Overview of the Course. Basic Counting

Discrete Mathematics and Optimization
Bioinformatics

Overview of the course

- Part 1 Basic Counting. Recurrence equations.
- Part 2 Graph Theory and Algorithms
- Part 3 Optimization: Linear Programing
- Part 4 Calculus with Several Variables and Nonlinear Optimization

Basic Counting

Samples of k individuals from a population of n individuals:

	Ordered	Unordered
With replacement	n^k	$\binom{n+k-1}{k}$
No replacement	$(n)_k = n(n-1)\cdots(n-k+1)$	$\binom{n}{k}$

- Words: ordered samples from an alphabet with repetition.
- Permutations: ordered samples with no repetition.
- Subsets: unordered samples from a set with no repetition.
- Multisets: unordered samples from a set with repetition.
- $\cdots$

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- Words: ordered samples from an alphabet with repetition.
  - Binary words of length n: $\{0,1\}^n \to 2^n$
  - ARN sequences of length n: $\{A, C, G, T\}^n \to 4^n$
  - Hexadecimal sequences of length n: $\{0, .., F\}^n \to (2^4)^n$
  - Car plates (Spain): $XXXXYY \to 10^4 \cdot 24^2$ (roughly)
- Permutations: ordered samples with no repetition.
  - Words with no repeated letters of length k on an alphabet of n letters

$$n(n-1)(n-2)...(n-k+1) = (n)_k = n!/(n-k)!$$

  - Permutations of n letters: $n!$ (convention: $0! = 1$)

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- Subsets: unordered samples from a set with no repetition.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$ Binomial coefficient

  - Committee of 3 people from a group of 6: $\binom{6}{3} = 20$.
  - Binary sequences of length 6 with 3 ones: $\binom{6}{3} = 20$.
  - Number of codes with 5 digits, two of them letters and three numbers: $\binom{5}{2} 24^2 10^3 = 576 \cdot 10^4$.

Basic Counting

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- Multisets: unordered samples from a set with repetition: samples of length k out of a set of size n.

$$\binom{n+k-1}{k}$$

  - Number of distinct values with three coins of value $1, 10, 100$: $\binom{5}{3} = 10$.
  - Number of multisets on $\{1, 2, 3\}$ of size 2: $\binom{4}{2} = 6$.

Properties of binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n)_k}{k!}.$$

- (Boundary values) $\binom{n}{0} = \binom{n}{n} = 1$.
- (Symmetry) $\binom{n}{k} = \binom{n}{n-k}$.
- (Recursion) $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$
- (Newton binomial formula) $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.
- (Number of subsets) $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.
- (Alternate sums) $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.
- (Bounds) $\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$.

Multinomial coefficients

$$\binom{n}{k_1, \ldots, k_r} = \frac{n!}{k_1! \cdots k_r!}, n = k_1 + \cdots + k_r.$$

Number of permutations of r symbols $a_1, \ldots, a_r$ where a_i is repeated k_i times.

Multinomial formula

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{k_1, \cdots, k_r : k_1 + \cdots + k_r = n} \binom{n}{k_1, \ldots, k_r} x_1^{k_1} \cdots x_r^{k_r}.$$

Proof by induction

Sometimes we need to prove things with respect to a certain size.

Strategy:

0) Guess the result.
1) Check it for $n = 1$ (or the first case).
2) Assume the result is true for n, check it for $n + 1$.

Example: Show that $1 + 2 + \cdots + 2^N = 2^{N+1} - 1$.

Recurrence equations

A sequence $a_0, a_1, \ldots, a_n, \ldots$ satisfies a recurrence equation if for some k (order of the recurrence) there is a function f such that

$$a_{n+k} = f(a_{n+k-1}, a_{n-k-2}, \ldots, a_n),\ n \geq 0$$

Examples:

- Fibonacci numbers $F_{n+2} = F_{n+1} + F_n,\ n \geq 0,\ F_0 = 0, F_1 = 1$
- Triangular numbers $T_{n+1} = T_n + n,\ n \geq 0,\ T_0 = 0$
- Factorial numbers $(n+1)! = (n+1)n!,\ n \geq 0,\ 0! = 1$
- $\cdots$

Goal: find an explicit formula for a_n as a function of n, or get some good estimates.

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A simple attempt: guessing the solution and checking the answer by induction
Examples:

- $t_{n+1} = 2t_n + 1, t_0 = 0\ \to 0, 1, 3, 7, 15, 31, 63, ... \to t_n = 2^n - 1.$
- $T_{n+1} = T_n + n, T_0 = 0\ \to 0, 1, 3, 6, 10, 15, 21, ... \to T_n = \binom{n+1}{2}$

Goal: find an explicit formula for a_n as a function of n, or get some good estimates.

Linear Recurrence equations

A sequence $a_0, a_1, \ldots, a_n, \ldots$ satisfies a linear recurrence equation of order k with constant coefficients if

$$a_{n+k} = c_1 a_{n+k-1} + c_2 a_{n+k-2} + \cdots + c_k a_n,\ n \geq 0$$

The polynomial

$$x^k - c_1 x^{k-1} - c_2 x^{k-2} - \cdots - c_k$$

is the characteristic polynomial of the recurrence equation.

Theorem

Let $\alpha_1, \ldots \alpha_t$ be the distinct roots of the characteristic polynomial of the linear recurrence relation. Then the general solution is

$$a_n = p_1(n)\alpha_1^n + \cdots p_t(n)\alpha_t^n, n \geq k,$$

where $p_i(n)$ is a polynomial with degree less than the multiplicity of α_i.

We use then the initial conditions to determine the $p_i(n)$.

Linear Recurrence equations: The Fibonacci numbers

$$F_{n+2} = F_{n+1} + F_n \; n \geq 0$$

- Characteristic polynomial $x^2 - x - 1$
- Roots $\phi = \frac{1+\sqrt{5}}{2}, \bar{\phi} = \frac{1-\sqrt{5}}{2}$
- General solution $F_n = A\phi^n + B\bar{\phi}^n$.
- Initial values $F_0 = 0$ and $F_1 = 1$ lead to

$$0 = A + B$$
$$1 = A\phi + B\bar{\phi}$$

$$\boxed{F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)}$$

Linear Recurrence equations: The Fibonacci numbers

$$F_{n+2} = F_{n+1} + F_n \ n \geq 0$$

Some consequences:

- $F_n / F_{n-1} \to \phi \approx 1.618$ ($n \to \infty$) (Golden Ratio)
- $F_n \sim \frac{1}{\sqrt{5}} \phi^n$ (asymptotic value)
- Solution of problems: number of binary words with no two consecutive ones.

$$\boxed{F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)}$$

Nonhomogeneus Linear Recurrence equations

A sequence $a_0, a_1, \ldots, a_n, \ldots$ satisfies a nonhomogeneous linear recurrence equation of order k with constant coefficients if

$$a_{n+k} = c_1 a_{n+k-1} + c_2 a_{n+k-2} + \cdots + c_k a_n + g(n), \;\; n \geq 0$$

The solution has the form

$$a_n = b_n + c_n$$

where

- b_n is the general solution of the homogeneous equation

$$a_{n+k} = c_1 a_{n-k-1} + c_2 a_{n-k-2} + \cdots + c_k a_n, \;\; n \geq 0$$

- c_n is a particular solution of the nonhomogeneous solution
    - If $g(n)$ is a polynomial, guess a polynomial solution of degree $\geq k$
    - If $g(n) = \alpha^n$, guess a solution $c \cdot \alpha^n$
    - If $g(n) = g_1(n) + g_2(n)$, add guesses from $g_1(n)$ and $g_2(n)$.

Nonhomogeneous Linear Recurrence equations

Example: $\boxed{a_{n+1} = 2a_n + n, n \geq 1, a_1 = 1}$

- General solution of the homogeneous solution: $b_n = b \cdot 2^n$,
- Guess a particular solution $c_n = cn + d$: substitute into the equation

  $$c(n+1)+d = 2(cn+d)+n \to cn+c+d = 2cn+2d+n \to 0 = (c+1)n+d-c.$$

  gives $c = -1, d = -1$.
- General solution: $a_n = b \cdot 2^n - n - 1$.
- Insert initial condition $a_1 = 1 = b \cdot 2^1 - 2 \to b = \frac{3}{2}$.

$$\boxed{a_n = \frac{3}{2}2^n - n - 1}$$

Check!