Bioinformatics

Discrete Mathematics and Optimisation

Problem Sheet Counting and Enumeration

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1. We wish to encode the 38 alphanumeric symbols (A-Z and 0-9) with binary words of length k. How large should k be?

2. The Morse code consists of words of dots ($\cdot$) and dashes ($-$) of variable length. Which maximum length is needed to encode 28 letters?

3. How many words of length n can be formed in the alphabet $\{A, C, T, G\}$ without repeated consecutive letters?

4. Let a_n be the number of binary words on the alphabet $\{A, B\}$ of length n without two consecutive A's. Find a recurrence equation for a_n. Same question avoiding AAA.

5. Find a recurrence for the number of words of length n in the alphabet $\{A, C, T, G\}$ without two consecutive A's.

6. How many different words can be formed by permuting the letters of the word MISSISSIPPI? And the word GUADALQUIVIR?

7. Prove the identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \; n \geq k \geq 1,$$

by using the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

8. Prove the identity

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}, \; n \geq k \geq 1,$$

by induction on n.

9. Assume $2n$ ballots are cast in an election and everyone tosses a (fair) coin to decide which of the two options they are going to vote for. What is the likelihood of a tie in the outcome? Determine that probability for $n = 3\,761\,500$.

10. Recall the linear recurrence from exercise 4, where a_n counted the number of binary words (with the alphabet $\{A, B\}$) of length n without two consecutive A's. Solve it using the tools from the lecture.

11. Solve the linear recurrence from exercise 5 of the previous sheet. Here we were counting the number of words of length n with letters from the alphabet $\{A, C, T, G\}$ not containing two consecutive A's.

12. Consider a rectangle 2 units high and n units long. You are given an infinite supply of smaller rectangles of size 1×2 (also orientable as 2×1) and 2×2. State the recurrence relation for the number of ways one can tile the big rectangle using then smaller ones. Solve this recurrence.

13. Verify that $x^3 + x^2 - 4x - 4 = (x + 1)(x + 2)(x - 2)$. Solve the linear recurrence $a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$ where $a_0 = 8$, $a_1 = 6$ and $a_2 = 26$. Can you also solve the non-homogeneous linear recurrence $a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3} + 12$?

14. Consider the growth of a population of bacteria. Initially there are 30 young bacteria. At the end of each month the young bacteria in the population reach maturity and the mature bacteria in the population have offspring at a rate of two young bacteria per mature bacterium. Formulate the recurrence relationship of this population. Solve it using the tools from the lecture.