **global** one.

- Let \mathbf{x}_0 be a local minimum, $f(\mathbf{x}_0) \leq f(\mathbf{y})$ for all \mathbf{y} close to \mathbf{x}_0 .
- Let $\mathbf{x} \in C$. Choose $0 < \lambda < 1$ such that $(1 - \lambda)\mathbf{x}_0 + \lambda\mathbf{x}$ is close to \mathbf{x}_0 .
- By convexity, $f(\mathbf{x}_0) \leq f((1 - \lambda)\mathbf{x}_0 + \lambda\mathbf{x}) \leq (1 - \lambda)f(\mathbf{x}_0) + \lambda f(\mathbf{x})$.
- It follows that $f(\mathbf{x}_0) \leq f(\mathbf{x})$.

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A word of caution

If f is convex its global minimum is contained in the critical points ($\nabla f(\mathbf{x}) = 0$)
or on the boundary of C .

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Examples of convex functions:

- Linear functions $f(x_1, \dots, x_n) = a_1 x_1 + \dots + a_n x_n$ are convex (and concave).
- x^2, x^4, \dots are convex.
- Odd powers x^3, x^5, \dots are **not** convex in \mathbb{R} .
- Exponential e^x is convex.
- Logarithm $\log(x)$ is concave.

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How to check if f is convex?

Theorem

Let $f \in \mathcal{C}^2(C)$. Then f is **convex** on C if and only if the Hessian matrix of f is **positive semidefinite** at every point $x \in C$.

Example: $f(x, y) = x^2 - xy + y^2$

- $\nabla f(x, y) = (2x - y, 2y - x)$
- $Hf(x, y) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

2. Convex functions

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An idea of the proof.

- f is convex if and only if $f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x})$.

- Use quadratic approximation

$$f(\mathbf{y}) = f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) + (\mathbf{y} - \mathbf{x})^T Hf(\mathbf{x})(\mathbf{y} - \mathbf{x}) + o(\|\mathbf{y} - \mathbf{x}\|^2).$$

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How to check if f is convex?

- If $f_1(\mathbf{x}), f_2(\mathbf{x})$ are convex on C then $f(\mathbf{x}) = f_1(\mathbf{x}) + f_2(\mathbf{x})$ is convex on C .
- If $f(\mathbf{x})$ is convex on C and $\alpha \in \mathbb{R}, \alpha > 0$, then $(\alpha f)(\mathbf{x})$ is convex on C .
- If f is convex on C and $g : f(C) \subset \mathbb{R} \rightarrow \mathbb{R}$ is increasing then $g(f(\mathbf{x}))$ is convex on C .

Example: $f(x, y) = e^{(x^2+y^2)} - \log(xy)$ is convex.

Summary

- **Convex** functions on **convex** sets have a global minimum if they have a local one.
- Convexity of a function can be discovered either by analyzing its components or by positivity of the Hessian.
- **Convex optimization** is a large area including topics as:
 - ▶ Least squares
 - ▶ Linear programming
 - ▶ Convex quadratic minimization with linear constraints
 - ▶ Quadratic minimization with convex quadratic constraints
 - ▶ Geometric programming
 - ▶ Semidefinite programming
 - ▶ Entropy maximization with appropriate constraints
 - ▶

A reference

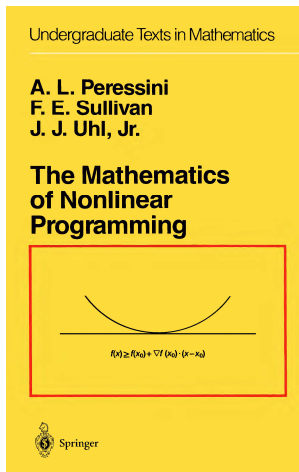


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