Bioinformatics

Discrete Mathematics and Optimization. Midterm Exam October 2023

Time: 90 minutes

Problem 1 (Enumeration, 2.25 pt)**.** Solve each question separately:

a) (0.75 pt) 5 boys and 6 girls stand on a line, and the 5 boys are consecutive. In how many ways this configuration may happen?

b) (0.75 pt) In how many ways we can order the letters of the word RECONOCER?

c) (0.75 pt) The number of ways we can sit n people on a table (the only point that matters is the cyclic order on the table).

Solution.

a) We first order the boys (5! ways) and we put them in one of the 7 possible positions defined by 6 girls. Finally we order the girls. We get then $5! \times 7 \times 6!$.

b) This is obtained by computing

$$\frac{9!}{2!2!2!2!1!} = 22680.$$

c) We can cut the circle in n different points, getting a linear ordering of the people and giving $n!$. As we can do this in n different ways, the number of circular orderings is then $n!/n = (n-1)!$.

Problem 2 (Recurrences, 2.25 pt)**.** Solve each question separately:

a) (1 pt) You have access to 1×1 tiles which come in 2 different colors and 1×2 tiles which come in 3 different colors. We want to figure out how many different $1 \times n$ path designs we can make out of these tiles. Find a recursive definition for the sequence a_n of paths of length n.

b) (1.25 pt) Solve the recurrence relation $a_1 = 2$, $a_2 = 5$ and $a_n = 6a_{n-1} - 9a_{n-2}$ for $n \geq 3$.

Solution.

a) Looking at the last tile we get that $a_n = 2a_{n-1} + 3a_{n-2}$, with initial conditions $a_1 = 2$ and $a_2 = 3 + 3 = 6$ (we can cover a 1×2 tile with 1 tile of length 2 or with 2 tiles of lenght 1 each)

b) The corresponding quadratic equation is $r^2 - 6r + 9 = 0$, which has a unique double root 3. So our solution should have the form $a_n = (c_1 + nc_2)3^n$. We must now solve the system of equations:

$$\begin{cases} 2 = 3c_1 + 3c_2 \\ 5 = 9c_1 + 18c_2 \end{cases}$$

Subtracting 3 times the first equation from the second, we obtain $-1 = 9c_2$, or $c_2 = -\frac{1}{9}$. So $c_1 = \frac{1}{9} + \frac{2}{3} = \frac{7}{9}$. Therefore, our solution is:

$$a_n = \frac{7}{9} \cdot 3^n - \frac{1}{9} \cdot n3^n$$

or

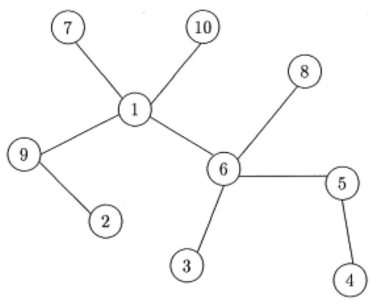
$$a_n = 73^{n-2} - n3^{n-2}.$$

Problem 3 (Graphs, 4 pt)**.** Answer the following questions. They can be answered independently.

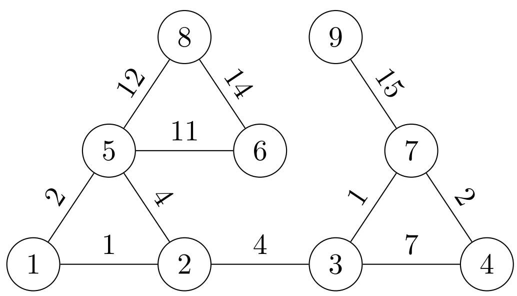
a) (1 pt) A graph with 8 vertices satisfies that each vertex has degree 3. How many edges does it have? Draw an example.

b) (1 pt) Let $V = \{1, 2, \ldots, n\}$. How are the graphs whose Prüfer codes are of the form $(a, a, \ldots, a)$ (namely, all entries are equal)?

c) (1 pt) Find the Prüfer code associated to the following tree



d) (1 pt) Apply Kruskal's algorithm to the following graph:



Solution.

a) Apply the Handshaking lemma to get that $\sum_{v \in V} d(v) = 3 \times 8 = 24$. Then the number of edges is half of it, namely 12. The cube is an example for this case.

b) Those trees are stars where the central vertex is labelled with symbol a.

c) In order to define the code we need, on each step of the algorithm, 1) identify the leaf with smallest label 2) identify its neighbour (and write it in the Prüfer code) 3) delete the corresponding leaf and repeat. We will need to run the procedure for 6 steps.

So, we have:

(a) **Initial list of vertices:** $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(b) **Start with an empty Prüfer code:** []

(c) Find the leaf with the smallest label (vertex 2), remove it, add its neighbor (vertex 9) to the Prüfer code, and remove the edge $\{2, 9\}$.

(d) **Prüfer Code:** [9], **Vertices Left:** $\{1, 3, 4, 5, 6, 7, 8, 10\}$

(e) Find the leaf with the smallest label (vertex 3), remove it, add its neighbor (vertex 6) to the Prüfer code, and remove the edge $\{3, 6\}$.

(f) **Prüfer Code:** $[9, 6]$, **Vertices Left:** $\{1, 4, 5, 7, 8, 10\}$

(g) Find the leaf with the smallest label (vertex 4), remove it, add its neighbor (vertex 5) to the Prüfer code, and remove the edge $\{4, 5\}$.

(h) **Prüfer Code:** $[9, 6, 5]$, **Vertices Left:** $\{1, 7, 8, 10\}$

(i) Find the leaf with the smallest label (vertex 5), remove it, add its neighbor (vertex 6) to the Prüfer code, and remove the edge $\{5, 6\}$.

(j) **Prüfer Code:** $[9, 6, 5, 6]$, **Vertices Left:** $\{1, 7, 8, 10\}$

(k) Find the leaf with the smallest label (vertex 7), remove it, add its neighbor (vertex 1) to the Prüfer code, and remove the edge $\{1, 7\}$.

(l) **Prüfer Code:** $[9, 6, 5, 6, 1]$, **Vertices Left:** $\{8, 10\}$

(m) Find the leaf with the smallest label (vertex 8), remove it, add its neighbor (vertex 6) to the Prüfer code, and remove the edge $\{6, 8\}$.

(n) **Prüfer Code:** $[9, 6, 5, 6, 1, 6]$, **Vertices Left:** $\{10\}$

(o) Find the leaf with the smallest label (vertex 10), remove it, add its neighbor (vertex 1) to the Prüfer code, and remove the edge $\{1, 10\}$.

(p) **Prüfer Code:** $[9, 6, 5, 6, 1, 6, 1]$, **Vertices Left:** $\{\}$

(q) The last two vertices are 1 and 6. Add them to the Prüfer code.

(r) **Final Prüfer Code:** $[9, 6, 5, 6, 1, 6, 1, 1, 6]$

d) To apply Kruskal's algorithm we need, at every step, to add a new edge with lower weight (not creating a cycle). In some cases we have some different possibilities

(a) **Initial State:** The graph with vertices and edges.

(b) **Step 1:** Add the smallest edge, $(1, 2)$ with weight 1.

(c) **Step 2:** Add the next smallest edge, $(3, 7)$ with weight 1.

(d) **Step 3:** Add the next smallest edge, $(4, 7)$ with weight 2.

(e) **Step 4:** Add the next smallest edge, $(2, 3)$ with weight 4.

(f) **Step 5:** Add the next smallest edge, $(5, 8)$ with weight 12.

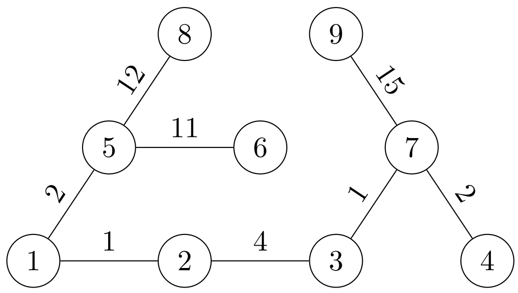
(g) **Step 6:** Add the next smallest edge, $(6, 8)$ with weight 14.

(h) **Step 7:** Add the next smallest edge, $(5, 6)$ with weight 11.

(i) **Step 8:** Add the next smallest edge, $(7, 9)$ with weight 15.

(j) **Result:** The minimum spanning tree is formed.

The resulting spanning tree of minimum weight is then


8
9
12
15
11
5
6
7
2
1
2
1
4
1
2
3
4

Problem 4 (Linear Programming, 1.5pt)**.** Consider the following Linear Program.

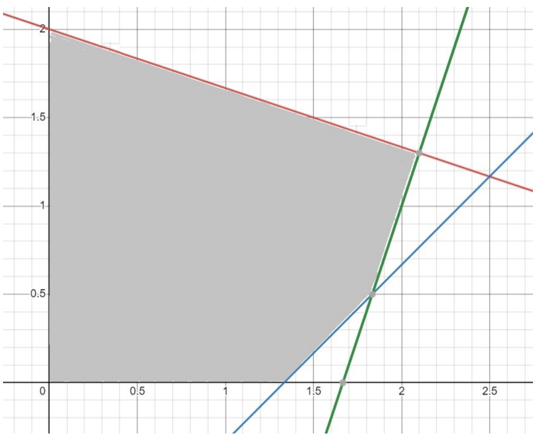
$$\begin{array}{ll} \text{maximize} & x \\ \text{subject to} & x+3y \leq 6 \\ & 3x-3y \leq 4 \\ & 3x-y \leq 5 \\ & x,y \geq 0. \end{array}$$

a) (1.0 pt) Draw the region of feasible solutions, and identify all relevant points in the boundary of the region.

b) (0.5 pt) Give a geometric interpretation for the solution of the problem. How does this change if we want to maximize y instead of x?

Solution.

a) We draw all the lines and consider the corresponding inequalities. Doing that we get the following region:



The important points to be considered in this context are the ones arising from the intersection of two (or more) lines. In this case we have $(0,0)$, $(0,2)$, $(4/3,0)$, $(21/10,13/10)$ and $(11/6,1/2)$.

b) Starting from a), we want to find the largest C such that the line $x=C$ intersects the region of feasible solutions. Hence we have to look at the largest possible value of x, which is 2.1 (which is taken at the point $(21/10,13/10)$). To do the same with y,we simply need to look at the highest point in the set of feasible solutions, which is in this case $(0,2)$ (so, x is maximized with the value $21/10$ and y is maximized with the value 2).