

$$\lim_{n\rightarrow\infty}n\frac{\left(\sum_{k=1}^n\sqrt{k}\right)^2}{\left(\sum_{k=1}^n\sqrt[3]{k}\right)^3}+\sum_{k=1}^{n-1}\left[\ln\left(1+\frac{1}{n+k}\right)\sin\left(\ln\left(1+\frac{k}{n}\right)\right)\right]=$$

$$\lim_{n\rightarrow\infty}\frac{\left(\frac{1}{n}\sum_{k=1}^n\sqrt{\frac{k}{n}}\right)^2}{\left(\frac{1}{n}\sum_{k=1}^n\sqrt[3]{\frac{k}{n}}\right)^3}+\frac{1}{n}\sum_{k=1}^{n-1}\frac{1}{1+\frac{k}{n}}\sin\left(\ln\left(1+\frac{k}{n}\right)\right)=$$

$$\lim_{n\rightarrow\infty}\frac{\left(\int_0^1x^{\frac{1}{2}}\mathrm{d}x\right)^2}{\left(\int_0^1x^{\frac{1}{3}}\mathrm{d}x\right)^3}+\int_0^1\frac{\sin(\ln(1+x))}{1+x}\mathrm{d}x=\frac{\frac{4}{9}}{\frac{27}{64}}+\int_0^1\sin(\ln(1+x))\mathrm{d}(\ln(x+1))=$$

$$\frac{256}{243}-\cos u|_0^{\ln 2}=\frac{499}{243}-\cos(\ln 2)$$

$$p(x)=\frac{\mathrm{d}^n}{\mathrm{d}x^n}(1-x^m)^n=\left(\sum_{k=0}^nC_n^k(-x^m)^k\right)^{(n)}=(1+C_n^1(-x^m)+C_n^2(-x^m)^2+...)^{(n)}$$

$$\exists k_0\in\{1,2,3,\ldots\},mk_0>n$$

$$\int_0^{\frac{\pi}{2}}\ln\left[\left(\sin^2x+99\cos^2x\right)\left(999\sin^2x+\cos^2x\right)\right]\mathrm{d}x$$

$$\left\{\begin{array}{l}x^2+y^2=z\\y=x\tan z\end{array}\right.=\left\{\begin{array}{l}x=\sqrt{\theta}\cos\theta\\y=\sqrt{\theta}\sin\theta\\z=\theta\end{array}\right.,\theta\in(0,c)$$

$$s=\int\mathrm{d}s=\int_0^c\sqrt{\left(\frac{\cos\theta}{2\sqrt{\theta}}-\sqrt{\theta}\sin\theta\right)^2+\left(\frac{\sin\theta}{2\sqrt{\theta}}+\sqrt{\theta}\cos\theta\right)^2+1}\mathrm{d}\theta=\\ \int_0^c\sqrt{\frac{1}{4\theta}+\theta+1}\mathrm{d}\theta=\int_0^c\frac{2\theta+1}{2\sqrt{\theta}}\mathrm{d}\theta=\frac{2}{3}\theta^{\frac{3}{2}}+\theta^{\frac{1}{2}}\Big|_0^c=\frac{2}{3}c^{\frac{3}{2}}+c^{\frac{1}{2}}=\sqrt{c}\left(\frac{2c}{3}+1\right)$$

$$f(x)=\sec x, x\in(-\frac{\pi}{4},\frac{\pi}{4})$$

$$S(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos 4nx$$

$$a_0=\frac{8}{\pi}\int_0^{\frac{\pi}{4}}\sec x\mathrm{d}x=\frac{8}{\pi}\ln|\sec x+\tan x||_0^{\frac{\pi}{4}}=\frac{8}{\pi}\ln(\sqrt{2}+1)$$

$$a_n=\frac{8}{\pi}\int_0^{\frac{\pi}{4}}\sec x\cos 4nxdx=\frac{4}{n\pi}\int_0^{\frac{\pi}{4}}\frac{\mathrm{d}(\sin 4nx)}{\cos x}=\frac{4\sin 4nx}{n\pi\cos x}\Big|_0^{\frac{\pi}{4}}-\frac{4}{n\pi}\int_0^{\frac{\pi}{4}}\sin 4nx\sec x\tan xdx=$$

$$\begin{aligned}
2m &= n, \frac{x^2}{2} - \frac{\sqrt{2}xy}{m} + \frac{3y^2}{2m} = 1 \\
A &= \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{2}}{2m} \\ \frac{-\sqrt{2}}{2m} & \frac{3}{2m} \end{bmatrix} \quad (x, y)^T = Q(u, v)^T \\
|\lambda E - A| &= (\lambda - \frac{1}{2})(\lambda - \frac{3}{2m}) - \frac{1}{2m^2} = \lambda^2 - \frac{m+3}{2m}\lambda + \frac{3m-2}{4m^2} = 0 \\
\lambda &= \frac{\frac{m+3}{2m} \pm \sqrt{(\frac{m+3}{2m})^2 - 4\frac{3m-2}{4m^2}}}{2} \notin Z
\end{aligned}$$

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{\int_0^x t \cos t dt - 1 + \cos x}{\sqrt{1+x \tan x} - \sqrt{1+x \sin x}} = \\
& \lim_{x \rightarrow 0} \frac{2(\int_0^x t \cos t dt - 1 + \cos x)}{x(\tan x - \sin x)} = \lim_{x \rightarrow 0} \frac{2(x \cos x - \sin x)}{2x^3} = -\frac{1}{3} \\
& \sum_{n=0}^{\infty} \frac{n^2+1}{(\frac{1}{2})^n n!} x^n = \sum_{n=0}^{\infty} \frac{n^2(2x)^n}{n!} + \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} \\
& \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} 2 \frac{(n+1)^2+1}{(n^2+1)(n+1)} = 0, x \in (-\infty, +\infty) \\
& S(x) = \sum_{n=1}^{\infty} \frac{n(2x)^n}{(n-1)!} + e^{2x} = \sum_{n=0}^{\infty} \frac{(n+1)(2x)^{n+1}}{n!} + e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^{n+2}}{n!} + \sum_{n=0}^{\infty} \frac{(2x)^{n+1}}{n!} + e^{2x} \\
& = (4x^2 + 2x + 1)e^{2x}, x \in (-\infty, +\infty) \\
& \begin{cases} x+y-z = e^z \\ xe^x = \tan t \\ y = \cos t \end{cases} \\
& x'e^x + xe^x x' = \sec t \rightarrow e^x(x' + x'x) = \sec t, e^x(x' + x'x + x'' + x''x + x'x') = \sec t \tan t \\
& y' = -\sin t, y'' = -\cos t \\
& x' + y' - z' = e^z z' \rightarrow z' = \frac{x' + y'}{e^z + 1} \\
& t = 0, x = 0, y = 1, z = 0, x' = 1, y' = 0, z' = \frac{1}{2} \\
& 1 + 0 + x'' + 0 + 1 = 0, x'' = -2, y'' = -1 \\
& z'' = \frac{(x'' + y'')(e^z + 1) - (x' + y')(e^z z')}{(e^z + 1)^2} = \frac{-3 * 2 - \frac{1}{2}}{4} = -\frac{13}{8} \\
& \iint_D r^2 \sin \theta \sqrt{1 - r^2 \cos(2\theta)} d\sigma, D = \{(r, \theta), 0 \leq r \leq \sec \theta, 0 \leq \theta \leq \frac{\pi}{4}\} \\
& D = \{(x, y), 0 \leq y \leq x, 0 \leq x \leq 1\} \\
& \iint_D y \sqrt{1 - x^2 + y^2} d\sigma = \frac{1}{2} \int_0^1 \int_{1-x^2}^{x^2+1-x^2} \sqrt{1 - x^2 + y^2} dy dx = \frac{1}{3} \int_0^1 u^{\frac{3}{2}}|_{1-x^2}^1 dx \\
& = \frac{1}{3} \int_0^1 1 - (1 - x^2)^{\frac{3}{2}} dx = \frac{1}{3} - \int_0^{\frac{\pi}{2}} \cos^4 t dt = \frac{1}{3} - \frac{1 * 3 * 1 * \pi}{3 * 4 * 2 * 2} = \frac{1}{3} - \frac{\pi}{16} \\
& u = ax^2 + by^2 + cz^2, x + y + z = 1, (ax^2 + by^2 + cz^2)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}) \geq (x + y + z)^2 = 1 \\
& \min_{x>0, y>0, z>0} u(x, y, z) = \frac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}
\end{aligned}$$

$$\begin{aligned}
a_n &= \int_0^1 \frac{x^{n-1}}{1+x} dx = \int_0^1 x^{n-1} d(\ln(x+1)) = x^{n-1} \ln(x+1) \Big|_0^1 - (n-1) \int_0^1 \ln(x+1) x^{n-2} dx \\
a_n &= \frac{1}{n} \int_0^1 \frac{d(x^n)}{1+x} = \frac{x^n}{n(1+x)} \Big|_0^1 + \int_0^1 \frac{x^n dx}{n(1+x)^2} = \frac{1}{2n} + \int_0^1 \frac{d(x^{n+1})}{(1+x)^2 n(n+1)} = \\
&\frac{1}{2n} + \frac{x^{n+1}}{(1+x)^2 n(n+1)} \Big|_0^1 + \int_0^1 \frac{2x^{n+1} dx}{(n+1)(1+x)^3} = \frac{1}{2n} + \frac{1}{4n^2} + o\left(\frac{1}{n^2}\right) mn \rightarrow \infty
\end{aligned}$$

$$I_1 = \iint_S \frac{xz}{a^2} dy dz + \frac{yz}{b^2} dz dx + \frac{z^2}{c^2} dx dy$$

$$I_2 = 0, S_1 = \{(x, y, z), \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1, z = 0\}, down 0$$

$$D_{xy} = \{(x, y), \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$$

$$I_1 + I_2 = \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{2}{c^2} \right) \iiint_{\Omega} z dv = \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{2}{c^2} \right) \frac{c^2}{2} \iint_{D_{xy}} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right) d\sigma$$

$$\frac{x}{a} = u, \frac{y}{b} = v, J = ab$$

$$I_1 + I_2 = \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{2}{c^2} \right) \frac{c^2}{2} ab \iint_{D_{uv}} (1 - u^2 - v^2) d\sigma$$

$$\iint_{D_{uv}} (1 - u^2 - v^2) d\sigma = \int_0^1 (1 - r^2) r dr \int_0^{2\pi} d\theta = \frac{\pi}{2}$$

$$I_1 = \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{2}{c^2} \right) \frac{c^2 \pi}{4} ab$$

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left(\frac{1 + \sqrt[n]{3}}{2} \right)^n \\
& \lim_{x \rightarrow +\infty} ((x+1)^a - x^a) = a \\
& \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2} \\
& \lim_{n \rightarrow \infty} \frac{(n+1)^2}{n^2} = 1, x \in (-1, 1), x \neq \pm 1 \\
& \sum_{n=1}^{\infty} n^2 x^{n-1} = \left(\sum_{n=1}^{\infty} n x^n \right)' = \left(\frac{1}{(1-x)^2} - 1 - \left(\frac{1}{1-x} - 1 \right) \right)' = \frac{2}{(1-x)^3} - \frac{1}{(1-x)^2} \\
& f(x) = \ln(1+x) + a x e^{-x}, x_1 \in (-1, 0), x_2 \in (0, +\infty) \\
& f(x), x \in (-1, +\infty) \\
& f'(x) = \frac{1}{x+1} + a(1-x)e^{-x} = \frac{1 + a(1-x^2)e^{-x}}{x+1} \\
& g(x) = 1 + a(1-x^2)e^{-x}, x \in (-1, +\infty) \\
& g'(x) = a(x^2 - 2x - 1)e^{-x} = a(x-1+\sqrt{2})(x-1-\sqrt{2})e^{-x} \\
& \exists a = 0, g(x) \equiv 1, f'(x) > 0, f(x) \uparrow, a \neq 0 \\
& \exists a < 0, g'(x) \leq 0, x \in (-1, +\infty), g(x) \downarrow \\
& \exists a > 0, g'(x) \geq 0, x \in (-1, +\infty), g(x) \uparrow \\
& g(-1) = 1, g(0) = 1 + a, g(+\infty) = 1 \\
& \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{1}{1 + \frac{i}{n}} = \int_0^1 \frac{1}{1+x} dx = \ln(x+1)|_0^1 = \ln 2
\end{aligned}$$

$$\lim_{x \rightarrow 0} \frac{1}{x} \left(\frac{1}{x} - \cot x \right) = \lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} = \frac{1}{3}$$

$$\lim_{n \rightarrow \infty} \tan n, n = k\pi \lim = 0, n \rightarrow \left(k\pi + \frac{\pi}{2}\right)^-, \lim = +\infty, n \rightarrow \left(k\pi + \frac{\pi}{2}\right)^+, \lim = -\infty$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x), \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2$$

$$\oint_L \frac{1}{x} \arctan \frac{y}{x} dx + \frac{2}{y} \arctan \frac{x}{y} dy = \iint_D \frac{2}{y} \frac{\frac{1}{y}}{\left(\frac{x}{y}\right)^2 + 1} - \frac{1}{x} \frac{\frac{1}{x}}{\left(\frac{y}{x}\right)^2 + 1} d\sigma = \iint_D \frac{1}{x^2 + y^2} d\sigma$$

$$= \int_1^2 \frac{1}{r} dr \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\theta = \frac{\pi \ln 2}{12}$$

$$\vec{n} = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right), dS = \sqrt{(-1)^2 + 1^2 + 1} dx dy = \sqrt{3} dx dy$$

$$\iint_{\Sigma} (x-y+z) \frac{1}{\sqrt{3}} dS = S_{xy} = \frac{1}{2}$$

$$\iiint_{\Omega} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dv = \frac{1}{3} \iiint_{\Omega} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} + \frac{x^2}{c^2} + \frac{z^2}{a^2} + \frac{x^2}{b^2} + \frac{y^2}{c^2} \right) dv$$

$$\frac{1}{3} \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \int_0^1 r^4 dr \int_0^\pi \sin \phi d\phi \int_0^{2\pi} d\theta = \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \frac{1 * 1 * 2 * 2\pi}{3 * 5} = \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \frac{4\pi}{15}$$

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{\sin \frac{i\pi}{n}}{n+i} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{\sin \left(\frac{i}{n} \right)}{1 + \frac{i}{n}} = \int_0^1 \frac{\sin(\pi x)}{1+x} dx$$

$$S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+2}}{3n+2}, S(0) = 0$$

$$S'(x) = \sum_{n=0}^{\infty} (-1)^n x^{3n+1} = x \sum_{n=0}^{\infty} (-x^3)^n = \frac{x}{1+x^3}$$

$$S(x) = \int_0^x S(t) dt + S(0) = \int_0^x \frac{t}{(t+1)(t^2-t+1)} dt + 0 = \frac{1}{3} \int_0^x -\frac{1}{t+1} + \frac{t+1}{t^2-t+1} dt$$

$$At^2 - At + A + Bt^2 + Ct + Bt + C \rightarrow A + B = 0, B + C - A = 1, A + C = 0, A = -\frac{1}{3}, B = C = \frac{1}{3}$$

$$S(x) = -\frac{\ln(x+1)}{3} + \frac{1}{3} \int_0^x \frac{1}{2} \frac{d(t^2-t+1)}{t^2-t+1} + \frac{1}{\sqrt{3}} \frac{d(t-\frac{1}{2})(\frac{2}{\sqrt{3}})}{[(t-\frac{1}{2})(\frac{2}{\sqrt{3}})]^2+1}$$

$$= -\frac{\ln(x+1)}{3} + \frac{\ln(t^2-t+1)}{6} + \frac{1}{\sqrt{3}} \arctan \left(\frac{2t-1}{\sqrt{3}} \right) + C|_0^x$$

$$= -\frac{\ln(x+1)}{3} + \frac{\ln(x^2-x+1)}{6} + \frac{1}{\sqrt{3}} \arctan \left(\frac{2x-1}{\sqrt{3}} \right) + \frac{\pi}{6\sqrt{3}}$$

$$S(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{3n+2} = -\frac{\ln 2}{3} + \frac{\pi\sqrt{3}}{9}$$

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\sqrt{k(n-k+1)}} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{\sqrt{\frac{k}{n} - \frac{k^2}{n^2} + \frac{k}{n^2}}} = \int_0^1 \frac{dx}{\sqrt{x-x^2}} = \pi$$

$$I = \int \sec^3 x dx = \int \sec x (\tan^2 x + 1) dx = \int \tan x d(\sec x) + \ln |\sec x + \tan x|$$

$$= \ln |\sec x + \tan x| + \sec x \tan x - \int \sec^3 x dx$$

$$\int \sec^3 x dx = \frac{1}{2} (\ln |\sec x + \tan x| + \sec x \tan x) + C$$

$$y(x) = \sqrt{x + \sqrt[3]{x + \sqrt[4]{x}}}, \ln y(x) = \frac{1}{2} \ln(x + g(x))$$

$$\frac{y'(x)}{y(x)} = \frac{1}{2} \frac{1 + g'(x)}{x + g(x)}$$

$$g(x) = \sqrt[3]{x + \sqrt[4]{x}}, \ln(g(x)) = \frac{1}{3} \ln\left(x + x^{\frac{1}{4}}\right)$$

$$\frac{g'(x)}{g(x)} = \frac{1}{3} \frac{1 + \frac{1}{4}x^{-\frac{3}{4}}}{x + x^{\frac{1}{4}}}$$

$$g'(x) = \frac{4x^{\frac{3}{4}} + 1}{12(x + x^{\frac{1}{4}})} g(x) = \frac{4x^{\frac{3}{4}} + 1}{12(x + x^{\frac{1}{4}})^{\frac{2}{3}} x^{\frac{3}{4}}}$$

$$\begin{aligned} y'(x) &= \frac{1 + g'(x)}{2(x + g(x))} y(x) = \frac{1 + \frac{4x^{\frac{3}{4}} + 1}{12(x + x^{\frac{1}{4}})^{\frac{2}{3}} x^{\frac{3}{4}}}}{2(x + (x + x^{\frac{1}{4}})^{\frac{1}{3}})^{\frac{1}{2}}} \\ &= \frac{12(x + x^{\frac{1}{4}})^{\frac{2}{3}} x^{\frac{3}{4}} + 4x^{\frac{3}{4}} + 1}{24(x + x^{\frac{1}{4}})^{\frac{2}{3}} x^{\frac{3}{4}} (x + (x + x^{\frac{1}{4}})^{\frac{1}{3}})^{\frac{1}{2}}} \end{aligned}$$

$$I = \lim_{x \rightarrow +\infty} y(x) - x^{\frac{1}{2}} = \lim_{x \rightarrow +\infty} \frac{\sqrt[3]{x + \sqrt[4]{x}}}{\sqrt{x + \sqrt[3]{x + \sqrt[4]{x}} + \sqrt{x}}} = 0$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{1}{1 + \left(\frac{i}{n}\right)^2} = \frac{\pi}{4}$$

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^x - x}{\ln x - x + 1} &= \lim_{x \rightarrow 1} \frac{x^{x-1} - 1}{\ln(x-1+1) - x + 1} = \lim_{t \rightarrow 0} \frac{(t+1)^t - 1}{\ln(t+1) - t} \\ &= \lim_{t \rightarrow 0} \frac{e^{t \ln(t+1)} - 1}{-\frac{t^2}{2} + o(t^2)} = -2 \end{aligned}$$

$$\int_0^{+\infty} (x+1)e^{-x^2} dx = -\frac{1}{2} \int_0^{+\infty} e^{-x^2} d(-x^2) + \frac{\sqrt{\pi}}{2} = \frac{1 + \sqrt{\pi}}{2}$$

$$\begin{aligned}
X_1 = X_2 &\sim N(0, 1), F_{X_1}(x) = F_{X_2}(x) = \Phi(x) \\
F(x, y) &= P\{X_1 \leq x, Y \leq y\} = P\{X_1 \leq x, X_3X_1 + (1 - X_3)X_2 \leq y\} \\
&= P\{X_1 \leq x, \{X_3X_1 + (1 - X_3)X_2 \leq y\}, X_3 = 0\} + \\
&\quad P\{X_1 \leq x, \{X_3X_1 + (1 - X_3)X_2 \leq y\}, X_3 = 1\} \\
&= P\{X_1 \leq x, X_2 \leq y\}P\{X_3 = 0\} + P\{X_1 \leq x, X_1 \leq y\}P\{X_3 = 1\} \\
&= \frac{1}{2} (P\{X_1 \leq x\}P\{X_2 \leq y\} + P\{X_1 \leq x, X_1 \leq y\}) \\
\exists x < y, \quad F(x, y) &= \frac{1}{2} (\Phi(x)\Phi(y) + \Phi(x)) \\
\exists x \geq y, \quad F(x, y) &= \frac{1}{2} (\Phi(x)\Phi(y) + \Phi(y)) \\
F_Y(y) = P\{Y \leq y\} &= P\{X_3X_1 + (1 - X_3)X_2 \leq y\} = \frac{1}{2} (P\{X_2 \leq y\} + P\{X_1 \leq y\}) \\
&= \frac{1}{2} (\Phi(y) + \Phi(y)) = \Phi(y), Y \sim N(0, 1)
\end{aligned}$$

$$\begin{aligned}
D &= \{(x, y) | x^2 + y^2 \leq 1\}, \alpha^2 + \beta^2 = 1 \\
I &= \iint_D \frac{d\sigma}{(1 - \alpha x + \beta y)^2 + (\beta x + \alpha y)^2} = \iint_D \frac{d\sigma}{(x - \alpha)^2 + (y + \beta)^2} \\
&\quad \begin{cases} x - \alpha = u \\ y + \beta = v \end{cases} \\
D_{uv} &= \{(u, v) | (u + \alpha)^2 + (v - \beta)^2 \leq 1\} \rightarrow D_{uv} = \{(u, v) | r^2 + 2\alpha r \cos \theta - 2\beta r \sin \theta \leq 0\} \\
I &= \iint_{D_{uv}} \frac{1}{r} dr d\theta = \int_{\sqrt{\alpha^2 + \beta^2} - 1}^{2(\beta \sin \theta - \alpha \cos \theta)} \frac{1}{r} dr \int_{\theta}^{\theta} d\theta \\
&\quad \int_{-2}^2 x^3 \cos \frac{x}{2} \sqrt{4 - x^2} + \frac{1}{2} \sqrt{4 - x^2} dx = \frac{1}{2 * 2} \pi 4 = \pi \approx 3.1415926 \\
I &= \int \frac{\sqrt{e^{2x} + 1}}{e^x + 1} dx \\
e^x + 1 &= u, e^{2x} + 2e^x + 1 = u^2, e^{2x} + 1 = u^2 - 2(u - 1), \\
x &= \ln(u - 1), dx = \frac{du}{u - 1} \\
I &= \int \frac{\sqrt{u^2 - 2u + 2} du}{u(u - 1)} = \int \frac{\sqrt{(u - 1)^2 + 1}}{u - 1} - \frac{\sqrt{u^2 - 2u + 2}}{u} du \\
&= \int \frac{\sqrt{(u - 1)^2 + 1}}{u - 1} d(u - 1) - \int \frac{\sqrt{(u - 1)^2 + 1}}{u} du \\
v &= u - 1, I = I_1 - I_2 = \int \frac{\sqrt{v^2 + 1}}{v} dv - \int \frac{\sqrt{v^2 + 1}}{v + 1} dv \\
v &= \tan t, I_1 = \int \frac{\sec^3 t dt}{\tan t} = \int \frac{1}{\cos^2 x \sin x} dt \\
&\quad I_2 =
\end{aligned}$$

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$$f(x, y) = Ae^{-2x^2+2xy-y^2}$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) dx dy = 1$$

$$A \int_{-\infty}^{+\infty} e^{-2(x^2-xy+\frac{y^2}{4}-\frac{y^2}{4})-y^2} dx \int_{-\infty}^{+\infty} dy = A \int_{-\infty}^{+\infty} e^{-2(x-\frac{y}{2})^2} dx \int_{-\infty}^{+\infty} e^{-\frac{y^2}{2}} dy$$

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1, \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}, \int_{-\infty}^{+\infty} e^{-2u^2} du = \sqrt{\frac{\pi}{2}}$$

$$A = \frac{1}{\pi}$$

$$\lim_{x \rightarrow 0^+} \tan \left(2x^2 \arctan \frac{1}{5x} \right) \sim \tan \left(2x^2 \frac{\pi}{2} \right) \sim \pi x^2 + o(x^2)$$

$$x \rightarrow 0^+, \arctan 5x \sim 5x, x \rightarrow 0^+, \arctan \left(\frac{1}{5x} \right) = \frac{\pi}{2}$$

$$I = \int \frac{1}{\sqrt{x}+1} dx, \sqrt{x} = t, x = t^2, dx = 2t dt$$

$$I = 2 \int \frac{t+1-1}{t+1} dt = 2t - 2 \ln(t+1) + C = 2\sqrt{x} - 2 \ln(\sqrt{x}+1) + C$$

$$y = x^{2022}, y^{(1)} = 2022x^{2021} \dots y^{(2022)} = 2022!x^0, y^{(2023)} = 0$$

$$y = \frac{x^{1+x}}{(1+x)^x}, x > 0$$

$$k = \lim_{x \rightarrow +\infty} \frac{y}{x} = \lim_{x \rightarrow +\infty} \frac{x^x}{(1+x)^x} = \frac{1}{\lim_{x \rightarrow +\infty} (1+\frac{1}{x})^x} = \frac{1}{e}$$

$$\lim_{x \rightarrow +\infty} e - \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow +\infty} e - e^{x \ln(1+\frac{1}{x})} = \lim_{x \rightarrow +\infty} e \left(1 - e^{x \ln(1+\frac{1}{x})-1}\right)$$

$$= \lim_{x \rightarrow +\infty} e \left(1 - x \ln \left(1 + \frac{1}{x}\right)\right) = \lim_{x \rightarrow +\infty} e \left(1 - x \left(\frac{1}{x} - \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)\right)\right) = \frac{e}{2x} + o\left(\frac{1}{x}\right)$$

$$b = \lim_{x \rightarrow +\infty} (y - kx) = \lim_{x \rightarrow +\infty} \left(\frac{x^{1+x}}{(1+x)^x} - \frac{x}{e}\right) = \lim_{x \rightarrow +\infty} \frac{x}{e} \left(\frac{ex^x - (1+x)^x}{(1+x)^x}\right)$$

$$= \lim_{x \rightarrow +\infty} \frac{x}{e} \frac{e - (1+\frac{1}{x})^x}{(1+\frac{1}{x})^x} = \lim_{x \rightarrow +\infty} \frac{x \frac{e}{2x}}{e^2} = \frac{1}{2e}$$

$$y = \frac{x}{e} + \frac{1}{2e}$$

$$A = \int_0^4 f(x) dx, f(x) = \sin 2x + 3A$$

$$A = \int_0^4 (\sin 2x + 3A) dx$$

$$A = -\frac{1}{2} \cos 2x|_0^4 + 12A \rightarrow \int_0^4 f(x) dx = A = \frac{\cos 8 - 1}{22}$$

$$\begin{aligned}
tx^2 = u, x = \sqrt{\frac{u}{t}}, dx &= \frac{1}{2\sqrt{ut}} du \\
\lim_{t \rightarrow 0} \frac{\int_0^t \sin(tx^2) dx}{t^4} &= \lim_{t \rightarrow 0} \frac{\int_0^{t^3} \frac{\sin u}{\sqrt{u}} du}{2t^{\frac{9}{2}}} = \lim_{t \rightarrow 0} \frac{3t^2 \sin t^3}{9t^{\frac{7}{2}} t^{\frac{3}{2}}} = \frac{1}{3} \\
\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n \rightarrow \infty} \frac{n}{n+2} = 1, x \in (-1, 1) \\
S(x) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+1}}{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-x)^{n+1}}{n} - \sum_{n=1}^{\infty} \frac{(-x)^{n+1}}{n+1} \\
&= (-x) \sum_{n=1}^{\infty} \frac{(-x)^n}{n} - \sum_{n=2}^{\infty} \frac{(-x)^n}{n} = x \ln(x+1) - (-\ln(x+1) - (-x)) = (x+1) \ln(x+1) - x \\
\sum_{n=0}^{\infty} x^n &= \frac{1}{1-x}, \sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x), \sum_{n=1}^{\infty} \frac{(-x)^n}{n} = -\ln(1+x) \\
x = -1, S(x) &= \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 1 \\
x = 1, S(x) &= 2 \ln 2 - 1 \\
S(x) &= (x+1) \ln(x+1) - x, x \in [-1, 1] \\
\lim_{n \rightarrow \infty} n^2 \left[1 - \frac{\pi}{2n} \sum_{k=1}^n \sin \frac{(2k-1)\pi}{4n} \right] & \\
\sin \frac{(2k-1)\pi}{4n} &= \sin \frac{\pi}{2} \left(\frac{k-1}{2n} + \frac{k}{2n} \right) \\
I_{n(k)} &= \lim_{n \rightarrow \infty} \frac{\pi}{2} \left(\frac{k}{n} - \frac{k-1}{n} \right) \sum_{k=1}^n \sin \left(\frac{\pi}{2} \frac{k-1+k}{2n} \right) = \frac{\pi}{2} \int_0^1 \sin \left(\frac{\pi x}{2} \right) dx = 1 \\
I_{n(k)} &= 1 + \frac{\pi^2}{96n^2} + o\left(\frac{1}{n^2}\right) \\
\int_0^1 \left(\ln \frac{1}{x} \right)^5 dx &= - \int_0^1 (\ln x)^5 dx = -x(\ln x)^5 \Big|_0^1 + 5 \int_0^1 (\ln x)^4 dx = 5x(\ln x)^4 \Big|_0^1 - 5 \int_0^1 4(\ln x)^3 dx \\
&= 20x(\ln x)^3 \Big|_0^1 - 20 \int_0^1 3(\ln x)^2 dx = -60x(\ln x)^2 \Big|_0^1 + 60 \int_0^1 2 \ln x dx = -120 \int_0^1 dx = -120 \\
\lim_{x \rightarrow -1} \frac{x^3 - ax^2 - x + 4}{x+1} &= \lim_{x \rightarrow -1} \frac{(x+1)^3 - 3x^2 - 3x - 1 - ax^2 - x + 4}{x+1} \\
&= \lim_{x \rightarrow -1} \frac{(x+1)^3 - (3+a)(x+1)^2 + 2(a+1)(x+1) + 4-a}{x+1} = \lim_{t \rightarrow 0} \frac{t^3 - (3+a)t^2 + 2(a+1)t + 4-a}{t} \\
&\quad 2a+2=l, 4-a=0 \rightarrow a=4, l=10
\end{aligned}$$

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{\sqrt{1+x \sin x} - \sqrt{\cos x}}{x \tan x} &= \lim_{x \rightarrow 0} \frac{1+x \sin x - \cos x}{(\sqrt{1+x \sin x} + \sqrt{\cos x})x \tan x} \\
&= \lim_{x \rightarrow 0} \frac{1+x(x+o(x)) - (1 - \frac{1}{2}x^2 + o(x^2))}{2x \tan x} = \frac{3}{4} \\
\sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n} &= x \sum_{n=1}^{\infty} \frac{x^{2n}}{2n} = x \int_0^x \sum_{n=1}^{\infty} t^{2n-1} dt = x \int_0^x \frac{t}{1-t^2} dt \\
&= \frac{x}{2} \int_0^x \frac{1}{1-t} - \frac{1}{1+t} dt = -\frac{x}{2} \ln(1-x^2), x \in (-1, 1) \\
&\quad x = \tan u, dx = \sec^2 u du \\
\int_0^{\sqrt{3}} \frac{1}{\sqrt{x^2+1}} dx &= \int_0^{\frac{\pi}{3}} \frac{\sec^2 u du}{\sec u} = \ln |\sec u + \tan u| \Big|_0^{\frac{\pi}{3}} = \ln(2 + \sqrt{3}) \\
\int x f''(x) dx &= \int x d(f'(x)) = x f'(x) - \int f'(x) dx + C = \\
&\quad x \left(\frac{1 - \ln x}{x^2} \right) - \frac{\ln x}{x} + C = \frac{1 - 2 \ln x}{x} + C
\end{aligned}$$

Consistent Continuity

$$f(x), x \in I$$

$$\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0, \forall x_1, x_2 \in I, |x_1 - x_2| < \delta, |f(x_1) - f(x_2)| < \epsilon$$

$$\forall \epsilon > 0, \exists \delta > 0, |x - x_0| < \delta, |f(x) - f(x_0)| < \epsilon$$

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left(\frac{\sqrt[n]{a} + \sqrt[n]{b} + \sqrt[n]{c}}{3} \right)^n &= e^{n \ln \left(\frac{\frac{1}{a} - 1 + b \frac{1}{n} - 1 + c \frac{1}{n} - 1}{3} + 1 \right)} = e^{n \frac{(\frac{1}{a} - 1 + b \frac{1}{n} - 1 + c \frac{1}{n} - 1)}{3}} = \\
&= e^{\ln a \frac{1}{3} + \ln b \frac{1}{3} + \ln c \frac{1}{3}} = \sqrt[3]{abc} \\
\lim_{x \rightarrow 0} \frac{x \ln(1+x) - ax^2 + bx^3}{x - \tan x} &= \lim_{x \rightarrow 0} \frac{x(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3)) - ax^2 + bx^3}{-\frac{1}{3}x^3 + o(x^3)} = 0 \\
1 - a = 0, b - \frac{1}{2} &= 0 \rightarrow a = 1, b = \frac{1}{2} \\
\iint_{\Sigma} x dy dz + y dz dx + z dx dy &= 3 \iiint_{\Omega} dv = 3 \iint_D \int_{\sqrt{x^2+y^2}}^{\sqrt{R^2-x^2-y^2}} dz d\sigma \\
&= 3 \iint_D (\sqrt{R^2-x^2-y^2} - \sqrt{x^2+y^2}) d\sigma, \quad D = \{(x, y) | x^2 + y^2 \leq \frac{R^2}{2}\} \\
&= 3 \int_0^{\frac{R}{\sqrt{2}}} (\sqrt{R^2-r^2} - r) r dr \int_0^{2\pi} d\theta = 6\pi \left(-\frac{1}{2} \int_{R^2}^{\frac{R^2}{2}} \sqrt{R^2-r^2} d(R^2-r^2) - \frac{R^3}{6\sqrt{2}} \right) \\
&= 6\pi \left(\frac{1}{3} u^{\frac{3}{2}} \Big|_{\frac{R^2}{2}}^{R^2} - \frac{R^3}{6\sqrt{2}} \right) = 2\pi R^3 \left(1 - \frac{1}{2\sqrt{2}} \right) - \pi R^3 \frac{1}{\sqrt{2}} = (2 - \sqrt{2})\pi R^3
\end{aligned}$$

$$\begin{aligned}
& \lim_{\Delta x \rightarrow 0} \frac{\ln(x + \Delta x) - \ln x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\ln(1 + \frac{\Delta x}{x})}{\Delta x} = \frac{1}{x} \\
& \lim_{\Delta x \rightarrow 0} \frac{\tan(x + \Delta x) - \tan x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{\tan x + \tan \Delta x}{1 - \tan x \tan \Delta x} - \tan x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{(\tan^2 x + 1) \tan \Delta x}{\Delta x} = \sec x \\
& \lim_{\Delta x \rightarrow 0} \frac{\cos(x + \Delta x) - \cos x}{\Delta x} = -\sin x \\
& \lim_{\Delta x \rightarrow 0} \frac{\sec(x + \Delta x) - \sec x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\cos x - \cos(x + \Delta x)}{\Delta x \cos(x + \Delta x) \cos x} = \tan x \sec x \\
& \lim_{\Delta x \rightarrow 0} \frac{\log_a(x + \Delta x) - \log_a x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\log_a(1 + \frac{\Delta x}{x}) \log_e a}{\Delta x \ln a} = \frac{1}{x \ln a} \\
& \lim_{\Delta x \rightarrow 0} \frac{\arctan(x + \Delta x) - \arctan x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\arctan(\frac{\Delta x}{1 + x^2 + x \Delta x})}{\Delta x} = \frac{1}{1 + x^2} \\
& \lim_{\Delta x \rightarrow 0} \frac{\arcsin(x + \Delta x) - \arcsin x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\arcsin((x + \Delta x)\sqrt{1 - x^2} - x\sqrt{1 - (x + \Delta x)^2})}{\Delta x} \\
& = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)\sqrt{1 - x^2} - x\sqrt{1 - (x + \Delta x)^2}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{x(\sqrt{1 - x^2} - \sqrt{1 - (x + \Delta x)^2})}{\Delta x} + \sqrt{1 - x^2} \\
& = \lim_{\Delta x \rightarrow 0} \frac{x(2x\Delta x + \Delta^2 x)}{\Delta x(\sqrt{1 - x^2} + \sqrt{1 - (x + \Delta x)^2})} + \sqrt{1 - x^2} = \frac{x^2}{\sqrt{1 - x^2}} + \sqrt{1 - x^2} = \frac{1}{\sqrt{1 - x^2}} \\
& \lim_{x \rightarrow 0^+} \frac{x^2 \sin \frac{1}{x} - 0}{x - 0} = \lim_{x \rightarrow 0^+} x \sin \frac{1}{x} = (x + o(x)) * M = 0, |M| \leq 1 \\
& f'(x) = 2x \sin \frac{1}{x} + x^2 \left(\frac{-1}{x^2}\right) \cos \frac{1}{x} = 2x \sin \frac{1}{x} - \cos \frac{1}{x}, (x \neq 0) \\
& \lim_{x \rightarrow 0^+} f'(x) = \lim_{x \rightarrow 0^+} 2x \sin \frac{1}{x} - \cos \frac{1}{x} = 0 - \nexists = \nexists \\
& \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{f(x)}{x^2} * x^2 = A * \lim_{x \rightarrow 0} x^2 = 0 \\
& \lim_{x \rightarrow 0} f(x) = f(0) = 0 \\
& \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{f(x)}{x} = f'(0) \\
& \lim_{x \rightarrow 0} \frac{1 - \cos x \sqrt{\cos 2x} \sqrt[3]{\cos 3x}}{x^2} \\
& x \rightarrow 0, \ln(1 + x) = x - \frac{x^2}{2} + o(x^2), \ln(1 + 2x) = 2x - \frac{4x^2}{2} + o(x^2) \\
& \lim_{x \rightarrow 0} \frac{xf(x) - \ln(1 + 2x)}{x - \ln(1 + x)} = \lim_{x \rightarrow 0} \frac{xf(x) - 2x + 2x^2 + o(x^2)}{\frac{x^2}{2} + o(x^2)} = 4 \\
& x \rightarrow 0, f(x) = 2 + o(1), f'(0) = 0
\end{aligned}$$

$$\begin{aligned}
f(x) &= \frac{1}{x} - \frac{1}{e^x - 1}, \lim_{x \rightarrow 0} \frac{f(x) - \lim_{x \rightarrow 0} f(x)}{x} = ? \\
\lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x(e^x - 1)} = \lim_{x \rightarrow 0} \frac{1 + x + \frac{x^2}{2} + o(x^2) - 1 - x}{x^2 + o(x^2)} = \frac{1}{2} \\
&\quad \lim_{x \rightarrow 0} \frac{f(x) - \frac{1}{2}}{x} \\
&= \lim_{x \rightarrow 0} \frac{2e^x - 2 - 2x - xe^x + x}{2x^2(e^x - 1)} = \lim_{x \rightarrow 0} \frac{2(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)) - 2 - 2x - x(1 + x + \frac{x^2}{2} + o(x^2)) + x}{2x^3 + o(x^3)} \\
&\quad = \lim_{x \rightarrow 0} \frac{-\frac{1}{6}x^3 + o(x^3)}{2x^3 + o(x^3)} = -\frac{1}{12} \\
\lim_{x \rightarrow 0} \frac{\tan \tan x - \sin \sin x}{x^3} &= \lim_{x \rightarrow 0} \frac{\tan \tan x - \tan \sin x}{x^3} + \frac{\tan \sin x - \sin \sin x}{x^3} = \frac{1}{2} + \frac{1}{2} = 1
\end{aligned}$$

$$\lim_{x \rightarrow +\infty} \sqrt{x^2 + x + 1} \frac{x - \ln(e^x + x)}{x} \sim \frac{\infty * \infty}{\infty} = \lim_{x \rightarrow +\infty} \sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} \ln \left(\frac{e^x}{e^x + x} \right) = 1 * 0 = 0$$

$$2 \sin(x + 2y - 3z) = x + 2y - 3z$$

$$2 \cos(x + 2y - 3z)(1 - 3z'_x) = 1 - 3z'_x$$

$$2 \cos(x + 2y - 3z)(2 - 3z'_y) = 2 - 3z'_y$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{1}{3} + \frac{2}{3} = 1$$

$$F(x) = \int_0^x f(t) dt$$

$$\lim_{x \rightarrow 0} \frac{2 \int_0^x (x-t)f(t)dt}{x \int_0^x f(x-t)dt} = \lim_{x \rightarrow 0} \frac{2x \int_0^x f(t)dt - 2 \int_0^x tf(t)dt}{x \int_0^x f(u)du} = \lim_{x \rightarrow 0} \frac{2F(x) + 2xf(x) - 2xf(x)}{F(x) + xf(x)}$$

$$= \lim_{x \rightarrow 0, \xi \rightarrow 0} \frac{2xf(\xi)}{xf(\xi) + xf(x)} = 1$$

$$ABA^{-1} = BA^{-1} + 3I, AB = B + 3A, (A - I)B = 3A, B = 3(A - I)^{-1}A$$

$$A^*A = |A|I, |A^*| = 16, |A| = 4, A^* = 4A^{-1}, A = 4(A^*)^{-1}$$

$$A = diag(4, \frac{1}{4}, 4), B = 3diag(\frac{1}{3}, -\frac{4}{3}, \frac{1}{3})diag(4, \frac{1}{4}, 4) = diag(4, -1, 4)$$

$$\begin{aligned} \int \frac{x^2}{1+x^2} \arctan x dx &= \int \left(1 - \frac{1}{1+x^2}\right) \arctan x dx = \int \arctan x dx - \int \arctan x d(\arctan x) \\ &= x \arctan x - \int \frac{x}{1+x^2} dx - \frac{1}{2}(\arctan x)^2 = x \arctan x - \frac{1}{2} \ln(1+x^2) - \frac{1}{2}(\arctan x)^2 + C \end{aligned}$$

$$\begin{aligned}
\int \frac{dx}{e^x - e^{-x}} &= \int \frac{de^x}{e^{2x} - 1} = \frac{1}{2} \left(\int \frac{d(e^x - 1)}{e^x - 1} - \int \frac{d(e^x + 1)}{e^x + 1} \right) = \frac{1}{2} \ln \left| \frac{e^x - 1}{e^x + 1} \right| + C \\
\int \frac{x}{(1-x)^3} dx &= \int \frac{1 - (1-x)}{(1-x)^3} dx = \int \frac{1}{(1-x)^3} - \frac{1}{(1-x)^2} dx = \frac{1}{2(1-x)^2} - \frac{1}{1-x} + C \\
\int \frac{x^2}{a^6 - x^6} dx &= \int \frac{dx^3}{3(a^3 - x^3)(a^3 + x^3)} = \int -\frac{d(a^3 - x^3)}{6a^3(a^3 - x^3)} + \int \frac{d(a^3 + x^3)}{6a^3(a^3 + x^3)} = \frac{1}{6a^3} \ln \left| \frac{a^3 + x^3}{a^3 - x^3} \right| + C \\
&\int \frac{1 + \cos x}{x + \sin x} dx = \ln |x + \sin x| + C \\
\int \frac{\ln(\ln x)}{x} dx &= \int \ln(\ln x) d \ln x = \ln x \ln |\ln x| - \int \frac{\ln x}{x \ln x} dx = \ln x \ln |\ln x| - \ln x + C \\
\int \frac{\sin x \cos x}{1 + \sin^4 x} dx &= \int \frac{\sin 2x}{2(1 + \frac{(1 - \cos 2x)^2}{4})} dx = \int \frac{1}{2} \frac{d(\frac{1 - \cos 2x}{2})}{1 + \left(\frac{1 - \cos 2x}{2}\right)^2} = \frac{1}{2} \arctan \sin^2 x + C \\
\int \tan^4 x dx &= \int (\sec^2 x - 1)^2 dx = \int (\tan^2 x + 1) d \tan x - 2 \tan x + x = \frac{1}{3} \tan^2 x - \tan x + x + C \\
&\cos(a + b) - \cos(a - b) = -2 \sin a \sin b, \sin(a + b) + \sin(a - b) = 2 \sin a \cos b \\
\int \sin x \sin 2x \sin 3x dx &= \frac{1}{2} \int (\cos x - \cos 3x) \sin 3x dx = -\frac{1}{4} \int \sin 6x dx + \frac{1}{4} \int (\sin 4x + \sin 2x) dx \\
&= \frac{\cos 6x}{24} - \frac{\cos 4x}{16} - \frac{\cos 2x}{8} + C \\
\int \frac{dx}{x(x^6 + 4)} &= \frac{1}{4} \int \frac{1}{x} - \frac{x^5}{x^6 + 4} dx = \frac{1}{4} \ln |x| - \frac{1}{24} \ln(x^6 + 4) + C \\
a > 0, \int \sqrt{\frac{a+x}{a-x}} dx &= \int \frac{a+x}{\sqrt{a^2 - x^2}} dx = \int \frac{ad\frac{x}{a}}{\sqrt{1 - (\frac{x}{a})^2}} - \frac{d(a^2 - x^2)}{2\sqrt{a^2 - x^2}} = a \arcsin \frac{x}{a} - \sqrt{a^2 - x^2} + C
\end{aligned}$$

$$\begin{aligned}
\int \frac{dx}{\sqrt{x(1+x)}} &= \int \frac{d(2x+1)}{\sqrt{(2x+1)^2-1}} = \ln(2x+1+\sqrt{(2x+1)^2-1}) + C = \ln|2x+\sqrt{4x^2+4x+1}| + C \\
\frac{1}{2} \int x(\cos 2x+1)dx &= \frac{1}{4} \int x d \sin 2x + \frac{x^2}{4} = \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + \frac{x^2}{4} + C \\
I = \int e^{ax} \cos bx dx &= \frac{1}{a} \int \cos bxd(e^{ax}) = \frac{\cos bxe^{ax}}{a} + \frac{b}{a} \int e^{ax} \sin bxdx = A(x, a, b) + \frac{b}{a^2} \int \sin bxd(e^{ax}) \\
&= A(x, a, b) + \frac{b \sin bxe^{ax}}{a^2} - \frac{b^2}{a^2} \int e^{ax} \cos bxdx = A(x, a, b) + B(x, a, b) - \frac{b^2}{a^2} I + C_1 = I \\
\frac{b^2+a^2}{a^2} I &= \frac{ae^{ax} \cos bx + be^{ax} \sin bx}{a^2} + C_2, \quad I = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2+b^2} + C \\
I &= \int \frac{dx}{\sqrt{1+e^x}}, \sqrt{1+e^x} = u, x = \ln(u^2-1), dx = \frac{2udu}{u^2-1} \\
I &= \int \frac{2du}{(u+1)(u-1)} = \int \frac{du}{u-1} - \int \frac{du}{u+1} = \ln \left| \frac{u-1}{u+1} \right| + C = \ln \left| \frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1} \right| + C \\
I &= \int \frac{dx}{x^2 \sqrt{x^2-1}}, x = \sec u, dx = \sec u \tan u du, \cos u = \frac{1}{x}, \sin u = \frac{\sqrt{x^2-1}}{x} \\
I &= \int \frac{\sec u \tan u du}{\sec^2 u \tan u} = \sin u + C = \frac{\sqrt{x^2-1}}{x} + C \\
I &= \int \frac{dx}{(a^2-x^2)^{\frac{5}{2}}}, x = a \sin u, dx = a \cos u du, \tan u = \frac{x}{\sqrt{a^2-x^2}} \\
I &= \int \frac{a \cos u du}{a^5 \cos^5 u} = \frac{1}{a^4} \int \sec^4 u du = \frac{1}{a^4} \left(\frac{1}{3} \tan^3 u + \tan u \right) + C = \frac{x^3}{3a^4(a^2-x^2)^{\frac{3}{2}}} + \frac{x}{a^4 \sqrt{a^2-x^2}} + C \\
I &= \int \frac{dx}{x^4 \sqrt{1+x^2}}, x = \tan u, dx = \sec^2 u du, (\csc x)' = -\csc x \cot x, 1 + \cot^2 x = \csc^2 x, \csc u = \frac{\sqrt{x^2+1}}{x} \\
I &= \int \frac{\sec^2 u du}{\tan^4 u \sec u} = \int \frac{\cos^3 u du}{\sin^4 u} = \int \cot^3 u \csc u du = \int (1 - \csc^2 u) d(\csc u) = \csc u - \frac{\csc^3 u}{3} + C \\
I &= \frac{\sqrt{x^2+1}}{x} - \frac{(x^2+1)^{\frac{3}{2}}}{3x^3} + C \\
I &= \int \sqrt{x} \sin \sqrt{x} dx, \sqrt{x} = u, x = u^2, dx = 2udu \\
I &= \int 2u^2 \sin u du = \int -2u^2 d(\cos u) = -2u^2 \cos u + \int 4u d \sin u = -2u^2 \cos u + 4u \sin u + 4 \cos u + C \\
I &= -2x \cos \sqrt{x} + 4\sqrt{x} \sin \sqrt{x} + 4 \cos \sqrt{x} + C \\
\int \ln(1+x^2) dx &= x \ln(1+x^2) - 2 \int \left(1 - \frac{1}{1+x^2} \right) dx = x \ln(1+x^2) - 2x + 2 \arctan x + C \\
\int \frac{\sin^2 x}{\cos^3 x} dx &= \int \tan^2 x \sec x dx = \int \tan x d \sec x = \tan x \sec x - \int \sec^3 x dx \\
&= \frac{1}{2} (\tan x \sec x - \ln |\sec x + \tan x|) + C
\end{aligned}$$

$$\begin{aligned}
I &= \int \arctan \sqrt{x} dx, \sqrt{x} = u, x = u^2, dx = 2u du \\
I &= \int 2u \arctan u du = \int \arctan u d(u^2 + 1) = (u^2 + 1) \arctan u - u + C = (x + 1) \arctan \sqrt{x} - \sqrt{x} + C \\
\sin x &= \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}} = \frac{2 \tan \frac{x}{2}}{\tan^2 \frac{x}{2} + 1}, \tan x = \frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}}, \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \\
\tan \frac{x}{2} &= u, x = 2 \arctan u, dx = \frac{2 du}{u^2 + 1} \\
I &= \int \frac{\sqrt{1 + \cos x}}{\sin x} dx = \int \frac{\sqrt{1 + \frac{1 - u^2}{1 + u^2}} \cdot \frac{2}{u^2 + 1}}{\frac{2u}{u^2 + 1}} du = \int \frac{\sqrt{2}}{u \sqrt{u^2 + 1}} du \\
u &= \tan k, \cot k = \frac{1}{u}, \csc k = \frac{\sqrt{u^2 + 1}}{u}, du = \sec^2 k dk \\
I &= \sqrt{2} \int \csc k dk = -\sqrt{2} \ln |\cot k + \csc k| + C = \sqrt{2} \ln \left| \frac{u}{\sqrt{u^2 + 1} + 1} \right| + C \\
&= \sqrt{2} \ln \left| \frac{\tan \frac{x}{2}}{\sqrt{\tan^2 \frac{x}{2} + 1} + 1} \right| + C \quad \text{or, } k = \frac{x}{2}, I = -\sqrt{2} \ln \left| \cot \frac{x}{2} + \csc \frac{x}{2} \right| + C \\
\int \frac{x^3}{(1 + x^8)^2} dx &= \int \frac{dx^4}{4(1 + (x^4)^2)^2} x^4 \xrightarrow{u} I = \int \frac{(1 + u^2) + (1 - u^2) du}{8(1 + u^2)^2} = \int \frac{du}{8(1 + u^2)} - \int \frac{d(\frac{1}{u} + u)}{8(\frac{1}{u} + u)^2} \\
&= \frac{1}{8} \arctan u + \frac{u}{8(1 + u^2)} + C = \frac{1}{8} \arctan x^4 + \frac{x^4}{8(1 + x^8)} + C \\
\int \frac{x^{11}}{x^8 + 3x^4 + 2} dx &= \int \frac{x^{11}}{(x^4 + 1)(x^4 + 2)} dx = \int \frac{d(x^{12})}{12(x^4 + 1)} - \int \frac{d(x^{12})}{12(x^4 + 2)} x^4 \xrightarrow{u} \\
&\frac{1}{4} \left(\int \frac{(u + 1)(u - 1) + 1}{u + 1} du - \int \frac{(u + 2)(u - 2) + 4}{u + 2} du \right) = \frac{1}{4} \left(u + \ln \left| \frac{u + 1}{(u + 2)^4} \right| \right) + C \\
&= \frac{x^4}{4} + \frac{1}{4} \ln \left| \frac{x^4 + 1}{(x^4 + 2)^4} \right| + C \\
\int \frac{dx}{16 - x^4} &= \int \frac{dx}{(4 - x^2)(4 + x^2)} = \int \frac{dx}{32(2 - x)} + \int \frac{dx}{32(2 + x)} + \int \frac{d(\frac{x}{2})}{16(1 + (\frac{x}{2})^2)} \\
&= \frac{1}{32} \left(\ln \left| \frac{2 + x}{2 - x} \right| + 2 \arctan \frac{x}{2} \right) + C \\
\int \frac{\sin x}{1 + \sin x} dx &\xrightarrow{\tan \frac{x}{2} = u} \int \frac{\frac{4u}{(u^2 + 1)^2}}{1 + \frac{2u}{u^2 + 1}} du = \int \frac{4u du}{(u^2 + 1)(u + 1)^2} = \int \frac{2 du}{u^2 + 1} - \int \frac{2 du}{(u + 1)^2} \\
&= 2 \arctan u + \frac{2}{u + 1} + C = x + \frac{2}{\tan \frac{x}{2} + 1} + C
\end{aligned}$$

$$\begin{aligned}
\int \frac{x + \sin x}{1 + \cos x} dx &= \int x d(\tan \frac{x}{2}) - \int \frac{d(1 + \cos x)}{1 + \cos x} = x \tan \frac{x}{2} - \int \tan \frac{x}{2} dx - \ln |1 + \cos x| \\
&= x \tan \frac{x}{2} + 2 \ln |\cos \frac{x}{2}| - \ln |1 + \cos x| + C = x \tan \frac{x}{2} + C \\
\int e^{\sin x} \frac{x \cos^3 x - \sin x}{\cos^2 x} dx &= \int x d(e^{\sin x}) - \int e^{\sin x} d(\sec x) = x e^{\sin x} - \int e^{\sin x} dx - \sec x e^{\sin x} + \int e^{\sin x} dx \\
&= (x - \sec x) e^{\sin x} + C \\
\int \frac{x^{\frac{1}{3}}}{x(x^{\frac{1}{2}} + x^{\frac{1}{3}})} dx, x^{\frac{1}{6}} = u, x = u^6, dx &= 6u^5 du \\
\int \frac{6u^7}{u^6(u^3 + u^2)} du &= \int \frac{6du}{u(u+1)} = \int \frac{6du}{u} - \int \frac{6du}{u+1} = 6 \ln \left| \frac{u}{u+1} \right| + C = 6 \ln \left| \frac{x^{\frac{1}{6}}}{x^{\frac{1}{6}} + 1} \right| + C \\
I &= \int \frac{dx}{(1 + e^x)^2}, e^x = u, x = \ln u, dx = \frac{du}{u} \\
I &= \int \frac{du}{u(1+u)^2} = \int \frac{du}{u} - \int \frac{(u+2)du}{(1+u)^2} = \ln |u| + \frac{2}{1+u} + \int u d\left(\frac{1}{1+u}\right) = \ln |u| + \frac{2+u}{1+u} - \ln |1+u| \\
&= \ln \left| \frac{e^x}{1+e^x} \right| + \frac{1}{1+e^x} + C \\
\int \frac{e^{3x} + e^x}{e^{4x} - e^{2x} + 1} dx &= \int \frac{e^x + e^{-x}}{e^{2x} + e^{-2x} - 1} dx = \int \frac{d(e^x - e^{-x})}{(e^x - e^{-x})^2 + 1} = \arctan(e^x - e^{-x}) + C \\
\int \frac{x e^x}{(1 + e^x)^2} dx &= - \int \ln u d\left(\frac{1}{1+u}\right) = -\frac{\ln u}{1+u} + \int \frac{1}{u} - \frac{1}{1+u} du = \ln \left| \frac{u}{1+u} \right| - \frac{\ln u}{1+u} + C \\
&= \ln \frac{e^x}{1+e^x} - \frac{x}{1+e^x} + C \\
\int \ln^2(x + \sqrt{1+x^2}) dx &= x \ln^2(x + \sqrt{1+x^2}) - 2 \int \ln(x + \sqrt{1+x^2}) d(\sqrt{1+x^2}) \\
&= x \ln^2(x + \sqrt{1+x^2}) - 2\sqrt{1+x^2} \ln^2(x + \sqrt{1+x^2}) + 2x + C
\end{aligned}$$

$$\begin{aligned}
x &= \tan u, u = \arctan x, dx = \sec^2 u du, \sin u = \frac{x}{\sqrt{1+x^2}}, \cos u = \frac{1}{\sqrt{1+x^2}} \\
\int \frac{\ln x}{(1+x^2)^{\frac{3}{2}}} dx &= \int (\ln(\sin u) - \ln(\cos u)) d \sin u = \sin u (\ln(\sin u) - 1) - \sin u \ln(\cos u) \\
&\quad - \int \frac{\sin^2 u}{\cos u} du = A(u) + \int \cos u - \sec u du = A(u) + \sin u - \ln |\sec u + \tan u| + C \\
&= \frac{x}{\sqrt{1+x^2}} \ln x - \ln(x + \sqrt{1+x^2}) + C \\
x &= \sin u, u = \arcsin x, dx = \cos u du, \sin 2u = 2x\sqrt{1-x^2}, \cos 2u = 1-2x^2 \\
\int \sqrt{1-x^2} \arcsin x dx &= \int u \cos^2 u du = \frac{u \sin 2u}{4} + \frac{\cos 2u}{8} + \frac{u^2}{4} + C \\
&= \frac{x\sqrt{1-x^2} \arcsin x}{2} - \frac{x^2}{4} + \frac{\arcsin^2 x}{4} + C \\
x &= \cos u, u = \arccos x, dx = -\sin u du \\
I_{36} &= \int \frac{x^3 \arccos x}{\sqrt{1-x^2}} dx = -\int \frac{u \cos^3 u \sin u}{\sin u} du = -\int u \cos^3 u du = -\frac{1}{2} \int u \cos u (\cos 2u + 1) du \\
&= -\frac{1}{4} \int u (\cos 3u + \cos u) du - \frac{1}{2} \int u \cos u du = -\frac{1}{36} \int 3u \cos 3u d(3u) - \frac{3}{4} \int u \cos u du \\
&= -\frac{1}{36} (3u \sin 3u + \cos 3u) - \frac{3}{4} (u \sin u + \cos u) + C \\
&= -\frac{1}{12} ((4x^2 - 1)\sqrt{1-x^2} \arccos x) - \frac{x(4x^2 - 3)}{36} - \frac{3}{4} (\sqrt{1-x^2} \arccos x + x) + C \\
\sin 3u &= \sin 2u \cos u + \cos 2u \sin u = (4x^2 - 1)\sqrt{1-x^2} \\
\cos 3u &= \cos 2u \cos u - \sin 2u \sin u = (4x^2 - 3)x \\
I_{36} &= -\frac{1}{3} \sqrt{1-x^2} (x^2 + 2) \arccos x - \frac{x(x^2 + 6)}{9} + C \\
\int \frac{\cot x}{1 + \sin x} dx &= \int \frac{d(\sin x)}{\sin x(1 + \sin x)} \xrightarrow{u} \int \frac{du}{u} - \int \frac{du}{1+u} = \ln \left| \frac{\sin x}{1 + \sin x} \right| + C \\
\tan x &= u, x = \arctan u, dx = \frac{du}{u^2 + 1} \\
\int \frac{dx}{\sin^3 x \cos x} &= \int \frac{4}{(1 - \cos 2x) \sin 2x} dx = \int \frac{4du}{(u^2 + 1)(1 - \frac{1-u^2}{1+u^2}) \frac{2u}{1+u^2}} = \int \frac{(1+u^2)du}{u^3} \\
&= -\frac{1}{2u^2} + \ln u + C = -\frac{\cot^2 x}{2} + \ln |\tan x| + C \\
\int \frac{dx}{(2 + \cos x) \sin x} &= \int \frac{\frac{2}{u^2+1} du}{(2 + \frac{1-u^2}{1+u^2}) \frac{2u}{u^2+1}} = \int \frac{(u^2 + 1)du}{(u^2 + 3)u} = \int \frac{du}{3u} + \int \frac{d(u^2 + 3)}{3(u^2 + 3)} \\
&= \frac{1}{3} \ln |u| + \frac{1}{3} \ln(u^2 + 3) + C = \frac{1}{3} \ln \left(\tan \frac{x}{2} (\tan^2 \frac{x}{2} + 3) \right) + C
\end{aligned}$$

$$\begin{aligned}
\sin x &= \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}} = \frac{2u}{u^2 + 1}, \tan x = \frac{2u}{1 - u^2}, \cos x = \frac{1 - u^2}{1 + u^2} \\
\tan \frac{x}{2} &= u, x = 2 \arctan u, dx = \frac{2du}{u^2 + 1} \\
\int \frac{\sin x \cos x}{\sin x + \cos x} dx &= \int \frac{\frac{2u}{u^2+1} \frac{1-u^2}{1+u^2} \frac{2}{u^2+1}}{\frac{2u}{u^2+1} + \frac{1-u^2}{1+u^2}} du = \int \frac{4u(1-u^2)}{(u^2+1)^2(1-u^2+2u)} du \\
&= \int \frac{au^2 + bu + c}{(u^2+1)^2} + \frac{d}{1-u^2+2u} du \xrightarrow{a=-1, b=2, c=1, d=-1} \int \frac{-u^2+2u+1}{(u^2+1)^2} - \frac{1}{1-u^2+2u} du \\
&= \int \frac{d(u^2+1)}{(u^2+1)^2} - \int \frac{d(u+\frac{1}{u})}{(u+\frac{1}{u})^2} + \int \frac{du}{(u-1+\sqrt{2})(u-1-\sqrt{2})} \\
&= \frac{u-1}{u^2+1} + \frac{1}{2\sqrt{2}} \left(\int \frac{d(u-1-\sqrt{2})}{u-1-\sqrt{2}} - \int \frac{d(u-1+\sqrt{2})}{u-1+\sqrt{2}} \right) \\
&= \frac{u-1}{u^2+1} + \frac{\sqrt{2}}{4} \ln \left| \frac{u-1-\sqrt{2}}{u-1+\sqrt{2}} \right| + C = \frac{\tan \frac{x}{2} - 1}{\tan^2 \frac{x}{2} + 1} + \frac{\sqrt{2}}{4} \ln \left| \frac{\tan \frac{x}{2} - 1 - \sqrt{2}}{\tan \frac{x}{2} - 1 + \sqrt{2}} \right| + C
\end{aligned}$$