Basic Equation

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \sum_{k=0}^{n} C_n^k a^{n-k} b^k$$

$$a^n - b^n = (a-b) \cdot \sum_{k=0}^{n-1} a^{n-1-k} b^k$$

$$\arcsin x + \arccos x = \arctan x + \mathrm{arccot} x = \frac{\pi}{2}$$

$$f(x) = \int_a^x f'(t)\mathrm{d}t + f(a)$$

$$K = \frac{|y''x' - y'x''|}{(x'^2 + y'^2)^{\frac{3}{2}}}, \quad \rho = \frac{1}{K}$$

$$\int_0^1 f(x)\mathrm{d}x = \sum_{i=1}^{n} \left(\frac{i+1}{n} - \frac{i}{n}\right) f\left(\frac{i}{n}\right) = \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{2i+1}{2n}\right)$$

$$\int_{t1}^{t2} f(x(t), y(t))\mathrm{d}s = \int_a^b f(x, y(x))\sqrt{x'^2 + y'^2}\mathrm{d}x = \int_\alpha^\beta f(r, \theta)\sqrt{r^2 + r'^2}\mathrm{d}\theta$$

$$S = \int_a^b y(x)\mathrm{d}x = \int_{t1}^{t2} y(t)x'(t)\mathrm{d}t = \frac{1}{2}\int_\alpha^\beta r^2(\theta)\mathrm{d}\theta$$

$$S_{side} = \int_a^b 2\pi y(x)\sqrt{x'^2(x) + y'^2(x)}\mathrm{d}x = \int_\alpha^\beta 2\pi r(\theta)\sin\theta\sqrt{r^2(\theta) + r'^2(\theta)}\mathrm{d}\theta$$

$$V_x = \int_a^b \pi y^2(x)\mathrm{d}x \quad V_y = \int_a^b 2\pi x y(x)\mathrm{d}x$$

$$\mathbf{1's} \quad \mathbf{Taylor} \quad f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} x^k + R_{k+1}(\xi)$$

$$\mathbf{2's} \quad \mathbf{Taylor} \quad f(x,y) = f(x_0, y_0) + f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0)$$
$$+\frac{1}{2}f''_{xy}(x_0, y_0)(x - x_0)^2 + \frac{1}{2}f''_{yy}(x_0, y_0)(y - y_0)^2 + \frac{1}{2}f''_{xy}(x_0, y_0)(x - x_0)(y - y_0) + o(\rho)$$

$$\mathbf{important} \quad \mathbf{integral} \quad \mathbf{ABS}: \quad \int u\mathrm{d}u = \ln|u| + C$$

$$\lim \mathbf{type}: \frac{0}{0} \quad \frac{\infty}{\infty} \quad 0 * \infty \quad \infty - \infty \quad \infty^0 \quad 0^0 \quad 1^\infty$$

Multivariate Integral

$$\iiint_{\Omega_{xyz}} f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{\Omega_{uvw}} f(u,v,w)\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}\mathrm{d}u\mathrm{d}v\mathrm{d}w$$

$$\left.\frac{\partial f}{\partial l}\right|_{P_0=(x_0,y_0,z_0)} = f_x(P_0)\cos\alpha + f_y(P_0)\cos\beta + f_z(P_0)\cos\gamma$$

$$\mathrm{grad}f(x_0,y_0,z_0) = f_x(x_0,y_0,z_0)\tilde{\mathbf{i}} + f_y(x_0,y_0,z_0)\tilde{\mathbf{j}} + f_z(x_0,y_0,z_0)\tilde{\mathbf{k}}$$

$$(\bar{x},\bar{y},\bar{z}) = \left(\frac{\iiint_\Omega x\rho\mathrm{d}v}{\iiint_\Omega \rho\mathrm{d}v}, \frac{\iiint_\Omega y\rho\mathrm{d}v}{\iiint_\Omega \rho\mathrm{d}v}, \frac{\iiint_\Omega z\rho\mathrm{d}v}{\iiint_\Omega \rho\mathrm{d}v}\right), \quad J_{k_j} = \iiint_\Omega ((\sum_{i=1}^n k_i^2) - k_j^2)\rho\mathrm{d}v$$

$$S = \iint_\Sigma \mathrm{d}S = \iint_{D_{xy}} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\mathrm{d}x\mathrm{d}y$$

$$S = \iint_{D_{zx}} \sqrt{1+\left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2}\mathrm{d}z\mathrm{d}x = \iint_{D_{yz}} \sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\mathrm{d}y\mathrm{d}z$$

$$\iint_{-\Sigma} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y = -\iint_\Sigma P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y$$

$$\oint_\Sigma P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y = \iiint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}v, \quad \mathrm{div}\tilde{\mathbf{A}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\Sigma : F(x,y,z) = 0 \quad \vec{n} = \left(\frac{F_x}{\sqrt{F_x^2+F_y^2+F_z^2}}, \frac{F_y}{\sqrt{F_x^2+F_y^2+F_z^2}}, \frac{F_z}{\sqrt{F_x^2+F_y^2+F_z^2}}\right) = (\cos\alpha, \cos\beta, \cos\gamma)$$

$$\oint_\Sigma P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y = \iint_\Sigma (P\cos\alpha + Q\cos\beta + R\cos\gamma)\mathrm{d}S$$

$$\oint_\Gamma P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = \iint_\Sigma \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathrm{d}y\mathrm{d}z + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathrm{d}z\mathrm{d}x + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

$$\mathrm{rot}\tilde{\mathbf{A}} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\tilde{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\tilde{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\tilde{\mathbf{k}}$$

$$\iint_\Sigma f(x,y,z)\mathrm{d}z\mathrm{d}x = \iint_\Sigma -f(x,y,z)\mathrm{d}x\mathrm{d}z \quad \iint_D f(x,y)\mathrm{d}z\mathrm{d}x = \iint_D f(x,y)\mathrm{d}x\mathrm{d}z$$

Gamma Function Integral

$$\left(-\frac{1}{2}\right)! = \sqrt{\pi}$$

$$\left(\frac{2n+1}{2}\right)! = \left(\Pi_{i=n}^{0} \frac{2i+1}{2}\right)\sqrt{\pi} = \frac{2n+1}{2}\frac{2n-1}{2}\dots\frac{1}{2}\sqrt{\pi}$$

$$\int_0^\infty x^a \mathrm{e}^{-x} \mathrm{d}x = a!$$

$$\int_0^\infty x^{-\frac{1}{2}} \mathrm{e}^{-x} \mathrm{d}x = \sqrt{\pi}$$

$$\int_0^\infty x^3 \mathrm{e}^{-x} \mathrm{d}x = (3)! = 6$$

$$\int_0^\infty x^{\frac{5}{2}} \mathrm{e}^{-x} \mathrm{d}x = \frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}$$

$$\int_0^\infty x^a \mathrm{e}^{-x^2} \mathrm{d}x = \frac{1}{2}\left(\frac{a-1}{2}\right)!$$

$$\int_0^\infty x^1 \mathrm{e}^{-x^2} \mathrm{d}x = \frac{1}{2}\left(\frac{1-1}{2}\right)! = \frac{1}{2}$$

$$\int_0^\infty x^4 \mathrm{e}^{-x^2} \mathrm{d}x = \frac{1}{2}\left(\frac{4-1}{2}\right)! = \frac{1}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}$$

$$\int_0^\infty x^7 \mathrm{e}^{-x^2} \mathrm{d}x = \frac{1}{2}\left(\frac{7-1}{2}\right)! = \frac{1}{2}(3)! = 3$$

Infinite Series

$$\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|=\rho \quad ,R=\frac{1}{\rho}$$

$$S(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l})$$

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)\mathrm{d}x,\quad a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}\mathrm{d}x,\quad b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}\mathrm{d}x$$

$$S(x)=\begin{cases} f(x), x\in consecutive \\ \frac{f(x-0)+f(x+0)}{2}, x\in discontinuity \\ \frac{f(l-0)+f(l+0)}{2}, x=\{-l,l\} \end{cases}$$

Constant Series

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}\quad \sum_{k=1}^{\infty}\frac{1}{k^4}=\frac{\pi^4}{90}\quad \sum_{k=1}^{\infty}\frac{1}{k^6}=\frac{\pi^6}{945}$$

$$\sum_{i=0}^{n}a_i\cdot\sum_{j=0}^{n}b_j=\sum_{i=0}^{n}\sum_{j=0}^{n}(a_i\cdot b_j)\quad \sum_{n=s}^{t}\ln f(n)=\ln\prod_{n=s}^{t}f(n)$$

$$\sum_{i=0}^{n}i=\frac{n(n+1)}{2}\quad \sum_{i=1}^{n}i(i+1)(i+2)=\frac{n(n+1)(n+2)(n+3)}{4}$$

$$\sum_{i=0}^{n}i^2=\frac{n(n+1)(2n+1)}{6}\quad \sum_{i=0}^{n}i^3=\left(\sum_{i=0}^{n}i\right)^2=\frac{n^2(n+1)^2}{4}$$

$$\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}=\sum_{i=0}^{\infty}\frac{1}{(2i+1)(2i+2)}=\sum_{i=1}^{\infty}\frac{1}{2^i i}=\sum_{i=1}^{\infty}(\frac{1}{3^i}+\frac{1}{4^i})\frac{1}{i}=\ln 2$$

Power Series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \ln(1+x)$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x)$$

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x$$

$$\sum_{n=0}^{\infty} (n+1) x^n = \frac{1}{(1-x)^2}$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) x^n = \frac{2}{(1-x)^3}$$

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{n} = -\ln(1-x^2)$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = \frac{1}{2}\ln(\frac{1+x}{1-x})$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin x$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \mathrm{e}^x$$

Basel Problem

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

$$f(x)=x^2 \quad \xrightarrow{Fourier Expansion} \quad S(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\cos nx)$$

$$a_0=\frac{2}{3}\pi^2 \quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx\mathrm{d}x=(-1)^n\frac{4}{n^2}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\sin nx\mathrm{d}x=0$$

$$S(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx=\frac{1}{3}\pi^2+\sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}\cos nx=\frac{1}{3}\pi^2+\sum_{n=1}^{n}\frac{4}{n^2}$$

$$\exists x=0, S(0)=\frac{1}{3}\pi^2+\sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}=f(0)=0\Longrightarrow\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=\frac{\pi^2}{12}$$

$$\exists x=\pi, S(\pi)=\frac{1}{3}\pi^2+\sum_{n=1}^{\infty}\frac{4}{n^2}=f(\pi)=\pi^2\Longrightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8} \quad \sum_{n=1}^{\infty}\frac{1}{(2n)^2}=\frac{\pi^2}{24}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{(2n-1)^2}+\frac{1}{(2n)^2}\right)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

Transcendental Equation

$$\sum_{i=0}^{n}a_ix^i=0\Longrightarrow\prod_{i=0}^{n}(x-x_i)=0$$

$$\prod_{i=0}^{n}x_i=(-1)^n\frac{a_0}{a_n}, \quad \sum_{i=0}^{n}\frac{\prod_{i=0}^{n}x_i}{x_i}=(-1)^{n-1}\frac{a_1}{a_n}$$

$$\sum_{i=0}^{n}\frac{1}{x_i}=-\frac{a_1}{a_0}$$

eg. $\tan x=x\Longrightarrow\sin x=\cos x\cdot x$

$$x-\frac{1}{6}x^3+\frac{1}{120}x^5+...=x\cdot(1-\frac{1}{2}x^2+\frac{1}{24}x^4+...)$$

$$\frac{1}{3}x^3-\frac{1}{30}x^5+...=0 \quad \xrightarrow{x\neq 0} \quad \frac{1}{3}x^2-\frac{1}{30}x^4+...=0$$

$$\sum_{i=0}^{n}\frac{1}{t_i}=\frac{1}{10} \quad \xrightarrow{t=x^2} \quad \lim_{n\to\infty}\sum_{i=0}^{n}\frac{1}{x_i^2}=\frac{1}{10}$$