Exercise 4

a)

$$Write\ f(x,\ y) = 2x^2 + y^2 - xy + 2x + y + 4\ as\ a\ quadratic\ function:$$

$$f(x,\ y)\ =\ \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + c$$

$$We\ identify\ the\ quadratic\ terms:$$

$$q_1\ = x{*}x\ = x^2$$
$$q_2\ = x{*}y\ = xy$$
$$q_3\ = y{*}x\ = xy$$
$$q_4\ = y{*}y\ = y^2$$

$$We\ extract\ the\ coefficients\ corresponding\ to\ our\ terms:$$

$$q_1\ = 2x^2 \implies 2$$
$$q_2\ =\ -xy \implies\ -1$$
$$q_3\ =\ -xy \implies\ -1$$
$$q_4\ = y^2 \implies 1$$

$$We\ identify\ the\ linear\ terms:$$
$$b_1\ = x$$
$$b_2\ = y$$

$$We\ extract\ the\ coefficients\ corresponding\ to\ our\ terms:$$
$$b_1\ = 2x \implies 2$$
$$b_2\ = y \implies 1$$

$$We\ identify\ our\ constant:$$
$$c = 4$$

$$We\ express\ f(x, y)\ as\ a\ quadratic\ function:$$

$$f(x,\ y)\ =\ \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 4$$

b)

Find partial derivatives and set them equal to 0 :

$$f(x,\ y) = 2x^2 + y^2 - xy + 2x + y + 4$$

$$f_x = \frac{\partial f}{\partial x} = 4x - y + 2$$
$$f_y = \frac{\partial f}{\partial y} = 2y - x + 1$$

$$f_x = 4x - y + 2 = 0$$
$$f_y = 2y - x + 1 = 0$$

Find second partial derivatives :

$$f_{xx} = 4$$
$$f_{yy} = 2$$
$$f_{xy} = f_{yx} = -1$$

Hessian matrix :

$$\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$$

Prove that $f(x, y)$ *has a global minimum* :

If the discriminant $D > 0$ *and* $f_{xx} > 0$, *there is a global minimum*

Calculate the discriminant D :

$$D = (f_{xx})(f_{yy})\ -\ (f_{xy})^2$$
$$D = (4)(2)\ -\ (-1)^2$$
$$D = 8 - 1 = 7$$

Since $D = 7 > 0$ *and* $f_{xx} = 4 > 0$, *we can confirm that* $f(x, y)$ *has a global minimum*

Find the minimum using Newton method with initial point $(-1,\ -1)$:

$$\textit{Newton method formula}:$$
$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - (H_f(x,y))^{-1} \nabla (f_x, f_y)$$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \left(\begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}\right)^{-1} \begin{bmatrix} 4x - y + 2 \\ 2y - x + 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}\right)^{-1} \begin{bmatrix} 4(-1) - (-1) + 2 \\ 2(-1) - (-1) + 1 \end{bmatrix}$$

$$\textit{Iteration } 1$$
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} - \begin{bmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{4}{7} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} - \begin{bmatrix} -\frac{2}{7} \\ -\frac{1}{7} \end{bmatrix} = \begin{bmatrix} -\frac{5}{7} \\ -\frac{6}{7} \end{bmatrix}$$

$$\textit{Iteration } 2$$
$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{7} \\ -\frac{6}{7} \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{7} \\ -\frac{6}{7} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{5}{7} \\ -\frac{6}{7} \end{bmatrix}$$

$$\textit{Global minimum}: \begin{bmatrix} -\frac{5}{7} \\ -\frac{6}{7} \end{bmatrix}$$

c)

$$\textit{Steepest descent method}:$$

$$x_{k+1} = x_k - t_k \nabla f(x_k)$$

$$\Phi_k(t) = f(x_k - t \triangledown f(x_k))$$

$$\textit{Initial point } x_0 = (-1,\ -1)$$
$$\textit{As we computed earlier}: \triangledown f(-1,\ -1) = (-1,\ 0)$$

$$\Phi_{x_0}(t) = f((-1,\ -1) - t(-1,\ 0))$$
$$\Phi'_{x_0}(t) = \triangledown f(-1+t,\ -1) * (-1,\ 0)$$

$$\triangledown f(-1+t,\ -1) = \begin{bmatrix} 4(-1+t) - (-1) + 2 \\ 2(-1) - (-1+t) + 1 \end{bmatrix} = \begin{bmatrix} -1+4t \\ -t \end{bmatrix}$$

$$\Phi'_{x_0}(t) = (-1+4t,\ -t) * (-1,\ 0)$$
$$\Phi'_{x_0}(t) = 1 - 4t$$

$$\textit{Set } \Phi'_{x_0}(t) \textit{ to } 0:$$

$$1 - 4t = 0$$
$$1 = 4t$$
$$t_0 = \frac{1}{4}$$

$$\textit{First iteration}:$$

$$x_1 = x_0 - \left(\frac{1}{4}\right) \triangledown f(x_0)$$
$$x_1 = (-1,\ -1) - \left(\frac{1}{4}\right)(-1,\ 0)$$
$$x_1 = (-1,\ -1) - \left(-\frac{1}{4}, 0\right)$$
$$x_1 = \left(-1 + \frac{1}{4},\ -1 + 0\right)$$
$$x_1 = \left(-\frac{3}{4},\ -1\right)$$