Elektronik

Basic Components and Circuit Theory 4

EITA10

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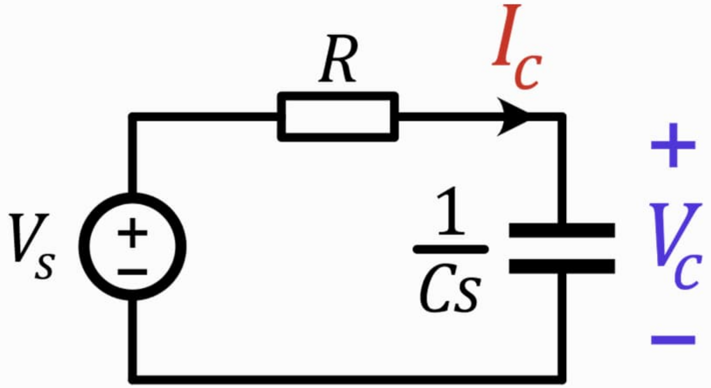
Today we learn

- Initial and final values
- Step Response of RC circuits
- Maximum operating frequency
- Limitations of circuit theory

First-order RC Circuits

- **Frequency-domain analysis**
  - Simple voltage division if we consider impedances:



$$V_c(s) = \frac{Z_c}{Z_c + Z_R} V_s(s) = \frac{\frac{1}{Cs}}{\frac{1}{Cs} + R} V_s(s) = \frac{1}{1 + RCs} V_s(s)$$

  - Impulse response ($V_s(s) = 1$):

$$\boxed{H(s) = \frac{1}{1 + \tau s}}$$

The same result

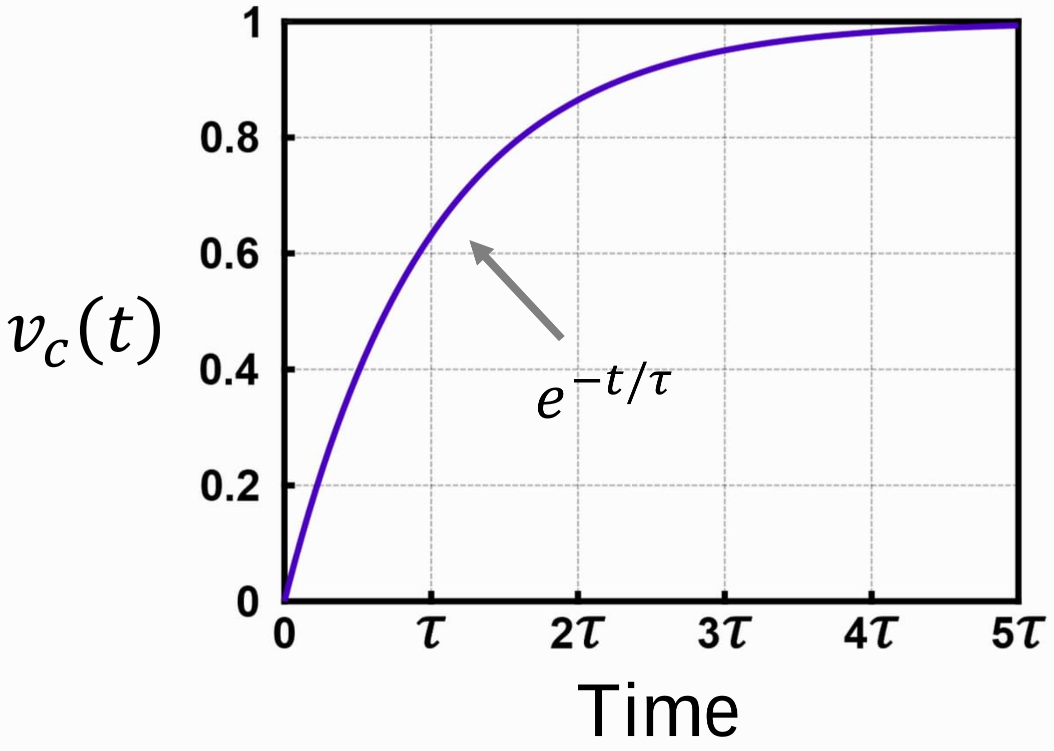
First-order RC Circuits

- **Step Response**

$$\boxed{v_c(t) = \left(1 - \exp\left(-\frac{t}{\tau}\right)\right) u(t)}$$

Initial value: $v_c(0) = 0$

Final value: $v_c(\infty) = 1$



First-order RC Circuits

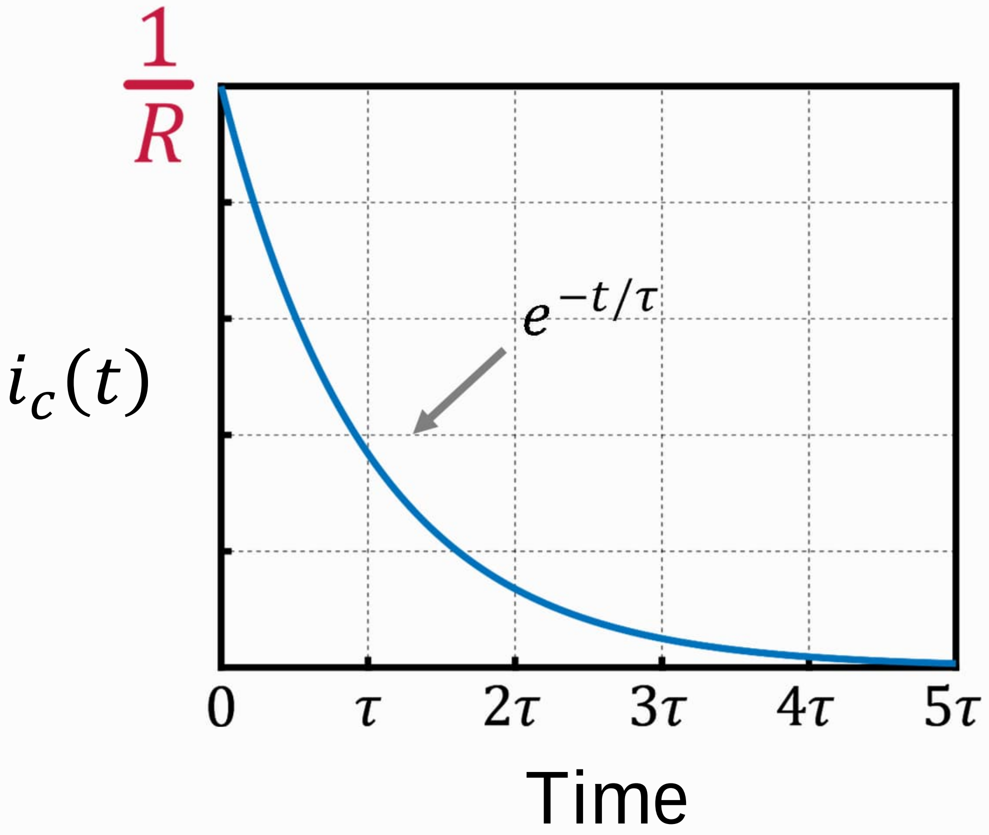
- **Current's Step Response**

$$i_c(t) = c\frac{dv_c}{dt}$$

$$\boxed{i_c(t) = \frac{1}{R}\exp\left(-\frac{t}{\tau}\right)u(t)}$$

Initial value: $i_c(0) = \frac{1}{R}$

Final value: $i_c(\infty) = 0$



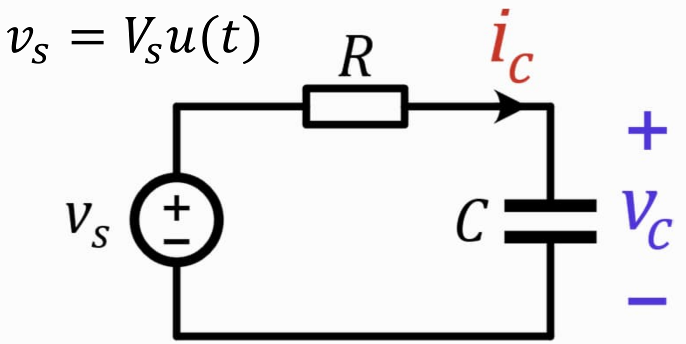
Capacitor's Model when $t \to 0^+$

- Capacitor's **voltage** must be **continuous**.

$$i_c(t) = c\frac{dv_c}{dt} \Rightarrow if\ v_c = \Delta V u(t): \quad i_c(t) \to \infty$$

Physically Impossible

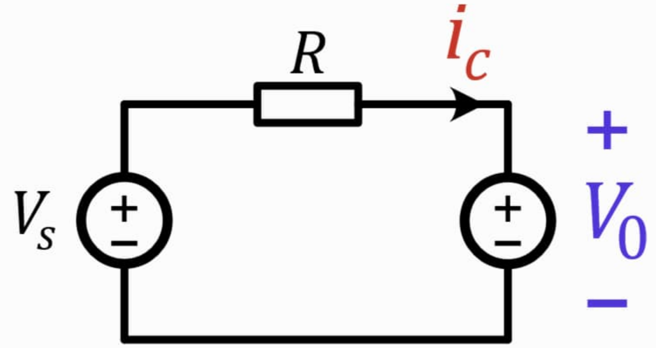
$$\Rightarrow v_c(0^+) = v_c(0^-) = V_0$$



- What about current?

$$i_c(0^+) = \frac{V_s - V_0}{R}$$

Capacitor acts like a voltage source, showing zero impedance

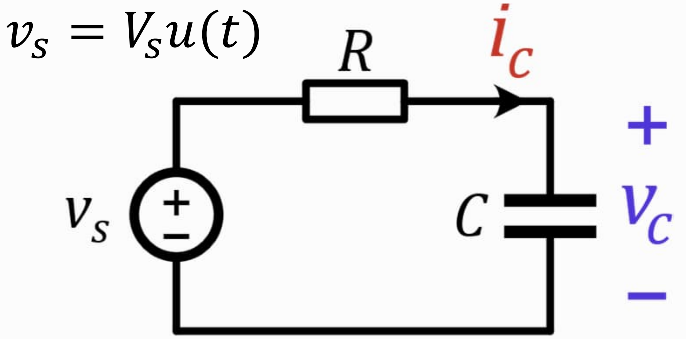


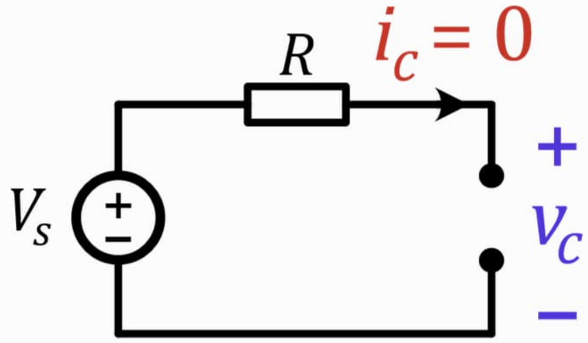
Capacitor’s Model when $t \to \infty$

- After a sufficiently long time ($t > 5\tau$), the voltage across a capacitor reaches a stable state.
- This duration is called **settling time**.
- **No current** flows into the circuit anymore.

$$\frac{dv_c}{dt}\Big|_{t\to\infty} = 0 \;\Rightarrow i_c(\infty) = 0, v_c(\infty) = V_s$$

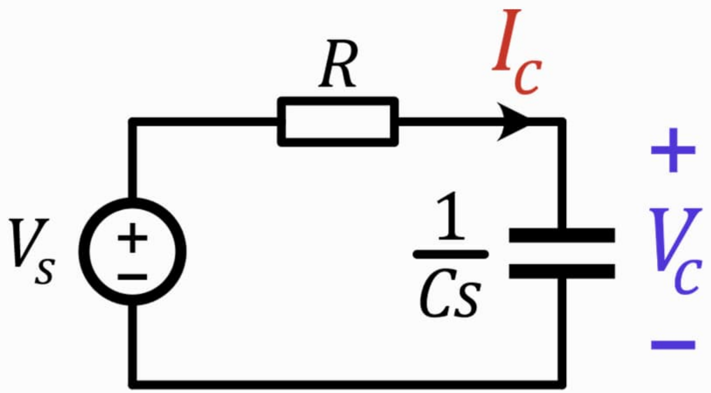
Capacitor acts like an open-circuit, showing infinite impedance





Step Response of RC Circuits

- Only three parameters need to be calculated:
  - **Time constant ($\tau = RC$)**
  - **Initial value ($x(0^+)$)**
  - **Final value ($x(\infty)$)**

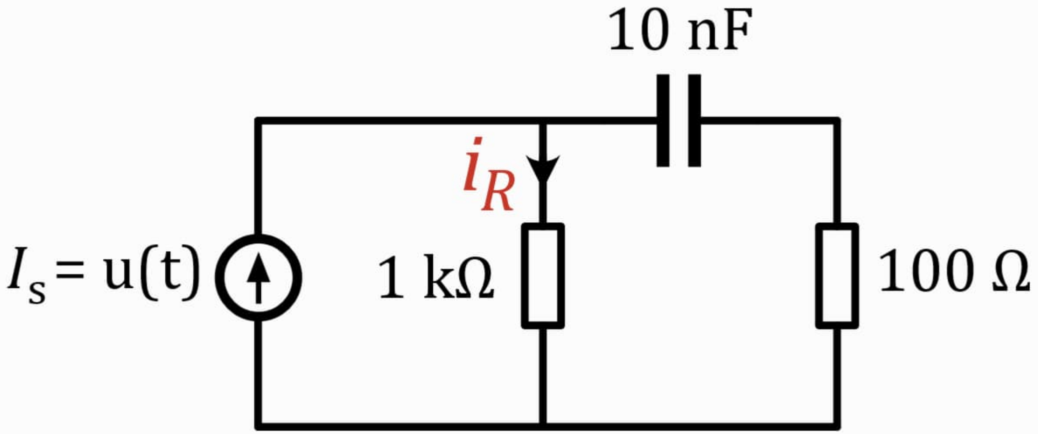


$$\boxed{x(t) = x(\infty) + \left(x(0^+) - x(\infty)\right) \exp\left(-\frac{t}{\tau}\right)}$$

$x(t)$ can be any voltage or current

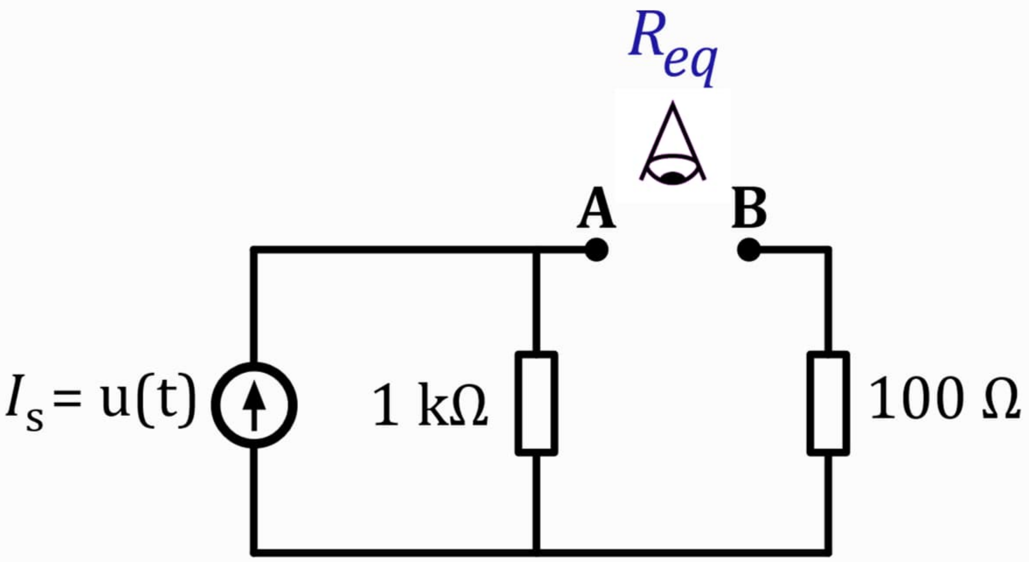
Example 1

- Find $i_R(t)$.
    - IC: $v_c(0^-) = 0$



Example 1 – step 1: calculating τ

- We employ **Thevenin's theorem** to find the equivalent resistance seen from the terminals A and B.

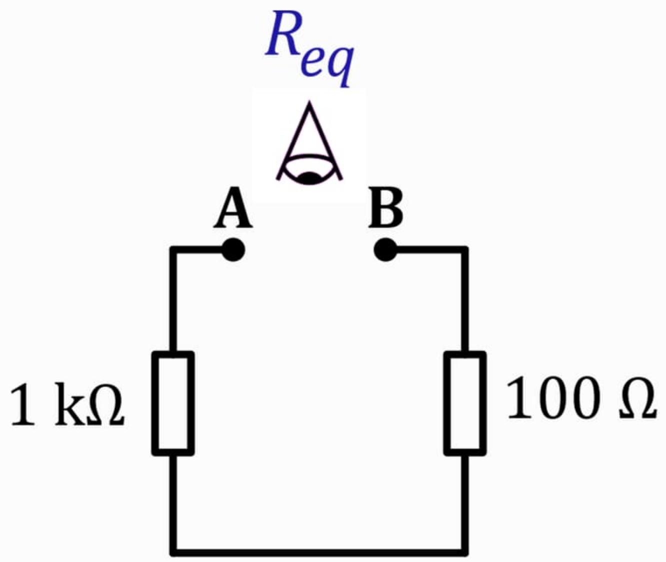


Example 1 – step 1: calculating τ

- We employ **Thevenin's theorem** to find the equivalent resistance seen from the terminals A and B.
- Turn off independent sources.
- Series connection of two resistors:

$$R_{Th} = 1\ \text{k}\Omega + 100\ \Omega = 1.1\ \text{k}\Omega$$

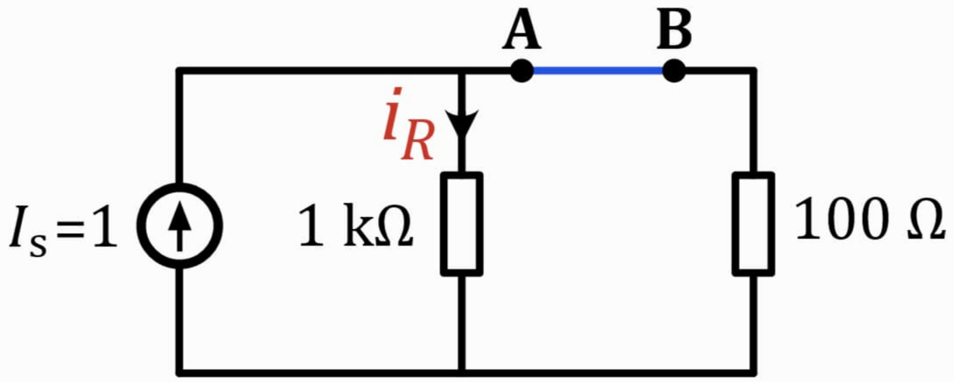
$$\Rightarrow \tau = R_{Th}C = 1.1\ \text{k}\Omega \times 10\ \text{nF} = 11\ \mu s$$



$$\boxed{\tau = 11\ \mu s}$$

Example 1 – step 2: calculating $i_R(0^+)$

- We model the capacitor with a **voltage source** whose value is $v_c(0^-)$.
- In this case, it is turned into short-circuit ($v_c(0^-) = 0$)
- Current division

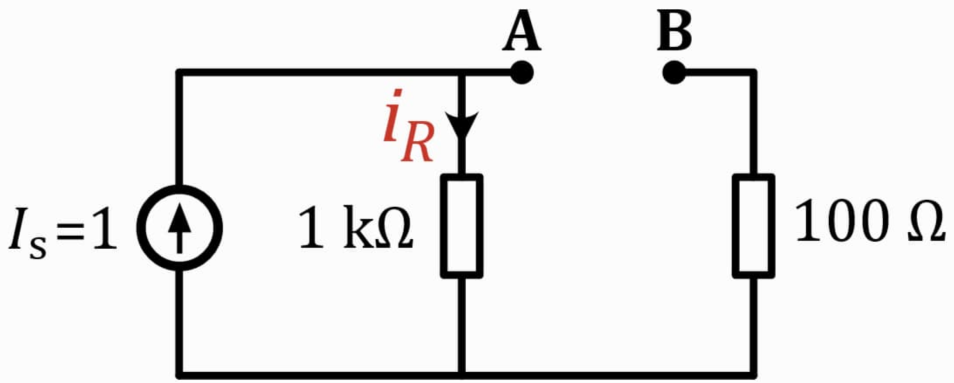


$$i_R(0^+) = \frac{100}{100 + 1k} I_s$$

$$\boxed{i_R(0^+) = \frac{1}{11} \text{A}}$$

Example 1 – step 3: calculating $i_R(\infty)$

- We model the capacitor as an **open-circuit.**
- No current flows into 100 Ω resistor.



$$i_R(\infty) = I_s = 1$$

$$\boxed{i_R(\infty) = 1\ \text{A}}$$

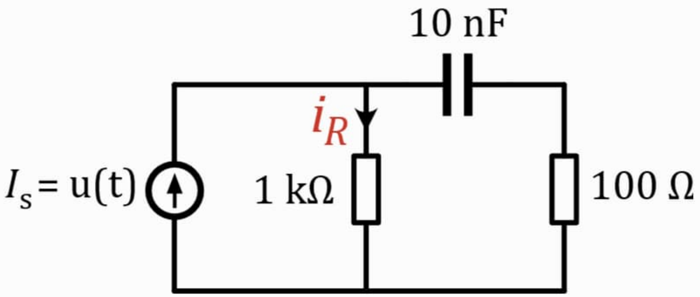
Example 1 – step 4: Conclusion

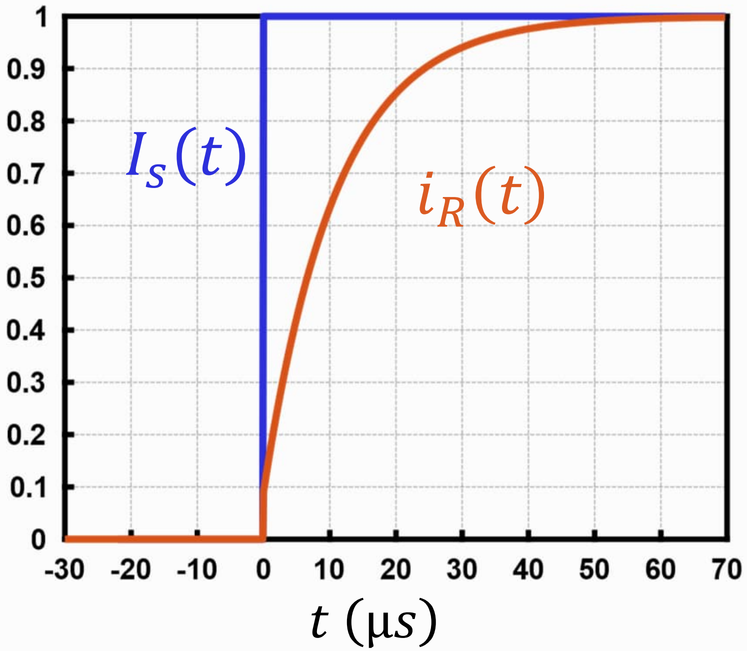
- Substituting these values in the formula:

$$i_R(t) = x(\infty) + \left(x(0^+) - x(\infty)\right) \exp\left(-\frac{t}{\tau}\right)$$

$$i_R(t) = 1 + \left(\frac{1}{11} - 1\right) \exp\left(-\frac{t}{11\mu s}\right)$$

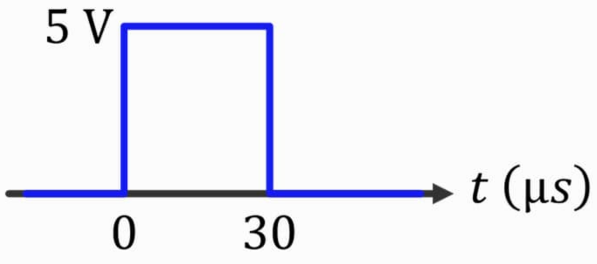
$$\boxed{i_R(t) = 1 - \frac{10}{11} \exp\left(-\frac{t}{11\mu s}\right)}$$





Pause and Ponder 1

- Plot the response of the circuit when the current source generates a pulse with a duration of 30 µs and an amplitude of 5 V.

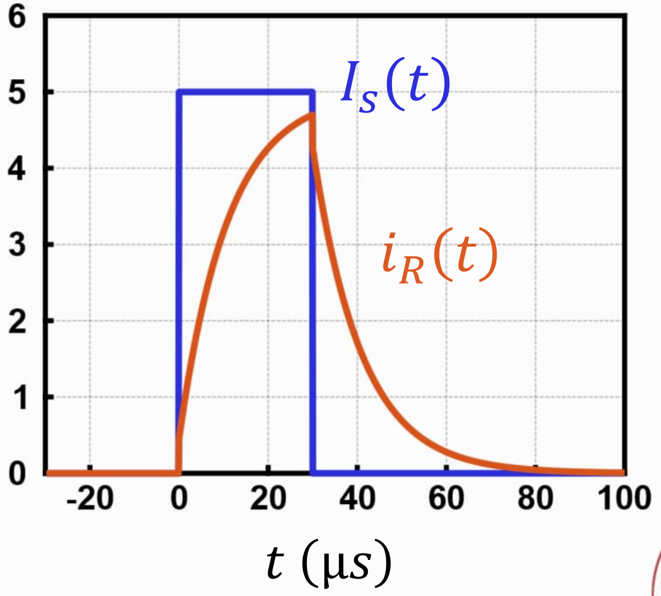


$$I_s(t) = 5\big(u(t) - u(t - 30\mu\text{s})\big)$$

$$\Rightarrow i_{R,new}\ (t) = 5\big(i_{R,old}(t) - i_{R,old}(t - 30\mu\text{s})\ \big)$$

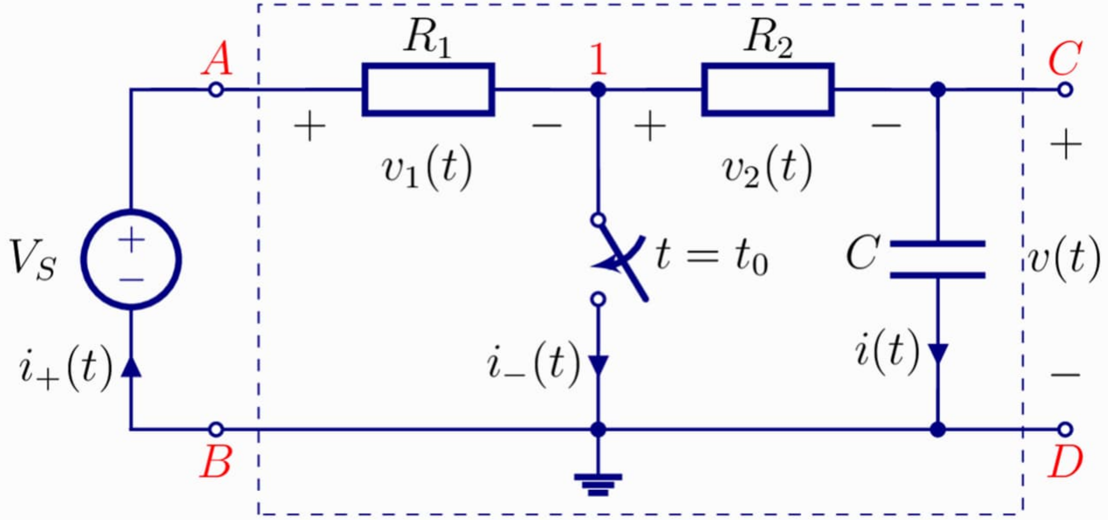
$$i_{R,old}(t) = \left(1 - \frac{10}{11}\exp\left(-\frac{t}{11\mu s}\right)\right)u(t)$$

$$i_{R,old}(t - 30\mu\text{s}) = \left(1 - \frac{10}{11}\exp\left(-\frac{t - 30\mu\text{s}}{11\mu s}\right)\right)u(t - 30\mu\text{s})$$



Example 2

- Find $v_c(t)$.
  - C has been fully charged before $t = t_0$.
  - At $t = t_0$ we the switch is closed.
  - Source voltage: 5 V



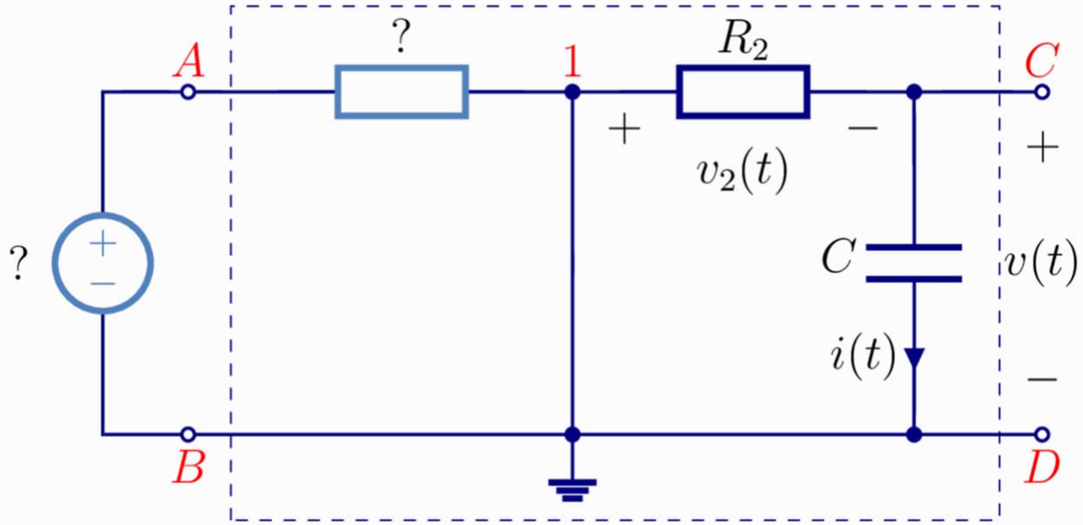
Component	Value
R_1	50 kΩ
R_2	50 kΩ
C	1.4 nF

Example 2 – step 1: calculating τ

- When the switch is closed.

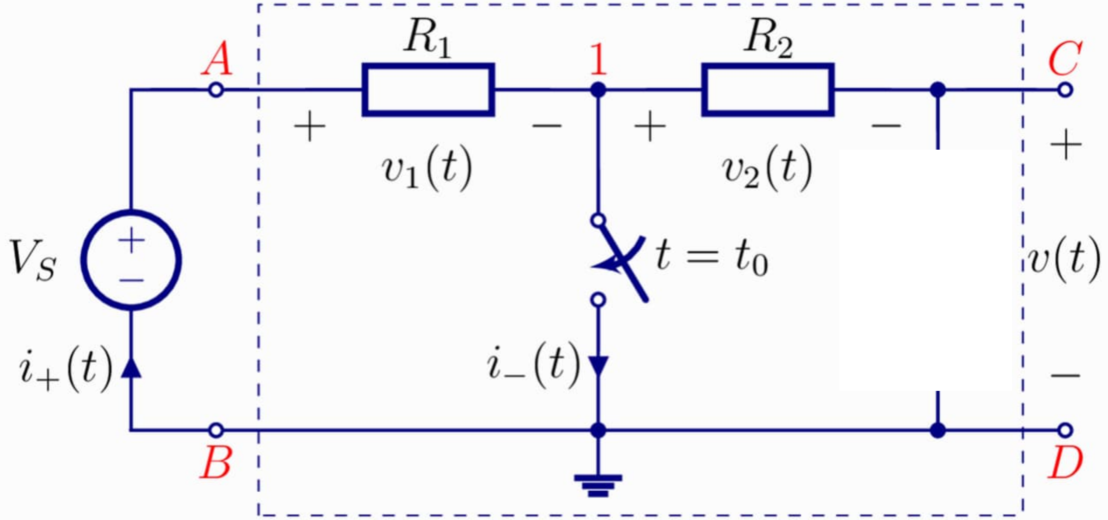
$$R_{Th} = R_2 = 50\ \text{k}\Omega$$

$$\boxed{\tau_1 = R_2 C = 70\ \mu\text{s}}$$



Example 2 – step 2: calculating $v_c(t_0^+)$

- $v_c(t)$ must be continuous.
- First, we calculate $v_c(t_0^-)$.
- $v_c(t_0^-)$ is the final value of the previous state ($t < t_0$).
- Then: $v_c(t_0^+) = v_c(t_0^-)$.

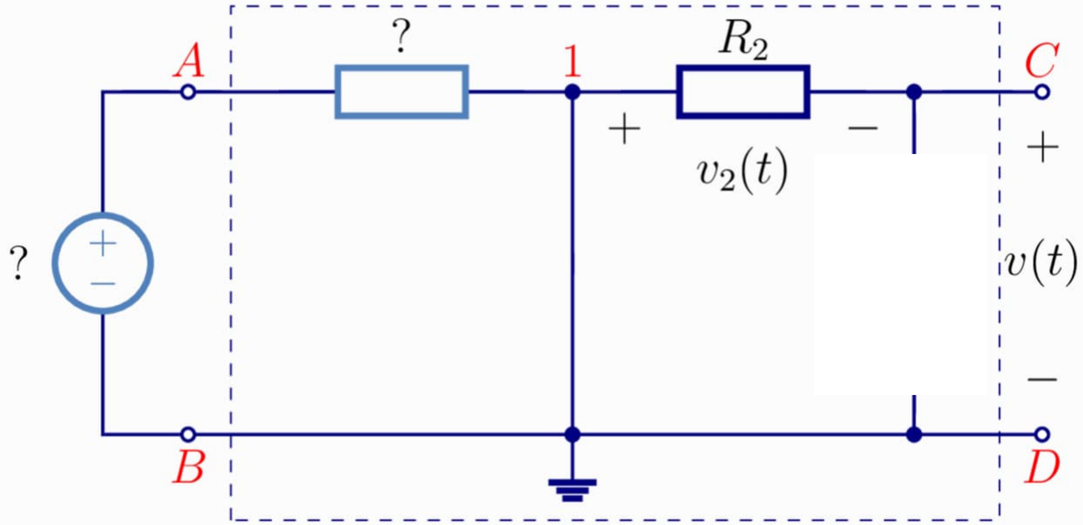


$$v_c(t_0^-) = V_s = 5\text{ V} \Rightarrow \boxed{v_c(t_0^+) = 5\text{ V}}$$

Example 2 – step 3: calculating $v_c(\infty)$

- In the end, C will be completely discharged through R_2.

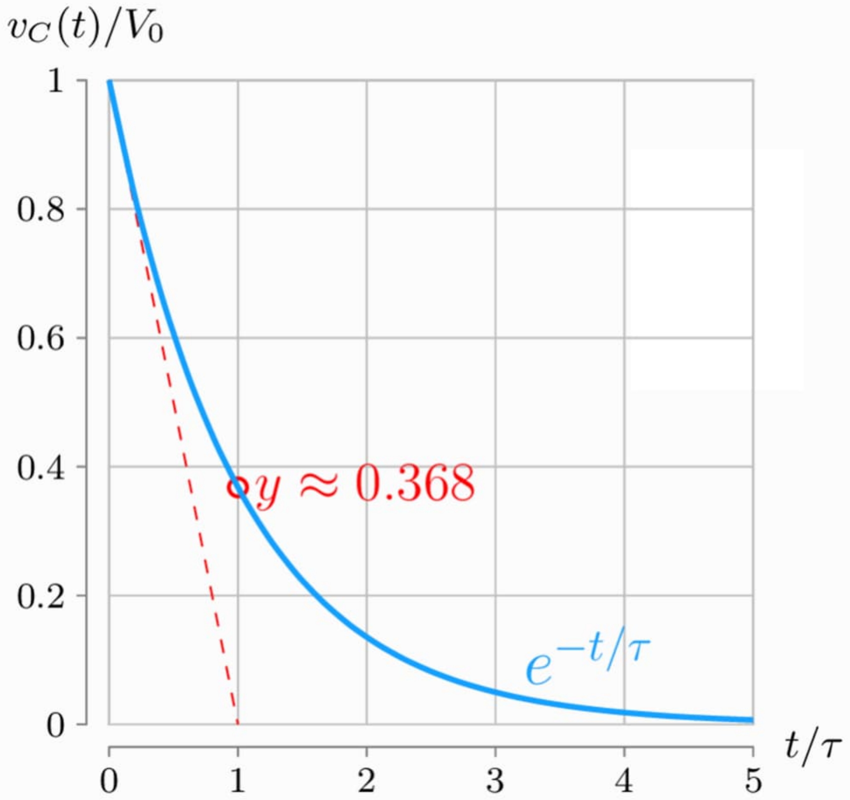
$$\boxed{v_c(\infty) = 0\ \text{V}}$$



Example 2 – step 4: Conclusion

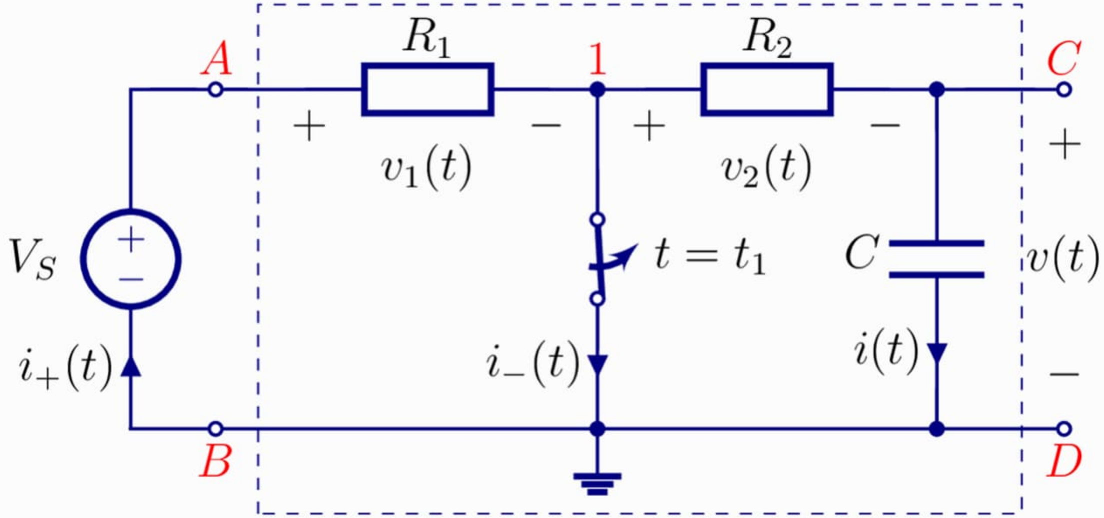
$$v_c(t) = 5\,\exp\left(-\frac{t - t_0}{70\mu s}\right) u(t - t_0)$$

Notice: at $\boldsymbol{t = t_0 + \tau}$ the voltage decreased by **63%**.



What happens if we open the switch again?

- We open the switch later, at $t = t_1$.
- If $t_1 - t_0 \gg \tau_1$, we can assume that the initial value in this state equals the final value of the previous state:



$$\boxed{v_c({t_1}^+) = v_c({t_1}^-) = 0\text{ V}}$$

- In the end, C will be charged to V_s = 5 V.

$$\boxed{v_c(\infty) = 5\text{ V}}$$

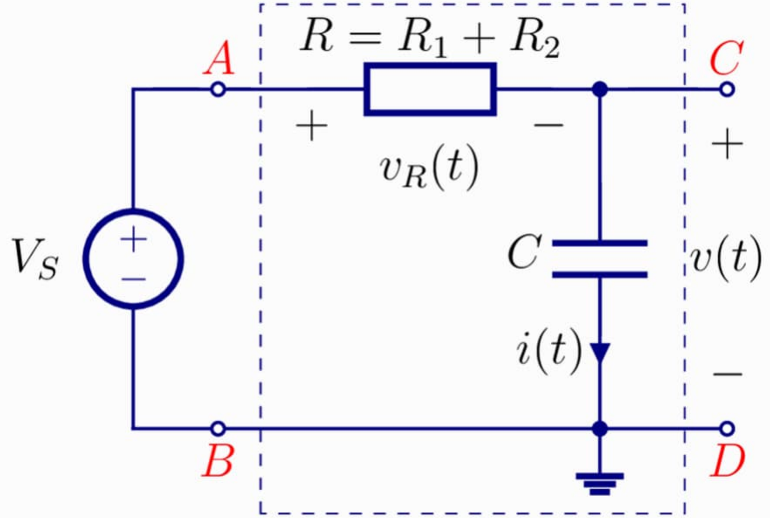
What happens if we open the switch again?

- What about the time constant?

$$R_{Th} = R_1 + R_2 = 100\ \text{k}\Omega$$

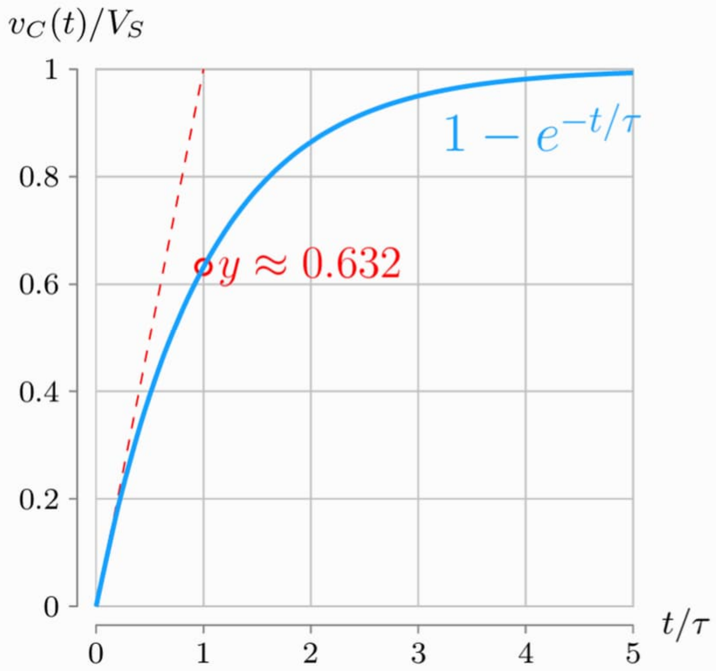
$$\boxed{\tau_1 = R_{Th}C = 140\ \mu\text{s}}$$

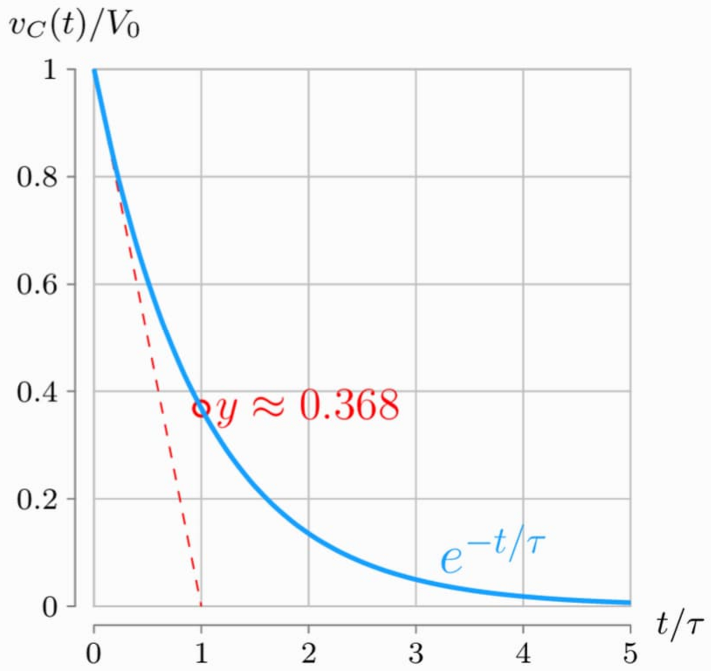
Charging happens slower.



How long should we wait before switching?

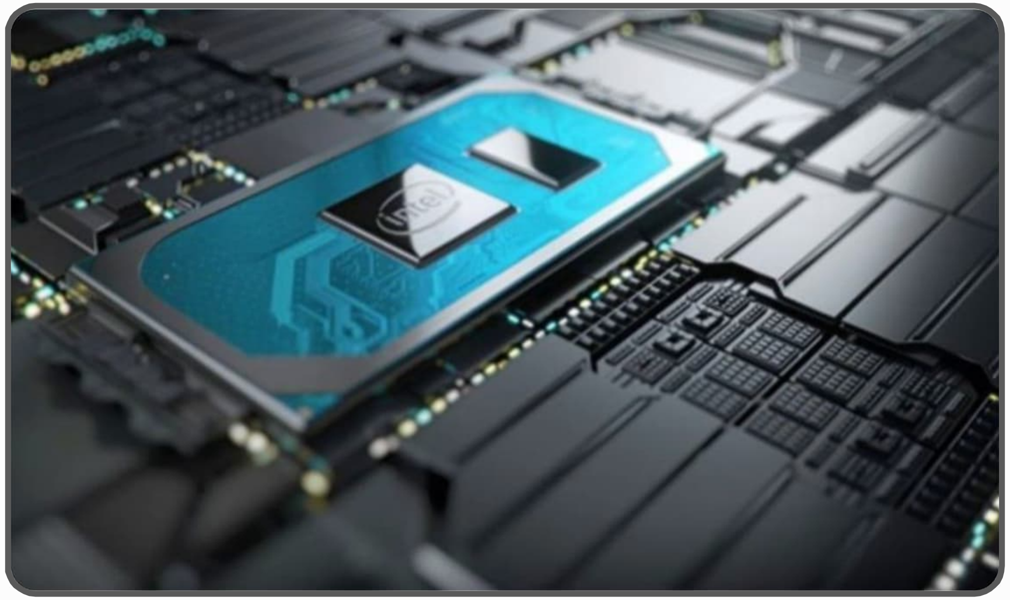
- Reaches **63%** of the final value after $\boldsymbol{1\tau}$.
- We should wait for at least $\boldsymbol{5\tau}$ to get to **99%** of the final value.



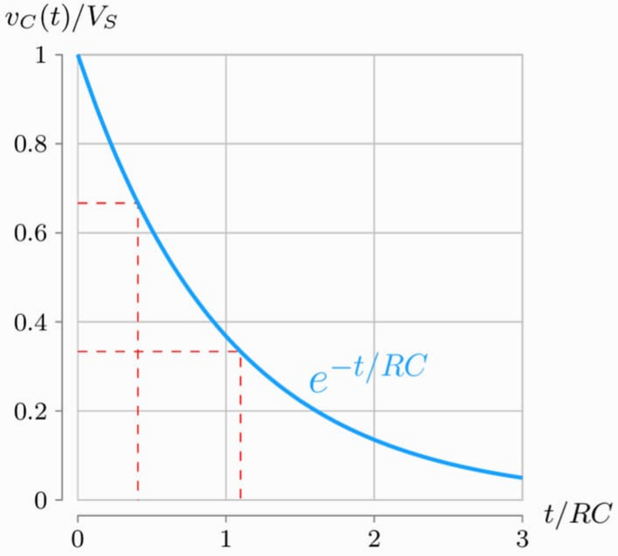


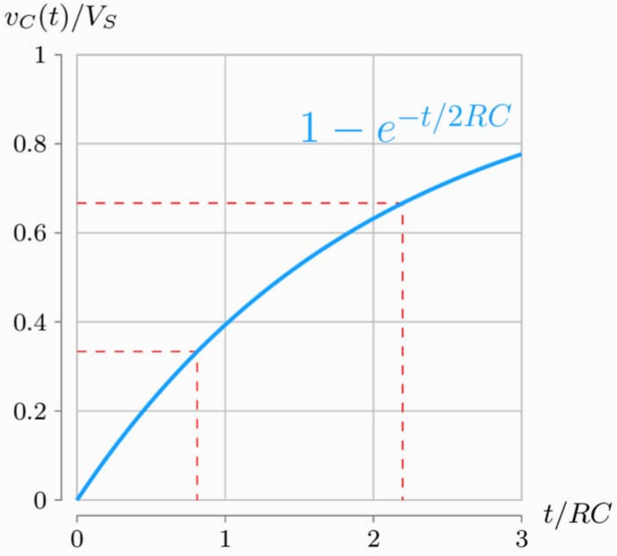
Maximum Operating Frequency

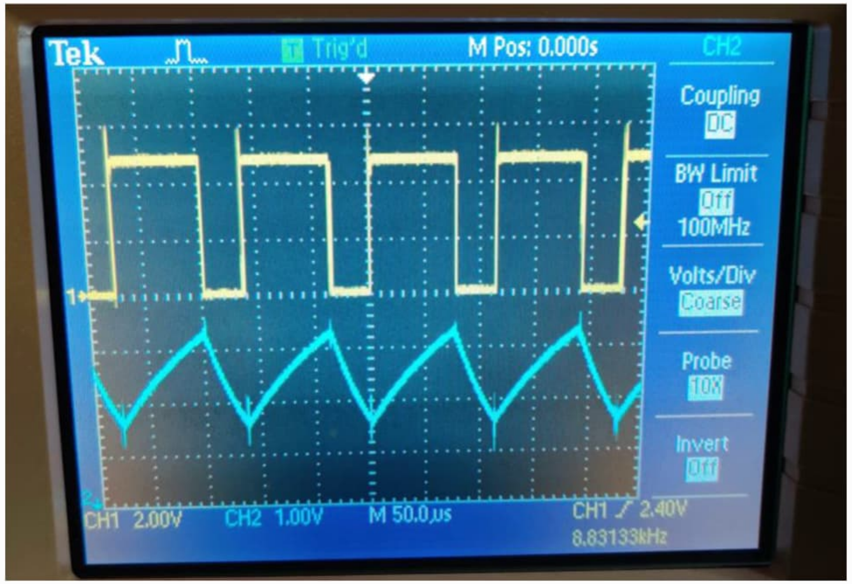
- An overly simplified view!
- For MOS transistors (~10 nm)
- In the switched-on mode:
  - $Vdd \approx 3\,\mathrm{V}$, i_d = 0.3 mA
  - Equivalent to R_{sw} = 10 kΩ
  - Capacitance: $C_g \approx$ 10 fF
- Time constant: $\tau \approx 100\ \mathrm{ps}$
- $f_{max} \approx$ 3 GHz



Clock Generation, we'll see later







Limitations of Circuit Theory

- **Lumped elements**
  - Dimensions of the circuit must be at least ten times lower than the minimum wavelength.

$$\lambda_{min} \approx \frac{c}{f_{max}}$$

c: light speed = 300,000 km/s

Example:

$$f_{max} = 3\ \text{GHz} \Rightarrow \lambda_{min} = 10\ \text{cm} \Rightarrow d_{circuit} < 1\ \text{cm}$$

- **Nonlinearity**
- **Parasitic elements**