Chapter 12
Change of Numéraire and Forward Measures

In this chapter we introduce the notion of numéraire. This allows us to consider pricing under random discount rates using forward measures, with the pricing of exchange options (Margrabe formula) and foreign exchange options (Garman-Kohlagen formula) as main applications. A short introduction to the computation of self-financing hedging strategies under change of numéraire is also given in Section 12.5. The change of numéraire technique and associated forward measures will also be applied to the pricing of bonds and interest rate derivatives such as bond options in Chapter 14.

12.1 Notion of Numéraire

A *numéraire* is any strictly positive $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted stochastic process $(N_t)_{t\in\mathbb{R}_+}$ that can be taken as a unit of reference when pricing an asset or a claim.

In general, the price S_t of an asset, when quoted in terms of the numéraire N_t, is given by

$$\hat{S}_t := \frac{S_t}{N_t}, \qquad t \in \mathbb{R}_+.$$

Deterministic numéraires transformations are easy to handle as a change of numéraire by a deterministic factor is a formal algebraic transformation that does not involve any risk. This can be the case for example when a currency is pegged to another currency, *e.g.* the exchange rate 6.55957 from Euro to French Franc has been fixed on January 1st, 1999.

On the other hand, a random numéraire may involve risk and allow for arbitrage opportunities.

Examples of numéraire processes $(N_t)_{t\in\mathbb{R}_+}$ include:

- *Money market account.*

Given $(r_t)_{t\in\mathbb{R}_+}$ a possibly random, time-dependent and $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted risk-free interest rate process, let*

$$N_t := \exp\left(\int_0^t r_s ds\right).$$

In this case,

$$\hat{S}_t = \frac{S_t}{N_t} = \mathrm{e}^{-\int_0^t r_s ds} S_t, \qquad t \in \mathbb{R}_+,$$

represents the discounted price of the asset at time 0.

- *Currenty exchange rates.*

In this case, $N_t := R_t$ denotes the SGD/EUR exchange rate between a domestic currency (SGD) and a foreign currency (EUR), *i.e.* one unit of local currency (SGD) corresponds to R_t units in foreign currency (EUR). Let

$$\hat{S}_t := \frac{S_t}{R_t}, \qquad t \in \mathbb{R}_+,$$

denote the price of a foreign (EUR) asset quoted in units of the local currency (SGD). For example, if $R_t = 0.59$ and $S_t = €1$, then $\hat{S}_t = S_t/R_t = S_t/0.59 \simeq S\$1.7$, and $1/R_t$ is the foreign EUR/SGD exchange rate.

My foreign currency account S_t grew by 5% this year.

Q: Did I achieve a positive return?

A:

(a) Scenario A.

My foreign currency account S_t grew by 5% this year.

The foreign exchange rate dropped by 10%.

Q: Did I achieve a positive return?

A:

(b) Scenario B.

Fig. 12.1: Why change of numéraire?

- *Forward numéraire.*

* "Anyone who believes exponential growth can go on forever in a finite world is either a madman or an economist", Kenneth E. Boulding, in: Energy Reorganization Act of 1973: Hearings, Ninety-third Congress, First Session, on H.R. 11510, page 248, United States Congress, U.S. Government Printing Office, 1973.

The price $P(t,T)$ of a bond paying $P(T,T) = \$1$ at maturity T can be taken as numéraire. In this case we have

$$N_t := P(t,T) = \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right], \qquad 0 \leqslant t \leqslant T.$$

Recall that

$$t \longmapsto \mathrm{e}^{-\int_0^t r_s ds} P(t,T) = \mathbb{E}^*\left[\mathrm{e}^{-\int_0^T r_s ds} \mid \mathcal{F}_t \right], \qquad 0 \leqslant t \leqslant T,$$

is an $\mathcal{F}_t$- martingale.

- *Annuity numéraires.*

Processes of the form

$$N_t := \sum_{k=1}^{n} (T_k - T_{k-1}) P(t, T_k), \qquad 0 \leqslant t \leqslant T_1,$$

where $P(t,T_1), P(t,T_2), \ldots, P(t,T_n)$ are bond prices with maturities $T_1 < T_2 < \cdots < T_n$ arranged according to a *tenor structure*.

- *Combinations of the above*: for example a foreign money market account $\mathrm{e}^{\int_0^t r_s^{\mathrm{f}} ds} R_t$, expressed in local (or domestic) units of currency, where $(r_t^{\mathrm{f}})_{t \in \mathbb{R}_+}$ represents a short term interest rate on the foreign market.

When the numéraire is a random process, the pricing of a claim whose value has been transformed under change of numéraire, *e.g.* under a change of currency, has to take into account the risks existing on the foreign market.

In particular, in order to perform a fair pricing, one has to determine a probability measure (for example on the foreign market), under which the transformed (or forward, or deflated) process $\hat{S}_t = S_t / N_t$ will be a martingale.

For example in case $N_t := \mathrm{e}^{\int_0^t r_s ds}$ is the money market account, the risk-neutral measure $\mathbb{P}^*$ is a measure under which the discounted price process

$$\hat{S}_t = \frac{S_t}{N_t} = \mathrm{e}^{-\int_0^t r_s ds} S_t, \qquad t \in \mathbb{R}_+,$$

is a martingale.

In the next section we will see that this property can be extended to any kind of numéraire.

12.2 Change of Numéraire

In this section we review the pricing of options by a change of measure associated to a numéraire N_t, cf. *e.g.* [GKR95] and references therein.

Most of the results of this chapter rely on the following assumption, which expresses absence of arbitrage. In the foreign exchange setting where $N_t = R_t$, this condition states that the price of one unit of foreign currenty is a martingale when quoted and discounted in the domestic currency.

Assumption (A) Under the risk-neutral measure $\mathbb{P}^*$, the discounted numéraire

$$t \longmapsto M_t := \mathrm{e}^{-\int_0^t r_s ds} N_t$$

is an $\mathcal{F}_t$-martingale.

Definition 12.1. *Given* $(N_t)_{t\in[0,T]}$ *a numéraire process, the associated* forward measure $\hat{\mathbb{P}}$ *is defined by*

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*} := \frac{M_T}{M_0} = \mathrm{e}^{-\int_0^T r_s ds} \frac{N_T}{N_0}. \tag{12.1}$$

Recall that from Section 6.3 the above Relation (12.1) rewrites as

$$d\hat{\mathbb{P}} = \frac{M_T}{M_0} d\mathbb{P}^* = \mathrm{e}^{-\int_0^T r_s ds} \frac{N_T}{N_0} d\mathbb{P}^*,$$

which is equivalent to stating that

$$\int_\Omega X(\omega) d\hat{\mathbb{P}}(\omega) = \int_\Omega \mathrm{e}^{-\int_0^T r_s ds} \frac{N_T}{N_0} X d\mathbb{P}^*$$

for any (bounded) random variable S or, under a different notation,

$$\hat{\mathbb{E}}[X] = \mathbb{E}^*\left[\mathrm{e}^{-\int_0^T r_s ds} \frac{N_T}{N_0} X\right],$$

for all integrable $\mathcal{F}_T$-measurable random variables X.

More generally, by (12.1) and the fact that the process

$$t \longmapsto M_t := \mathrm{e}^{-\int_0^t r_s ds} N_t$$

is an $\mathcal{F}_t$-martingale under $\mathbb{P}^*$ under Assumption **(A)**, we find that

$$\mathbb{E}^*\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*}\ \Big|\ \mathcal{F}_t\right] = \mathbb{E}^*\left[\frac{N_T}{N_0}\,\mathrm{e}^{-\int_0^T r_s ds}\ \Big|\ \mathcal{F}_t\right] = \frac{N_t}{N_0}\,\mathrm{e}^{-\int_0^t r_s ds} = \frac{M_t}{M_0}, \quad (12.2)$$

$0 \leqslant t \leqslant T$. In Proposition 12.3 we will show, as a consequence of next Lemma 12.2 below, that for any integrable random claim C we have

$$\mathbb{E}^*\left[C\,\mathrm{e}^{-\int_t^T r_s ds} N_T \mid \mathcal{F}_t\right] = N_t \hat{\mathbb{E}}[C \mid \mathcal{F}_t], \qquad 0 \leqslant t \leqslant T.$$

Note that (12.2), which is $\mathcal{F}_t$-measurable, should not be confused with (12.3), which is $\mathcal{F}_T$-measurable. In the next Lemma 12.2 we compute the probability density $d\hat{\mathbb{P}}_{|\mathcal{F}_t}/d\mathbb{P}^*_{|\mathcal{F}_t}$ of $\hat{\mathbb{P}}_{|\mathcal{F}_t}$ with respect to $\mathbb{P}^*_{|\mathcal{F}_t}$.

Lemma 12.2. *We have*

$$\frac{d\hat{\mathbb{P}}_{|\mathcal{F}_t}}{d\mathbb{P}^*_{|\mathcal{F}_t}} = \frac{M_T}{M_t} = \mathrm{e}^{-\int_t^T r_s ds}\frac{N_T}{N_t}, \qquad 0 \leqslant t \leqslant T. \quad (12.3)$$

Proof. The proof of (12.3) relies on the abstract version of the Bayes formula. we start by noting that for all integrable $\mathcal{F}_t$-measurable random variable G, by (12.2) we have

$$\begin{aligned}
\hat{\mathbb{E}}[G\hat{X}] &= \mathbb{E}^*\left[G\hat{X}\,\mathrm{e}^{-\int_0^T r_s ds}\frac{N_T}{N_0}\right] \\
&= \mathbb{E}^*\left[G\frac{N_t}{N_0}\,\mathrm{e}^{-\int_0^t r_s ds}\,\mathbb{E}^*\left[\hat{X}\,\mathrm{e}^{-\int_t^T r_s ds}\frac{N_T}{N_t}\ \Big|\ \mathcal{F}_t\right]\right] \\
&= \mathbb{E}^*\left[G\,\mathbb{E}^*\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*}\ \Big|\ \mathcal{F}_t\right]\mathbb{E}^*\left[\hat{X}\,\mathrm{e}^{-\int_t^T r_s ds}\frac{N_T}{N_t}\ \Big|\ \mathcal{F}_t\right]\right] \\
&= \mathbb{E}^*\left[G\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*}\,\mathbb{E}^*\left[\hat{X}\,\mathrm{e}^{-\int_t^T r_s ds}\frac{N_T}{N_t}\ \Big|\ \mathcal{F}_t\right]\right] \\
&= \hat{\mathbb{E}}\left[G\,\mathbb{E}^*\left[\hat{X}\,\mathrm{e}^{-\int_t^T r_s ds}\frac{N_T}{N_t}\ \Big|\ \mathcal{F}_t\right]\right],
\end{aligned}$$

for all integrable random variable $\hat{X}$, which shows that

$$\hat{\mathbb{E}}[\hat{X} \mid \mathcal{F}_t] = \mathbb{E}^*\left[\hat{X}\,\mathrm{e}^{-\int_t^T r_s ds}\frac{N_T}{N_t}\ \Big|\ \mathcal{F}_t\right],$$

i.e. (12.3) holds. □

We note that in case the numéraire $N_t = \mathrm{e}^{\int_0^t r_s ds}$ is equal to the money market account we simply have $\hat{\mathbb{P}} = \mathbb{P}^*$.

Pricing using Change of Numéraire

The change of numéraire technique is specially useful for pricing under random interest rates, in which case an expectation of the form

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \;\Big|\; \mathcal{F}_t \right]$$

becomes a *path integral*, see *e.g.* [Das04] for a recent account of path integral methods in quantitative finance. The next proposition is the basic result of this section, it provides a way to price an option with arbitrary payoff C under a random discount factor $\mathrm{e}^{-\int_t^T r_s ds}$ by use of the forward measure. It will be applied in Chapter 14 to the pricing of bond options and caplets, cf. Propositions 14.1, 14.3 and 14.4 below.

Proposition 12.3. *An option with integrable claim payoff $C \in L^1(\mathbb{P}^*, \mathcal{F}_T)$ is priced at time t as*

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \;\Big|\; \mathcal{F}_t \right] = N_t \hat{\mathbb{E}}\left[\frac{C}{N_T} \;\Big|\; \mathcal{F}_t \right], \qquad 0 \leqslant t \leqslant T, \tag{12.4}$$

provided that $C/N_T \in L^1(\hat{\mathbb{P}}, \mathcal{F}_T)$.

Proof. By Relation (12.3) in Lemma 12.2 we have

$$\begin{aligned}
\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \;\Big|\; \mathcal{F}_t \right] &= \mathbb{E}^*\left[\frac{d\hat{\mathbb{P}}_{|\mathcal{F}_t}}{d\mathbb{P}^*_{|\mathcal{F}_t}} \frac{N_t}{N_T} C \;\Big|\; \mathcal{F}_t \right] \\
&= N_t \, \mathbb{E}^*\left[\frac{d\hat{\mathbb{P}}_{|\mathcal{F}_t}}{d\mathbb{P}^*_{|\mathcal{F}_t}} \frac{C}{N_T} \;\Big|\; \mathcal{F}_t \right] \\
&= N_t \hat{\mathbb{E}}\left[\frac{C}{N_T} \;\Big|\; \mathcal{F}_t \right], \qquad 0 \leqslant t \leqslant T.
\end{aligned}$$

Equivalently we can write

$$\begin{aligned}
N_t \hat{\mathbb{E}}\left[\frac{C}{N_T} \;\Big|\; \mathcal{F}_t \right] &= N_t \mathbb{E}^*\left[\frac{C}{N_T} \frac{d\hat{\mathbb{P}}_{|\mathcal{F}_t}}{d\mathbb{P}^*_{|\mathcal{F}_t}} \;\Big|\; \mathcal{F}_t \right] \\
&= \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \;\Big|\; \mathcal{F}_t \right], \qquad 0 \leqslant t \leqslant T.
\end{aligned}$$

□

Each application of the formula (12.4) will require to

a) identify a suitable numéraire $(N_t)_{t \in \mathbb{R}_+}$, and to
b) make sure that the ratio C/N_T takes a sufficiently simple form,

in order to allow for the computation of the expectation in the right-hand side of (12.4).

Next, we consider further examples of numéraires and associated examples of option prices.

Examples:

a) *Money market account.*

We have $N_t := \mathrm{e}^{\int_0^t r_s ds}$, where $(r_t)_{t\in\mathbb{R}_+}$ is a possibly random and time-dependent risk-free interest rate. In this case we have $\hat{\mathbb{P}} = \mathbb{P}^*$ and (12.4) simply reads

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \mid \mathcal{F}_t\right] = \mathrm{e}^{\int_0^t r_s ds}\, \mathbb{E}^*\left[\mathrm{e}^{-\int_0^T r_s ds} C \mid \mathcal{F}_t\right], \qquad 0 \leqslant t \leqslant T,$$

which yields no particular information.

b) *Forward numéraire.*

Here, $N_t := P(t,T)$ is the price $P(t,T)$ of a bond maturing at time T, $0 \leqslant t \leqslant T$, and the discounted bond price process $\left(\mathrm{e}^{-\int_0^t r_s ds} P(t,T)\right)_{t\in[0,T]}$ is an $\mathcal{F}_t$-martingale under $\mathbb{P}^*$, *i.e.* Assumption **(A)** is satisfied and $N_t = P(t,T)$ can be taken as numéraire. In this case, (12.4) shows that a random claim C can be priced as

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \,\Big|\, \mathcal{F}_t\right] = P(t,T)\hat{\mathbb{E}}\left[C \mid \mathcal{F}_t\right], \qquad 0 \leqslant t \leqslant T, \tag{12.5}$$

since $P(T,T) = 1$, where the forward measure $\hat{\mathbb{P}}$ satisfies

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*} = \mathrm{e}^{-\int_0^T r_s ds} \frac{P(T,T)}{P(0,T)} = \frac{\mathrm{e}^{-\int_0^T r_s ds}}{P(0,T)} \tag{12.6}$$

by (12.1).

c) *Annuity numéraires.*

We take

$$N_t := \sum_{k=1}^{n} (T_k - T_{k-1}) P(t, T_k)$$

where $P(t,T_1), \ldots, P(t,T_n)$ are bond prices with maturities $T_1 < T_2 < \cdots < T_n$. Here, (12.4) shows that a swaption on the cash flow $P(T,T_n) - P(T,T_1) - \kappa N_T$ can be priced as

$$
\begin{aligned}
&\mathbb{E}^* \left[\mathrm{e}^{-\int_t^T r_s ds} (P(T,T_n) - P(T,T_1) - \kappa N_T)^+ \mid \mathcal{F}_t \right] \\
&\qquad = N_t \widehat{\mathbb{E}} \left[\left(\frac{P(T,T_n) - P(T,T_1)}{N_T} - \kappa \right)^+ \mid \mathcal{F}_t \right],
\end{aligned}
$$

$0 \leqslant t \leqslant T$, where $(P(T,T_n) - P(T,T_1))/N_T$ becomes a *swap rate*, cf. (13.46) in Proposition 13.9 and Section 14.5.

In the sequel, given $(X_t)_{t\in\mathbb{R}_+}$ an asset price process, we define the process of forward (or deflated) prices

$$
\hat{X}_t := \frac{X_t}{N_t}, \qquad 0 \leqslant t \leqslant T, \tag{12.7}
$$

which represents the values at times t of X_t, expressed in units of the numéraire N_t. It will be useful to determine the dynamics of $(\hat{X}_t)_{t\in\mathbb{R}_+}$ under the forward measure $\hat{\mathbb{P}}$.

Proposition 12.4. *Let $(X_t)_{t\in\mathbb{R}_+}$ denote a continuous $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted asset price process such that*

$$
t \longmapsto \mathrm{e}^{-\int_0^t r_s ds} X_t, \qquad t \in \mathbb{R}_+,
$$

is a martingale under $\mathbb{P}^*$. Then, under change of numéraire,

the process $(\hat{X}_t)_{t\in[0,T]}$ of forward prices is an $\mathcal{F}_t$-martingale under $\hat{\mathbb{P}}$, provided that it is integrable under $\hat{\mathbb{P}}$.

Proof. We need to show that

$$
\widehat{\mathbb{E}} \left[\frac{X_t}{N_t} \mid \mathcal{F}_s \right] = \frac{X_s}{N_s}, \qquad 0 \leqslant s \leqslant t, \tag{12.8}
$$

and we achieve this using a standard characterization of conditional expectation. Namely, for all bounded $\mathcal{F}_s$-measurable random variables G we note that under Assumption **(A)** we have

$$
\begin{aligned}
\widehat{\mathbb{E}} \left[G \frac{X_t}{N_t} \right] &= \mathbb{E}^* \left[G \frac{X_t}{N_t} \frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*} \right] \\
&= \mathbb{E}^* \left[G \frac{X_t}{N_t} \mathbb{E}^* \left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*} \mid \mathcal{F}_t \right] \right] \\
&= \mathbb{E}^* \left[G \,\mathrm{e}^{-\int_0^t r_u du} \frac{X_t}{N_0} \right] \\
&= \mathbb{E}^* \left[G \,\mathrm{e}^{-\int_0^s r_u du} \frac{X_s}{N_0} \right]
\end{aligned}
$$

$$= \hat{\mathbb{E}}\left[G\frac{X_s}{N_s}\right], \qquad 0 \leqslant s \leqslant t,$$

because

$$t \longmapsto e^{-\int_0^t r_s ds} X_t$$

is an $\mathcal{F}_t$-martingale. Finally, the identity

$$\hat{\mathbb{E}}\left[G\frac{X_t}{N_t}\right] = \hat{\mathbb{E}}\left[G\frac{X_s}{N_s}\right], \qquad 0 \leqslant s \leqslant t,$$

for all bounded $\mathcal{F}_s$-measurable G, implies (12.8). □

Next we will rephrase Proposition 12.4 in Proposition 12.6 using the Girsanov theorem, which is briefly recalled below.

Girsanov theorem

Recall that letting

$$\Phi_t := \mathbb{E}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*} \;\Big|\; \mathcal{F}_t\right], \qquad t \in [0, T], \tag{12.9}$$

the Girsanov theorem* shows that the process $(\hat{W}_t)_{t\in\mathbb{R}_+}$ defined by

$$d\hat{W}_t := dW_t - \frac{1}{\Phi_t} d\Phi_t \cdot dW_t, \qquad t \in \mathbb{R}_+, \tag{12.10}$$

is a standard Brownian motion under $\hat{\mathbb{P}}$. In case the martingale $(\Phi_t)_{t\in[0,T]}$ takes the form

$$\Phi_t = \exp\left(-\int_0^t \psi_s dW_s - \frac{1}{2}\int_0^t |\psi_s|^2 ds\right), \qquad t \in \mathbb{R}_+,$$

we have

$$d\Phi_t = -\psi_t \Phi_t dW_t, \qquad t \in \mathbb{R}_+,$$

and by the Itô multiplication Table 4.1, (12.10) reads

$$\begin{aligned} d\hat{W}_t &= dW_t - \frac{1}{\Phi_t} d\Phi_t \cdot dW_t \\ &= dW_t - \frac{1}{\Phi_t}(-\psi_t \Phi_t dW_t) \cdot dW_t \\ &= dW_t + \psi_t dt, \qquad t \in \mathbb{R}_+, \end{aligned}$$

* See *e.g.* Theorem III-35 page 132 of [Pro04].

and shows that the shifted process $(\hat{W}_t)_{t\in\mathbb{R}_+} = \left(W_t + \int_0^t \psi_s ds\right)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\hat{\mathbb{P}}$, which is consistent with the Girsanov Theorem 6.2. The next result is another application of the Girsanov theorem.

Proposition 12.5. *The process* $(\hat{W}_t)_{t\in\mathbb{R}_+}$ *defined by*

$$d\hat{W}_t := dW_t - \frac{1}{N_t} dN_t \cdot dW_t, \qquad t \in \mathbb{R}_+, \tag{12.11}$$

is a standard Brownian motion under $\hat{\mathbb{P}}$.

Proof. Relation (12.2) shows that Φ_t defined in (12.9) satisfies

$$\begin{aligned}
\Phi_t &= \mathbb{E}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}^*} \;\Big|\; \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\frac{N_T}{N_0} \mathrm{e}^{-\int_0^T r_s ds} \;\Big|\; \mathcal{F}_t\right] \\
&= \frac{N_t}{N_0} \mathrm{e}^{-\int_0^t r_s ds}, \qquad 0 \leqslant t \leqslant T,
\end{aligned}$$

hence

$$d\Phi_t = -\Phi_t r_t dt + \frac{\Phi_t}{N_t} dN_t,$$

which, by (12.10), yields

$$\begin{aligned}
d\hat{W}_t &= dW_t - \frac{1}{\Phi_t} d\Phi_t \cdot dW_t \\
&= dW_t - \frac{1}{\Phi_t}\left(-\Phi_t r_t dt + \frac{\Phi_t}{N_t} dN_t\right) \cdot dW_t \\
&= dW_t - \frac{1}{N_t} dN_t \cdot dW_t \\
&= dW_t - \frac{1}{\Phi_t}\left(-\Phi_t r_t dt + \frac{\Phi_t}{N_t} dN_t\right) \cdot dW_t \\
&= dW_t - \frac{1}{N_t} dN_t \cdot dW_t,
\end{aligned}$$

which is (12.11), from Relation (12.10) and the Itô multiplication Table 4.1. □

The next proposition confirms the statement of Proposition 12.4, and in addition it determines the precise dynamics of $(\hat{X}_t)_{t\in\mathbb{R}_+}$ under $\hat{\mathbb{P}}$. See Exercise 12.1 for another calculation based on geometric Brownian motion, and Exercise 12.6 for an extension to correlated Brownian motions. As a consequence, we have the next proposition.

Proposition 12.6. *Assume that* $(X_t)_{t\in\mathbb{R}_+}$ *and* $(N_t)_{t\in\mathbb{R}_+}$ *satisfy the stochastic differential equations*

$$dX_t = r_t X_t dt + \sigma_t^X X_t dW_t, \quad \text{and} \quad dN_t = r_t N_t dt + \sigma_t^N N_t dW_t, \qquad (12.12)$$

where $(\sigma_t^X)_{t\in\mathbb{R}_+}$ *and* $(\sigma_t^N)_{t\in\mathbb{R}_+}$ *are* $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$*-adapted volatility processes. Then we have*

$$d\hat{X}_t = (\sigma_t^X - \sigma_t^N)\hat{X}_t d\hat{W}_t. \qquad (12.13)$$

Proof. First we note that by (12.11) and (12.12),

$$d\hat{W}_t = dW_t - \frac{1}{N_t} dN_t \cdot dW_t = dW_t - \sigma_t^N dt, \qquad t \in \mathbb{R}_+,$$

is a standard Brownian motion under $\hat{\mathbb{P}}$. Next, by Itô's calculus and the Itô multiplication Table 4.1 and (12.12) we have

$$\begin{aligned}
d\left(\frac{1}{N_t}\right) &= -\frac{1}{N_t^2} dN_t + \frac{1}{N_t^3}(dN_t)^2 \\
&= -\frac{1}{N_t^2}(r_t N_t dt + \sigma_t^N N_t dW_t) + \frac{|\sigma_t^N|^2}{N_t} dt \\
&= -\frac{1}{N_t^2}(r_t N_t dt + \sigma_t^N N_t (d\hat{W}_t + \sigma_t^N dt)) + \frac{|\sigma_t^N|^2}{N_t} dt \\
&= -\frac{1}{N_t}(r_t dt + \sigma_t^N d\hat{W}_t), \qquad (12.14)
\end{aligned}$$

hence

$$\begin{aligned}
d\hat{X}_t &= d\left(\frac{X_t}{N_t}\right) \\
&= \frac{dX_t}{N_t} + X_t d\left(\frac{1}{N_t}\right) + dX_t \cdot d\left(\frac{1}{N_t}\right) \\
&= \frac{1}{N_t}(r_t X_t dt + \sigma_t^X X_t dW_t) - \frac{X_t}{N_t}(r_t dt + \sigma_t^N dW_t) \\
&\quad + X_t \frac{|\sigma_t^N|^2 N_t^2}{N_t^3} dt - \frac{X_t N_t}{N_t^2} \sigma_t^X \sigma_t^N dt \\
&= \frac{X_t}{N_t}\sigma_t^X dW_t - \frac{X_t}{N_t}\sigma_t^N dW_t - \frac{X_t}{N_t}\sigma_t^X \sigma_t^N dt + X_t \frac{|\sigma_t^N|^2}{N_t} dt \\
&= \frac{X_t}{N_t}\left(\sigma_t^X dW_t - \sigma_t^N dW_t - \sigma_t^X \sigma_t^N dt + |\sigma_t^N|^2 dt\right) \\
&= \hat{X}_t(\sigma_t^X - \sigma_t^N) dW_t - \hat{X}_t(\sigma_t^X - \sigma_t^N)\sigma_t^N dt \\
&= \hat{X}_t(\sigma_t^X - \sigma_t^N) d\hat{W}_t,
\end{aligned}$$

since $d\hat{W}_t = dW_t - \sigma_t^N dt$, $t \in \mathbb{R}_+$. □

We end this section with a comment on inverse changes of measure.

Inverse Changes of Measure

In the next proposition we compute conditional inverse density $d\mathbb{P}^*/d\hat{\mathbb{P}}$.

Proposition 12.7. *We have*

$$\hat{\mathbb{E}}\left[\frac{d\mathbb{P}^*}{d\hat{\mathbb{P}}} \;\Big|\; \mathcal{F}_t\right] = \frac{N_0}{N_t}\exp\left(\int_0^t r_s ds\right) \qquad 0 \leqslant t \leqslant T, \tag{12.15}$$

and the process

$$t \longmapsto \frac{N_0}{N_t}\exp\left(\int_0^t r_s ds\right), \qquad 0 \leqslant t \leqslant T,$$

is an $\mathcal{F}_t$-martingale under $\hat{\mathbb{P}}$.

Proof. For all bounded and $\mathcal{F}_t$-measurable random variables F we have,

$$\begin{aligned}
\hat{\mathbb{E}}\left[F\frac{d\mathbb{P}^*}{d\hat{\mathbb{P}}}\right] &= \mathbb{E}^*[F] \\
&= \mathbb{E}^*\left[F\frac{N_t}{N_t}\right] \\
&= \mathbb{E}^*\left[F\frac{N_T}{N_t}\exp\left(-\int_t^T r_s ds\right)\right] \\
&= \hat{\mathbb{E}}\left[F\frac{N_0}{N_t}\exp\left(\int_0^t r_s ds\right)\right].
\end{aligned}$$

□

By (12.14) we also have

$$d\left(\frac{1}{N_t}\exp\left(\int_0^t r_s ds\right)\right) = -\frac{1}{N_t}\exp\left(\int_0^t r_s ds\right)\sigma_t^N d\hat{W}_t,$$

which recovers the second part of Proposition 12.7, *i.e.* the martingale property of

$$t \longmapsto \frac{1}{N_t}\exp\left(\int_0^t r_s ds\right)$$

under $\hat{\mathbb{P}}$.

This version: December 22, 2017
http://www.ntu.edu.sg/home/nprivault/indext.html

12.3 Foreign Exchange

Currency exchange is a typical application of change of numéraire that illustrate the principle of absence of arbitrage.

Let R_t denote the foreign exchange rate, *i.e.* R_t is the (possibly fractional) quantity of local currency that correspond to one unit of foreign currency.

Consider an investor that intends to exploit an "overseas investment opportunity" by

a) at time 0, changing one unit of local currency into $1/R_0$ units of foreign currency,
b) investing $1/R_0$ on the foreign market at the rate r^{f} to make the amount $\mathrm{e}^{tr^{\mathrm{f}}}/R_0$ until time t,
c) changing back $\mathrm{e}^{tr^{\mathrm{f}}}/R_0$ into a quantity $\mathrm{e}^{tr^{\mathrm{f}}}R_t/R_0$ of his local currency.

In other words, the foreign money market account $\mathrm{e}^{tr^{\mathrm{f}}}$ is valued $\mathrm{e}^{tr^{\mathrm{f}}}R_t$ on the local (or domestic) market, and its discounted value on the local market is

$$\mathrm{e}^{-tr+tr^{\mathrm{f}}}R_t, \qquad t \in \mathbb{R}_+.$$

The outcome of this investment will be obtained by a martingale comparison of $\mathrm{e}^{tr^{\mathrm{f}}}R_t/R_0$ to the amount e^{rt} that could have been obtained by investing on the local market.

Taking

$$\boxed{N_t := \mathrm{e}^{tr^{\mathrm{f}}}R_t, \qquad t \in \mathbb{R}_+, \qquad (12.16)}$$

as *numéraire*, absence of arbitrage is expressed by stating that the discounted numéraire process

$$t \longmapsto \mathrm{e}^{-rt}N_t = \mathrm{e}^{-t(r-r^{\mathrm{f}})}R_t$$

is an $\mathcal{F}_t$-martingale under $\mathbb{P}^*$, which is Assumption **(A)**.

Next, we find a characterization of this arbitrage condition under the parameters of the model, and for this we will model foreign exchange rates R_t according to a geometric Brownian motion (12.17). Major currencies have started floating against each other since 1973, following the end of the system of fixed exchanged rates agreed upon at the Bretton Woods Conference, July 1-22, 1944.

Proposition 12.8. *Assume that the foreign exchange rate R_t satisfies a stochastic differential equation of the form*

$$dR_t = \mu R_t dt + \sigma R_t dW_t, \tag{12.17}$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$. Under the absence of arbitrage Assumption **(A)** *for the numéraire (12.16), we have*

$$\mu = r - r^{\rm f}, \tag{12.18}$$

hence the exchange rate process satisfies

$$dR_t = (r - r^{\rm f}) R_t dt + \sigma R_t dW_t. \tag{12.19}$$

under $\mathbb{P}^*$.

Proof. The equation (12.17) has solution

$$R_t = R_0 \, \mathrm{e}^{\mu t + \sigma W_t - \sigma^2 t/2}, \qquad t \in \mathbb{R}_+,$$

hence the discounted value of the foreign money market account $\mathrm{e}^{t r^{\rm f}}$ on the local market is

$$\mathrm{e}^{-tr + tr^{\rm f}} R_t = R_0 \, \mathrm{e}^{t(r^{\rm f} - r + \mu) + \sigma W_t - \sigma^2 t/2}, \qquad t \in \mathbb{R}_+.$$

Under absence of arbitrage, $\mathrm{e}^{-t(r-r^{\rm f})} R_t = \mathrm{e}^{-tr} N_t$ should be an $\mathcal{F}_t$-martingale under $\mathbb{P}^*$ and this holds provided that $r^{\rm f} - r + \mu = 0$, which yields (12.18) and (12.19). □

As a consequence of Proposition 12.8, under absence of arbitrage a local investor who buys a unit of foreign currency in the hope of a higher return $r^{\rm f} >> r$ will have to face a lower (or even more negative) drift

$$\mu = r - r^{\rm f} << 0$$

in his exchange rate R_t,

The local money market account $X_t := \mathrm{e}^{rt}$ is valued e^{rt}/R_t on the foreign market, and its discounted value on the foreign market is

$$\begin{aligned} t \longmapsto \frac{\mathrm{e}^{t(r-r^{\rm f})}}{R_t} &= \frac{X_t}{N_t} \qquad (12.20) \\ &= \frac{1}{R_0} \, \mathrm{e}^{t(r-r^{\rm f}) - \mu t - \sigma W_t + \sigma^2 t/2} \\ &= \frac{1}{R_0} \, \mathrm{e}^{t(r-r^{\rm f}) - \mu t - \sigma \hat{W}_t - \sigma^2 t/2}, \end{aligned}$$

where

$$d\hat{W}_t = dW_t - \frac{1}{N_t} dN_t \cdot dW_t$$

This version: December 22, 2017
http://www.ntu.edu.sg/home/nprivault/indext.html

$$
\begin{aligned}
&= dW_t - \frac{1}{R_t} dR_t \cdot dW_t \\
&= dW_t - \sigma dt, \qquad t \in \mathbb{R}_+,
\end{aligned}
$$

is a standard Brownian motion under $\hat{\mathbb{P}}$ by (12.11). Under absence of arbitrage $\mathrm{e}^{-t(r-r^{\mathrm{f}})} R_t$ is an $\mathcal{F}_t$-martingale under $\mathbb{P}^*$ and (12.20) is an $\mathcal{F}_t$-martingale under $\hat{\mathbb{P}}$ by Proposition 12.4, which recovers (12.18).

Proposition 12.9. *Under the absence of arbitrage condition* (12.18), *the inverse exchange rate* $1/R_t$ *satisfies*

$$
d\left(\frac{1}{R_t}\right) = \frac{r^{\mathrm{f}} - r}{R_t} dt - \frac{\sigma}{R_t} d\hat{W}_t, \tag{12.21}
$$

under $\hat{\mathbb{P}}$, *where* $(R_t)_{t \in \mathbb{R}_+}$ *is given by* (12.19).

Proof. By (12.18), the exchange rate $1/R_t$ is written by Itô's calculus as

$$
\begin{aligned}
d\left(\frac{1}{R_t}\right) &= -\frac{1}{R_t^2}(\mu R_t dt + \sigma R_t dW_t) + \frac{1}{R_t^3}\sigma^2 R_t^2 dt \\
&= -(\mu - \sigma^2)\frac{1}{R_t} dt - \frac{\sigma}{R_t} dW_t \\
&= -\frac{\mu}{R_t} dt - \frac{\sigma}{R_t} d\hat{W}_t \\
&= (r^{\mathrm{f}} - r)\frac{1}{R_t} dt - \frac{\sigma}{R_t} d\hat{W}_t,
\end{aligned}
$$

where $\hat{W}_t$ is a standard Brownian motion under $\hat{\mathbb{P}}$. $\square$

Consequently, under absence of arbitrage, a foreign investor who buys a unit of the local currency in the hope of a higher return $r >> r^{\mathrm{f}}$ will have to face a lower (or even more negative) drift $-\mu = r^{\mathrm{f}} - r$ in his exchange rate $1/R_t$ as written in (12.21) under $\hat{\mathbb{P}}$.

Foreign exchange options

We now price a foreign exchange option with payoff $(R_T - \kappa)^+$ under $\mathbb{P}^*$ on the exchange rate R_T by the Black-Scholes formula as in the next proposition, also known as the Garman-Kohlagen [GK83] formula.

Proposition 12.10. *(Garman-Kohlagen formula). Consider an exchange rate process* $(R_t)_{t \in \mathbb{R}_+}$ *given by* (12.19). *The price of the foreign exchange call option on* R_T *with maturity* T *and strike price* κ *is given by*

$$
\boxed{\mathrm{e}^{-(T-t)r}\, \mathbb{E}^*[(R_T - \kappa)^+ \mid R_t] = \mathrm{e}^{-(T-t)r^{\mathrm{f}}} R_t \Phi_+(t, R_t) - \kappa\, \mathrm{e}^{-(T-t)r} \Phi_-(t, R_t),} \tag{12.22}
$$

$0 \leqslant t \leqslant T$, *where*

$$\Phi_+(t,x) = \Phi\left(\frac{\log(x/\kappa) + (T-t)(r - r^{\mathrm{f}} + \sigma^2/2)}{\sigma\sqrt{T-t}}\right),$$

and

$$\Phi_-(t,x) = \Phi\left(\frac{\log(x/\kappa) + (T-t)(r - r^{\mathrm{f}} - \sigma^2/2)}{\sigma\sqrt{T-t}}\right).$$

Proof. As a consequence of (12.19) we find the numéraire dynamics

$$\begin{aligned}
dN_t &= d(\mathrm{e}^{tr^{\mathrm{f}}} R_t) \\
&= r^{\mathrm{f}} \mathrm{e}^{tr^{\mathrm{f}}} R_t dt + \mathrm{e}^{tr^{\mathrm{f}}} dR_t \\
&= r\, \mathrm{e}^{tr^{\mathrm{f}}} R_t dt + \sigma\, \mathrm{e}^{tr^{\mathrm{f}}} R_t dW_t \\
&= r N_t dt + \sigma N_t dW_t.
\end{aligned}$$

Hence a standard application of the Black-Scholes formula yields

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\, \mathbb{E}^*[(R_T - \kappa)^+ \mid \mathcal{F}_t] = \mathrm{e}^{-(T-t)r}\, \mathbb{E}^*[(\mathrm{e}^{-Tr^{\mathrm{f}}} N_T - \kappa)^+ \mid \mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\, \mathrm{e}^{-Tr^{\mathrm{f}}}\, \mathbb{E}^*[(N_T - \kappa\, \mathrm{e}^{Tr^{\mathrm{f}}})^+ \mid \mathcal{F}_t] \\
&= \mathrm{e}^{-Tr^{\mathrm{f}}} \left(N_t \Phi\left(\frac{\log(N_t\, \mathrm{e}^{-Tr^{\mathrm{f}}}/\kappa) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)\right. \\
&\qquad \left. -\kappa\, \mathrm{e}^{Tr^{\mathrm{f}} - (T-t)r} \Phi\left(\frac{\log(N_t\, \mathrm{e}^{-Tr^{\mathrm{f}}}/\kappa) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)\right) \\
&= \mathrm{e}^{-Tr^{\mathrm{f}}} \left(N_t \Phi\left(\frac{\log(R_t/\kappa) + (T-t)(r - r^{\mathrm{f}} + \sigma^2/2)}{\sigma\sqrt{T-t}}\right)\right. \\
&\qquad \left. -\kappa\, \mathrm{e}^{Tr^{\mathrm{f}} - (T-t)r} \Phi\left(\frac{\log(R_t/\kappa) + (T-t)(r - r^{\mathrm{f}} - \sigma^2/2)}{\sigma\sqrt{T-t}}\right)\right) \\
&= \mathrm{e}^{-(T-t)r^{\mathrm{f}}} R_t \Phi_+(t, R_t) - \kappa\, \mathrm{e}^{-(T-t)r} \Phi_-(t, R_t).
\end{aligned}$$

□

Similarly, from (12.21) rewritten as

$$d\left(\frac{\mathrm{e}^{rt}}{R_t}\right) = r^{\mathrm{f}} \frac{\mathrm{e}^{rt}}{R_t} dt - \sigma \frac{\mathrm{e}^{rt}}{R_t} d\hat{W}_t,$$

a foreign exchange call option with payoff $(1/R_T - \kappa)^+$ can be priced under $\hat{\mathbb{P}}$ in a Black-Scholes model by taking e^{rt}/R_t as underlying price, r^{f} as risk-free interest rate, and $-\sigma$ as volatility parameter. In this framework the Black-Scholes formula (5.17) yields

$$\mathrm{e}^{-(T-t)r^{\mathrm{f}}}\hat{\mathbb{E}}\left[\left(\frac{1}{R_T}-\kappa\right)^+ \;\Big|\; R_t\right] \tag{12.23}$$

$$= \mathrm{e}^{-(T-t)r^{\mathrm{f}}}\,\mathrm{e}^{-rT}\hat{\mathbb{E}}\left[\left(\frac{\mathrm{e}^{rT}}{R_T}-\kappa\,\mathrm{e}^{rT}\right)^+ \;\Big|\; R_t\right] \tag{12.24}$$

$$= \frac{\mathrm{e}^{-r(T-t)}}{R_t}\Phi_+\left(t,\frac{1}{R_t}\right) - \kappa\,\mathrm{e}^{-(T-t)r^{\mathrm{f}}}\Phi_-\left(t,\frac{1}{R_t}\right),$$

which is the symmetric of (12.22) by exchanging R_t with $1/R_t$ and r with r^f, where

$$\Phi_+(t,x) = \Phi\left(\frac{\log(x/\kappa)+(T-t)(r^{\mathrm{f}}-r+\sigma^2/2)}{\sigma\sqrt{T-t}}\right),$$

and

$$\Phi_-(t,x) = \Phi\left(\frac{\log(x/\kappa)+(T-t)(r^{\mathrm{f}}-r-\sigma^2/2)}{\sigma\sqrt{T-t}}\right).$$

Call/put duality for foreign exchange options

Let $N_t = \mathrm{e}^{tr^{\mathrm{f}}}R_t$, where R_t is an exchange rate with respect to a foreign currency and r_f is the foreign market interest rate.

From Proposition 12.3 and (12.4) we have

$$\hat{\mathbb{E}}\left[\frac{1}{\mathrm{e}^{Tr^{\mathrm{f}}}R_T}\left(\frac{1}{\kappa}-R_T\right)^+ \;\Big|\; R_t\right] = \frac{1}{N_t}\,\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{1}{\kappa}-R_T\right)^+ \;\Big|\; R_t\right],$$

and this yields the call/put duality

$$\begin{aligned}
\mathrm{e}^{-(T-t)r^{\mathrm{f}}}\hat{\mathbb{E}}\left[\left(\frac{1}{R_T}-\kappa\right)^+ \;\Big|\; R_t\right] &= \mathrm{e}^{-(T-t)r^{\mathrm{f}}}\hat{\mathbb{E}}\left[\frac{\kappa}{R_T}\left(\frac{1}{\kappa}-R_T\right)^+ \;\Big|\; R_t\right]\\
&= \kappa\,\mathrm{e}^{tr^{\mathrm{f}}}\hat{\mathbb{E}}\left[\frac{1}{\mathrm{e}^{Tr^{\mathrm{f}}}R_T}\left(\frac{1}{\kappa}-R_T\right)^+ \;\Big|\; R_t\right]\\
&= \frac{\kappa}{N_t}\,\mathrm{e}^{tr^{\mathrm{f}}-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{1}{\kappa}-R_T\right)^+ \;\Big|\; R_t\right]\\
&= \frac{\kappa}{R_t}\,\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{1}{\kappa}-R_T\right)^+ \;\Big|\; R_t\right],
\end{aligned} \tag{12.25}$$

between a call option with strike price κ and a (possibly fractional) quantity κ/R_t of put option(s) with strike price $1/\kappa$.

In the Black-Scholes case the duality (12.25) can be directly checked by verifying that (12.23) coincides with

$$\begin{aligned}
&\frac{\kappa}{R_t}\,\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{1}{\kappa}-R_T\right)^+ \Bigm| R_t\right]\\
&\quad=\frac{\kappa}{R_t}\,\mathrm{e}^{-(T-t)r}\,\mathrm{e}^{-Tr^{\mathrm{f}}}\,\mathbb{E}^*\left[\left(\frac{\mathrm{e}^{Tr^{\mathrm{f}}}}{\kappa}-\mathrm{e}^{Tr^{\mathrm{f}}}R_T\right)^+ \Bigm| R_t\right]\\
&\quad=\frac{\kappa}{R_t}\,\mathrm{e}^{-(T-t)r}\,\mathrm{e}^{-Tr^{\mathrm{f}}}\,\mathbb{E}^*\left[\left(\frac{\mathrm{e}^{Tr^{\mathrm{f}}}}{\kappa}-N_T\right)^+ \Bigm| R_t\right]\\
&\quad=\frac{\kappa}{R_t}\left(\frac{\mathrm{e}^{-(T-t)r}}{\kappa}\Phi^p_-(t,R_t)-\mathrm{e}^{-(T-t)r^{\mathrm{f}}}R_t\Phi^p_+(t,R_t)\right)\\
&\quad=\frac{\mathrm{e}^{-(T-t)r}}{R_t}\Phi^p_-(t,R_t)-\kappa\,\mathrm{e}^{-(T-t)r^{\mathrm{f}}}\Phi^p_+(t,R_t)\\
&\quad=\frac{\mathrm{e}^{-r(T-t)}}{R_t}\Phi_+\left(t,\frac{1}{R_t}\right)-\kappa\,\mathrm{e}^{-(T-t)r^{\mathrm{f}}}\Phi_-\left(t,\frac{1}{R_t}\right),
\end{aligned}$$

where

$$\Phi^p_-(t,x):=\Phi\left(-\frac{\log(x\kappa)+(T-t)(r-r^{\mathrm{f}}-\sigma^2/2)}{\sigma\sqrt{T-t}}\right),$$

and

$$\Phi^p_+(t,x):=\Phi\left(-\frac{\log(x\kappa)+(T-t)(r-r^{\mathrm{f}}+\sigma^2/2)}{\sigma\sqrt{T-t}}\right).$$

12.4 Pricing of Exchange Options

Based on Proposition 12.4 we model the process $\hat{X}_t$ of forward prices as a continuous martingale under $\hat{\mathbb{P}}$, written as

$$d\hat{X}_t=\hat{\sigma}_t d\hat{W}_t,\qquad t\in\mathbb{R}_+, \tag{12.26}$$

where $(\hat{W}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\hat{\mathbb{P}}$ and $(\hat{\sigma}_t)_{t\in\mathbb{R}_+}$ is an $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted process. The following lemma is a consequence of the Markov property of the process $(\hat{X}_t)_{t\in\mathbb{R}_+}$ and leads to the Margrabe formula of Proposition 12.11 below.

Assume that $(\hat{X}_t)_{t\in\mathbb{R}_+}$ has the dynamics

This version: December 22, 2017
http://www.ntu.edu.sg/home/nprivault/indext.html

$$d\hat{X}_t = \hat{\sigma}_t(\hat{X}_t)d\hat{W}_t, \tag{12.27}$$

where the function $x \longmapsto \hat{\sigma}_t(x)$ is uniformly Lipschitz in $t \in \mathbb{R}_+$. The Markov property of the diffusion process $(\hat{X}_t)_{t\in\mathbb{R}_+}$, cf. Theorem V-6-32 of [Pro04], shows that the conditional expectation $\hat{\mathbb{E}}[\hat{g}(\hat{X}_T) \mid \mathcal{F}_t]$ can be written using a (measurable) function $\hat{C}(t,x)$ of t and $\hat{X}_t$, as

$$\hat{\mathbb{E}}[\hat{g}(\hat{X}_T) \mid \mathcal{F}_t] = \hat{C}(t,\hat{X}_t), \qquad 0 \leqslant t \leqslant T.$$

Consequently, a vanilla option with claim payoff $C := N_T\hat{g}(\hat{X}_T)$ can be priced as

$$\begin{aligned}\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} N_T\hat{g}(\hat{X}_T) \;\middle|\; \mathcal{F}_t\right] &= N_t\hat{\mathbb{E}}[\hat{g}(\hat{X}_T) \mid \mathcal{F}_t] \\ &= N_t\hat{C}(t,\hat{X}_t), \qquad 0 \leqslant t \leqslant T. \end{aligned}\tag{12.28}$$

The next proposition states the Margrabe [Mar78] formula for the pricing of exchange options by the zero interest rate Black-Scholes formula. It will be applied in particular in Proposition 14.3 below for the pricing of bond options.

Proposition 12.11. *(Margrabe formula). Assume that $\hat{\sigma}_t(\hat{X}_t) = \hat{\sigma}(t)\hat{X}_t$, i.e. the martingale $(\hat{X}_t)_{t\in[0,T]}$ is a (driftless) geometric Brownian motion under $\hat{\mathbb{P}}$ with deterministic volatility $(\hat{\sigma}(t))_{t\in[0,T]}$. Then we have*

$$\boxed{\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(X_T - \kappa N_T)^+ \;\middle|\; \mathcal{F}_t\right] = X_t\Phi_+^0(t,\hat{X}_t) - \kappa N_t\Phi_-^0(t,\hat{X}_t),} \tag{12.29}$$

$t \in [0,T]$, where

$$\Phi_+^0(t,x) = \Phi\left(\frac{\log(x/\kappa)}{v(t,T)} + \frac{v(t,T)}{2}\right), \quad \Phi_-^0(t,x) = \Phi\left(\frac{\log(x/\kappa)}{v(t,T)} - \frac{v(t,T)}{2}\right),$$

and $v^2(t,T) = \int_t^T \hat{\sigma}^2(s)ds$.

Proof. Taking $g(x) = (x-\kappa)^+$ in (12.28), the call option with payoff

$$(X_T - \kappa N_T)^+ = N_T(\hat{X}_T - \kappa)^+,$$

and floating strike price κN_T is priced by (12.28) as

$$\begin{aligned}\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(X_T - \kappa N_T)^+ \;\middle|\; \mathcal{F}_t\right] &= \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} N_T(\hat{X}_T - \kappa)^+ \;\middle|\; \mathcal{F}_t\right] \\ &= N_t\hat{\mathbb{E}}\left[(\hat{X}_T - \kappa)^+ \;\middle|\; \mathcal{F}_t\right]\end{aligned}$$

$$= N_t \hat{C}(t, \hat{X}_t),$$

where the function $\hat{C}(t, \hat{X}_t)$ is given by the Black-Scholes formula

$$\hat{C}(t, x) = x\Phi_+^0(t, x) - \kappa\Phi_-^0(t, x),$$

with zero interest rate, since $(\hat{X}_t)_{t\in[0,T]}$ is a driftless geometric Brownian motion which is an $\mathcal{F}_t$-martingale under $\hat{\mathbb{P}}$, and $\hat{X}_T$ is a lognormal random variable with variance coefficient $v^2(t,T) = \int_t^T \hat{\sigma}^2(s)ds$. Hence we have

$$\begin{aligned}\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(X_T - \kappa N_T)^+ \mid \mathcal{F}_t\right] &= N_t \hat{C}(t, \hat{X}_t) \\ &= N_t \hat{X}_t \Phi_+^0(t, \hat{X}_t) - \kappa N_t \Phi_-^0(t, \hat{X}_t),\end{aligned}$$

$t \in \mathbb{R}_+$. □

In particular, from Proposition 12.6 and (12.13), we can take $\hat{\sigma}(t) = \sigma_t^X - \sigma_t^N$ when $(\sigma_t^X)_{t\in\mathbb{R}_+}$ and $(\sigma_t^N)_{t\in\mathbb{R}_+}$ are deterministic.

Examples:

a) When the short rate process $(r(t))_{t\in[0,T]}$ is a *deterministic* function and $N_t = \mathrm{e}^{-\int_t^T r(s)ds}$, $0 \leqslant t \leqslant T$, we have $\hat{\mathbb{P}} = \mathbb{P}^*$ and Proposition 12.11 yields Merton's [Mer73] "zero interest rate" version of the Black-Scholes formula

$$\begin{aligned}&\mathrm{e}^{-\int_t^T r(s)ds}\,\mathbb{E}^*\left[(X_T - \kappa)^+ \mid \mathcal{F}_t\right] \\ &\quad = X_t \Phi_+^0\left(t, \mathrm{e}^{\int_t^T r(s)ds} X_t\right) - \kappa\,\mathrm{e}^{-\int_t^T r(s)ds}\Phi_-^0\left(t, \mathrm{e}^{\int_t^T r(s)ds} X_t\right),\end{aligned}$$

where $(X_t)_{t\in\mathbb{R}_+}$ satisfies the equation

$$\frac{dX_t}{X_t} = r(t)dt + \hat{\sigma}(t)dW_t, \quad i.e. \quad \frac{d\hat{X}_t}{\hat{X}_t} = \hat{\sigma}(t)dW_t, \quad 0 \leqslant t \leqslant T.$$

b) In the case of pricing under a *forward numéraire*, *i.e.* $N_t = P(t,T)$, $0 \leqslant t \leqslant T$, we get

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(X_T - \kappa)^+ \mid \mathcal{F}_t\right] = X_t \Phi_+(t, \hat{X}_t) - \kappa P(t,T)\Phi_-(t, \hat{X}_t),$$

$t \in \mathbb{R}_+$, since $P(T,T) = 1$. In particular, when $X_t = P(t,S)$ the above formula allows us to price a bond call option on $P(T,S)$ as

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(P(T,S) - \kappa)^+ \mid \mathcal{F}_t\right] = P(t,S)\Phi_+(t, \hat{X}_t) - \kappa P(t,T)\Phi_-(t, \hat{X}_t),$$

This version: December 22, 2017
http://www.ntu.edu.sg/home/nprivault/indext.html

$0 \leqslant t \leqslant T$, provided that the martingale $\hat{X}_t = P(t,S)/P(t,T)$ under $\hat{\mathbb{P}}$ is given by a geometric Brownian motion, cf. Section 14.2.

12.5 Hedging by Change of Numéraire

In this section we reconsider and extend the Black-Scholes self-financing hedging strategies found in (6.29)-(6.30) and Proposition 6.11 of Chapter 6. For this, we use the stochastic integral representation of the forward claim payoffs and change of numéraire in order to compute self-financing portfolio strategies. Our hedging portfolios will be built on the assets (X_t, N_t), not on X_t and the money market account $B_t = \mathrm{e}^{\int_0^t r_s ds}$, extending the classical hedging portfolios that are available in from the Black-Scholes formula, using a technique from [Jam96], cf. also [PT12].

Consider a claim with random payoff C, typically an interest rate derivative, cf. Chapter 14. Assume that the forward claim payoff $C/N_T \in L^2(\Omega)$ has the stochastic integral representation

$$\frac{C}{N_T} = \hat{\mathbb{E}}\left[\frac{C}{N_T}\right] + \int_0^T \hat{\phi}_t d\hat{X}_t, \tag{12.30}$$

where $(\hat{X}_t)_{t\in[0,T]}$ is given by (12.26) and $(\hat{\phi}_t)_{t\in[0,T]}$ is a square-integrable adapted process under $\hat{\mathbb{P}}$, from which it follows that the forward claim price

$$\hat{V}_t := \frac{V_t}{N_t} = \frac{1}{N_t}\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \;\middle|\; \mathcal{F}_t\right] = \hat{\mathbb{E}}\left[\frac{C}{N_T} \;\middle|\; \mathcal{F}_t\right], \qquad 0 \leqslant t \leqslant T,$$

is an $\mathcal{F}_t$-martingale under $\hat{\mathbb{P}}$, that can be decomposed as

$$\hat{V}_t = \hat{\mathbb{E}}\left[\frac{C}{N_T}\right] + \int_0^t \hat{\phi}_s d\hat{X}_s, \qquad 0 \leqslant t \leqslant T. \tag{12.31}$$

The next proposition extends the argument of [Jam96] to the general framework of pricing using change of numéraire. Note that this result differs from the standard formula that uses the money market account $B_t = \mathrm{e}^{\int_0^t r_s ds}$ for hedging instead of N_t, cf. *e.g.* [GKR95] pages 453-454. The notion of self-financing portfolio is similar to that of Definition 5.1.

Proposition 12.12. *Letting $\hat{\eta}_t := \hat{V}_t - \hat{X}_t\hat{\phi}_t$, $0 \leqslant t \leqslant T$, the portfolio allocation $(\hat{\phi}_t, \hat{\eta}_t)_{t\in[0,T]}$ with value*

$$V_t = \hat{\phi}_t X_t + \hat{\eta}_t N_t, \qquad 0 \leqslant t \leqslant T,$$

is self-financing in the sense that

$$dV_t = \hat{\phi}_t dX_t + \hat{\eta}_t dN_t,$$

and it hedges the claim C, i.e.

$$V_t = \hat{\phi}_t X_t + \hat{\eta}_t N_t = \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \;\middle|\; \mathcal{F}_t \right], \qquad 0 \leqslant t \leqslant T. \tag{12.32}$$

Proof. In order to check that the portfolio allocation $(\hat{\phi}_t, \hat{\eta}_t)_{t\in[0,T]}$ hedges the claim C it suffices to check that (12.32) holds since by (12.4) the price V_t at time $t \in [0,T]$ of the hedging portfolio satisfies

$$V_t = N_t \hat{V}_t = \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \;\middle|\; \mathcal{F}_t \right], \qquad 0 \leqslant t \leqslant T.$$

Next, we show that the portfolio allocation $(\hat{\phi}_t, \hat{\eta}_t)_{t\in[0,T]}$ is self-financing. By numéraire invariance, cf. *e.g.* page 184 of [Pro01], we have

$$\begin{aligned}
dV_t &= d(N_t \hat{V}_t) \\
&= \hat{V}_t dN_t + N_t d\hat{V}_t + dN_t \cdot d\hat{V}_t \\
&= \hat{V}_t dN_t + N_t \hat{\phi}_t d\hat{X}_t + \hat{\phi}_t dN_t \cdot d\hat{X}_t \\
&= \hat{\phi}_t \hat{X}_t dN_t + N_t \hat{\phi}_t d\hat{X}_t + \hat{\phi}_t dN_t \cdot d\hat{X}_t + (\hat{V}_t - \hat{\phi}_t \hat{X}_t) dN_t \\
&= \hat{\phi}_t d(N_t \hat{X}_t) + \hat{\eta}_t dN_t \\
&= \hat{\phi}_t dX_t + \hat{\eta}_t dN_t.
\end{aligned}$$

□

We now consider an application to the forward Delta hedging of European type options with payoff $C = \hat{g}(\hat{X}_T)$ where $\hat{g} : \mathbb{R} \longrightarrow \mathbb{R}$ and $(\hat{X}_t)_{t\in\mathbb{R}_+}$ has the Markov property as in (12.27), where $\hat{\sigma} : \mathbb{R}_+ \times \mathbb{R}$. Assuming that the function $\hat{C}(t,x)$ defined by

$$\hat{V}_t := \hat{\mathbb{E}}\left[g(\hat{X}_T) \;\middle|\; \mathcal{F}_t \right] = \hat{C}(t, \hat{X}_t)$$

is $\mathcal{C}^2$ on $\mathbb{R}_+$, we have the following corollary of Proposition 12.12, which extends the Black-Scholes Delta hedging technique to the general change of numéraire setup.

Corollary 12.13. *Letting* $\hat{\eta}_t = \hat{C}(t, \hat{X}_t) - \hat{X}_t \dfrac{\partial \hat{C}}{\partial x}(t, \hat{X}_t)$, $0 \leqslant t \leqslant T$, *the portfolio allocation* $\left(\dfrac{\partial \hat{C}}{\partial x}(t, \hat{X}_t), \hat{\eta}_t \right)_{t\in[0,T]}$ *with value*

$$V_t = \hat{\eta}_t N_t + X_t \frac{\partial \hat{C}}{\partial x}(t, \hat{X}_t), \qquad t \in \mathbb{R}_+,$$

is self-financing and hedges the claim $C = N_T \hat{g}(\hat{X}_T)$.

Proof. This result follows directly from Proposition 12.12 by noting that by Itô's formula, and the martingale property of $\hat{V}_t$ under $\hat{\mathbb{P}}$ the stochastic

This version: December 22, 2017
http://www.ntu.edu.sg/home/nprivault/indext.html

integral representation (12.31) is given by

$$\hat{\phi}_t = \frac{\partial \hat{C}}{\partial x}(t, \hat{X}_t), \qquad 0 \leqslant t \leqslant T.$$

□

In the case of an exchange option with payoff function

$$C = (X_T - \kappa N_T)^+ = N_T(\hat{X}_T - \kappa)^+$$

on the geometric Brownian motion $(\hat{X}_t)_{t\in[0,T]}$ under $\hat{\mathbb{P}}$ with

$$\hat{\sigma}_t(\hat{X}_t) = \hat{\sigma}(t)\hat{X}_t,$$

where $(\hat{\sigma}(t))_{t\in[0,T]}$ is a deterministic function, we have the following corollary on the hedging of exchange options based on the Margrabe formula (12.29).

Corollary 12.14. *The decomposition*

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(X_T - \kappa N_T)^+ \,\middle|\, \mathcal{F}_t \right] = X_t \Phi_+^0(t, \hat{X}_t) - \kappa N_t \Phi_-^0(t, \hat{X}_t)$$

yields a self-financing portfolio allocation $(\Phi_+^0(t, \hat{X}_t), -\kappa\Phi_-^0(t, \hat{X}_t))_{t\in[0,T]}$ *in the assets* (X_t, N_t)*, that hedges the claim* $C = (X_T - \kappa N_T)^+$*.*

Proof. We apply Corollary 12.13 and the relation

$$\frac{\partial \hat{C}}{\partial x}(t, x) = \Phi_+^0(t, x), \qquad x \in \mathbb{R},$$

for the function $\hat{C}(t, x) = x\Phi_+^0(t, x) - \kappa\Phi_-^0(t, x)$, cf. Relation (5.20) in Proposition 5.13. □

Note that the Delta hedging method requires the computation of the function $\hat{C}(t, x)$ and that of the associated finite differences, and may not apply to path-dependent claims.

Examples:

a) When the short rate process $(r(t))_{t\in[0,T]}$ is a *deterministic* function and $N_t = \mathrm{e}^{\int_t^T r(s)ds}$, Corollary 12.14 yields the usual Black-Scholes hedging strategy

$$\begin{aligned}&\left(\Phi_+(t, \hat{X}_t), -\kappa\,\mathrm{e}^{\int_0^T r(s)ds}\Phi_-(t, X_t)\right)_{t\in[0,T]}\\ &= \left(\Phi_+^0(t, \mathrm{e}^{\int_t^T r(s)ds}\hat{X}_t), -\kappa\,\mathrm{e}^{\int_0^T r(s)ds}\Phi_-^0(t, \mathrm{e}^{\int_t^T r(s)ds}X_t)\right)_{t\in[0,T]},\end{aligned}$$

in the assets $(X_t, \mathrm{e}^{\int_0^t r(s)ds})$, that hedges the claim $C = (X_T - \kappa)^+$, with

$$\Phi_+(t,x) := \Phi\left(\frac{\log(x/\kappa) + \left(\int_t^T r(s)ds + (T-t)\sigma^2/2\right)}{\sigma\sqrt{T-t}}\right),$$

and

$$\Phi_-(t,x) := \Phi\left(\frac{\log(x/\kappa) + \left(\int_t^T r(s)ds - (T-t)\sigma^2/2\right)}{\sigma\sqrt{T-t}}\right).$$

b) In case $N_t = P(t,T)$ and $X_t = P(t,S)$, $0 \leqslant t \leqslant T < S$, Corollary 12.14 shows that a bond call option with payoff $(P(T,S) - \kappa)^+$ can be hedged as

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(P(T,S)-\kappa)^+ \mid \mathcal{F}_t\right] = P(t,S)\Phi_+(t,\hat{X}_t) - \kappa P(t,T)\Phi_-(t,\hat{X}_t)$$

by the self-financing portfolio allocation

$$(\Phi_+(t,\hat{X}_t), -\kappa\Phi_-(t,\hat{X}_t))_{t\in[0,T]}$$

in $(P(t,S), P(t,T))$, *i.e.* one needs to hold the quantity $\Phi_+(t,\hat{X}_t)$ of the bond maturing at time S, and to short a quantity $\kappa\Phi_-(t,\hat{X}_t)$ of the bond maturing at time T.

Exercises

Exercise 12.1 Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion started at 0 under the risk-neutral measure $\mathbb{P}^*$. Consider a numéraire $(N_t)_{t\in\mathbb{R}_+}$ given by

$$N_t := N_0\,\mathrm{e}^{\eta B_t - \eta^2 t/2}, \qquad t \in \mathbb{R}_+,$$

and a risky asset $(X_t)_{t\in\mathbb{R}_+}$ given by

$$X_t := X_0\,\mathrm{e}^{\sigma B_t - \sigma^2 t/2}, \qquad t \in \mathbb{R}_+.$$

Let $\hat{\mathbb{P}}$ denote the forward measure relative to the numéraire $(N_t)_{t\in\mathbb{R}_+}$, under which the process $\hat{X}_t := X_t/N_t$ of forward prices is known to be a martingale.

a) Using the Itô formula, compute

$$d\hat{X}_t = d(X_t/N_t) = (X_0/N_0)d\left(\mathrm{e}^{(\sigma-\eta)B_t - (\sigma^2-\eta^2)t/2}\right).$$

b) Explain why the exchange option price $\mathbb{E}[(X_T - \lambda N_T)^+]$ at time 0 has the Black-Scholes form

$$\begin{aligned} & e^{-rT}\,\mathbb{E}[(X_T - \lambda N_T)^+] \qquad (12.33) \\ &= X_0 \Phi\left(\frac{\log(\hat{X}_0/\lambda)}{\hat{\sigma}\sqrt{T}} + \frac{\hat{\sigma}\sqrt{T}}{2}\right) - \lambda N_0 \Phi\left(\frac{\log(\hat{X}_0/\lambda)}{\hat{\sigma}\sqrt{T}} - \frac{\hat{\sigma}\sqrt{T}}{2}\right). \end{aligned}$$

Hints:

(i) Use the change of numéraire identity

$$e^{-rT}\,\mathbb{E}[(X_T - \lambda N_T)^+] = N_0 \hat{\mathbb{E}}[(\hat{X}_T - \lambda)^+].$$

(ii) The forward price $\hat{X}_t$ is a martingale under the forward measure $\hat{\mathbb{P}}$ relative to the numéraire $(N_t)_{t\in\mathbb{R}_+}$.

c) Give the value of $\hat{\sigma}$ in terms of σ and η.

Exercise 12.2 Consider two zero-coupon bond prices of the form $P(t,T) = F(t,r_t)$ and $P(t,S) = G(t,r_t)$, where $(r_t)_{t\in\mathbb{R}_+}$ is a short term interest rate process. Taking $N_t := P(t,T)$ as a numeraire defining the forward measure $\hat{\mathbb{P}}$, compute the dynamics of $(P(t,S))_{t\in[0,T]}$ under $\hat{\mathbb{P}}$ using a standard Brownian motion $(\hat{W}_t)_{t\in[0,T]}$ under $\hat{\mathbb{P}}$.

Exercise 12.3 Forward contract. Using a change of numeraire argument for the numeraire $N_t := P(t,T)$, $t \in [0,T]$, compute the price at time $t \in [0,T]$ of a forward (or future) contract with payoff $P(T,S) - K$ in a bond market with short term interest rate $(r_t)_{t\in\mathbb{R}_+}$. How would you hedge this forward contract?

Exercise 12.4 Bond options. Consider two bonds with maturities T and S, with prices $P(t,T)$ and $P(t,S)$ given by

$$\frac{dP(t,T)}{P(t,T)} = r_t dt + \zeta_t^T dW_t,$$

and

$$\frac{dP(t,S)}{P(t,S)} = r_t dt + \zeta_t^S dW_t,$$

where $(\zeta^T(s))_{s\in[0,T]}$ and $(\zeta^S(s))_{s\in[0,S]}$ are deterministic functions.

a) Show, using Itô's formula, that

$$d\left(\frac{P(t,S)}{P(t,T)}\right) = \frac{P(t,S)}{P(t,T)}(\zeta^S(t) - \zeta^T(t))d\hat{W}_t,$$

where $(\hat{W}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\hat{\mathbb{P}}$.

b) Show that

$$P(T,S) = \frac{P(t,S)}{P(t,T)} \exp\left(\int_t^T (\zeta^S(s) - \zeta^T(s)) d\hat{W}_s - \frac{1}{2}\int_t^T |\zeta^S(s) - \zeta^T(s)|^2 ds\right).$$

Let $\hat{\mathbb{P}}$ denote the forward measure associated to the numéraire

$$N_t := P(t,T), \qquad 0 \leqslant t \leqslant T.$$

c) Show that for all $S,T > 0$ the price at time t

$$\mathbb{E}\left[\mathrm{e}^{-\int_t^T r_s ds}(P(T,S)-\kappa)^+ \mid \mathcal{F}_t\right]$$

of a bond call option on $P(T,S)$ with payoff $(P(T,S)-\kappa)^+$ is equal to

$$\begin{aligned}&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(P(T,S)-\kappa)^+ \mid \mathcal{F}_t\right] \qquad (12.34)\\ &= P(t,S)\Phi\left(\frac{v}{2} + \frac{1}{v}\log\frac{P(t,S)}{\kappa P(t,T)}\right) - \kappa P(t,T)\Phi\left(-\frac{v}{2} + \frac{1}{v}\log\frac{P(t,S)}{\kappa P(t,T)}\right),\end{aligned}$$

where

$$v^2 = \int_t^T |\zeta^S(s) - \zeta^T(s)|^2 ds.$$

d) Compute the self-financing hedging strategy that hedges the bond option using a portfolio based on the assets $P(t,T)$ and $P(t,S)$.

Exercise 12.5 Consider two risky assets S_1 and S_2 modeled by the geometric Brownian motions

$$S_1(t) = \mathrm{e}^{\sigma_1 W_t + \mu t} \qquad \text{and} \qquad S_2(t) = \mathrm{e}^{\sigma_2 W_t + \mu t}, \qquad t \in \mathbb{R}_+, \qquad (12.35)$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}$.

a) Find a condition on r, μ and σ_2 so that the discounted price process $\mathrm{e}^{-rt}S_2(t)$ is a martingale under $\mathbb{P}$.
b) Assume that $r - \mu = \sigma_2^2/2$, and let

$$X_t = \mathrm{e}^{(\sigma_2^2-\sigma_1^2)t/2} S_1(t), \qquad t \in \mathbb{R}_+.$$

Show that the discounted process $\mathrm{e}^{-rt}X_t$ is a martingale under $\mathbb{P}$.

c) Taking $N_t = S_2(t)$ as numéraire, show that the forward process $\hat{X}(t) = X_t/N_t$ is a martingale under the forward measure $\hat{\mathbb{P}}$ defined by

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \mathrm{e}^{-rT}\frac{N_T}{N_0}.$$

This version: December 22, 2017
http://www.ntu.edu.sg/home/nprivault/indext.html

Recall that

$$\hat{W}_t := W_t - \sigma_2 t$$

is a standard Brownian motion under $\hat{\mathbb{P}}$.

d) Using the relation

$$e^{-rT}\,\mathbb{E}[(S_1(T) - S_2(T))^+] = N_0\hat{\mathbb{E}}[(S_1(T) - S_2(T))^+/N_T],$$

compute the price

$$e^{-rT}\,\mathbb{E}[(S_1(T) - S_2(T))^+]$$

of the exchange option on the assets S_1 and S_2.

Exercise 12.6 Extension of Proposition 12.6 to correlated Brownian motions. Assume that $(S_t)_{t\in\mathbb{R}_+}$ and $(N_t)_{t\in\mathbb{R}_+}$ satisfy the stochastic differential equations

$$dS_t = r_t S_t dt + \sigma_t^S S_t dW_t^S, \quad \text{and} \quad dN_t = \eta_t N_t dt + \sigma_t^N N_t dW_t^N,$$

where $(W_t^S)_{t\in\mathbb{R}_+}$ and $(W_t^N)_{t\in\mathbb{R}_+}$ have the correlation

$$dW_t^S \cdot dW_t^N = \rho dt,$$

where $\rho \in [-1, 1]$.

a) Show that $(W_t^N)_{t\in\mathbb{R}_+}$ can be written as

$$W_t^N = \rho W_t^S + \sqrt{1-\rho^2} W_t, \qquad t \in \mathbb{R}_+,$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$, independent of $(W_t^S)_{t\in\mathbb{R}_+}$.

b) Letting $X_t = S_t/N_t$, show that dX_t can be written as

$$dX_t = (r_t - \eta_t + (\sigma_t^N)^2 - \rho\sigma_t^N\sigma_t^S)X_t dt + \hat{\sigma}_t X_t dW_t^X,$$

where W_t^X is a standard Brownian motion under $\mathbb{P}^*$ and $\hat{\sigma}_t$ is to be computed.

Exercise 12.7 Quanto options (Exercise 9.5 in [Shr04]). Consider an asset priced S_t at time t, with

$$dS_t = rS_t dt + \sigma^S S_t dW_t^S,$$

and an exchange rate $(R_t)_{t\in\mathbb{R}_+}$ given by

$$dR_t = (r - r^{\mathrm{f}})R_t dt + \sigma^R R_t dW_t^R,$$

from (12.18) in Proposition 12.8, where $(W_t^R)_{t\in\mathbb{R}_+}$ is written as

$$W_t^R = \rho W_t^S + \sqrt{1-\rho^2} W_t, \qquad t \in \mathbb{R}_+,$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$, independent of $(W_t^S)_{t\in\mathbb{R}_+}$, *i.e.* we have

$$dW_t^R \cdot dW_t^S = \rho dt,$$

where ρ is a correlation coefficient.

a) Let

$$a = r - r^{\mathrm{f}} + \rho\sigma^R\sigma^S - (\sigma^R)^2$$

and $X_t = \mathrm{e}^{at} S_t / R_t$, $t \in \mathbb{R}_+$, and show by Exercise 12.6 that dX_t can be written as

$$dX_t = rX_t dt + \hat{\sigma} X_t dW_t^X,$$

where $(W_t^X)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$ and $\hat{\sigma}$ is to be computed.

b) Compute the price

$$\mathrm{e}^{-r(T-t)} \mathbb{E}^* \left[\left(\frac{S_T}{R_T} - \kappa \right)^+ \Bigm| \mathcal{F}_t \right]$$

of the quanto option at time $t \in [0, T]$.

This version: December 22, 2017
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