Assignment 4 of Algorithm

Zuyao Chen 201728008629002

1. Linear-inequality feasibility

Proof:

suppose that we have an algorithm for linear programming, that means we can solve

$$\begin{aligned} \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \end{aligned}$$

given a linear inequality feasibility problem , our goal is to check whether there exist $\boldsymbol{x}^*$,

$$\boldsymbol{A}'\boldsymbol{x}^* \leq \boldsymbol{b}'$$

which equals to

$$\begin{aligned} \max \quad & \boldsymbol{0} \cdot \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}'\boldsymbol{x} \leq \boldsymbol{b}' \end{aligned}$$

this problem can be solved using the same algorithm.

2. Airplane Landing Problem

Let $x_1, x_2, ..., x_n$ be the exact landing time of each airplane respectively, the problem can be written as

$$\begin{aligned} \max \quad & \min(x_2 - x_1, x_3 - x_2, ..., x_n - x_{n-1}) \\ \text{s.t.} \quad & s_i \leq x_i \leq t_i, i = 1, 2, \cdots, n \end{aligned}$$

LP form

$$\begin{aligned} \max \quad & d \\ \text{s.t.} \quad & s_i \leq x_i \leq t_i, i = 1, 2, \cdots, n \\ & x_k - x_{k-1} \geq d, k = 2, 3, \cdots, n \\ & d \geq 0 \end{aligned}$$

Now we show that how to obtain the dual form of this question, first we minimize $-d$,

$$\begin{aligned} \min \quad & z = -d \\ \text{s.t.} \quad & s_i \leq x_i \leq t_i, i = 1, 2, \cdots, n \\ & -x_{k+1} + x_k \leq -d = z, k = 1, 2, \cdots, n-1 \\ & z \leq 0 \end{aligned}$$

using Lagrange multiplier,

$$L(z,x,\lambda,\alpha,\varphi,\phi) = z + \sum_{i=1}^{n} \lambda_i(x_i - t_i) + \sum_{i=1}^{n} \alpha_i(-x_i + s_i) + \sum_{i=1}^{n-1} \varphi_i(-x_{i+1} + x_i - z) + \phi z$$

$$\begin{aligned}
\frac{\partial L}{\partial z} &= 1 - \sum_{i=1}^{n-1} \varphi_i + \phi = 0 \\
\frac{\partial L}{\partial x_1} &= \lambda_1 - \alpha_1 + \varphi_1 = 0 \\
\frac{\partial L}{\partial x_n} &= \lambda_n - \alpha_n - \varphi_{n-1} = 0 \\
\frac{\partial L}{\partial x_i} &= \lambda_i - \alpha_i + \varphi_i - \varphi_{i-1} = 0, i = 2, 3, \cdots, n-1
\end{aligned}$$

thus

$$\begin{aligned}
g(\lambda,\alpha,\varphi,\phi) &= \inf_{z,x} L(z,x,\lambda,\alpha,\varphi,\phi) \\
&= z(1 - \sum_{i=1}^{n} \varphi_i + \phi) + x_1(\lambda_1 - \alpha_1 + \varphi_1) + x_n(\lambda_n - \alpha_n - \varphi_{n-1}) + \\
&\quad \sum_{i=2}^{n-1} x_i(\lambda_i - \alpha_i + \varphi_i - \varphi_{i-1}) - \sum_{i=1}^{n} \lambda_i t_i + \sum_{i=1}^{n} \alpha_i s_i \\
&= -\sum_{i=1}^{n} \lambda_i t_i + \sum_{i=1}^{n} \alpha_i s_i
\end{aligned}$$

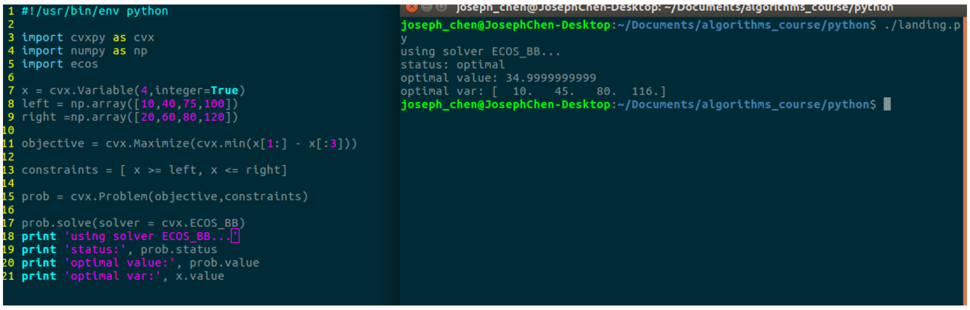
Minimizing $-d$ is equivalent to

$$\begin{aligned}
\max \quad & -\sum_{i=1}^{n} \lambda_i t_i + \sum_{i=1}^{n} \alpha_i s_i \\
\text{s.t.} \quad & \lambda_i \geq 0, \alpha \geq 0, i = 1, 2, \cdots, n \\
& \varphi_i \geq 0, i = 1, 2, \cdots, n-1 \\
& \lambda_1 - \alpha_1 + \varphi_1 = 0 \\
& \lambda_n - \alpha_n - \varphi_{n-1} = 0 \\
& \lambda_i - \alpha_i + \varphi_i - \varphi_{i-1} = 0, i = 2, 3, \cdots, n-1 \\
& 1 - \sum_{i=1}^{n-1} \varphi_i \leq 0
\end{aligned}$$

Hence the original problem that Maximizing d is equivalent to

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} \lambda_i t_i - \sum_{i=1}^{n} \alpha_i s_i \\
\text{s.t.} \quad & \lambda_i \geq 0, \alpha_i \geq 0, i = 1, 2, \cdots, n \\
& \varphi_i \geq 0, i = 1, 2, \cdots, n-1 \\
& \lambda_1 - \alpha_1 + \varphi_1 = 0 \\
& \lambda_n - \alpha_n - \varphi_{n-1} = 0 \\
& \lambda_i - \alpha_i + \varphi_i - \varphi_{i-1} = 0, i = 2, 3, \cdots, n-1 \\
& 1 - \sum_{i=1}^{n-1} \varphi_i \leq 0
\end{aligned}
$$

for instance, we have $n = 4, [10, 20], [40, 60], [75, 80], [100, 120]$ (here the minute is the metric of time), using tool cvxpy we can obtain the optimal solution 35 with optimal variables $x_1 = 10, 45, 80, 116$.



3. Interval Scheduling Problem

Let x_{ij} be the status of course j scheduled in classroom i, if $x_{ij} = 1$ status is Yes, otherwise status is No.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \\
\text{s.t.} \quad & x_{ij} \in \{0, 1\}, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n \\
& \sum_{i=1}^{m} x_{ij} \leq 1, j = 1, 2, \cdots, n \\
& F_j(x_{ij} + x_{ik} - 1) \leq S_k, 1 \leq j \leq k \leq n
\end{aligned}
$$

4. Gas Station Placement

the question is to solve

$$
\begin{aligned}
\min \quad & \max(x_2 - x_1, x_3 - x2, \cdots, x_n - x_{n-1}) \\
\text{s.t.} \quad & |x_i - d_i| \leq r, i = 1, 2, \cdots, n
\end{aligned}
$$

the LP form

$$
\begin{aligned}
\min \quad & z \\
\text{s.t.} \quad & x_i - d_i \le r, i = 1, 2, \cdots, n \\
& -x_i + d_i \le r, i = 1, 2, \cdots, n \\
& x_{k+1} - x_k \le z, k = 1, 2, \cdots, n-1
\end{aligned}
$$

5. Stable Matching Problem

Let x_{ij} be the stable matching status of man i and women j,if $x_{ij} = 1$ the status is stable;otherwise the status is unstable. ILP form

$$
\begin{aligned}
\max \quad & 0 \\
\text{s.t.} \quad & x_{ij} \in \{0, 1\}, i = 1, 2, \cdots, n, j = 1, 2, \cdots, n \\
& \sum_{j=1}^{n} x_{ij} = 1, i = 1, 2, \cdots, n \\
& \sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, \cdots, n \\
& x_{ij} + x_{kl} \le S_{i,j,k,l} + 1
\end{aligned}
$$

6. Dual Simplex Algorithm

the result is showed below.

```
directly solving...
problem has 3 equalities and 0 inequalities
starting optimization...
status: feasible!
optimal value: -16.5
optimal x: [  0.    0.    0.    1.5   2.5  16.5]
solving dual problem...
problem has 0 equalities and 6 inequalities
starting optimization...
status: feasible!
optimal value: -16.5
optimal y: [ 2.5  6.   0. ]
using tool cvxpy
optimal value: -16.5
optimal x: [ -5.45737158e-22   6.11583402e-22  -4.40096362e-22   1.50000000e+00
   2.50000000e+00   1.65000000e+01]
```