

$$\begin{aligned}
& Y = F_X(X), X \sim F_X(x), \quad Y \sim? \\
& F_Y(y) = P\{Y \leq y\} = P\{F_X(X) \leq y\} = P\{X \leq F_X^{-1}(y)\} = F_X(F_X^{-1}(y)) = y \\
& \quad Y \sim U(0, 1) \\
& E(X) = \int_{-\infty}^{+\infty} x dF(x) \\
& E(X) = \begin{cases} \int_{-\infty}^{+\infty} x f(x) dx, & \text{if } F'(x) = f(x) \\ \sum_{k=1}^{+\infty} x_k (F(x_k) - F(x_k^-)) = \sum_{k=1}^{+\infty} x_k P\{X = x_k\}, & \text{if } X \text{ is discrete} \end{cases} \\
& \quad \text{Riemann Integral, } Ex[R] = \sum_{\omega \in S} R(\omega) \cdot Pr[\omega] \\
& \quad \text{Lebesgue Integral, } Ex[R] = \sum_{x \in \text{range}(R)} x \cdot Pr[R = x]
\end{aligned}$$

Abstract Probability SUM

$$\mathbb{E}(X)=\left\{\begin{array}{ll}\sum_{\omega\in S}R(\omega)\Pr(\omega), & Discrete \\ \int_{\Omega}X\mathrm{d}P, & Continuous\end{array}\right.$$

Numeric Sum And Integral

$$\mathbb{E}(X)=\left\{\begin{array}{ll}\sum_{i=1}^nx_iP(x_i), & Discrete \\ \int_{-\infty}^{+\infty}x\mathrm{d}F(x), & Continuous\end{array}\right.$$

$$\Omega\implies_{h(\omega)}^{MAP}R$$

$$h(\omega)=\sum_{k=1}^na_kI_{A_k}(\omega)\quad A_1\cup A_2\cup\cdots\cup A_n=\Omega$$

$$\mathbb{E}(h)=\sum_{k=1}^na_kP(A_k)$$

$$def. \quad X^+=X\vee 0 \quad X^-=-X\vee 0 \quad X=X^+-X^-$$

$$\int_{\Omega}X\mathrm{d}P=\int_{\Omega}X^+\mathrm{d}P-\int_{\Omega}X^-\mathrm{d}P$$

Dimension Extend

$$\vec{X}=(X_1,X_2,\cdots,X_n)\in\Re^n$$

$$g(\vec{X})=\int\cdots\int g(x_1,x_2,\cdots,x_n)\mathrm{d}F(x_1,x_2,\cdots,x_n)$$

**Random Variable Characteristic Function**

Distr	Mark	$\varphi_X(t)$
Bin	$B(n, p)$	$(e^{it}p + 1 - p)^n$
Poi	$\pi(\lambda)$	$e^{(e^{it}-1)\lambda}$
Geo	$G(p)$	
Hyp	$H(n, M, N)$	
Uni	$U(a, b)$	$\frac{\sin tb - \sin ta}{(b-a)t} - i \frac{\cos tb - \cos ta}{(b-a)t}$
Exp	$E(\lambda)$	$\frac{\lambda^2}{\lambda^2 + t^2} + i \frac{\lambda t}{\lambda^2 + t^2}$
Nor	$N(\mu, \sigma^2)$	$e^{-\frac{1}{2}\sigma^2 t^2 + i\mu t}$

$$\exists E(|X|^n) < +\infty, k \in [1, n] \quad \varphi_X(t) = E(e^{itX})$$

$$E(X^k) = (-i)^k \varphi_X^{(k)}(0)$$

$$\varphi_X^{(n)}(t) = [E(e^{itX})]^{(n)} = E(e^{itX} (iX)^n)$$

$$t = 0, \varphi_X^{(n)}(0) = E(X^n (i)^n) = E(X^n)(i)^n$$

$$\longrightarrow E(X^n) = (i)^{-n} \varphi_X^{(n)}(0)$$

$$S_n = X_1 + X_2 + \cdots + X_n$$

$$\varphi_{S_n}(t) = E(e^{itS_n}) = E(e^{it(X_1+X_2+\cdots+X_n)})$$

$$\longrightarrow E(e^{it(\sum_{k=1}^n X_k)}) = E\left(\prod_{k=1}^n e^{itX_k}\right) = \prod_{k=1}^n E(e^{itX_k})$$

$$\phi_X(t) = E(e^{itx})$$

$$f(x) = \frac{1}{b-a}, x \in (a, b) \quad \phi_x(t) = \int_a^b e^{itx} \frac{1}{b-a} dx = \frac{-i}{(b-a)t} e^{itx} \Big|_a^b = \frac{-i(e^{itb} - e^{ita})}{(b-a)t}$$

$$t \sim N(0, 1) \quad \phi(t) = e^{-\frac{t^2}{2}}$$

$$Y = aX + b$$

$$\phi_Y(t) = \phi_{(aX+b)}(t) = E(e^{it(ax+b)}) = e^{itb} \phi_X(at)$$

Markov inequality

$$P\{|X| \geq \epsilon\} \leq \frac{E|X|^r}{\epsilon^r}, \quad \forall \epsilon > 0, \quad E|X|^r < +\infty, \quad r > 0$$

$$P\{|X| \geq \epsilon\} = \int_{\frac{|x|}{\epsilon} \geq 1} 1 dF(x) \leq \int_{\frac{|x|}{\epsilon} \geq 1} \frac{|x|^r}{\epsilon^r} dF(x) = \frac{E|X|^r}{\epsilon^r}$$

Dispersion Formula

$\exists X = X - E(X), r = 2,$  Chebyshev's inequality

$$P\{|X - E(X)| \geq \epsilon\} \leq \frac{E|X - E(X)|^2}{\epsilon^2} = \frac{D(X)}{\epsilon^2}$$

$$\frac{S_n - n\mu}{\sqrt{n}\sigma} \sim N(0, 1)$$

# Multiple Random Variable PDF

$$\vec{\mu} = (\mu_1, \mu_2, \dots)^T, \quad \vec{X} = (x_1, x_2, \dots)^T, \quad \Sigma = (\text{Cov}(x_i, x_j))_{n \times n}$$

$$f(\vec{X}) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}(\vec{X} - \vec{\mu})^T \Sigma^{-1} (\vec{X} - \vec{\mu})}$$