Experimental survey of discrete minimizers of the p-frame energy (extended abstract)

Dmitriy Bilyk (University of Minnesota), Alexey Glazyrin (University of Texas Rio Grande Valley), Ryan Matzke (Vanderbilt University), Josiah Park (TAMU), Oleksandr Vlasiuk (Vanderbilt University)

dbilyk@umn.edu, Alexey.Glazyrin@utrgv.edu, ryan.w.matzke@vanderbilt.edu, jmdpark@berkeley.edu, oleksandr.vlasiuk@vanderbilt.edu

***Abstract*—We provide a detailed analysis of results from a large-scale computational exploration of real and complex (weighted) point configurations that minimize p-frame energies, uncovering phase transition behavior exhibited by the minimizers. We utilize numerical linear programming methodologies to offer complementary lower bounds that support our experimentally obtained upper bounds on minimal energy values. Furthermore, we present the development of an exceptionally symmetric weighted design consisting of 85 points, which outperforms the current best known lower bounds for a minimal-sized weighted design in the realm of five-dimensional complex projective space. In conclusion, based on our thorough observations and in-depth analysis, we conjecture that the support of this novel weighted design is universally optimal.**

***Index Terms*—Equiangular tight frames, line packings, complex projective codes, discrete geometry, manifold optimization, MIMO.**

I. Introduction

Point configurations that maximize the pairwise spherical distances over all point sets of fixed size are called *optimal codes*, reflecting their role in coding theory. Exact solutions to the optimal packing problem are generally known only for small numbers of points and in low dimensions, with the exception of some highly symmetric point sets.

Examples of such highly symmetric spherical sets are the vertices of the icosahedron on $\mathbb{S}^2$ or the minimal vectors of the Leech lattice Λ_{24} on $\mathbb{S}^{23}$. These configurations appear not only as optimizers of the harmonic energy, or maximizing pairwise distances, but also for other energies, such as p-frame energies [A], [BGMPV], [KY2], [KY1], [PSZ], [Y1], [Y2].

For a finite configuration of points on the sphere $\mathcal{C} \subset \mathbb{S}^{d-1}$ (also known as a *code*) the discrete f-potential energies are given by

$$E_f(\mathcal{C}) = \frac{1}{|\mathcal{C}|^2} \sum_{x,y\in\mathcal{C}} f(\langle x, y\rangle). \tag{I.1}$$

Universally optimal point configurations are collections of points $\mathcal{C}$ minimizing the discrete energies E_f among all point sets of fixed cardinality $|\mathcal{C}|$, for all absolutely monotonic functions f on $[-1, 1)$ [CK].

In this paper we detail the results of a numerical study of p-frame energies. These energies are an example of a continuous analog of the discrete energy where instead of point configurations we optimize over measures. Given a kernel function $f \in C[-1, 1]$ and a Borel probability measure $\mu \in \mathcal{P}(\mathbb{S}^{d-1})$, our energy then takes the form

$$I_f(\mu) = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} f(\langle x, y\rangle) d\mu(x) d\mu(y). \tag{I.2}$$

So when we say that a configuration $\mathcal{C}$ minimizes the energy $I_f(\mu)$ we mean a probability measure supported on the configuration minimizes the energy. Taking $f(t) = |t|^p$, $p > 0$, in this equation yields the *p-frame energies*:

$$I_f(\mu) = \int_{\mathbb{S}_{\mathbb{F}}^{d-1}} \int_{\mathbb{S}_{\mathbb{F}}^{d-1}} |\langle x, y\rangle|^p d\mu(x) d\mu(y), \tag{I.3}$$

where $\mathbb{S}_{\mathbb{F}}^{d-1} = \{x \in \mathbb{F}^d : \|x\| = 1\}$ and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

The discrete version of this energy for $p = 2$, known simply as the *frame energy* or *frame potential*, was introduced by Benedetto and Fickus [BeF] and has as minimizers precisely unit norm *tight frames*. These configurations play an important role in signal processing and other branches of applied mathematics.

A finite collection of vectors $\mathcal{C} \subset \mathbb{F}^d$ is a tight frame, if for any $x \in \mathbb{F}^d$, and some constant $A > 0$, one has an analog of Parseval's identity holding for $\mathcal{C}$,

$$\sum_{y\in\mathcal{C}} |\langle x, y\rangle|^2 = A\|x\|^2. \tag{I.4}$$

A phenomenon that is the central focus of this paper is that discrete symmetric objects occur as minimizers of the continuous energy (I.3) over *measures*. This incidentally results in allowing us to make new conclusions about the minimizing configurations of the discrete energies (I.1) for certain values of the cardinality N.

Extensive numerical experiments were conducted in the course of our investigations. The results of these experiments are collected in tables. Unlike the case of tight designs (treated in [BGMPV]), optimal weights for these configurations are generally not equal and thus must be computed for each relevant value of p. Each table gives the minimal support size of a conjectured or known optimal point set: when a configuration on the sphere is origin-symmetric, this minimal support size equals half of the size of the named configuration. For example, the icosahedron has twelve vertices, however 6 vertices on one hemisphere suffice to give a minimizer of the 3-frame energy on $\mathbb{S}^2$. We give additional details for these conjectured minimizers of the p-frame energies. Notably several of these configurations are not universally optimal, and further, several universally optimal configurations are nowhere to be found in this table. We discuss common features of minimizers in the paper.

Our experimental results support the hypothesis that discreteness of minimizers is a general phenomenon when p is not an even integer.

Conjecture 1.1. *In all dimensions $d \geq 2$ and for all $p > 0$ such that $p \notin 2\mathbb{N}$, the minimizing measures of the p-frame energy* (I.3) *are discrete.*

This conjecture is supported by the fact that discreteness of minimizers is known for certain attractive-repulsive potentials on $\mathbb{R}^d$ and Riemannian manifolds [CFP], [Vl].

It is worth noting that the classical paper [Bj] shows that for $F(x, y) = -\|x - y\|^\alpha$ with $\alpha > 2$ and any compact domain $\Omega \subset \mathbb{R}^d$, the energy minimizers are discrete and their support consists of at most $d + 1$ points (just two antipodal points if $\Omega = \mathbb{S}^{d-1}$). Moreover, in [CFP] discreteness has been established for mildly repulsive potentials, i.e. those that behave as $-\|x - y\|^\alpha$ with $\alpha > 2$ when $\|x - y\|$ is small. Observe that for the p-frame potential, we have $|\langle x, y\rangle|^p \approx 1 - \frac{p}{2}\|x - y\|^2$ when $x, y \in \mathbb{S}^{d-1}$ are close, hence the p-frame energy falls into the endpoint case $\alpha = 2$, and, according to the discussion above, this case is more subtle.

While we have yet to establish Conjecture 1.1 and prove discreteness, in our companion paper [BGM+] we show that on $\mathbb{S}^{d-1}$, whenever p is not even, the support of the measure minimizing the p-frame potential necessarily has empty interior.

In addition to the conjectured discreteness of minimizers our initial study gave rise to surprisingly symmetric minimizers for p-frame energies, suggesting that further investigation might give new interesting spherical codes. While nearly all of the minimizing configurations arising from our numerical experiments have appeared before in the coding theory literature, we did however discover a new code in $\mathbb{C}^5$ of 85 vectors which in turn gives a new bound for a minimal sized weighted projective 3-design. We detail a construction of this code and its properties in the paper.

We would like to point out that in many papers, the term *p-frame potential* is usually used to denote the p-frame energy (I.3) or its discrete counterpart. We find the term "energy" to be more appropriate in this context and reserve the term "potential" for the kernel $f(t)$ of the energy I_f.

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