Elementary Function

$$(a+b)^n=\sum_{k=0}^{n}\mathbf{C_n^k}a^{n-k}b^k$$

$$a^n-b^n=(a-b)\cdot\sum_{k=0}^{n-1}a^{n-1-k}b^k$$

$$\arcsin x+\arccos x=\arctan x+\mathrm{arccot}x=\frac{\pi}{2}$$

$$f(x)=\int_a^x f'(t)\mathrm{d}t+f(a)$$

$$K=\frac{|y''x'-y'x''|}{(x'^2+y'^2)^{\frac{3}{2}}},\quad \rho=\frac{1}{K}$$

$$\int_0^1 f(x)\mathrm{d}x=\sum_{i=1}^{n}f\left(\frac{i}{n}\right)\left(\frac{i+1}{n}-\frac{i}{n}\right)=\sum_{i=1}^{n}f\left(\frac{2i+1}{2n}\right)\frac{1}{n}$$

$$\int_{t1}^{t2}f(x(t),y(t))\mathrm{d}s=\int_a^b f(x,y(x))\sqrt{x'^2+y'^2}\mathrm{d}x=\int_\alpha^\beta f(r,\theta)\sqrt{r^2+r'^2}\mathrm{d}\theta$$

$$S=\int_a^b y(x)\mathrm{d}x=\int_{t1}^{t2}y(t)x'(t)\mathrm{d}t=\frac{1}{2}\int_\alpha^\beta r^2(\theta)\mathrm{d}\theta$$

$$S_{side}=\int_a^b 2\pi y(x)\sqrt{x'^2(x)+y'^2(x)}\mathrm{d}x=\int_\alpha^\beta 2\pi r(\theta)\sin\theta\sqrt{r^2(\theta)+r'^2(\theta)}\mathrm{d}\theta$$

$$[\overrightarrow{abc}]=\begin{vmatrix}a_x & a_y & a_z\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{vmatrix}$$

Gamma Function Integral

$$\left(-\frac{1}{2}\right)! = \sqrt{\pi}$$

$$\left(\frac{2n+1}{2}\right)! = \Pi_{i=n}^{0} \frac{2i+1}{2}\sqrt{\pi} = \frac{2n+1}{2}\frac{2n-1}{2}\dots\frac{1}{2}\sqrt{\pi}$$

$$\int_0^\infty x^a \mathrm{e}^{-x} = a!$$

$$\int_0^\infty x^{-\frac{1}{2}} \mathrm{e}^{-x} = \sqrt{\pi}$$

$$\int_0^\infty x^3 \mathrm{e}^{-x} = (3)! = 6$$

$$\int_0^\infty x^{\frac{5}{2}} \mathrm{e}^{-x} = \frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}$$

$$\int_0^\infty x^a \mathrm{e}^{-x^2} = \frac{1}{2}\left(\frac{a-1}{2}\right)!$$

$$\int_0^\infty x^1 \mathrm{e}^{-x^2} = \frac{1}{2}\left(\frac{1-1}{2}\right)! = \frac{1}{2}$$

$$\int_0^\infty x^4 \mathrm{e}^{-x^2} = \frac{1}{2}\left(\frac{4-1}{2}\right)! = \frac{1}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}$$

$$\int_0^\infty x^7 \mathrm{e}^{-x^2} = \frac{1}{2}\left(\frac{7-1}{2}\right)! = \frac{1}{2}(3)! = \frac{1}{2}6$$

Multivariate Integral

$$\iiint_{\Omega_{xyz}} f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{\Omega_{uvw}} f(u,v,w)\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \mathrm{d}u\mathrm{d}v\mathrm{d}w$$

$$\left.\frac{\partial f}{\partial l}\right|_{P_0=(x_0,y_0,z_0)} = f_x(P_0)\cos\alpha + f_y(P_0)\cos\beta + f_z(P_0)\cos\gamma$$

$$\mathbf{grad} f(x_0,y_0,z_0) = f_x(x_0,y_0,z_0)\vec{\boldsymbol{i}} + f_y(x_0,y_0,z_0)\vec{\boldsymbol{j}} + f_z(x_0,y_0,z_0)\vec{\boldsymbol{k}}$$

$$(\bar{x},\bar{y},\bar{z}) = \left(\frac{\iiint_\Omega x\rho\mathrm{d}v}{\iiint_\Omega \rho\mathrm{d}v}, \frac{\iiint_\Omega y\rho\mathrm{d}v}{\iiint_\Omega \rho\mathrm{d}v}, \frac{\iiint_\Omega z\rho\mathrm{d}v}{\iiint_\Omega \rho\mathrm{d}v}\right), \quad J_{k_j} = \iiint_\Omega ((\sum_{i=1}^n k_i^2) - k_j^2)\rho\mathrm{d}v$$

$$S = \iint_\Sigma \mathrm{d}S = \iint_{D_{xy}} \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}\mathrm{d}x\mathrm{d}y$$

$$S = \iint_{D_{zx}} \sqrt{1 + (\frac{\partial y}{\partial z})^2 + (\frac{\partial y}{\partial x})^2}\mathrm{d}z\mathrm{d}x = \iint_{D_{yz}} \sqrt{1 + (\frac{\partial x}{\partial y})^2 + (\frac{\partial x}{\partial z})^2}\mathrm{d}y\mathrm{d}z$$

$$\iint_{-\Sigma} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y = -\iint_\Sigma P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y$$

$$\oint_\Sigma P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y = \iiint_\Omega (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z})\mathrm{d}v, \quad \mathrm{div}\vec{\boldsymbol{A}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\Sigma : F(x,y,z) = 0 \quad \vec{n} = \left(\frac{F_x}{\sqrt{F_x^2+F_y^2+F_z^2}}, \frac{F_y}{\sqrt{F_x^2+F_y^2+F_z^2}}, \frac{F_z}{\sqrt{F_x^2+F_y^2+F_z^2}}\right) = (\cos\alpha, \cos\beta, \cos\gamma)$$

$$\oint_\Sigma P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y = \iint_\Sigma (P\cos\alpha + Q\cos\beta + R\cos\gamma)\mathrm{d}S$$

$$\oint_\Gamma P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = \iint_\Sigma (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\mathrm{d}y\mathrm{d}z + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\mathrm{d}z\mathrm{d}x + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\mathrm{d}x\mathrm{d}y$$

$$\mathbf{rot}\vec{\boldsymbol{A}} = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\vec{\boldsymbol{i}} + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\vec{\boldsymbol{j}} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\vec{\boldsymbol{k}}$$

Infinite Series

$$\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|=\rho \quad ,R=\frac{1}{\rho}$$

$$S(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l})$$

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)\mathrm{d}x,\quad a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}\mathrm{d}x,\quad b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$

$$S(x)=\begin{cases} f(x),x\in consecutive \\ \frac{f(x-0)+f(x+0)}{2},x\in discontinuity \\ \frac{f(l-0)+f(l+0)}{2},x=\{-l,l\}\end{cases}$$

Matrix Calculation

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \alpha^{\mathbf{T}}=\begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix} \quad |\mathbf{A}|=\det\mathbf{A}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$|\mathbf{A}|=\Pi_{i=1}^{n}\lambda_i \quad |k\mathbf{A}|=k^n|\mathbf{A}| \quad |\mathbf{AB}|=|\mathbf{A}||\mathbf{B}| \quad |\mathbf{A}^n|=|\mathbf{A}|^n$$

$$|\mathbf{A}^{\mathbf{T}}|=|\mathbf{A}| \quad (k\mathbf{A})^{\mathbf{T}}=k\mathbf{A}^{\mathbf{T}} \quad (\mathbf{A}^{\mathbf{T}})^{\mathbf{T}}=\mathbf{A} \quad (\mathbf{A}^n)^{\mathbf{T}}=(\mathbf{A}^{\mathbf{T}})^n$$

$$|\mathbf{A}^{-1}|=\frac{1}{|\mathbf{A}|} \quad (k\mathbf{A})^{-1}=\frac{1}{k}\mathbf{A}^{-1} \quad (\mathbf{A}^{-1})^{-1}=\mathbf{A} \quad (\mathbf{A}^n)^{-1}=(\mathbf{A}^{-1})^n$$

$$|\mathbf{A}^*|=|\mathbf{A}|^{n-1} \quad (k\mathbf{A})^*=k^{n-1}\mathbf{A}^* \quad (\mathbf{A}^*)^*=|\mathbf{A}|^{n-2}\mathbf{A} \quad \mathbf{AA}^*=\mathbf{A}^*\mathbf{A}=|\mathbf{A}|\mathbf{E}$$

$$(\mathbf{ABC})^T=\mathbf{C}^{\mathbf{T}}\mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}} \quad (\mathbf{ABC})^{-1}=\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} \quad (\mathbf{ABC})^*=\mathbf{C}^*\mathbf{B}^*\mathbf{A}^*$$

$$\mathbf{diag}(a_1,...,a_n)\mathbf{diag}(b_1,...,b_n)=\mathbf{diag}(a_1b_1,...,a_nb_n)$$

$$\Lambda=\mathbf{diag}(\lambda_1,...,\lambda_n),|\Lambda|=\Pi_{i=1}^{n}\lambda_i,\varphi(\Lambda)=\mathbf{diag}(\varphi(\lambda_1),...,\varphi(\lambda_n))$$

$$\Lambda=\begin{pmatrix} 0 & 0 & \lambda_1 \\ 0 & \lambda_i & 0 \\ \lambda_n & 0 & 0 \end{pmatrix} \quad |\Lambda|=(-1)^{\frac{n(n-1)}{2}}\Pi_{i=1}^{n}\lambda_i \quad \Lambda^{-1}=\begin{pmatrix} 0 & 0 & \frac{1}{\lambda_1} \\ 0 & \frac{1}{\lambda_i} & 0 \\ \frac{1}{\lambda_n} & 0 & 0 \end{pmatrix}$$

$$\begin{vmatrix} \mathbf{A} & * \\ \mathbf{O} & \mathbf{B} \end{vmatrix}=\begin{vmatrix} \mathbf{A} & \mathbf{O} \\ * & \mathbf{B} \end{vmatrix}=|\mathbf{A}||\mathbf{B}|$$

$$\begin{vmatrix} \mathbf{O} & \mathbf{A}_{mm} \\ \mathbf{B}_{nn} & * \end{vmatrix}=\begin{vmatrix} * & \mathbf{A}_{mm} \\ \mathbf{B}_{nn} & \mathbf{O} \end{vmatrix}=(-1)^{mn}|\mathbf{A}||\mathbf{B}|$$

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{\mathbf{T}}=\begin{pmatrix} \mathbf{A}^{\mathbf{T}} & \mathbf{B}^{\mathbf{T}} \\ \mathbf{C}^{\mathbf{T}} & \mathbf{D}^{\mathbf{T}} \end{pmatrix} \quad \begin{pmatrix} \mathbf{A} & 0 & 0 \\ 0 & \mathbf{B} & 0 \\ 0 & 0 & \mathbf{C} \end{pmatrix}^{-1}=\begin{pmatrix} \mathbf{A}^{-1} & 0 & 0 \\ 0 & \mathbf{B}^{-1} & 0 \\ 0 & 0 & \mathbf{C}^{-1} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}^{\mathbf{T}}=(\mathbf{A}^{\mathbf{T}},\mathbf{B}^{\mathbf{T}}) \quad \begin{pmatrix} 0 & 0 & \mathbf{A} \\ 0 & \mathbf{B} & 0 \\ \mathbf{C} & 0 & 0 \end{pmatrix}^{-1}=\begin{pmatrix} 0 & 0 & \mathbf{A}^{-1} \\ 0 & \mathbf{B}^{-1} & 0 \\ \mathbf{C}^{-1} & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{O} & \mathbf{B} \end{pmatrix}^{-1}=\begin{pmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1} \\ \mathbf{O} & \mathbf{B}^{-1} \end{pmatrix} \quad \begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{C} & \mathbf{B} \end{pmatrix}^{-1}=\begin{pmatrix} \mathbf{A}^{-1} & \mathbf{O} \\ -\mathbf{B}^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{B}^{-1} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{C} & \mathbf{A} \\ \mathbf{B} & \mathbf{O} \end{pmatrix}^{-1}=\begin{pmatrix} \mathbf{O} & \mathbf{B}^{-1} \\ \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1} \end{pmatrix} \quad \begin{pmatrix} \mathbf{O} & \mathbf{A} \\ \mathbf{B} & \mathbf{C} \end{pmatrix}^{-1}=\begin{pmatrix} -\mathbf{B}^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{B}^{-1} \\ \mathbf{A}^{-1} & \mathbf{O} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{B} \end{pmatrix}^{*}=\begin{pmatrix} |\mathbf{B}|\mathbf{A}^* & \mathbf{O} \\ \mathbf{O} & |\mathbf{A}|\mathbf{B}^* \end{pmatrix} \quad \begin{pmatrix} \mathbf{O} & \mathbf{A_{mm}} \\ \mathbf{B_{nn}} & \mathbf{O} \end{pmatrix}^{*}=(-1)^{mn}\begin{pmatrix} \mathbf{O} & |\mathbf{A}|\mathbf{B}^* \\ |\mathbf{B}|\mathbf{A}^* & \mathbf{O} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{B} \end{pmatrix}^{n}=\begin{pmatrix} \mathbf{A}^n & \mathbf{O} \\ \mathbf{O} & \mathbf{B}^n \end{pmatrix} \quad \begin{pmatrix} \mathbf{O} & \mathbf{A} \\ \mathbf{B} & \mathbf{O} \end{pmatrix}^{n}=\begin{pmatrix} \textbf{not} & \textbf{important} \\ \textbf{at} & \textbf{all} \end{pmatrix}$$

Matrix Rank

$$r(\mathbf{A}^*) = \begin{cases} n & \text{if } r(\mathbf{A}) = n, \\ 1 & \text{if } r(\mathbf{A}) = n-1, \\ 0 & \text{if } r(\mathbf{A}) < n-1. \end{cases}$$

$$0 \le r(\mathbf{A_{mn}}) \le \max\{m, n\}$$

$$\max\{r(\mathbf{A}), r(\mathbf{B})\} \le \mathbf{r}(\mathbf{A}, \mathbf{B}) \le r(\mathbf{A}) + r(\mathbf{B})$$

$$\max\{r(\mathbf{A}), r(\mathbf{B})\} \le r \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} \le r(\mathbf{A}) + r(\mathbf{B})$$

$$r(\mathbf{A} \pm \mathbf{B}) \le r(\mathbf{A} \pm \mathbf{B}, \mathbf{B}) = r(\mathbf{A}, \mathbf{B}) \le r(\mathbf{A}) + r(\mathbf{B})$$

$$r(\mathbf{A}, \mathbf{B}) \le r \begin{pmatrix} \mathbf{A^T} \\ \mathbf{B^T} \end{pmatrix} \ne r \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} = r(\mathbf{A}, \mathbf{B})$$

$$r(\mathbf{A}) + r(\mathbf{B}) - n \le r(\mathbf{AB}) \le \min\{r(\mathbf{A}), r(\mathbf{B})\}$$

$$r(\mathbf{A^T A}) = r(\mathbf{A A^T}) = r(\mathbf{A^T}) = r(\mathbf{A}) = r(k\mathbf{A}), (\forall k \ne 0)$$

$$\exists \mathbf{A_{mn} B_{ns}} = \mathbf{O}, r(\mathbf{A}) + r(\mathbf{B}) \le n$$

$$\exists \mathbf{A_{mn} B_{ns}} = \mathbf{C}, \exists r(\mathbf{A} = n), r(\mathbf{B}) = r(\mathbf{C})$$

$$\exists \mathbf{A_{mn} B_{ns}} = \mathbf{C}, \exists r(\mathbf{B} = n), r(\mathbf{A}) = r(\mathbf{C})$$

$$\exists \mathbf{A_{nn}}, \forall k \in \mathbf{N}^*, r(\mathbf{A}^n) = r(\mathbf{A}^{n+k}) \Longrightarrow r(\mathbf{A}) = ... = r(\mathbf{A}^n)$$

$$\boldsymbol{Similarity \quad Theory \quad and \quad Feature \quad Vector}$$

$$|\lambda\mathbf{E}-\mathbf{A}| = 0 \Rightarrow \lambda_i, i \in [1,n]$$

$$|\lambda\mathbf{E}-\mathbf{A}| = \begin{pmatrix} \lambda - a_{11} & ... & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & ... & \lambda - a_{nn} \end{pmatrix} = \lambda^n + \sum_{i=1}^{n} a_{ii}\lambda^{n-1} + ...$$

$$\forall i \in [0,n], \exists \lambda_i, f(\lambda_i) = 0 \Longrightarrow \Pi_{i=1}^{n}(\lambda - \lambda_i) = 0$$

$$\lambda^n + \sum_{i=1}^{n} \lambda_i \lambda^{n-1} + ... + (-1)^n \Pi_{i=1}^{n} \lambda_i = 0 \Rightarrow \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$$

$$\exists \lambda = 0, (-1)^n \Pi_{i=1}^{n} \lambda_i = |-\mathbf{A}| = (-1)^n |\mathbf{A}| \Rightarrow \Pi_{i=1}^{n} \lambda_i = |\mathrm{A}|$$

$$\boldsymbol{Specially \quad n = 3}$$

$$|\lambda\mathbf{E}-\mathbf{A}| = \begin{pmatrix} \lambda - a_{11} & a_{12} & a_{13} \\ a_{21} & \lambda - a_{22} & a_{23} \\ a_{31} & a_{32} & \lambda - a_{33} \end{pmatrix} = \lambda^3 - \left(\sum_{i=1}^{3} a_{ii}\right)\lambda^2 + \left(\sum_{i=1}^{3} \mathbf{A_{ii}}\right)\lambda - |\mathbf{A}|$$

$$\lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + (\mathbf{A_{11}} + \mathbf{A_{22}} + \mathbf{A_{33}})\lambda - |\mathbf{A}| = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + (\lambda_2\lambda_3 + \lambda_1\lambda_3 + \lambda_1\lambda_2)\lambda - (\lambda_1\lambda_2\lambda_3)$$

$$\sum_{i=1}^{3} \mathbf{A_{ii}} = \mathbf{tr}(\mathbf{A}^*) = (\lambda_2\lambda_3 + \lambda_1\lambda_3 + \lambda_1\lambda_2)$$

$$\mathbf{A} \sim \mathbf{B} \Longrightarrow \mathbf{A^T} \sim \mathbf{B^T}, \mathbf{A}^{-1} \sim \mathbf{B}^{-1}, \mathbf{A}^* \sim \mathbf{B}^*, f(\mathbf{A}) \sim f(\mathbf{B})$$

$$f(\mathbf{A}) = 0 \Rightarrow f(\lambda) = 0 (E \sim 1)$$

$$\lambda_{\mathbf{A}_i^*} \lambda_{\mathbf{A}_i} = |\mathbf{A}|, i \in [1,n]$$

$\mathbf{A}$	$\mathbf{A^T}$	$\mathbf{A}^{-1}$	$\mathbf{A}^*$	$f(\mathbf{A})$	$\mathbf{P}^{-1}\mathbf{AP}$	$\mathbf{PAP}^{-1}$
λ	λ	$\frac{1}{\lambda}$	$\frac{\lvert\mathbf{A}\rvert}{\lambda}$	$f(\lambda)$	λ	λ
α		α	α	α	$\mathbf{P}^{-1}\alpha$	$\mathbf{P}\alpha$

Base $\boldsymbol{\sigma}$

$$(\eta_1, \eta_2, ...\eta_n) = (\xi_1, \xi_2, ...\xi_n)\mathbf{M}$$

$$\left\{\begin{array}{ccccc} \eta_1 = a_{11}\xi_1 + a_{21}\xi_2 + ... + a_{n1}\xi n \\ \vdots \quad \vdots \quad \ddots & & & \vdots & \vdots \\ \eta_n = a_{1n}\xi_1 + a_{2n}\xi_2 + ... + a_{nn}\xi n \end{array}\right. \Longrightarrow \mathbf{AM} = \mathbf{B} \Longleftrightarrow \mathbf{M} = \mathbf{A^{-1}B}$$

$$\mathbf{A}\xi_{\mathbf{A}} = \mathbf{B}\xi_{\mathbf{B}} \Longrightarrow \xi_{\mathbf{A}} = \mathbf{A}^{-1}\mathbf{B}\xi_{\mathbf{B}}$$

$$\boldsymbol{Traditional \quad Probability \quad Theory}$$

$$\boldsymbol{Opposition}: P(A) + P(\bar{A}) = 1 \quad \boldsymbol{Exclusive}: A \cap B = \varnothing \Rightarrow P(AB) = 0$$

$$\boldsymbol{Independent}: P(AB) = P(A)P(B) \quad \boldsymbol{Equal}: A = B \Rightarrow P(A) = P(B)$$

$$A - B = A\bar{B} = A \cap \bar{B} \Rightarrow P(A - B) = P(A) - P(AB)$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(BC) - P(AC) - P(AB) + P(ABC)$$

$$P(B|A) = \frac{P(AB)}{P(A)} \quad P(A_1A_2...A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2)...P(A_n|A_1...A_{n-1})$$

$$\boldsymbol{Bayes} \quad P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}$$

Variable Digital Properties

Distr	Mark	EX	DX	Addition
Bin	$B(n,p)$	np	$np(1-p)$	$P\{X=k\}=\mathbf{C_n^k}(1-p)^{n-k}p^k$
Poi	$P(\lambda)$	λ	λ	$P\{X=k\}=\frac{\lambda^k}{k!}\mathrm{e}^{-\lambda}, k=0,1,2,...$
Geo	$G(p)$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$P\{X=k\}=(1-p)^{k-1}p, k=1,2,...$
Hyp	$H(n,M,N)$	$\frac{nM}{N}$	$\frac{nM}{N}(1-\frac{M}{N})(\frac{N-n}{N-1})$	$P\{X=i\}=\frac{\mathbf{C}_M^i\mathbf{C}_{N-M}^{n-i}}{\mathbf{C}_N^n}$
Uni	$U(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Exp	$E(\lambda)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	
Nor	$N(\mu,\sigma^2)$	μ	σ^2	

$$\textbf{Uni}: f(x)=\begin{cases}\frac{1}{b-a}, a\le x\le b\\ 0, others\end{cases}\qquad F(x)=\begin{cases}0, x<a\\ \frac{x-a}{b-a}, a\le x<b\\ 1, x\ge b\end{cases}$$

$$\textbf{Exp}: f(x)=\begin{cases}\lambda\mathrm{e}^{-\lambda x}, x>0\\ 0, x\le 0\end{cases}\qquad F(x)=\begin{cases}1-\lambda\mathrm{e}^{-\lambda x}, x>0\\ 0, x\le 0\end{cases}$$

$$\textbf{Nor}: f(x)=\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}, F(x)=\int_{-\infty}^{x}f(t)\mathrm{d}t, x\in(-\infty,+\infty)$$

$$f(\mu+x)=f(\mu-x), F(\mu+x)+F(\mu-x)=1, F(\mu)=\frac{1}{2}$$

$$X\sim N(0,1), \phi(x)=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}, \Phi(x)=\int_{-\infty}^{x}\phi(t)\mathrm{d}t$$

$$\phi(-x)=\phi(x),\quad \Phi(a)+\Phi(-a)=1, \Phi(0)=\frac{1}{2},\quad F(x)=\Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$f(x,y)\ge 0\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\mathrm{d}x\mathrm{d}y=1$$

$$f_X(x)=\int_{-\infty}^{\infty}f(x,y)\mathrm{d}y>0\quad f_Y(y)=\int_{-\infty}^{\infty}f(x,y)\mathrm{d}x>0$$

$$f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}, f_{Y|X}(y|x)=\frac{f(x,y)}{f_X(x)}$$

$$\textbf{\textit{Discrete}}: P\{Z=g(x_i,y_j)\}=P\{X=x_i,Y=y_j\}=P_{ij}$$

$$F_Z(z)=P\{Z\le z\}=P\{g(X,Y)\le z\}=\sum_{g(x_i,y_j)\le z}P\{X=x_i,Y=y_j\}$$

$$\textbf{\textit{Continuous}}: Z=g(X,Y)\quad F_Z(z)=P\{Z\le z\}=P\{g(X,Y)\le z\}=\iint_{g(x,y)\le z}f(x,y)\mathrm{d}x\mathrm{d}y$$

$$Z=\max(X,Y), F_{\max}(z)=F_X(z)F_Y(z)\quad Z=\min(X,Y), F_{\min}(z)=1-[1-F_X(z)][1-F_Y(z)]$$

$$(X,Y)\sim U(D), f(x,y)=\begin{cases}\frac{1}{S_D}, (x,y)\in D\\ 0, others\end{cases}$$

$$(X,Y)\sim N(\mu_1,\mu_2;\sigma_1,\sigma_2,\rho), f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\mathrm{e}^{-\frac{1}{2(1-\rho^2)}[(\frac{x-\mu_1}{\sigma_1})^2-\frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}+(\frac{y-\mu_2}{\sigma_2})^2]}$$

$$E(X)=\sum_{i=1}^{\infty}x_ip_i \quad E(X)=\int_{-\infty}^{\infty}xf(x)\mathrm{d}x$$

$$E[g(X)]=\sum_{i=1}^{\infty}g(x_i)p_i \quad E[g(X)]=\int_{-\infty}^{\infty}g(x)f(x)\mathrm{d}x$$

$$E[g(X,Y)]=\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}g(x_i,y_j)P_{ij} \quad E[g(X,Y)]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x,y)f(x,y)\mathrm{d}x\mathrm{d}y$$

$$E(C)=C \quad E(CX)=CE(X) \quad E(X+C)=E(X)+C \quad E(X+Y)=E(X)+E(Y)$$

$$D(X)=\sum_{i}[x_i-E(X)]^2p_i \quad D(X)=\int_{-\infty}^{\infty}[x-E(X)]^2f(x)\mathrm{d}x$$

$$D(X)=E(X^2)-[E(X)]^2 \quad D(C)=0 \quad D(CX)=C^2D(X) \quad D(X+C)=D(X)$$

$$Cov(X,Y)=E(XY)-E(X)E(Y) \quad D(X\pm Y)=D(X)+D(Y)\pm 2Cov(X,Y)$$

$$Cov(X,Y)=Cov(Y,X) \quad Cov(X,X)=D(X) \quad Cov(X,c)=0$$

$$Cov(aX,bY)=abCov(X,Y) \quad Cov(X_1+X_2,Y)=Cov(X_1,Y)+Cov(X_2,Y)$$

$$\rho_{XY}=\frac{Cov(X,Y)}{\sqrt{D(X)}\sqrt{D(Y)}}$$

Large Number Law Central Limit Theorem

$$P\{|X-E(X)|\geq\epsilon\}\leq\frac{D(X)}{\epsilon^2}$$

$$\lim_{n\to\infty}P\{|\frac{X_n}{n}-p|<\epsilon\}=1$$

$$\lim_{n\to\infty}P\{|\frac{1}{n}\sum_{k=1}^{n}x_k-\mu|<\epsilon\}=1$$

$$\lim_{n\to\infty}P\{|\frac{1}{n}\sum_{k=1}^{n}X_k-\frac{1}{n}\sum_{k=1}^{n}E(X_k)|<\epsilon\}=1$$

$$\sum_{i=1}^{n}X_i\sim N(n\mu,n\sigma^2),\lim_{n\to\infty}P\{\frac{\sum_{i=1}^{n}X_i-n\mu}{\sqrt{n}\sigma}\leq x\}=F_n(x)=\Phi(x)$$

$$X\sim N(np,np(1-p)),\lim_{n\to\infty}P\{\frac{X_n-np}{\sqrt{np(1-p)}}\leq x\}=\Phi(x)$$

Mathematical Statistics

$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n}), \frac{\bar{X}-u}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2 = \frac{1}{n-1}[\sum_{i=1}^{n}X_i^2 - n\bar{X}^2]$$

$$E(S^2)=\sigma^2, D(S^2)=\frac{2\sigma^4}{n-1}$$

$$\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\sigma^2} \sim \chi^2(n)$$

$$\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma^2} \sim \chi^2(n-1)$$

$$\chi^2 \sim \chi^2(n), E(\chi^2(n))=n, D(\chi^2(n))=2n$$

$$T \sim t(n), T=\frac{X}{\sqrt{Y/n}}\frac{\sim}{\sim}\frac{N(0,1)}{\sqrt{\chi^2(n)/n}}$$

$$F \sim F(n1,n2), F=\frac{X/n1}{Y/n2}\frac{\sim}{\sim}\frac{\chi^2(n1)/n1}{\chi^2(n2)/n2}$$

$$(X,Y) \sim N(\mu_1,\mu_2;\sigma_1^2,\sigma_2^2,0)$$

$$\frac{\frac{(n_1-1)S_1^2}{\sigma_1^2}/(n_1-1)}{\frac{(n_2-1)S_2^2}{\sigma_2^2}/(n_2-1)} \sim F(n_1-1,n_2-1)$$

$$\frac{\frac{\sum_{i=1}^{n_1}(X_i-\mu_1)^2}{\sigma_1^2}/n_1}{\frac{\sum_{i=1}^{n_2}(X_i-\mu_2)^2}{\sigma_2^2}/n2} \sim F(n1,n2)$$

$$\frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n2}}} \sim N(0,1)$$

$$\exists\sigma_1=\sigma_2, \frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{S_\omega\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \sim t(n_1+n_2-2)$$

$$S_\omega=\sqrt{\frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}}$$

Constant Series

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}\qquad\sum_{k=1}^{\infty}\frac{1}{k^4}=\frac{\pi^4}{90}\qquad\sum_{k=1}^{\infty}\frac{1}{k^6}=\frac{\pi^6}{945}$$

$$\sum_{i=0}^{n}a_i\cdot\sum_{j=0}^{n}b_j=\sum_{i=0}^{n}\sum_{j=0}^{n}(a_i\cdot b_j)\qquad\sum_{n=s}^{t}\ln f(n)=\ln\prod_{n=s}^{t}f(n)$$

$$\sum_{i=0}^{n}i=\frac{n(n+1)}{2}\qquad\sum_{i=0}^{n}i^2=\frac{n(n+1)(2n+1)}{6}\qquad\sum_{i=0}^{n}i^3=(\sum_{i=0}^{n}i)^2=\frac{n^2(n+1)^2}{4}$$

$$\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}=\sum_{i=0}^{\infty}\frac{1}{(2i+1)(2i+2)}=\sum_{i=1}^{\infty}\frac{1}{2^i i}=\sum_{i=1}^{\infty}(\frac{1}{3^i}+\frac{1}{4^i})\frac{1}{i}=\ln 2$$

Power Series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \ln(1+x)$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x)$$

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x$$

$$\sum_{n=0}^{\infty} (n+1) x^n = \frac{1}{(1-x)^2}$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) x^n = \frac{2}{(1-x)^3}$$

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{n} = -\ln(1-x^2)$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = \frac{1}{2} \ln(\frac{1+x}{1-x})$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin x$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \mathrm{e}^x$$

Transcendental Equation

$$\sum_{i=0}^{n} a_i x^i = 0 \Longrightarrow \prod_{i=0}^{n} (x - x_i) = 0$$

$$\prod_{i=0}^{n} x_i = (-1)^n \frac{a_0}{a_n}, \quad \sum_{i=0}^{n} \frac{\prod_{i=0}^{n} x_i}{x_i} = (-1)^{n-1} \frac{a_1}{a_n}$$

$$\sum_{i=0}^{n} \frac{1}{x_i} = -\frac{a_1}{a_0}$$

eg. $\tan x = x \Longrightarrow \sin x = \cos x \cdot x$

$$x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + ... = x \cdot (1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + ...)$$

$$\frac{1}{3}x^3 - \frac{1}{30}x^5 + ... = 0 \quad \xrightarrow{x \neq 0} \quad \frac{1}{3}x^2 - \frac{1}{30}x^4 + ... = 0$$

$$\sum_{i=0}^{n} \frac{1}{t_i} = \frac{1}{10} \quad \xrightarrow{t = x^2} \quad \lim_{n \to \infty} \sum_{i=0}^{n} \frac{1}{x_i^2} = \frac{1}{10}$$

Beyond Integral

$$\sum_{n=0}^{\infty}\frac{1}{(3n)!}\Longrightarrow S(x)=\sum_{n=0}^{\infty}\frac{x^{3n}}{(3n)!},S(1)=\frac{\mathrm{e}}{3}+\frac{2}{3}\cos\frac{\sqrt{3}}{2}\mathrm{e}^{-\frac{1}{2}}$$

$$\iint_{D_{xy}}(\mathrm{e}^x\cos y)\mathrm{d}\sigma, D=\{(x,y)|x^2+y^2\leq 1\}$$

$$\iint_{D_{xy}}(x+y)\mathrm{d}\sigma, D=\{(x,y)|y^2\leq x+2, x^2\leq y+2\}$$

$$\boldsymbol{A_0}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{m=0}^{\infty}\frac{1}{n2^m+1}$$

$$\boldsymbol{A_1}=\int_0^{x^2}\pi(\sqrt[4]{1+t}-1)\sin t^4\mathrm{d}x$$

$$\boldsymbol{A_2}=\sum_{n=1}^{\infty}\frac{((n-1)!)^2(2t)^{2n}}{(2n)!}$$

$$\boldsymbol{A_3}=\int_0^1\frac{(1-2x)\ln(1-x)}{x^2-x+1}\mathrm{d}x$$

$$\boldsymbol{A_4}=x^2(x-\tan x)\ln(x^2+1)[\left(\frac{2\arctan\frac{y}{x}}{\pi}\right)^y-1]$$

$$\lim_{x\to 0^+}\lim_{y\to+\infty}\frac{\boldsymbol{A_0}\boldsymbol{A_1}}{\boldsymbol{A_2}\boldsymbol{A_3}\boldsymbol{A_4}}=\frac{27}{32}$$

$$\int\frac{\sec^3 x}{1-\tan^6 x}\mathrm{d}x$$

$$\int\frac{1}{\csc x+\sec x+\tan x+\cot x}\mathrm{d}x$$

$$\lim_{N\to\infty}\sum_{n=1}^{N}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}-\ln 2=\ln 2-\frac{1}{2}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1-\cos\frac{\pi}{\sqrt{n}}}{1+\cos\frac{i\pi}{\sqrt{2n}}}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{[\sqrt{n}]}}{n}\qquad\sum_{n=1}^{\infty}\frac{(n)!}{(2n+1)!}\frac{1}{(n+1)}$$

$$t\frac{\mathrm{d}^3x}{\mathrm{d}t^3}+3\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-t\frac{\mathrm{d}x}{\mathrm{d}t}-x=0$$

$$\boldsymbol{Basel\quad Problem}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

$$f(x)=x^2\quad\xrightarrow{FourierExpansion}\quad S(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\cos nx)$$

$$a_0=\frac{2}{3}\pi^2\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx\mathrm{d}x=(-1)^n\frac{4}{n^2}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\sin nx\mathrm{d}x=0$$

$$S(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx=\frac{1}{3}\pi^2+\sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}\cos nx=\frac{1}{3}\pi^2+\sum_{n=1}^{n}\frac{4}{n^2}$$

$$\exists x=0,S(0)=\frac{1}{3}\pi^2+\sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}=f(0)=0\Longrightarrow\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=\frac{\pi^2}{12}$$

$$\exists x=\pi,S(\pi)=\frac{1}{3}\pi^2+\sum_{n=1}^{\infty}\frac{4}{n^2}=f(\pi)=\pi^2\Longrightarrow\sum_{n=1}^{\infty}\frac{4}{n^2}=\frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}\qquad\sum_{n=1}^{\infty}\frac{1}{(2n)^2}=\frac{\pi^2}{24}$$