Bioinformatics

Discrete Mathematics and Optimisation

Problem Sheet: Extrema in Several Variables

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1. Let $f(x, y) = x^4 + \frac{1}{4}y^4 - 8x^2 - y$.
   (a) Find the critical points of f.
   (b) Determine the nature of the critical points of f using Sylvester's criterion. [1].

2. Let $f(x, y) = x^2 + y^3$ and $g(x, y) = x^2 + y^4$.
   (a) Show that both functions have a critical point at $(0, 0)$, and that it is a *degenerate point* in both cases, i.e. the determinant of the Hessian is zero at that point.
   (b) Show that only one of the functions has a local minimum at $(0, 0)$.

3. Let $f(x, y, z) = e^x + e^y + e^z + 2e^{-x-y-z}$.
   (a) Find a local minimum of f in $\mathbb{R}^3$ (*hint: check points for which* $x = y = z$).
   (b) Show that this point is furthermore the unique global minimum of f in $\mathbb{R}^3$.

4. Let $f(x, y) = x^4 - 4xy + y^4$.
   (a) Find the critical points of f and determine their nature if possible.
   (b) Show that $f(x, y) \to \infty$, whenever $|(x, y)| = \sqrt{x^2 + y^2} \to \infty$.
   Hint: observe that $4\max(x^4, y^4) \geq f(x, y) \geq \max(x^4, y^4) - 4\max(x^2, y^2)$.
   (c) Deduce that f has at least one global minimum, and determine it or them.

5. Suppose that a particular gene occurs as one of two alleles (A and a), where allele A has frequency θ in the population. That is, a random copy of the gene is A with probability θ and a with probability $1 - \theta$. Since a diploid genotype consists of two genes, the probability of each genotype is given by:

genotype	AA	Aa	aa
probability	θ^2	$2\theta(1-\theta)$	$(1-\theta)^2$

   Suppose we test a random sample of people and find that k_1 are AA, k_2 are Aa, and k_3 are aa. Find the maximum likelihood estimate of θ.

6. Let $f(x, y) = x^3 - 12xy + 8y^3$.
   (a) Find the critical points of f.
   (b) Determine the nature of the critical points of f.

[1] `https://en.wikipedia.org/wiki/Sylvester%27s_criterion`