$$Y = F_X(X), X \sim F_X(x), \quad Y \sim ?$$

$$F_Y(y) = P\{Y \le y\} = P\{F_X(X) \le y\} = P\{X \le F_X^{-1}(y)\} = F_X(F_X^{-1}(y)) = y$$

$$Y \sim U(0,1)$$

$$E(X) = \int_{-\infty}^{+\infty} x \mathrm{d}F(x)$$

$$E(X) = \begin{cases} \int_{-\infty}^{+\infty} x f(x) \mathrm{d}x, \quad if \quad F'(x) = f(x) \\ \sum_{k=1}^{+\infty} x_k (F(x_k) - F(x_k^-)) = \sum_{k=1}^{+\infty} x_k P\{X = x_k\}, \quad if \quad X \quad is \quad discrete \end{cases}$$

$$\text{Riemann} \quad \text{Integral}, \quad Ex[R] = \sum_{\omega \in S} R(\omega) \cdot Pr[\omega]$$

$$\text{Lebesgue} \quad \text{Integral}, \quad Ex[R] = \sum_{x \in range(R)} x \cdot Pr[R = x]$$

$$\text{Abstract} \quad \text{Probability} \quad \text{SUM}$$

$$\mathrm{E(X)} = \begin{cases} \sum_{\omega \in S} R(\omega)\mathrm{Pr}(\omega), & Discrete \\ \int_{\Omega} X \mathrm{d}P, & Continuous \end{cases}$$

$$\text{Numeric} \quad \text{Sum} \quad \text{And} \quad \text{Integral}$$

$$\mathrm{E(X)} = \begin{cases} \sum_{i=1}^{n} x_i P(x_i), & Discrete \\ \int_{-\infty}^{+\infty} x \mathrm{d}F(x), & Continuous \end{cases}$$

$$\Omega \quad \Longrightarrow_{h(\omega)}^{MAP} \quad R$$

$$h(\omega) = \sum_{k=1}^{n} a_k I_{A_k}(\omega) \quad A_1 \cup A_2 \cup \cdots \cup A_n = \Omega$$

$$\mathrm{E}(h) = \sum_{k=1}^{n} a_k P(A_k)$$

$$def. \quad X^+ = X \vee 0 \quad X^- = -X \vee 0 \quad X = X^+ - X^-$$

$$\int_{\Omega} X \mathrm{d}P = \int_{\Omega} X^+ \mathrm{d}P - \int_{\Omega} X^- \mathrm{d}P$$

$$\text{Dimension} \quad \text{Extend}$$

$$\vec{X} = (X_1, X_2, \cdots, X_n) \in \Re^n$$

$$g(\vec{X}) = \int \cdots \int g(x_1, x_2, \cdots, x_n) \mathrm{d}F(x_1, x_2, \cdots, x_n)$$

Random Variable Characteristic Function

Distr	Mark	$\varphi_X(t)$
Bin	$B(n,p)$	$(\mathrm{e}^{it}p+1-p)^n$
Poi	$\pi(\lambda)$	$\mathrm{e}^{(\mathrm{e}^{it}-1)\lambda}$
Geo	$G(p)$	
Hyp	$H(n,M,N)$	
Uni	$U(a,b)$	$\frac{\sin tb-\sin ta}{(b-a)t}-i\frac{\cos tb-\cos ta}{(b-a)t}$
Exp	$E(\lambda)$	$\frac{\lambda^2}{\lambda^2+t^2}+i\frac{\lambda t}{\lambda^2+t^2}$
Nor	$N(\mu,\sigma^2)$	$\mathrm{e}^{-\frac{1}{2}\sigma^2t^2+i\mu t}$

$$\exists\mathrm{E}(|X|^n)<+\infty, k\in[1,n] \quad \varphi_X(t)=E(\mathrm{e}^{itX})$$

$$E(X^k)=(-i)^k\varphi_X^{(k)}(0)$$

$$\varphi_X^{(n)}(t)=\left[\mathrm{E}(\mathrm{e}^{itX})\right]^{(n)}=\mathrm{E}(\mathrm{e}^{itX}(iX)^n)$$

$$t=0, \varphi_X^{(n)}(0)=\mathrm{E}(X^n(i)^n)=\mathrm{E}(X^n)(i)^n$$

$$\longrightarrow \mathrm{E}(X^n)=(i)^{-n}\varphi_X^{(n)}(0)$$

$$S_n=X_1+X_2+\cdots+X_n$$

$$\varphi_{S_n}(t)=\mathrm{E}(\mathrm{e}^{itS_n})=\mathrm{E}(\mathrm{e}^{it(X_1+X_2+\cdots+X_n)})$$

$$\longrightarrow \mathrm{E}(\mathrm{e}^{it\left(\sum_{k=1}^n X_k\right)})=\mathrm{E}(\prod_{k=1}^n \mathrm{e}^{itX_k})=\prod_{k=1}^n \mathrm{E}(\mathrm{e}^{itX_k})$$

$$\phi_X(t)=\mathrm{E}(\mathrm{e}^{itx})$$

$$f(x)=\frac{1}{b-a}, x\in(a,b) \quad \phi_x(t)=\int_a^b \mathrm{e}^{itx}\frac{1}{b-a}\mathrm{d}x=\frac{-i}{(b-a)t}\mathrm{e}^{itx}|_a^b=\frac{-i(\mathrm{e}^{itb}-\mathrm{e}^{ita})}{(b-a)t}$$

$$t\sim N(0,1) \quad \phi(t)=\mathrm{e}^{-\frac{t^2}{2}}$$

$$Y=aX+b$$

$$\phi_Y(t)=\phi_{(aX+b)}(t)=\mathrm{E}(\mathrm{e}^{it(ax+b)})=\mathrm{e}^{itb}\phi_X(at)$$

$$\text{Markov} \quad \text{inequality}$$
$$P\{|X| \geq \epsilon\} \leq \frac{\mathrm{E}|X|^r}{\epsilon^r}, \quad \forall \epsilon > 0, \quad \mathrm{E|X|^r} < +\infty, \quad r > 0$$
$$P\{|X| \geq \epsilon\} = \int_{\frac{|x|}{\epsilon} \geq 1} 1 \mathrm{d}F(x) \leq \int_{\frac{|x|}{\epsilon} >= 1} \frac{|x|^r}{\epsilon^r} \mathrm{d}F(x) = \frac{\mathrm{E|X|}^r}{\epsilon^r}$$
$$\text{Despersion} \quad \text{Fromula}$$
$$\exists X = X - E(X), r = 2, \quad \text{Chebyshev's} \quad \text{inequality}$$
$$P\{|X - \mathrm{E}(X)| \geq \epsilon\} \leq \frac{\mathrm{E}|X - E(X)|^2}{\epsilon^2} = \frac{\mathrm{D}(X)}{\epsilon^2}$$
$$\frac{S_n - n\mu}{\sqrt{n}\sigma} \sim N(0,1)$$

$$\text{De} \quad \text{Moivre} \quad \text{Laplace} \quad \text{Integral} \quad \text{Limit} \quad \text{Theory}$$
$$\lim_{n \to \infty} P\{a < \frac{\mu_n - np}{\sqrt{npq}} \leq b\} = \frac{1}{\sqrt{2\pi}} \int_a^b \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x = \Phi(b) - \Phi(a)$$
$$\mu_n \sim B(n,p), \quad \phi_n(t) = (p\mathrm{e}^{it} + q)^n$$
$$X_n = \frac{\mu_n - np}{\sqrt{npq}}, \quad \varphi_n(t) = \mathrm{E}(\mathrm{e}^{iX_n t}) = \left(p\mathrm{e}^{\frac{iqt}{\sqrt{npq}}} + q\mathrm{e}^{-\frac{ipt}{\sqrt{npq}}}\right)^n$$
$$\mathrm{e}^x = 1 + x + \frac{x^2}{2} + o(x^2)$$
$$p\mathrm{e}^{\frac{iqt}{\sqrt{npq}}} = p + it\sqrt{\frac{pq}{n}} - \frac{qt^2}{2n} + o(\frac{t^2}{n})$$
$$q\mathrm{e}^{-\frac{ipt}{\sqrt{npq}}} = q - it\sqrt{\frac{pq}{n}} - \frac{pt^2}{2n} + +o(\frac{t^2}{n})$$
$$\varphi_n(t) = \left[1 - \frac{t^2}{2n} + o(\frac{t^2}{n})\right]^n \to \mathrm{e}^{-\frac{t^2}{2}}, (n \to \infty)$$

Multiple Random Variable PDF

$$\vec{\mu}=(\mu_1,\mu_2,\dots)^{\mathrm{T}},\quad \vec{X}=(x_1,x_2,\dots)^{\mathrm{T}},\quad \Sigma=(\mathrm{Cov}(x_i,x_j))_{n*n}$$

$$f(\vec{X})=\frac{1}{(2\pi)^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}}\mathrm{e}^{-\frac{1}{2}(\vec{X}-\vec{\mu})^{\mathrm{T}}\Sigma^{-1}(\vec{X}-\vec{\mu})}$$

$$eg.X\sim N(\mu_1,\sigma_1^2),Y\sim N(\mu_2,\sigma_2^2)$$

$$\Sigma=\begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix},\quad \Sigma^{-1}=\frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\begin{pmatrix}\sigma_2^2 & -\rho\sigma_1\sigma_2\\ -\rho\sigma_1\sigma_2 & \sigma_1^2\end{pmatrix}$$

$$|\Sigma|=\sigma_1^2\sigma_2^2(1-\rho^2)$$

$$\vec{\mu}=\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}\quad \vec{X}=\begin{pmatrix}x\\ y\end{pmatrix}$$

$$f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\mathrm{e}^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2}+\frac{(y-\mu_2)^2}{\sigma_2^2}-\frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}\right]}$$