

```
> with(DifferentialGeometry):
with(Tensor):
```

## Experiment 5.1

```
> DGEEnvironment[Manifold]([x,y],R2);
Manifold: R2 (1.1)
```

```
R2 > gR2:=evalDG(dx &t dx + dy &t dy)
gR2 := dx ⊗ dx + dy ⊗ dy (1.2)
```

```
R2 > X:=evalDG(-y*D_x + x*D_y);
X := -y ∂x + x ∂y (1.3)
```

```
R2 > Xhat:=(0.0012*x-0.7075*y+0.0002)*D_x + (0.7067*x-0.0058*
y-0.0005)*D_y;
Xhat := (0.0012 x - 0.7075 y + 0.0002) ∂x + (0.7067 x - 0.0058 y - 0.0005) ∂y (1.4)
```

```
R2 > innerprod:=proc(X,Xhat,g)
ContractIndices(g,evalDG(X &t Xhat),[[1,1],[2,2]]);
end;
innerprod := proc(X, Xhat, g)
DifferentialGeometry:-Tensor:-ContractIndices(g, DifferentialGeometry:-
evalDG(X&t Xhat), [[1, 1], [2, 2]])
end proc (1.5)
```

```
R2 > integrand:=innerprod(X,Xhat,gR2)/sqrt(innerprod(Xhat,
Xhat,gR2)*innerprod(X,X,gR2));
integrand := (-0.0070 y x + 0.7075 y2 - 0.0002 y + 0.7067 x2 - 0.0005 x) /
((0.49942633 x2 - 0.00989572 y x - 0.00070622 x
+ 0.50058989 y2 - 0.00027720 y + 2.9 10-7) (x2 + y2))1/2 (1.6)
```

```
R2 > V1:=Int(Int(integrand,x),y);
V1 := ∫∫ (-0.0070 y x + 0.7075 y2 - 0.0002 y + 0.7067 x2 - 0.0005 x) /
((0.49942633 x2 - 0.00989572 y x - 0.00070622 x
+ 0.50058989 y2 - 0.00027720 y + 2.9 10-7) (x2 + y2))1/2 dx dy (1.7)
```

```
R2 > V2:=IntegrationTools:-Change(V1,{x=r*cos(theta),y=r*sin
(theta)});
V2 := ∫∫ (r(-0.0070 cos(θ) r sin(θ) + 0.7075 r sin(θ)2 - 0.0002 sin(θ)
+ 0.7067 r cos(θ)2 - 0.0005 cos(θ)) |r cos(θ)2 + r sin(θ)2|) / (1.8)
```

$$\begin{aligned} & \left( (0.49942633 r^2 \cos(\theta)^2 - 0.00989572 \sin(\theta) \cos(\theta) r^2 \right. \\ & \quad \left. - 0.00070622 r \cos(\theta) + 0.50058989 r^2 \sin(\theta)^2 - 0.00027720 r \sin(\theta) \right. \\ & \quad \left. + 2.9 \cdot 10^{-7} \right) r^2 (\sin(\theta)^2 + \cos(\theta)^2)^{1/2} dr d\theta \end{aligned}$$

**R2 > Integrand:=simplify(simplify(op(1,op(1,V2)),symbolic),  
useassumptions) assuming r::positive;**

$$\begin{aligned} \text{Integrand} := & \left( r \left( -0.0070 \cos(\theta) r \sin(\theta) - 0.0002 \sin(\theta) - 0.0008 r \cos(\theta)^2 \right. \right. \\ & \left. \left. + 0.70750 r - 0.00050 \cos(\theta) \right) \right) / \\ & \left( 2.900 \cdot 10^{-7} - 0.00989572 \sin(\theta) \cos(\theta) r^2 - 0.0002772 r \sin(\theta) \right. \\ & \left. - 0.00116356 r^2 \cos(\theta)^2 - 0.000706220 r \cos(\theta) + 0.50058989 r^2 \right)^{1/2} \end{aligned} \quad (1.9)$$

### ▼ Experiment 5.3

**> DGEEnvironment[Manifold]([x1,x2,x3,x4,x5,x6],R6);**  
*Manifold: R6* (2.1)

**R6 > gR6:=convert(LinearAlgebra:-IdentityMatrix(6),DGtensor,[  
"cov\_bas","cov\_bas"],[]);**

$$gR6 := dx1 \otimes dx1 + dx2 \otimes dx2 + dx3 \otimes dx3 + dx4 \otimes dx4 + dx5 \otimes dx5 + dx6 \otimes dx6 \quad (2.2)$$

**R6 > plane:=x1+x4/2-x5/sqrt(2)+x6/2-1/2;**

$$\text{plane} := x1 + \frac{1}{2} x4 - \frac{1}{2} x5 \sqrt{2} + \frac{1}{2} x6 - \frac{1}{2} \quad (2.3)$$

**R6 > f6:=eval(1/(1+exp(-z)),z=plane);**

$$f6 := \frac{1}{1 + e^{-x1 - \frac{1}{2} x4 + \frac{1}{2} x5 \sqrt{2} - \frac{1}{2} x6 + \frac{1}{2}}} \quad (2.4)$$

**R6 > eqns:=[x1 = x\*\*2, x2 = sqrt(2)\*x\*y, x3 = sqrt(2)\*x\*z, x4  
= y\*\*2, x5 = sqrt(2)\*y\*z, x6 = z\*\*2];**

$$\text{eqns} := [x1 = x^2, x2 = \sqrt{2} x y, x3 = \sqrt{2} x z, x4 = y^2, x5 = \sqrt{2} y z, x6 = z^2] \quad (2.5)$$

**R6 > DGEEnvironment[Manifold]([x,y,z],M);**

*Manifold: M* (2.6)

**M > Phi:=Transformation(M,R6,eqns);**

$$\Phi := x1 = x^2, x2 = \sqrt{2} x y, x3 = \sqrt{2} x z, x4 = y^2, x5 = \sqrt{2} y z, x6 = z^2 \quad (2.7)$$

**M > g:=Pullback(Phi,gR6);**

$$g := (4 x^2 + 2 y^2 + 2 z^2) dx \otimes dx + 2 y x dx \otimes dy + 2 z x dx \otimes dz + 2 y x dy \otimes dx \quad (2.8)$$

$$+(2x^2+4y^2+2z^2)dy\otimes dy+2zydy\otimes dz+2zx dz\otimes dx+2zy dz\otimes dy \\ +(2x^2+2y^2+4z^2)dz\otimes dz$$

$$\mathbf{M} > \text{detg} := \text{LinearAlgebra:-Determinant}(\text{convert}(\mathbf{g}, \text{DGMatrix})); \\ \text{detg} := 16x^6 + 48x^4y^2 + 48x^4z^2 + 48x^2y^4 + 96x^2y^2z^2 + 48x^2z^4 + 16y^6 \\ + 48y^4z^2 + 48y^2z^4 + 16z^6 \quad (2.9)$$

$$\mathbf{M} > \mathbf{kv} := \text{KillingVectors}(\mathbf{g}); \\ \mathbf{kv} := \left[ -\frac{z}{2}\partial_y + \frac{y}{2}\partial_z, -\frac{z}{2}\partial_x + \frac{x}{2}\partial_z, -\frac{y}{2}\partial_x + \frac{x}{2}\partial_y \right] \quad (2.10)$$

$$\mathbf{M} > \mathbf{Xs} := \text{evalDG}(2*\mathbf{kv}); \\ \mathbf{Xs} := [-z\partial_y + y\partial_z, -z\partial_x + x\partial_z, -y\partial_x + x\partial_y] \quad (2.11)$$

$$\mathbf{M} > \mathbf{f} := \text{eval}(\mathbf{f6}, \mathbf{eqns}); \\ \mathbf{f} := \frac{1}{1 + e^{-x^2 - \frac{1}{2}y^2 + zy - \frac{1}{2}z^2 + \frac{1}{2}}} \quad (2.12)$$

$$\mathbf{M} > \mathbf{Xabc} := \text{evalDG}(\mathbf{a}*\mathbf{Xs}[1] + \mathbf{b}*\mathbf{Xs}[2] + \mathbf{c}*\mathbf{Xs}[3]); \\ \mathbf{Xabc} := -(bz + cy)\partial_x - (az - cx)\partial_y + (ay + bx)\partial_z \quad (2.13)$$

$$\mathbf{M} > \mathbf{X} := \text{eval}(\mathbf{Xabc}, [\mathbf{a}=0, \mathbf{b}=-1, \mathbf{c}=1]); \\ \mathbf{X} := -(y-z)\partial_x + x\partial_y - x\partial_z \quad (2.14)$$

$$\mathbf{M} > \text{simplify}(\text{LieDerivative}(\mathbf{X}, \mathbf{f})); \\ 0 \quad (2.15)$$

$$\mathbf{M} > \mathbf{Xhat} := \text{eval}(\mathbf{Xabc}, [\mathbf{a}=0.0055, \mathbf{b}=0.7076, \mathbf{c}=-0.7066]); \\ \text{\#The sign for b is corrected, since the second Killing} \\ \text{\#vector varies here by a sign.} \\ \mathbf{Xhat} := (0.7066y - 0.7076z)\partial_x - (0.7066x + 0.0055z)\partial_y + (0.7076x \\ + 0.0055y)\partial_z \quad (2.16)$$

$$\mathbf{M} > \text{innerprod} := \text{proc}(\mathbf{X}, \mathbf{Xhat}, \mathbf{g}) \\ \text{ContractIndices}(\mathbf{g}, \text{evalDG}(\mathbf{X} \& \mathbf{Xhat}), [[1, 1], [2, 2]]); \\ \text{end}; \\ \text{innerprod} := \text{proc}(\mathbf{X}, \mathbf{Xhat}, \mathbf{g}) \\ \text{\`DifferentialGeometry:-Tensor:-ContractIndices}(\mathbf{g}, \\ \text{DifferentialGeometry:-evalDG}(\mathbf{X} \& \mathbf{Xhat}), [[1, 1], [2, 2]]) \\ \text{end proc} \quad (2.17)$$

$$\mathbf{M} > \text{integrand} := \text{innerprod}(\mathbf{X}, \mathbf{Xhat}, \mathbf{g}) / \sqrt{\text{innerprod}(\mathbf{Xhat}, \mathbf{Xhat}, \mathbf{g}) * \\ \text{innerprod}(\mathbf{X}, \mathbf{X}, \mathbf{g})}; \\ \text{integrand} := (2.82840zx^2y - 1.41320y^4 + 2.82840y^3z - 2.82840z^2y^2 \\ + 2.82840yz^3 - 4.2436z^2x^2 - 1.4152z^4 - 2.8284x^4 - 4.2416y^2x^2 \\ - 0.0110x^3z - 0.01100y^2xz - 0.0110xz^3 - 0.01100z^2xy - 0.01100x^3y \\ - 0.01100xy^3) / \quad (2.18)$$

$$\left( \begin{aligned}
 & (-1.999960640 \, z \, x^2 \, y + 3.00141866 \, z^2 \, x^2 + 0.9986276200 \, y^4 \\
 & - 1.999960640 \, y^3 \, z + 2.000083640 \, z^2 \, y^2 - 1.999960640 \, y \, z^3 \\
 & + 1.00145602 \, z^4 + 0.0155452000 \, y^2 \, x \, z + 1.99996264 \, x^4 \\
 & + 2.99859026 \, y^2 \, x^2 + 0.01554520 \, x^3 \, z + 0.01554520 \, x \, z^3 \\
 & + 0.015567200 \, z^2 \, x \, y + 0.01556720 \, x^3 \, y + 0.01556720 \, x \, y^3) \, (4 \, x^4 + 6 \, x^2 \, y^2 \\
 & - 4 \, x^2 \, y \, z + 6 \, x^2 \, z^2 + 2 \, y^4 - 4 \, y^3 \, z + 4 \, y^2 \, z^2 - 4 \, y \, z^3 + 2 \, z^4) \big)^{1/2}
 \end{aligned} \right.$$