Matrix Calculation

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \mathbf{A^T}=\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$$\mathbf{A}^*=\begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{21}} & \mathbf{A_{31}} \\ \mathbf{A_{12}} & \mathbf{A_{22}} & \mathbf{A_{32}} \\ \mathbf{A_{13}} & \mathbf{A_{23}} & \mathbf{A_{33}} \end{bmatrix} \quad \alpha^{\mathbf{T}}=\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad |\mathbf{A}|=\det\mathbf{A}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$|\mathbf{A}|=\Pi_{i=1}^n\lambda_i \quad |k\mathbf{A}|=k^n|\mathbf{A}| \quad |\mathbf{AB}|=|\mathbf{A}||\mathbf{B}| \quad |\mathbf{A}^n|=|\mathbf{A}|^n$$

$$|\mathbf{A^T}|=|\mathbf{A}| \quad (k\mathbf{A})^{\mathbf{T}}=k\mathbf{A^T} \quad (\mathbf{A^T})^{\mathbf{T}}=\mathbf{A} \quad (\mathbf{A}^n)^{\mathbf{T}}=(\mathbf{A^T})^n$$

$$|\mathbf{A}^{-1}|=\frac{1}{|\mathbf{A}|} \quad (k\mathbf{A})^{-1}=\frac{1}{k}\mathbf{A}^{-1} \quad (\mathbf{A}^{-1})^{-1}=\mathbf{A} \quad (\mathbf{A}^n)^{-1}=(\mathbf{A}^{-1})^n$$

$$|\mathbf{A}^*|=|\mathbf{A}|^{n-1} \quad (k\mathbf{A})^*=k^{n-1}\mathbf{A}^* \quad (\mathbf{A}^*)^*=|\mathbf{A}|^{n-2}\mathbf{A} \quad \mathbf{AA}^*=\mathbf{A}^*\mathbf{A}=|\mathbf{A}|\mathbf{E}$$

$$(\mathbf{ABC})^T=\mathbf{C^TB^TA^T} \quad (\mathbf{ABC})^{-1}=\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} \quad (\mathbf{ABC})^*=\mathbf{C}^*\mathbf{B}^*\mathbf{A}^*$$

$$\mathbf{diag}(a_1,...,a_n)\mathbf{diag}(b_1,...,b_n)=\mathbf{diag}(a_1b_1,...,a_nb_n)$$

$$\Lambda=\mathbf{diag}(\lambda_1,...,\lambda_n),|\Lambda|=\Pi_{i=1}^n\lambda_i,\varphi(\Lambda)=\mathbf{diag}(\varphi(\lambda_1),...,\varphi(\lambda_n))$$

$$\Lambda=\begin{pmatrix} 0 & 0 & \lambda_1 \\ 0 & \lambda_i & 0 \\ \lambda_n & 0 & 0 \end{pmatrix} \quad |\Lambda|=(-1)^{\frac{n(n-1)}{2}}\Pi_{i=1}^n\lambda_i \quad \Lambda^{-1}=\begin{pmatrix} 0 & 0 & \frac{1}{\lambda_n} \\ 0 & \frac{1}{\lambda_i} & 0 \\ \frac{1}{\lambda_1} & 0 & 0 \end{pmatrix}$$

$$\begin{vmatrix} \mathbf{A} & * \\ \mathbf{O} & \mathbf{B} \end{vmatrix}=\begin{vmatrix} \mathbf{A} & \mathbf{O} \\ * & \mathbf{B} \end{vmatrix}=|\mathbf{A}||\mathbf{B}|$$

$$\begin{vmatrix} \mathbf{O} & \mathbf{A}_{mm} \\ \mathbf{B}_{nn} & * \end{vmatrix}=\begin{vmatrix} * & \mathbf{A}_{mm} \\ \mathbf{B}_{nn} & \mathbf{O} \end{vmatrix}=(-1)^{mn}|\mathbf{A}||\mathbf{B}|$$

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{\mathbf{T}}=\begin{pmatrix} \mathbf{A^T} & \mathbf{C^T} \\ \mathbf{B^T} & \mathbf{D^T} \end{pmatrix} \quad \begin{pmatrix} \mathbf{A} & 0 & 0 \\ 0 & \mathbf{B} & 0 \\ 0 & 0 & \mathbf{C} \end{pmatrix}^{-1}=\begin{pmatrix} \mathbf{A}^{-1} & 0 & 0 \\ 0 & \mathbf{B}^{-1} & 0 \\ 0 & 0 & \mathbf{C}^{-1} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}^{\mathbf{T}}=(\mathbf{A^T},\mathbf{B^T}) \quad \begin{pmatrix} 0 & 0 & \mathbf{A} \\ 0 & \mathbf{B} & 0 \\ \mathbf{C} & 0 & 0 \end{pmatrix}^{-1}=\begin{pmatrix} 0 & 0 & \mathbf{C}^{-1} \\ 0 & \mathbf{B}^{-1} & 0 \\ \mathbf{A}^{-1} & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{O} & \mathbf{B} \end{pmatrix}^{-1}=\begin{pmatrix} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1} \\ \mathbf{O} & \mathbf{B}^{-1} \end{pmatrix} \quad \begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{C} & \mathbf{B} \end{pmatrix}^{-1}=\begin{pmatrix} \mathbf{A}^{-1} & \mathbf{O} \\ -\mathbf{B}^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{B}^{-1} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{C} & \mathbf{A} \\ \mathbf{B} & \mathbf{O} \end{pmatrix}^{-1}=\begin{pmatrix} \mathbf{O} & \mathbf{B}^{-1} \\ \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1} \end{pmatrix} \quad \begin{pmatrix} \mathbf{O} & \mathbf{A} \\ \mathbf{B} & \mathbf{C} \end{pmatrix}^{-1}=\begin{pmatrix} -\mathbf{B}^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{B}^{-1} \\ \mathbf{A}^{-1} & \mathbf{O} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{B} \end{pmatrix}^*=\begin{pmatrix} |\mathbf{B}|\mathbf{A}^* & \mathbf{O} \\ \mathbf{O} & |\mathbf{A}|\mathbf{B}^* \end{pmatrix} \quad \begin{pmatrix} \mathbf{O} & \mathbf{A_{mm}} \\ \mathbf{B_{nn}} & \mathbf{O} \end{pmatrix}^*=(-1)^{mn}\begin{pmatrix} \mathbf{O} & |\mathbf{A}|\mathbf{B}^* \\ |\mathbf{B}|\mathbf{A}^* & \mathbf{O} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{B} \end{pmatrix}^n=\begin{pmatrix} \mathbf{A}^n & \mathbf{O} \\ \mathbf{O} & \mathbf{B}^n \end{pmatrix} \quad \begin{pmatrix} \mathbf{O} & \mathbf{A} \\ \mathbf{B} & \mathbf{O} \end{pmatrix}^n=\begin{pmatrix} \textbf{not} & \textbf{important} \\ \textbf{at} & \textbf{all} \end{pmatrix}$$

Matrix Rank

$$r(\mathbf{A}^*) = \begin{cases} n & \text{if } r(\mathbf{A}) = n, \\ 1 & \text{if } r(\mathbf{A}) = n-1, \\ 0 & \text{if } r(\mathbf{A}) < n-1. \end{cases}$$

$$0 \leq r(\mathbf{A_{mn}}) \leq \min\{m, n\}$$

$$\max\{r(\mathbf{A}), r(\mathbf{B})\} \leq \mathbf{r}(\mathbf{A}, \mathbf{B}) \leq r(\mathbf{A}) + r(\mathbf{B})$$

$$r(\mathbf{A}) \leq \mathbf{r}(\mathbf{A}, \mathbf{b}) \leq r(\mathbf{A}) + 1$$

$$\max\{r(\mathbf{A}), r(\mathbf{B})\} \leq r\left(\begin{array}{c} \mathbf{A} \\ \mathbf{B} \end{array}\right) \leq r(\mathbf{A}) + r(\mathbf{B})$$

$$r(\mathbf{A} \pm \mathbf{B}) \leq r\left(\begin{array}{c} \mathbf{A} \pm \mathbf{B} \\ \mathbf{B} \end{array}\right) = r(\mathbf{A} \pm \mathbf{B}, \mathbf{B}) = r(\mathbf{A}, \mathbf{B}) \leq r(\mathbf{A}) + r(\mathbf{B})$$

$$r(\mathbf{A}, \mathbf{B}) \leq r\left(\begin{array}{c} \mathbf{A^T} \\ \mathbf{B^T} \end{array}\right) \neq r\left(\begin{array}{c} \mathbf{A} \\ \mathbf{B} \end{array}\right) = r(\mathbf{A}, \mathbf{B})$$

$$r(\mathbf{A}) + r(\mathbf{B}) - n \leq r(\mathbf{AB}) \leq \min\{r(\mathbf{A}), r(\mathbf{B})\}$$

$$r(\mathbf{A^T A}) = r(\mathbf{A A^T}) = r(\mathbf{A^T}) = r(\mathbf{A}) = r(k\mathbf{A}), (\forall k \neq 0)$$

$$\exists \mathbf{A_{mn}} \mathbf{B_{ns}} = \mathbf{O}, r(\mathbf{A}) + r(\mathbf{B}) \leq n$$

$$\exists \mathbf{A_{mn}} \mathbf{B_{ns}} = \mathbf{C_{ms}}, \exists r(\mathbf{A}) = n, r(\mathbf{B}) = r(\mathbf{C})$$

$$\exists \mathbf{A_{mn}} \mathbf{B_{ns}} = \mathbf{C_{ms}}, \exists r(\mathbf{B}) = n, r(\mathbf{A}) = r(\mathbf{C})$$

$$\exists \mathbf{A_{nn}}, \forall k \in \mathbf{N}^*, r(\mathbf{A}^n) = r(\mathbf{A}^{n+k}) \Longrightarrow r(\mathbf{A}) = r(\mathbf{A}^2) = ... = r(\mathbf{A}^n)$$

$$r\left(\begin{array}{cc} \mathbf{A}_{mm} & \mathbf{O} \\ \mathbf{O} & \mathbf{B}_{nn} \end{array}\right) = r(\mathbf{A}) + r(\mathbf{B})$$

Similarity Theory and Feature Vector

$$|\lambda\mathbf{E}-\mathbf{A}|=0\Rightarrow\lambda_i,i\in[1,n]$$

$$|\lambda\mathbf{E}-\mathbf{A}|=\begin{pmatrix}\lambda-a_{11} & ... & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{n1} & ... & \lambda-a_{nn}\end{pmatrix}=\lambda^n+\sum_{i=1}^{n}a_{ii}\lambda^{n-1}+...$$

$$\forall i\in[0,n],\exists\lambda_i,f(\lambda_i)=0\Longrightarrow\Pi_{i=1}^{n}(\lambda-\lambda_i)=0$$

$$\lambda^n+\sum_{i=1}^{n}\lambda_i\lambda^{n-1}+...+(-1)^n\Pi_{i=1}^{n}\lambda_i=0\Rightarrow\sum_{i=1}^{n}a_{ii}=\sum_{i=1}^{n}\lambda_i$$

$$\exists\lambda=0,(-1)^n\Pi_{i=1}^{n}\lambda_i=|-\mathbf{A}|=(-1)^n|\mathbf{A}|\Rightarrow\Pi_{i=1}^{n}\lambda_i=|\mathrm{A}|$$

Specially $\mathbf{n=3}$

$$|\lambda\mathbf{E}-\mathbf{A}|=\begin{pmatrix}\lambda-a_{11} & a_{12} & a_{13}\\ a_{21} & \lambda-a_{22} & a_{23}\\ a_{31} & a_{32} & \lambda-a_{33}\end{pmatrix}=\lambda^3-\left(\sum_{i=1}^{3}a_{ii}\right)\lambda^2+\left(\sum_{i=1}^{3}\mathbf{A_{ii}}\right)\lambda-|\mathbf{A}|$$

$$\lambda^3-(a_{11}+a_{22}+a_{33})\lambda^2+(\mathbf{A_{11}}+\mathbf{A_{22}}+\mathbf{A_{33}})\lambda-|\mathbf{A}|=(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$$

$$(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)=\lambda^3-(\lambda_1+\lambda_2+\lambda_3)\lambda^2+(\lambda_2\lambda_3+\lambda_1\lambda_3+\lambda_1\lambda_2)\lambda-(\lambda_1\lambda_2\lambda_3)$$

$$\sum_{i=1}^{3}\mathbf{A_{ii}}=\mathbf{tr}(\mathbf{A}^*)=(\lambda_2\lambda_3+\lambda_1\lambda_3+\lambda_1\lambda_2)$$

$$\alpha^{\mathbf{T}}\alpha=\mathbf{tr}(\mathbf{A})\qquad r(\alpha\alpha^{\mathbf{T}})=1$$

$$\alpha\alpha^{\mathbf{T}}\sim\begin{pmatrix}\mathbf{tr}(\mathbf{A}) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}_{nn}$$

$$\mathbf{A}\sim\mathbf{B}\Longrightarrow\mathbf{A^T}\sim\mathbf{B^T},\mathbf{A}^{-1}\sim\mathbf{B}^{-1},\mathbf{A}^*\sim\mathbf{B}^*,f(\mathbf{A})\sim f(\mathbf{B})$$

$$f(\mathbf{A})=0\Rightarrow f(\lambda)=0(E\sim 1)$$

$$\lambda_{\mathbf{A}_i^*}\lambda_{\mathbf{A}_i}=|\mathbf{A}|,i\in[1,n]$$

$\mathbf{A}$	$\mathbf{A^T}$	$\mathbf{A}^{-1}$	$\mathbf{A}^*$	$f(\mathbf{A})$	$\mathbf{P}^{-1}\mathbf{AP}$	$\mathbf{PAP}^{-1}$
λ	λ	$\frac{1}{\lambda}$	$\frac{\lvert\mathbf{A}\rvert}{\lambda}$	$f(\lambda)$	λ	λ
α	*	α	α	α	$\mathbf{P}^{-1}\alpha$	$\mathbf{P}\alpha$

Base σ

$$(\eta_1, \eta_2, ...\eta_n) = (\xi_1, \xi_2, ...\xi_n)\mathbf{M}$$

$$\left\{\begin{array}{ccccc} \eta_1 = a_{11}\xi_1 + a_{21}\xi_2 + ... + a_{n1}\xi n \\ \vdots \quad \vdots \quad \ddots \qquad\qquad\qquad \vdots \quad \vdots \\ \eta_n = a_{1n}\xi_1 + a_{2n}\xi_2 + ... + a_{nn}\xi n \end{array}\right. \Longrightarrow \mathbf{AM} = \mathbf{B} \Longleftrightarrow \mathbf{M} = \mathbf{A^{-1}B}$$

$$\mathbf{A}\xi_{\mathbf{A}} = \mathbf{B}\xi_{\mathbf{B}} \Longrightarrow \xi_{\mathbf{A}} = \mathbf{A}^{-1}\mathbf{B}\xi_{\mathbf{B}} = \mathbf{M}\xi_B$$