

### *Elementary Function*

$$(a+b)^n=\sum_{k=0}^n\mathbf{C}_n^{\mathbf{k}}a^{n-k}b^k$$

$$a^n-b^n=(a-b)\cdot\sum_{k=0}^{n-1}a^{n-1-k}b^k$$

$$\arcsin x+\arccos x=\arctan x+\operatorname{arccot} x=\frac{\pi}{2}$$

$$f(x)=\int_a^xf'(t)\mathrm{d}t+f(a)$$

$$K=\frac{|y''x'-y'x''|}{(x'^2+y'^2)^{\frac{3}{2}}},\quad \rho=\frac{1}{K}$$

$$\int_0^1f(x)\mathrm{d}x=\sum_{i=1}^n\left(\frac{i+1}{n}-\frac{i}{n}\right)f\left(\frac{i}{n}\right)=\frac{1}{n}\sum_{i=1}^nf\left(\frac{2i+1}{2n}\right)$$

$$\int_{t1}^{t2}f(x(t),y(t))\mathrm{d}s=\int_a^bf(x,y(x))\sqrt{x'^2+y'^2}\mathrm{d}x=\int_{\alpha}^{\beta}f(r,\theta)\sqrt{r^2+r'^2}\mathrm{d}\theta$$

$$S=\int_a^by(x)\mathrm{d}x=\int_{t1}^{t2}y(t)x'(t)\mathrm{d}t=\frac{1}{2}\int_{\alpha}^{\beta}r^2(\theta)\mathrm{d}\theta$$

$$S_{side}=\int_a^b2\pi y(x)\sqrt{x'^2(x)+y'^2(x)}\mathrm{d}x=\int_{\alpha}^{\beta}2\pi r(\theta)\sin\theta\sqrt{r^2(\theta)+r'^2(\theta)}\mathrm{d}\theta$$

$$[\overrightarrow{abc}]=\begin{vmatrix}a_x&a_y&a_z\\b_x&b_y&b_z\\c_x&c_y&c_z\end{vmatrix}$$

***Gamma Function Integral***

$$\left(-\frac{1}{2}\right)! = \sqrt{\pi}$$

$$\left(\frac{2n+1}{2}\right)! = \left(\prod_{i=n}^0 \frac{2i+1}{2}\right) \sqrt{\pi} = \frac{2n+1}{2} \frac{2n-1}{2} \dots \frac{1}{2} \sqrt{\pi}$$

$$\int_0^{\infty} x^a e^{-x} dx = a!$$

$$\int_0^{\infty} x^{-\frac{1}{2}} e^{-x} dx = \sqrt{\pi}$$

$$\int_0^{\infty} x^3 e^{-x} dx = (3)! = 6$$

$$\int_0^{\infty} x^{\frac{5}{2}} e^{-x} dx = \frac{5}{2} \frac{3}{2} \frac{1}{2} \sqrt{\pi}$$

$$\int_0^{\infty} x^a e^{-x^2} dx = \frac{1}{2} \left(\frac{a-1}{2}\right)!$$

$$\int_0^{\infty} x^1 e^{-x^2} dx = \frac{1}{2} \left(\frac{1-1}{2}\right)! = \frac{1}{2}$$

$$\int_0^{\infty} x^4 e^{-x^2} dx = \frac{1}{2} \left(\frac{4-1}{2}\right)! = \frac{1}{2} \frac{3}{2} \frac{1}{2} \sqrt{\pi}$$

$$\int_0^{\infty} x^7 e^{-x^2} dx = \frac{1}{2} \left(\frac{7-1}{2}\right)! = \frac{1}{2} (3)! = 3$$

# Multivariate Integral

$$\iiint_{\Omega_{xyz}} f(x, y, z) dx dy dz = \iiint_{\Omega_{uvw}} f(u, v, w) \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} du dv dw$$

$$\left. \frac{\partial f}{\partial l} \right|_{P_0=(x_0, y_0, z_0)} = f_x(P_0) \cos \alpha + f_y(P_0) \cos \beta + f_z(P_0) \cos \gamma$$

$$\mathbf{grad} f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0) \vec{i} + f_y(x_0, y_0, z_0) \vec{j} + f_z(x_0, y_0, z_0) \vec{k}$$

$$(\bar{x}, \bar{y}, \bar{z}) = \left( \frac{\iiint_{\Omega} x \rho dv}{\iiint_{\Omega} \rho dv}, \frac{\iiint_{\Omega} y \rho dv}{\iiint_{\Omega} \rho dv}, \frac{\iiint_{\Omega} z \rho dv}{\iiint_{\Omega} \rho dv} \right), \quad J_{k_j} = \iiint_{\Omega} ((\sum_{i=1}^n k_i^2) - k_j^2) \rho dv$$

$$S = \iint_{\Sigma} dS = \iint_{D_{xy}} \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} dx dy$$

$$S = \iint_{D_{zx}} \sqrt{1 + \left( \frac{\partial y}{\partial z} \right)^2 + \left( \frac{\partial y}{\partial x} \right)^2} dz dx = \iint_{D_{yz}} \sqrt{1 + \left( \frac{\partial x}{\partial y} \right)^2 + \left( \frac{\partial x}{\partial z} \right)^2} dy dz$$

$$\iint_{-\Sigma} P dy dz + Q dz dx + R dx dy = - \iint_{\Sigma} P dy dz + Q dz dx + R dx dy$$

$$\oint_{\Sigma} P dy dz + Q dz dx + R dx dy = \iiint_{\Omega} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dv, \quad \operatorname{div} \vec{A} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\Sigma : F(x, y, z) = 0 \quad \vec{n} = \left( \frac{F_x}{\sqrt{F_x^2 + F_y^2 + F_z^2}}, \frac{F_y}{\sqrt{F_x^2 + F_y^2 + F_z^2}}, \frac{F_z}{\sqrt{F_x^2 + F_y^2 + F_z^2}} \right) = (\cos \alpha, \cos \beta, \cos \gamma)$$

$$\oint_{\Sigma} P dy dz + Q dz dx + R dx dy = \iint_{\Sigma} (P \cos \alpha + Q \cos \beta + R \cos \gamma) dS$$

$$\oint_{\Gamma} P dx + Q dy + R dz = \iint_{\Sigma} \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy dz + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz dx + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$\mathbf{rot} \vec{A} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

***Infinite Series***

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho, R = \frac{1}{\rho}$$

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx, \quad a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx, \quad b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx$$

$$S(x) = \begin{cases} f(x), x \in \text{consecutive} \\ \frac{f(x-0)+f(x+0)}{2}, x \in \text{discontinuity} \\ \frac{f(l-0)+f(l+0)}{2}, x = \{-l, l\} \end{cases}$$

### Matrix Calculation

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \mathbf{A}^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \\
\mathbf{A}^* &= \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{21} & \mathbf{A}_{31} \\ \mathbf{A}_{12} & \mathbf{A}_{22} & \mathbf{A}_{32} \\ \mathbf{A}_{13} & \mathbf{A}_{23} & \mathbf{A}_{33} \end{bmatrix} \quad \alpha^T = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad |\mathbf{A}| = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
|\mathbf{A}| &= \prod_{i=1}^n \lambda_i \quad |k\mathbf{A}| = k^n |\mathbf{A}| \quad |\mathbf{AB}| = |\mathbf{A}||\mathbf{B}| \quad |\mathbf{A}^n| = |\mathbf{A}|^n \\
|\mathbf{A}^T| &= |\mathbf{A}| \quad (k\mathbf{A})^T = k\mathbf{A}^T \quad (\mathbf{A}^T)^T = \mathbf{A} \quad (\mathbf{A}^n)^T = (\mathbf{A}^T)^n \\
|\mathbf{A}^{-1}| &= \frac{1}{|\mathbf{A}|} \quad (k\mathbf{A})^{-1} = \frac{1}{k} \mathbf{A}^{-1} \quad (\mathbf{A}^{-1})^{-1} = \mathbf{A} \quad (\mathbf{A}^n)^{-1} = (\mathbf{A}^{-1})^n \\
|\mathbf{A}^*| &= |\mathbf{A}|^{n-1} \quad (k\mathbf{A})^* = k^{n-1} \mathbf{A}^* \quad (\mathbf{A}^*)^* = |\mathbf{A}|^{n-2} \mathbf{A} \quad \mathbf{AA}^* = \mathbf{A}^* \mathbf{A} = |\mathbf{A}| \mathbf{E} \\
(\mathbf{ABC})^T &= \mathbf{C}^T \mathbf{B}^T \mathbf{A}^T \quad (\mathbf{ABC})^{-1} = \mathbf{C}^{-1} \mathbf{B}^{-1} \mathbf{A}^{-1} \quad (\mathbf{ABC})^* = \mathbf{C}^* \mathbf{B}^* \mathbf{A}^* \\
&\quad \mathbf{diag}(a_1, \dots, a_n) \mathbf{diag}(b_1, \dots, b_n) = \mathbf{diag}(a_1 b_1, \dots, a_n b_n) \\
\Lambda &= \mathbf{diag}(\lambda_1, \dots, \lambda_n), |\Lambda| = \prod_{i=1}^n \lambda_i, \varphi(\Lambda) = \mathbf{diag}(\varphi(\lambda_1), \dots, \varphi(\lambda_n)) \\
\Lambda &= \begin{pmatrix} 0 & 0 & \lambda_1 \\ 0 & \lambda_i & 0 \\ \lambda_n & 0 & 0 \end{pmatrix} \quad |\Lambda| = (-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^n \lambda_i \quad \Lambda^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{\lambda_n} \\ 0 & \frac{1}{\lambda_i} & 0 \\ \frac{1}{\lambda_1} & 0 & 0 \end{pmatrix} \\
\begin{vmatrix} \mathbf{A} & * \\ \mathbf{O} & \mathbf{B} \end{vmatrix} &= \begin{vmatrix} \mathbf{A} & \mathbf{O} \\ * & \mathbf{B} \end{vmatrix} = |\mathbf{A}||\mathbf{B}| \\
\begin{vmatrix} \mathbf{O} & \mathbf{A}_{mm} \\ \mathbf{B}_{nn} & * \end{vmatrix} &= \begin{vmatrix} * & \mathbf{A}_{mm} \\ \mathbf{B}_{nn} & \mathbf{O} \end{vmatrix} = (-1)^{mn} |\mathbf{A}||\mathbf{B}| \\
\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^T &= \begin{pmatrix} \mathbf{A}^T & \mathbf{C}^T \\ \mathbf{B}^T & \mathbf{D}^T \end{pmatrix} \quad \begin{pmatrix} \mathbf{A} & 0 & 0 \\ 0 & \mathbf{B} & 0 \\ 0 & 0 & \mathbf{C} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} & 0 & 0 \\ 0 & \mathbf{B}^{-1} & 0 \\ 0 & 0 & \mathbf{C}^{-1} \end{pmatrix} \\
\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}^T &= (\mathbf{A}^T, \mathbf{B}^T) \quad \begin{pmatrix} 0 & 0 & \mathbf{A} \\ 0 & \mathbf{B} & 0 \\ \mathbf{C} & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & \mathbf{C}^{-1} \\ 0 & \mathbf{B}^{-1} & 0 \\ \mathbf{A}^{-1} & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{O} & \mathbf{B} \end{pmatrix}^{-1} &= \begin{pmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1} \mathbf{C} \mathbf{B}^{-1} \\ \mathbf{O} & \mathbf{B}^{-1} \end{pmatrix} \quad \begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{C} & \mathbf{B} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{A}^{-1} & \mathbf{O} \\ -\mathbf{B}^{-1} \mathbf{C} \mathbf{A}^{-1} & \mathbf{B}^{-1} \end{pmatrix} \\
\begin{pmatrix} \mathbf{C} & \mathbf{A} \\ \mathbf{B} & \mathbf{O} \end{pmatrix}^{-1} &= \begin{pmatrix} \mathbf{O} & \mathbf{B}^{-1} \\ \mathbf{A}^{-1} & -\mathbf{A}^{-1} \mathbf{C} \mathbf{B}^{-1} \end{pmatrix} \quad \begin{pmatrix} \mathbf{O} & \mathbf{A} \\ \mathbf{B} & \mathbf{C} \end{pmatrix}^{-1} = \begin{pmatrix} -\mathbf{B}^{-1} \mathbf{C} \mathbf{A}^{-1} & \mathbf{B}^{-1} \\ \mathbf{A}^{-1} & \mathbf{O} \end{pmatrix} \\
\begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{B} \end{pmatrix}^* &= \begin{pmatrix} |\mathbf{B}| \mathbf{A}^* & \mathbf{O} \\ \mathbf{O} & |\mathbf{A}| \mathbf{B}^* \end{pmatrix} \quad \begin{pmatrix} \mathbf{O} & \mathbf{A}_{mm} \\ \mathbf{B}_{nn} & \mathbf{O} \end{pmatrix}^* = (-1)^{mn} \begin{pmatrix} \mathbf{O} & |\mathbf{A}| \mathbf{B}^* \\ |\mathbf{B}| \mathbf{A}^* & \mathbf{O} \end{pmatrix} \\
\begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{B} \end{pmatrix}^n &= \begin{pmatrix} \mathbf{A}^n & \mathbf{O} \\ \mathbf{O} & \mathbf{B}^n \end{pmatrix} \quad \begin{pmatrix} \mathbf{O} & \mathbf{A} \\ \mathbf{B} & \mathbf{O} \end{pmatrix}^n = \begin{pmatrix} \text{not} & \text{important} \\ \text{at} & \text{all} \end{pmatrix}
\end{aligned}$$

**Matrix Rank**

$$r(\mathbf{A}^*) = \begin{cases} n & \text{if } r(\mathbf{A}) = n, \\ 1 & \text{if } r(\mathbf{A}) = n - 1, \\ 0 & \text{if } r(\mathbf{A}) < n - 1. \end{cases}$$

$$0 \leq r(\mathbf{A}_{mn}) \leq \min\{m, n\}$$

$$\max\{r(\mathbf{A}), r(\mathbf{B})\} \leq \mathbf{r}(\mathbf{A}, \mathbf{B}) \leq r(\mathbf{A}) + r(\mathbf{B})$$

$$\max\{r(\mathbf{A}), r(\mathbf{B})\} \leq r \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} \leq r(\mathbf{A}) + r(\mathbf{B})$$

$$r(\mathbf{A} \pm \mathbf{B}) \leq r(\mathbf{A} \pm \mathbf{B}, \mathbf{B}) = r(\mathbf{A}, \mathbf{B}) \leq r(\mathbf{A}) + r(\mathbf{B})$$

$$r(\mathbf{A}, \mathbf{B}) \leq r \begin{pmatrix} \mathbf{A}^T \\ \mathbf{B}^T \end{pmatrix} \neq r \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} = r(\mathbf{A}, \mathbf{B})$$

$$r(\mathbf{A}) + r(\mathbf{B}) - n \leq r(\mathbf{AB}) \leq \min\{r(\mathbf{A}), r(\mathbf{B})\}$$

$$r(\mathbf{A}^T \mathbf{A}) = r(\mathbf{AA}^T) = r(\mathbf{A}^T) = r(\mathbf{A}) = r(k\mathbf{A}), (\forall k \neq 0)$$

$$\exists \mathbf{A}_{mn} \mathbf{B}_{ns} = \mathbf{O}, r(\mathbf{A}) + r(\mathbf{B}) \leq n$$

$$\exists \mathbf{A}_{mn} \mathbf{B}_{ns} = \mathbf{C}_{ms}, \exists r(\mathbf{A}) = n, r(\mathbf{B}) = r(\mathbf{C})$$

$$\exists \mathbf{A}_{mn} \mathbf{B}_{ns} = \mathbf{C}_{ms}, \exists r(\mathbf{B}) = n, r(\mathbf{A}) = r(\mathbf{C})$$

$$\exists \mathbf{A}_{nn}, \forall k \in \mathbb{N}^*, r(\mathbf{A}^n) = r(\mathbf{A}^{n+k}) \implies r(\mathbf{A}) = \dots = r(\mathbf{A}^n)$$

**Similarity Theory and Feature Vector**

$$|\lambda \mathbf{E} - \mathbf{A}| = 0 \Rightarrow \lambda_i, i \in [1, n]$$

$$|\lambda \mathbf{E} - \mathbf{A}| = \begin{pmatrix} \lambda - a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & \lambda - a_{nn} \end{pmatrix} = \lambda^n + \sum_{i=1}^n a_{ii} \lambda^{n-1} + \dots$$

$$\forall i \in [0, n], \exists \lambda_i, f(\lambda_i) = 0 \Rightarrow \prod_{i=1}^n (\lambda - \lambda_i) = 0$$

$$\lambda^n + \sum_{i=1}^n \lambda_i \lambda^{n-1} + \dots + (-1)^n \prod_{i=1}^n \lambda_i = 0 \Rightarrow \sum_{i=1}^n a_{ii} = \sum_{i=1}^n \lambda_i$$

$$\exists \lambda = 0, (-1)^n \prod_{i=1}^n \lambda_i = |-\mathbf{A}| = (-1)^n |\mathbf{A}| \Rightarrow \prod_{i=1}^n \lambda_i = |\mathbf{A}|$$

**Specially  $n = 3$**

$$|\lambda \mathbf{E} - \mathbf{A}| = \begin{pmatrix} \lambda - a_{11} & a_{12} & a_{13} \\ a_{21} & \lambda - a_{22} & a_{23} \\ a_{31} & a_{32} & \lambda - a_{33} \end{pmatrix} = \lambda^3 - \left( \sum_{i=1}^3 a_{ii} \right) \lambda^2 + \left( \sum_{i=1}^3 \mathbf{A}_{ii} \right) \lambda - |\mathbf{A}|$$

$$\lambda^3 - (a_{11} + a_{22} + a_{33}) \lambda^2 + (\mathbf{A}_{11} + \mathbf{A}_{22} + \mathbf{A}_{33}) \lambda - |\mathbf{A}| = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3) \lambda^2 + (\lambda_2 \lambda_3 + \lambda_1 \lambda_3 + \lambda_1 \lambda_2) \lambda - (\lambda_1 \lambda_2 \lambda_3)$$

$$\sum_{i=1}^3 \mathbf{A}_{ii} = \text{tr}(\mathbf{A}^*) = (\lambda_2 \lambda_3 + \lambda_1 \lambda_3 + \lambda_1 \lambda_2)$$

$$\alpha^T \alpha = \text{tr}(\mathbf{A}) \quad r(\alpha \alpha^T) = 1$$

$$\alpha \alpha^T = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \end{pmatrix}_{nn} \sim \begin{pmatrix} \text{tr}(\mathbf{A}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{nn}$$

$$\mathbf{A} \sim \mathbf{B} \Rightarrow \mathbf{A}^T \sim \mathbf{B}^T, \mathbf{A}^{-1} \sim \mathbf{B}^{-1}, \mathbf{A}^* \sim \mathbf{B}^*, f(\mathbf{A}) \sim f(\mathbf{B})$$

$$f(\mathbf{A}) = 0 \Rightarrow f(\lambda) = 0 (E \sim 1)$$

$$\lambda_{\mathbf{A}_i^*} \lambda_{\mathbf{A}_i} = |\mathbf{A}|, i \in [1, n]$$

$\mathbf{A}$	$\mathbf{A}^T$	$\mathbf{A}^{-1}$	$\mathbf{A}^*$	$f(\mathbf{A})$	$\mathbf{P}^{-1} \mathbf{A} \mathbf{P}$	$\mathbf{P} \mathbf{A} \mathbf{P}^{-1}$
$\lambda$	$\lambda$	$\frac{1}{\lambda}$	$\frac{ \mathbf{A} }{\lambda}$	$f(\lambda)$	$\lambda$	$\lambda$
$\alpha$	$*$	$\alpha$	$\alpha$	$\alpha$	$\mathbf{P}^{-1} \alpha$	$\mathbf{P} \alpha$

$$\begin{array}{l}
\text{Base } \sigma \\
(\eta_1, \eta_2, \dots, \eta_n) = (\xi_1, \xi_2, \dots, \xi_n) \mathbf{M} \\
\left\{ \begin{array}{l} \eta_1 = a_{11}\xi_1 + a_{21}\xi_2 + \dots + a_{n1}\xi_n \\ \vdots \quad \vdots \quad \ddots \quad \quad \quad \vdots \quad \vdots \\ \eta_n = a_{1n}\xi_1 + a_{2n}\xi_2 + \dots + a_{nn}\xi_n \end{array} \right. \implies \mathbf{A}\mathbf{M} = \mathbf{B} \iff \mathbf{M} = \mathbf{A}^{-1}\mathbf{B} \\
\mathbf{A}\xi_{\mathbf{A}} = \mathbf{B}\xi_{\mathbf{B}} \implies \xi_{\mathbf{A}} = \mathbf{A}^{-1}\mathbf{B}\xi_{\mathbf{B}} = \mathbf{M}\xi_{\mathbf{B}}
\end{array}$$



**Traditional Probability Theory**

**Opposition** :  $P(A) + P(\bar{A}) = 1$     **Exclusive** :  $A \cap B = \emptyset \Rightarrow P(AB) = 0$

**Independent** :  $P(AB) = P(A)P(B)$     **Equal** :  $A = B \Rightarrow P(A) = P(B)$

$A - B = A\bar{B} = A \cap \bar{B} \Rightarrow P(A - B) = P(A) - P(AB)$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(BC) - P(AC) - P(AB) + P(ABC)$

$P(B|A) = \frac{P(AB)}{P(A)}$      $P(A_1 A_2 \dots A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 A_2) \dots P(A_n|A_1 \dots A_{n-1})$

**Bayes**     $P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^n P(A|B_i)P(B_i)}$

**Variable Digital Properties**

Distr	Mark	EX	DX	Addition
Bin	$B(n, p)$	$np$	$np(1-p)$	$P\{X = k\} = \mathbf{C}_n^k (1-p)^{n-k} p^k$
Poi	$P(\lambda)$	$\lambda$	$\lambda$	$P\{X = k\} = \frac{\lambda^k}{k!} e^{-\lambda}, k = 0, 1, 2, \dots$
Geo	$G(p)$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$P\{X = k\} = (1-p)^{k-1} p, k = 1, 2, \dots$
Hyp	$H(n, M, N)$	$\frac{nM}{N}$	$\frac{nM}{N} (1 - \frac{M}{N}) (\frac{N-n}{N-1})$	$P\{X = i\} = \frac{\mathbf{C}_M^i \mathbf{C}_{N-M}^{n-i}}{\mathbf{C}_N^n}$
Uni	$U(a, b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Exp	$E(\lambda)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	
Nor	$N(\mu, \sigma^2)$	$\mu$	$\sigma^2$	

$$\mathbf{Uni} : f(x) = \begin{cases} \frac{1}{b-a}, a \leq x \leq b \\ 0, \text{others} \end{cases} \quad F(x) = \begin{cases} 0, x < a \\ \frac{x-a}{b-a}, a \leq x < b \\ 1, x \geq b \end{cases}$$

$$\mathbf{Exp} : f(x) = \begin{cases} \lambda e^{-\lambda x}, x > 0 \\ 0, x \leq 0 \end{cases} \quad F(x) = \begin{cases} 1 - e^{-\lambda x}, x > 0 \\ 0, x \leq 0 \end{cases}$$

$$\mathbf{Nor} : f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, F(x) = \int_{-\infty}^x f(t)dt, x \in (-\infty, +\infty)$$

$$f(\mu+x) = f(\mu-x), F(\mu+x) + F(\mu-x) = 1, F(\mu) = \frac{1}{2}$$

$$X \sim N(0, 1), \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \Phi(x) = \int_{-\infty}^x \phi(t)dt$$

$$\phi(-x) = \phi(x), \quad \Phi(a) + \Phi(-a) = 1, \Phi(0) = \frac{1}{2}, \quad F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$f(x, y) \geq 0 \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy > 0 \quad f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx > 0$$

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}, f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)}$$

$$\mathbf{Discrete} : P\{Z = g(x_i, y_j)\} = P\{X = x_i, Y = y_j\} = P_{ij}$$

$$F_Z(z) = P\{Z \leq z\} = P\{g(X, Y) \leq z\} = \sum_{g(x_i, y_j) \leq z} P\{X = x_i, Y = y_j\}$$

$$\mathbf{Continuous} : Z = g(X, Y) \quad F_Z(z) = P\{Z \leq z\} = P\{g(X, Y) \leq z\} = \iint_{g(x, y) \leq z} f(x, y) dx dy$$

$$Z = \max(X, Y), F_{\max}(z) = F_X(z)F_Y(z) \quad Z = \min(X, Y), F_{\min}(z) = 1 - [1 - F_X(z)][1 - F_Y(z)]$$

$$(X, Y) \sim U(D), f(x, y) = \begin{cases} \frac{1}{S_D}, (x, y) \in D \\ 0, \text{others} \end{cases}$$

$$(X, Y) \sim N(\mu_1, \mu_2; \sigma_1, \sigma_2, \rho), f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}[(\frac{x-\mu_1}{\sigma_1})^2 - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + (\frac{y-\mu_2}{\sigma_2})^2]}$$

$$\begin{aligned}
E(X) &= \sum_{i=1}^{\infty} x_i p_i & E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\
E[g(X)] &= \sum_{i=1}^{\infty} g(x_i) p_i & E[g(X)] &= \int_{-\infty}^{\infty} g(x) f(x) dx \\
E[g(X, Y)] &= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} g(x_i, y_j) P_{ij} & E[g(X, Y)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) dx dy \\
E(C) &= C & E(CX) &= CE(X) & E(X + C) &= E(X) + C & E(X + Y) &= E(X) + E(Y) \\
D(X) &= \sum_i^n [x_i - E(X)]^2 p_i & D(X) &= \int_{-\infty}^{\infty} [x - E(X)]^2 f(x) dx \\
D(X) &= E(X^2) - [E(X)]^2 & D(C) &= 0 & D(CX) &= C^2 D(X) & D(X + C) &= D(X) \\
Cov(X, Y) &= E(XY) - E(X)E(Y) & D(X \pm Y) &= D(X) + D(Y) \pm 2Cov(X, Y) \\
Cov(X, Y) &= Cov(Y, X) & Cov(X, X) &= D(X) & Cov(X, c) &= 0 \\
Cov(aX, bY) &= abCov(X, Y) & Cov(X_1 + X_2, Y) &= Cov(X_1, Y) + Cov(X_2, Y) \\
\rho_{XY} &= \frac{Cov(X, Y)}{\sqrt{D(X)}\sqrt{D(Y)}}
\end{aligned}$$

**Large Number Law Central Limit Theorem**

$$\begin{aligned}
P\{|X - E(X)| \geq \epsilon\} &\leq \frac{D(X)}{\epsilon^2} \\
\lim_{n \rightarrow \infty} P\{|\frac{X_n}{n} - p| < \epsilon\} &= 1 \\
\lim_{n \rightarrow \infty} P\{|\frac{1}{n} \sum_{k=1}^n X_k - \mu| < \epsilon\} &= 1 \\
\lim_{n \rightarrow \infty} P\{|\frac{1}{n} \sum_{k=1}^n X_k - \frac{1}{n} \sum_{k=1}^n E(X_k)| < \epsilon\} &= 1 \\
\sum_{i=1}^n X_i \sim N(n\mu, n\sigma^2), \lim_{n \rightarrow \infty} P\{\frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n}\sigma} \leq x\} &= F_n(x) = \Phi(x) \\
X \sim N(np, np(1-p)), \lim_{n \rightarrow \infty} P\{\frac{X_n - np}{\sqrt{np(1-p)}} \leq x\} &= \Phi(x)
\end{aligned}$$

**Mathematical Statistics**

$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n}), \frac{\bar{X} - u}{\sigma/\sqrt{n}} \sim N(0, 1)$$

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 = \frac{1}{n-1} [\sum_{i=1}^n X_i^2 - n\bar{X}^2]$$

$$E(S^2) = \sigma^2, D(S^2) = \frac{2\sigma^4}{n-1}$$

$$\frac{\sum_{i=1}^n (X_i - \mu)^2}{\sigma^2} \sim \chi^2(n)$$

$$\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2} \sim \chi^2(n-1)$$

$$\chi^2 \sim \chi^2(n), E(\chi^2(n)) = n, D(\chi^2(n)) = 2n$$

$$T \sim t(n), T = \frac{X}{\sqrt{Y/n}} \sim \frac{N(0, 1)}{\sqrt{\chi^2(n)/n}}$$

$$F \sim F(n_1, n_2), F = \frac{X/n_1}{Y/n_2} \sim \frac{\chi^2(n_1)/n_1}{\chi^2(n_2)/n_2}$$

$$(X, Y) \sim N(\mu_1, \mu_2; \sigma_1^2, \sigma_2^2, 0)$$

$$\frac{\frac{(n_1-1)S_1^2}{\sigma_1^2}/(n_1-1)}{\frac{(n_2-1)S_2^2}{\sigma_2^2}/(n_2-1)} \sim F(n_1-1, n_2-1)$$

$$\frac{\frac{\sum_{i=1}^{n_1} (X_i - \mu_1)^2}{\sigma_1^2}/n_1}{\frac{\sum_{i=1}^{n_2} (X_i - \mu_2)^2}{\sigma_2^2}/n_2} \sim F(n_1, n_2)$$

$$\frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0, 1)$$

$$\exists \sigma_1 = \sigma_2, \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{S_\omega \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t(n_1 + n_2 - 2)$$

$$S_\omega = \sqrt{\frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1 + n_2 - 2}}$$

*Constant Series*

$$\begin{aligned}
 \sum_{k=1}^{\infty} \frac{1}{k^2} &= \frac{\pi^2}{6} & \sum_{k=1}^{\infty} \frac{1}{k^4} &= \frac{\pi^4}{90} & \sum_{k=1}^{\infty} \frac{1}{k^6} &= \frac{\pi^6}{945} \\
 \sum_{i=0}^n a_i \cdot \sum_{j=0}^n b_j &= \sum_{i=0}^n \sum_{j=0}^n (a_i \cdot b_j) & \sum_{n=s}^t \ln f(n) &= \ln \prod_{n=s}^t f(n) \\
 \sum_{i=0}^n i &= \frac{n(n+1)}{2} & \sum_{i=0}^n i^2 &= \frac{n(n+1)(2n+1)}{6} & \sum_{i=0}^n i^3 &= \left(\sum_{i=0}^n i\right)^2 = \frac{n^2(n+1)^2}{4} \\
 \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} &= \sum_{i=0}^{\infty} \frac{1}{(2i+1)(2i+2)} = \sum_{i=1}^{\infty} \frac{1}{2^i i} = \sum_{i=1}^{\infty} \left(\frac{1}{3^i} + \frac{1}{4^i}\right) \frac{1}{i} = \ln 2
 \end{aligned}$$

*Power Series*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \ln(1+x)$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x)$$

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x$$

$$\sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}$$

$$\sum_{n=0}^{\infty} (n+1)(n+2)x^n = \frac{2}{(1-x)^3}$$

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{n} = -\ln(1-x^2)$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin x$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

**Basel Problem**

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$f(x) = x^2 \xrightarrow{\text{Fourier Expansion}} S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{2}{3}\pi^2 \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx dx = (-1)^n \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin nx dx = 0$$

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos nx = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$$

$$\exists x = 0, S(0) = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} = f(0) = 0 \implies \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$$

$$\exists x = \pi, S(\pi) = \frac{1}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2} = f(\pi) = \pi^2 \implies \sum_{n=1}^{\infty} \frac{4}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} \quad \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{24}$$

***Transcendental Equation***

$$\sum_{i=0}^n a_i x^i = 0 \implies \prod_{i=0}^n (x - x_i) = 0$$

$$\prod_{i=0}^n x_i = (-1)^n \frac{a_0}{a_n}, \quad \sum_{i=0}^n \frac{\prod_{j=0}^n x_j}{x_i} = (-1)^{n-1} \frac{a_1}{a_n}$$

$$\sum_{i=0}^n \frac{1}{x_i} = -\frac{a_1}{a_0}$$

**eg.**  $\tan x = x \implies \sin x = \cos x \cdot x$

$$x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \dots = x \cdot \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \dots\right)$$

$$\frac{1}{3}x^3 - \frac{1}{30}x^5 + \dots = 0 \quad \xrightarrow{x \neq 0} \quad \frac{1}{3}x^2 - \frac{1}{30}x^4 + \dots = 0$$

$$\sum_{i=0}^n \frac{1}{t_i} = \frac{1}{10} \quad t = x^2 \quad \lim_{n \rightarrow \infty} \sum_{i=0}^n \frac{1}{x_i^2} = \frac{1}{10}$$



***Beyond Integral***

$$\sum_{n=0}^{\infty} \frac{1}{(2n)!} \Rightarrow S(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, S(1) = \frac{e^{-1} + e}{2}$$

$$\sum_{n=0}^{\infty} \frac{1}{(3n)!} \Rightarrow S(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}, S(1) = \frac{e}{3} + \frac{2}{3} \cos\left(\frac{\sqrt{3}}{2}\right) e^{-\frac{1}{2}}$$

$$\sum_{n=0}^{\infty} \frac{1}{(4n)!} \Rightarrow S(x) = \sum_{n=0}^{\infty} \frac{x^{4n}}{(4n)!}, S(1) = \frac{e + e^{-1}}{4} + \frac{\cos 1}{2}$$

$$\iint_{D_{xy}} (x+y) d\sigma, D = \{(x,y) | y^2 \leq x+2, x^2 \leq y+2\}$$

$$\mathbf{A_0} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{m=0}^{\infty} \frac{1}{n2^m + 1}$$

$$\mathbf{A_1} = \int_0^{x^2} \pi(\sqrt[4]{1+t} - 1) \sin t^4 dx$$

$$\mathbf{A_2} = \sum_{n=1}^{\infty} \frac{((n-1)!)^2 (2t)^{2n}}{(2n)!}$$

$$\mathbf{A_3} = \int_0^1 \frac{(1-2x) \ln(1-x)}{x^2 - x + 1} dx$$

$$\mathbf{A_4} = x^2(x - \tan x) \ln(x^2 + 1) \left[ \left( \frac{2 \arctan \frac{y}{x}}{\pi} \right)^y - 1 \right]$$

$$\lim_{x \rightarrow 0^+} \lim_{y \rightarrow +\infty} \frac{\mathbf{A_0 A_1}}{\mathbf{A_2 A_3 A_4}} = \frac{27}{32}$$

$$\int \frac{\sec^3 x}{1 - \tan^6 x} dx$$

$$\int \frac{1}{\csc x + \sec x + \tan x + \cot x} dx$$

$$\lim_{N\rightarrow\infty}\sum_{n=1}^N\sum_{k=1}^n\frac{(-1)^{k-1}}{k}-\ln 2=\ln 2-\frac{1}{2}$$

$$\iint_{D_{xy}}(e^x\cos y)\mathrm{d}\sigma,D=\{(x,y)|x^2+y^2\leq 1\}$$

$$\lim_{n\rightarrow\infty}\sum_{i=1}^n\frac{1-\cos\frac{\pi}{\sqrt{n}}}{1+\cos\frac{i\pi}{\sqrt{2n}}}$$

$$\sum_{n=1}^\infty \frac{(-1)^{[\sqrt{n}]}}{n} \quad \sum_{n=1}^\infty \frac{n!}{(2n+1)!} \frac{1}{(n+1)}$$

$$t\frac{\mathrm{d}^3x}{\mathrm{d}t^3}+3\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-t\frac{\mathrm{d}x}{\mathrm{d}t}-x=0$$

$$\begin{aligned}
& \mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3 + \mathbf{L}_4 \\
& \mathbf{L}_1 : x \in (0, 1), y = 0 \quad \mathbf{L}_2 : y \in (0, 1), x = 1 \quad \mathbf{L}_3 : x \in (1, 0), y = 1 \quad \mathbf{L}_4 : y \in (1, 0), x = 0 \\
& I_1 = \oint_L -xyf'_x(x, y)dx + xyf'_y(x, y)dy = \iint_D \left( \frac{\partial (xyf'_y(x, y))}{\partial x} - \frac{\partial (-xyf'_x(x, y))}{\partial y} \right) dx dy \\
& \quad = 2 \iint_D xyf''_{xy}(x, y)dx dy + \iint_D (xf'_x(x, y) + yf'_y(x, y)) dx dy \\
& I_1 = \oint_L -xyf'_x(x, y)dx + xyf'_y(x, y)dy = \int_{L_1} + \int_{L_2} + \int_{L_3} + \int_{L_4} \\
& \quad = 0 + \int_0^1 yf'_y(1, y)dy + \int_1^0 -xf'_x(x, 1)dx + 0 \\
& \quad = yf(1, y)|_0^1 - \int_0^1 f(1, y)dy + xf(x, 1)|_0^1 - \int_0^1 f(x, 1)dx \\
& \quad = 0 + 0 + 0 + 0 = 0 \\
& I_2 = \iint_D xf'_x(x, y) + yf'_y(x, y)dx dy = I_3 - I_4 \\
& I_3 = \oint_L -yf(x, y)dx + xf(x, y)dy = \iint_D (xf'_x(x, y) + yf'_y(x, y)) dx dy + 2 \iint_D f(x, y)dx dy \\
& I_3 = \oint_L -yf(x, y)dx + xf(x, y)dy = \int_{L_1} + \int_{L_2} + \int_{L_3} + \int_{L_4} \\
& \quad = 0 + \int_0^1 f(1, y)dy + \int_1^0 -f(x, 1)dx + 0 = 0 + 0 + 0 + 0 = 0 \\
& I_4 = 2 \iint_D f(x, y)dx dy = 2a \\
& I = \frac{1}{2}I_1 - \frac{1}{2}I_2 = \frac{1}{2}I_1 - \frac{1}{2}(I_3 - I_4) = 0 - \frac{1}{2}(0 - 2a) = a
\end{aligned}$$