Discrete Mathematics and Optimisation.
Partial exam: Fall 2018

1. **Graph Theory (10 points)**

   (a) Define the *degree* of a vertex in a graph.

   (b) Describe what 1-regular graphs look like.

   (c) Describe what 2-regular graphs look like.

   (d) State the *Handshaking Lemma*.

2. **Counting words (20 points)**

   (a) How many different words can be formed with the letters in $AACTGGATACGA$?

   (b) How many different words can be formed with exactly n occurrences of the letter A and m occurrences of the letter B?

   (c) How many words of n letters can be made with an alphabet of k symbols without two repeated consecutive letters?

3. **Tilings and recurrences (20 points)**

   (a) Consider tilings of a board of size $2 \times n$ with large squares (2×2 tiles) and small squares (1×1 tiles). Find a recurrence equation for the number a_n of different tilings, and determine the initial conditions. Solve this recurrence.

   (b) Restate the recurrence equation and initial conditions for the case where the large squares can have s different colours and the small squares can have t different colours. You do not have to solve this recurrence.

4. **Linear Optimisation (25 points)**

   A small business enterprise makes dresses and trousers. To make a dress requires 1/2 hour of cutting and 20 minutes of stitching and to make a pair of trousers requires 15 minutes of cutting 1/2 hour of stitching. The profit on a dress is 40 Euros and on a pair of trousers 50 Euros. The business employs one person each for cutting and stitching and operates for a maximum of 8 hours per day. The goal is to determine how many dresses and trousers should be made in a day to maximise profit.

   (a) Define the decision variables and formulate a linear program for the problem.

   (b) Determine the number of dresses and pants the enterprise should make and what profit it should expect graphically.

Solutions

Exercise 1.

(a) The degree $\deg(v)$ of a vertex v is the number of its neighbours in a graph.

(b) A 1-regular graph is a collection of disjoint edges.

(c) A 2-regular graph is a collection of disjoint cycles.

(d) For any graph $G = (V, E)$, the Handshaking Lemma states that

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Exercise 2.

(a) Note that $AACTGGATACGA$ is 12 letters long and contains five As, two Cs, two Ts and three Gs. It follows that you can you can form

$$\binom{12}{5}\binom{7}{2}\binom{5}{2} = \frac{12!}{5!\,2!\,2!\,3!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{2 \cdot 2 \cdot 6} = 166\,320$$

different words.

(b) You can form

$$\binom{n+m}{m} = \binom{n+m}{n}$$

different words.

(c) You can form $k\,(k-1)^{n-1}$ different words.

Exercise 3.

(a) The recurrence relation satisfies

$$a_n = a_{n-2} + a_{n-1}$$

with initial values $a_1 = 1$ and $a_2 = 2$. We note that this is the Fibonacci recurrence with different in initial values, so that

$$a_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

for appropriate values of A and B. Due to the initial values we must have

$$\begin{aligned} 1 &= a_0 = A + B, \\ 1 &= a_1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) \end{aligned}$$

which implies that $A = (5+1)/(2\sqrt{5})$ and $B = 1 - A = (\sqrt{5}-1)/(2\sqrt{5})$ so that

$$a_n = \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{-\sqrt{5}}{2}\right)^n.$$

(b) In this case the recurrence relation is

$$a_n = t^2\, a_{n-1} + s\, a_{n-2}$$

with initial values $a_1 = t^2$ and $a_2 = s + t^4$.

Exercise 4.

(a) Let x and y respectively denote the number of dresses and trousers made by the small business. The optimisation problem can be formulated as

$$\begin{aligned} \text{maximise} \quad & 40x + 50y \\ \text{subject to} \quad & 30x + 15y \leq 480 \\ & 20x + 30y \leq 480 \\ & x, y \geq 0. \end{aligned}$$

(b) In the following plot, the red line represents the restriction $30x + 15y \leq 480$ coming from the cutter and the blue line represents the restriction $20x+30y \leq 480$ coming from the stitcher. The green line represents the maximised payoff. The light yellow region is the feasible region. The lines meet in the point $(12, 8)$, so the enterprise should make 12 dresses and 8 trousers for a profit of 880 Euros.