Discrete Mathematics and Optimization.
Review exam January 2020

1. **Enumeration. (25 points)** Consider the growth of a population of bacteria. Initially there are 20 young bacteria. At the end of each day the young bacteria in the population reach maturity and the mature bacteria in the population have offspring at a rate of two young bacteria per mature bacterium. Let a_n be the number of bacteria at the end of the n–th day (so $a_0 = 20$).

   (a) Find a recurrence relation for a_n.

   (b) Solve the recurrence and find an explicit formula for a_n.

2. **Graph Theory. (25 points)** Clustering is the process of grouping points in the plane by proximity. One can use Kruskal algorithm for clustering n points in the plane by building a complete graph K_n and weighting edge ij by their distance in the plane. If we want to classify the points in k groups we can run Kruskal algorithm in this weighted graph and stop it at the moment where we have a spanning forest with k connected components.

   Consider the set $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ in the plane where their mutual distances are given in the following table

	a_1	a_2	a_3	a_4	a_5	a_6
a_1		$2\sqrt{5}$	$\sqrt{26}$	2	1	$\sqrt{17}$
a_2	$2\sqrt{5}$		$\sqrt{2}$	4	$\sqrt{13}$	1
a_3	$\sqrt{26}$	$\sqrt{2}$		$\sqrt{26}$	$\sqrt{17}$	1
a_4	2	4	$\sqrt{26}$		$\sqrt{5}$	$\sqrt{26}$
a_5	1	$\sqrt{13}$	$\sqrt{17}$	$\sqrt{5}$		$\sqrt{10}$
a_6	$\sqrt{17}$	1	1	$\sqrt{26}$	$\sqrt{10}$	

   (a) Run Kruskal algorithm on the complete graph weighted by the mutual distances until we reach a spanning forest with two connected components.

   (b) Can we ensure that the obtained forest is the minimum weight spanning forest of the given weighted graph with two connected components? Explain.

3. **Linear Optimisation. (25 points)**

   Consider the Linear Program given by

$$\begin{aligned} \text{maximize} \quad & 3x + 2y \\ \text{subject to} \quad & x + 2y \leq 2 \\ & 2x + y \leq 3 \\ & x, y \geq 0. \end{aligned}$$

   (a) Write it in equational standard form and find a basic feasible solution.

   (b) Apply the simplex method in order to find an optimal solution.

4. **Non-linear Optimisation. (25 points)**

   Let $f(x, y) = 2x^2 + y^2 - xy + 2x + y + 4$.

   (a) Write f as a quadratic function $f(x, y) = (x, y)\, Q \binom{x}{y} + (x, y) \binom{b_1}{b_2} + c$.

(b) Explain why f has a global minimum and find it by the Newton Method with initial point $(-1,-1)$.

(c) Find the first iteration of the Steepest Descent Method with initial point $(-1,-1)$.

Solutions

1. **Enumeration.** Let a_n denote the number of bacteria at the end of month n, then $a_0 = 30$ and $a_1 = 30$. Denote by M_n the number of mature bacteria, and by Y_n the number of young bacteria, both at the end of month n. Clearly, $a_n = M_n + Y_n$. Now notice that any mature bacterium was either already mature in the previous month, or it was young, and hence

$$M_n = M_{n-1} + Y_{n-1} = a_{n-1}.$$

On the other hand, a young bacterium was produced by a mature one at a rate of 2 to 1, and hence

$$Y_n = 2M_{n-1} = 2a_{n-2},$$

where the last equality follows from our previous argument. Putting this together, we get the recurrence relation

$$a_0 = 30,\ a_1 = 30,\ \text{ and } a_n = a_{n-1} + 2a_{n-2} \text{ for all } n \geq 2.$$

To solve this, we see that the characteristic polynomial is $x^2 - x - 2$, which has roots $\lambda_1 = 2$ and $\lambda_2 = -1$, and hence the general form of the recurrence is

$$a_n = A \cdot 2^n + B \cdot (-1)^n,$$

where

$$\begin{aligned} A + B &= 30, \\ 2A - B &= 30. \end{aligned}$$

Solving this, we get $A = 20$ and $B = 10$, and hence

$$a_n = 20 \cdot 2^n + 10 \cdot (-1)^n.$$

2. **Linear Optimisation.**

(a) The equational standard form is given by

$$\begin{aligned} \text{maximize} \quad & 3x + 2y \\ \text{subject to} \quad & x + 2y + s_1 = 2 \\ & 2x + y + s_2 = 3 \\ & x, y, s_1, s_2 \geq 0. \end{aligned}$$

A basic feasible solution is given by $(0, 0, 2, 3)$.

(b) We write the system as

$$\begin{aligned} s_1 &= 2 - x - 2y \\ s_2 &= 3 - 2x - y \\ z &= 0 + 3x + 2y. \end{aligned}$$

We choose x as the entering variable and s_2 as the leaving variable and get

$$\begin{aligned} s_1 &= 1/2 + 1/2s_2 - 3/2y \\ x &= 3/2 - 1/2s_2 - 1/2y \\ z &= 9/2 - 3/2s_2 + 1/2y. \end{aligned}$$

We choose y as the entering variable and s_1 as the leaving variable and get

$$\begin{aligned} y &= 1/3 + 1/3s_2 - 2/3s_1 \\ x &= 4/3 - ... \\ z &= 14/3 - 4/3s_2 - 1/3s_1. \end{aligned}$$

As all coefficients in the last row are negative this cannot be optimised any further and the solution to the original system is $14/3$ obtained by $x = 1/3$ and $y = 4/3$.

3. **Non-linear Optimisation**

(a) We have $Q = \begin{pmatrix} 2 & -1/2 \\ -1/2 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $c = 4$.

(b) The Hessian of f is $2Q$ which is positive definite at every point: the function is strictly convex and has a global minimum.

Starting at $(x_0, y_0) = (-1, -1)$ the first iteration of the Newton method (for finding the critical point of $\nabla f(x, y)$) gives:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \frac{1}{2} Q^{-1} \nabla f(x_0, y_0)^T = \begin{pmatrix} -1 \\ -1 \end{pmatrix} - \frac{2}{7} \begin{pmatrix} 1 & 1/2 \\ 1/2 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} -5 \\ -6 \end{pmatrix}.$$

Since the Newton method applied to a quadratic function reaches the critical point in one step, $(-5/7, -6/7)$ is the global minimum.

(c) The first iteration of the steepest descent method starting at $(-1, -1)$ gives

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - t_0 \nabla f(x_0, y_0)^T = \begin{pmatrix} -1 \\ -1 \end{pmatrix} - t_0 \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

where t_0 is chosen to minimize

$$f((x_0, y_0) - t\nabla f(x_0, y_0)) = f((-1, -1) - t(-1, 0)) = f(t - 1, -1) = 2t^2 - t + 3.$$

The minimum is taken at the solution of $4t - 1 = 0$, namely $t = 1/4$. Therefore,

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3/4 \\ -1 \end{pmatrix}.$$