A Faster Practical Approximation Scheme for the Permanent
Supplement

Juha Harviainen, Mikko Koivisto
University of Helsinki
juha.harviainen@helsinki.fi, mikko.koivisto@helsinki.fi

Proofs of Lemmas

This section gives detailed proofs of the lemmas that were deferred from the paper.

Proof of Lemma 2

By using the upper bound (Huber 2006)

$$h(k) \leq h(2) + \frac{k + 0.5 \ln k - 2 - 0.5 \ln 2}{\mathrm{e}}$$

and a lower bound

$$\gamma(k) \geq (2\pi k)^{0.5/k} \frac{k}{\mathrm{e}}$$

from Stirling's formula, we get that

$$\begin{aligned}
&h(k) - \gamma(k) \\
\leq & h(2) + \frac{k + 0.5 \ln k - 2 - 0.5 \ln 2}{\mathrm{e}} - \frac{(2\pi k)^{0.5/k} k}{\mathrm{e}} \\
< & \frac{k + 0.5 \ln k - (2\pi k)^{0.5/k} k}{\mathrm{e}} + 0.603 \\
= & \frac{k + 0.5 \ln k - \mathrm{e}^{\ln(2\pi k)/(2k)} k}{\mathrm{e}} + 0.603.
\end{aligned}$$

The result follows from applying inequality $1 + x < \mathrm{e}^x$.

Proof of Lemma 4

Order statistics of the entries of the matrix have the property that $a^*_{i,k} \sim \mathrm{Beta}(n + 1 - k, k)$ (see, e.g., Gentle (2009)), which means that

$$\mathbf{E}[a^*_{i,k}] = \frac{n + 1 - k}{n + 1}.$$

Therefore, by the properties of the expected value,

$$\begin{aligned}
\mathbf{E}[U^*(A)] &= \left(\sum_{k=1}^{n} (u(k) - u(k-1)) \mathbf{E}[a^*_{1,k}] \right)^n \\
&= \left(\frac{\sum_{k=1}^{n} (u(k) - u(k-1))(n + 1 - k)}{n + 1} \right)^n \\
&= \left(\frac{\sum_{k=1}^{n} u(k)}{n + 1} \right)^n.
\end{aligned}$$

For any $n \times n$ matrix of ones, the permanent equals

$$n! > \sqrt{2\pi n} \left(\frac{n}{\mathrm{e}} \right)^n$$

and has an upper bound $u(n)^n$. Thus $u(n) > n/\mathrm{e}$, whence

$$\begin{aligned}
\mathbf{E}[U^*(A)] &= \left(\frac{\sum_{k=1}^{n} u(k)}{n + 1} \right)^n \\
&> \left(\frac{\sum_{k=1}^{n} k/\mathrm{e}}{n + 1} \right)^n \\
&= \left(\frac{n}{2\mathrm{e}} \right)^n.
\end{aligned}$$

Proof of Lemma 5

The order statistics for each row are independent, so the variance of U^* is the variance of u applied on a single row raised to the power of n. Recall that we denote $u(k) - u(k-1)$ by Δ_k and assume $\Delta_k \leq 1$. Since the order statistics of the entries are beta-distributed, we get that

$$\begin{aligned}
&\mathrm{Var}\left[\sum_{k=1}^{n} \Delta_k a^*_{ik} \right] \\
= & \sum_{k=1}^{n} \Delta_k^2 \mathrm{Var}\left[a^*_{ik}\right] + 2 \sum_{1 \leq k < k' \leq n} \Delta_k \Delta_{k'} \mathrm{Cov}[a^*_{ik}, a^*_{ik'}] \\
\leq & \sum_{k=1}^{n} \mathrm{Var}\left[a^*_{ik}\right] + 2 \sum_{1 \leq k < k' \leq n} \mathrm{Cov}[a^*_{ik}, a^*_{ik'}] \\
= & \sum_{k=1}^{n} \frac{(n + 1 - k)k}{(n+1)^2(n+2)} + 2 \sum_{1 \leq k < k' \leq n} \frac{(n + 1 - k')k}{(n+1)^2(n+2)} \\
= & \frac{n^2(n+1)^2}{4(n+1)^2(n+2)} \\
\leq & \frac{n}{4}.
\end{aligned}$$

Proof of Lemma 8

Huber (2006) gives an upper bound

$$g(k) \leq k + (1/2) \ln k + 1.64$$

for $k \geq 1$, which is less than $\ell(k)$. For $k = 0$, both $g(k)$ and $\ell(k)$ are zeros. The result $U_{\mathrm{H}}(B) \leq U_{\mathrm{HL}}(B)$ follows from the fact that B is a binary matrix.

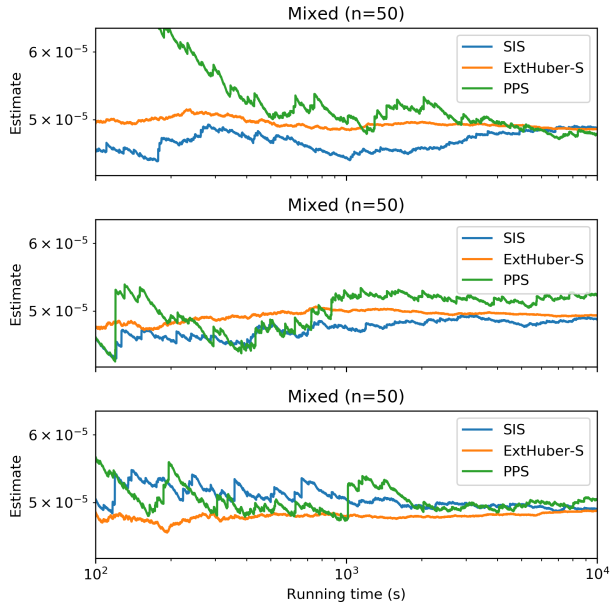

Figure 1: Evolution of the estimates in three independent runs of the methods on the same *Mixed* instance.

Empirical Results

Computing Infrastructure

The experiment for Figure 1 (main paper) was computed on a HPC cluster on a Sandy Bridge processor (2.66 GHz) with 2 gigabytes of memory. The results of Table 2 and Figure 2 were computed on an Ubuntu-based laptop with a Tiger Lake processor (2.40 GHz) and 16 gigabytes of memory.

Additional Evaluations of *Mixed*

Figure 1 shows additional evalutations of the experiment for importance samplers on the same *Mixed* instance as in the main paper. The behavior of the schemes is consistent here: EXTHUBER-S is rather stable, while SIS and PPS take longer to converge. This demonstrates that rejection samplers are able to beat importance samplers on some instances in addition to having accuracy guarantees.

References

Gentle, J. E. 2009. *Computational statistics*, volume 308. Springer.

Huber, M. 2006. Exact Sampling from Perfect Matchings of Dense Regular Bipartite Graphs. *Algorithmica*, 44(3): 183–193.