Discrete Mathematics and Optimization. Fall 2017

1. **Linear Optimisation 1. (25 points)**

   Consider the Linear Program given by

$$\begin{aligned} \text{maximize} \quad & 3x + 2y \\ \text{subject to} \quad & x + 2y \leq 2 \\ & 2x + y \leq 3 \\ & x, y \geq 0. \end{aligned}$$

   (a) Write it in equational standard form and find a basic feasible solution.

   (b) Apply the simplex method in order to find an optimal solution.

2. **Linear Optimisation 2. (25 points)**

   Suppose WaterTech manufactures three products, requiring time on two machines and two types of labour (skilled and unskilled). The amount of machine time and labor (in hours) needed to produce a unit of each product and the sales prices in dollars per unit of each product are given in the following table:

product	**machine 1**	**machine 2**	**skilled labour**	**unskilled labour**	**unit sale price**
1	11	4	8	7	300
2	7	6	5	8	260
3	6	5	5	7	220

   Each month, 700 hours are available on machine 1 and 500 hours on machine 2. Each month, WaterTech can purchase up to 600 hours of skilled labour at \$8 per hour and up to 650 hours of unskilled labour at \$6 per hour. The company wants to determine how much of each product it should produce each month and how much labor to purchase in order to maximise its profit (i.e., revenue from sales minus labour costs).

   Formulate the problem as a Linear Program. Define the decision variables carefully and translate the constraints into corresponding inequalities. You do **not** have to solve it.

3. **Non-linear Optimisation 1. (25 points)**

   Let $f(x, y) = 2x^2 + y^2 - xy + 2x + y + 4$.

   (a) Write f as a quadratic function $f(x, y) = (x, y)\, Q \binom{x}{y} + (x, y) \binom{b_1}{b_2} + c$.

   (b) Explain why f has a global minimum and find it by the Newton Method with initial point $(-1, -1)$.

   (c) Find the first iteration of the Steepest Descent Method with initial point $(-1, -1)$.

4. **Non-linear Optimisation 2. (25 points)**

   Let $g(x, y) = x^4 + 2x^2 - xy + y^2 + y^4$ and $f(x, y) = e^{g(x,y)}$.

   (a) Compute the gradient $\nabla g(x, y)$ and the Hessian $Hg(x, y)$ of g.

   (b) Show that both g and f are convex functions.

   (c) Find the global minimum of f.

Solutions

1. **Linear Optimisation 1. (25 points)**

   (a) The equational standard form is given by

$$\begin{aligned} \text{maximize} \quad & 3x + 2y \\ \text{subject to} \quad & x + 2y + s_1 = 2 \\ & 2x + y + s_2 = 3 \\ & x, y, s_1, s_2 \geq 0. \end{aligned}$$

   A basic feasible solution is given by $(0, 0, 2, 3)$.

   (b) We write the system as

$$\begin{aligned} s_1 &= 2 - x - 2y \\ s_2 &= 3 - 2x - y \\ z &= 0 + 3x + 2y. \end{aligned}$$

   We choose x as the entering variable and s_2 as the leaving variable and get

$$\begin{aligned} s_1 &= 1/2 + 1/2s_2 - 3/2y \\ x &= 3/2 - 1/2s_2 - 1/2y \\ z &= 9/2 - 3/2s_2 + 1/2y. \end{aligned}$$

   We choose y as the entering variable and s_1 as the leaving variable and get

$$\begin{aligned} y &= 1/3 + 1/3s_2 - 2/3s_1 \\ x &= 4/3 - \ldots \\ z &= 14/3 - 4/3s_2 - 1/3s_1. \end{aligned}$$

   As all coefficients in the last row are negative this cannot be optimised any further and the solution to the original system is $14/3$ obtained by $x = 1/3$ and $y = 4/3$.

2. **Linear Optimisation 2. (25 points)**

   x_i = number of units of product i to manufacture

   y_1 = number of purchased hours of skilled labour

   y_2 = number of purchased hours of unskilled labour

$$\begin{aligned} \text{maximize} \quad & 300x_1 + 260x_2 + 220x_3 - 8y_1 - 6y_2 \\ \text{subject to} \quad & 11x_1 + 7x_2 + 6x_3 \leq 700 \\ & 4x_1 + 6x_2 + 5x_3 \leq 500 \\ & 8x_1 + 5x_2 + 5x_3 \leq y_1 \\ & 7x_1 + 8x_2 + 7x_3 \leq y_2 \\ & y_1 \leq 600 \\ & y_2 \leq 650 \\ & x_1, x_2, x_3, y_1, y_2 \geq 0 \end{aligned}$$

   There is the additional restriction that the x_i should be integer but we don't need to take this into account.

3. **Non-linear Optimisation 1. (25 points)**

(a) We have $Q = \begin{pmatrix} 2 & -1/2 \\ -1/2 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $c = 4$.

(b) The Hessian of f is $2Q$ which is positive definite at every point: the function is strictly convex and has a global minimum.

Starting at $(x_0, y_0) = (-1, -1)$ the first iteration of the Newton method (for finding the critical point of $\nabla f(x, y)$) gives:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \frac{1}{2} Q^{-1} \nabla f(x_0, y_0)^T = \begin{pmatrix} -1 \\ -1 \end{pmatrix} - \frac{2}{7} \begin{pmatrix} 1 & 1/2 \\ 1/2 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} -5 \\ -6 \end{pmatrix}.$$

Since the Newton method applied to a quadratic function reaches the critical point in one step, $(-5/7, -6/7)$ is the global minimum.

(c) The first iteration of the steepest descent method starting at $(-1, -1)$ gives

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - t_0 \nabla f(x_0, y_0)^T = \begin{pmatrix} -1 \\ -1 \end{pmatrix} - t_0 \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

where t_0 is chosen to minimize

$$f((x_0, y_0) - t\nabla f(x_0, y_0)) = f((-1, -1) - t(-1, 0)) = f(t-1, -1) = 2t^2 - t + 3.$$

The minimum is taken at the solution of $4t - 1 = 0$, namely $t = 1/4$. Therefore,

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3/4 \\ -1 \end{pmatrix}.$$

4. **Non-linear Optimisation 2. (25 points)**

(a) $\nabla f(x, y) = (4x^3 + 4x - y, 4y^3 + 2y - x)$ and $Hf(x, y) = \begin{pmatrix} 12x^2 + 4 & -1 \\ -1 & 12y^2 + 2 \end{pmatrix}$.

(b) Since $\Delta_1 = 12x^4 + 4 > 0$ and $\det Hf(x, y) = \Delta_2 = (12x^4 + 4)(12y^2 + 2) - 1 \geq 7 > 0$ for all $(x, y) \in \mathbb{R}^2$, the Hessian Matrix is positive definite and g is strictly convex. Since $h(t) = e^t$ is a strictly increasing (and convex) function, it follows that $e^{g(x,y)}$ is strictly convex as well.

(c) The point $(0, 0)$ is clearly a critical point of g, where the Hessian is positive definite and therefore a local minimum of $g(x, y)$. Note that, by the convexity of $g(x, y)$, there can not be other critical points, so $(0, 0)$ is a global minimum of $g(x, y)$. Since $h(t) = e^t$ is an increasing function then $e^{g(x,y)}$ has also a global minimum at $(0, 0)$.