INTEGRÁLY $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

PZ 1 a) $\int 3x^2+2x-4\,dx = 3\frac{x^{2+1}}{2+1}+2\cdot\frac{x^{1+1}}{1+1}-4\cdot\frac{x^{0+1}}{0+1}+c=$

$= \frac{3x^3}{3}+\frac{2x^2}{2}-4x+c = x^3+x^2-4x+c$

b) $\int \frac{x^3}{3}-\frac{x}{5}\,dx = \frac{1}{3}\frac{x^4}{4}-\frac{1}{5}\frac{x^2}{2}+c = \frac{x^4}{12}-\frac{x^2}{10}+c$

c) $\int \sqrt{x^3}-\frac{1}{\sqrt{x}}\,dx = \int x^{\frac{3}{2}}-x^{-\frac{1}{2}}\,dx = \frac{x^{\frac{3}{2}+\frac{2}{2}}}{\frac{5}{2}}-\frac{x^{-\frac{1}{2}+\frac{2}{2}}}{\frac{1}{2}}+c$

$= \frac{2x^{\frac{5}{2}}}{5}-2x^{\frac{1}{2}}+c = \frac{2}{5}\sqrt{x^5}-2\sqrt{x}+c$

d) $\int \frac{\sqrt{x^4+2+x^{-4}}}{x^3}\,dx = \int \frac{\sqrt{x^4+2+\frac{1}{x^4}}}{x^3}\,dx =$

$= \int \frac{\sqrt{\frac{x^8+2x^4+1}{x^4}}}{\frac{x^3}{1}}\,dx = \int \frac{\sqrt{x^8+2x^4+1}}{x^2\cdot x^3}\,dx =$

$x^4 = t \qquad x^8+2x^4+1 = t^2+2t+1 = (t+1)^2 = (x^4+1)^2$

$= \int \frac{\sqrt{(x^4+1)^2}}{x^5}\,dx = \int \frac{x^4+1}{x^5}\,dx = \int \frac{1}{x}+\frac{1}{x^5}\,dx$

$= \ln x + \frac{x^{-4}}{-4}+c = \ln x - \frac{1}{4x^4}+c$

$a^x \cdot a^y = a^{x+y}$

e) $\int \frac{x(\sqrt[3]{x} - x\sqrt[3]{x})}{\sqrt[4]{x}}\,dx = \int \frac{x^1 \cdot x^{\frac{1}{3}} - x^2 \cdot x^{\frac{1}{3}}}{x^{\frac{1}{4}}}\,dx =$

$= \int x^{\frac{4}{3}-\frac{1}{4}} - x^{\frac{7}{3}-\frac{1}{4}}\,dx = \int x^{\frac{13}{12}} - x^{\frac{25}{12}}\,dx =$

$= \frac{x^{\frac{13}{12}+\frac{12}{12}}}{\frac{25}{12}} - \frac{x^{\frac{25}{12}+\frac{12}{12}}}{\frac{37}{12}} + c = \frac{12x^{\frac{25}{12}}}{25} - \frac{12x^{\frac{37}{12}}}{37} + c =$

$= \frac{12 \cdot x^2 \cdot \sqrt[12]{x}}{25} - \frac{x^3\sqrt[12]{x}}{37} + c$

$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

f) $\int \frac{x^3-1}{x-1}\,dx = \int \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}}\,dx = \int x^2+x+1\,dx =$

$= \frac{x^3}{3} + \frac{x^2}{2} + x + c$

h) $\int 5\cos x - 2x^5 + \frac{3}{1+x^2}\,dx = 5 \cdot \sin x - \cancel{2}^{1}\frac{x^6}{\cancel{6}_{3}} +$

$+ 3 \cdot \operatorname{arctg} x + c$

j) $\int 2\sin x - 3\cos x\,dx = -2\cos x - 3\sin x + c$

k) $\int \frac{1}{\sqrt{3-3x^2}}\,dx = \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{1-x^2}}\,dx = \frac{1}{\sqrt{3}}\arcsin x + c$

l) $\int \frac{3 \cdot 2^x - 2 \cdot 3^x}{2^x}\,dx = \int 3 \cdot \frac{2^x}{2^x}\,dx - 2\int \left(\frac{3}{2}\right)^x dx =$

$= 3x - 2\frac{\left(\frac{3}{2}\right)^x}{\ln\left(\frac{3}{2}\right)} + c$

$\cos^2 x + \sin^2 x = 1$

m) $\int \frac{1+\cos^2 x}{1+\cos(2x)} dx = \int \frac{1+\cos^2 x}{1+\cos^2 x - \sin^2 x} dx =$

$\cos^2 x$

$= \int \frac{1+\cos^2 x}{2\cos^2 x} dx = \int \frac{1}{2\cos^2 x} + \frac{1}{2} dx = \frac{1}{2} \operatorname{tg} x + \frac{1}{2} x + C$

$\cos 2x = \cos^2 x - \sin^2 x$

n) $\int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx =$

$= \int \frac{\cos^2 x - \sin^2 x}{\cos^2 x \cdot \sin^2 x} dx = \int \frac{\cancel{\cos^2 x}}{\cancel{\cos^2 x} \cdot \sin^2 x} dx - \int \frac{\cancel{\sin^2 x}}{\cos^2 x \cdot \cancel{\sin^2 x}} dx =$

$\int \frac{1}{\sin^2 x} = -\operatorname{cotg} x \qquad \int \frac{1}{\cos^2 x} = \operatorname{tg} x$

$= \int \frac{1}{\sin^2 x} dx - \int \frac{1}{\cos^2 x} dx = -\operatorname{cotg} x - \operatorname{tg} x + C$

$\cos^2 x + \sin^2 x = 1$

o) $\int \operatorname{tg}^2 x \, dx = \int \left(\frac{\sin x}{\cos x}\right)^2 dx = \int \frac{\sin^2 x}{\cos^2 x} dx =$

$= \int \frac{1-\cos^2 x}{\cos^2 x} dx = \int \frac{1}{\cos^2 x} dx - \int 1 \, dx = \operatorname{tg} x - x + C$

p) $\int \cot g^2 x \, dx = \int \left(\frac{\cos x}{\sin x}\right)^2 dx = \int \frac{\cos^2 x}{\sin^2 x} dx =$

$= \int \frac{1-\sin^2 x}{\sin^2 x} dx = \int \frac{1}{\sin^2 x} - \int 1 \, dx = -\cot g\, x - x + c$

r) $\int \frac{1}{\cos 2x + \sin^2 x} dx = \int \frac{1}{\cos^2 x - \sin^2 x + \sin^2 x} dx = \int \frac{1}{\cos^2 x} dx =$

$= \operatorname{tg} x + c$

POUŽITIE PARCIÁLNYCH ZLOMKOV

q) $\int \frac{1+2x^2}{x^2(1+x^2)} dx = \int \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{1+x^2} dx =$

$$\frac{1+2x^2}{x^2(1+x^2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$$\frac{1+2x^2}{x^2(1+x^2)} = \frac{Ax(1+x^2) + B(1+x^2) + (Cx+D)x^2}{x^2(1+x^2)} \quad / \cdot x^2(1+x^2)$$

$$1 + 2x^2 = Ax + Ax^3 + B + Bx^2 + Cx^3 + Dx^2$$

$0 = A + C \Rightarrow C = 0$

$2 = B + D \Rightarrow 2 = 1 + D \Rightarrow D = 1$

$0 = A$

$1 = B$

$= \int \frac{0}{x} + \frac{1}{x^2} + \frac{0 \cdot x + 1}{1+x^2} dx = \frac{x^{-1}}{-1} + \operatorname{arctg} x + c$

g) $\int \frac{(1+x)^2}{x(1+x^2)} dx = \int \frac{x^2+2x+1}{x(1+x^2)} = \int \frac{A}{x} dx + \int \frac{Bx+C}{1+x^2} dx \overset{\text{①}}{=}$

$$\frac{x^2+2x+1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} \quad / x\cdot(1+x^2)$$

$$x^2+2x+1 = A(1+x^2) + (Bx+C)x$$

$$x^2+2x+1 = A + Ax^2 + Bx^2 + Cx$$

$1 = A + B$

$2 = C$

$1 = A \Rightarrow 1 = 1 + B \Rightarrow B = 0$

$\overset{\text{①}}{=} \int \frac{1}{x} dx + \int \frac{0\cdot x+2}{1+x^2} dx =$

$\int \frac{1}{x} = \ln|x| + c$

$\int \frac{1}{a^2+x^2} = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + c$

$= \ln|x| + 2\cdot \operatorname{arctg} x + c$

SUBSTITUČNÁ METÓDA

$\int a^x dx = \frac{a^x}{\ln a}$

i) $\int 10^{-x} + \frac{x^2+3}{x^2+1} dx = \underbrace{\int 10^{-x} dx}_{\text{I.}} + \underbrace{\int \frac{x^2+3}{x^2+1} dx}_{\text{II.}} \overset{\text{②}}{=}$

I. $\int 10^{-x} dx \quad \left| \begin{array}{l} -x = t \\ -1\,dx = 1\cdot dt \end{array} \right| \quad - \int -10^{\overset{t}{-x}} dx = - \int 10^t dt =$

$$= -\frac{10^t}{\ln 10} = -\frac{10^{-x}}{\ln 10} + c$$

11. $$\int \frac{x^2+1+2}{x^2+1}\,dx = \int \frac{x^2+1}{x^2+1}dx + 2\int \frac{1}{x^2+1}\,dx =$$

$$= \int 1\,dx + 2\int \frac{1}{x^2+1}\,dx = x + 2\,\mathrm{arctg}\,x$$

$$\int \frac{1}{x^2+a^2} = \frac{1}{a}\,\mathrm{arctg}\,\frac{x}{a}$$

$$\overset{(*)}{=} -\frac{10^{-x}}{\ln 10} + x + 2\,\mathrm{arctg}\,x + c \qquad c \in \mathbb{R}$$

PR 2

a) $$\int \frac{1}{3+4x^2}\,dx = \int \frac{1}{3+(2x)^2}\,dx = \left| \begin{array}{l} 2x = t \\ 2dx = 1dt \end{array} \right| =$$

$$= \int \frac{1}{2}\,\frac{\overset{dt}{2\,dx}}{3+(\underset{t}{2x})^2} = \frac{1}{2}\int \frac{dt}{3+t^2} \boxed{=}$$

$$\int \frac{1}{x^2+a^2} = \frac{1}{a}\,\mathrm{arctg}\,\frac{x}{a}$$

$$\boxed{=} = \frac{1}{2}\left(\frac{1}{\sqrt{3}}\,\mathrm{arctg}\,\frac{t}{\sqrt{3}}\right) = \frac{1}{2}\left(\frac{1}{\sqrt{3}}\,\mathrm{arctg}\,\frac{2x}{\sqrt{3}}\right) + c$$

b) $$\int \frac{x}{3+4x^2}\,dx = \left| \begin{array}{l} 3+4x^2 = t \\ 0+8x\,dx = dt \end{array} \right| = \frac{1}{8}\int \frac{\overset{dt}{8x\,dx}}{\underset{t}{3+4x^2}} =$$

$$= \frac{1}{8}\int \frac{1}{t}\,dt = \frac{1}{8}\ln|t| = \frac{1}{8}\ln|3+4x^2| + c$$

d) $$\int \underbrace{e^x\,\mathrm{tg}\,e^x\,dx}_{dt} = \left| \begin{array}{l} e^x = t \\ e^x dx = dt \end{array} \right| = \int \mathrm{tg}\,t\,dt = -\int \frac{\overset{dw}{-\sin t\,dt}}{\underset{w}{\cos t}}$$

$$\left| \begin{aligned} \cos t &= w \\ -\sin t \, dt &= dw \end{aligned} \right| = -\int \frac{dw}{w} = -\ln|w| = -\ln|\cos t| =$$

$$= -\ln|\cos e^x| + c$$

e) $$\int \frac{3x+2}{x^2+4x+5}\,dx = \int \frac{\frac{3}{2}\cdot\frac{2}{3}\cdot(3x+2)}{x^2+4x+5}\,dx =$$

$$\frac{3}{2}\int \frac{\frac{2}{3}\cdot 3x + \frac{2}{3}\cdot 2}{x^2+4x+5}\,dx = \frac{3}{2}\int \frac{2x + \frac{4}{3} + 4 - 4}{x^2+4x+5}\,dx =$$

$$= \frac{3}{2}\Bigg(\underbrace{\int \frac{2x+4}{x^2+4x+5}\,dx}_{\text{I.}} + \underbrace{\int \frac{-4+\frac{4}{3}}{x^2+4x+5}\,dx}_{\text{II.}} \Bigg) =$$

I. $$\int \frac{\overset{f'}{2x+4}}{\underset{f}{x^2+4x+5}}\,dx \;\boxed{=}\; \int \frac{f'(x)}{f(x)} = \ln|f(x)| + c$$

$$(x^2+4x+5)' = 2x+4$$

$$\boxed{=}\; \ln|x^2+4x+5|$$

II. $$x^2+4x+5 = x^2+4x+4+1 = (x+2)^2+1$$

$$\int \frac{-\frac{12+4}{3}}{(x+2)^2+1}\,dx = \frac{-8}{3}\int \frac{1}{(\underset{t}{x+2})^2+1}\,\overset{dt}{dx} \quad \left| \begin{aligned} x+2 &= t \\ dx &= dt \end{aligned} \right| =$$

$$= -\frac{8}{3}\int \frac{1}{t^2+1}\,dt = -\frac{8}{3}\operatorname{arctg} t = -\frac{8}{3}\operatorname{arctg}(x+2) + c$$

$$= \frac{3}{2}\left(\ln|x^2+4x+5| - \frac{8}{3}\operatorname{arctg}(x+2) \right) + c$$

f) $\int \frac{2x+1}{x^2+2x+5}\,dx \overset{\nabla}{=}$ $\qquad \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|$

$(x^2+2x+5)' = 2x+2$

$\overset{\nabla}{=} \int \frac{2x+1+1-1}{x^2+2x+5}\,dx = \underbrace{\int \frac{2x+2}{x^2+2x+5}}_{I.} - \underbrace{\int \frac{1}{x^2+2x+5}\,dx}_{II.} \overset{\triangle}{=}$

I. $\int \frac{2x+2}{x^2+2x+5}\,dx = \left| \begin{matrix} x^2+2x+5 = t \\ 2x+2\,dx = 1 \cdot dt \end{matrix} \right| = \int \frac{1}{t}\,dt =$

$= \ln|t| = \ln|x^2+2x+5| + c$

II. $\int \frac{1}{x^2+2x+1+4}\,dx = \int \frac{1\,dx}{(x+1)^2+4} \quad \left| \begin{matrix} x+1 = w \\ 1\,dx = 1 \cdot dw \end{matrix} \right| =$

$\int \frac{1}{w^2+4}\,dw = \frac{1}{\sqrt{4}} \operatorname{arctg} \frac{w}{\sqrt{4}} = \frac{1}{2} \operatorname{arctg} \frac{w}{2} = \frac{1}{2} \operatorname{arctg} \frac{x+1}{2} + c$

$\overset{\triangle}{=} \ln|x^2+2x+5| - \frac{1}{2} \operatorname{arctg} \frac{x+1}{2} + c$

g) $\int \frac{5x-1}{x^2+2x+3}\,dx = \int \frac{\frac{5}{2} \cdot \frac{2}{5}(5x-1)}{x^2+2x+3}\,dx =$

$\frac{5}{2} \int \frac{2x - \frac{2}{5} + 2 - 2}{x^2+2x+3}\,dx = \frac{5}{2}\left(\underbrace{\int \frac{2x+2\,dx}{x^2+2x+3}}_{I.} - \underbrace{\int \frac{\frac{2}{5}+2}{x^2+2x+3}\,dx}_{II.} \right) =$

$$\text{I.}\ \int \frac{2x+2}{x^2+2x+3}\,dx \quad \left| \begin{array}{l} x^2+2x+3 = t \\ 2x+2\,dx = dt \end{array} \right| = \int \frac{1}{t}\,dt = \ln|t| =$$

$$= \ln|x^2+2x+3| + c$$

$$\text{II.}\ \frac{12}{5}\int \frac{1}{x^2+2x+3}\,dx = \frac{12}{5}\int \frac{1}{x^2+2x+1+2}\,dx = \frac{12}{5}\int \underbrace{\frac{1}{(x+1)^2+2}\,dx}_{dz}$$

$$\left| \begin{array}{l} x+1 = z \\ 1\cdot dx = dz \end{array} \right| = \frac{12}{5}\int \frac{1}{t^2+2}\,dt = \frac{12}{5}\left(\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{t}{\sqrt{2}}\right) =$$

$$= \frac{12}{5}\left(\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x+1}{\sqrt{2}}\right) + c$$

$$\boxed{=}\ \text{I.} - \text{II.} = \ln|x^2+2x+3| - \frac{12}{5}\left(\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x+1}{\sqrt{2}}\right) + c$$

$$h)\ \int \frac{2^x}{(2^x+3)^7}\,dx \quad \left| \begin{array}{l} 2^x = t \\ 2^x\cdot\ln 2\,dx = dt \end{array} \right| = \frac{1}{\ln 2}\int \frac{\overbrace{2^x\cdot\ln 2\,dx}^{dt}}{(\underbrace{2^x}_{=t}+3)^7} =$$

$$= \int \frac{dt}{(t+3)^7} = \left| \begin{array}{l} t+3 = z \\ 1\cdot dt = 1\cdot dz \end{array} \right| = \int \frac{dz}{z^7} = \frac{z^{-7+1}}{-7+1} = \frac{1}{-6z^6} =$$

$$= -\frac{1}{6(t+3)^6} = \frac{-1}{6(2^x+3)^6} + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + c$$

$$i)\ \int \frac{1}{\sqrt{3-4x^2}}\,dx = \int \frac{1}{\sqrt{3-(2x)^2}}\,dx$$

$$\left| \begin{array}{l} 2x = t \\ 2\,dx = dt \end{array} \right| = \frac{1}{2}\int \frac{\overbrace{2\,dx}^{dt}}{\sqrt{3-(\underbrace{2x}_{t})^2}} = \frac{1}{2}\int \frac{dt}{\sqrt{3-t^2}} =$$

$$= \frac{1}{2}\arcsin\frac{t}{\sqrt{3}} = \frac{1}{2}\arcsin\frac{2x}{\sqrt{3}} + c$$

j) $\int \frac{x}{\sqrt{3-4x^2}}\,dx = \left| \begin{matrix} 3-4x^2=t \\ -8x\,dx=dt \end{matrix} \right| = -\frac{1}{8}\int \frac{\overbrace{-8x\,dx}^{dt}}{\sqrt{\underbrace{3-4x^2}_{t}}} =$

$= -\frac{1}{8}\int \frac{1}{\sqrt{t}}\,dt = \frac{t^{-\frac{1}{2}+\frac{2}{2}}}{\frac{1}{2}} = 2t^{\frac{1}{2}} = 2\sqrt{t} = 2\sqrt{3-4x^2}+c$

k) $\int \frac{\overbrace{e^x(e^x-1)\,dx}^{dt}}{e^{2x}+1} = \left| \begin{matrix} e^x=t \\ e^x\,dx=dt \end{matrix} \right| = \int \frac{t-1}{t^2+1}\,dt =$

$\underbrace{\int \frac{t}{t^2+1}\,dt}_{I.} - \underbrace{\int \frac{1}{t^2+1}\,dt}_{II.}$ $\heartsuit$

I. $\int \frac{t}{t^2+1}\,dt = \left| \begin{matrix} t^2+1=w \\ 2t\,dt=dw \end{matrix} \right| = \frac{1}{2}\int \frac{\overbrace{2t\,dt}^{dw}}{t^2+1} = \frac{1}{2}\int \frac{dw}{w} =$

$= \frac{1}{2}\ln|w| = \frac{1}{2}\ln|t^2+1| = \frac{1}{2}\ln|e^{2x}+1|+c$

II. $\int \frac{1}{t^2+1}\,dt = \operatorname{arctg} t = \operatorname{arctg}(e^t)+c$

$\heartsuit$ I. − II. $= \frac{1}{2}\ln|e^{2x}+1| - \operatorname{arctg} e^t + c$

l) $\int \frac{\sin x\,\overbrace{\cos x\,dx}^{dt}}{\sin^2 x+\sin x+3} = \left| \begin{matrix} \sin x=t \\ \cos x\,dx=dt \end{matrix} \right| = \int \frac{t\,dt}{t^2+t+3}\,dt =$

$= \left| \begin{matrix} t^2+t+3=k \\ 2t+1\,dt=dk \end{matrix} \right| = \frac{1}{2}\int \frac{2t+1-1}{t^2+t+3}\,dt = \frac{1}{2}\underbrace{\int \frac{2t+1}{t^2+t+1}\,dt}_{I.}$

$-\frac{1}{2}\underbrace{\int \frac{1}{t^2+t+3}\,dt}_{II.}$

I. $= \frac{1}{2}\int \frac{dk}{k} = \ln|k| = \frac{1}{2}\ln|t^2+t+3| = \frac{1}{2}\ln|\sin^2 x+\sin x+3|$

II. $= \frac{1}{2}\int \frac{1}{t^2+t+\frac{1}{4}+\frac{11}{4}}\,dt = \frac{1}{2}\int \frac{1}{(t+\frac{1}{2})^2+\frac{11}{4}}\,dt = \begin{vmatrix} t+\frac{1}{2} = w \\ dt = dw \end{vmatrix}$

$= \frac{1}{2}\int \frac{1}{w^2+\frac{11}{4}}\,dw = \frac{1}{2}\cdot\frac{1}{\sqrt{\frac{11}{4}}}\operatorname{arctg}\frac{w}{\sqrt{\frac{11}{4}}} =$

$= \frac{1}{\cancel{2}}\,\frac{\cancel{2}}{\sqrt{11}}\operatorname{arctg}\frac{2w}{\sqrt{11}} = \frac{1}{\sqrt{11}}\operatorname{arctg}\frac{2(t+\frac{1}{2})}{\sqrt{11}} =$

$= \frac{1}{\sqrt{11}}\operatorname{arctg}\frac{2\sin x+1}{\sqrt{11}} + c$

$\text{I.} - \text{II.} = \frac{1}{2}\ln|\sin^2 x+\sin x+3| - \frac{1}{\sqrt{11}}\operatorname{arctg}\frac{2\sin x+1}{\sqrt{11}} + c$

m) $\int \frac{\sqrt{x}\,dx}{x(\sqrt[3]{x}+\sqrt{x})} = \begin{vmatrix} x = t^6 \\ \sqrt[6]{x} = t \\ dx = 6t^5\,dt \end{vmatrix} = \int \frac{\sqrt{t^6}\cdot 6t^5\,dt}{t^6(\sqrt[3]{t^6}+\sqrt{t^6})} =$

$= \int \frac{t^3\cdot 6t^5}{t^6(t^2+t^3)}\,dt = \int \frac{6t^8}{t^8+t^9}\,dt = \int \frac{6t+6-6}{t+1}\,dt =$

$= \underbrace{\int \frac{6t+6}{t+1}\,dt}_{\text{I.}} - \underbrace{\int \frac{6}{t+1}\,dt}_{\text{II.}} =$

I. $\int \frac{6(\cancel{t+1})}{\cancel{t+1}}\,dt = \int 6\,dt = 6t = 6\sqrt[6]{x}$

II. $6\int \frac{1}{t+1}\,dt \quad \begin{vmatrix} t+1 = q \\ dt = dq \end{vmatrix} = 6\int \frac{1}{q}\,dq = 6\ln|q| =$

$$= 6\ln|t+1| = 6\ln|\sqrt[6]{x}+1|$$

$$\text{I.} - \text{II.} = 6\sqrt[6]{x} - 6\ln|\sqrt[6]{x}+1| + c$$

σ) $\displaystyle\int \frac{x}{1+\sqrt{x-1}}\,dx \quad \left| \begin{array}{l} x-1=t^2 \\ x=t^2+1 \\ dx = 2t\,dt \end{array} \right| = \int \frac{(t^2+1)\,2t\,dt}{1+\sqrt{t^2}} =$

$$= \int \frac{2t^3+2t}{1+t}\,dt = \quad \left| \begin{array}{l} (2t^3+2t) : (t+1) = 2t^2-2t+4 \\ -2t^3 \pm 2t^2 \\ \hline 0 - 2t^2 + 2t \\ \quad +2t^2 + 2t \\ \hline \quad 0 + 4t \\ \quad\quad -4t \pm 4 \\ \hline \quad\quad 0 - 4 \end{array} \right|$$

$$= \underbrace{\int 2t^2 - 2t + 4\,dt}_{\text{I.}} - \underbrace{\int \frac{4}{1+t}\,dt}_{\text{II.}}$$

$x-1=t^2 \Rightarrow t=\sqrt{x-1}$

$$\text{I.} = \frac{2t^3}{3} - \frac{2t^2}{2} + 4t = \frac{2t^2\cdot t}{3} - t^2 + 4t + c =$$

$$= \frac{2(x-1)\sqrt{x-1}}{3} - (x-1) + 4\sqrt{x-1} + c$$

$$\text{II.} = 4\int \frac{1}{1+t}\,dt = \left| \begin{array}{l} 1+t = w \\ dt = dw \end{array} \right| = 4\int \frac{1}{w}\,dw =$$

$$= 4\ln|w| = 4\ln|1+t| = 4\ln|1+\sqrt{x-1}| + c$$

$$= \text{I.} - \text{II.} = \frac{2(x-1)\sqrt{x-1}}{3} - x + 1 + 4\sqrt{x-1} - 4\ln|1+\sqrt{x-1}| + c$$

MÉTODO PER-PARTES

$$\int u' \cdot v = u \cdot v - \int u v'$$
$$\int u \cdot v' = u v - \int u' v$$

PR 1

g) $\int x a^x \, dx \quad \left| \begin{array}{ll} u = x & u' = 1 \\ v' = a^x & v = \int a^x dx = \frac{a^x}{\ln a} \end{array} \right| = \frac{x \cdot a^x}{\ln a} - \int \frac{a^x}{\ln a} dx =$

$$= \frac{x a^x}{\ln a} - \frac{1}{\ln a} \cdot \frac{a^x}{\ln a} = \frac{x a^x}{\ln a} - \frac{a^x}{\ln^2 a} + c$$

PR 3

a) $\int \underbrace{x \cdot \sin x}_{u \cdot v'} \, dx \quad \left| \begin{array}{ll} u = x & u' = 1 \\ v' = \sin x & v = \int \sin x \, dx = -\cos x \end{array} \right| =$

$$= \underbrace{x}_{u} \cdot \underbrace{(-\cos x)}_{v} - \int \underbrace{1}_{u'} \cdot \underbrace{(-\cos x)}_{v} \, dx =$$

$$= -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + c$$

b) $\int \underbrace{x}_{u} \underbrace{e^{2x}}_{v'} dx = \left| \begin{array}{ll} u = x & u' = 1 \\ v' = e^{2x} & v = \int e^{2x} dx = \frac{e^{2x}}{2} \end{array} \right| =$

$$= x \cdot \frac{e^{2x}}{2} - \frac{1}{2} \int e^{2x} dx = \frac{x e^{2x}}{2} - \frac{1}{2} \cdot \frac{e^{2x}}{2} =$$

$$= \frac{x e^{2x}}{2} - \frac{1}{4} e^{2x} + c$$

c) $\int \underbrace{(x^3-x-1)}_{u}\,\underbrace{e^{2x}}_{v'}\,dx = \left| \begin{array}{ll} u = x^3-x-1 & u' = 3x^2-1 \\ v' = e^{2x} & v = \int e^{2x}\,dx \end{array} \right| \stackrel{(=)}{}$

$\int e^{2x}dx = \left| \begin{array}{l} 2x = t \\ 2dx = dt \end{array} \right| = \frac{1}{2}\int 2e^{2x}dx \overset{dt}{} = \frac{1}{2}\int e^t\,dt = \frac{1}{2}e^t = \frac{1}{2}e^{2x}+c$

$(=)\ \underbrace{(x^3-x-1)}_{u}\cdot\underbrace{\frac{1}{2}e^{2x}}_{v} - \int \underbrace{(3x^2-1)}_{u'}\cdot\underbrace{\frac{e^{2x}}{2}}_{v}\,dx \quad \left| \begin{array}{ll} u = 3x^2-1 & u' = 6x \\ v' = e^{2x} & v = \frac{1}{2}e^{2x} \end{array} \right|$

$= \frac{1}{2}(x^3-x-1)\,e^{2x} - \frac{1}{2}\left((3x^2-1)\frac{e^{2x}}{2} - \int \cancel{6}^{3}x\cdot\frac{e^{2x}}{\cancel{2}}\,dx\right)$

$\left| \begin{array}{ll} u = x & u' = 1 \\ v' = e^{2x} & v = \frac{1}{2}e^{2x} \end{array} \right| = \frac{1}{2}(x^3-x-1)e^{2x} - \frac{1}{4}(3x^2-1)e^{2x} +$

$+\ 3\left(x\cdot\frac{e^{2x}}{2} - \int 1\cdot\frac{e^{2x}}{2}\,dx\right) = (x^3-x-1)e^{2x} - \frac{1}{4}(3x^2-1)e^{2x} +$

$+\ \frac{3}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$

d) $\int \ln x\,dx = \int \underbrace{1}_{u'}\cdot\underbrace{\ln x}_{v}\,dx = \left| \begin{array}{ll} u' = 1 & u = x \\ v = \ln x & v' = \frac{1}{x} \end{array} \right|$

$= \underbrace{x}_{u}\cdot\underbrace{\ln x}_{v} - \int \underbrace{x}_{u}\cdot\underbrace{\frac{1}{x}}_{v'}\,dx = x\ln x - \int 1\,dx =$

$= x\ln x - x + c$

e) $\int \underset{u'}{x}\, \underset{v}{\log_{10} 2x}\, dx$ $\left| \begin{array}{ll} u' = x & u = \int x\,dx = \frac{x^2}{2} \\ v = \log_{10} 2x & v' = \frac{1}{\cancel{2}x \cdot \ln 10} \cdot \cancel{2} \end{array} \right| =$

$$= \underset{u}{\frac{x^2}{2}}\, \underset{v}{\log_{10} 2x} - \int \underset{u}{\frac{x^{\cancel{2}}}{2}} \cdot \underset{v'}{\frac{1}{\cancel{x} \cdot \ln 10}}\, dx = \frac{x^2}{2} \log_{10} 2x -$$

$$- \frac{1}{2 \ln 10} \int x\,dx = \frac{x^2}{2} \log_{10} 2x - \frac{1}{2 \cdot \ln 10} \cdot \frac{x^2}{2} =$$

$$= \frac{x^2}{2} \log_{10} 2x - \frac{1}{4 \cdot \ln 10} \cdot x^2 + c$$

f) $\overbrace{\int \underset{u}{e^x}\, \underset{v'}{\sin x}\, dx}^{IN}$ $\left| \begin{array}{ll} u = e^x & u' = e^x \\ v' = \sin x & v = -\cos x \end{array} \right| =$

$$= \underset{u \cdot v}{-e^x \cos x} - \int \underset{u' \cdot v}{e^x (-\cos x)}\, dx = -e^x \cos x + \int e^x \cos x\, dx$$

$\left| \begin{array}{ll} u = e^x & u' = e^x \\ v' = \cos x & v = \sin x \end{array} \right| = -e^x \cos x + e^x \sin x - \underbrace{\int e^x \sin x\, dx}_{IN}$

$$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx \quad \Big/ + \int e^x \sin x\, dx$$

$$2 \int e^x \sin x\, dx = -e^x (\cos x - \sin x) + c \quad \Big/ : 2$$

$$\int e^x \sin x\, dx = -\frac{e^x}{2} (\cos x - \sin x) + c$$

g) $\int \frac{x}{\cos^2 x}\,dx = \int \underbrace{x \cdot \frac{1}{\cos^2 x}}_{u \cdot v'}\,dx = \left| \begin{matrix} u = x & u' = 1 \\ v' = \frac{1}{\cos^2 x} & v = \operatorname{tg} x \end{matrix} \right| =$

$= \underbrace{x \cdot \operatorname{tg} x}_{u \cdot v} - \int \underbrace{1 \cdot \operatorname{tg} x}_{u' \cdot v}\,dx = x \operatorname{tg} x + \int \frac{-\sin x}{\cos x}\,dx \quad (dt) \quad \left| \begin{matrix} \cos x = t \\ -\sin x\,dx = dt \end{matrix} \right|$

$= x \operatorname{tg} x + \int \frac{1}{t}\,dt = x \operatorname{tg} x + \ln|t| = x \operatorname{tg} x + \ln|\cos x| + c$

i) $\int \operatorname{arccotg} x\,dx = \int \underbrace{1}_{u'} \cdot \underbrace{\operatorname{arccotg} x}_{v}\,dx \quad \left| \begin{matrix} u' = 1 & u = x \\ v = \operatorname{arccotg} x & v' = -\frac{1}{1+x^2} \end{matrix} \right|$

$= \underbrace{x \cdot \operatorname{arccotg} x}_{u \cdot v} - \int \underbrace{x}_{u} \cdot \underbrace{\frac{-1}{1+x^2}}_{v'}\,dx = x \operatorname{arccotg} x + \frac{1}{2} \int \frac{2x}{1+x^2} =$

$= x \operatorname{arccotg} x + \frac{1}{2} \ln|1+x^2| + c$

j) $\int x \ln(x^2+3x-10)\,dx \quad \left| \begin{matrix} u' = x & u = \frac{x^2}{2} \\ v = \ln(x^2+3x-10) & v' = \frac{1}{x^2+3x-10} \cdot (2x+3) \end{matrix} \right|$

$= \frac{x^2}{2} \ln(x^2+3x-10) - \int \frac{x^2}{2} \cdot \frac{(2x+3)}{x^2+3x-10}\,dx =$

$= \frac{x^2}{2} \ln(x^2+3x-10) - \int \underbrace{\frac{2x^3+3x^2}{2x^2+6x-20}}_{(*)}\,dx =$

$(2x^3+3x^2) : (2x^2+6x-10) = x - \frac{3}{2}$

$-2x^3 \pm 6x^2 \mp 10x$

$0 \quad -3x^2 + 10x$

$\quad +3x^2 + 9x \pm 15$

$\quad 0 \quad 19x + 15$

$$⊗ = \underbrace{\int x - \frac{3}{2}\,dx}_{I.} + \underbrace{\int \frac{19x+15}{2x^2+3x-10}\,dx}_{II.} =$$

$$I.\quad \int x + \frac{3}{2}\,dx = \frac{x^2}{2} + \frac{3}{2}x + c$$

$$II.\quad \int \frac{19x+15}{2x^2+3x-10}\,dx = \frac{19}{4}\int \frac{\frac{4}{19}(19x+15)}{2x^2+3x-10}\,dx =$$

$$= \frac{19}{4}\int \frac{4x+3-3+\frac{60}{19}}{2x^2+3x-10}\,dx = \frac{19}{4}\int \frac{4x+3}{2x^2+3x-10} + \frac{\frac{3}{19}}{2x^2+3x-10}\,dx =$$

$$= \frac{19}{4}\left(\underbrace{\ln|2x^2+3x-10|}_{III.} + \frac{3}{19}\underbrace{\int \frac{1}{2}\cdot\frac{1}{x^2+\frac{3}{2}x-5}\,dx}_{IV.}\right) =$$

$$IV.\quad \frac{3}{38}\int \frac{1}{\left(x+\frac{3}{4}\right)^2 - \frac{9}{16} - 5} = \frac{-3}{38}\int \frac{1}{\frac{89}{16} - \left(x+\frac{3}{4}\right)^2}\,dx =$$

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|$$

$$= \left|\begin{matrix} x+\frac{3}{4} = t \\ dx = dt \end{matrix}\right| = -\frac{3}{38}\int \frac{1}{\frac{89}{16} - t^2}\,dt =$$

$$= -\frac{3}{38}\,\frac{1}{\cancel{2}\sqrt{\frac{89}{\cancel{4}}}}\ln\left|\frac{\sqrt{\frac{89}{4}}+t}{\sqrt{\frac{89}{4}}-t}\right| = \frac{-3}{38\sqrt{89}}\ln\left|\frac{\sqrt{\frac{89}{4}}+x+\frac{3}{4}}{\sqrt{\frac{89}{4}}-x-\frac{3}{4}}\right|$$

$$= \frac{x^2}{2}\ln(2x^2+3x-10) + \underline{I.} + \underline{II.} = I. + \frac{19}{4}\left(\underline{III.} + \underline{IV.}\right) =$$

$$\frac{x^2}{2}\ln|2x^2+3x-10| +$$

$$+\frac{x^2}{2}+\frac{3}{2}x+\frac{19}{4}\left(\ln|2x^2+3x-10|-\frac{3}{38\sqrt{89}}\ln\left|\frac{\frac{2\sqrt{89}+3}{4}+x}{\frac{2\sqrt{89}-3}{4}-x}\right|\right)+c$$

2) $\int \ln(x^2-4x+6)\,dx = \int 1\cdot\ln(x^2-4x+6)\,dx =$

$$\left|\begin{matrix} u'=1 & u=x \\ v=\ln(x^2-4x+6) & v'=\frac{2x-4}{x^2-4x+6}\end{matrix}\right| =$$

$$x\cdot\ln(x^2-4x+6)-\int x\cdot\frac{2x-4}{x^2-4x+6}\,dx =$$

$$=x\ln(x^2-4x+6)-\int\frac{2x^2-4x-4x+12+4x-12}{x^2-4x+6}\,dx =$$

$$=x\ln(x^2-4x+6)-\int\frac{2(x^2-4x+6)}{x^2-4x+6}\,dx+\int\frac{4x-12}{x^2-4x+6}\,dx=$$

$$=x\ln(x^2-4x+6)-\int 2\,dx+\underbrace{2\int\frac{2x-4-2}{x^2-4x+6}\,dx}_{\Delta}=$$

$$\Delta = 2\int\frac{2x-4}{x^2-4x+6}\,dx-\underbrace{4\int\frac{1}{(x-2)^2+2}\,dx}_{\Delta} =$$

$$\Delta=\left|\begin{matrix} x-2=t \\ dx=dt\end{matrix}\right| = \int\frac{1}{t^2+2}\,dt=\frac{1}{\sqrt2}\operatorname{arctg}\frac{t}{\sqrt2}=$$

$$=\frac{1}{\sqrt2}\operatorname{arctg}\frac{x-2}{\sqrt2}+c$$

$= 2\ln|x^2-4x+6| - 4\cdot\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x-2}{\sqrt{2}} + c$

$= x\ln(x^2-4x+6) - 2x + 2\ln|x^2-4x+6| -$

$-\frac{4}{\sqrt{2}}\operatorname{arctg}\frac{x-2}{\sqrt{2}} + c$

e) $\int x \operatorname{arctg}(x+3)\,dx$ $\left|\begin{array}{ll} u' = x & u = \frac{x^2}{2} \\ v = \operatorname{arctg}(x+3) & v' = \frac{1}{(x+3)^2+1} \end{array}\right|$

$= \frac{x^2}{2}\operatorname{arctg}(x+3) - \underbrace{\int \frac{x^2}{2}\cdot\frac{1}{(x+3)^2+1}\,dx}_{✡} =$

$✡ = \frac{1}{2}\int \frac{x^2}{x^2+6x+9+1}\,dx = \frac{1}{2}\int 1\,dx + \frac{1}{2}\int \frac{-6x-10}{x^2+6x+10}\,dx =$

$x^2 : (x^2+6x+10) = 1$

$-x^2 + 6x + 10$

$-6x-10$

$= \frac{1}{2}x - \int \frac{3x+5}{x^2+6x+10}\,dx = \frac{x}{2} - \frac{1}{2}\int \frac{2(3x+5)}{x^2+6x+10}\,dx =$

$= \frac{x}{2} - \frac{1}{2}\int \frac{6x+10+8-8}{x^2+6x+10}\,dx = \frac{x}{2} - \frac{3}{2}\int \frac{2x+6}{x^2+6x+10}\,dx$

$+ \underbrace{\frac{1}{2}\int \frac{8}{x^2+6x+10}}_{@}\,dx =$

$$4\int \frac{1}{(x+3)^2+1} \quad \left| \begin{array}{l} x+3 = w \\ dx = dw \end{array} \right| = 4\int \frac{1}{w^2+1}\, dw =$$

$$= 4 \cdot \operatorname{arctg} w = 4 \operatorname{arctg}(x+3) + c$$

$$= \frac{x}{2} - \frac{3}{2} \ln|x^2+6x+10| + 4 \operatorname{arctg}(x+3) + c$$

$$= \frac{x^2}{2} \operatorname{arctg}(x+3) - \frac{x}{2} + \frac{3}{2} \ln|x^2+6x+10| - 4 \operatorname{arctg}(x+3) + c$$