Problem 1 (Enumeration, 1pt)**.** How many words of length 8 on the alphabet $\{A, C, G, T\}$ are there

(a) containing no A's;

(b) containing exactly two A's;

(c) containing every letter twice.

Solution. (a) This is the number of words of length 8 on the alphabet of the three remaining letters: $3^8 = 6561$

(b) There are $\binom{8}{2}$ ways to choose the positions of the A's. For each choice the remaining six positions are filled with letters other than A: $\binom{8}{2} \cdot 3^6 = 20412$

(c) This is the number of permutations with repetitions, each symbol repeated twice: $\frac{8!}{2!2!2!2!} =$ 2520.

Problem 2 (Recurrences, 2pt)**.** Consider the linear recurrence

$$a_n = 2a_{n-1} + 3a_{n-2},\ n \geq 2,\ a_0 = 1, a_1 = 1.$$

(a) Solve the recurrence. Check your answer for $n = 4$.

(b) Consider the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Let $A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. Find a linear recurrence relation for a_n. Solve the recurrence.

Solution. (a) The characteristic polynomial of the recurrence is

$$x^2 - 2x - 3$$

which has roots $\alpha = 3$ and $\beta = -1$.

The general solution is

$$a_n = a3^n + b(-1)^n.$$

By using the initial conditions we get

$$\begin{aligned} 1 &= a + b \\ 1 &= 3a - b \end{aligned}$$

from which we get $a = b = 1/2$.

The first terms of the recurrence are $(1, 1, 5, 13, 41, ...)$. For $n = 4$ we obtain from the formula

$$a_4 = (1/2)3^4 + (1/2)(-1)^4 = 41$$

in agreement with the sequence.

(b) We have

$$A^{n+1} = A^n \cdot A = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} a_n + 2b_n & 2a_n + b_n \\ c_n + 2d_n & 2c_n + d_n \end{pmatrix},$$

from which we get

$$a_{n+1} = a_n + 2b_n.$$

On the other hand, from A^n we get

$$\begin{aligned} a_n &= a_{n-1} + 2b_{n-1} \\ b_n &= 2a_{n-1} + b_{n-1} \end{aligned}$$

From the first equation we obtain $b_{n-1} = (a_n - a_{n-1})/2$, which substituted in the second one gives

$$b_n = 2a_{n-1} + (a_n - a_{n-1})/2) = a_n + (3/2)a_{n-1},$$

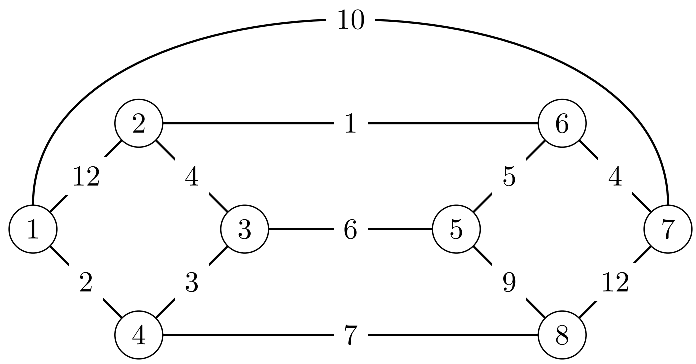
so we get

$$a_{n+1} = 2a_n + 3a_{n-1}.$$

The first two terms of the sequence are $a_0 = a_1 = 1$. This is the same recurrence as in the former part, so the solution is

$$a_n = (3^n + (-1)^n))/2.$$

Problem 3 (Graphs, 4pt)**.** Consider the graph G depicted in Figure 1.



G

Figure 1:

(a) Verify the Handshaking Lemma in this graph G.

(b) Is the graph G bipartite?

(c) Find the diameter of G.

(d) Give a minimum spanning tree of G by using Kruskal algorithm. Is the output of this algorithm unique in this graph? Can you modify the weight of one edge so that the output of the algorithm is not uniquely determined? What is the subgraph obtained by running the algorithm if we stop when there are three connected components?

(e) Give the Prüfer code of the minimum spanning tree obtained above.

Solution. (a) The graph G is 3–regular and has 8 vertices. The Handshaking Lemma gives $2|E(G)| = \sum_{x \in V(G)} d(x) = 3 \cdot 8 = 24$. The graph has indeed 12 edges.

(b) The graph has no odd cycles, so it is bipartite. The sets $A = \{1, 3, 6, 8\}$ and $B = \{2, 4, 5, 7\}$ form a bipartition of the graph.

(c) All paths from vertices 1 and 5 have at least length 3, so the diameter is at least 3. From every vertex in the 4–cycle formed by the vertices $\{1, 2, 3, 4\}$ we reach the remaining vertices in this 4–cycle in at most two steps and one additional step reaches every vertex in the 4–cycle $\{5, 6, 7, 8\}$. The argument is symmetric by exchanging the two 4–cycles, so the diameter is 3.

(d) We apply Kruskal algorithm. The edges ordered by weight are

1	2	3	4	4	5	6	7	9	10	12	12
{2,6}	{1,4}	{3,4}	{2,3}	{6,7}	{5,6}	{3,5}	{4,8}	{5,8}	{1,7}	{1,2}	{7,8}

The algorithm selects the edges of weight up to 5. The edge $\{3, 5\}$ is skipped as it creates a cycle with the current graph at this step. The resulting spanning tree is depicted in Figure 3 and has weight 26. The output is uniquely defined. However, if the weight of edge $\{3, 5\}$ is 5, then the algorithm should choose one of the two edges $\{5, 6\}$ or $\{3, 5\}$ and each choice produces a different minimum spanning tree.

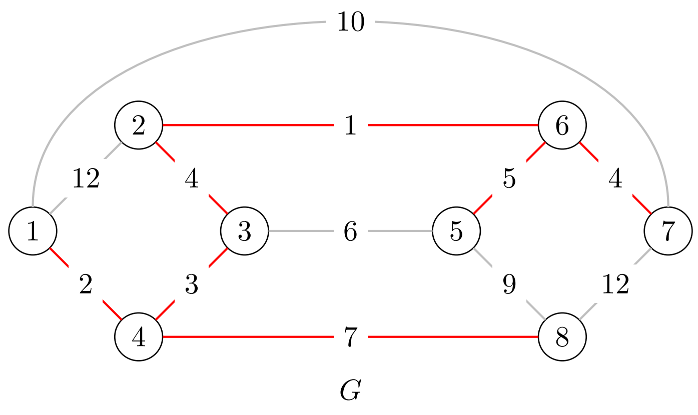


Figure 2:

At the start of the algorithm there are 8 connected components, each of one vertex. Every additional edge in the process of the algorithm decreases by one the number of connected components so the graph H obtained at the time where there are 3 connected components has edge–set

$$E(H) = \{\{2, 6\}, \{1, 4\}, \{3, 4\}, \{2, 3\}, \{6, 7\}\}.$$

(e) By running the Prüfer algorithm on the tree obtained in the preceding item we get

smallest leave	Prüfer code
1	4
5	46
7	466
6	4662
2	46623
3	466234

The resulting Prüfer code is (466234).

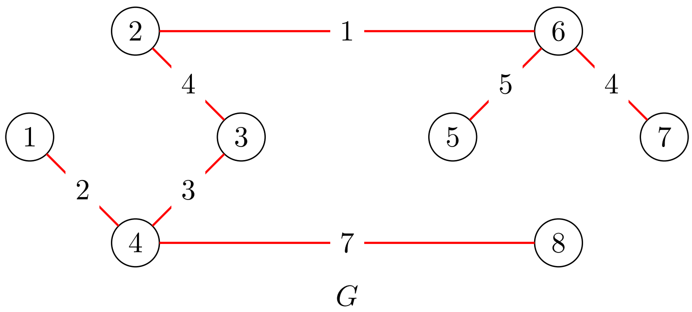


Figure 3:

Problem 4 (Linear Programming, 3pt)**.** A pharmaceutical laboratory produces chemical components C_1 and C_2 and has two assistants A_1 and A_2. Producing one gram of C_1 requires 3 hours of work by A_1 and another 6 hours of work by A_2. Similarly, one gram of C_2 requires 4 hours of work from A_1 and another 2 hours of work from A_2.

Chemical C_1 can be sold for \$ 12 per gram, and chemical C_2 can be sold for \$7 per gram. Suppose that at least 1 gram of chemical C_2 must be made (due to some prior commitment), but any amount of chemical C_1 is allowed. Assistant A_1 has 12 available hours and A_2 has 18 available hours.

What is the optimal amount of each component to be produced?

(a) Write a Linear Program to solve the problem and put it in equational form.

(b) Identify the region of feasible solutions and give a geometric solution of the problem.

(c) Write the problem in appropriate form for the Simplex algorithm, initialize the algorithm and give its first step.

Solution. (a) We must specify the quantities x_1 and x_2 in grams of each component to be produced. The constraints are given by the time availability of the two assistants. We can optimize the revenues from the production. Therefore we want to

$$\begin{array}{ll} \text{Maximize} & 12x_1 + 7x_2 \\ \text{Subject to} & 3x_1 + 4x_2 \leq 12 \\ & 6x_1 + 2x_2 \leq 18 \\ & x_2 \geq 1 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$

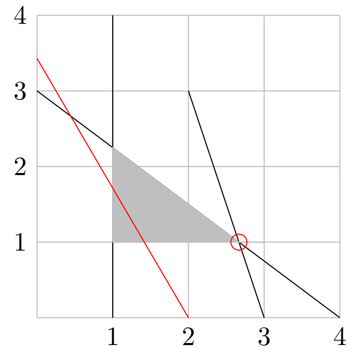


Figure 4: The gray area is the feasible region and the red line is the optimizing line.

(b) The feasible region is depicted in Figure 4:

The vertex of the feasible region which allows for a larger value of the maximizing function is $(8/3, 1)$ giving a gain of \$39.

(c) The problem can be formulated to apply the simplex method as

$$
\begin{array}{ll}
\text{Maximize} & 12x_1 + 7x_2 \\
\text{Subject to} & 3x_1 + 4x_2 + x_3 = 12 \\
& 6x_1 + 2x_2 + x_4 = 18 \\
& -x_2 + x_5 = 1 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{array}
$$

One basic feasible solution to initialize the simplex is $(0, 0, 12, 18, 1)$. The starting tableau is

$$
\begin{array}{llll|l}
x_3 & = 12 & -3x_1 & -4x_2 & \\
x_4 & = 18 & -6x_1 & -2x_2 & \\
x_5 & = 1 & & +x_2 & \\
\hline
z & = 12x_1 & +7x_2 & & z = 0
\end{array}
$$

Either x_1 or x_2 can enter, let us choose x_1 which has larger coefficient in the cost function (and has smaller subscript).

The constraints to the increase of x_1 are

$$x_1 \leq 4 \text{ from } x_3 \geq 0$$
$$x_1 \leq 3 \text{ from } x_4 \geq 0$$

The stronger constraint comes from x_4 so x_4 leaves and we get the tableau

$$
\begin{array}{llll|l}
x_3 & = 12 & -3x_1 & -4x_2 & \\
x_1 & = 3 & & -(1/3)x_2 & -(1/6)x_4 \\
x_5 & = 1 & & +x_2 & \\
\hline
z & = 36 & +3x_2 - 2x_4 & & z = 36
\end{array}
$$

with the current basic feasible solution $(3, 0, 12, 0, 1)$.