Discrete Mathematics and Optimisation Exam. Fall 2018

1. **Graph Theory (10 points)**
   (a) Define the *degree* of a vertex in a graph.
   (b) Describe what 1-regular graphs look like.
   (c) Describe what 2-regular graphs look like.
   (d) State the *Handshaking Lemma*.

2. **Counting words (20 points)**
   (a) How many different words can be formed with the letters in $AACTGGATACGA$?
   (b) How many different words can be formed with exactly n occurrences of the letter A and m occurrences of the letter B?
   (c) How many words of n letters can be made with an alphabet of k symbols without two repeated consecutive letters?

3. **Tilings and recurrences (20 points)**
   (a) Consider tilings of a board of size $2 \times n$ with large squares (2×2 tiles) and small squares (1×1 tiles). Find a recurrence equation for the number a_n of different tilings, and determine the initial conditions. Solve this recurrence.
   (b) Restate the recurrence equation and initial conditions for the case where the large squares can have s different colours and the small squares can have t different colours. You do not have to solve this recurrence.

4. **Linear Optimisation (25 points)**

   A small business enterprise makes dresses and trousers. To make a dress requires 1/2 hour of cutting and 20 minutes of stitching and to make a pair of trousers requires 15 minutes of cutting 1/2 hour of stitching. The profit on a dress is 40 Euros and on a pair of trousers 50 Euros. The business employs one person each for cutting and stitching and operates for a maximum of 8 hours per day. The goal is to determine how many dresses and trousers should be made in a day to maximise profit.

   (a) Define the decision variables and formulate a linear program for the problem.
   (b) Determine the number of dresses and pants the enterprise should make and what profit it should expect graphically.

5. **Non-linear Optimisation (25 points)**

   Let $f(x, y)$ be the function which gives the sum of the squares of the distances from a point (x, y) in the plane to the points $\mathbf{a} = (0, 0)$, $\mathbf{b} = (4, 2)$ and $\mathbf{c} = (1, 4)$, that is

   $$f(x, y) = (x^2 + y^2) + ((x - 4)^2 + (y - 2)^2) + ((x - 1)^2 + (y - 4)^2).$$

   (a) Show that $f(x, y)$ is a strictly convex function on $\mathbb{R}^2$.
   (b) Find the point $\mathbf{d} = (d_1, d_2)$ which minimizes the sum of the squares of the distances to $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.
   (c) Execute the first step of the gradient method for the function f starting at $(1, 1)$.
   (d) Find the point in the first iteration of the Newton method for the function $g(x, y) = f(x, y) + x^3 y^3$ starting at the point $(1, 1)$.

Solutions

Exercise 1.

(a) The degree $\deg(v)$ of a vertex v is the number of its neighbours in a graph.

(b) A 1-regular graph is a collection of disjoint edges.

(c) A 2-regular graph is a collection of disjoint cycles.

(d) For any graph $G = (V, E)$, the Handshaking Lemma states that

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Exercise 2.

(a) Note that $AACTGGATACGA$ is 12 letters long and contains five As, two Cs, two Ts and three Gs. It follows that you can you can form

$$\binom{12}{5}\binom{7}{2}\binom{5}{2} = \frac{12!}{5!\,2!\,2!\,3!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{2 \cdot 2 \cdot 6} = 166\,320$$

different words.

(b) You can form

$$\binom{n+m}{m} = \binom{n+m}{n}$$

different words.

(c) You can form $k\,(k-1)^{n-1}$ different words.

Exercise 3.

(a) The recurrence relation satisfies

$$a_n = a_{n-2} + a_{n-1}$$

with initial values $a_1 = 1$ and $a_2 = 2$. We note that this is the Fibonacci recurrence with different in initial values, so that

$$a_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

for appropriate values of A and B. Due to the initial values we must have

$$\begin{aligned} 1 &= a_0 = A + B, \\ 1 &= a_1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) \end{aligned}$$

which implies that $A = (5+1)/(2\sqrt{5})$ and $B = 1 - A = (\sqrt{5}-1)/(2\sqrt{5})$ so that

$$a_n = \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{-\sqrt{5}}{2}\right)^n.$$

(b) In this case the recurrence relation is

$$a_n = t^2\, a_{n-1} + s\, a_{n-2}$$

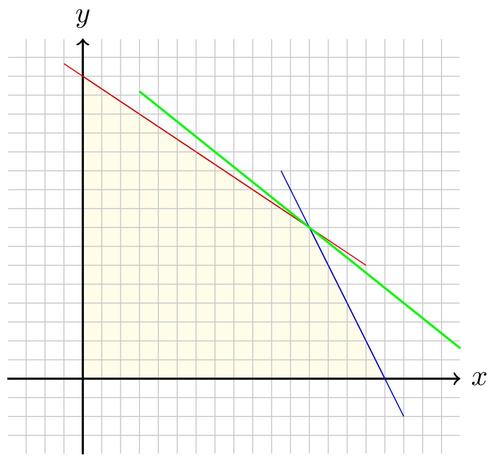
with initial values $a_1 = t^2$ and $a_2 = s + t^4$.

Exercise 4.

(a) Let x and y respectively denote the number of dresses and trousers made by the small business. The optimisation problem can be formulated as

$$\begin{aligned} \text{maximise} \quad & 40x + 50y \\ \text{subject to} \quad & 30x + 15y \le 480 \\ & 20x + 30y \le 480 \\ & x, y \ge 0. \end{aligned}$$

(b) In the following plot, the red line represents the restriction $30x + 15y \le 480$ coming from the cutter and the blue line represents the restriction $20x+30y \le 480$ coming from the stitcher. The green line represents the maximised payoff. The light yellow region is the feasible region. The lines meet in the point $(12, 8)$, so the enterprise should make 12 dresses and 8 trousers for a profit of 880 Euros.



Exercise 5.

(a) We note that the function $g(x, y) = (x - x_0)^2 + (y - y_0)^2$ is convex in $\mathbb{R}^2$ for each fixed $(x_0, y_0) \in \mathbb{R}^2$. Since the function f is the sum of three such functions and it is therefore convex in $\mathbb{R}^2$. To see that it is strictly convex, we compute the Hessian of f and show that it is positive definite.

$$\begin{aligned} \nabla f(x, y) &= (2x + 2(x - 4) + 2(x - 1), 2y + 2(y - 2) + 2(y - 4) \\ &= (6x - 10, 6y - 12). \end{aligned}$$

and

$$Hf(x, y) = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}.$$

The principal determinants of H are 6 and 36, both positive. Hence, Hf is positive definite.

(b) Since f is strictly convex in $\mathbb{R}^2$, if it has a critical point then this must be a global minimum. From the above computation of the gradient, we have

$$\begin{aligned} 6x - 10 &= 0 \\ 6y - 12 &= 0 \end{aligned}$$

which gives the point $d = (5/3, 2)$.

(c) By starting at the point $\mathbf{x}_0 = (1, 1)$ the first point in the iteration of the gradient method gives

$$\mathbf{x}_1 = \mathbf{x}_0 + t\nabla f(\mathbf{x}_0) = (1, 1) - t(-4, -6),$$

where t is the point which minimizes the function

$$g(t) = f(\mathbf{x}_0 - t\nabla f(\mathbf{x}_0)) = f(1 + 4t, 1 + 6t) = 156t^2 - 52t + 21,$$

which is the solution of $g'(t) = 312t - 52 = 0$, namely $t = 1/6$. Therefore the first point in the iteration of the gradient method is

$$\mathbf{x}_1 = (1, 1) - 1/6\,(-4, -6) = (5/3, 2).$$

(d) We now have

$$h(x, y) = \nabla g(x, y) = (3x^2y^3 + 6x - 10, 3x^3y^2 + 6y - 12).$$

and we use the Newton method to approximate a solution of the equation $h(x, y) = (0, 0)$. We have

$$\nabla h(x, y) = Hg(x, y) = \begin{pmatrix} 6xy^3 + 6 & 9x^2y^2 \\ 9x^2y^2 & 6x^3y + 6 \end{pmatrix}$$

and, starting at the point $(1, 1)$, the next point in the iteration of the Newton method is the solution of the linear system

$$Hg(1, 1)\begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix},$$

or

$$\begin{aligned} 12(x - 1) + 9(y - 1) &= 1 \\ 9(x - 1) + 12(y - 1) &= 3 \end{aligned}$$

which gives $\mathbf{x}_1 = (16/21, 10/7) \approx (0.76, 1.43)$.