Bayesian Inference

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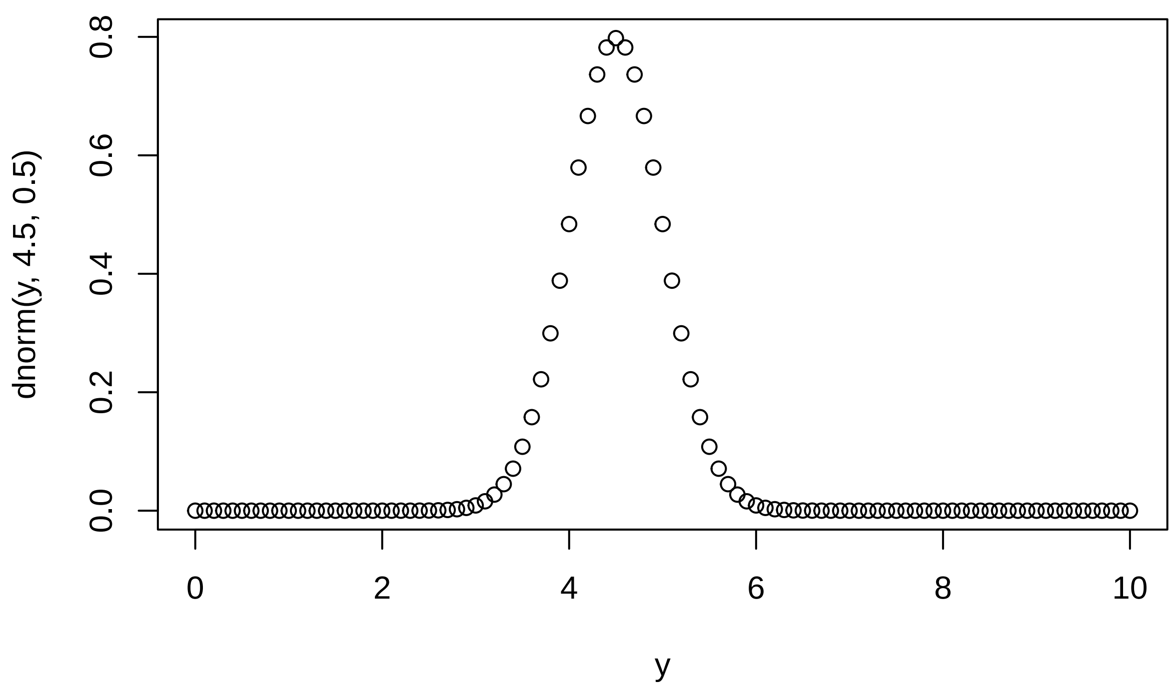
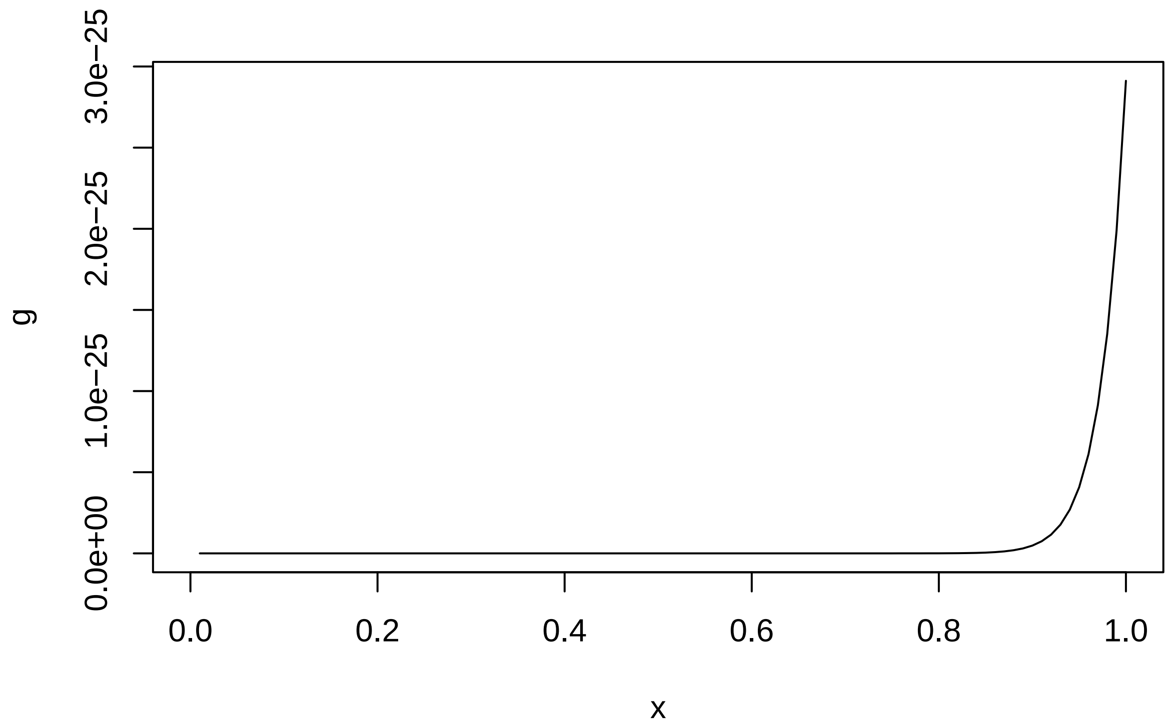
2. The authors of a paper claim that the mean life expectancy of this bacterium is 4.5±0.5h (reading the paper we see that 0.5h is just the standard deviation). Using a Gaussian density for implementing the prior information for parameter , compute the posterior density, the expectation and a 95% credibility interval. You can follow the following steps: #prior normal distribution

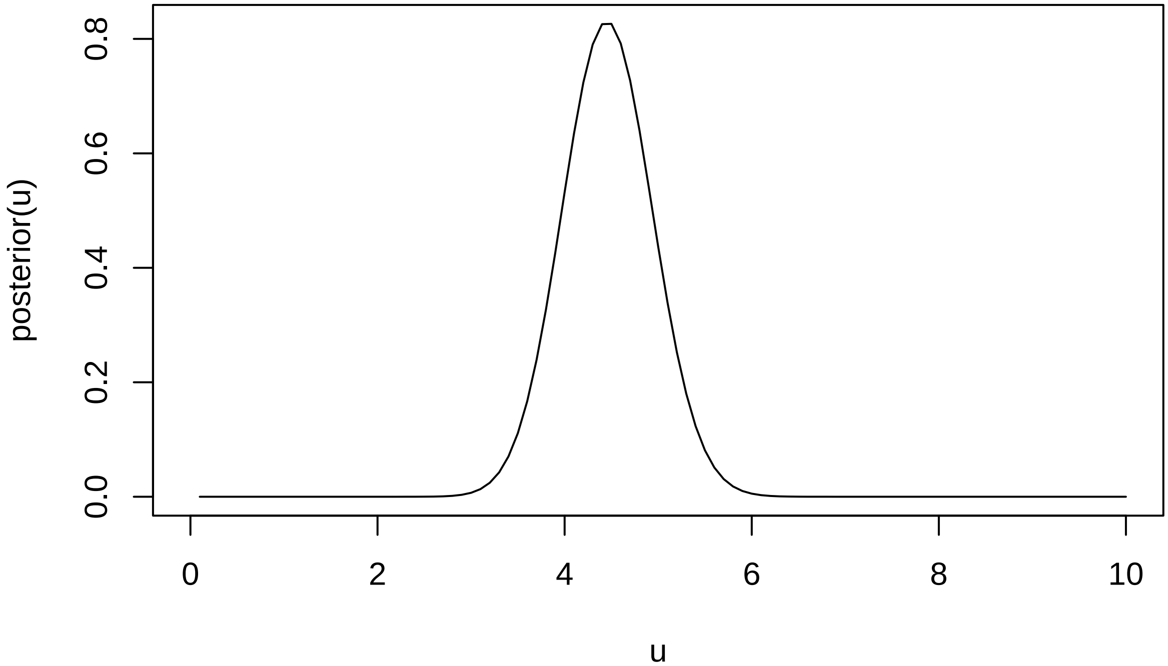
```
Likelihood_Function <- function(x){(1/(x^8))*exp(-31.77/x)}
y = seq(0,10,0.1)
plot(y, dnorm(y,4.5,0.5))
```




```
#now define f(nu|x) (posterior function)
g = function(x){dnorm(x,4.5,0.5)*(1/x^8)*exp(-31.77/x)}
plot(g)
```

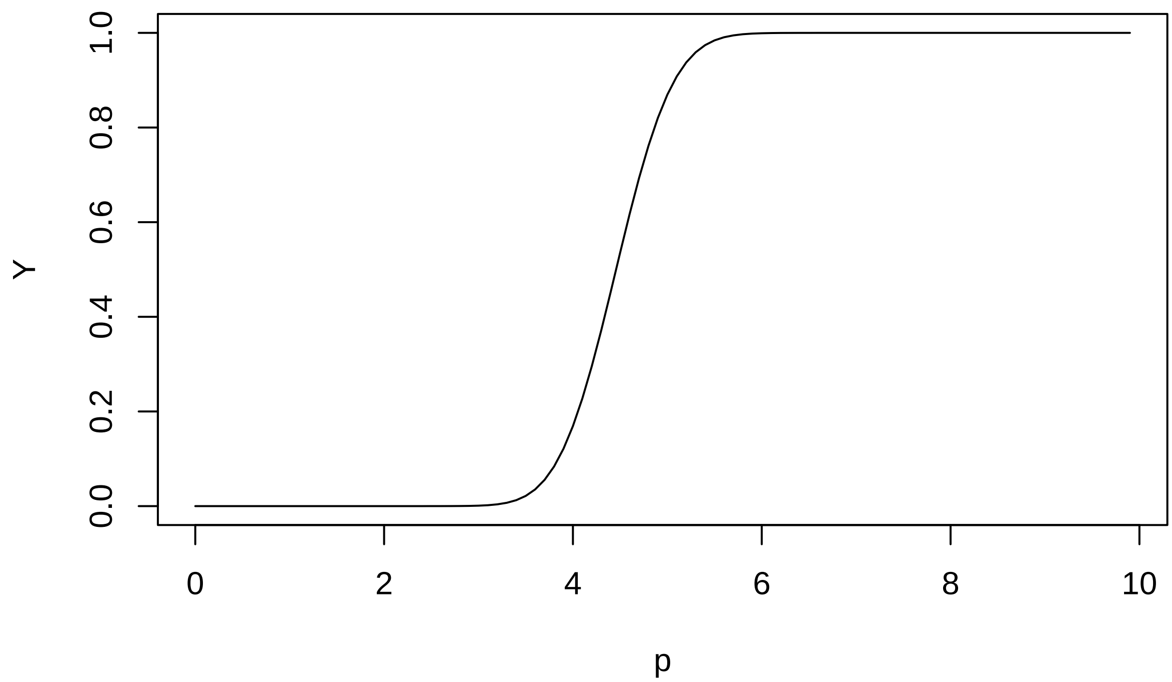



```
#integrate to obtain Constant
c <- integrate(g,0,Inf, rel.tol = 1e-18)$value #rel.tol is to have a lower error and be more sure
help("integrate")
#declare posterior with constant
posterior <- function(x){(dnorm(x,4.5,0.5)*(1/x^8)*exp(-31.77/x))/c}
u = seq(0,10,0.1)
Y = numeric(length(u))
for (i in 1:length(u)){
  Y[i] = posterior(u[i])
}
plot(u,posterior(u), type = 'l')
```




```
#it gives a error with seq because seq is a vector an this function does not accept vectors

#define the distribution
h = function(x){integrate(posterior,0,x)$value}
p = seq(0.000001,10,0.1)
Y = numeric(length(p))
for (i in 1:length(p)){
  Y[i] = h(p[i])
}
plot(p,Y,type = 'l')
```




```r
#F(u) = 0.025
lower_bound <- uniroot(function(x) h(x) - 0.025, interval = c(2, 6))

#F(u) = 0.975
upper_bound <- uniroot(function(x) h(x) - 0.975, interval = c(4, 8))
```

####3. Test if your results agree with those of the authors of the paper. In other words, consider H0: = 4.5 and H1: 4.5. Be non informative for the prior probabilities of H0 and H1 , and consider for H1 the same Gaussian prior than in part 2. Compute P(H0|X), P(H1|X), and the Bayes factor. Which are the conclusions?

```r
pH0 <- (Likelihood_Function(4.5))/(Likelihood_Function(4.5)+ integrate(g, 0, Inf)$value)
pH1 <- 1 - pH0
#Bayesian factor (in favor of H0):
bayesian <- pH0/pH1
cat('The bayesian factor in favor of H0:',bayesian)
```

```
## The bayesian factor in favor of H0: 1.259583
```

```r
#Bayesian factor (in favor of H1):
cat('The bayesian factor in favor of H1:',1/bayesian)
```

```
## The bayesian factor in favor of H1: 0.7939133
```

Practical Bayesian inference

① Lifetime of bacterium at certain T follows exponential distribution with density

$$f(x|\mu) = \frac{1}{\mu} e^{-x/\mu}, \quad x > 0 \quad \mu > 0$$

$n = 8$

$(2'56, 0'42, 3'72, 0'64, 10'7, 1'57, 9'62, 2'54)$ → mean $= \mu$

a) Likelihood function. Estimate mean life expectancy μ using classical inference procedure

$$L(x|\mu) = \prod_{i=0}^{n} \frac{1}{\mu} e^{\frac{-x}{\mu}} = \frac{1}{\mu^n} e^{\frac{-\sum^n x_i}{\mu}} = \frac{1}{\mu^8} e^{-\frac{(2'56+0'42+3'72+0'64+10'7+1'57+9'62+2'54)}{\mu}}$$

$$= \frac{1}{\mu^8} e^{\frac{-31'77}{\mu}}$$

Estimate mean life expectancy μ

$$\ell(x|\mu) = \ln\left(\frac{e^{-\sum^n x_i/\mu}}{\mu^n}\right) = \frac{-\sum x_i}{\mu} - \ln \mu^n = \frac{-31'77}{\mu} - \ln \mu^8$$

$$\frac{d}{d\mu} \ell(x|\mu) = \frac{d}{d\mu}\left(-\frac{31'77}{\mu} - \ln(\mu^8)\right) = \frac{31'77}{\mu^2} - \frac{1}{\mu^8} 8\mu^7 = \frac{31'77}{\mu^2} - \frac{8}{\mu}$$

$$\frac{d}{d\mu} \ell(x|\mu) = 0 \rightarrow \frac{31'77}{\mu^2} - \frac{8}{\mu} = 0 = \frac{31'77 - 8\mu}{\mu^2} \rightarrow 31'77 - 8\mu = 0 \rightarrow \frac{31'77}{8} = \mu$$

$\mu = 3'97$

② a) $\pi(\mu) L(x|\mu)$