Solutions to Problem Sheet Linear Programming

Exercise 1. The equational standard form is

$$\begin{aligned} \text{maximize} \quad & 4x_1 + 3x_2 + 2x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + 3x_3 + x_4 = 6 \\ & 2x_1 + x_2 + x_3 + x_5 = 3 \\ & x_1 + x_2 + x_3 + x_6 = 2 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \end{aligned}$$

Guessing a basic feasible solution is easy when you have slack variables:

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 6, 3, 2).$$

Using this, we create the tableau and apply the simplex method.

$$\begin{array}{rcrrrr} x_4 & = & \mathbf{6} & \boldsymbol{-x_1} & -2x_2 & -3x_3 \\ \boldsymbol{x_5} & = & \mathbf{3} & \boldsymbol{-2x_1} & -x_2 & -x_3 \\ x_6 & = & \mathbf{2} & \boldsymbol{-x_1} & -x_2 & -x_3 \\ \hline z & = & \mathit{0} & \boldsymbol{+4x_1} & +3x_2 & +2x_3 \end{array}$$

We choose x_1 as the entering variable and since $-6 < -2 < -3/2$ the leaving variable is x_5.

$$\begin{array}{rcrrrr} x_1 & = & \mathbf{3/2} & \boldsymbol{-1/2x_2} & -1/2x_3 & -1/2x_5 \\ x_4 & = & \mathbf{9/2} & \boldsymbol{-3/2x_2} & -5/2x_3 & +1/2x_5 \\ \boldsymbol{x_6} & = & \mathbf{1/2} & \boldsymbol{-1/2x_2} & -1/2x_3 & +1/2x_5 \\ \hline z & = & \mathit{6} & \boldsymbol{+x_2} & & -2x_5 \end{array}$$

We choose x_2 as our entering variable and the leaving variable is x_6.

$$\begin{array}{rcrrrr} x_2 & = & 1 & -x_3 & +x_5 & -2x_6 \\ x_1 & = & 1 & & -x_5 & +x_6 \\ x_4 & = & 3 & -x_3 & -x_5 & -3x_6 \\ \hline z & = & \mathit{7} & \boldsymbol{-x_3} & \boldsymbol{-x_5} & \boldsymbol{-2x_6} \end{array}$$

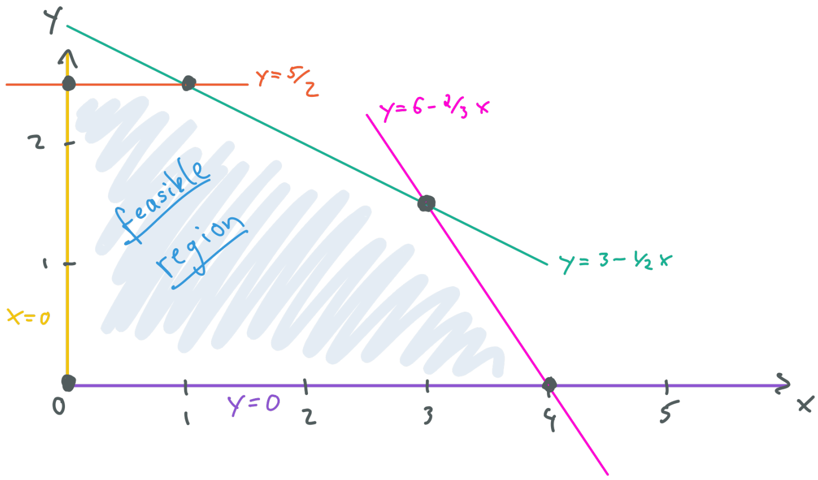
As there is no candidate for an entering variable, it follows that the maximum payoff is $\mathit{7}$ and it is attained by $(1, 1, 0, 3, 0)$.

Exercise 2.

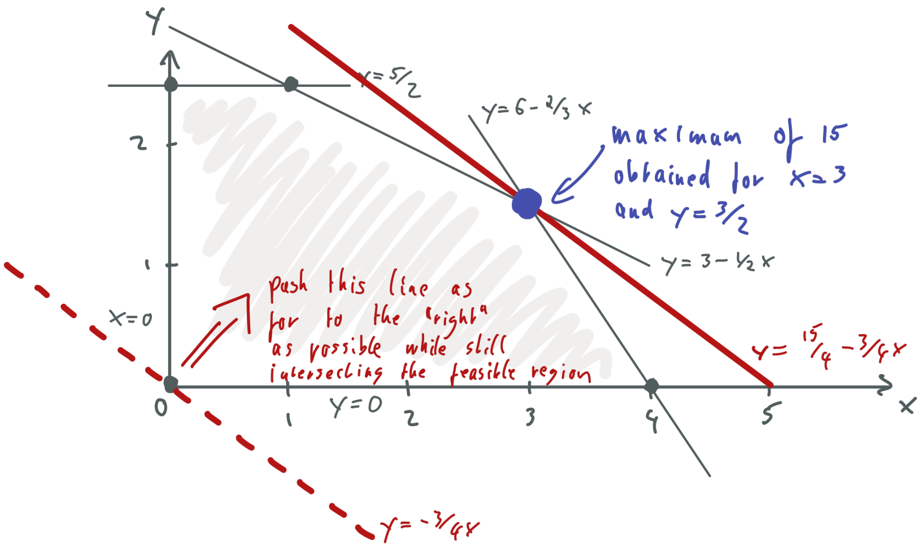
(a) Let us rewrite the conditions so that they are a function of x.

$$\begin{aligned} 3x + 2y \leq 12 \quad & \Leftrightarrow \quad y \leq 6 - 3/2\,x \\ 5x + 10y \leq 30 \quad & \Leftrightarrow \quad y \leq 3 - 1/2\,x \\ 2y \leq 5 \quad & \Leftrightarrow \quad y \leq 5/2. \end{aligned}$$

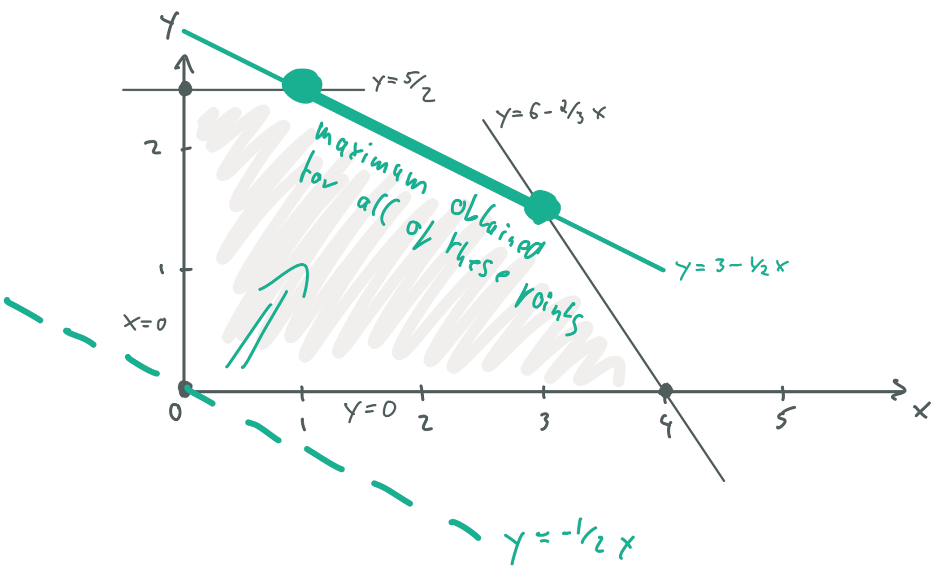
Drawing these conditions along with $x, y \geq 0$ gives us the feasible region:



Within this region, we want to maximize $3x + 4y$, which looks like this:



It follows that the maximum is 15 which is obtained for $(x, y) = (3, 3/2)$. If we change payoff we want to maximize to $6x + 12y$, then the maximum becomes 36. However, it is no longer obtained by a unique point but instead on the entire set $\{(x, y) : y = 3 - 1/2\, x \text{ and } 1 \leq x \leq 3\}$ as the following shows:



(b) Let us write x and y as x_1 and x_2 and introduce the three *slack variables* x_3, x_4, x_5 for the three conditions that are inequalities. The system in standard equational form becomes

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 4x_2 \\ \text{subject to} \quad & 3x_1 + 2x_2 + x_3 = 12 \\ & 5x_1 + 10x_2 + x_4 = 30 \\ & 2x_2 + x_5 = 5 \\ & x_1, x_2, x_3, x_4, x_4 \geq 0. \end{aligned}$$

Clearly $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 12, 30, 5)$ is a basic feasible solution. Starting from this, we form the tableau and apply the simplex algorithm. The payoff that we want to maximize will be denoted by the variable z.

$\boldsymbol{x_3}$	=	**12**	$\boldsymbol{-3x_1}$	$-2x_2$
x_4	=	**30**	$\boldsymbol{-5x_1}$	$-10x_2$
x_5	=	5		$-2x_2$
z	=	*0*	$\boldsymbol{+3x_1}$	$+4x_2$

Using Bland's rule, x_1 becomes the entering variable and since $-12/3 = -4 > -6 = -30/5$, the leaving variable should be x_3.

x_1	=	**4**	$\boldsymbol{-2/3x_2}$	$-1/3x_3$
$\boldsymbol{x_4}$	=	**10**	$\boldsymbol{-20/3x_2}$	$+5/3x_3$
x_5	=	**5**	$\boldsymbol{-2x_2}$	
z	=	*12*	$\boldsymbol{+2x_2}$	$-x_3$

We have increased the payoff from *0* to *12* by going to the feasible solution $(4, 0, 0, 10, 5)$. Next, x_2 becomes the entering variable and since $-6 < -5/2 < -3/2$, the leaving variable is x_4.

x_2	=	$3/2$	$-3/20x_4$	$+1/4x_3$
x_1	=	3	$+1/10x_4$	$-1/2x_3$
x_5	=	2	$+3/10x_4$	$-1/2x_3$
z	=	*15*	$\boldsymbol{-3/10x_4}$	$\boldsymbol{-1/2x_3}$

As there are no candidates with positive coefficients for the entering variable, we are done and the maximum payoff is *15* obtained by $(3, 3/2, 0, 0, 2)$.

Now if the payoff becomes $z = 6x_1 + 12x_2$ and we start at the previously calculated point $(3, 3/2, 0, 0, 2)$, then the tableau becomes

$$\begin{array}{ccccc} x_2 & = & 3/2 & -3/20x_4 & +1/4x_3 \\ x_1 & = & 3 & +1/10x_4 & -1/2x_3 \\ x_5 & = & 2 & +3/10x_4 & -1/2x_3 \\ \hline z & = & \mathit{36} & \boldsymbol{-6/5x_4} & \end{array}$$

There is no entering variable that would increase the payoff function and so it follows that we have already obtained a (not unique) maximum with payoff *36*. We could introduce x_3 as an entering variable and this will give the other vertex which has the same payoff.

Exercise 3. Note that for $y \to \infty$ the payoff $11x + y$ also goes to infinity while all conditions are still met. The feasible region is unbounded and there is no optimal solution.

Exercise 4.

(a) We flip the minimization, introduce two *slack variables* $s_1, s_2 \geq 0$ to turn the two inequalities into equalities and set $0 \leq y_1 = -x_1$, $0 \leq y_2 = x_2$ and $0 \leq y_3 = x_3$ to ensure that all variables are ≥ 0. We get

$$\begin{aligned} \text{maximize} \quad & 2y_1 - 2y_2 + 4y_3 \\ \text{subject to} \quad & -2y_1 + 2y_2 + 2y_3 = 10 \\ & -2y_1 - 6y_2 + y_3 - s_1 = 10 \\ & y_1 + 3y_2 - s_2 = 3 \\ & y_1, y_2, y_3, s_1, s_2 \geq 0. \end{aligned}$$

(b) We introduce two slack variables $s_1, s_2 \geq 0$ and set $0 \leq y_1 = -x_1$, $0 \leq x_2 = x_2$. Furthermore, we we introduce $y_3, y_4 \geq 0$ and set $x_3 = y_3 - y_4$. This allows us to replace the unbounded variable x_3 by two ≥ 0 variables. We get

$$\begin{aligned} \text{maximize} \quad & -3y_1 - 7y_2 + 5y_3 - 5y_4 \\ \text{subject to} \quad & y_2 - y_3 + y_4 + s_1 = -9 \\ & -y_1 - 2y_3 + 2y_4 - s_2 = 5 \\ & -4y_1 - y_2 = 6 \\ & y_1, y_2, y_3, y_4, s_1, s_2 \geq 0. \end{aligned}$$

Exercise 5.

(a) We introduce slack variables x_3, x_4, x_5, then the equational standard form is given by

$$\begin{aligned} \text{maximize} \quad & x_1 + 3x_2 + 2x_3 \\ \text{subject to} \quad & x_1 + 3x_2 + 3x_3 + x_4 = 6 \\ & 2x_1 + x_2 + 2x_3 + x_5 = 2 \\ & x_1 + 2x_2 + x_3 + x_6 = 3 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \end{aligned}$$

A basic feasible solution is given by $(0, 0, 0, 6, 2, 3)$.

(b) We can now write the tableau and apply the simplex algorithm. The payoff that we want to maximize will be denoted by the variable z, and we will be using Bland's rule to determine the entering and leaving variables.

$$\begin{array}{rcrrrr} x_4 & = & \mathbf{6} & -\mathbf{x_1} & -3x_2 & -3x_3 \\ \mathbf{x_5} & = & \mathbf{2} & -\mathbf{2x_1} & -x_2 & -2x_3 \\ x_6 & = & \mathbf{3} & -\mathbf{x_1} & -2x_2 & -x_3 \\ \hline z & = & \mathit{0} & +\mathbf{x_1} & +3x_2 & +2x_3 \end{array}$$

By Bland's rule, x_1 is the entering variable – it has the least index among all variables with positive coefficient in the objective function – and since $-1 = -2/2 > -3 > -6$, x_5 is the leaving variable. Rearranging the tableau we get

$$\begin{array}{rcrrrr} x_1 & = & \mathbf{1} & -\mathbf{1/2x_2} & -x_3 & -1/2x_5 \\ x_4 & = & \mathbf{5} & -\mathbf{5/2x_2} & -2x_3 & +1/2x_5 \\ x_6 & = & \mathbf{2} & -\mathbf{3/2x_2} & & +1/2x_5 \\ \hline z & = & \mathit{1} & +\mathbf{5/2x_2} & +x_3 & -1/2x_5 \end{array}$$

We see that x_2 enters, and since $-4/3 > -2$, the leaving variable is x_6.

$$\begin{array}{rcrrrr} \mathbf{x_1} & = & \mathbf{1/3} & -\mathbf{x_3} & -2/3x_5 & +1/3x_6 \\ x_2 & = & 4/3 & & +1/3x_5 & -2/3x_6 \\ x_4 & = & \mathbf{5/3} & -\mathbf{2x_3} & -1/3x_5 & +5/3x_6 \\ \hline z & = & \mathit{13/3} & +x_3 & +1/3x_5 & -5/3x_6 \end{array}$$

The entering variable is x_3, and since $-1/3 > -5/6$, x_1 leaves.

$$\begin{array}{rcrrrr} x_2 & = & 4/3 & & +1/3x_5 & -2/3x_6 \\ x_3 & = & 1/3 & -x_1 & -2/3x_5 & +1/3x_6 \\ x_4 & = & 1 & +2x_1 & +x_5 & +x_6 \\ \hline z & = & \mathit{14/3} & -\mathbf{x_1} & -\mathbf{1/3x_5} & -\mathbf{4/3x_6} \end{array}$$

All coefficients in the objective function are negative, so we cannot improve any further. So $(x_1, x_2, x_3) = (0, 4/3, 1/3)$ is an optimal solution for the original linear program, with objective function value 14/3.

Exercise 6.

(a) Let x_{ij} denote the tons of cargo C1/C2/C3/C4 ($i = 1, 2, 3, 4$ respectively) placed in the front/centre/rear ($j = F, C, R$ respectively) of the plane. The linear program is given by

$$\begin{aligned} \text{maximize} \quad & 310\sum_j x_{1j} + 380\sum_j x_{2j} + 350\sum_j x_{3j} + 285\sum_j x_{4j} \\ \text{subject to} \quad & \sum_i x_{iF} \le 10, \ \sum_i x_{iC} \le 16, \ \sum_i x_{iR} \le 8 \\ & 480x_{1F} + 650x_{2F} + 580x_{3F} + 390x_{4F} \le 6800, \\ & 480x_{1C} + 650x_{2C} + 580x_{3C} + 390x_{4C} \le 8700, \\ & 480x_{1R} + 650x_{2R} + 580x_{3R} + 390x_{4R} \le 5300, \\ & \sum_j x_{1j} \le 18, \ \sum_j x_{2j} \le 15, \ \sum_j x_{3j} \le 23, \ \sum_j x_{4j} \le 12, \\ & \sum_i x_{1F} = \frac{10}{34}\sum_{i,j} x_{ij}, \ \sum_i x_{1C} = \frac{16}{34}\sum_{i,j} x_{ij}, \ \sum_i x_{1R} = \frac{8}{34}\sum_{i,j} x_{ij}, \\ & x_{1F}, x_{1C}, x_{1R}, \ x_{2F}, x_{2C}, x_{2R}, \ x_{3F}, x_{3C}, x_{3R}, \ x_{4F}, x_{4C}, x_{4R} \ge 0. \end{aligned}$$

(b) We have assumed that we can choose arbitrarily small fractions of each cargo.

Exercise 7. Let $x_{\text{Mon}}, x_{\text{Tue}}, x_{\text{Wed}}, x_{\text{Thu}}, x_{\text{Fri}}, x_{\text{Sat}}, x_{\text{Sun}}$ respectively denote the number of recovery rooms starting to be in use on that specific day of the week. It follows that the room management service of the hospital needs to minimize the following linear program.

$$\begin{aligned}
\text{minimize} \quad & x_{\text{Mon}} + x_{\text{Tue}} + x_{\text{Wed}} + x_{\text{Thu}} + x_{\text{Fri}} + x_{\text{Sat}} + x_{\text{Sun}} \\
\text{subject to} \quad & x_{\text{Mon}} + x_{\text{Thu}} + x_{\text{Fri}} + x_{\text{Sat}} + x_{\text{Sun}} \geq 17 \\
& x_{\text{Mon}} + x_{\text{Tue}} + x_{\text{Fri}} + x_{\text{Sat}} + x_{\text{Sun}} \geq 13 \\
& x_{\text{Mon}} + x_{\text{Tue}} + x_{\text{Wed}} + x_{\text{Sat}} + x_{\text{Sun}} \geq 15 \\
& x_{\text{Mon}} + x_{\text{Tue}} + x_{\text{Wed}} + x_{\text{Thu}} + x_{\text{Sun}} \geq 19 \\
& x_{\text{Mon}} + x_{\text{Tue}} + x_{\text{Wed}} + x_{\text{Thu}} + x_{\text{Fri}} \geq 14 \\
& x_{\text{Tue}} + x_{\text{Wed}} + x_{\text{Thu}} + x_{\text{Fri}} + x_{\text{Sat}} \geq 16 \\
& x_{\text{Wed}} + x_{\text{Thu}} + x_{\text{Fri}} + x_{\text{Sat}} + x_{\text{Sun}} \geq 11.
\end{aligned}$$

Note that the solution might not consist of integer values. In that case, the room management service of the hospital could use fractions of rooms. If they can't, however, they will have to round up. In that case, optimality of the solution is no longer guaranteed, but solving an integer valued optimization is significantly harder.

Exercise 8.

(a) Let x and y be the number of executions of P on M_1 and M_2 respectively. The Linear Program is

$$\begin{aligned}
\text{Minimize:} \quad & 3x + y && \text{Energy consumption} \\
\text{Subject to:} \quad & 4x + y \leq 37 && \text{Time} \\
& x + 4y \leq 28 && \text{Cost} \\
& x + y \geq 10 && \text{Number of executions} \\
& x \geq 0, y \geq 0 && \text{Nonnegativity}
\end{aligned}$$

(b) The feasible region is indicated in Figure 1, the intersection of the semiplanes defined by the restrictions. The objective function is represented as a dashed line. Its minimum is at the intersection of a parallel line to the dashed line which first touches the feasinle region. Therefore its minimum occurs at the point $A = (4, 6)$ in the figure. The minimum is $18mW$.

(c) In the equational form we should maximize a linear function subject to constraints that are all equalities except for the variables, which must be nonnegative. We introduce slack variables z, w, t and write the problem as

$$\begin{aligned}
\text{Maximize:} \quad & -3x - y \\
\text{Subject to:} \quad & 4x + y + z = 37 \\
& x + 4y + w = 28 \\
& -x - y + t = -10 \\
& x, y, z, w, t \geq 0
\end{aligned}$$

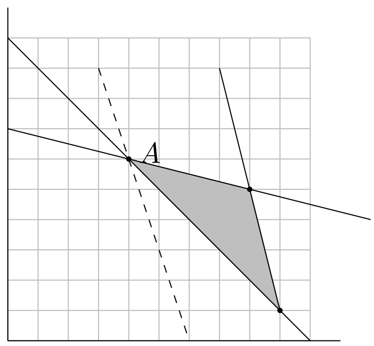


Figure 1: The feasible region and the objective function.

The matrix of the system is

$$A = \begin{pmatrix} 4 & 1 & 1 & 0 & 0 \\ 1 & 4 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

In order to find a basic feasible solution to start the simplex algorithm, we select two variables and set them to 0 as long as the corresponding solution to the system has nonnegative entries. The choice $w = t = 0$ gives the basic feasible solution $(4, 6, 15, 0, 0)$ and the tableau

$$\begin{array}{llll|l} x & = 4 & -(1/3)w & +(4/3)t & \\ y & = 6 & -(1/3)w & -(1/3)t & \\ z & = 15 & -w & -5t & \\ \hline c & = -18 & -(2/3)w & -(11/3)t & -18 \end{array}$$

We can not increase the objective function by replacing the values $w = t = 0$ by any positive values (because the coefficients of c are negative), so that the given basic feasible condition is already the optimal one.