Bioinformatics
Discrete Mathematics and Optimization. Final Exam: 13.12.2023
Write problems in separate sheets

Problem 1 (Linear Programming, 3 points)**.** A manufacturer of printed circuits has an stock of 300 resistors, 120 transistors and 150 capacitors. They are required to to produce two types of circuits: Type A requires 20 resistors, 10 transistors and 10 capacitors, while Type B requires 10 resistors, 20 transistors and 30 capacitors. We know that the profit for each Type A circuit is of 5 euros, while the profit for each Type B circuit is of 12 euros. We want to maximize the profit once we sell our circuits.

a) (1.5 pt) Write a linear program to model this problem. Draw also the feasable region.

b) (1.5 pt) Find the solution of the problem by applying the simplex algorithm.

Solution.

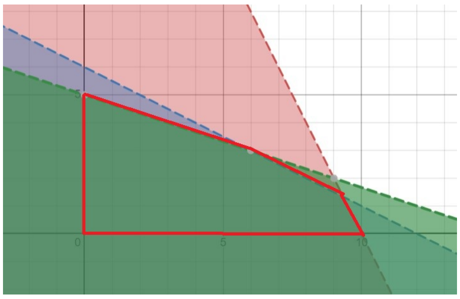
a) We denote by x and y the number of circuits of Type A and Type B, respectively. The Linear Program is then

$$\begin{aligned} \text{Maximize:} \quad & 5x + 12y \\ \text{Subject to:} \quad & 20x + 10y \leq 200 \\ & 10x + 20y \leq 120 \\ & 10x + 30y \leq 150 \\ & x, y \geq 0, \end{aligned}$$

which can be simplified to

$$\begin{aligned} \text{Maximize:} \quad & 5x + 12y \\ \text{Subject to:} \quad & 2x + 1y \leq 20 \\ & x + 2y \leq 12 \\ & x + 3y \leq 15 \\ & x, y \geq 0. \end{aligned}$$

Drawing each of the three equations we get the following region:



b) We introduce 3 slack variables s_1, s_2 and s_3 to get

$$\begin{aligned} \text{Maximize:}\quad & 5x + 12y \\ \text{Subject to:}\quad & 2x + y + s_1 = 20 \\ & x + 2y + s_2 = 12 \\ & x + 3y + s_3 = 15 \\ & x, y, s_1, s_2, s_3 \geq 0. \end{aligned}$$

We start with the basic feasible solution $(0, 0, 20, 12, 15)$ on this problem and organize it in a tableau

$$\begin{array}{llll|l} s_1 & = 20 & -2x & -y & \\ s_2 & = 12 & -x & -2y & \\ s_3 & = 15 & -x & -3y & \\ \hline z & = 0 & +5x & +12y & z = 0 \end{array}$$

Choose x as entering variable. The remaining variables put the restrictions $2x \leq 20$ from s_1, $x \leq 12$ form s_2 and $x \leq 15$ for s_3, the most restrictive is for s_1. So now x and s_1 are switched. The first equation is changed to $x = 10 - \frac{1}{2}s_1 - \frac{1}{2}y$ and the new tableau

$$\begin{array}{llll|l} x & = 10 & -\frac{1}{2}s_1 & -\frac{1}{2}y & \\ s_2 & = 2 & +\frac{1}{2}s_1 & -\frac{3}{2}y & \\ s_3 & = 5 & +\frac{1}{2}s_1 & -\frac{5}{2}y & \\ \hline z & = 50 & -\frac{5}{2}s_1 & +\frac{19}{2}y & z = 50 \end{array}$$

with solution $(10, 0, 0, 2, 5)$. In the next iteration we pick variable y, and then the equation for s_2 give us the most restrictive condition. Hence, $y = \frac{4}{3} + \frac{1}{3}s_1 - \frac{2}{3}s_2$, and the tableau is now

$$\begin{array}{llll|l} x & = 8 & -\frac{5}{4}s_1 & +\frac{1}{4}s_2 & \\ y & = \frac{4}{3} & +\frac{1}{3}s_1 & -\frac{2}{3}s_2 & \\ s_3 & = \frac{10}{6} & -\frac{1}{3}s_1 & +\frac{5}{3}s_s & \\ \hline z & = \frac{188}{3} & +\frac{2}{3}s_1 & -\frac{19}{3}s_2 & z = \frac{188}{3} \end{array}$$

So, we need to make an extra iteration, by entering s_1. This is the last one as the most restrictive condition is given by the equation for s_3. In this case, by substitution we obtain that $z = 66 - 2s_3 - 3s_2$, and so $z = 66$ is the maximum value for our problem.

Problem 2 (Nonlinear optimisation and convexity, 3 points)**.**

a) (1.25 pt) Consider the function $f(x, y) = xe^{-x}(y^2 - 4y)$. Find all the critical points of f and describe their local nature.

b) (1.75 pt) Write $g(x, y) = x^2 - 4xy + 5y^2 - 2x + 5y - 1$ in matrix form and show that it admits a unique global minimum in $\mathbb{R}^2$. For which values of a we can assure that the quadratic form depending on parameter a, $g_a(x, y) = g(x, y) = ax^2 - 4xy + 5y^2 - 2x + 5y - 1$ admits a unique global minimum?

Solution.

a) The gradient of $f(x, y) = xe^{-x}(y^2 - 4y)$ is

$$\nabla f(x, y) = e^{-x}\left((1 - x)(y^2 - 4y), x(2y - 4)\right).$$

The critical points are solutions to the following system of equations

$$\begin{aligned}(1-x)(y^2-4y) &= 0,\\ x(2y-4) &= 0,\end{aligned}$$

that is $(0,0)$, $(0,4)$, and $(1,2)$. The Hessian is

$$\mathrm{H}_f(x,y) = e^{-x}\begin{bmatrix}(x-2)(y^2-4y) & (1-x)(2y-4)\\ (1-x)(2y-4) & 2x\end{bmatrix}.$$

At the critical points, we obtain that

$$\begin{aligned}\mathrm{H}_f(0,0) &= \begin{bmatrix}0 & -4\\ -4 & 0\end{bmatrix}\text{(indefinite)}\\ \mathrm{H}_f(0,4) &= \begin{bmatrix}0 & 4\\ 4 & 0\end{bmatrix}\text{(indefinite)}\\ \mathrm{H}_f(1,2) &= \begin{bmatrix}4e^{-1} & 0\\ 0 & 2e^{-1}\end{bmatrix}\text{(positive definite)}\end{aligned}$$

So, $(0,0)$ and $(0,4)$ are saddle points, and $(1,2)$ is a local minimum.

b) The quadratic form of $g(x,y)$ is given by

$$g(x,y) = [x,y]\begin{bmatrix}1 & -2\\ -2 & 5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} + [x,y]\begin{bmatrix}-2\\ 5\end{bmatrix} - 1.$$

Using this form, we obtain

$$\nabla g(x,y) = \begin{bmatrix}1 & -2\\ -2 & 5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}-2\\ 5\end{bmatrix}$$

and

$$H_g(x,y) = \begin{bmatrix}2 & -4\\ -4 & 10\end{bmatrix}.$$

In particular, for any $(x,y)\in\mathbb{R}^2$ $H_g(x,y)$ is definite positive: its two leading principal minors are positive. That's true because the top-left corner is equal to 2 while the determinant equals 4. This implies that $g(x,y)$ is strictly convex in $\mathbb{R}^2$ and thus admits a unique global minimum.

In the case of g_a, the hessian matrix is equal to

$$H_g(x,y) = \begin{bmatrix}2a & -4\\ -4 & 10\end{bmatrix},$$

so in order to assure that the matrix is definite positive we need that $2a > 0$ and $20a-16 > 0$. So if $a > 5/4$, we can assure the existence of a global minimum.

Problem 3 (Iterative methods, 4 points)**.**

a) (2 pt) Let $f(x,y) = x^3 + y^3 - x - y$. Perform one iteration of the steepest descent method on $f(x,y)$ with initial point $(1,1)$.

b) (2 pt) Let $g(x,y) = x^2 + xy + 3y^2 - x + y + 5$. Find the global minimum of $g(x,y)$ via the Newton method with initial point $(1,1)$. What can you say for the function $g(x,y) - 7$?

Solution.

(a) The gradient of $f(x,y)$ is $\nabla f(x,y) = (3x^2 - 1, 3y^2 - 1)$, and evaluated at $(1,1)$ is equal to $(2,2)$. The first iteration of the steepest descent method applied to $f(x,y)$ and starting at $(1,1)$ is given by the point

$$(x_1, y_1) = (1,1) - t_0 \nabla f(1,1),$$

such that t_0 minimizes the function $\phi_0(t) = f((1,1) - t\nabla f(1,1)) = f(1-2t, 1-2t)$. In particular, t_0 is determined by $\phi_0'(t) = 0$ and $\phi_0''(t) \geq 0$. Using the chain rule, we first have

$$\begin{aligned}\phi_0'(t) &= -\nabla f(1,1) \cdot \nabla f((1,1) - t\nabla f(1,1)) \\ &= -(2,2) \cdot \nabla f(1-2t, 1-2t) \\ &= -4(3(1-2t)^2 - 1).\end{aligned}$$

Thus $\phi_0'(t) = 0 \implies t = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$. Furthermore,

$$\phi_0''(t) = 48(1-2t) = 24(\frac{1}{2} - t),$$

where $\phi_0''(1/2 - \sqrt{3}/6) > 0$, while $\phi_0''(1/2 + \sqrt{3}/6) < 0$. This means that $t_0 = 1/2 - \sqrt{3}/6$. And finally,

$$(x_1, y_1) = (1,1) - (\frac{1}{2} - \frac{\sqrt{3}}{6})(2,2) = (\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}).$$

b) The quadratic form of $g(x,y)$ is given by

$$g(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 5.$$

Using this form, we obtain

$$\nabla g(x,y) = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad H_g(x,y) = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}.$$

In particular, for any $(x,y) \in \mathbb{R}^2$ and by Sylvester's criterion, $H_g(x,y)$ is definite positive as its two leading principal minors are positive. This implies that $g(x,y)$ is striclty convex in $\mathbb{R}^2$ and thus admits a unique global minimum. Starting at $(1,1)$, the next iteration is the point

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - H_g(1,1)^{-1} \nabla g(1,1),$$

where

$$\nabla g(1,1) = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix},$$

and

$$H_g(1,1)^{-1} = \frac{1}{11}\begin{bmatrix} 6 & -1 \\ -1 & 2 \end{bmatrix}.$$

So that the global minimum of $g(x,y)$ is reached at

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{11}\begin{bmatrix} 6 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{11}\begin{bmatrix} 4 \\ 14 \end{bmatrix} = \begin{bmatrix} 7/11 \\ -3/11 \end{bmatrix}.$$

So this is the point we were looking for. When dealing with $g(x,y) - 7$, the computation is exactly the same, so would give exactly the same point.