Problem 1 (Linear Programing)**.** A factory can make three products, P_1, P_2 and P_3. To make 1 tonne of P_1 takes 2 tonnes of raw material, 1 tonne of P_2 takes 1.5 tonnes of raw material, 1 tonne of P_3 takes 3 tonnes of raw material. The factory can import raw materials at a rate of 12 tonnes per hour. The power required to make a tonne of each product is $3KWh$ for P_1, $4KWh$ for P_2 and $1KWh$ for P_3. The factory has a power supply which can provide $10KW$. The factory can export at most 4.5 tonnes of all products combined each hour. The factory wants to make as much money as possible and the profit per tonne in thousands of euros is 2 for P_1, 3 for P_2 and 2 for P_3.

(a) Formulate this as a linear program.

(b) Suppose that the machine producing P_2 breaks down. Reformulate the linear program to account for this and solve it graphically.

(c) Solve the linear program from a) with the simplex method.

(d) Interpret your answer to part c): What does the factory need more of in order to make more profit? Why?

Solution.

(a) Let x, y, z be the number of tones produced of products P_1, P_2, P_3 respectively. The linear program is given by

$$\begin{array}{ll} \text{Maximize} & 2x + 3y + 2z \\ \text{Subject to} & 2x + 1.5y + 3z \leq 12 \\ & 3x + 4y + z \leq 10 \\ & x + y + z \leq 4.5 \\ & x, y, z \geq 0. \end{array}$$

(b) We remove the variable y as this corresponds to P_2. The new linear program is

$$\begin{array}{ll} \text{Maximize} & 2x + 2z \\ \text{Subject to} & 2x + 3z \leq 12 \\ & 3x + z \leq 10 \\ & x + z \leq 4.5 \\ & x, z \geq 0. \end{array}$$

The feasible region is indicated in the Figure 1. The solution is not unique, any point on the line $x + z = 4.5$ with $x \in [1.5, 2.75]$ is one, and the optimum value is 9.

You can watch an animation at:

`https://www.desmos.com/calculator/mm0w35bo4a`

(c) Solution 65/6 at $(0, 11/6, 8/3)$ in two iterations.

Introduce the slack variables t, u, v to get

$$\begin{array}{ll} \text{Maximize} & 2x + 3y + 2z \\ \text{Subject to} & 2x + (3/2)y + 3z + t = 12 \\ & 3x + 4y + z + u = 10 \\ & x + y + z + v = 9/2 \\ & x, y, z, t, u, v \geq 0. \end{array}$$

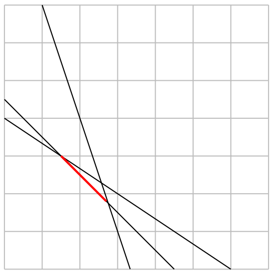

Figure 1: A graphical solution of (b). The red line is the optimizing function.

A basic feasible solution is $(0, 0, 0, 12, 10, 9/2)$. The initial tableau is

$$
\begin{array}{l|l}
t = 12 - 2x - (3/2)y - 3z & \\
u = 10 - 3x - 4y - z & \\
v = (9/2) - x - y - z & \\
\hline
2x + 3y + 2z & 0
\end{array}
$$

We update the tableau by letting y enter (larger coefficient in cost function) and u leave (the one restricting most the value of y). The new basic feasible solution is $(0, 5/2, 0, 33/4, 0, 2)$ and the new tableau

$$
\begin{array}{l|l}
t = (33/4) - (7/8)x - (21/8)z - (3/8)u & \\
y = (5/2) - (3/4)x - (1/4)z - (1/4)u & \\
v = 2 - (1/4)x - (3/4)z - (1/4)u & \\
\hline
(15/2) - (1/4)x + (5/4)z - (1/4)u & 15/2
\end{array}
$$

Entering z (the only variable with positive coefficient in the cost function) and leaving v (the one restricting most the value of z) we get the basic feasible solution $(0, 11/6, 8/3, 5/4, 0, 0)$ and the new tableau

$$
\begin{array}{l|l}
t = (5/4) + (1/2)u + (7/2)v & \\
y = (11/6) - (2/3)x - (1/6)u - (1/3)v & \\
z = (8/3) - (1/3)x - (1/3)u - (4/3)v & \\
\hline
65/6 - (8/12)x - (8/12)u - (5/3)v & 65/6
\end{array}
$$

Since the coefficients of the cost function are all negative, the algorithm stops giving the claimed solution.

(d) The factory should increase their power and/or export capacity as the slack variables corresponding to these constraints are 0 in the solution to the LP.

Problem 2 (Extrema of functions)**.** The regression line of a set $D = \{(x_1, y_1), \ldots, (x_n, y_n)\}$, $n \geq 2$, of data points in the plane is the line $y = ax + b$ where a, b are chosen to minimize the function

$$f(a, b) = \sum_{i=1}^{n} (ax_i + b - y_i)^2.$$

(a) Explain why the function f has a global minimum. What is the resulting line if $n = 2$? It is unique?

(b) Find the global minimum of f for the set of points $\{(0,0), (2,1), (4,3)\}$. Give the equation of the regression line of these points and sketch it graphically.

(c) Apply the Newton method with initial value $(1, -1)$ to the function in part (b) to obtain the global minimum.

(d) Apply two iterations of the gradient method with initial value $(1, -1)$ to the function in part (b). (You can leave the operations indicated in the second interation).

Solution.

(a) Each summand is the composition of an affine function $h_i(a, b) = ax_i + b - y_i$ with $g(t) = t^2$, both convex. The function f is the sum of convex functions and therefore convex. Therefore it has a global minimum in the convex set $\mathbb{R}^2$.

When $n = 2$ the line through the two points corresponds to values a, b such that $f(a, b) = 0$ and it is indeed a global minimum of f as $f(a, b) \geq 0$. The values of a, b are uniquely determined by the two points.

(b) For the given set of points the function is

$$f(a, b) = b^2 + (2a + b - 1)^2 + (4a + b - 3)^2.$$

The gradient of the function is

$$\begin{aligned} \nabla f(a, b) &= (4(2a + b - 1) + 8(4a + b - 3), 2b + 2(2a + b - 1) + 2(4a + b - 3)) \\ &= (40a + 12b - 28, 12a + 6b - 8). \end{aligned}$$

The critical point is

$$(a_0, b_0) = (3/4, -1/6).$$

We know that this point minimizes the function (also because the Hessian is $Hf(a_0, b_0) = \begin{pmatrix} 40 & 12 \\ 12 & 6 \end{pmatrix}$ which is positive definite). The equation of the regression line is $y = (3/4)x - 1/6$.

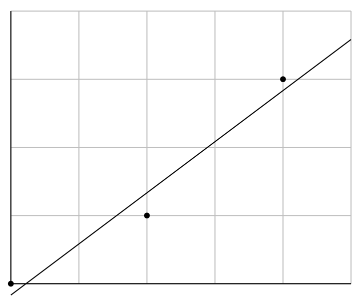

Figure 2: The regression line of the three points.

(c) Starting at $(u_0, v_0) = (1, -1)$ the first step of the Newton method gives the next point (u_1, v_1) satisfying

$$\begin{pmatrix} 40 & 12 \\ 12 & 6 \end{pmatrix} \begin{pmatrix} u_1 - 1 \\ v_1 + 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

from which we readily get $(u_1, v_1) = (3/4, -1/6)$. Since f is a quadratic function the Newton method gives the solution in one step.

(d) Starting at $(u_0, v_0) = (1, -1)$ the first step of the gradient function gives the next point (u_1, v_1) satisfying

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - t_1 \begin{pmatrix} 0 \\ -2 \end{pmatrix},$$

where t_1 is chosen to minimize the function

$$g_1(t) = f((1, -1) - t(0, -2)) = 12t^2 - 4t + 1,$$

namely, $t = 1/6$ and we get $(u_1, v_1) = (1, -2/3)$. The next iteration gives a point (u_2, v_2) satisfying

$$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2/3 \end{pmatrix} - t_2 \begin{pmatrix} 4 \\ 0 \end{pmatrix},$$

where t_2 is chosen to minimize the function

$$g_2(t) = f((1, -2/3) - t(4, 0)) = 320t^2 - 16t + 2/3,$$

which gives $t_2 = 1/20$ and $(u_2, v_2) = (4/5, -2/3)$.

Problem 3 (Questions). (a) The linear constraints of a linear programing problem are given by the inequalities $x + y \leq 3, -3x + y \leq 24, x \geq 0, y \geq 0$. Which are the possible solutions to the LP problem with a linear cost function different from $x + y, -3x + y, x$ or y? ?

(b) The gradient of a function $f : \mathbb{R}^2 \to \mathbb{R}$ is $\nabla f(x, y) = (2x^2 - e^y, \cos(xy))$. What is the direction in which the function has the largest decrease (locally) at the point $(1, 0)$?

(c) The Hessian of a function $f : \mathbb{R}^3 \to \mathbb{R}$ at the critical point $(1, 0)$ is $Hf(1, 0) = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$. What can be said about this critical point?

(d) One of the two following sequences can not be the output of the gradient method. Identify it and explain why.

$$x_0 = (4, 4), x_1 = (2, 2), x_2 = (4, 0), x_3 = (0, 0); \quad x_0 = (4, 4), x_1 = (2, 2), x_2 = (4, 0), x_3 = (3, -1)$$

Problem 4 (Counting and Graphs, Optional). Let $G = (V, E)$ a graph whose vertex set is the set of all words of length 3 on the alphabet $\{A, B, C\}$ which have precisely two repeated letters. Two vertices are adjacent if the corresponding words differ in precisely one entry. For instance, the vertices ABA and ACA are adjacent, while the vertices ABA and BAA are not.

(a) Give the number of vertices, the degree of a vertex and the number of edges of the graph.

(b) Consider the subgraph $G' \subset G$ induced by the vertices whose first letter is **not** C. Starting on the vertex AAC number the vertices of G' according to the first time the Breath First Search algorithm explores them. Draw the corresponding spanning tree and give its Prüfer code according to the obtained numbering.

(c) Give a weight to the edges of G' by the time you traversed them in the BFS algorithm, so when you first visit the i–th vertex the edge used to reach it has weight i. Give weight 0 to the edges not in the BFS tree. Run the Kruskal algorithm on the resulting weighted graph and give a minimum spanning tree of G' with this weighting. Give the minimum weight of a spanning tree of G'.

Solution.

(a) There are $\binom{3}{2}$ choices for the positions of the repeated letter, three choices for the repeated letter and two choices for the remaining one. The number of vertices is $n = \binom{3}{2} \cdot 3 \cdot 2 = 18$.

A vertex is adjacent to the ones obtained by (i) replacing one repeated letter by the nonrepeated one or (ii) by replacing the nonrepeated letter by the one not appearing in it. Thus every vertex has degree three.

The number of edges, by the handshaking lemma, is $m = 3n/2 = 27$.

(b) One possible output of the BFS algorithm is shown in the figure

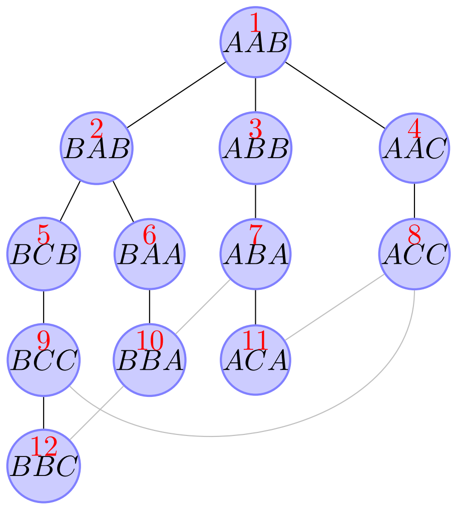

Figure 3: An output of the BFS algorithm on G'.

For the above numbering the BFS spanning tree has Prüfer code

$$(4, 1, 6, 2, 7, 3, 1, 2, 5, 9)$$

(c) The minimum spanning tree with weighting displayed is shown in Figure 4 The minimum weight of a spanning tree in the weighted graph G' is 35.

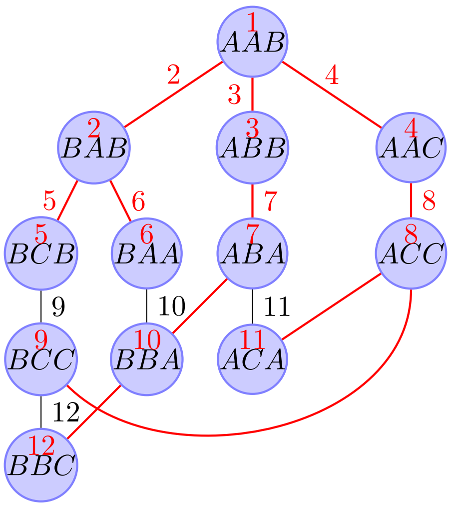

Figure 4: An output of the Kruskal algorithm.