Experimental survey of discrete minimizers of the p-frame energy

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***Abstract*—We detail results from a large computational study of real and complex (weighted) point configurations minimizing p-frame energies, documenting interesting phase transition behavior of minimizers. Using numerical linear programming we give complementary lower bounds to our experimentally obtained upper bounds on minimal energy. We also describe the construction of a highly symmetric weighted design of 85 points, improving the best known lower bounds for a minimal sized weighted design in $\mathbb{CP}^4$. We finally conjecture the support of this weighted design is universally optimal.**

***Index Terms*—Equiangular tight frames, line packings, complex projective codes, discrete geometry, manifold optimization, MIMO.**

I. Introduction

Point configurations that maximize the pairwise spherical distances over all point sets of fixed size are called *optimal codes*, reflecting their role in coding theory. Exact solutions to the optimal packing problem are generally known only for small numbers of points and in low dimensions, with the exception of some highly symmetric point sets.

Examples of such highly symmetric spherical sets are the vertices of the icosahedron on $\mathbb{S}^2$ or the minimal vectors of the Leech lattice Λ_{24} on $\mathbb{S}^{23}$. These configurations appear not only as optimizers of the harmonic energy, or maximizing pairwise distances, but also for other energies, such as p-frame energies [A], [BGMPV], [KY2], [KY1], [**?**], [Y1], [Y2].

For a finite configuration of points on the sphere $\mathcal{C} \subset \mathbb{S}^{d-1}$ (also known as a *code*) the discrete f-potential energies are given by

$$E_f(\mathcal{C}) = \frac{1}{|\mathcal{C}|^2} \sum_{x,y \in \mathcal{C}} f(\langle x, y \rangle). \tag{I.1}$$

In this paper we detail the results of a numerical study of p-frame energies. These energies are an example of a continuous analog of the discrete energy where instead of point configurations we optimize over measures. Given a kernel function $f \in C[-1,1]$ and a Borel probability measure $\mu \in \mathcal{P}(\mathbb{S}^{d-1})$, our energy then takes the form

$$I_f(\mu) = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} f(\langle x, y \rangle) d\mu(x) d\mu(y). \tag{I.2}$$

So when we say that a configuration $\mathcal{C}$ minimizes the energy $I_f(\mu)$ we mean a probability measure supported on the configuration minimizes the energy. Taking $f(t) = |t|^p$, $p > 0$, in this equation yields the *p-frame energies*:

$$I_f(\mu) = \int_{\mathbb{S}_{\mathbb{F}}^{d-1}} \int_{\mathbb{S}_{\mathbb{F}}^{d-1}} |\langle x, y \rangle|^p d\mu(x) d\mu(y), \tag{I.3}$$

where $\mathbb{S}_{\mathbb{F}}^{d-1} = \{x \in \mathbb{F}^d : \|x\| = 1\}$ and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

The discrete version of this energy for $p = 2$, known simply as the *frame energy* or *frame potential*, was introduced by Benedetto and Fickus [BeF] and has as minimizers precisely unit norm *tight frames*. These configurations play an important role in signal processing and other branches of applied mathematics.

A finite collection of vectors $\mathcal{C} \subset \mathbb{F}^d$ is a tight frame, if for any $x \in \mathbb{F}^d$, and some constant $A > 0$, one has an analog of Parseval's identity holding for $\mathcal{C}$,

$$\sum_{y \in \mathcal{C}} |\langle x, y \rangle|^2 = A \|x\|^2. \tag{I.4}$$

A phenomenon that is the central focus of this paper is that discrete symmetric objects occur as minimizers of the continuous energy (I.3) over *measures*. This incidentally results in allowing us to make new conclusions about the minimizing configurations of the discrete energies (I.1) for certain values of the cardinality N.

Extensive numerical experiments were conducted in the course of our investigations. The results of these experiments are collected in Tables III and V for the real case and Tables IV and VI for the complex case. Unlike the case of tight designs (treated in [BGMPV]), optimal weights for these configurations are generally not equal and thus must be computed for each relevant value of p. Each table gives the minimal support size of a conjectured or known optimal point set: when a configuration on the sphere is origin-symmetric, this minimal support size equals half of the size of the named configuration. For example, the icosahedron has twelve vertices, however 6 vertices on one hemisphere suffice to give a minimizer of the 3-frame energy on $\mathbb{S}^2$. We give additional details for these conjectured minimizers of the p-frame energies. Notably several of these configurations are not universally optimal, and further, several universally optimal configurations are nowhere to be found in this table. We discuss common features of minimizers in Section VI.

Our experimental results support the hypothesis that discreteness of minimizers is a general phenomenon when p is not an even integer. In all dimensions $d \geq 2$ and for all $p > 0$ such that $p \notin 2\mathbb{N}$, the minimizing measures of the p-frame energy

(I.3) are discrete. This conjecture is supported by the fact that discreteness of minimizers is known for certain attractive-repulsive potentials on $\mathbb{R}^d$ and Riemannian manifolds [CFP], [Vl].

It is worth noting that the classical paper [Bj] shows that for $F(x,y) = -\|x-y\|^\alpha$ with $\alpha > 2$ and any compact domain $\Omega \subset \mathbb{R}^d$, the energy minimizers are discrete and their support consists of at most $d+1$ points (just two antipodal points if $\Omega = \mathbb{S}^{d-1}$). Moreover, in [CFP] discreteness has been established for mildly repulsive potentials, i.e. those that behave as $-\|x-y\|^\alpha$ with $\alpha > 2$ when $\|x-y\|$ is small. Observe that for the p-frame potential, we have $|\langle x, y\rangle|^p \approx 1 - \frac{p}{2}\|x-y\|^2$ when $x, y \in \mathbb{S}^{d-1}$ are close, hence the p-frame energy falls into the endpoint case $\alpha = 2$, and, according to the discussion above, this case is more subtle.

While we have yet to establish Conjecture I and prove discreteness, in our companion paper [BGM+] we show that on $\mathbb{S}^{d-1}$, whenever p is not even, the support of the measure minimizing the p-frame potential necessarily has empty interior.

In addition to the conjectured discreteness of minimizers our initial study gave rise to surprisingly symmetric minimizers for p-frame energies, suggesting that further investigation might give new interesting spherical codes. While nearly all of the minimizing configurations arising from our numerical experiments have appeared before in the coding theory literature, we did however discover a new code in $\mathbb{C}^5$ of 85 vectors which in turn gives a new bound for a minimal sized weighted projective 3-design. We detail a construction of this code and its properties in Section V-A.

We would like to point out that in many papers, the term *p-frame potential* is usually used to denote the p-frame energy (I.3) or its discrete counterpart. We find the term "energy" to be more appropriate in this context and reserve the term "potential" for the kernel $f(t)$ of the energy I_f.

II. Background

A. Projective Spaces and Jacobi Polynomials

The projective spaces $\mathbb{FP}^{d-1}$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathcal{H}$, are the spaces of lines passing through the origin in $\mathbb{F}^d$,

$$x\mathbb{F} = \{x\lambda \mid \lambda \in \mathbb{F} \setminus \{0\}\}. \tag{II.1}$$

Using this identification, one can associate each element of $\mathbb{FP}^{d-1}$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}$) with a unit vector $x \in \mathbb{F}^d$, and we shall often abuse notation by doing so.

Each of these spaces can be equipped with a geodesic metric ϑ, which takes values in $[0, \pi]$, and a chordal metric, ρ.

Each of the spaces $(\mathbb{FP}^{d-1}, \vartheta)$ is *two-point homogeneous*, meaning that for any $x_1, x_2, y_1, y_2 \in \mathbb{FP}^{d-1}$ such that $\vartheta(x_1, x_2) = \vartheta(y_1, y_2)$ there exists an isometry of $\mathbb{FP}^{d-1}$, mapping x_i to y_i, $i = 1, 2$.

B. Designs

We now discuss some basics of designs on our projective spaces $\mathbb{FP}^{d-1}$. A finite, nonempty set (code) $\mathcal{C} \subset \mathbb{FP}^{d-1}$ with a set of weights $w_{\mathcal{C}} = \{w_x : x \in \mathcal{C}\} \subset [0,1]$, satisfying $\sum_{x\in\mathcal{C}} w_x = 1$, is called a *weighted M-design* if

$$\sum_{x\in\mathcal{C}} w_x C_n^{\alpha,\beta}(\cos(\vartheta(x,y))) = \int_\Omega C_n^{\alpha,\beta}(\cos(\vartheta(x,y)))\, d\sigma(x) = 0 \tag{II.2}$$

for all $n \in \{1, ..., M\}$. When the weights are all the same, $w_x = \frac{1}{|\mathcal{C}|}$, then these are simply referred to as M-designs. The *strength* of a (weighted) design is the maximum value of M for which identity (II.2) holds.

Any k design $\mathcal{C}$ corresponds to a minimizer of the $2k$-frame energy $\mu_{\mathcal{C},w_{\mathcal{C}}} = \sum_{x\in\mathcal{C}} w_x \delta_x$, though these are not the only discrete minimizers.

A weighted M-design is called *tight* if its cardinality meets the lower bound, and in such cases, the weights must all be equal (i.e. weighted tight designs are simply designs) [T], [Le2].

For the all projective spaces, the vertices of a cross-polytope (i.e. an orthonormal basis in the projective space) always provide a tight 1-design. Tight M-designs on real projective spaces correspond to (symmetric) tight $(2M+1)$-designs on the real unit spheres. In the complex setting, tight 2-designs, also known as *symmetric, informationally complete, positive operator-valued measures (SIC-POVMs)*, are known to exist for $d \le 16$, $d = 19, 24, 28, 35, 48$, and numerical experiments suggest that they may exist in every dimension [ABBEGL], [RBSC], [SG], [Z]. Explicit constructions of the remaining designs in Table VII are given in [H1], [CKM]. In all three settings, it is known that no tight M-designs exist whenever $M \ge 4$ and $d \ge 3$, except for the Leech Lattice on $\mathbb{RP}^{23}$ [BD1], [BD2], [BH], [H2], [L].

C. Linear Programming

Both our numerical methods here as well as the theoretical methods in [BGMPV] to determine optima of the p-frame energies make use of linear programming. Our application of the method can be summed up in the following lemma, which is a measure-theoretic counterpart of the linear programming bound of Delsarte and Yudin [De], [Y1].

Lemma 2.1. *Let $h \in C[-1,1]$ be a positive definite function, i.e. $h(t) = \sum_{n=0}^{\infty} \widehat{h}_n C_n^{\alpha,\beta}(t)$ and $\widehat{h}_n \ge 0$ for all $n \ge 0$. (i)*

(i) *If $h(t) \le f(t)$ for all $t \in [-1,1]$, then for any $\mu \in \mathcal{P}(\mathbb{FP}^{d-1})$,*
$$I_f(\mu) \ge I_h(\mu) \ge I_h(\sigma) = \widehat{h}_0.$$

(ii) *Assume further that h is a polynomial of degree k and that there exists a weighted k-design $\mathcal{C} \subset \mathbb{FP}^{d-1}$, with weights w_x, such that $h(t) = f(t)$ for each $t \in \{\cos(\vartheta(x,y)) : x, y \in \mathcal{C}\}$. Then for any $\mu \in \mathcal{P}(\mathbb{FP}^{d-1})$,*
$$I_f(\mu) \ge I_f\left(\sum_{x\in\mathcal{C}} w_x \delta_x\right).$$

In [BGMPV], we constructed positive definite polynomials as Hermite interpolants of the p-frame potentials at the points of $\{\cos(\vartheta(x,y)) : x, y \in \mathcal{C}\}$ for tight designs $\mathcal{C}$, and used them as our h in 2 in order to show optimality of such configurations. The requirements of equality of f and h on $\{\cos(\vartheta(x,y)) : x, y \in \mathcal{C}\}$, the positive definiteness of h, and the constraint on the degree of h limits how much a method could be used outside of tight designs. However, with 1, we can determine bounds on the p-frame energy by bounding from below by continuous positive definite functions, generally using positive definite polynomials of bounded degree, and optimizing over $\widehat{h}_0$, as we will discuss below.

III. Numerical LP bounds

If a suitable candidate is not available, one can still rely on part (1) of Lemma 2.1 and attempt to optimize the value of the energy $I_h(\sigma)$ over auxiliary positive definite polynomials h, obtaining a lower bound for the energy over all probability measures. If the degree of an auxiliary function h is bounded by D, we have $D+1$ non-negative variables $\widehat{h}_i$, $0 \le i \le D$, and infinitely many linear constraints $h(t) \le f(t)$ for all $t \in [-1, 1]$. In order to get the best possible lower bound, we need to maximize $\widehat{h}_0$ given these linear conditions.

This problem is, generally, intractable as a linear optimization problem. However, when f is a polynomial, the condition $f(t) - h(t) \ge 0$ for all $t \in [-1, 1]$ may be represented as a finite-size positive semi-definite constraint on the coefficients $\widehat{h}_i$. In particular, the polynomial inequality may be rewritten as a sum-of-squares optimization problem (see, for instance, [N]) and thus solved as a semi-definite program.

By using sum-of-squares optimization described above, we obtain lower bounds on the p-frame energies over measures on projective spaces when p is an odd integer. A table of such bounds for real projective spaces $\mathbb{RP}^{d-1}$, $3 \le d \le 24$, and $p = 3, 5, 7$, is shown in Table VIII in the Appendix. The concrete bounds are computed by a series of steps. For the first step, we fix the degree D of the auxiliary polynomial and solve the sum-of-squares problem. The numerical solver outputs a polynomial which is feasible up to a small tolerance. By rounding coefficients, it is then possible to obtain polynomials which are less than f and positive definite.

Since the choice of the maximal degree D is arbitrary, not much is lost by rounding, and our bounds in the appendix are thus rounded down to four significant figures. The last condition $f - h \ge 0$ can be checked using interval arithmetic, or by hand.

It is interesting to compare the values of conjectured energy minimizers with the lower bounds obtained using the approach above. We make comparison of these bounds in Table I below for all conjectured optimizers from Tables III,IV,V, and VI: observe that the values are indeed close, which motivates our conjectures about the minimizers. Tight designs are excluded from this table since for them the lower and the upper bounds coincide.

A. Other weighted designs

1) 11 points in $\mathbb{R}^3$: It seems that as p goes to 6 from below, the limiting minimizing configuration on the sphere $\mathbb{S}^2$ is of the following form. Concisely, the system consists of all combinations of signs of the 6 vectors below,

TABLE I: Comparison of p-frame energies for conjectured optimal configurations on $\mathbb{RP}^{d-1}$ and $\mathbb{CP}^{d-1}$ with LP lower bounds. Energies are evaluated at the odd integer midpoint of the conjectured optimality interval.

d	$\mathbb{F}$	Energy	LP bound	p	Name
3	$\mathbb{R}$	0.1249	0.1248	7	icosahedron and dodecahedron
4	$\mathbb{R}$	0.09628	0.09607	5	D_4 root vectors
5	$\mathbb{R}$	0.1183	0.1170	3	hemicube
5	$\mathbb{R}$	0.06184	0.06169	5	Stroud design
6	$\mathbb{R}$	0.09056	0.08970	3	cross-polytope and hemicube
6	$\mathbb{R}$	0.04249	0.04240	5	E_6 and E_6^* roots
7	$\mathbb{R}$	0.03065	0.03060	5	E_7 and E_7^* roots
8	$\mathbb{R}$	0.05910	0.05852	3	mid-edges of regular simplex
3	$\mathbb{C}$	0.01261	0.01258	5	union equiangular lines
5	$\mathbb{C}$	0.04200	0.04184	5	O_{10} and $W(K_5)$ minimal vectors

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{2}{\sqrt{7}} & \sqrt{\frac{3}{7}} & 0 \\ \frac{2}{\sqrt{7}} & 0 & \sqrt{\frac{3}{7}} \\ \sqrt{\frac{1}{7}} & \sqrt{\frac{3}{7}} & \sqrt{\frac{3}{7}} \end{bmatrix}$$

with the weights,

$$\frac{2}{27}, \frac{1}{10}, \frac{1}{10}, \frac{49}{540}, \frac{49}{540}, \frac{49}{540}$$

on each line. The off-diagonal inner products are then

$$1/7, -1/7, 5/7, -5/7, \sqrt{3/7}, -\sqrt{3/7}, 0, \sqrt{1/7}, -\sqrt{1/7},$$
$$4/7, -4/7, \sqrt{4/7}, -\sqrt{4/7}$$

appearing in number, $(10, 18, 10, 10, 14, 10, 14, 6, 2, 4, 4, 6, 2)$ respectively. From these facts, one may check that the 11 lines defined by these vectors forms a projective 3-design. Notably, this is the same extremal code, which forms a minimal cubature formula and is found also in [Rez, page 135].

2) 16 points in $\mathbb{R}^3$: Lines through antipodal points in the union of a regular icosahedron with its dual dodecahedron. The frequencies of absolute values of inner products are $N(\sqrt{\frac{1}{15}(5-2\sqrt{5})}) = 60$, $N(\frac{\sqrt{75+30\sqrt{5}}}{15}) = 60$, $N(\frac{1}{3}) = 60$, $N(\frac{1}{\sqrt{5}}) = 30$, $N(\sqrt{\frac{5}{9}}) = 30$, and $N(1) = 60$. The weights making this configuration a projective 4-design are $\omega_1 = 5/84$ and $\omega_2 = 9/140$ for the icosahedron and dodecahedron vertices respectively.

3) 11 points in $\mathbb{R}^4$: See Table II for what appears to be the limiting minimizing configuration as p goes to 6 from below when minimizing over $\mathbb{S}^3$.

4) 24 points in $\mathbb{R}^4$: The regular 24 cell, or alternatively the D_4 root system. The frequencies of absolute values of inner products are $N(0) = 216$, $N(\frac{1}{\sqrt{2}}) = 144$, $N(\frac{1}{2}) = 192$, and $N(1) = 24$. The configuration is unweighted as a projective 3-design.

TABLE II: The Gram matrix of the weighted projective 2-design in $\mathbb{RP}^3$ which appears as a minimizer as $p \to 4^-$ along with ordered weights, with each weight corresponding to the vector with inner products given in the adjacent row. In the matrix, a and b are $\frac{\sqrt{5}+1}{6}$ and $\frac{1}{6}\sqrt{(6-2\sqrt{5})}$, respectively.

$$\left[\begin{array}{ccccccccccc|c}
1 & -\frac{2}{3} & a & a & a & a & b & b & b & b & \frac{\sqrt{6}}{6} & \frac{3}{40} \\
-\frac{2}{3} & 1 & -b & -b & -b & -b & -a & -a & -a & -a & \frac{\sqrt{6}}{6} & \frac{3}{40} \\
a & -b & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{6}}{6} & \frac{3}{32} \\
a & -b & \frac{1}{3} & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & \frac{\sqrt{6}}{6} & \frac{3}{32} \\
a & -b & \frac{1}{3} & -\frac{1}{3} & 1 & \frac{1}{3} & -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{6}}{6} & \frac{3}{32} \\
a & -b & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & \frac{\sqrt{6}}{6} & \frac{3}{32} \\
b & -a & -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{\sqrt{6}}{6} & \frac{3}{32} \\
b & -a & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{1}{3} & 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{\sqrt{6}}{6} & \frac{3}{32} \\
b & -a & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & \frac{1}{3} & -\frac{1}{3} & 1 & \frac{1}{3} & -\frac{\sqrt{6}}{6} & \frac{3}{32} \\
b & -a & \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 & -\frac{\sqrt{6}}{6} & \frac{3}{32} \\
\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & 1 & \frac{1}{10}
\end{array}\right]$$

5) 16 points in $\mathbb{R}^5$: Lines through antipodal points in the following construction. Take all permutations of $\pm\frac{1}{\sqrt{30}}(-5,1,1,1,1,1,1)$ and $\frac{1}{\sqrt{6}}(1,1,1,-1,-1,-1)$ and consider these as vectors in the copy of $\mathbb{S}^4$ in $\mathbb{S}^5$ on the plane perpendicular to $(1,1,1,1,1,1)$. The frequencies of absolute values of inner products are $N(\frac{1}{3}) = 90$, $N(\frac{1}{5}) = 30$, $N(\frac{1}{\sqrt{5}}) = 120$, and $N(1) = 16$. The weights making this a projective 2-design are $\omega_1 = \frac{5}{84}$ and $\omega_2 = \frac{9}{140}$ for the above parts respectively.

6) 41 points in $\mathbb{R}^5$: An example of a design construction appearing in [Str]. The configuration comprises of lines through antipodal points in the following construction. Let A be the set of vectors which are permutations of $(\pm 1,0,0,0,0)$, B permutations of $(\pm\sqrt{\frac{1}{2}}, \pm\sqrt{\frac{1}{2}}, 0,0,0)$, and C permutations of $(\pm\sqrt{\frac{1}{5}}, \pm\sqrt{\frac{1}{5}}, \pm\sqrt{\frac{1}{5}}, \pm\sqrt{\frac{1}{5}}, \pm\sqrt{\frac{1}{5}})$. The frequencies of absolute values of inner products are $N(0) = 600$, $N(\frac{1}{5}) = 160$, $N(\frac{3}{5}) = 80$, $N(\sqrt{\frac{1}{5}}$, $N(\sqrt{\frac{2}{5}}) = 320$, and $N(1) = 41$. The weights making this a projective 3-design are $\omega_1 = \frac{2}{105}$, $\omega_2 = \frac{8}{315}$, and $\omega_3 = \frac{25}{1008}$, on A, B, and C respectively.

7) 22 points in $\mathbb{R}^6$: Lines through antipodal points in a hemicube/cross polytope compound, where the hemicube is within the cube dual to the cross polytope. The frequencies of absolute values of inner products are $N(0) = 30$, $N(\frac{1}{\sqrt{6}}) = 192$, $N(\frac{1}{3}) = 240$, and $N(1) = 22$. The weights making this a projective 2-design are $\omega_1 = 3/64$ on the hemicube and $\omega_2 = 1/24$ on the cross-polytope.

8) 63 points in $\mathbb{R}^6$: Lines through antipodal points in the union of minimal vectors of E_6 and its dual lattice, E_6^*. The frequencies of absolute values of inner products are $N(0) = 1620$, $N(\frac{1}{4}) = 432$, $N(\frac{1}{2}) = 990$, $N(\sqrt{\frac{3}{8}}) = 864$, and $N(1) = 63$. The weights making this a projective 3-design are $\omega_1 = 1/60$ and $\omega_2 = 2/135$ on the minimal vectors of E_6 and its dual, respectively.

9) 91 points in $\mathbb{R}^7$: The configuration is projectively composed of the union of the minimal vectors of E_7 and its dual lattice, E_7^*. The frequencies of absolute values of inner products are $N(0) = 3906$, $N(\frac{1}{27}) = 756$, $N(\frac{1}{8}) = 2016$, $N(\frac{\sqrt{3}}{9}) = 1512$, and $N(1) = 91$. The weights making this a projective 3-design are $\omega_1 = 8/693$ and $\omega_2 = 3/308$ on the E_7 part and its dual, respectively. The cubature formula appears also in [NoS].

10) 36 points in $\mathbb{R}^8$: The edge midpoints of a regular simplex. The frequencies of absolute values of inner products are $N(\frac{2}{7}) = 756$, $N(\frac{5}{14}) = 504$, and $N(1) = 36$. This code is a projective 1-design with equal weights.

11) 21 points in $\mathbb{C}^3$: A structured union of a maximal (tight) simplex (equiangular tight frame, or ETF) of 9 vectors and 4 mutually unbiased bases (a 4-MUB) of 12 vectors. The frequencies of absolute values of inner products are $N(0) = 96$, $N(\frac{1}{2}) = 72$, $N(\frac{1}{\sqrt{3}}) = 108$, $N(\frac{1}{\sqrt{2}}) = 144$, $N(1) = 21$. The weights making this a projective 3-design are $\omega_1 = 4/90$ on the 9-ETF and $\omega_2 = \frac{1}{20}$ on the 4-MUB.

IV. Proposed Method

We give additional details on how we made the conjectures found in Tables III, IV, V, and VI. The numerical method employed to find conjectured minimizers involved two steps. Initially we used conjugate gradient method to minimize energies. Afterwards we implemented an arbitrary precision library with a second order method, Limited Memory Broyden-Fletcher-Goldfarb-Shanno algorithm (L-BFGS) [**?**] to check our conjectures and test endpoint behavior. L-BFGS stores a modified version of the Hessian to avoid prohibitive memory storage costs.

Algorithm 1 L-BFGS [NW][Alg. 7.5]

Choose starting point x_0, integer $m > 0$;
$k \leftarrow 0$;
repeat
  choose H_k^0 (†)
  compute $p_k \leftarrow -H_k \nabla f_k$ (††)
  (using two-loop recursion, alg. 7.4 in [NW])
  compute $x_{k+1} \leftarrow x_k + \alpha_k p_k$
  (where α_k is chosen to satisfy the Wolfe conditions)
if $k > m$ **then**
  Discard the vector pair $\{s_{k-m}, y_{k-m}\}$ from storage;
  Compute and save $s_k \leftarrow x_{k+1} - x_k$, $y_k = \nabla f_{k+1} - \nabla f_k$;
end if
until convergence.
(†): $H_k^0 = \gamma_k I$, $\gamma_k = s_{k-1}^T y_{k-1} / y_{k-1}^T y_{k-1}$ scaling factor
(††): H_k inverse hessian approximation

We note that the Wolfe conditions in L-BFGS are given below

$$f(x_k + \alpha_k p_k) \leq f(x_k) + c_1 \alpha_k \nabla f_k^T p_k,$$

$$\nabla f(x_k + \alpha_k p_k)^T \geq c_2 \nabla f_k^T p_k,$$

with $0 < c_1 < c_2 < 1$. The first condition gives that α_k gives sufficient decrease in the objective function while the second is a curvature condition.

TABLE III: Dimension and support size for optimal and conjectured optimal configurations for p-frame energies on $\mathbb{RP}^{d-1}$. The energies are evaluated at odd integers.

d	N	Energy	p
2	N	$(*)$	$2N-3$
d	d	$1/d$	1
3	6	0.241202265916660	3
3	11	0.142857142857143	6−
3	16	0.124867143799450	7
4	11	0.125000000000000	4−
4	24	0.096277507157493	5
4	60	0.047015486159502	9
5	16	0.118257675970387	3
5	41	0.061838820473855	5
6	22	0.090559619406078	3
6	63	0.042488105634495	5
7	28	0.071428571428571	3
7	91	0.030645893660944	5
8	36	0.059098639455782	3
8	120	0.022916666666667	5
23	276	0.011594202898551	3
23	2300	0.002028985507246	5
24	98280	0.000103419439357	9

TABLE IV: Dimension and support size for optimal and conjectured optimal configurations for p-frame energies on $\mathbb{CP}^{d-1}$. The energies are evaluated at odd integers.

d	N	Energy	p
d	d	$1/d$	1
3	9	0.222222222222223	3
3	21	0.012610934678518	5
4	16	0.146352549156242	3
4	40	0.068301270189222	5
5	25	0.105319726474218	3
5	85	0.041997097378053	5
6	36	0.080272843473504	3
6	126	0.027777777777778	5
d	d^2	$\frac{1+(d^2-1)(1/(d+1))^{3/2}}{d^2}$	3

TABLE V: Optimal and conjectured optimal configurations for p-frame energies on $\mathbb{RP}^{d-1}$. Energies are evaluated in most cases at the odd integer which is the midpoint of the interval given. The range $q-$ configurations are obtained as limiting configurations as p tends to q from below. For these configurations, the energy is evaluated for the even limit value. Among the configurations which are not tight, the 600-cell is the only configuration which is proved to be optimal in the table. The corresponding dimension and support size of the optimizers appear in Table III.

Range of p	Tight	Name
$[2N-4, 2N-2]$	t	regular $2N$-gon
$[0, 2]$	t	orthonormal basis
$[2, 4]$	t	icosahedron
6−		Reznick design
$[6, 8]$		icosahedron and dodecahedron
4−		small weighted design
$[4, 6]$		D_4 root vectors
$[8, 10]$		600-cell
$[2, 4]$		hemicube
$[4, 6]$		Stroud design
$[2, 4]$		cross-polytope and hemicube
$[4, 6]$		E_6 and E_6^* roots
$[2, 4]$	t	kissing E_8
$[4, 6]$		E_7 and E_7^* roots
3		mid-edges of regular simplex
$[4, 6]$	t	E_8 roots
$[2, 4]$	t	equiangular lines
$[4, 6]$	t	kissing Leech lattice
$[8, 10]$	t	Leech lattice minimal vectors

TABLE VI: Optimal and conjectured optimal configurations for p-frame energies on $\mathbb{CP}^{d-1}$. Similar to the real table the corresponding dimension and support size of the optimizers appear in Table IV.

Range of p	Tight	Name
$[0, 2]$	t	orthonormal basis
$[2, 4]$	t	SIC-POVM
$[4, 6]$		union equiangular lines
$[2, 4]$	t	SIC-POVM
$[4, 6]$	t	Eisenstein structure on E_8
$[2, 4]$	t	SIC-POVM
$[4, 6]$		O_{10} and $W(K_5)$ minimal vectors
$[2, 4]$	t	SIC-POVM
$[4, 6]$	t	Eisenstein structure on K_{12}
$[2, 4]$	t	SIC-POVM (conjectured)

V. Experimental Data and Results

Table VIII collects linear programming lower bounds corresponding to small values of d and odd values p for the p-frame energy on $\mathbb{S}^{d-1}$.

A. New small weighted projective design

We now collect facts on the 85 vector system which was found while numerically minimizing the $p = 5$ frame potential in $\mathbb{C}^5$. This system of vectors forms a weighted design of strength 3, or equivalently, for the functional $\sum_{i,j} |\langle v_i, v_j\rangle|^6 \omega_i\omega_j$, the weighted system takes the value $1/35$, thus minimizing this quantity over all probability measures $\mu = \sum_i \delta_{v_i}\omega_i$, $\sum_i \omega_i = 1$ supported on unit vectors $\|v_i\| = 1$ in $\mathbb{C}^5$ [We]. The above construction appears to be new especially when comparing its size to previously obtained bounds from [LS] for smallest known 3 weighted designs in $\mathbb{C}^5$.

One part of the system is well studied, given by the root vectors corresponding to the 45 2-reflections which generate the unitary reflection group $W(K_5)$ of 51840 elements [LT]. This group is alternatively described as the group $G_3(10) \simeq (C_6 \times SU_4(2)) : C_2$, one of the maximal finite irreducible subgroups of $GL_{10}(\mathbb{Z})$ [So]. $SU_4(2)$ here is just the special linear group of 4×4 matrices, unitary matrices over $\mathbb{F}_{2^2}$, with determinant one.

Choosing the representation of the root vectors in $W(K_5)$ as $X_1 = \{\sigma((1,0,0,0,0))\} \cup \{\sigma(\frac{1}{2}(0,1,\pm\omega,\pm\omega,\pm1))\}$ under cyclic coordinate permutations, σ, the new weighted design arises when this system is joined with some other 40 vectors. The second system may be described as $\Psi = \{\sigma(\frac{1}{\sqrt{3}}(1,0,\pm\omega,\pm\omega,0))\} \cup \{\sigma(\frac{1}{\sqrt{3}}(1,\pm\omega,\pm1,0,0))\}$ also generated under cyclic coordinate permutations. The projective

TABLE VII: A list of parameters for which projective tight designs are known to exist (besides designs in $\mathbb{FP}^1$ for $\mathbb{F} \neq \mathbb{R}$). Here M denotes the strength of the design, d the dimension of the ambient space $\mathbb{F}^d$, and N is the size of the design. For SIC-POVMs, these configurations exist for certain values of d, and may or may not exist for all values.

d	N	M	$\mathbb{F}$	Name
d	$d+1$	1	$\mathbb{R}$	cross-polytope/ONB
2	N	$N-1$	$\mathbb{R}$	regular $2N$-gon
3	6	2	$\mathbb{R}$	icosahedron
7	28	2	$\mathbb{R}$	kissing configuration for E_8
8	120	3	$\mathbb{R}$	roots of E_8 lattice
23	276	2	$\mathbb{R}$	equiangular lines
23	2300	3	$\mathbb{R}$	kissing configuration for Λ_{24}
24	98280	5	$\mathbb{R}$	minimal vectors of Λ_{24}
d	$d+1$	1	$\mathbb{C}$	cross-polytope/ONB
d	d^2	2	$\mathbb{C}$	SIC-POVM
4	40	3	$\mathbb{C}$	Eisenstein structure on E_8
6	126	3	$\mathbb{C}$	Eisenstein structure on K_{12}

TABLE VIII: Numeric linear programming lower bounds for odd-valued p-frame energies.

d	$p=3$	$p=5$	$p=7$
3	0.2412	0.1655	0.1248
4	0.1612	0.09607	0.06454
5	0.1170	0.06169	0.03740
6	0.08970	0.04240	0.02344
7	0.07142	0.03060	0.01556
8	0.05852	0.02291	0.01080
9	0.04902	0.01770	0.007768
10	0.04180	0.01401	0.005750
11	0.03616	0.01131	0.004360
12	0.03166	0.009290	0.003375
13	0.02801	0.007737	0.002658
14	0.02499	0.006524	0.002125
15	0.02248	0.005561	0.001721
16	0.02035	0.004785	0.001413
17	0.01853	0.004152	0.001171
18	0.01696	0.003630	0.0009813
19	0.01559	0.003195	0.0008280
20	0.01440	0.002830	0.0007054
21	0.01335	0.002520	0.0006047
22	0.01242	0.002256	0.0005217
23	0.01159	0.002028	0.0004529
24	0.01085	0.001832	0.0003952

design is finally given by assigning weights to the $W(K_5)$ system joined with the 40 vector system after giving Ψ the orientation $X_2 = U\Psi$, where

$$U = \frac{1}{2}\begin{bmatrix} 1 & -\omega & -\omega & 1 & 0 \\ -1 & 1 & -\omega^2 & 0 & -\omega^2 \\ \omega^2 & 0 & -\omega^2 & 1 & 1 \\ 0 & 1 & \omega & -\omega & -1 \\ \omega^2 & \omega & 0 & -\omega & \omega^2 \end{bmatrix}, \tag{V.1}$$

is unitary ($\omega = e^{2\pi i/3}$). With the above orientation the 40 points in X_2 appear to fit so that each point is a maximizer of the projective distance from each of the 45 vectors in the $W(K_5)$ system and vice versa. If so, the additional 40 points satisfy that they are the points at greatest distance from the original 45, in particular.

To form a weighted 3-design, the corresponding weights for X_1, the 45 vector system, are $\omega_1 = \frac{4}{315}$, and for the remaining

TABLE IX: Table of inner products between vectors in parts X_1, X_2 of the new cubature formula of 85-vectors. N counts the number of times a value occurs as an entry in $|X_i'X_j|$, $i, j = 1, 2$.

	$\lvert\langle x, y\rangle\rvert$	N
$\lvert X_1'X_1\rvert$	$0, 1/2, 1$	$540, 1440, 45$
$\lvert X_2'X_2\rvert$	$1/3, 1/\sqrt{3}, 1$	$1080, 480, 40$
$\lvert X_1'X_2\rvert$	$0, 1/\sqrt{3}$	$720, 1080$
$\lvert X_2'X_1\rvert$	$0, 1/\sqrt{3}$	$720, 1080$

40 vectors in X_2, the weights are $\omega_2 = \frac{3}{280}$. In total the distribution of absolute values of inner products that appears in the unweighted 85 vector system is given in Table IX.

The above construction hides the relation between its two parts. The 85 vectors in $\mathbb{C}^5$ may be seen, after canonically embedding the vectors in $\mathbb{R}^{10}$, as the weighted union of vectors coming from two 10 dimensional lattices. Under this identification, the 45 vectors in the $W(K_5)$ system may be selected as, up to projective equivalence (modulo multiples of sixth roots of unity), the 270 minimal vectors of the lattice called $(C_6 \times SU_4(2)) : C_2$ in the database [NS], and the other 40 points are taken one from each antipodal pair of the 80 minimal vectors of the shorter Coxeter-Todd lattice, O_{10} detailed in [RS]. The relationship between these two lattices is that $(C_6 \times SU_4(2)) : C_2$ is similar to the maximal even sub-lattice of O_{10}. In our tables, we choose to name these the $W(K_5)$ and O_{10} lattices. We prefer an alternative name for the first since the automorphism group of each lattice is $(C_6 \times SU_4(2)) : C_2$.

Altogether, upon splitting the weights across minimal vectors in appropriately scaled and oriented copies of these lattices and then complexifying everything, one arrives at the cubature formula, which when viewed projectively, is a system of 85 vectors improving on the best previous known bound of size 320 for such a formula (see [Sh]). Some experiments suggest this might be the smallest sized weighted projective 3-design in $\mathbb{CP}^4$. Expecting that this code might be optimal in a few other settings, we conjecture:

The code constructed in this section of 85 points in $\mathbb{C}^5$ is universally optimal.

This is an example of one of the 'highly symmetric tight frames', as was later demonstrated in [MW].

VI. Further remarks

We have many remaining questions about the p-frame energies, and many curiosities were brought to our attention through our numerical study. One immediate question concerns uniqueness of the 600-cell as a minimizer for $\mathbb{RP}^3$ and $p \in (8, 10)$, which we expect to hold. Note that tight designs, generally, are not unique (not even up to unitary equivalence). It is interesting whether it is more often the case that infinite families arise or that such configurations are isolated, as is known to be the case when $d = 2$ [Z].

An interesting observation is that some configurations minimize p-frame energies for a range of p (the 600-cell for example), while others, like the $p = 3$ minimizer in $\mathbb{RP}^7$, do not minimize on an entire range between even integers. When minimizers have the same support for a range $p \in (2k-2, 2k)$, it indicates that the supporting configuration has to be a weighted k-design.

This suggests another phenomenon similar to the notion of universal optimality, and we are tempted to conjecture that in the real case for $d > 2$ there are only finitely many configurations which optimize the p-frame energy on a whole range of $p \in [2k-2, 2k]$.

Looking at the tables, one can note that as the value of p increases, for p not even, the support size of a candidate appears to be monotonically increasing. Further, for a fixed dimension, the support size seems to grow polynomially in p. We do not have an explanation for this phenomenon.

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