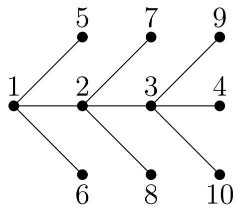
Solutions to Problem Sheet Graphs & Networks I

Exercise 1.

(a) Recall that to compute the Prüfer code, you choose the leaf with the least label, remove it, and note the label of its neighbour. Doing this for the tree here results in the code $(4, 5, 6, 6, 4, 5, 5, 6)$.

(b) For the second part, first note that the given sequence has length 8, and so the tree has 10 vertices. This gives the following tree:



Exercise 2.

(a) Let T be a labelled tree with $n \geq 2$ vertices. Its vertices fall into two categories: the *leaves*, of degree 1, and the *inner vertices*, of degree at least 2. Notice that if T has ℓ leaves, then it has $n - \ell$ inner vertices. We will now show that $\ell \geq 2$. To that end, recall that T has $n - 1$ edges. So that by the *Handshaking Lemma* we get

$$2(n-1) = \sum_{v \in V(T)} \deg(v) = \sum_{\substack{v \text{ is a} \\ \text{leaf of } T}} \deg(v) + \sum_{\substack{v \text{ is an inner} \\ \text{vertex of } T}} \deg(v) \geq \sum_{\substack{v \text{ is a} \\ \text{leaf of } T}} 1 + \sum_{\substack{v \text{ is an inner} \\ \text{vertex of } T}} 2$$
$$= \ell + 2(n - \ell).$$

Thus $2n - 2 \geq \ell + 2n - 2k$, which implies that $\ell \geq 2$.

(b) We use the same method as above, but now the inner vertices of T have degree at least 3 as there is no vertex of degree 2. Hence,

$$2(n-1) \geq \sum_{\substack{v \text{ is a} \\ \text{leaf of } T}} 1 + \sum_{\substack{v \text{ is an inner} \\ \text{vertex of } T}} 3 = \ell + 3(n - \ell).$$

We then have $2n - 2 \geq \ell + 3n - 3\ell$, which imples that $n \leq 2\ell - 2$.

Bonus: *This is best possible, as can be seen by the following construction. Start with a single central vertex connected to three leaves. For $\ell > 3$, choose any of these leaves and connect two new vertices to it, see Figure 1 for an illustration of the first three steps. Finally, note that the case $\ell = 2$ is not possible, since the only tree with exactly two leaves is a path, and hence every vertex besides the leaves has degree two.*

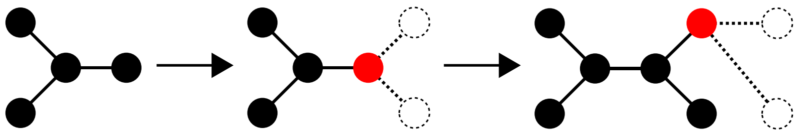


Figure 1: The first three steps in the extremal construction

(c) **Fact**: *each letter of the Prüfer code associated to a tree T is a different inner vertex of T.*

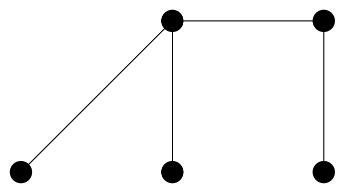
If a tree T on n vertices has $n-2$ leaves, it only has 2 inner vertices. Thus, its associated Prüfer code will be a word with only two different letters.

We are hence interested in words of length $n-2$ from a 2 letter alphabet, of which there are 2^{n-2}. Of those, the two that contain only a single letter are not relevant Prüfer codes, since they would correspond to trees with $n-1$ leaves. So we are left with $2^{n-2}-2$ codes. Finally, any of the $\binom{n}{2}$ choices of labels for the two inner vertices leads to a different Prüfer code, so all in all there are

$$\binom{n}{2}(2^{n-2}-2)$$

different labelled trees with n vertices and $n-2$ leaves.

For $n=5$, this gives 60 different labelled trees. Note though that these trees are actually all the same base tree, but with the 60 different labellings of its vertices.



Exercise 3.

(a) Assume that $n \geq 2$. Then, each pair of vertices of K_n is connected by an edge, i.e. a path of length 1. Thus all shortest paths between pairs of vertices have length 1, and so the longest of the shortest paths has length 1. By definition, this implies that the diameter of K_n is 1.

(b) If $n=m=1$, $K_{1,1}$ is reduced to a single edge and its diameter is 1.

Else, let A and B be the two parts of $V(K_{n,m})$. If $a \in A$ and $b \in B$, because the edge $\{a,b\} \in E(K_{n,m})$ then the shortest path joining a and b has length 1. If $u, v \in A$, then for any vertex $w \in B$ both edges $\{u,w\}, \{v,w\} \in E(K_{n,m})$, and a sorthest path between u and v is $(\{u,w\},\{w,v\})$, which has length 2. Thus the diameter of $K_{n,m}$ is 2.

(c) Assume that $n \geq 2$. Then, between any two vertices of P_n there is a unique path joining them. The longest of all those paths is the one connecting the two leaves. It has length $n-1$. Hence the diameter of P_n is $n-1$.

(d) Between any two vertices of C_n there are exactly two paths P_1 and P_2 joining them, with respective lengths ℓ_1 and ℓ_2. Notice that those lengths can take any value between 1 and $n-1$, as long as $\ell_1+\ell_2=n$. Assume that P_1 is shorter than P_2, i.e. $\ell_1 \leq \ell_2$. Then ℓ_1 is maximized when it is equal to $\lfloor n/2 \rfloor$, and in that case $\ell_2 = \lceil n/2 \rceil$. This implies that the longest of the shortest paths in C_n has length $\lfloor n/2 \rfloor$.

Exercise 4.

(a) The vertices of Q_n are binary words of length n and there are exactly 2^n of them. To count the number of edges we are going to use the *Handshaking Lemma.* Note that two vertices of Q_n are connected by an edge if and only if they differ in exactly one entry, and hence each vertex has degree n. Applying the lemma we thus have

$$|E(Q_n)| = \frac{1}{2}\sum_{v\in Q_n} \text{degree}(v) = \frac{1}{2}\sum_{v\in Q_n} n = \frac{n}{2}\sum_{v\in Q_n} 1 = n|V(Q_n)| = \frac{n}{2}2^n = n2^{n-1}.$$

(b) Two vertices of Q_n clearly differ in at most n entries, and so they can be connected by at most n edges, which implies $\operatorname{diam}(Q_n) \leq n$. On the other hand, in order to reach vertex $(1, \ldots, 1)$ from vertex $(0, \ldots, 0)$ n edges are required, so $\operatorname{diam}(Q_n) = n$.

(c) *Base case* ($n = 2$): $(00, 10, 11, 01)$ is a cycle of Q_2 of length $4 = 2^2$.

Induction hypothesis: Let $n \geq 3$ and suppose that Q_{n-1} admits a cycle of length 2^{n-1}.

Observe now that the induced subgraph of Q_n containing all the binary words $x_1 \ldots x_n$ with $x_n = 0$ is an $(n-1)$–cube (more precisely, it is *isomorphic* to Q_{n-1}), and the same is true for the induced subgraph on the binary words with last digit $x_n = 1$. By the induction hypothesis, there is a cycle C in Q_{n-1} of length 2^{n-1}. This implies the existence of two *identical* cycles C_0 and C_1 in Q_n. If $\{c_1 \ldots c_{n-1},\ d_1 \ldots d_{n-1}\}$ is an edge of C then

$$\{c_1 \ldots c_{n-1}0,\ d_1 \ldots d_{n-1}0\} \quad \text{and} \quad \{c_1 \ldots c_{n-1}1,\ d_1 \ldots d_{n-1}1\}$$

are the corresponding edges in C_0 and C_1, respectively. Replacing these two edges by

$$\{c_1 \ldots c_{n-1}0,\ c_1 \ldots c_{n-1}1\} \quad \text{and} \quad \{d_1 \ldots d_{n-1}0,\ d_1 \ldots d_{n-1}1\}$$

we get a cycle of length $2 \cdot 2^{n-1} = 2^n$ in Q_n. Below is an illustration in the case $n = 3$.

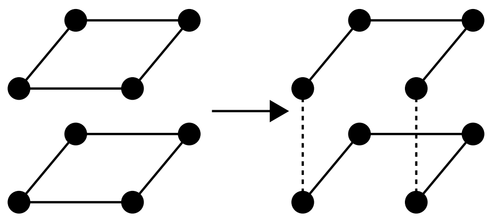
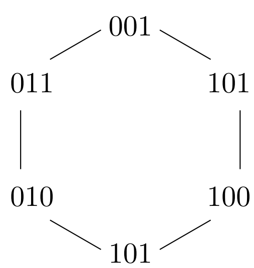


Figure 2: Joining the two cycles

Exercise 5.

(a) When $m = 1$, the vertex set of G_1 is composed of the binary words of length $3 = 2 \cdot 1 + 1$ with either 1 or 2 ones, i.e. $V(G_1) = \{001, 010, 100, 011, 101, 110\}$. A graphical representation of G_1 is for instance given by:



(b) Let V_1 and V_2 be the subsets of vertices of G_m containing m and $m+1$ `ones`, respectively. Vertices in V_1 will contain $2m + 1 - m = m + 1$ `zeros`, while those in V_2 will contain m `zeros`. Thus, for a vertex of V_1 there are exactly $m + 1$ ways to flip a `zero` to a `one` and obtain a different vertex of V_2. Similarly, for a vertex of V_2 there are $m+1$ ways to flip a `one` to a `zero` and obtain a different vertex of V_1.

Furthermore, notice that there cannot be an edge between two vertices in V_1 or in V_2, as we would require to at least filp a `one` to a `zero` and another `zero` to a `one`. Thus the distance between two vertices in V_1 or in V_2 is at least 2. Together with the argument from the previous paragraph, we conclude that vertices in G_m all have the same degree: $m+1$.

(c) We use the notation from (b). The number of vertices in V_1 is the number of ways to choose m positions for the `ones` in a binary word of length $2m+1$. This number is given by $\binom{2m+1}{m+1}$, which is equal to $\binom{2m+1}{m}$ by symmetry of the binomial numbers. It is the same for the number of vertices in V_2, just changing `ones` to `zeros` in the argument. Hence,

$$|V(G_m)| = 2\binom{2m+1}{m}.$$

By the Handshaking Lemma and (b), we thus have

$$|E(G_m)| = \frac{1}{2}\sum_{v\in V(G_m)} \text{degree}(v) = (m+1)\binom{2m+1}{m}.$$

(d) We already argued in (b) that there cannot be any edge between vertices in V_1 or V_2. In fact, all the edges of G_m are of the form $\{a,b\}$ with $a \in V_1$ and $b \in V_2$. Furthermore, we have by construction that $V(G_m) = V_1 \cup V_2$. And finally, because a vertex cannot have both m and $m+1$ `ones`, the sets V_1 and V_2 are disjoint. All together, this means that G_m is a bipartite graph where (V_1, V_2) is the bipartition of $V(G_m)$.

(e) We use the notation from (b). Let $u \in V_1$. Notice that by applying an alternating sequence of flips, where we successively flip a `zero` to a `one`, then another `one` to a `zero`, etc., we construct a path in G_m in which the vertices alternate between the sets V_1 and V_2. If $u \in V_2$, then we apply the same sequence but we start by flipping a `one` to a `zero` instead and alternate from there. Remark that actually, every shortest path in G_m can be uniquely encoded as a sequence of flips alternating between a *`one`-to-`zero` flip* and a *`zero`-to-`one` flip*. And we can reach any vertex of G_m from any other vertex by a sequence of such flips, which implies that G_m is connected.

Now, let P be a path in G_m. The number of `one`-to-`zero` flips of the sequence associated to P is m, if the starting vertex v_0 was in V_1, and $m+1$ if $v_0 \in V_2$, while the number of `zero`-to-`one` flips of the sequenceassociated to P is $m+1$ if $v_0 \in V_1$, and m if $v_0 \in V_2$. Thus the length of P is at most $2m+1$. So that the diameter of G_m is at most $2m+1$.

Consider finally the vertex $u \in V_1$, whose associated binary word has its m first entries set to `one` and the $m+1$ last set to `zero`, and the vertex $v \in V_2$, whose associated binary word has its m first entries set to `zero` and the $m+1$ last set to `one`. The shortest path between u and v is asoociated to a sequence requiring $m+1+m$ alternating flips. It has thus length $2m+1$. This means that the diameter of G_m is at least $2m+1$.

Exercise 6. Let G be a simple graph (no multiple edge nor loop) with n vertices. The possible degrees that each vertex of G can take ranges thus from 0 to $n-1$, i.e. n possible values. Remark that if a vertex were to have degree $n-1$, then there can be no vertex in G with degree 0. This means that the actual number of different values that the degrees of the vertices of G can take is $n-1$: either it ranges from 1 to $n-1$, or from 0 to $n-2$. By the *pigeon-hole principle*[1], as we

[1] See the Wikipedia entry https://en.wikipedia.org/wiki/Pigeonhole_principle

have to choose the different values for the degrees of the n vertices of G among a range of $n-1$ values, there are at least two vertices of G with the same degree.

Exercise 7.

(a) Let F be a forest with n vertices and k connected components $C_1, C_2, \dots, C_k$, with $n_1, n_2, \dots, n_k$ vertices respectively, so that $\sum_{i=1}^{k} n_i = n$. Because F is a forest, each C_i is a tree, for $i \in \{1, \dots, k\}$, and has hence $n_i - 1$ many edges. This gives:

$$|E(F)| = \sum_{i=1}^{k} |E(C_i)| = \sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k.$$

(b) We start by proving a formula for general n and then apply it to the case $n = 6$. If the forest has two connected components, one will have ℓ vertices, while the other has $n - \ell$ vertices, for some $1 \le \ell \le n-1$. Now for a fixed ℓ, there are $\binom{n}{\ell}$ ways to split up the n labels among these two components, and by Cayley's formula, for a fixed choice there are $\ell^{\ell-2}$ labelled trees with ℓ vertices and $(n-\ell)^{n-\ell-2}$ with $n - \ell$ vertices. Putting this together, we thus have

$$\binom{n}{\ell} \ell^{\ell-2} (n-\ell)^{n-\ell-2}$$

different labelled trees with one component of size ℓ and one of size $n - \ell$. To finish, we must sum this over all relevant values of ℓ. Be cautious that in order to not count some forests twice, we should not sum over all $\ell = 1, \dots, n-1$, but rather only over $\ell = 1, \dots, \lfloor n/2 \rfloor$. So the final formula for the number of labelled forests with n vertices and two connected components is

$$\sum_{\ell=1}^{\lfloor n/2 \rfloor} \binom{n}{\ell} \ell^{\ell-2} (n-\ell)^{n-\ell-2}.$$

Computing this quantity for $n = 6$, we get 1170 many forests.

Exercise 8.

(a) Each non-empty subset of the vertices of G will give rise to a unique induced subgraph. This is because by definition, for each such subset the edge set of the corresponding induced subgraph is fixed. G has n vertices, so it has $2^n - 1$ induced subgraphs.

(b) Because the vertex set of a spanning subgraph is that of G, any subset of the edges of G will give rise to a unique spanning subgraph. G has m edges, and has thus 2^m spanning subgraphs.