Kapitel 3

Computation of the implied and local volatility: elementary approaches

Die Welt hat sich verändert, sie ist volatiler geworden

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In this chapter we consider methods to find the implied volatility and to reconstruct the local volatility. The considerations are of elementary nature and the results can be considered more or less as recipes, in the following chapters some of the problems discussed are further analyzed. But nevertheless, already these elementary problems lead to very interesting mathematical considerations.

3.1 Computation of the implied volatility

The Black-Scholes model assumes that the volatility is constant across strikes and maturity dates. However as we know, in the world of options, this is a very unrealistic assumption. Option prices for different maturities change drastically, and option prices for different strikes also experience significant variations. In this section we consider the numerical problem to compute the implied volatilities and the implied volatility surface.

3.1.1 Some properties of arbitrage-free pricing

Here is a list of assumptions concerning the market of financial products considered.

- There are market participants.
- There are no transactions costs.
- Interest rates are constant and known.
- Borrowing as well as lending is both possible at the risk-free interest rate.
- All market participants will be prepared to take advantage of arbitrage opportunities as and when they arise. Arbitrage is defined as the possibility to do a set of trades that leaves you with no position, but money in the pocket.
- There are no arbitrage opportunities.

- The Black-Scholes model assumes that markets are perfectly liquid and it is possible to purchase or sell any amount of stock or options or their fractions at any given time.

We assume that the market satisfies the assumptions above. Suppose that we have an European call option. The option price is denoted by $C = C(S,t;K,T)$. Here, S is the value of the underlying at time t, K is the strike at expiration time T. If we have also an European put on the same underlying and if $P = P(S,t;K,T)$ is the price of the put where S,t,K,T have the same meaning then, as we already mentioned (see (**??**)), the put-call-parity is the identity

$$P(S,t;K,T) + Se^{-q(T-t)} = C(S,t;K,T) + Ke^{-r(T-t)}\,,\ S > 0, t \in [0,T]\,, \tag{3.1}$$

where q is the current dividend and r is the interest rate of a riskless bond.

Suppose that at date t there are two European options on the same underlying with value S quoted at the market: a call with strike K and expiry date T and a put with strike K and expiry date T. The two options should generate the same implied volatility value to exclude arbitrage. Recall the put-call parity:

$$C(S,t;K,T) - P(S,t;K,T) = e^{-r(T-t)}(F-K)\,, \tag{3.2}$$

where $F := e^{(r-q)(T-t)}$ is the forward value of the underlying. Thus, the difference between the call and the put does not depend on the model assumptions and the implied volatility; it is observable on the market. Therefore, when the identity (3.2) is violated the assumptions concerning the market cannot hold. One may argue that since the put option is a natural hedging instrument, investors may be willing to pay more for it, and therefore, its implied volatility would be higher than the call counterpart.

On the other side, the put-call-parity can be used, to retrieve the implied dividend yield q and interest rate r from the market by simultaneously solving the above system of equations (3.2). In total, there are four prices that need to be used.

Here are some properties of the call prices without dividend whose proofs can be based on the assumptions of the market alone.

Rules 3.1

(a) $C(S,T;K,T) = (S-K)^+\,,\ P(S,t;K,T) = (K-S)^+\,.$

(b) $(S - Ke^{-r(T-t)})^+ \leq C(S,T;K,T) \leq S\,,\ Ke^{-r(T-t)} \leq P(S,t;K,T) \leq Ke^{-r(T-t)}\,.$

(c) $C(S,t;K,\cdot)$ *monoton nondecreasing.*

(d) $C(S,t,\cdot,T)$ *is monotone non increasing and convex.*

(e) $C(\cdot,t;K,T)$ *is monotone non decreasing and convex.*

(f) $\lim_{K\to\infty} C(S,t;K,T) = 0\,.$

(g) *For all* $k > 0$ *we have* $C(kS,t;kK,T) = kC(S,t;K,T)\,.$

We give the proof of the first assertion in 4.), for the other proofs see [48].

Let $K_1 \leq K_2$ and assume $C(S,t;K_1,T) < C(S,t;K_2,T)\,.$

Portfolio: By a call with strike K_1 (long call) and sell a call with strike K_2 (short call).

At the beginning: $C(S,t;K_2,T) - C(S,t;K_1,T) > 0\,.$

At the end: $(S_T - K_1)^+ - (S_T - K_2)^+ \geq 0\,.$

This is a contradiction to the assumption that the market is arbitrage-free.

3.1.2 Analytical formulae for the computation of the implied volatility

Almost always, the inversion of the Black-Scholes formula to get the implied volatility is done with some sort of solver method, for example, the Newton-Raphson method; see next subsection. These methods work very well for a single option, usually producing very accurate estimates in negligible computing time. However, frequently one has to invert millions or hundreds of thousands of options at the same time. In these situations, a solver method might prove to be slow, especially for real-time applications. The need to overcome the slow-speed problem in the solver methods have led researchers to consider an alternative to the solver methods, namely, analytical computable approximations.

Most of the approximate closed-form inversion methods perform some Taylor expansion to the Black-Scholes formula and then analytically invert the expansion to obtain a formula for the implied volatility. The usual assumption being made to justify the Taylor expansion is that the strike price is close to the forward price. Because of the local nature of the Taylor expansion, these methods work relatively well for very near-the-money options but their performances deteriorate as the absolute value of the moneyness gets away from zero.

We sketch one such analytical approximation of the implied volatility. Let us first define the normalized call option price $c(\cdot;\cdot)$ by

$$C(S;t;K,T,r,\sigma) = S\,c(\ln(Se^{r(T-t)}/K), \sigma\sqrt{T-t})\,.$$

The expression for the normalized call price $c(x,\nu)$ is given by

$$c(x;\nu) = \mathcal{N}\left(\frac{x}{\nu}+\frac{\nu}{2}\right) - e^{-x}\mathcal{N}\left(\frac{x}{\nu}-\frac{\nu}{2}\right)\,. \tag{3.3}$$

The variable x is the logarithmic forward-moneyness and we shall call ν the integrated volatility. They are given by

$$x = \ln(Se^{r(T-t)}/K) = \ln(F/K)\,,\ \nu = \sigma\sqrt{T-t}, \tag{3.4}$$

where $F = Se^{r(T-t)}$ is the forward price. An option with $x = 0$ is said to be **at-the-money-forward**. We shall call equation (3.3) the dimensionless Black-Scholes formula because all the three quantities entering the equation are dimensionless. It clearly states that the Black-Scholes formula is essentially a relation between three dimensionless quantities, namely the normalized price c, the integrated volatility ν and the moneyness x. (3.3) implicitly gives ν as a function of c and x and is our starting point for the approximate inversion.

Given observed values of S, t, K, T, r and an option price C, we first calculate p according to $p = C/S$ and x according to equation (3.4) and then use an approximate formula to get ν from the equation $p = c(x,\nu)$. The implied volatility σ can then be obtained by dividing ν by $\sqrt{T-t}$. When the option is at-the-money-forward, that is, when $x = 0$, the inversion for $\nu(x,c) = p$ can be done explicitly as follows:

$$\nu(0,p) = \sqrt{8}\,\mathrm{erf}^{-1}(p) \approx \sqrt{2\pi}\,p + \frac{\pi^{\frac{3}{2}}}{6\sqrt{2}}\,p^3 \tag{3.5}$$

where erf^{-1} is the inverse of the so called error-function. We see that $\nu(0,p)$ is almost linear in the normalized call price p.

3.1.3 Newton's method

For European options under the Black-Scholes model, calculation of the implied volatility seems to be a straightforward exercise since a closed-form presentation exists for the price. However, this closed-form allows not an analytical computation of the implied volatility. The computational method we use is the classical Newton-procedure.

Let us recall the Black-Scholes-formula for the price of an European call-option.

$$C(S,t;K,\tau,r,q,\sigma) := e^{-q\tau}S\mathcal{N}(d_+(\sigma)) - Ke^{-r\tau}\mathcal{N}(d_-(\sigma)), \tag{3.6}$$

where

$$d_\pm(\sigma) := \frac{\ln(\frac{S}{K}) + (r - q \pm \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}},$$

and $\mathcal{N}$ is the distribution function of the standard normal distribution. Notice that we have used the remaining time τ as a „new" variable. In the following we use the identity

$$d_-(\sigma) = d_+(\sigma) - \sigma\sqrt{\tau}.$$

The procedure for the „daily" computation of the implied volatility of an underlying is as follows:

> Given at the time t_* the the price S_* of the underlying, the interest rate r and the current divident q, both constant in the future, an european call option with strike price K, maturity T and term to maturity τ.
> Compute the volatility $\sigma := \sigma_{K,\tau}$ by solving the equation
>
> $$C(S_*,t_*;K,\tau,r,q,\sigma) = \nu \tag{3.7}$$
>
> where $\nu := C_{market}$ is the observed market-price of the option.

To abbreviate the notation, we set

$$f(\sigma) := C(S_*,t_*;K,\tau,r,q,\sigma),\ d_+(\sigma) := d_+(\sigma,S,K,\tau,r) \tag{3.8}$$

As a result, we have to compute a solution of

$$f(\sigma) - \nu = 0 \tag{3.9}$$

Obviously, f is a smooth function but it depends on the variable σ in a highly nonlinear manner. Therefore there is no closed-form solution, but we can solve it iteratively. Since f is differentiable, we can apply each variant of the Newton method. The classical Newton-procedure is the following one:

Algorithm 3.1 Computation of the implied volatility by Newton's method

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IN Price S_* of the underlying at time t_*, strike price K, term to maturity τ, interest rate r, current dividend q.
Market price ν.
Initial guess σ^0 for the implied volatility.
Stopping-bound $N \in \mathbb{N}$ for the iteration steps.

step 0 Set $f(\sigma) := C(S_*,t_*;K,\tau,r,q,\sigma)$, $f'(\sigma) := \frac{\partial C}{\partial\sigma}(S_*,t_*;K,\tau,r,q,\sigma)$.

step 1 Compute for $n = 0,\dots,N-1$

$$\sigma^{n+1} = \sigma^n - \frac{f(\sigma^n) - \nu}{f'(\sigma^n)} \tag{3.10}$$

OUT Approximate value σ^N for the implied volatility $\sigma_{imp}(K,\tau)$.

Beispiel 3.2 *Consider the case of an European call option whose price is* 1.875 € *when*

$$S = 21, K = 20, r = 0.1, q = 0, \tau = 0.25 .$$

Suppose we plug in $\sigma_0 := 0.20$ *into the Black-Scholes formula. Then we obtain the value* $C_0 := C(S, 0; K, \tau, r, \sigma_0) = 1.76$, *a value which is lower than the market price* 1.875 . *The same procedure with* $\sigma_1 := 0.30$ *gives the value* $C_1 := 2.10$ *which is larger than the market price* 1.875 . *Since the Black-Scholes formula is monotone increasing with respect to* σ *we know that the correct value of the implied volatility is between* σ_0 *and* σ_1 . *When we plug in the value* $\sigma_2 := 0.25$ *we obtain a value* C_2 *again larger than the market price. Continuing with this „bisection method"we obtain the implied volatility* $\sigma_{imp} = 0.235$. □

3.1.4 Analysis of the computation scheme

Now, we want to analyze whether the conditions are satisfied which guarantee the quadratic convergence of the method. We do that under the assumptions

$$S > 0, \tau > 0 ,$$

which are not restrictive in practice.

Differentiability Obviously, the function f has infintely many derivatives; the first derivative is given as follows:

$$f'(\sigma) = S\sqrt{\tau}\mathcal{N}'(d_+(\sigma)) .$$

Since f' is positive the Newton iteration is feasible. The derivative f' is the is the vega, i.e. $f' = \mathcal{V} = \frac{\partial C}{\partial \sigma}$.

Monotonicity f is strictly monotone increasing due to the fact that the first derivative of f is positive. This implies that the solution of (3.9) is uniquely determined.

Bounds We know from (a) in 3.1 that $(S - Ke^{-r\tau})^+ \leq f(\sigma) \leq S$. Notice that $f(\sigma)$ depends on S, t, K, τ, r, q .

Curvature The second derivative of f is given as follows:

$$f''(\sigma) = \frac{\tau}{2\pi} S e^{-q\tau} e^{-\frac{1}{2} d_+(\sigma)^2} \frac{d_+(\sigma) d_-(\sigma)}{\sigma} .$$

f'' is called the **Volga**, i.e. Volga $= \frac{\partial^2 C}{\partial \sigma^2}$. [1] Volga is positive for options away from the money, and initially increases with distance from the money. Specifically, volga is positive where $d_+(\sigma)$ and $d_-(\sigma)$ terms are of the same sign, which is true when $d_+(\sigma) > 0$ or $d_-(\sigma) < 0$.

Existence of a solution A solution is guaranteed when we can show that

$$r_l := \lim_{\sigma \to 0} f(\sigma) - \nu \leq 0 \, , \; r_u := \lim_{\sigma \to \infty} f(\sigma) - \nu \geq 0 \qquad (3.11)$$

is satisfied since due to the monotonicity one of the following inequalities $r_l < 0, r_u > 0$ holds true; continuity of f implies the solvability of (3.9).
Obviously, $\lim_{\sigma \to 0} f(\sigma) = (S - Ke^{-r\tau})^+$ and $\lim_{\sigma \to \infty} f(\sigma) = S$.

[1] The interest of volga is to measure the convexity of an option with respect to volatility. An option with high volga benefits from volatility of volatility. Hence, for options with substantial volga, pricing with a stochastic volatility model with high volatility of volatility may change the price dramatically. The concept of volga is integral part of the whole volatility management process. For complex exotics, traders cannot just simply ignore the risk due to the change of vega. With positive volga, a position will become long vega as implied volatility increases and short vega as it decreases.

Initial guess An initial guess can be determined by the bisection method: find an intervall $[\sigma_l, \sigma_u]$ with $f(\sigma_l) - \nu \le 0, f(\sigma_u) - \nu \ge 0$, and choose $\sigma^0 \in [\sigma_l, \sigma_u]$. (If we have found the intervall $[\sigma_l, \sigma_u]$ we could use the Bisection Method to compute implied volatilities. The advantage of this method is that we don't need the Vega.)

Order of convergence The conditions for the quadratic convergence are given when the initial guess is sufficient close to the solution; see the following theorem which is now applicable.

Theorem 3.1 *Let* $g : [a, b] \longrightarrow \mathbb{R}$ *be twice continuously differentiable and suppose that*

$$|g'(x)| \ge m\,,\ |g''(x)| \le M \text{ for all } x \in [a, b] \tag{3.12}$$

with $m > 0, M > 0$. *Then*

(a) g *has in* $[a, b]$ *at most one zero.*

(b) *If* z *is a zero in* (a, b) *then the iteration*

$$x^{n+1} := x^n - \frac{g(x^n)}{g'(x^n)}\,,\ n \in \mathbb{N}_0\,, \tag{3.13}$$

is defined for every $x^0 \in U_r(z) := (z - r, z + r)$ *where* $r := \min(2mM^{-1}, b - z, z - a)$. *Moreover, we have with* $q := M(2m)^{-1}|x^0 - z| < 1$ *for every* $n \in \mathbb{N}$*:*

1. $|z - x^{n+1}| \le \frac{M}{2m}|z - x^n|^2$; ***(quadratic convergence)***
2. $|z - x^n| \le \frac{2m}{M}q^{2^n}$ ***(a priori estimate)****;*
3. $|z - x^{n+1}| \le \frac{1}{m}|g(x^{n+1})| \le \frac{M}{2m}|x^n - x^{n+1}|^2$ ***(a posteriori estimate)****.*

Proof:
Suppose that z^1, z^2 are zeros of g. From

$$0 = |g(z^1) - g(z^2)| = |g'(\eta)||z^1 - z^2| \ge m|z^1 - z^2|$$

we obtain $z^1 = z^2$ and a) is proved.
Using Taylor's development we have with η between z and x^n

$$\begin{aligned} 0 &= g(z) = g(x^n) + g'(x^n)(z - x^n) + \frac{1}{2}g''(\eta)(z - x^n)^2\,, \\ 0 &= g(x^n) + g'(x^n)(x^{n+1} - x^n)\,, \end{aligned}$$

and this gives by subtraction

$$0 = (z - x^{n+1})g'(x^n) + \frac{1}{2}g''(\eta)(z - x^n)^2;$$

Since $\eta \in [a, b]$ this proves

$$|z - x^{n+1}| \le \frac{M}{2m}|z - x^n|^2.$$

Let $x^0 \in U_r(z)$. Then

$$|z - x^1| \le \frac{M}{2m}|z - x^0|^2 = \left(\frac{M}{2m}|z - x^0|\right)^2 \frac{2m}{M} = \frac{2m}{M}q^2\,.$$

Inductively we obtain the a priori estimate.
We have with $\eta \in [a, b]$

$$|g(x^{n+1})| = |g(z) - g(x^{n+1})| = |g'(\eta)||z - x^{n+1}| \geq m|z - x^{n+1}|$$

and with $\xi \in [a, b]$

$$g(x^{n+1}) = g(x^n - \frac{g(x^n)}{g'(x^n)}) = \frac{1}{2}g''(\xi)(x^{n+1} - x^n)^2$$

which implies the a posteriori estimate. ■

The result of Theorem 3.1 can be summarized by saying that the convergence of the iteration is guaranteed and is of order two if the starting point is good enough ($x^0 \in U_r(z)$!) and the root of g is of multiplicity one ($g'(z) \neq 0$!).

Example 3.3 *Consider the function $g(x) := x^2, x \in \mathbb{R}$. Here the root $z = 0$ of f is of multiplicity two, a situation which does not occur in the case of the Black-Scholes-formula. Newton's method applied to this function with starting point $x^0 \neq 0$ we have the iteration*

$$x^{n+1} = \frac{1}{2}x^n, \; n \in \mathbb{N}_0.$$

Therefore,

$$|x^{n+1} - 0| = \frac{1}{2}|x^n - 0|$$

and the rate of convergence is linear only. □

Remark 3.4 *We can avoid to evaluate the derivative of the function in each iteration step by fixing the derivative:*

$$x^{n+1} := x^n - \frac{g(x^n)}{g'(x^0)}, \; x^0 \textit{ given}. \tag{3.14}$$

This iteration method is called modified Newton method. *The convergence of the method is guaranteed for good starting values x^0 but the order of convergence is not two.* □

Remark 3.5 *The iteration sequence $(x^n)_{n\in\mathbb{N}}$ has not necessarily the property that $|g(x^{n+1})| \leq |g(x^n)|, n \in \mathbb{N}$. This fact motivates the introduction of a damping factor:*

$$x^{n+1} := x^n - \lambda_n \frac{g(x^n)}{g'(x^n)}, \; x^0 \textit{ given}. \tag{3.15}$$

Here λ_n is chosen in such a way that $|g(x^{n+1})| \leq |g(x^n)|$. The convergence of the method is guaranteed for larger subset of starting point x^0 but the order of convergence is not two (at least in the beginning of the iteration). □

The solvability of the equation (3.9) is in doubt when the market prices are not in agreement with the Black-Scholes model, a fact that cannot be excluded; see the smile-observation. Therefore, we cannot be sure that the market price of an option is in the interval $((S - Ke^{-r\tau})^+, S)$. From the numerical point of view, it is already delicate when the observed market price v is in the near of the boundary of this interval. A high instability of a solution of (3.9) is the consequence since we have that the vega vanishes on on the boundary of the interval. This corresponds to the fact that $\lim_{\sigma\to 0} \mathcal{V}(\sigma) = \lim_{\sigma\to\infty} \mathcal{V}(\sigma) = 0$ holds.

Remark 3.6 *Let us make a remark concerning the problem „inverse crime" in the implementation of the Newton-method. The main work in the iteration is done in the evaluation of the functions* f, f'. *Since this evaluation cannot done anallytically, an approximation procedure has to be used. In a simulation with artificial data the evaluation in getting the artificial data and the evaluation in the iteration should be realized by different methods in order to test the robustness of the method.* □

3.2 Pricing surface

Given a set of market prices, we consider the problem to determine a complete surface of implied volatilities and of option prices.

3.2.1 Implied volatility surface

By varying the strike price K and the term to maturity τ we can create a table whose elements represent volatilities for different strikes and maturities. Under this practice, the implied volatility parameters will be different for options with different time-to-expiration τ and strike price K. As we know from Section **??**, this collection of implied volatilities for different strikes and maturities is called the implied volatility surface and the market participants use this table as the option price quotation. For many applications (calibration, pricing of nonliquid or nontraded options,...) we are interested in an implied volatility surface which is complete, i.e. which contains an implied volatility for each pair (K, T) in a reasonable large enough set $[0, K_{\max}] \times [0, T_{\max}]$.

However, in a typical option market, one often observes the prices of a few options with the same time-to-expiration but different strike levels only. To make things worse, some of these option contracts are not liquid at all, i.e. are not traded to an adequatable extent. Therefore, we are faced with the problem of how to interpolate/extrapolate the table of implied volatilities. Such methods for completing the table of implied volatilities are well known: polynomials in two variables, linear, quadratic or cubic splines in one or two variables, parametrization of the surface and fitting of the parameters; we will come back to this problem in the context of the local volatility. But it seems to be appropriate to complete the pricing table to a pricing surface insted of completing the volatilities table since the properties of the pricing surface are deeper related to the assumptions concerning the market.

3.2.2 Option price surface: the interpolation problem

The reality of missing prices requires a good method to interpolate the option price as a continuous function of the strike price (and the remaining life time of the option). Cubic spline volatilities interpolation is the method used for almost all option price inverse models. Unfortunately, we cannot choose freely an interpolation method. The no-arbitrage principle determines that a call option price must be a monotonically decreasing and convex function of the strike price; see (d) in 3.1. So, when we use the Black-Scholes formula and the interpolated volatilities to price options, we need to make sure that the final option price function satisfies the decreasingness and convexity restrictions. Violation of these restrictions will result into arbitrage opportunities. Additional, standard interpolation techniques may give rise to arbitrage in the interpolated volatilities surface even if „there is no arbitrage in the original set".

In order to ensure a pricing (using the implied volatility surface) which is arbitrage-free we have to find sufficient conditions for arbitrage-freeness. These conditions are related to the so called **pricing surface**. In the case of a call option this is a mapping

$$C : (0, \infty) \times (0, \infty) \ni (K, T) \longmapsto C(K, T) \in [0, \infty) . \tag{3.16}$$

This price surface is called **free of (static) arbitrage** if there can be no arbitrage oportunities „trading in the surface“. This is not a very mathematical definition. More precicely:

Definition 3.7 *The call price surface* (3.16) *is called* **free of (static) arbitrage** *if and only if there exists a nonnegative martingal, say* S, *such that* $C(K,T) = \mathbb{E}((S_T - K)^+)$ *for all* $(K,T) \in (0,\infty) \times [0,\infty)$.

□

We don't go into the details of the definition 3.7 (probability space, martingal, filtration,...). In [54] there are given sufficient and nearly necessary conditions for the fact that a implied volatility surface leads to a arbitrage-free pricing surface in the sense of Definition 3.7; see also [13, 14, 26]. In the next section we realize an interpolation method for a discrete table of option prices which guarantees a necessary condition for arbitrage-freeness, namely the convexity of the option price with respect to the strike price.

Remark 3.8 *If the pricing table is complete by interpolation we may compute the implied volatilities in order to complete the table of implied volatilities.* □

3.2.3 Interpolation by shape-preserving splines

We know from Subsection 3.1.1 that for a fixed expire-time convexity with respect the strikes should be realized. This can be done by the requirement, that the second derivative of the price-function is nonnegative. Here we describe a method which is closely related to the usual interpolationg cubic splines.

Let us start with a partition

$$0 < a = x_0 < x_1 < \cdots < x_{N+1} = b < \infty .$$

We set $h_i := x_i - x_{i-1}, i = 1, \ldots, N+1$.

Consider the following constrained interpolation problem:

Problem (I1)

$$\min_{g \in W_2^2[a,b]} \quad \frac{1}{2}\|g'' - \psi\|_2^2 \tag{3.17}$$

$$\text{such that} \quad g(x_i) = y_i\,,\ i = 0, \ldots, N+1, \tag{3.18}$$

$$g''(x) \geq 0 \text{ for a.e. } x \in [a,b]. \tag{3.19}$$

Here $\|\cdot\|_2$ is the Lebesgue $L_2[a,b]$-norm, $W_2^k[a,b]$ denotes the **Sobolev space** of functions with absolutely continuous k–1-th derivatives and k-th derivative in $L_2[a,b], k \in \mathbb{N}$. Therefore, each function g in $W_2^2[a,b]$ can be written as follows:

$$g(x) = g(a) + \int_a^x g'(\eta)d\eta + \int_a^x g''(\xi)d\xi\,,\ x \in [a,b]\,,$$

with $g'' \in L_2[a,b]$. The usual inner product in $W_2^2[a,b]$ is given as follows:

$$\langle g, h\rangle := g(a)h(a) + \int_a^b g''(\xi)h''(\xi)d\xi\,,\ g,h \in W_2^2[a,b]\,.$$

In the context of the option price surface, $\{(x_i, y_i) \in \mathbb{R}^2 | i = 0, 2, \dots, N+1\}$ are strike levels and the corresponding observed prices of the options with the same underlying S and the same time-to-expiration τ. Concerning the function ψ we assume

$$\psi \text{ is continuous differentiable} \tag{3.20}$$

It should be interpreted as an approximation of the second derivative of the pricing function; with a constant volatility σ we have

$$\psi(x) = e^{-r\tau} \frac{1}{x\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{(\ln(x/S) - (r - \frac{1}{2}\sigma^2)\tau)^2}{2\sigma^2\tau} \right), \; x \in [a, b]\,. \tag{3.21}$$

We want to introduce in (I1) the second derivative $u := g''$ as the variable which has to be determined. This is possible by using linear splines[2] as explained in the Appendix 3.6. With these splines the interpolation condition (3.19) can be written as follows:

$$\langle M_i, u \rangle = \int_a^b M_i(x) g''(x) dx = (Ky)_i\,, \; i = 1, 2, \dots, N\,.$$

The matrix $K \in \mathbb{R}^{N,N+2} = (k_{i,j})_{i=1,\dots,N, j=0,\dots,N+1}$ is defined as follows:

$$k_{i,i} = \frac{1}{(h_{i+1} + h_i)h_{i+1}}, k_{i,i+1} = \frac{-h_i}{(h_{i+1} + h_i)h_{i+1}} + \frac{-h_{i+1}}{(h_{i+1} + h_i)h_i}, k_{i,i+2} = \frac{1}{(h_{i+1} + h_i)h_i} \tag{3.22}$$

for $i = 1, \dots, N$ and $k_{i,j} = 0$ else. Notice that the matrix K is sparse and has full rank.

Now, the problem (I1) can rewritten as the following minimization problem:

Problem (I2)

$$\min_{u \in L_2[a,b]} \quad \frac{1}{2} \|u - \psi\|_2^2 \tag{3.23}$$

$$\text{such that} \quad \langle M_i, u \rangle = (Ky)_i\,, \; i = 1, 2, \dots, N\,. \tag{3.24}$$

$$u(x) \geq 0 \text{ for a.e. } x \in [a, b]\,. \tag{3.25}$$

This optimization problem is an infinite-dimensional problem. The constraint condition (3.24) can be reformulated by

$$u \in V := \{v \in L_2[a, b] | \langle M_i, v \rangle = (Ky)_i, i = 1, \dots, N\}\,. \tag{3.26}$$

V is an affine subspace of $L_2[a, b]$.

The constraints in (3.24), (3.25) are now reformulated by the admissible set $U_{ad} := V \cap U_+$ where U_+ is the cone $\{u \in L_2[a, b] | u \geq 0 \text{ a.e.}\}$. The main question is, whether the set U_{ad} is nonempty. Since the pricing formula is nonincreasing with respect to the strike we may assume that $y_0 \geq y_1 \geq \dots \geq y_{N+1}$ holds. Moreover, since the pricing formula is convex with respect to

[2] Isaac Schoenberg (1903-1990), the father of the spline theory. It was his paper of 1946 where the word „spline“ appeared. In 1930 he married E. Landau's daughter Charlotte in Berlin. This was not his only mathematical connection by marriage since his sister married Hans Rademacher.

the strike the second-order divided differences associated with the data, denoted by $(\mathsf{K}y)_i, i = 1, 2, \dots, N$, are not negative. Let us summarize this assumption in

$$(\mathsf{K}y)_i \geq 0, i = 1, \dots, N\,. \tag{3.27}$$

Therefore, a necessary condition for $U_{\text{ad}} \neq \emptyset$ is that (3.27) holds.

Theorem 3.2 *Suppose that the condition*

$$U_{ad} \neq \emptyset \tag{3.28}$$

holds. Then the problem (I2) has a unique solution.

Proof:
The objective function in (3.73) is lower semicontinuous, strictly convex, and coercive. Moreover, V is a closed convex subset. Therefore, the feasible set $U_{\text{ad}} := V \cap U_+$ is closed and convex. As a consequence, the problem (I2) is equivalent to find the projection of ψ onto the nonempty, closed and convex set U_{ad}. It is well known that in a Hilbert space such a projection exists and is uniquely determined; see Theorem **??**. ■

In general, the basis for characterizing and computing the solution of an optimization problem is to verify a necessary condition for a solution. The intendet necessary condition is a Kuhn-Tucker equation. As it is known, such an equation does not hold without a so called **constrained qualification** condition. Such a constraint qualification is the well known **Slater condition** which is in our case

$$\text{int}(U_+) \cap V \neq \emptyset \tag{3.29}$$

Here $\text{int}(U_+)$ denotes the interior (with respect to the norm-topology) of the cone U_+. Unfortunately, the condition (3.29) cannot hold since we know $\text{int}(U_+) = \emptyset$. Therefore, one has to look for a condition which does not use interior points of the cone U_+.

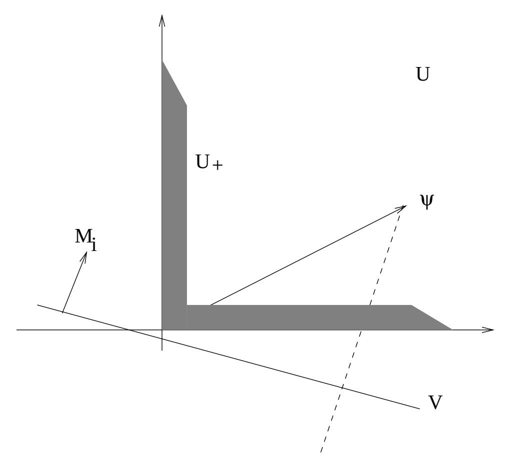

Abbildung 3.1: Lagrange multiplier–1

An improvement of (3.27) is the following condition

$$\text{There exists } \hat{u} \in V \text{ with } \hat{u}(x) > 0 \text{ a.e. in } [a, b]\,. \tag{3.30}$$

From this equation we conclude that the assumption (3.27) has to be chanced into

$$(\mathsf{K}y)_i > 0, i = 1, \dots, N\,. \tag{3.31}$$

Notice that (3.30) is not saying that $\hat{u}$ is an interior point of U_+. In the Appendix 3.5 we show that the condition (3.30) is related to the so called **quasi relative interior points**.

Theorem 3.3 *If assumption* (3.30) *holds then the problem (I2) has a unique solution in the form*

$$\hat{u}(x) = \left(\sum_{i=1}^{N} \lambda_i M_i(x) + \psi(x) \right)^+ , \; x \in [a, b]\,, \tag{3.32}$$

where the Lagrange parameter λ *is the solution of the equation*

$$G(\lambda) = d := Ky\,, \tag{3.33}$$

with

$$G_i(\lambda) = \langle M_i, (\sum_{j=1}^{N} \lambda_j M_j + \psi)^+\rangle\,,\ i = 1, \dots, N\,. \tag{3.34}$$

Proof:
Set $\rho := \inf_{u\in U_{ad}} \frac{1}{2}\|u-\psi\|^2$ and let $\hat{u} \in U_{ad}$ with $\rho = \frac{1}{2}\|\hat{u}-\psi\|^2$; see Theorem 3.2. The main step in the proof of our theorem is the following fact:
There exists a parameter $\lambda \in \mathbb{R}^N$ with

$$\rho = \min_{u\in U_+} \left(\frac{1}{2}\|u-\psi\|^2 - \langle Au - Ky, \lambda\rangle\right) = \frac{1}{2}\|\hat{u}-\psi\|^2 - \langle \hat{u}, A^*\lambda\rangle + \langle Ky, \lambda\rangle \tag{3.35}$$

where A^* is the adjoint of the linear mapping

$$A : U \ni u \longmapsto (M_1(u), \dots, M_N(u)) \in \mathbb{R}^N\,.$$

This fact is the result of a separation theorem which uses the assumption (3.30) in the form that Ky is a point in $A(\mathrm{qri}(U_+))$; see Appendix 3.5. From (3.47) we obtain

$$\begin{aligned}\|\hat{u}-\psi\|^2 - 2\langle \hat{u}, A^*\lambda\rangle + 2\langle Ky, \lambda\rangle &= \|\hat{u}-\psi\|^2 - 2\langle \hat{u}-\psi, A^*\lambda\rangle + 2\langle Ky - A\psi, \lambda\rangle \\ &= \|\hat{u}-\psi-A^*\lambda\|^2 - \|A^*\lambda\|^2 + 2\langle Ky - A\psi, \lambda\rangle\end{aligned}$$

This implies that $\hat{u}$ is the projection of $\psi + A^*\lambda$ onto the cone U_+. The equation (3.33) is a consequence of the fact that $\hat{u}$ has to be admissible.

For the complete proof of the theorem we refer to [19, 58, 30] in connection with [10]. An illustration of the proof in the case when we have just one condition in (3.24) is presented by the Figure 3.1 (the nonconsitency case) and by Figure 3.2 (the case when condition (3.30) hods). The figures are selfexplaining. ■

3.2.4 Interpolation by shape-preserving splines in the presence of noise

The shape-preserving interpolation method given in the previous subsection presumes that the observed data accurately reflect the behavior of the call option price function to be approximated. Unfortunately, in financial practice, the observed data usually have errors, thus the previous interpolation technique is not appropriate in many cases.

Again let

$$0 < a = x_0 < x_1 < \dots < x_{N+1} = b < \infty$$

be a partition of the interval $[a, b]$ let $y_0^\varepsilon, \dots, y_{N+1}^\varepsilon$ be the noisy data. In the context of finance, this means that $y_0^\varepsilon, \dots, y_{N+1}^\varepsilon$ are noisy prices of call options: $y_i^\varepsilon \approx C(S_*, t_*; K, \tau)$ for some pair (K, τ). As usually, $\varepsilon \geq 0$ is the size of the noise, i.e. there are data $y_0, \dots, y_{N+1}$ considered as noiseless, such that

$$|y_i^\varepsilon - y_i| \leq \varepsilon\,.$$

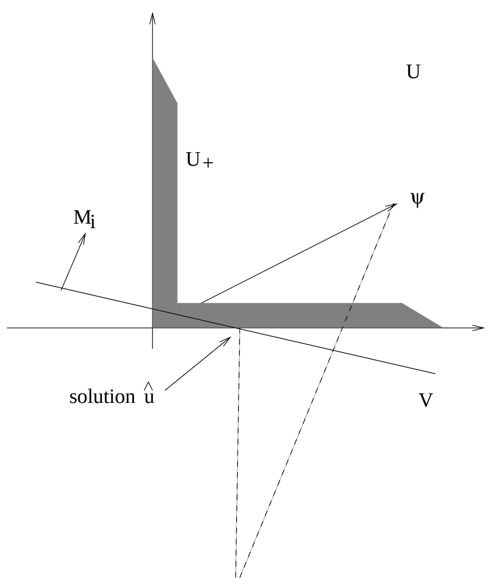

Abbildung 3.2: Lagrange multiplier–2

We set $y := (y_0, \ldots, y_{N+1}), y^\varepsilon := (y_0^\varepsilon, \ldots, y_{N+1}^\varepsilon)$.

Here is a first approach to regard the presence of noise; the convex interpolation model (I2) is changed to the following minimization problem:

Problem (I3)

$$\min_{u \in L_2[a,b], z=(z_0,\ldots,z_{N+1}) \in \mathbb{R}^N} \quad \frac{1}{2}\|u-\psi\|_2{}^2 + \frac{1}{2\alpha}|Q(z-y^\varepsilon)|^2 \tag{3.36}$$

$$\text{such that} \quad \langle M_i, u\rangle = (Kz)_i\,,\ i = 1, 2, \ldots, N\,. \tag{3.37}$$

$$u(x) \geq 0 \text{ for a.e. } x \in [a,b]\,. \tag{3.38}$$

Q is a nonsingular matrix in $\mathbb{R}^{N+2,N+2}$ and $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^N$. The matrix Q^*Q may be considered as a correlation matrix since $|Q(z-y^\varepsilon)|^2 = \langle z-y^\varepsilon, Q^*Q(z-y^\varepsilon)\rangle$ where $\langle\cdot,\cdot\rangle$ is the euclidean inner product in $\mathbb{R}^{N+2}$.

A solution of (I3) is called a **smoothing spline**. The term $\|u-\psi\|_2{}^2$ is a regularization quantity which is „enhanced“ by the positive **regularization parameter** α. α is a parameter which describes the tradeoff between the deviation of the smoothing spline and the a-priori guess and the deviation of the noiseless variable z and the noisy data y^ε.

Again, (I3) is an infinite-dimensional convex optimization problem. It can be reformulated as a problem of the type (I2). This can be seen as follows:
We define a Hilbert space W and its inner product $\langle\cdot,\cdot\rangle$ by

$$W := U \times \mathbb{R}^{N+2}\,,\ \langle (u,a),(v,b)\rangle := \langle u, v\rangle + \alpha^{-1}\langle Qa, Qb\rangle\,.$$

Let $\Psi := (\psi, y^\varepsilon) \in W$, $W_+ := U_+ \times \mathbb{R}^{N+2}$, and define the subspace V of W by

$$V := \{(u,z) | Au - Kz = \theta\}\,,$$

where θ denotes the null vector in $\mathbb{R}^{N+2}$. Then problem (I3) can be written as an interpolation problem in W:

$$\min\{\frac{1}{2}\|w-\Psi\|^2 | w \in W_+ \cap V\}$$

Now, it is clear that the following theorem can be proved similar to Theorem 3.3.

Theorem 3.4 *If assumption* (3.30) *holds then problem (I3) has a unique solution in the form*

$$\hat{u}(x) = \left(\sum_{i=1}^N \hat{\lambda}_i M_i(x) + \psi(x)\right)^+\,,\ x \in [a,b]\,, \tag{3.39}$$

$$\hat{z} = y^\varepsilon - \alpha(Q^*Q)^{-1}K^*\hat{\lambda} \tag{3.40}$$

where the Lagrange multiplier $\hat{\lambda}$ *is the solution of the system of equations*

$$G(\lambda) + \alpha K(Q^*Q)^{-1}K^*\lambda - y^\varepsilon = 0 \tag{3.41}$$

with

$$G_i(\lambda) = \langle M_i, (\sum\nolimits_{j=1}^N \lambda_j M_j + \psi)^+\rangle\,,\ i = 1, \ldots, N\,. \tag{3.42}$$

Proof:
For the complete proof of the theorem we refer to [30]. ■

We can see that, as $\alpha \to \infty$, the solution of minimization problem (I3) tends to the solution of the shape-preserving interpolation problem (I2).

Here is another approach which uses again the idea of regularization, but in a direct manner.

Problem (I4)

$$\min_{g \in W_2^2[a,b]} \quad \frac{1}{2}\sum_{i=1}^{N} |Q(g(x_i) - y^\varepsilon)|^2 + \frac{\beta}{2}\|g''\|_2^2 \tag{3.43}$$

$$\text{such that} \quad g(x_0) = y_0^\varepsilon, g(x_{N+1}) = y_{N+1}^\varepsilon \tag{3.44}$$

$$u(x) \geq 0 \text{ for a.e. } x \in [a,b]\,. \tag{3.45}$$

The positive parameter β is known as Lagrange or regularization parameter. Let the solution be denoted as $u^{\beta,\varepsilon}$.

The problem (I4) has a uniquely determined solution. It is called a smoothing convex cubic spline. Notice that we have not incorporated the initial guess ψ. Once the cubic splines $u^{\widehat{\beta},\varepsilon}$ is determined it can easily be differentiated analytically to compute approximations of $u^{\widehat{\beta},\varepsilon}{}_x, u^{\widehat{\beta},\varepsilon}{}_{xx}$ wherever needed. Of course, the regularization involed in (3.43) is a crucial issue, and we briefly comment on how to determine an appropriate regularization parameter β. It is desirable that the convergence $\lim_{\varepsilon\to 0} u^{\beta(\varepsilon),\varepsilon} = \hat{u}$ holds when the **regularization strategy** $\beta(\varepsilon)$ is chosen appropriate; here $\hat{u}$ is the cubic spline which interpolates the noiseless data.

Remark 3.9 *Several methods are known for the appropriate choice of the regularization parameter* β; *we refer to [32], especially for the case of numerical differentiation.* □

Remark 3.10 *Let us make a remark concerning the problem „inverse crime" in the implementation of Newton-method. The main work in the iteration is done in the evaluation of the functions* f, f'. *Since this evaluation cannot done analytically, an approximation procedure has to be used. In a simulation with artificial data the evaluation in getting the artificial data and the evaluation in the iteration should be realized by different methods in order to test the robustness of the method.* □

3.2.5 Computation of the shape preserving spline by the Newton-method

We have to solve the equation

$$G_i(\lambda) = (Ky)_i\,,\ i = 1, \ldots, N\,, \tag{3.46}$$

with

$$G_i(\lambda) = \langle M_i, \left(\sum_{j=1}^{N} \lambda_j M_j + \psi\right)^+ \rangle \tag{3.47}$$

Since the mappings

$$\mathbb{R}^n \ni \lambda \longmapsto g_i(\lambda, x) := \sum_{j=1}^{N} (\lambda_j M_j(x) + \psi(x)) M_i(x) \in \mathbb{R} \tag{3.48}$$

are continuous differentiable with respect to λ for all $x \in [a,b]$ and piecewise continuous differentiable functions with respect to $x \in a,b]$, $g_i(\cdot,\cdot)$ is Lipschitz continuous with respect to $\lambda \in \mathbb{R}^N$ uniformly in $x \in [a,b]$. Moreover, one can show that the following identity holds:

$$\delta G_i(\lambda) = \int_a^b \delta(g_i(\cdot,x)^+)(\lambda)dx\,.$$

Hence, the each G_i is semismooth at λ, and the composed map G is semismooth at λ too.

Thus, we have to compute

$$\delta(g_i(\cdot,x))^+(\lambda)\,.$$

We set

$$\mathbb{M}(x) := \begin{pmatrix} M_1(x) \\ \vdots \\ M_N(x) \end{pmatrix}\,,\ x \in [a,b]\,.$$

From the considerations in Appendix 3.5 we see that

$$\delta(g_i(\cdot,x))^+(\lambda) = \begin{cases} M_i(x)\mathbb{M}(x) & \text{if } g_i(\lambda,x) > 0 \\ \{\alpha M_i(x)\mathbb{M}(x) | \alpha \in [0,1]\} & \text{if } g_i(\lambda,x) = 0 \\ 0 & \text{if } g_i(\lambda,x) < 0 \end{cases}\,,\ \lambda \in \mathbb{R}^N, x \in [a,b]\,. \tag{3.49}$$

Let

$$q(\lambda,x) := \sum_{j=1}^N \lambda_j M_j(x) + \psi(x)\,,\ \lambda \in \mathbb{R}^N, x \in [a,b],$$

and define

$$T(\lambda) := \{x \in [a,b] | q(\lambda,x) = 0\}\,,\ \overline{T}(\lambda) = [a,b]\backslash T(\lambda)\,,\ \lambda \in \mathbb{R}^N\,.$$

The generalized Jacobian δG of G is now given as follows:

$$\delta G(\lambda) = \delta G^-(\lambda) + DG^+(\lambda)\,,\ \lambda \in \mathbb{R}^N, \tag{3.50}$$

where for $j = 1,\dots,N$

$$\begin{aligned} G_j^-(\lambda) &= \int_{T(\lambda)} q(\lambda,x)^+ M_j(x)dt \\ G_j^+(\lambda) &= \int_{\overline{T}(\lambda)} q(\lambda,x)^+ M_j(x)dt \end{aligned}$$

As a consequence,

$$DG^+(\lambda)_{ij} = \int_a^b q(\lambda,x)_+^0 M_i(x)M_j(x)dx\,,\ i,j = 1,\dots,N, \tag{3.51}$$

where

$$z_+^0 := \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{if } z \le 0 \end{cases}$$

Now, one can prove the following lemma which is the basis for the applicability of Newton's method to the equation (3.33).

Lemma 3.11 *Suppose that the assumption* (3.30) *holds. If* λ *is a solution of* $G(\lambda) = Ky$ *then every matrix* $Q \in \delta G(\lambda)$ *is positive definite.*

Algorithm 3.2 Interpolation (of the pricing function) by smoothing splines using Newton's method

IN Interpolation data $(x_i, y_i), i = 0, \dots, N+1$.
Initial approximation ψ for the second derivative (of the pricing fuction)
Initial guess λ^0 for the Lagrange multiplier.
Stopping-bound $L \in \mathbb{N}$ for the iteration steps.

step 0 Compute $M_1, \dots, M_N$, $y := (y_0, \dots, y_N)$, and $\nu := Ky$.

step 1 For $k = 0, \dots, L-1$ do

- Compute $G(\lambda^k)$
- Choose $Q^k \in \delta G(\lambda^k)$
- Solve the linear equation $Q^k s^k = -G(\lambda^k) + \nu$
- Set $\lambda^{k+1} := \lambda^k + s^k$

OUT Approximate Lagrange multiplier λ^L and approximate smoothing spline

$$\hat{u}^L := \left(\sum_{i=1}^{N} \lambda_i^L M_i(x) + \psi(x) \right)^+ , \ x \in [a, b].$$

3.2.6 Computation of the option price function

The solution to the original problem (3.17), (3.18),(3.19) can be obtained by integrating twice the solution $\hat{u}$ on $[a, b]$ with the result C. Here are some remarks how to do this; $\hat{u}$ may be replaced by the approximation $\hat{u}^L$.

C can be written as

$$C(x) = y_k + (x - x_k)C'(x_k) + \int_{x_k}^{x} \int_{x_k}^{s} \hat{u}(t) dt ds , \ x \in [x_k, x_{k+1}], k = 0, \dots, N. \tag{3.52}$$

In this formula, two terms need to be calculated, the first derivative of C at x_k and the integration

$$Q(x_k, x) := \int_{x_k}^{x} \int_{x_k}^{s} \hat{u}(t) dt ds .$$

If the integration $Q(x_k, x)$ is known, then

$$C'(x_k) = \frac{y_{k+1} - y_k - Q(x_k, x_{k+1})}{x_{k+1} - x_k} .$$

We have $\hat{u} = f_+$ with

$$f(x) = \sum_{i=1}^{N} \lambda_i M_i(x) + \psi(x) , \ x \in [a, b].$$

Obviously, f is a piecewise function. Then we can obtain for each $k = 0, \dots, N$ a finite number of node by solving the equation $f(x) = 0$ in $[x_k, x_{k+1}]$. This gives a new enlarged partition

$$a = t_0 < t_1 < \dots < t_{R+1} = b .$$

In every interval $[t_k, t_{k+1}]$ the solution $\hat{u}$ has a computable presentation. By considering several cases, the function $Q(x_k, x)$ can be computed by simple integration (under the assumption that a primitive of ψ is known).

3.3 Calibration

In this section, we describe a least squares algorithm to calibrate the local volatility by fitting the prices of a set vanilla european calls available on the market. As we will see, for these „simple“ derivates a numerical more efficient method is available, namely a least squares method based on Dupire's equation; see the next section. But nevertheless, for models with a complex structure these methods are not feasible because in this case no replacement for Dupire's equation seems to be possible.

3.3.1 The least squares problem

When the Black-Scholes formula fails to match the observed prices, the simplest remedy is to use a local volatility, i.e. to use a volatility which depends on the parametrs „strike, time-to-expire“. A possible way is to find σ among a family Σ of functions usually defined by a few parameters which fits the data in the least squares sense; see below.

Suppose we are given the market prices of vanilla european calls spanning a set of expiration dates $T_1, \dots, T_N$. Assume that vor each expiration date T_i, options are traded with strike prices $K_{i1}, \dots, K_{im_i}$, $m_i \in \mathbb{N}$. Let V^b_{ij} and V^a_{ij} denote the bid and ask prices, respectively, for an option with expiration date T_i and strike price K_{ij}.

Let

$$c_{ij}(\sigma) = C(S_*, t_*; K_{ij}, T_i, r, q, \sigma)\,,\ j = 1, \dots, m_i, i = 1, \dots, N\,, \tag{3.53}$$

be the prices of the options bases on a Black-Scholes model with local volatility $\sigma : [0, \infty) \times [0, T_{max}] \longrightarrow \mathbb{R}$ where $T_{max} := \max_{i=1,\dots,N} T_i$. On the market, the following prices are available:

$$v_{ij} \approx c_{ij} \text{ with } V^b_{ij} \le v_{ij} \le V^a_{ij}\,,\ j = 1, \dots, m_i, i = 1, \dots, N\,; \tag{3.54}$$

v_{ij} may be chosen as the arithmetic mean of the bid and ask prices.

Let Σ denote the space of continuous functions on the domain $[0, \infty) \times [0, T_{max}]$. The reconstruction problem can be written as an optimization problem:

$$\text{Minimize } G(\sigma) := \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{m_i} |c_{ij}(\sigma) - v_{ij}|^2 \text{ subject to } \sigma \in \Sigma\,. \tag{3.55}$$

Since the observation data is finite the problem of reconstruction σ from this data set is severely underdetermined. As a rule, there exist several volatility functions σ that matches the market option price data. This lack of uniqueness leads to the observation that the optimization problem cannot be solved in a stable way: small perturbations in the price data can result in large changes in the matching function. In general, such an unstable behavior is indicated by oszillations showing up in the matching volatility. To make the problem well-posed we must introduce some type of regularization which surpresses these oszillations. in other words, we have to restrict the set of admissible volatilities to a subset of smooth functions.

3.3.2 Tikhonov regularization

Up to now, we have not introduced the most important method to regularize an ill-posed problem. We do this here for the first time in order to stabilize the least squares problem (3.55). The regularization technique we propose changes the problem (3.55) in the following way:

$$\text{Minimize } J(\sigma) := \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{m_i} |c_{ij}(\sigma) - v_{ij}|^2 + \frac{\alpha}{2} |||\nabla\sigma|||_2^2 \quad \text{subject to } \sigma \in \Sigma_1\,. \tag{3.56}$$

Here Σ_1 is a subset of continously differentiable functions. In order to use Hilbert space arguments we assume for Σ_1:

$$\Sigma_1 \subset W_2^1([0,\infty) \times [0, T_{\max}])\,.$$

As a consequence, $\|\cdot\|$ denotes the usual L_2-norm of functions of the two variables S and t. Due to the regularization term in J it is common to choose $\Sigma_1 = W_2^1([0,\infty) \times [0, T_{\max}])$.

α is a constant which weights the regularization term and may be interpreted as a Lagrange multiplier. As we see, the smoothness of σ can be enforced quantitatively by the size of α.

The finding of a minimizer $\hat{\sigma}$ can be realized by a gradient method or in other words by the idea of steepest descent. This method can be derived from the fixed point equation for the equation $J(\hat{\sigma}) = \theta$:

$$\hat{\sigma} = \hat{\sigma} - \text{grad } J(\hat{\sigma}) \tag{3.57}$$

The resulting iteration

$$\sigma^{k+1} := \sigma^k - \text{grad } J(\sigma^k)\,,\; k \in \mathbb{N}_0\,, \tag{3.58}$$

where σ^0 is the initial guess for $\hat{\sigma}$.

To follow this idea we have to be able to compute several derivatives. Without going into the details of a rigourous deduction – in Chapter **??** we will study related problems – the gradient is given by the identity

$$\langle \text{grad } J(\sigma), h\rangle = \sum_{i=1}^{N} \sum_{j=1}^{m_i} (c_{ij}(\sigma) - \nu_{ij}) \frac{\partial c_{ij}}{\partial \sigma}(\sigma)(h) + \alpha \langle \nabla \sigma, \nabla h\rangle\,. \tag{3.59}$$

Of course, the computation of the gradient is very costly since for each $j = 1, \ldots, m_i, i = 1, \ldots, N$ the number $\frac{\partial c_{ij}}{\partial \sigma}(\sigma)(h)$ has to be computed by solving the Black-Scholes equation; see below.

An alternative which is considered in [42] is a gradient method which uses Gateaux-derivatives along the so called „spike variations“ replacing $\frac{\partial c_{ij}}{\partial \sigma}(\sigma)(h)$. The descent direction for the functional J is constructed as follows:

$$\frac{\partial c_{ij}}{\partial \sigma}(S,t)(\xi, s) := g^{ij}(S,t;\xi,s) := \frac{d}{d\varepsilon} C(S,t;K_{ij},T_i,r,q,\sigma + \varepsilon h)\Big|_{\varepsilon=0}\,, \tag{3.60}$$

where the perturbation $h = h_{(\xi,,s)}$ is a product of Dirac delta „functions“

$$h(S,t) := \delta_{(S-\xi,)}\delta_{(t-s)} \quad (S \in (0,\infty), s \in (0, T_{\max})) \tag{3.61}$$

Clearly, this choice of the variation h contradicts the smoothness we have to require for the consideration of the Black-Scholes equation with the volatility $\sigma + h$. But by formal considerations, this variational derivative can be evaluated by solving the partial differential equation

$$\mathcal{L}_{BS}\, g^{ij}(\cdot,\cdot;\xi,s)(S,t) \quad = \quad -\delta_{(S-\xi,)}\delta_{(t-s)} \frac{1}{2} \sigma(S,t)^2 S^2 \frac{\partial^2 C}{\partial S^2}(S,t;K_{ij},T_i,r,q,\sigma) \tag{3.62}$$

with homogenous boundary and initial conditions. We do this by choosing a grid

$$\Xi := \{(S_m, t_n) | m = 0, \ldots, N_S, n = 0, \ldots, N_t\}$$

in $[0, S_{\max}] \times [0, T_{\max}]$ for the choice of the spike-variations.

The computation of the solution of the Black-Scholes equation and the variational equation (3.57) can be accomplished conveniently using a finite-difference technique. The domain of the

Black-Scholes equation is semiinfinite concerning the S-variable. Therefore, this domain has to be truncated into a bounded domain $[0, S_{\max}]$. A reasonable choice for $S_{\max}$ is $S_{\max} := 2S_0$.

The discrete implementation of this idea can be summarized along the following algorithmic description:

Algorithm 3.3 Computation of the local volatility by a regularized least squares method

---

IN Prices of options $v_{ij}, \jmath = 1, \dots, m_i, j = 1, \dots, N$, with expiration dates T_i and strike prices $K_{i1}, \dots, K_{im_i}, j = 1, \dots, m_i, i = 1, \dots, N$.
Regularization parameter α.
Bound L for the iteration steps.
Select a grid $\Xi := \{(S_m, t_n) | m = 0, \dots, N_S, n = 0, \dots, N_t\}$ in $[0, S_{\max}] \times [0, T_{\max}]$.
Choose an initial guess σ^0_{ij} for the values of σ at these grid points.

step 0 For $k = 0, \dots, L-1$

step 1 For $j = 1, \dots, m_i, i = 1, \dots, N$, compute the prices

$$C(S_m, t_n; K_{ij}, T_i, r, q, \sigma^k_{mn}), m = 0, \dots, N_S, n = 0, \dots, N_t.$$

step 2 Evaluate the pricing functional G: $\gamma^k := G(\sigma^k)$

step 3 If γ^k is less than a given tolerance, stopp the procedure otherwise go to the next step.

step 4 Solve the equation (3.62) with homogeneous boundary and initial conditions for $j = 1, \dots, m_i, i = 1, \dots, N, m = 0, \dots, N_S, n = 0, \dots, N_t$, and set for $m = 0, \dots, N_S, n = 0, \dots, N_t$

$$\sigma^{k+1}_{mn} := \sigma^k_{mn} - \sum_{i=1}^{N} \sum_{j=1}^{M_i} g^{ij}(S_0, 0; S_m, t_n)(c_{ij}(\hat{\sigma^k}) - v_{ij}) \frac{c_{ij}}{\partial \sigma}(\hat{\sigma^k}) - \alpha(\nabla^2 \sigma^k)_{mn}. \tag{3.63}$$

OUT Approximate value σ^L_{mn}, $m = 0, \dots, N_S, n = 0, \dots, N_t$, for the local volatilities at the grid points.

The iterative procedure is clearly computational demanding: a large number of solutions of a partial differential equation has to be computed. However, the amount of work can be reduced by exploiting the fact that each solution involves the same partial differential equation, just the source term in the equation is different. This can be exploited when the solution of the equation is computed by a finite-difference scheme because then the resulting linear system can be solved by a single matrix decomposition/inversion per iteration step.

The choice of the parameter α is crucial for the procedure. There are several proposals in the literature for this choice but all these suggestions are costly in their implementation; a trial and error in connection with rxperience with the applied situation may be helpful.

Remark 3.12 *To simulate with the method above we have to compute option prices* $v_{ij}, \jmath = 1, \dots, M_i, j = 1, \dots, N$, *with expiration dates* T_i *and strike prices* $K_{i1}, \dots, K_{im_i}, i = 1, \dots, N$. *Notice that we should avoid the inverse crime. Therefore we should compute these prices by a method which is different from the procedure suggested in algorithm 3.3.2. A good choice for such a simulation is to use a constant volatility in the Black-Scholes model and to compute the option* *prices via the Black-Scholes pricing formula. To test the robustness of the method we should perturbe the results by adding a random error.* □

3.4 Local volatility via Dupire's formula

Here we continue the considerations in Section **??** concerning Dupire's equation, especially its use for finding the local volatility.

3.4.1 Dupire's formula again

We consider the Black-Scholes equation for an european call option with local volatility:

$$\begin{aligned} \mathcal{L}^B(C) &= \frac{\partial C}{\partial t} + \frac{1}{2}\sigma(S,t)^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-q)S\frac{\partial C}{\partial S} - rC &= 0\,,\, S \in (0,\infty), t \in (0,T)\,, & \quad (3.64)\\ C(0,t) &= 0\,,\, t \in (0,T)\,, & & \quad (3.65)\\ C(S,T) &= (S-K)^+\,,\, S > 0\,. & & \quad (3.66) \end{aligned}$$

For the moment, we asume that $\sigma : [0,\infty) \times [0,T] \longrightarrow (0,\infty)$ is a continuous function.

Suppose that the option price of this Black-Scholes model is denoted by

$$C = C(S,t) = C(S,t;K,T,r,q,\sigma)\,.$$

Given $S_* > 0, t_* > 0$, we introduce a new variable U as follows:

$$U(K,T) := U(K,T;S_*,t_*,r,q,\sigma) := C(S_*,t_*;K,T,r,q,\sigma)\,.$$

Then one can show that the function U satisfies the following system:

$$\begin{aligned} \mathcal{L}_{DU}(U) &:= \frac{\partial U}{\partial T} - \frac{1}{2}\sigma(K,T)^2 K^2 \frac{\partial^2 U}{\partial K^2} + (r-q)(U - K\frac{\partial U}{\partial K}) + qU \\ &= 0, (K,T) \in (0,\infty) \times (t_*,\infty)\,, \quad (3.67)\\ U(0,T) = S_*\,,\, \lim_{K\to\infty} U(K,T) &= 0\,,\, T \in (t_*,\infty)\,, \quad (3.68)\\ U(K,t_*) &= (S_* - K)^+\,,\, K > 0\,. \quad (3.69) \end{aligned}$$

Now, we can solve (3.67) for the volatility to obtain

$$\sigma(K,T)^2 = 2\,\frac{\frac{\partial U}{\partial T}(K,T) + (r-q)K\frac{\partial U}{\partial K}(K,T) + qU(K,T)}{K^2\frac{\partial^2 U}{\partial K^2}(K,T)} \quad (3.70)$$

We want to propose methods to calibrate the local volatility function of a European call from observed option prices of the underlying. In the focous of these methods is the stabilization of the numerical differentiation used to evaluate the formula (3.70).

Remark 3.13 *Now, we can compute the same quantity, namely the option price* $U(S_*,t_*;K,T,r,q,\sigma)$ *as a function of* K*, with the help of two different systems. First, along the Black-Scholes equation: solve severals Black-Scholes equations for different strikes* K.
Second, along Dupire's equation: solve Dupire's equation once to obtain the price for several strikes. Theoretically, the values have to be identical, but under numerical methods there may be a difference, especially, since we have to truncate the domain for the variables S, K *respectively.* □

3.4.2 Local volatility via numerical differentiation

Despite its simplicity, the approach to compute the local volatility via (3.70) has severe practical shortcomings which reflect the ill-posedness of the problem. First, financial markets typically allow only few and prefixed maturity dates, and just a discrete sample of strikes are on sell, too; therefore some sort of numerical diferentiation is required to evaluate the fraction in (3.70). Second, the Black-Scholes equation is just a model of real market dynamics, so that (??) is at best an approximate identity. Therefore it can easily happen that the computed fractions in (3.70) change sign, and taking the square root to obtain σ is prohibited. Even if the fraction remains finite and positive the volatility functions computed from (3.70) exhibit rough oscillations, if the data have not been preprocessed properly.

3.4.3 Dupire's formula and splines

We assume that the following market data are available:

- Expiration dates $T_1, \ldots, T_N$.
- Traded options with strike prices $K_{i1}, \ldots, K_{im}$, for each expiration date T_i.
- Market prices V_{ij} for an option with expiration date T_i and strike price K_{ij}.

To evaluate (3.70) we need to differentiate these discrete data with respect to the strike variable K and the maturity time T.

Using the interpolation methods in Section 3.2 we may assume that for each maturity time T_i we have a pricing function $u(\cdot, T_i) \in W_2^2[0, K_{\max}]$. Once these functions are determined they can easily be differentiated analytically to compute approximations of $\frac{\partial u}{\partial K}(K,T)$, $\frac{\partial^2 u}{\partial K^2}(K,T)$ wherever needed. The assumption that for each i the same number m of option prices V_{ij} are known is therefore no serious restriction.

Differentiation with respect to time is less delicate because of the relatively large time gaps. While this implies that the numerical differentiation cannot be very accurate, it also means that lack of stability is not really an issue here. Therefore we can use simple difference schemes to approximate the time derivatives at the maturity times. More precisely, we use centered differences at the inner maturities $T_2, \ldots, T_N$, and one-sided differences at the extremal maturities T_1 and T_N. Note that the maturities are not equispaced in general, so that the appropriate central difference quotient is

$$\frac{\partial u}{\partial T}(K_{ij}, T_i)) \approx \frac{1}{h_i + h_{i+1}} \left(\frac{h_i}{h_{i+1}} u(K_{ij}, T_{i+1}) + (\frac{h_{i+1}}{h_i} - \frac{h_i}{h_{i+1}}) u(K_{ij}, T_i) - \frac{h_i}{h_{i+1}} u(K_{ij}, T_{i-1}) \right) \tag{3.71}$$

with $h_i := T_i - T_{i-1}, i = 2, \ldots, N$.

Let

$$z_{ij} := \sigma(K_{ij}, T_i)^2 \,,\; j = 1, \ldots, m_i, i = 1, \ldots, N,$$

the numbers which we want to compute via the formula (3.70). To use the formula (3.70) we have to compute the nominator and the denominator. Let

$$b_{ij}, d_{ij} \text{ be these numbers, respectively}.$$

We stack all those values in one-dimensional vectors z, d, and $b \in \mathbb{R}^N$, $N = (m+1)n$, using a standard lexicographical ordering (with all abscissa for a single maturity in consecutive, increasing, order).The key idea is to consider the expression in (3.70) as a linear system

$$Dz = b \tag{3.72}$$

where D is the diagonal matrix whose diagonal coincides with d.

Since we cannot exclude tiny entries in the matrix D the linear system (3.72) may be ill-conditioned in general. Finally, although D is nonnegative by construction the system (3.72) may not have a nonnegative solution because of possible sign changes in b. For these reasons the linear system (3.72) needs to be regularized to obtain reasonable approximations of z. We use Tikhonov regularization and consider the problem

Problem (Dupire)

$$\min_{z\in\mathbb{R}^N} \quad \|Dz-b\|_2^2+\alpha\|Lz\|_2^2 \tag{3.73}$$

Again, $\alpha>0$ is an appropriate regularization parameter. This minimization problem is equivalent to solving the linear system

$$(D^*D+\alpha L^*L)z=D^*b \tag{3.74}$$

Notice that $D^*=D$.

Care has to be taken in the choice of L. With L equal to the identity matrix I we cannot guarantee that the solution of the equation (3.74) is positive. In [32] a choice L different from the identity matrix I is proposed.

3.5 Appendix: Some aspects of convex analysis

Quasi relative interior points

We imbed the problem (I2) into an abstract problem setting.

Problem (A)

Given Hilbert spaces U, Z, a closed convex subset C of U, $f: U \longrightarrow \mathbb{R}$, and a linear continuous operator $A: H \longrightarrow Z$. For a given $z^0\in Z$ we denote by

$$U_{ad} := \{u\in U | u\in \overset{-1}{A}(z^0)\cap C\}$$

the so called **feasible set**.
Herewith consider the optimization problem

$$\min_{u\in U_{ad}} \frac{1}{2}\|u-\psi\|^2$$

We will shortly discusss consistency conditions for the data of the problem (A) which ensure existence of a unique minimizer $\hat{u}$. Clearly, the feasible set has to be nonempty:

$$U_{ad}\neq\emptyset \tag{3.75}$$

Theorem 3.5 *Suppose that* (3.75) *holds. Then the problem (A) posesses a uniquely determined solution.*

Proof:
As we know, the norm in a Hilbert space is strictly convex. Therefore, we have uniqueness of a solution. To prove existence, we observe that the objective function is coercive ($\lim_{\|x\|\to infty, x\in U_{ad}} \|x-\psi\|=\infty$) and bounded from below. Therefore we obtain a minimal sequence $(x^k)_{k\in\mathbb{N}}$ in U_{ad} which

is weakly convergent; let $\hat{u}$ be the limit. Since the norm is weakly lower semicontinuous and because U_{ad} is weakly closed due to the fact that it si closed and convex, we see that $\hat{u}$ is a solution. ■

In order to find conditions which characterize the solution of (A) we need a Lagrange multiplier rule. Especially in the infinite dimensional case, such rules need a so called **constraint qualification**. Here are two such conditions. Of course, these conditions mainly depend on the data point z^0:

$$\mathrm{int}(C) \cap \overset{-1}{A}(z^0) \neq \emptyset \tag{3.76}$$

$$z^0 \in \mathrm{int}(A(C)) \tag{3.77}$$

Each u^0 in $\mathrm{int}(C) \cap \overset{-1}{A}(z^0)$ is called a **Slater point**. The condition (3.77) is a weaker condition is which is shown very easiliy.

Unfortunately, for our problem (I2) this condition is too sharp since the cone U_+ in $L_2[a,b]$ has empty interior. Therefore, weaker constraint qualifications should be used. Such a new constraint qualification has been introduced in [10] by Borwein and Lewis; see also [9]. It is much weaker than the standard Slater condition, and equally easy to check: rather than requiring an interior point of C to be feasible, we only need a feasible point in the **quasi relative interior** of C, an extension of the idea of relative interior in $\mathbb{R}^n$. The quasi relative interior will frequently be nonempty even when the interior of C is empty. Secondly, under suitable conditions on the space U and when the function f and the set C are sufficiently simple, the corresponding dual problem is finite-dimensional and of very simple form. This weaker constraint qualification is applicable for problem (I2) since the interpolation constraint is defined with $Z = \mathbb{R}^N$.

Let M be a subset of a topological linear space V. The smallest cone which contains M is denoted by cone(M) and the smallest affine subset which contains M is denoted by ri(M). For a convex set $M \subset V$, the **quasi relative interior** of M, denoted by qri(M), is the set of those $x \in M$ for which the closure cl(cone($M - x$)) is a subspace. An observation in the finite dimensional case $V = \mathbb{R}^n$ is the following: Let V be a normed vector space, and $M \subset V$ be convex and finitedimensional. Then $\mathrm{qri}(M) = \mathrm{ri}(M)$. The main observation is

Rules 3.14
If M is a convex subset of a normed space V and if $A : V \longrightarrow \mathbb{R}^n$ is a continuous linear mapping then

$$A(\mathit{qri}(C)) \subset \mathit{ri}(AC) \tag{3.78}$$

$$\mathit{qri}(M) \neq \emptyset \ \textit{implies}\ A(\mathit{qri}(M) = \mathit{ri}(AC). \tag{3.79}$$

Duality

In the following we shall adopt the following notation.

X is a normed space,
a function $f : X \longrightarrow (-\infty, \infty]$ is proper if it is not identically ∞,
$g : X \longrightarrow [\infty, \infty]$ convex, proper,
$h : X \longrightarrow [\infty, \infty]$ convex, proper,
$A : X \longrightarrow \mathbb{R}^n$ continuous, linear.

If $f : X \longrightarrow (-\infty, \infty]$ is a proper convex function its **(Fenchel-)conjugate** is defined by

$$f^*(\lambda) := \sup\{\lambda(x) - f(x) | x \in \mathrm{dom}(f)\}, \lambda \in X^*,$$

where

Semismooth mappings

For computing the solution in (3.39),(3.40) a Newton-type method can be advised. The problem consists in the fact that the projection on the cone U_+ is not differentiable in the classical sense but ist is **semismooth**, as we will shoh in the following.

Semismoothness was originally introduced by Mifflin [49] for real-valued functions. The convex functions, the piecewise linear functions, and of course the smooth functions are examples of semismooth functions; also, the composition of semismooth functions is semismooth. To define „semismoothness“ we need the subdifferential of a Lipschitz continuous function and related differentiability properties.

Definition 3.15 *Let X be a normed space with norm $\|\cdot\|$ and dual space X^* and let $f : X \longrightarrow (-\infty, \infty]$ be a function.*

(a) f *is* **locally Lipschitzian** *if for $x \in X$ there is a ball $B_r(x), r > 0$, in X such that, for some $\lambda \geq 0$,*

$$|f(x'') - f(x')| \leq \lambda \|x'' - x'\| \textit{ for all } x'', x' \in B_r(x)\,.$$

(b) f *is* **directionally differentiable** *at $x \in X$ in direction $d \in X$ if the following limit exists:*

$$f'(x; d) := \lim_{h \downarrow 0} \frac{f(x + td) - f(x)}{h}\,.$$

(c) f *is* **generalized directionally differentiable** *at $x \in X$ in direction $d \in X$ if the following limit exists:*

$$f^{\circ}(x; d) := \lim_{x' \to x, h \downarrow 0} \frac{f(x' + td) - f(x')}{h}\,.$$

(d) The (Clarke-)subdifferential of f at $x \in X$ is given as follows:

$$\partial f(x) := \{\lambda \in X^* | \langle \lambda, d \rangle \leq f^{\circ}(x; d) \textit{ for all } d \in X\}\,.$$

(e) f *is* **strictly differentiable** *at $x \in X$ if $\partial f(x)$ consits of a singular vector λ which is then the* **gradient** $\nabla f(x)$.

□

As we know from [?], the subdifferential $\partial f(x)$ of a Lipschitzian function f is nonempty.

Example 3.16 *Consider on $\mathbb{R}$ the functions*

$$f_1(x) := |x|\,,\ f_2(x) := -|x|\,,\ f_3(x) := \max(0, x) = (x)_+\,,\ x \in \mathbb{R}\,.$$

We have at $x = 0$:

$$f_1(x) = [-1, 1]\,,\ f_2(x) = [-1, 1]\,,\ f_3(x) = [0, 1]\,.$$

□

Example 3.17 *Consider on X functions $f_1, \ldots, f_N$ and set $f(x) := \max(f_1(x), \ldots, f_N(x)), x \in X$. Then we have at $x \in X$*

$$\partial f(x) \subset co\{\partial f_i(x) | i \in I(x)\}$$

where $I(x) = \{j | f_j(x) = f(x)\}$. Moreover, equality holds when for all $i = 1, \ldots, N$

$$f_i'(x; d) = f_i^{\circ}(x; d) \textit{ for all } d \in X\,.$$

□

Definition 3.18 *Let* X *be a normed space with norm* $\|\cdot\|$ *and dual space* X^* *and let* $G : X \longrightarrow (-\infty, \infty]$ *be a Lipschitz continuous function.*
G *is* **semismooth** *at* $x \in X$ *if for any* $d \in X$ *the following limit exists:*

$$\delta G(x) := \lim_{d' \to d, h \downarrow 0, V \in \partial G(x+hd')} Vd' .$$

$\square$

Definition 3.19 *Let* X *be a normed space with norm* $\|\cdot\|$ *and dual space* X^* *and let* $G = (G_1, \dots, G_N) : X \longrightarrow (-\infty, \infty]^N$ *be a mapping where each* G_j *is a Lipschitz continuous function.*
G *is* **semismooth** *at* $x \in X$ *if* $G_1, \dots, G_N$ *are semismooth at* x. *We set*

$$\delta G(x) := (\delta G_1(x), \dots, \delta G_N(x)) .$$

$\square$

Suppose we want to solve an equation

$$G(x) = \theta \tag{3.80}$$

where $G : \mathbb{R}^N \ni x \longmapsto G(x) \in \mathbb{R}^N$ is a semismooth mapping. To solve this equation we can use Newton's method generalized to the semismooth case. The iteration has the following form:

$$x^{k+1} := x^k - V_k^{-1} G(x^k), k \in \mathbb{N}_0, x^0 \text{ given} , \tag{3.81}$$

or

$$V_k(x^{k+1} - x^k) = G(x^k), k \in \mathbb{N}_0, x^0 \text{ given} , \tag{3.82}$$

In each step, V_k has to be chosen in $\delta G(x^k)$. The following local convergence result was established in [51] (Theorem 3.2).

Theorem 3.6 *Let* x *be a solution of the equation* (3.80) *and let* G *be a locally Lipschitz function which is semismooth at* x^*. *Assume that all* $V \in \delta G(x^*)$ *are nonsingular matrices. Then every sequence generated by the method* (3.82) *is superlinearly convergent to* x^* *provided that the starting point* x^* *is sufficiently close to* x^*.

3.6 Appendix: Peano kernel

Let

$$\Delta : a = x_0 < x_1 < \cdots < x_{N+1} = b$$

be a partition of the interval $[a, b]$ and let f be a function on the interval $[a, b]$. The **divided differences** according to the partition Δ are recursively defined as follows:

$$\begin{aligned} [x_i]\, f &:= f(x_i) \, , \; i = 0, \dots, N+1 ; \\ [x_i, x_{i+1}, \dots, x_{i+m}]\, f &:= \frac{1}{x_{i+m} - x_i}([x_{i+1}, \dots, x_{i+m}]\, f - [x_i, \dots, x_{i+m-1}]\, f) \\ & \qquad\qquad m > 1, i = 0, \dots, N+1-m . \end{aligned}$$

With the help of induction we can prove:

$$[x_i, x_{i+1}, \dots, x_{i+m}]\, f = \sum_{\nu=i}^{i+m} \omega_{\nu,i} f(x_\nu) \text{ where } \omega_{\nu,i} = \prod_{k=i, k \neq \nu} \frac{1}{x_\nu - x_k} . \tag{3.83}$$

Example 3.20 *Before continuing, we illustrate the Peano kernel theorem below and show its derivation in the case of linear splines.*
Let us define for $i = 1, \dots, N$

$$\tilde{B}_i(t) := \begin{cases} \frac{t - x_{i-1}}{x_i - x_{i-1}} & \textit{for } t \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - t}{x_{i+1} - x_i} & \textit{for } t \in (x_i, x_{i+1}] \\ 0 & \textit{else} \end{cases}$$

We have

$$\int_a^b \tilde{M}_i(t)dt = \frac{1}{2}(x_{i+1} - x_{i-1})\,,\ i = 1, \dots, N,$$

and use this identity to normalize:

$$M_i(t) := \frac{2}{x_{i+1} - x_{i-1}} \tilde{M}_i(t)\,,\ t \in [a, b], i = 1, \dots, N\,.$$

Now we obtain for twice continuous differentiable function g *on the interval* $[a, b]$

$$\int_a^b \tilde{M}_i(t) g''(t) dt = 2[x_{i-1}, x_i, x_{i+1}], i = 1, \dots, N\,.$$

□

Suppose now that f is suffiently smooth in order to allow the following steps. Then we have

$$f(x) = p_{m-1}(x) + \int_a^b \frac{(x-t)_+^{m-1}}{(m-1)!} f^{(m)} dt \text{ where } p_{m-1}(x) = \sum_{\nu=0}^{m-1} \frac{f^{(\nu)}(a)}{\nu!}(x-a)^\nu \tag{3.84}$$

As a consequence,

$$[x_i, x_1, \dots, x_{i+m}]f = \int_{x_i}^{x_{i+m}} \left(\sum_{\nu=0}^{m} \omega_{\nu,i} \frac{(x_\nu - t)_+^{m-1}}{(m-1)!} \right) f^{(m)}(t) dt \tag{3.85}$$

since $[x_i, x_{i+1}, \dots, x_{i+m}]p_{m-1} = 0$ due to the fact that p_{m-1} is a polynomial of degree $\le m - 1$.

We set

$$M_{m-1,i}(t) := \sum_{\nu=i}^{i+m} \omega_{\nu,i} \frac{(x_\nu - t)_+^{m-1}}{(m-1)!}\,,\ i = 0, \dots, N + 1 - m\,.$$

These functions are called **B-splines**. They are normalized as follows:

$$\int_a^b M_{m-1,i}(t) dt = 1\,.$$

Therefore, the B-splines can be interpreted as probability densities. The support of $M_{m-1,i}$ is contained in $[x_i, x_{i+m}]$ and $M_{m-1,i}$ is positive in (x_i, x_{i+m}).

Theorem 3.7 (Peano identity) *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *be twice differentiable. Then*

$$[x_i, x_{i+1}, \dots, x_{i+m}]f = \int_a^b \left(\sum_{\nu=0}^{m} \omega_{\nu,i} \frac{(x_\nu - t)_+^{m-1}}{(m-1)!} \right) f^{(m)}(t) dt \tag{3.86}$$

where

$$\omega_{\nu,i} = \prod_{k=i, k\neq\nu} \frac{1}{x_\nu - x_k}\,.$$

Proof:
See above. ■

The function

$$[a,b] \ni t \longmapsto \sum_{\nu=0}^{m} \omega_{\nu,i} \frac{(x_\nu - t)_+^{m-1}}{(m-1)!} \in \mathbb{R}$$

is called a **Peano kernel**. These kernels are functions which are piecewise polynomials of degree $m-1$ with respect to the nodes $x_0, x_1, \ldots, x_{N+1}$ and $m-2$ times continuous differentiable in $[a,b]$.

3.7 Bibliographical comments

The construction of the implied volatility is considered intensively discussed in [23]. Several exact closed-form solution for the implied volatility are developed under assumptions concerning the data of the option; see for instance [4, 55, 15, 16, 50].

The construction of the pricing surface via shape-preserving splines is considered in [12, 19, 29, 39, 43, 30] and [3, 2, 38, 1, 58, 54]. See also [5, 8, 6, 32, 24]. Material related to the Lagrange multiplier theorem for semismooth optimization problems can be found in [9, 10, 11, 44, 18, 28]. The theory of nonsmooth Newton-method is described in [47, 51, 52, 57, 59]; see also [22].

Tikhonov regularization to calibrate a volatility model was investigated by [17, 25, 21, 20, 26, 32, 33, 36, 35, 34, 40, 42, 45]. Convergence analysis and rates are also discussed in some of these papers; see also [7, 31, 41, 46, 53, 56, 37].

3.8 Exercises

3.1 Suppose that (X,d) is a metric space and assume that the triple $(x^\dagger, (x_n)_{n\in\mathbb{N}}, (x_{n,\tau})_{n\in\mathbb{N}})$ is consistent and local stable. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence satisfying

$$d(x_{n+2}, x_{n+1}) \le L d(x_{n+1}, x_n)^p$$

with $L, p \in \mathbb{R}, p > 1, Ld(x_2, x_1)^p < 1$. Define a stopping rule $(n(\tau))_{\tau>0}$ such that under reasonable assumptions the numerical algorithm $(x^\dagger, (x_n)_{n\in\mathbb{N}}, (x_{n,\tau})_{n\in\mathbb{N}}, (n(\tau))_{\tau>0})$ is convergent.

3.2 The following data correspond to calls on the DAX from May 5, 2008 and are taken from the homepage `www.optionsscheine.onvista.de`:

$$T - t = \frac{120}{366} = 0.32787, S_0 = 7133.06, r = 0.0487$$

K	6400	6700	7000	7300	7600	7900	8200	8500	8800
C	934.0	690.0	469.0	283.0	145.0	62.0	22.0	7.5	2.1

Calculate the implied volatilities σ_{imp} for these data. For each calculated value of σ_{imp} enter the point (K, σ_{imp}) into a figure and join the points with straight lines.

3.3 Let $x \in [0,1]$. Suppose that $X, X_1, \ldots, X_n, \ldots$ independent identical distributed random variables with

$$P(\{X = 1-x\}) = x\,,\ P(\{X = -x\}) = 1-x\,.$$

Let $\overline{X}_n := \sum_{i=1}^n X_i$. Prove

(a) $\cosh(\beta) + \alpha \sinh(\beta) \le \exp(\beta^2/2 + \alpha\beta)$ for all α, β with $|\alpha| \le 1$.

(b) $\theta \exp(\lambda(1-\theta)) + (1-\theta)\exp(-\lambda\theta) \leq \exp(-\lambda^2/8)$ for all $\theta \in [0,1], \lambda \in \mathbb{R}$.

(c) $\mathbb{E}(\lambda \overline{X}_n) \leq \exp(\lambda^2/8)$ for all $\lambda > 0$.

3.4 It is possible to differentiate by integration? Here is a curious approximate differentiation rule of Cornelius Lanczos; see [43, 29].[3]
Suppose that $f : [0,1] \longrightarrow \mathbb{R}$ is three times continuous differentiable and let $f^\varepsilon : [0,1] \longrightarrow \mathbb{R}$ be a (perturbed) function with $|f^\varepsilon(t) - f(t)| \leq \varepsilon, t \in [0,1]$. Let $\tau \in [0,1)$. Show:

(a) $f'(\tau) = L_h f(\tau) := \frac{3}{2h^3} \int_h^h s f(\tau + s) ds + O(h^2)$ where $O(\cdot)$ is the big-O Landau-Symbol.

(b) $|f'(\tau) - L_h f^\varepsilon(\tau)| = O(\varepsilon^{\frac{2}{3}})$.

3.5 Für das Black-Scholes-Modell ohne Dividende und mit dem Zinssatz $r = 0$ bezeichne $C(S,t)$ den Preis einer Europaischen Call-Option zu S und Ausübungszeit t. Für den Fall einer einfachen Option, d.h., der Ausübungspreis K ist gleich dem Basiswert S_0 zur Zeit $t = 0$, zeige man, dass für kleine t gilt:

$$C(s,t) \approx \frac{S_0 \sigma}{\sqrt{2\pi}} \sqrt{t}.$$

(Setzt man $\tilde{\sigma} := \sigma S_0$, als neue Volatilität, so ist die rechte Seite die von Bachelier (1900) gefundene Preisformel.)

3.6 Suppose that the price of the underlying is equal to a discounted exercise price, i.e. S is equal to the strike price Ke^{-rT}. Compute an exact formula for the implied volatility.

3.7 Use the Newton method to compute the implied volatility in the case of an European call option with the following data:

$$S = 18000, t = 0, K = 17500, T = 30/365, r = 0.004, q = 0.$$

3.8 Let U be the Hilbert space l_2 with the usual inner product $\langle \cdot, \cdot \rangle$. Consider in l_2 the cone $U_+ := \{(x^k)_{k\in\mathbb{N}} | x_k \geq 0, k \in \mathbb{N}\}$ and the mapping $J : l_2 \ni (x^k)_{k\in\mathbb{N}} \longmapsto x^1 + x^2 \in \mathbb{R}$.

(a) Show that U_+ is convex.

(b) Is the element $x := (\frac{1}{k})_{k\in\mathbb{N}}$ an interior point of U_+ ?

(c) As we know, the space of linear functionals l_2^* on l_2 can be identified with the space l_2. Find an element $(y_k)_{k\in\mathbb{N}}$ in l_2 with $J((x^k)_{k\in\mathbb{N}}) = \langle (y^k)_{k\in\mathbb{N}}, (x^k)_{k\in\mathbb{N}} \rangle$.

3.9 Let U be the Hilbert space l_2 with the usual inner product $\langle \cdot, \cdot \rangle$. Consider in l_2 the cone $U_+ := \{(x^k)_{k\in\mathbb{N}} | x_k \geq 0, k \in \mathbb{N}\}$ and the mapping $J : l_2 \ni (x^k)_{k\in\mathbb{N}} \longmapsto x^1 + x^2 \in \mathbb{R}$. Let $(\psi^k)_{k\in\mathbb{N}}$ be the sequence with $\psi^1 = 2, \psi^2 = 1, \psi^k = 0$ else. Solve the problem

$$\begin{array}{ll} \min\limits_{u \in l_2} & \frac{1}{2}\|u - \psi\|_2{}^2 \\ \text{such that} & J(u^k)_{k\in\mathbb{N}} = 1 \,,\; u \in U_+ . \end{array}$$

3.10 Berechne die Richtungsableitung $f'(\overline{x}; v)$ für alle v in den Fällen

(a) $f(x) := |x-1| + |x-2|, x \in \mathbb{R}; \overline{x} \in \mathbb{R}$;

[3] C. Lanczos has been an assistant in the physics department in the early years of the university in Frankfurt/Main (1924-1931). One year he was on leave in Berlin as a „Wissenschaftlicher Mitarbeiter“ of A. Einstein; see [27].

(b) $f(x_1, x_2) := (x_1^2 + x_2^2 - 1)^+$, $x = (x_1, x_2) \in \mathbb{R}^2$; $\overline{x} := (1, 1)$, $\overline{x} := (1, 0)$.

3.11 Sei $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. Wir setzen

$$\partial f(\overline{x}) = \operatorname{conv}\{d | \exists (x^k)_{k\in\mathbb{N}} \text{ mit: } \lim_k x^k = \overline{x}, \nabla f(x^k) \text{ existiert für alle } k, \lim_k \nabla f(x^k) = d\}$$

und nennen $\partial f(\overline{x})$ das Subdifferential in $\overline{x}$.
Berechne $\partial f(\overline{x})$ in $\overline{x} := (0, 1)$ für

$$f(x_1, x_2) := \int_0^{2\pi} (x_1 \sin(t) + x_2)^+ dt\,,\ (x_1, x_2) \in \mathbb{R}^2.$$

3.12 Amerikanische Optionen (auf ein Basisobjekt) führen auf ein freies Randwertproblem für die Black-Scholes-Gleichung; der freie Rand in $(0, \infty) \times (0, T)$ beschreibt die Ausübungskurve. Als illustrierendes Modell für ein freies Randwertproblem soll das Hindernisproblem für ein über ein Hindernis gespanntes Seil betrachtet werden.

Im Intervall $[-1, 1]$ sei $f \in C^2(-1, 1)$ das Hindernis, über das das Seil, eingespannt in $x = -1$ und $x = 1$, zu spannen ist:

$$a > -1, b < 1; f(-1) < 0, f(1) < 0; f(x) > 0, x \in (a, b); f''(x) < 0, x \in (a, b).$$

Das Seil werde durch den Graphen einer Funktion $u \in C^2(-1, 1)$ dargestellt:

$$u(-1) = u(1) = 0; u(x) \geq f(x), -u''(x) \geq 0, u''(x)(u(x) - f(x)) = 0, x \in (-1, 1). \tag{3.87}$$

Skizziere die Situation. Welche Punkte stehen für den *freien Rand*?
Zeige:
Ist $u \in C^2(-1, 1)$ ein Seil gemäß (3.87), so is u eine Lösung der Variationsungleichung

$$\int_{-1}^{1} u'(x)(v - f)'(x) dx \geq 0 \text{ für alle } v \in \mathcal{K} \tag{3.88}$$

wobei

$$\mathcal{K} := \{v \in C(-1, 1) | v \text{ stückweise differenzierbar}, v(-1) = v(1) = 0, v(x) - f(x) \geq 0, x \in (-1, 1)\}$$

(Man kann die Umkehrung auch beweisen. Damit sind wir dann die „zweifelhafte Voraussetzung“ $u \in C^2(-1, 1)$ in (3.87) los.)

3.13 Let X be a normed space with dual space X^*, let $M \subset X$ be convex and $\hat{x} \in M$. Then $\hat{x} \in \operatorname{qri}(M)$ if and only if

$$N_M(\hat{x}) := \{\phi \in X^* | \phi(x - \hat{x}) \leq 0 \text{ for all } x \in M\} \quad \text{(normal cone)}$$

is a subspace of X^*.

3.14 Let X be a normed space and $M \subset X$ be convex. Suppose $x_1 \in \operatorname{qri}(M)$ and $x_2 \in M$. Then $\lambda x_1 + (1 - \lambda) x_2 \in \operatorname{qri}(M)$ for all $0 < \lambda \leq 1$.

3.15 Let X be a normed space and $M \subset X$ be convex. If $\operatorname{qri}(M) \neq \emptyset$, then $\operatorname{cl}(\operatorname{qri}(M)) = \operatorname{cl}(M)$.

3.16 Let X, Y be a normed spaces, $M \subset X$ convex and $A : X \longrightarrow Y$ linear and continuous. Then $A(\operatorname{qri}(M)) \subset \operatorname{qri}(A(M))$.

3.17 Let X be a normed space and $M, N \subset X$ be convex. If $M \cap \operatorname{int}(M) \neq \emptyset$, then $\operatorname{qri}(M) \cap \operatorname{int}(N) = \operatorname{qri}(M \cap N)$.

3.18 Let X be a normed space and $M, N \subset X$ be convex. If $M \cap \operatorname{int}(M) \neq \emptyset$, then $\operatorname{qri}(M) \cap \operatorname{int}(N) = \operatorname{qri}(M \cap N)$.

3.19 Let $X := l_2$ and let $M := \{(x^k)_{k\in\mathbb{N}} \in l_2 | \sum_{k=1}^{\infty} |x^k| \leq 1\}$. Show that M is convex and

$$\operatorname{qri}(M) = M \backslash \{(x^k)_{k\in\mathbb{N}} \in l_2 | \sum_{k=1}^{\infty} |x^k| = 1 \text{ and } x^k = 0, k \geq n, \text{ for some } n\}.$$

3.20 Let $X := l_2$ and let $\hat{x} := (2^{-k})_{k\in\mathbb{N}}$. Consider $M_1 := \{(x^k)_{k\in\mathbb{N}} \in l_2 | \sum_{k=1}^{\infty} |x^k| \leq 1\}$, $M_2 := \{\alpha\hat{x} | \alpha \in \mathbb{R}\}$. Show

(a) $\operatorname{qri}(M_1) = M \backslash \{(x^k)_{k\in\mathbb{N}} \in l_2 | \sum_{k=1}^{\infty} |x^k| = 1 \text{ and } x^k = 0, k \geq n, \text{ for some } n\}$,
$\operatorname{qri}(M_2) = \{\alpha\hat{x} | \alpha \in [-1, 1]\}$.

(b) $\operatorname{qri}(M_1 \cap M_2) = \{\alpha\hat{x} | \alpha \in (-1, 1)\}$.

(c) $\operatorname{qri}(M_1) \cap \operatorname{qri}(M_2) \not\subset \operatorname{qri}(M_1 \cap M_2)$.

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