Elementary Function

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \sum_{k=0}^{n} C_n^k a^{n-k} b^k$$

$$a^n - b^n = (a-b) \cdot \sum_{k=0}^{n-1} a^{n-1-k} b^k$$

$$\arcsin x + \arccos x = \arctan x + \mathrm{arccot} x = \frac{\pi}{2}$$

$$f(x) = \int_a^x f'(t)\mathrm{d}t + f(a)$$

$$K = \frac{|y''x' - y'x''|}{(x'^2 + y'^2)^{\frac{3}{2}}}, \quad \rho = \frac{1}{K}$$

$$\int_0^1 f(x)\mathrm{d}x = \sum_{i=1}^{n} \left(\frac{i+1}{n} - \frac{i}{n}\right) f\left(\frac{i}{n}\right) = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{2i+1}{2n}\right)$$

$$\int_{t1}^{t2} f(x(t), y(t))\mathrm{d}s = \int_a^b f(x, y(x))\sqrt{x'^2 + y'^2}\mathrm{d}x = \int_\alpha^\beta f(r,\theta)\sqrt{r^2 + r'^2}\mathrm{d}\theta$$

$$S = \int_a^b y(x)\mathrm{d}x = \int_{t1}^{t2} y(t)x'(t)\mathrm{d}t = \frac{1}{2}\int_\alpha^\beta r^2(\theta)\mathrm{d}\theta$$

$$S_{side} = \int_a^b 2\pi y(x)\sqrt{x'^2(x) + y'^2(x)}\mathrm{d}x = \int_\alpha^\beta 2\pi r(\theta)\sin\theta\sqrt{r^2(\theta) + r'^2(\theta)}\mathrm{d}\theta$$

$$V_x = \int_a^b \pi y^2(x)\mathrm{d}x \quad V_y = \int_a^b 2\pi x y(x)\mathrm{d}x$$

$$\boldsymbol{1's} \quad \boldsymbol{Taylor} \quad f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}x^k + R_{k+1}(\xi)$$

$$\boldsymbol{2's} \quad \boldsymbol{Taylor} \quad f(x,y) = f(x_0,y_0) + f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0)$$
$$+\frac{1}{2}f''_{xy}(x_0,y_0)(x-x_0)^2 + \frac{1}{2}f''_{yy}(x_0,y_0)(y-y_0)^2 + \frac{1}{2}f''_{xy}(x_0,y_0)(x-x_0)(y-y_0) + o(\rho)$$

$$\boldsymbol{important} \quad \boldsymbol{integral} \quad \boldsymbol{ABS}: \quad \int u\mathrm{d}u = \ln|u| + C$$

$$\lim \boldsymbol{type}: \frac{0}{0} \quad \frac{\infty}{\infty} \quad 0*\infty \quad \infty - \infty \quad \infty^0 \quad 0^0 \quad 1^\infty$$

Gamma Function Integral

$$\left(-\frac{1}{2}\right)! = \sqrt{\pi}$$

$$\left(\frac{2n+1}{2}\right)! = \left(\Pi_{i=n}^{0}\frac{2i+1}{2}\right)\sqrt{\pi} = \frac{2n+1}{2}\frac{2n-1}{2}\dots\frac{1}{2}\sqrt{\pi}$$

$$\int_0^\infty x^a \mathrm{e}^{-x}\mathrm{d}x = a!$$

$$\int_0^\infty x^{-\frac{1}{2}} \mathrm{e}^{-x}\mathrm{d}x = \sqrt{\pi}$$

$$\int_0^\infty x^3 \mathrm{e}^{-x}\mathrm{d}x = (3)! = 6$$

$$\int_0^\infty x^{\frac{5}{2}} \mathrm{e}^{-x}\mathrm{d}x = \frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}$$

$$\int_0^\infty x^a \mathrm{e}^{-x^2}\mathrm{d}x = \frac{1}{2}\left(\frac{a-1}{2}\right)!$$

$$\int_0^\infty x^1 \mathrm{e}^{-x^2}\mathrm{d}x = \frac{1}{2}\left(\frac{1-1}{2}\right)! = \frac{1}{2}$$

$$\int_0^\infty x^4 \mathrm{e}^{-x^2}\mathrm{d}x = \frac{1}{2}\left(\frac{4-1}{2}\right)! = \frac{1}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}$$

$$\int_0^\infty x^7 \mathrm{e}^{-x^2}\mathrm{d}x = \frac{1}{2}\left(\frac{7-1}{2}\right)! = \frac{1}{2}(3)! = 3$$

Multivariate Integral

$$\iiint_{\Omega_{xyz}} f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{\Omega_{uvw}} f(u,v,w)\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \mathrm{d}u\mathrm{d}v\mathrm{d}w$$

$$\left.\frac{\partial f}{\partial l}\right|_{P_0=(x_0,y_0,z_0)} = f_x(P_0)\cos\alpha + f_y(P_0)\cos\beta + f_z(P_0)\cos\gamma$$

$$\mathbf{grad} f(x_0,y_0,z_0) = f_x(x_0,y_0,z_0)\vec{\boldsymbol{i}} + f_y(x_0,y_0,z_0)\vec{\boldsymbol{j}} + f_z(x_0,y_0,z_0)\vec{\boldsymbol{k}}$$

$$(\bar{x},\bar{y},\bar{z}) = \left(\frac{\iiint_\Omega x\rho\mathrm{d}v}{\iiint_\Omega \rho\mathrm{d}v}, \frac{\iiint_\Omega y\rho\mathrm{d}v}{\iiint_\Omega \rho\mathrm{d}v}, \frac{\iiint_\Omega z\rho\mathrm{d}v}{\iiint_\Omega \rho\mathrm{d}v}\right), \quad J_{k_j} = \iiint_\Omega ((\sum_{i=1}^n k_i^2) - k_j^2)\rho\mathrm{d}v$$

$$S = \iint_\Sigma \mathrm{d}S = \iint_{D_{xy}} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\mathrm{d}x\mathrm{d}y$$

$$S = \iint_{D_{zx}} \sqrt{1+\left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2}\mathrm{d}z\mathrm{d}x = \iint_{D_{yz}} \sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\mathrm{d}y\mathrm{d}z$$

$$\iint_{-\Sigma} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y = -\iint_\Sigma P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y$$

$$\oint_\Sigma P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y = \iiint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}v, \quad \mathrm{div}\vec{\boldsymbol{A}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\Sigma : F(x,y,z) = 0 \quad \vec{n} = \left(\frac{F_x}{\sqrt{F_x^2+F_y^2+F_z^2}}, \frac{F_y}{\sqrt{F_x^2+F_y^2+F_z^2}}, \frac{F_z}{\sqrt{F_x^2+F_y^2+F_z^2}}\right) = (\cos\alpha, \cos\beta, \cos\gamma)$$

$$\oint_\Sigma P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y = \iint_\Sigma (P\cos\alpha + Q\cos\beta + R\cos\gamma)\mathrm{d}S$$

$$\oint_\Gamma P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = \iint_\Sigma \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathrm{d}y\mathrm{d}z + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathrm{d}z\mathrm{d}x + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

$$\mathbf{rot}\vec{\boldsymbol{A}} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{\boldsymbol{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{\boldsymbol{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{\boldsymbol{k}}$$

$$\iint_\Sigma f(x,y,z)\mathrm{d}z\mathrm{d}x = \iint_\Sigma -f(x,y,z)\mathrm{d}x\mathrm{d}z \quad \iint_D f(x,y)\mathrm{d}z\mathrm{d}x = \iint_D f(x,y)\mathrm{d}x\mathrm{d}z$$

Infinite Series

$$\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|=\rho \quad ,R=\frac{1}{\rho}$$

$$S(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l})$$

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)\mathrm{d}x,\quad a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}\mathrm{d}x,\quad b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$

$$S(x)=\begin{cases} f(x), x\in consecutive \\ \frac{f(x-0)+f(x+0)}{2}, x\in discontinuity \\ \frac{f(l-0)+f(l+0)}{2}, x=\{-l,l\} \end{cases}$$

Matrix Calculation

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\quad \mathbf{A^T}=\begin{pmatrix} a_{11} & a_{21} & a_{31}\\ a_{12} & a_{22} & a_{32}\\ a_{13} & a_{23} & a_{33}\end{pmatrix}$$

$$\mathbf{A}^*=\begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{21}} & \mathbf{A_{31}}\\ \mathbf{A_{12}} & \mathbf{A_{22}} & \mathbf{A_{32}}\\ \mathbf{A_{13}} & \mathbf{A_{23}} & \mathbf{A_{33}}\end{bmatrix}\quad \alpha^{\mathbf{T}}=\begin{pmatrix} a_1\\ a_2\\ a_3\end{pmatrix}\quad |\mathbf{A}|=\det\mathbf{A}=\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$

$$|\mathbf{A}|=\Pi_{i=1}^n\lambda_i\quad |k\mathbf{A}|=k^n|\mathbf{A}|\quad |\mathbf{AB}|=|\mathbf{A}||\mathbf{B}|\quad |\mathbf{A}^n|=|\mathbf{A}|^n$$

$$|\mathbf{A^T}|=|\mathbf{A}|\quad (k\mathbf{A})^{\mathbf{T}}=k\mathbf{A^T}\quad (\mathbf{A^T})^{\mathbf{T}}=\mathbf{A}\quad (\mathbf{A}^n)^{\mathbf{T}}=(\mathbf{A^T})^n$$

$$|\mathbf{A}^{-1}|=\frac{1}{|\mathbf{A}|}\quad (k\mathbf{A})^{-1}=\frac{1}{k}\mathbf{A}^{-1}\quad (\mathbf{A}^{-1})^{-1}=\mathbf{A}\quad (\mathbf{A}^n)^{-1}=(\mathbf{A}^{-1})^n$$

$$|\mathbf{A}^*|=|\mathbf{A}|^{n-1}\quad (k\mathbf{A})^*=k^{n-1}\mathbf{A}^*\quad (\mathbf{A}^*)^*=|\mathbf{A}|^{n-2}\mathbf{A}\quad \mathbf{AA}^*=\mathbf{A}^*\mathbf{A}=|\mathbf{A}|\mathbf{E}$$

$$(\mathbf{ABC})^T=\mathbf{C^TB^TA^T}\quad (\mathbf{ABC})^{-1}=\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}\quad (\mathbf{ABC})^*=\mathbf{C}^*\mathbf{B}^*\mathbf{A}^*$$

$$\mathbf{diag}(a_1,...,a_n)\mathbf{diag}(b_1,...,b_n)=\mathbf{diag}(a_1b_1,...,a_nb_n)$$

$$\Lambda=\mathbf{diag}(\lambda_1,...,\lambda_n),|\Lambda|=\Pi_{i=1}^n\lambda_i,\varphi(\Lambda)=\mathbf{diag}(\varphi(\lambda_1),...,\varphi(\lambda_n))$$

$$\Lambda=\begin{pmatrix} 0 & 0 & \lambda_1\\ 0 & \lambda_i & 0\\ \lambda_n & 0 & 0\end{pmatrix}\quad |\Lambda|=(-1)^{\frac{n(n-1)}{2}}\Pi_{i=1}^n\lambda_i\quad \Lambda^{-1}=\begin{pmatrix} 0 & 0 & \frac{1}{\lambda_n}\\ 0 & \frac{1}{\lambda_i} & 0\\ \frac{1}{\lambda_1} & 0 & 0\end{pmatrix}$$

$$\begin{vmatrix} \mathbf{A} & *\\ \mathbf{O} & \mathbf{B}\end{vmatrix}=\begin{vmatrix} \mathbf{A} & \mathbf{O}\\ * & \mathbf{B}\end{vmatrix}=|\mathbf{A}||\mathbf{B}|$$

$$\begin{vmatrix} \mathbf{O} & \mathbf{A}_{mm}\\ \mathbf{B}_{nn} & *\end{vmatrix}=\begin{vmatrix} * & \mathbf{A}_{mm}\\ \mathbf{B}_{nn} & \mathbf{O}\end{vmatrix}=(-1)^{mn}|\mathbf{A}||\mathbf{B}|$$

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix}^{\mathbf{T}}=\begin{pmatrix} \mathbf{A^T} & \mathbf{C^T}\\ \mathbf{B^T} & \mathbf{D^T}\end{pmatrix}\quad \begin{pmatrix} \mathbf{A} & 0 & 0\\ 0 & \mathbf{B} & 0\\ 0 & 0 & \mathbf{C}\end{pmatrix}^{-1}=\begin{pmatrix} \mathbf{A}^{-1} & 0 & 0\\ 0 & \mathbf{B}^{-1} & 0\\ 0 & 0 & \mathbf{C}^{-1}\end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A}\\ \mathbf{B}\end{pmatrix}^{\mathbf{T}}=(\mathbf{A^T},\mathbf{B^T})\quad \begin{pmatrix} 0 & 0 & \mathbf{A}\\ 0 & \mathbf{B} & 0\\ \mathbf{C} & 0 & 0\end{pmatrix}^{-1}=\begin{pmatrix} 0 & 0 & \mathbf{C}^{-1}\\ 0 & \mathbf{B}^{-1} & 0\\ \mathbf{A}^{-1} & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A} & \mathbf{C}\\ \mathbf{O} & \mathbf{B}\end{pmatrix}^{-1}=\begin{pmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1}\\ \mathbf{O} & \mathbf{B}^{-1}\end{pmatrix}\quad \begin{pmatrix} \mathbf{A} & \mathbf{O}\\ \mathbf{C} & \mathbf{B}\end{pmatrix}^{-1}=\begin{pmatrix} \mathbf{A}^{-1} & \mathbf{O}\\ -\mathbf{B}^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{B}^{-1}\end{pmatrix}$$

$$\begin{pmatrix} \mathbf{C} & \mathbf{A}\\ \mathbf{B} & \mathbf{O}\end{pmatrix}^{-1}=\begin{pmatrix} \mathbf{O} & \mathbf{B}^{-1}\\ \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1}\end{pmatrix}\quad \begin{pmatrix} \mathbf{O} & \mathbf{A}\\ \mathbf{B} & \mathbf{C}\end{pmatrix}^{-1}=\begin{pmatrix} -\mathbf{B}^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{B}^{-1}\\ \mathbf{A}^{-1} & \mathbf{O}\end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A} & \mathbf{O}\\ \mathbf{O} & \mathbf{B}\end{pmatrix}^{*}=\begin{pmatrix} |\mathbf{B}|\mathbf{A}^* & \mathbf{O}\\ \mathbf{O} & |\mathbf{A}|\mathbf{B}^*\end{pmatrix}\quad \begin{pmatrix} \mathbf{O} & \mathbf{A_{mm}}\\ \mathbf{B_{nn}} & \mathbf{O}\end{pmatrix}^{*}=(-1)^{mn}\begin{pmatrix} \mathbf{O} & |\mathbf{A}|\mathbf{B}^*\\ |\mathbf{B}|\mathbf{A}^* & \mathbf{O}\end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A} & \mathbf{O}\\ \mathbf{O} & \mathbf{B}\end{pmatrix}^{n}=\begin{pmatrix} \mathbf{A}^n & \mathbf{O}\\ \mathbf{O} & \mathbf{B}^n\end{pmatrix}\quad \begin{pmatrix} \mathbf{O} & \mathbf{A}\\ \mathbf{B} & \mathbf{O}\end{pmatrix}^{n}=\begin{pmatrix} \textbf{not} & \textbf{important}\\ \textbf{at} & \textbf{all}\end{pmatrix}$$

$$\boldsymbol{Matrix \quad Rank}$$

$$r(\mathbf{A}^*) = \begin{cases} n & \text{if } r(\mathbf{A}) = n, \\ 1 & \text{if } r(\mathbf{A}) = n-1, \\ 0 & \text{if } r(\mathbf{A}) < n-1. \end{cases}$$

$$0 \le r(\mathbf{A_{mn}}) \le \min\{m, n\}$$

$$\max\{r(\mathbf{A}), r(\mathbf{B})\} \le \mathbf{r}(\mathbf{A}, \mathbf{B}) \le r(\mathbf{A}) + r(\mathbf{B})$$

$$r(\mathbf{A}) \le \mathbf{r}(\mathbf{A}, \mathbf{b}) \le r(\mathbf{A}) + 1$$

$$\max\{r(\mathbf{A}), r(\mathbf{B})\} \le r \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} \le r(\mathbf{A}) + r(\mathbf{B})$$

$$r(\mathbf{A} \pm \mathbf{B}) \le r \begin{pmatrix} \mathbf{A} \pm \mathbf{B} \\ \mathbf{B} \end{pmatrix} = r(\mathbf{A} \pm \mathbf{B}, \mathbf{B}) = r(\mathbf{A}, \mathbf{B}) \le r(\mathbf{A}) + r(\mathbf{B})$$

$$r(\mathbf{A}, \mathbf{B}) \le r \begin{pmatrix} \mathbf{A^T} \\ \mathbf{B^T} \end{pmatrix} \ne r \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} = r(\mathbf{A}, \mathbf{B})$$

$$r(\mathbf{A}) + r(\mathbf{B}) - n \le r(\mathbf{AB}) \le \min\{r(\mathbf{A}), r(\mathbf{B})\}$$

$$r(\mathbf{A^T A}) = r(\mathbf{A A^T}) = r(\mathbf{A^T}) = r(\mathbf{A}) = r(k\mathbf{A}), (\forall k \ne 0)$$

$$\exists \mathbf{A_{mn} B_{ns}} = \mathbf{O}, r(\mathbf{A}) + r(\mathbf{B}) \le n$$

$$\exists \mathbf{A_{mn} B_{ns}} = \mathbf{C_{ms}}, \exists r(\mathbf{A}) = n, r(\mathbf{B}) = r(\mathbf{C})$$

$$\exists \mathbf{A_{mn} B_{ns}} = \mathbf{C_{ms}}, \exists r(\mathbf{B}) = n, r(\mathbf{A}) = r(\mathbf{C})$$

$$\exists \mathbf{A_{nn}}, \forall k \in \mathbf{N}^*, r(\mathbf{A}^n) = r(\mathbf{A}^{n+k}) \Longrightarrow r(\mathbf{A}) = r(\mathbf{A}^2) = ... = r(\mathbf{A}^n)$$

$$r \begin{pmatrix} \mathbf{A}_{mm} & \mathbf{O} \\ \mathbf{O} & \mathbf{B}_{nn} \end{pmatrix} = r(\mathbf{A}) + r(\mathbf{B})$$

$$\boldsymbol{Similarity\quad Theory\quad and\quad Feature\quad Vector}$$

$$|\lambda\mathbf{E}-\mathbf{A}|=0\Rightarrow\lambda_i, i\in[1,n]$$

$$|\lambda\mathbf{E}-\mathbf{A}|=\begin{pmatrix}\lambda-a_{11} & ... & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{n1} & ... & \lambda-a_{nn}\end{pmatrix}=\lambda^n+\sum_{i=1}^{n}a_{ii}\lambda^{n-1}+...$$

$$\forall i\in[0,n],\exists\lambda_i, f(\lambda_i)=0\Longrightarrow\Pi_{i=1}^{n}(\lambda-\lambda_i)=0$$

$$\lambda^n+\sum_{i=1}^{n}\lambda_i\lambda^{n-1}+...+(-1)^n\Pi_{i=1}^{n}\lambda_i=0\Rightarrow\sum_{i=1}^{n}a_{ii}=\sum_{i=1}^{n}\lambda_i$$

$$\exists\lambda=0,(-1)^n\Pi_{i=1}^{n}\lambda_i=|-\mathbf{A}|=(-1)^n|\mathbf{A}|\Rightarrow\Pi_{i=1}^{n}\lambda_i=|\mathrm{A}|$$

$$\boldsymbol{Specially\quad n=3}$$

$$|\lambda\mathbf{E}-\mathbf{A}|=\begin{pmatrix}\lambda-a_{11} & a_{12} & a_{13}\\ a_{21} & \lambda-a_{22} & a_{23}\\ a_{31} & a_{32} & \lambda-a_{33}\end{pmatrix}=\lambda^3-\left(\sum_{i=1}^{3}a_{ii}\right)\lambda^2+\left(\sum_{i=1}^{3}\mathbf{A_{ii}}\right)\lambda-|\mathbf{A}|$$

$$\lambda^3-(a_{11}+a_{22}+a_{33})\lambda^2+(\mathbf{A_{11}}+\mathbf{A_{22}}+\mathbf{A_{33}})\lambda-|\mathbf{A}|=(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$$

$$(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)=\lambda^3-(\lambda_1+\lambda_2+\lambda_3)\lambda^2+(\lambda_2\lambda_3+\lambda_1\lambda_3+\lambda_1\lambda_2)\lambda-(\lambda_1\lambda_2\lambda_3)$$

$$\sum_{i=1}^{3}\mathbf{A_{ii}}=\mathbf{tr}(\mathbf{A}^*)=(\lambda_2\lambda_3+\lambda_1\lambda_3+\lambda_1\lambda_2)$$

$$\alpha^{\mathbf{T}}\alpha=\mathbf{tr}(\mathbf{A})\qquad r(\alpha\alpha^{\mathbf{T}})=1$$

$$\alpha\alpha^{\mathbf{T}}\sim\begin{pmatrix}\mathbf{tr}(\mathbf{A}) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}_{nn}$$

$$\mathbf{A}\sim\mathbf{B}\Longrightarrow\mathbf{A^T}\sim\mathbf{B^T},\mathbf{A}^{-1}\sim\mathbf{B}^{-1},\mathbf{A}^*\sim\mathbf{B}^*,f(\mathbf{A})\sim f(\mathbf{B})$$

$$f(\mathbf{A})=0\Rightarrow f(\lambda)=0(E\sim 1)$$

$$\lambda_{\mathbf{A}_i^*}\lambda_{\mathbf{A}_i}=|\mathbf{A}|, i\in[1,n]$$

$\mathbf{A}$	$\mathbf{A^T}$	$\mathbf{A}^{-1}$	$\mathbf{A}^*$	$f(\mathbf{A})$	$\mathbf{P}^{-1}\mathbf{AP}$	$\mathbf{PAP}^{-1}$
λ	λ	$\frac{1}{\lambda}$	$\frac{\lvert\mathbf{A}\rvert}{\lambda}$	$f(\lambda)$	λ	λ
α	*	α	α	α	$\mathbf{P}^{-1}\alpha$	$\mathbf{P}\alpha$

Base $\boldsymbol{\sigma}$

$$(\eta_1, \eta_2, ...\eta_n) = (\xi_1, \xi_2, ...\xi_n)\mathbf{M}$$

$$\left\{\begin{array}{cccc} \eta_1 = a_{11}\xi_1 + a_{21}\xi_2 + ... + a_{n1}\xi n \\ \vdots \quad \vdots \quad \ddots & & \vdots & \vdots \\ \eta_n = a_{1n}\xi_1 + a_{2n}\xi_2 + ... + a_{nn}\xi n \end{array}\right. \Longrightarrow \mathbf{AM} = \mathbf{B} \Longleftrightarrow \mathbf{M} = \mathbf{A^{-1}B}$$

$$\mathbf{A}\xi_{\mathbf{A}} = \mathbf{B}\xi_{\mathbf{B}} \Longrightarrow \xi_{\mathbf{A}} = \mathbf{A}^{-1}\mathbf{B}\xi_{\mathbf{B}} = \mathbf{M}\xi_B$$

$$\textbf{\textit{Traditional Probability Theory}}$$

$$\textbf{\textit{Opposition}}: P(A)+P(\bar{A})=1 \quad \textbf{\textit{Exclusive}}: A\cap B=\varnothing \Rightarrow P(AB)=0$$

$$\textbf{\textit{Independent}}: P(AB)=P(A)P(B) \quad \textbf{\textit{Equal}}: A=B \Rightarrow P(A)=P(B)$$

$$A-B=A\bar{B}=A\cap\bar{B} \Rightarrow P(A-B)=P(A)-P(AB)$$

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(BC)-P(AC)-P(AB)+P(ABC)$$

$$P(B|A)=\frac{P(AB)}{P(A)} \quad P(A_1A_2...A_n)=P(A_1)P(A_2|A_1)P(A_3|A_1A_2)...P(A_n|A_1...A_{n-1})$$

$$\textbf{\textit{Bayes}} \quad P(B_j|A)=\frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{n}P(A|B_i)P(B_i)}$$

Random Variable Digital Properties

Distr	Mark	EX	DX	Addition
Bin	$B(n,p)$	np	$np(1-p)$	$P\{X=k\}=\mathbf{C_n^k}(1-p)^{n-k}p^k$
Poi	$\pi(\lambda)$	λ	λ	$P\{X=k\}=\frac{\lambda^k}{k!}\mathrm{e}^{-\lambda}, k=0,1,2,...$
Geo	$G(p)$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$P\{X=k\}=(1-p)^{k-1}p, k=1,2,...$
Hyp	$H(n,M,N)$	$\frac{nM}{N}$	$\frac{nM}{N}(1-\frac{M}{N})(\frac{N-n}{N-1})$	$P\{X=i\}=\frac{\mathbf{C}_M^i\mathbf{C}_{N-M}^{n-i}}{\mathbf{C}_N^n}$
Uni	$U(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Exp	$E(\lambda)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	
Nor	$N(\mu,\sigma^2)$	μ	σ^2	

Random Variable Characteristic Function

Distr	Mark	$\phi_X(t) = \mathrm{E}(\mathrm{e}^{itX})$	$\mathrm{E}(\mathrm{X}^k) = (-i)^k \phi_X^{(k)}(0), \quad \mathrm{E}(\lvert X \rvert^n) < +\infty$
Bin	$B(n,p)$	$(\mathrm{e}^{it}p + 1 - p)^n$	
Poi	$\pi(\lambda)$	$\mathrm{e}^{(\mathrm{e}^{it}-1)\lambda}$	
Geo	$G(p)$		
Hyp	$H(n,M,N)$		
Uni	$U(a,b)$	$\frac{\sin tb - \sin ta}{(b-a)t} - i\frac{\cos tb - \cos ta}{(b-a)t}$	
Exp	$E(\lambda)$	$\frac{\lambda^2}{\lambda^2+t^2} + i\frac{\lambda t}{\lambda^2+t^2}$	
Nor	$N(\mu,\sigma^2)$	$\mathrm{e}^{-\frac{1}{2}\sigma^2 t^2 + i\mu t}$	

$$\text{Abstract} \quad \text{Probability} \quad \text{SUM}$$
$$\mathrm{E(X)} = \begin{cases} \sum_{\omega \in S} R(\omega)\mathrm{Pr}(\omega), & Discrete \\ \int_\Omega X \mathrm{d}P, & Continuous \end{cases}$$

$$\text{Numeric} \quad \text{Sum} \quad \text{And} \quad \text{Integral}$$
$$\mathrm{E(X)} = \begin{cases} \sum_{i=1}^{n} x_i P(x_i), & Discrete \\ \int_{-\infty}^{+\infty} x \mathrm{d}F(x), & Continuous \end{cases}$$

$$\Omega \quad \Longrightarrow_{h(\omega)}^{MAP} \quad R$$
$$h(\omega) = \sum_{k=1}^{n} a_k I_{A_k}(\omega) \quad A_1 \cup A_2 \cup \cdots \cup A_n = \Omega$$
$$\mathrm{E}(h) = \sum_{k=1}^{n} a_k P(A_k)$$
$$def. \quad X^+ = X \vee 0 \quad X^- = -X \vee 0 \quad X = X^+ - X^-$$
$$\int_\Omega X \mathrm{d}P = \int_\Omega X^+ \mathrm{d}P - \int_\Omega X^- \mathrm{d}P$$
$$\text{Dimension} \quad \text{Extend}$$
$$\vec{X} = (X_1, X_2, \cdots, X_n) \in R^n$$

$$\textbf{Uni}: f(x)=\begin{cases}\frac{1}{b-a}, a\le x\le b\\ 0, others\end{cases}\qquad F(x)=\begin{cases}0, x<a\\ \frac{x-a}{b-a}, a\le x<b\\ 1, x\ge b\end{cases}$$

$$\textbf{Exp}: f(x)=\begin{cases}\lambda \mathrm{e}^{-\lambda x}, x>0\\ 0, x\le 0\end{cases}\qquad F(x)=\begin{cases}1-\mathrm{e}^{-\lambda x}, x>0\\ 0, x\le 0\end{cases}$$

$$\textbf{Nor}: f(x)=\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}, F(x)=\int_{-\infty}^{x} f(t)\mathrm{d}t, x\in(-\infty,+\infty)$$

$$f(\mu+x)=f(\mu-x), F(\mu+x)+F(\mu-x)=1, F(\mu)=\frac{1}{2}$$

$$X\sim N(0,1), \phi(x)=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}, \Phi(x)=\int_{-\infty}^{x}\phi(t)\mathrm{d}t$$

$$\phi(-x)=\phi(x),\quad \Phi(a)+\Phi(-a)=1, \Phi(0)=\frac{1}{2},\quad F(x)=\Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$f(x,y)\ge 0\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\mathrm{d}x\mathrm{d}y=1$$

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)\mathrm{d}y>0\quad f_Y(y)=\int_{-\infty}^{\infty} f(x,y)\mathrm{d}x>0$$

$$f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}, f_{Y|X}(y|x)=\frac{f(x,y)}{f_X(x)}$$

$$\boldsymbol{Discrete}: P\{Z=g(x_i,y_j)\}=P\{X=x_i, Y=y_j\}=P_{ij}$$

$$F_Z(z)=P\{Z\le z\}=P\{g(X,Y)\le z\}=\sum_{g(x_i,y_j)\le z} P\{X=x_i, Y=y_j\}$$

$$\boldsymbol{Continuous}: Z=g(X,Y)\quad F_Z(z)=P\{Z\le z\}=P\{g(X,Y)\le z\}=\iint_{g(x,y)\le z} f(x,y)\mathrm{d}x\mathrm{d}y$$

$$Z=\max(X,Y), F_{\max}(z)=F_X(z)F_Y(z)\quad Z=\min(X,Y), F_{\min}(z)=1-[1-F_X(z)][1-F_Y(z)]$$

$$Z_1=\max(X,Y)=\frac{X+Y+|X-Y|}{2}\quad Z_2=\min(X,Y)=\frac{X+Y-|X-Y|}{2}\quad Z_1Z_2=XY$$

$$(X,Y)\sim U(D), f(x,y)=\begin{cases}\frac{1}{S_D}, (x,y)\in D\\ 0, others\end{cases}$$

$$(X,Y)\sim N(\mu_1,\mu_2;\sigma_1,\sigma_2,\rho), f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\mathrm{e}^{-\frac{1}{2(1-\rho^2)}[(\frac{x-\mu_1}{\sigma_1})^2-\frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}+(\frac{y-\mu_2}{\sigma_2})^2]}$$

$$F(x) = P\{X \leq x\} = \int_{-\infty}^{x} f(x)\mathrm{d}x$$
$$F(x,y) = P\{X \leq x, Y \leq y\} = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\mathrm{d}v\mathrm{d}u$$
$$1 = \sum_{i=1}^{\infty} x_i p_i \quad 1 = \int_{-\infty}^{+\infty} f(x)\mathrm{d}x$$
$$E(X) = \sum_{i=1}^{\infty} x_i p_i \quad E(X) = \int_{-\infty}^{+\infty} x f(x)\mathrm{d}x$$
$$E[g(X)] = \sum_{i=1}^{\infty} g(x_i) p_i \quad E[g(X)] = \int_{-\infty}^{+\infty} g(x) f(x)\mathrm{d}x$$
$$E[g(X,Y)] = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} g(x_i, y_j) P_{ij} \quad E[g(X,Y)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x,y) f(x,y)\mathrm{d}x\mathrm{d}y$$
$$\exists X \sim F(x), \forall X > 0, E(X) = \int_{0}^{+\infty} (1 - F(x))\mathrm{d}x = \int_{0}^{+\infty} P\{X > x\}\mathrm{d}x$$
$$E(C) = C \quad E(CX) = CE(X) \quad E(X+C) = E(X) + C \quad E(X+Y) = E(X) + E(Y)$$
$$D(X) = \sum_{i=1}^{n} [x_i - E(X)]^2 p_i \quad D(X) = \int_{-\infty}^{+\infty} [x - E(X)]^2 f(x)\mathrm{d}x$$
$$D(X) = E(X^2) - [E(X)]^2 \quad D(C) = 0 \quad D(CX) = C^2 D(X) \quad D(X+C) = D(X)$$
$$Cov(X,Y) = E(XY) - E(X)E(Y) \quad D(X \pm Y) = D(X) + D(Y) \pm 2Cov(X,Y)$$
$$Cov(X,Y) = Cov(Y,X) \quad Cov(X,X) = D(X) \quad Cov(X,c) = 0$$
$$Cov(aX,bY) = abCov(X,Y) \quad Cov(X_1 + X_2, Y) = Cov(X_1,Y) + Cov(X_2,Y)$$
$$\rho_{XY} = \frac{E(XY) - E(X)E(Y)}{\sqrt{D(X)}\sqrt{D(Y)}} = \frac{Cov(X,Y)}{\sqrt{D(X)}\sqrt{D(Y)}}$$
$$X \sim N(0,1) \quad E(X^{2k}) = (2k-1)!! \quad E(X^{2k-1}) = 0, k \in \{1,2,3,...\}$$

Large Number Law Central Limit Theorem

$$P\{|X-E(X)|\geq\epsilon\}\leq\frac{D(X)}{\epsilon^2}$$

$$\lim_{n\to\infty}P\{|\frac{X_n}{n}-p|<\epsilon\}=1$$

$$\lim_{n\to\infty}P\{|\frac{1}{n}\sum_{k=1}^{n}X_k-\mu|<\epsilon\}=1$$

$$\lim_{n\to\infty}P\{|\frac{1}{n}\sum_{k=1}^{n}X_k-\frac{1}{n}\sum_{k=1}^{n}E(X_k)|<\epsilon\}=1$$

$$\sum_{i=1}^{n}X_i\sim N(n\mu,n\sigma^2),\lim_{n\to\infty}P\{\frac{\sum_{i=1}^{n}X_i-n\mu}{\sqrt{n}\sigma}\leq x\}=F_n(x)=\Phi(x)$$

$$X\sim N(np,np(1-p)),\lim_{n\to\infty}P\{\frac{X_n-np}{\sqrt{np(1-p)}}\leq x\}=\Phi(x)$$

Mathematical Statistics

$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n}), \frac{\bar{X}-u}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2 = \frac{1}{n-1}[\sum_{i=1}^{n}X_i^2 - n\bar{X}^2]$$

$$E(S^2)=\sigma^2, D(S^2)=\frac{2\sigma^4}{n-1}$$

$$\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\sigma^2} \sim \chi^2(n)$$

$$\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma^2} \sim \chi^2(n-1)$$

$$\chi^2 \sim \chi^2(n), E(\chi^2(n))=n, D(\chi^2(n))=2n$$

$$T \sim t(n), T=\frac{X}{\sqrt{Y/n}}\frac{\sim}{\sim}\frac{N(0,1)}{\sqrt{\chi^2(n)/n}}$$

$$F \sim F(n1,n2), F=\frac{X/n1}{Y/n2}\frac{\sim}{\sim}\frac{\chi^2(n1)/n1}{\chi^2(n2)/n2}$$

$$(X,Y) \sim N(\mu_1,\mu_2;\sigma_1^2,\sigma_2^2,0)$$

$$\frac{\frac{(n_1-1)S_1^2}{\sigma_1^2}/(n_1-1)}{\frac{(n_2-1)S_2^2}{\sigma_2^2}/(n_2-1)} \sim F(n_1-1,n_2-1)$$

$$\frac{\frac{\sum_{i=1}^{n_1}(X_i-\mu_1)^2}{\sigma_1^2}/n_1}{\frac{\sum_{i=1}^{n_2}(X_i-\mu_2)^2}{\sigma_2^2}/n2} \sim F(n1,n2)$$

$$\frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n2}}} \sim N(0,1)$$

$$\exists\sigma_1=\sigma_2, \frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{S_\omega\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \sim t(n_1+n_2-2)$$

$$S_\omega = \sqrt{\frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}}$$

Constant Series

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6} \quad \sum_{k=1}^{\infty}\frac{1}{k^4}=\frac{\pi^4}{90} \quad \sum_{k=1}^{\infty}\frac{1}{k^6}=\frac{\pi^6}{945}$$

$$\sum_{i=0}^{n}a_i\cdot\sum_{j=0}^{n}b_j=\sum_{i=0}^{n}\sum_{j=0}^{n}(a_i\cdot b_j) \quad \sum_{n=s}^{t}\ln f(n)=\ln\prod_{n=s}^{t}f(n)$$

$$\sum_{i=0}^{n}i=\frac{n(n+1)}{2} \quad \sum_{i=1}^{n}i(i+1)(i+2)=\frac{n(n+1)(n+2)(n+3)}{4}$$

$$\sum_{i=0}^{n}i^2=\frac{n(n+1)(2n+1)}{6} \quad \sum_{i=0}^{n}i^3=\left(\sum_{i=0}^{n}i\right)^2=\frac{n^2(n+1)^2}{4}$$

$$\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}=\sum_{i=0}^{\infty}\frac{1}{(2i+1)(2i+2)}=\sum_{i=1}^{\infty}\frac{1}{2^i i}=\sum_{i=1}^{\infty}(\frac{1}{3^i}+\frac{1}{4^i})\frac{1}{i}=\ln 2$$

Power Series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \ln(1+x)$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x)$$

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x$$

$$\sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}$$

$$\sum_{n=0}^{\infty} (n+1)(n+2)x^n = \frac{2}{(1-x)^3}$$

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{n} = -\ln(1-x^2)$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = \frac{1}{2}\ln(\frac{1+x}{1-x})$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin x$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \mathrm{e}^x$$

Basel Problem

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

$$f(x)=x^2 \quad \xrightarrow{Fourier Expansion} \quad S(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\cos nx)$$

$$a_0=\frac{2}{3}\pi^2 \quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx\mathrm{d}x=(-1)^n\frac{4}{n^2}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\sin nx\mathrm{d}x=0$$

$$S(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx=\frac{1}{3}\pi^2+\sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}\cos nx=\frac{1}{3}\pi^2+\sum_{n=1}^{n}\frac{4}{n^2}$$

$$\exists x=0, S(0)=\frac{1}{3}\pi^2+\sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}=f(0)=0\Longrightarrow\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=\frac{\pi^2}{12}$$

$$\exists x=\pi, S(\pi)=\frac{1}{3}\pi^2+\sum_{n=1}^{\infty}\frac{4}{n^2}=f(\pi)=\pi^2\Longrightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8} \quad \sum_{n=1}^{\infty}\frac{1}{(2n)^2}=\frac{\pi^2}{24}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{(2n-1)^2}+\frac{1}{(2n)^2}\right)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

Transcendental Equation

$$\sum_{i=0}^{n} a_i x^i = 0 \Longrightarrow \prod_{i=0}^{n} (x - x_i) = 0$$

$$\prod_{i=0}^{n} x_i = (-1)^n \frac{a_0}{a_n}, \quad \sum_{i=0}^{n} \frac{\prod_{i=0}^{n} x_i}{x_i} = (-1)^{n-1} \frac{a_1}{a_n}$$

$$\sum_{i=0}^{n} \frac{1}{x_i} = -\frac{a_1}{a_0}$$

eg. $\tan x = x \Longrightarrow \sin x = \cos x \cdot x$

$$x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + ... = x \cdot (1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + ...)$$

$$\frac{1}{3}x^3 - \frac{1}{30}x^5 + ... = 0 \quad \xrightarrow{x \neq 0} \quad \frac{1}{3}x^2 - \frac{1}{30}x^4 + ... = 0$$

$$\sum_{i=0}^{n} \frac{1}{t_i} = \frac{1}{10} \quad \xrightarrow{t = x^2} \quad \lim_{n \to \infty} \sum_{i=0}^{n} \frac{1}{x_i^2} = \frac{1}{10}$$

Beyond Integral

$$\sum_{n=0}^{\infty}\frac{1}{(2n)!}\Longrightarrow S(x)=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!},S(1)=\frac{\mathrm{e}^{-1}+\mathrm{e}}{2}$$

$$\sum_{n=0}^{\infty}\frac{1}{(3n)!}\Longrightarrow S(x)=\sum_{n=0}^{\infty}\frac{x^{3n}}{(3n)!},S(1)=\frac{\mathrm{e}}{3}+\frac{2}{3}\cos\left(\frac{\sqrt{3}}{2}\right)\mathrm{e}^{-\frac{1}{2}}$$

$$\sum_{n=0}^{\infty}\frac{1}{(4n)!}\Longrightarrow S(x)=\sum_{n=0}^{\infty}\frac{x^{4n}}{(4n)!},S(1)=\frac{\mathrm{e}+\mathrm{e}^{-1}}{4}+\frac{\cos 1}{2}$$

$$\iint_{D_{xy}}(x+y)\mathrm{d}\sigma,D=\{(x,y)|y^2\le x+2,x^2\le y+2\}$$

$$\boldsymbol{A_0}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{m=0}^{\infty}\frac{1}{n2^m+1}$$

$$\boldsymbol{A_1}=\int_0^{x^2}\pi(\sqrt[4]{1+t}-1)\sin t^4\mathrm{d}x$$

$$\boldsymbol{A_2}=\sum_{n=1}^{\infty}\frac{((n-1)!)^2(2t)^{2n}}{(2n)!}$$

$$\boldsymbol{A_3}=\int_0^1\frac{(1-2x)\ln(1-x)}{x^2-x+1}\mathrm{d}x$$

$$\boldsymbol{A_4}=x^2(x-\tan x)\ln(x^2+1)[\left(\frac{2\arctan\frac{y}{x}}{\pi}\right)^y-1]$$

$$\lim_{x\to 0^+}\lim_{y\to+\infty}\frac{\boldsymbol{A_0A_1}}{\boldsymbol{A_2A_3A_4}}=\frac{27}{32}$$

$$\int\frac{\sec^3 x}{1-\tan^6 x}\mathrm{d}x$$

$$\int\frac{1}{\csc x+\sec x+\tan x+\cot x}\mathrm{d}x$$

$$\lim_{N\to\infty}\sum_{n=1}^{N}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}-\ln 2=\ln 2-\frac{1}{2}$$

$$\iint_D \mathrm{e}^x\cos y\mathrm{d}\sigma, D=\{(x,y)|x^2+y^2\leq 1\}$$

$$\iint_D \mathrm{e}^{-y^2}\mathrm{d}x\mathrm{d}y, D=\{(x,y)|x<y<1, 0<x<1\}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1-\cos\frac{\pi}{\sqrt{n}}}{1+\cos\frac{i\pi}{\sqrt{2n}}}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{[\sqrt{n}]}}{n}\qquad\sum_{n=1}^{\infty}\frac{n!}{(2n+1)!!}\frac{1}{(n+1)}$$

$$t\frac{\mathrm{d}^3x}{\mathrm{d}t^3}+3\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-t\frac{\mathrm{d}x}{\mathrm{d}t}-x=0$$

$$X\sim F(x)\quad is\quad Consecutive\quad Variable:\int_{-\infty}^{+\infty}[F(x+a)-F(x)]\mathrm{d}x=a$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{1+4n^2}=\frac{\pi}{2}\frac{1}{\mathrm{e}^{\frac{\pi}{2}}-\mathrm{e}^{-\frac{\pi}{2}}}-\frac{1}{2}$$

$$\mathbf{L} = \mathbf{L_1} + \mathbf{L_2} + \mathbf{L_3} + \mathbf{L_4}$$

$$\mathbf{L_1} : x \in (0,1), y = 0 \quad \mathbf{L_2} : y \in (0,1), x = 1 \quad \mathbf{L_3} : x \in (1,0), y = 1 \quad \mathbf{L_4} : y \in (1,0), x = 0$$

$$I_1 = \oint_L -xyf'_x(x,y)\mathrm{d}x + xyf'_y(x,y)\mathrm{d}y = \iint_D \left(\frac{\partial \left(xyf'_y(x,y)\right)}{\partial x} - \frac{\partial \left(-xyf'_x(x,y)\right)}{\partial y} \right) \mathrm{d}x\mathrm{d}y$$

$$= 2\iint_D xyf''_{xy}(x,y)\mathrm{d}x\mathrm{d}y + \iint_D \left(xf'_x(x,y) + yf'_y(x,y)\right)\mathrm{d}x\mathrm{d}y$$

$$I_1 = \oint_L -xyf'_x(x,y)\mathrm{d}x + xyf'_y(x,y)\mathrm{d}y = \int_{L_1} + \int_{L_2} + \int_{L_3} + \int_{L_4}$$

$$= 0 + \int_0^1 yf'_y(1,y)\mathrm{d}y + \int_1^0 -xf'_x(x,1)\mathrm{d}x + 0$$

$$= \left. yf(1,y)\right|_0^1 - \int_0^1 f(1,y)\mathrm{d}y + \left. xf(x,1)\right|_0^1 - \int_0^1 f(x,1)\mathrm{d}x$$

$$= 0 + 0 + 0 + 0 = 0$$

$$I_2 = \iint_D xf'_x(x,y) + yf'_y(x,y)\mathrm{d}x\mathrm{d}y = I_3 - I_4$$

$$I_3 = \oint_L -yf(x,y)\mathrm{d}x + xf(x,y)\mathrm{d}y = \iint_D \left(xf'_x(x,y) + yf'_y(x,y)\right)\mathrm{d}x\mathrm{d}y + 2\iint_D f(x,y)\mathrm{d}x\mathrm{d}y$$

$$I_3 = \oint_L -yf(x,y)\mathrm{d}x + xf(x,y)\mathrm{d}y = \int_{L_1} + \int_{L_2} + \int_{L_3} + \int_{L_4}$$

$$= 0 + \int_0^1 f(1,y)\mathrm{d}y + \int_1^0 -f(x,1)\mathrm{d}x + 0 = 0 + 0 + 0 + 0 = 0$$

$$I_4 = 2\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = 2a$$

$$I = \frac{1}{2}I_1 - \frac{1}{2}I_2 = \frac{1}{2}I_1 - \frac{1}{2}(I_3 - I_4) = 0 - \frac{1}{2}(0 - 2a) = a$$

$$\frac{1}{\log_a^3 x}+\frac{1}{\log_b^3 x}+\frac{1}{\log_c^3 x}=\frac{3}{\log_a x\log_b x\log_c x}$$

$$(\log_x a)^3+(\log_x b)^3+(\log_x c)^3=3(\log_x a)(\log_x b)(\log_x c)$$

$$\aleph \quad \log_x a=m,\log_x b=n,\log_x c=p\rightarrow m^3+n^3+p^3=3mnp$$

$$\log_a x\log_b x\log_c x\neq 0\longrightarrow mnp\neq 0$$

$$\exists p\neq 0\rightarrow (\frac{m}{p})^3+(\frac{n}{p})^3+1=3\frac{mn}{p^2}$$

$$\aleph \quad \frac{m}{p}=A,\frac{n}{p}=B\rightarrow A^3+B^3-3AB+1=0$$

$$\aleph \quad f(A,B)=A^3+B^3-3AB+1=0$$

$$f'_A(A,B)=3(A^2-B),f'_BA,B=3(B^2-A)$$

$$f''_{AA}(A,B)=6A,f''_{BB}(A,B)=6B,f''_{AB}(A,B)=-3$$

$$\exists f'_A(A,B)=f'_B(A,B)=0\longrightarrow A=B=1,f''_{AA}(A,B)f''_{BB}(1,1)>(f''_{AB}(1,1))^2$$

$$\exists only \quad A=B=1\in\mathbf{R^2},f(A,B)=0\longrightarrow f(A,B)=A^3+B^3-3AB+1=0$$

$$so \quad \exists m=n=p=1\longrightarrow \log_4\left(\frac{a+b}{c}\right)=\frac{1}{2}$$

$$\lim_{x\to 0}\frac{-2\sin x+\arctan x-\frac{x^3\cos x}{3}+\frac{\ln(\frac{1+x}{1-x})}{2}}{(\mathrm{e}^x-1)^2\sum_{n=0}^{\infty}\left(1-\frac{1}{x^2}\right)^n D(\chi^2(x))}=\frac{11}{40}$$

$$X\sim N(0,1)\quad E(X^{2k})=(2k-1)!!\quad E(X^{2k-1})=0$$

$$X\sim N(0,\sigma^2),Y\sim N(0,2\sigma^2)$$

$$\frac{X}{\sigma}\sim N(0,1)\frac{Y}{\sqrt{2}\sigma}\sim N(0,1)$$

My Method

$$E(\left(\frac{X}{\sigma}\right)^4)=3,E(\left(\frac{X}{\sigma}\right)^2)=1$$

$$E(\left(\frac{Y}{\sqrt{2}\sigma}\right)^4)=3,E(\left(\frac{Y}{\sqrt{2}\sigma}\right)^2)=1$$

$$E(X^4)=3\sigma^4,E(X^2)=\sigma^2,D(X^2)=2\sigma^4$$

$$E(Y^4)=12\sigma^4,E(Y^2)=2\sigma^2,D(Y^2)=8\sigma^4$$

$$D(\hat{\sigma}^2)=D(\frac{1}{2n}\sum_{i=1}^{n}X_i^2+\frac{1}{4n}\sum_{i=1}^{n}Y_i^2)=\frac{1}{4n^2}\sum_{i=1}^{n}D(X_i^2)+\frac{1}{16n^2}\sum_{i=1}^{n}D(Y_i^2)=\frac{\sigma^4}{n}$$

AnswerMethod

$$(\frac{X}{\sigma})^2\sim\chi^2(1),(\frac{Y}{\sqrt{2}\sigma})^2\sim\chi^2(1)$$

$$D(X^2)=2\sigma^4,D(Y^2)=8\sigma^4$$

$$D(\hat{\sigma}^2)=D(\frac{1}{2n}\sum_{i=1}^{n}X_i^2+\frac{1}{4n}\sum_{i=1}^{n}Y_i^2)=\frac{1}{4n^2}\sum_{i=1}^{n}D(X_i^2)+\frac{1}{16n^2}\sum_{i=1}^{n}D(Y_i^2)=\frac{\sigma^4}{n}$$

$$\lim_{x\to+\infty}\int_x^{2x}\sin\frac{1}{x+t}\mathrm{d}t=\lim_{x\to+\infty}\int_{2x}^{3x}\sin\frac{1}{u}\mathrm{d}u\quad\xrightarrow{\sin\frac{1}{u}\sim\frac{1}{u}+o(\frac{1}{u})}\ln(\frac{3}{2})$$

$$\mathrm{e}^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}=\sum_{k=0}^{\infty}\left(\frac{x^{2k-1}}{(2k-1)!}+\frac{x^{2k}}{(2k)!}\right)=\sum_{k=0}^{\infty}\frac{x^{2k-1}}{(2k-1)!}+\sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}$$

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{4}\sum_{k=0}^{n}\frac{1}{(ik)!}&=\sum_{k=0}^{n}\frac{1}{(k)!}+\sum_{k=0}^{n}\frac{1}{(2k)!}+\sum_{k=0}^{n}\frac{1}{(3k)!}+\sum_{k=0}^{n}\frac{1}{(4k)!}\\&=\mathrm{e}+\frac{\mathrm{e}+\mathrm{e}^{-1}}{2}+\frac{\mathrm{e}}{3}+\frac{2}{3}\cos\frac{\sqrt{3}}{2}\mathrm{e}^{-\frac{1}{2}}+\frac{\mathrm{e}+\mathrm{e}^{-1}}{4}+\frac{\cos 1}{2}\\&=\frac{2}{3}\cos\frac{\sqrt{3}}{2}\mathrm{e}^{-\frac{1}{2}}+\frac{\cos 1}{2}+\frac{3}{4}\mathrm{e}^{-1}+\frac{25}{12}\mathrm{e}\end{aligned}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n} k(1-p)^{k-1}p = 3$$

$$p\sum_{k=1}^{n} kq^{k-1} = pf(q) = (1-q)f(q) = g(q)$$

$$f(q) = \frac{g(q)}{1-q} \quad \xrightarrow{integral} \quad \int f(q)\mathrm{d}q = \int \frac{g(q)}{1-q}\mathrm{d}q$$

$$\sum_{k=1}^{n} q^k = \frac{q}{1-q} = \int \frac{g(q)}{1-q}\mathrm{d}q \quad \xrightarrow{derivation} \quad \frac{(1-q)+q}{(1-q)^2} = \frac{g(q)}{1-q}$$

$$p = \frac{1}{3}$$

$$\lim_{n\to\infty}\sum_{k=2}^{n} k[(1-p)^{k-1}p + p^{k-1}(1-p)] = 3$$

$$G(p) = p\sum_{k=2}^{n} k(1-p)^{k-1} + (1-p)\sum_{k=2}^{n} kp^{k-1}$$

$$F(x) = (1-x)\sum_{k=2}^{n} kx^{k-1}$$

$$\frac{F(x)}{1-x} = \sum_{k=2}^{n} kx^{k-1} \quad \xrightarrow{integral}$$

$$\int \frac{F(x)}{1-x}\mathrm{d}x = \frac{x^2}{1-x} \quad \xrightarrow{derivation} \quad \frac{F(x)}{1-x} = \frac{2x(1-x)+x^2}{(1-x)^2}$$

$$F(x) = \frac{(2-x)x}{1-x}, \quad G(p) = F(1-p) + F(p) = \frac{(1-p)(1+p)}{p} + \frac{(2-p)p}{1-p} = 3$$

$$p = \frac{1}{2}$$