Mean Value Coordinates

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Abstract: We derive a generalization of barycentric coordinates which allows a vertex in a planar triangulation to be expressed as a convex combination of its neighbouring vertices. The coordinates are motivated by the Mean Value Theorem for harmonic functions and can be used to simplify and improve methods for parameterization and morphing.

Keywords: barycentric coordinates, harmonic function, mean value theorem, parameterization, morphing.

1. Introduction

Let $v_0, v_1, \ldots, v_k$ be points in the plane with $v_1, \ldots, v_k$ arranged in an anticlockwise ordering around v_0, as in Figure 1. The points $v_1, \ldots, v_k$ form a star-shaped polygon with v_0 in its kernel. Our aim is to study sets of weights $\lambda_1, \ldots, \lambda_k \geq 0$ such that

$$\sum_{i=1}^{k} \lambda_i v_i = v_0, \tag{1.1}$$

$$\sum_{i=1}^{k} \lambda_i = 1. \tag{1.2}$$

Equation (1.1) expresses v_0 as a convex combination of the neighbouring points $v_1, \ldots, v_k$. In the simplest case $k = 3$, the weights λ_1, λ_2, λ_3 are uniquely determined by (1.1) and (1.2) alone; they are the barycentric coordinates of v_0 with respect to the triangle $[v_1, v_2, v_3]$, and they are positive. This motivates calling any set of non-negative weights satisfying (1.1–1.2) for general k, a set of *coordinates* for v_0 with respect to $v_1, \ldots, v_k$.

There has long been an interest in generalizing barycentric coordinates to k-sided polygons with a view to possible multisided extensions of Bézier surfaces; see for example [**8**]. In this setting, one would normally be free to choose $v_1, \ldots, v_k$ to form a convex polygon but would need to allow v_0 to be any point inside the polygon or on the polygon, i.e. on an edge or equal to a vertex.

More recently, the need for such coordinates arose in methods for parameterization [**2**] and morphing [**5**], [**6**] of triangulations. Here the points $v_0, v_1, \ldots, v_k$ will be vertices of a (planar) triangulation and so the point v_0 will never lie on an edge of the polygon formed by $v_1, \ldots, v_k$.

If we require no particular properties of the coordinates, the problem is easily solved. Because v_0 lies in the convex hull of $v_1, \ldots, v_k$, there must exist at least one triangle $T = [v_{i_1}, v_{i_2}, v_{i_3}]$ which contains v_0, and so we can take λ_{i_1}, λ_{i_2}, λ_{i_3} to be the three barycentric coordinates of v_0 with respect to T, and make the remaining coordinates zero. However, these coordinates depend randomly on the choice of triangle. An improvement is to take an average of such coordinates over certain covering triangles, as proposed in [**2**]. The resulting coordinates depend continuously on $v_0, v_1, \ldots, v_k$, yet still not smoothly. The

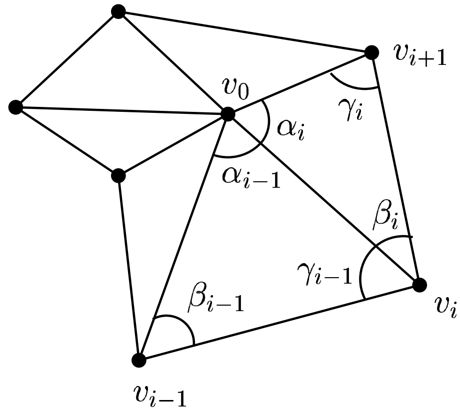

Figure 1. Star-shaped polygon.

main purpose of this paper is to address this latter problem. We derive coordinates which depend (infinitely) smoothly on the data points $v_0, v_1, \ldots, v_k$ through a simple algebraic formula.

Several researchers have studied closely related problems [**9,11,14,15**]. In the special case that the polygon $v_1, \ldots, v_k$ is convex, Wachspress [**14**] found a solution in which the coordinates can be expressed in terms of rational polynomials,

$$\lambda_i = \frac{w_i}{\sum_{j=1}^k w_j}, \qquad w_i = \frac{A(v_{i-1}, v_i, v_{i+1})}{A(v_{i-1}, v_i, v_0)A(v_i, v_{i+1}, v_0)} = \frac{\cot \gamma_{i-1} + \cot \beta_i}{||v_i - v_0||^2}, \tag{1.3}$$

where $A(a, b, c)$ is the signed area of triangle $[a, b, c]$ and γ_{i-1} and β_i are the angles shown in Figure 1. The latter formulation in terms of angles is due to Meyer, Lee, Barr, and Desbrun [**9**]. Of course these coordinates depend smoothly on the data points $v_0, v_1, \ldots, v_k$ and are therefore suitable when the polygon is convex. However, for star-shaped polygons the coordinate λ_i in (1.3) can be negative, and, in fact, will be so precisely when $\gamma_{i-1} + \beta_i > \pi$.

Another set of previously found weights can be expressed as

$$\lambda_i = \frac{w_i}{\sum_{j=1}^k w_j}, \qquad w_i = \cot \beta_{i-1} + \cot \gamma_i. \tag{1.4}$$

These weights arise from the standard piecewise linear finite element approximation to the Laplace equation and appear in several books on numerical analysis, e.g. [**7**], and probably go back to the work of Courant. They have since been used in the computer graphics literature [**10**], [**1**]. However, for our purposes these weights suffer from the same problem as the last ones, namely that they might be negative. The weight λ_i is negative if and only if $\beta_{i-1} + \gamma_i > \pi$.

Another possible set of coordinates might be Sibson's natural neighbour coordinates [**11**], if we treated the points $v_1, \ldots, v_k$ as a set of scattered data points. However, despite various other good properties, Sibson's coordinates, like those of [**2**], suffer from being defined piecewise, and have in general only C^1 dependence on the point v_0. Moreover, several of Sibson's coordinates might be zero, since the only non-zero ones would correspond to Voronoi neighbours of v_0.

2. Mean Value Coordinates

We now describe a set of coordinates which satisfy all the properties we would like. Let α_i, $0 < \alpha_i < \pi$, be the angle at v_0 in the triangle $[v_0, v_i, v_{i+1}]$, defined cyclically; see Figure 1.

Proposition 1. *The weights*

$$\lambda_i = \frac{w_i}{\sum_{j=1}^k w_j}, \qquad w_i = \frac{\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)}{||v_i - v_0||}, \tag{2.1}$$

are coordinates for v_0 with respect to $v_1, \dots, v_k$.

As will be explained in Section 3, these weights can be derived from an application of the mean value theorem for harmonic functions, which suggests calling them *mean value coordinates*. They obviously depend smoothly on the points $v_0, v_1, \dots, v_k$.

Proof: Since $0 < \alpha_i < \pi$, we see that $\tan(\alpha_i/2)$ is defined and positive, and therefore λ_i is well-defined and positive for $i = 1, \dots, k$, and by definition the λ_i sum to one. It remains to prove (1.1). From (2.1), equation (1.1) is equivalent to

$$\sum_{i=1}^k w_i(v_i - v_0) = 0. \tag{2.2}$$

We next use polar coordinates, centred at v_0, so that

$$v_i = v_0 + r_i(\cos\theta_i, \sin\theta_i).$$

Then we have

$$\frac{v_i - v_0}{||v_i - v_0||} = (\cos\theta_i, \sin\theta_i), \qquad \text{and} \qquad \alpha_i = \theta_{i+1} - \theta_i,$$

and equation (2.2) becomes

$$\sum_{i=1}^k (\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2))(\cos\theta_i, \sin\theta_i) = 0,$$

or equivalently

$$\sum_{i=1}^k \tan(\alpha_i/2)((\cos\theta_i, \sin\theta_i) + (\cos\theta_{i+1}, \sin\theta_{i+1})) = 0. \tag{2.3}$$

To establish this latter identity, observe that

$$\begin{aligned} 0 &= \int_0^{2\pi} (\cos\theta, \sin\theta)\, d\theta \\ &= \sum_{i=1}^k \int_{\theta_i}^{\theta_{i+1}} (\cos\theta, \sin\theta)\, d\theta \\ &= \sum_{i=1}^k \int_{\theta_i}^{\theta_{i+1}} \frac{\sin(\theta_{i+1} - \theta)}{\sin\alpha_i}(\cos\theta_i, \sin\theta_i) + \frac{\sin(\theta - \theta_i)}{\sin\alpha_i}(\cos\theta_{i+1}, \sin\theta_{i+1})\, d\theta, \end{aligned} \tag{2.4}$$

the last line following from the addition formula for sines. Since also

$$\int_{\theta_i}^{\theta_{i+1}} \sin(\theta_{i+1} - \theta)\, d\theta = \int_{\theta_i}^{\theta_{i+1}} \sin(\theta - \theta_i)\, d\theta = 1 - \cos\alpha_i, \tag{2.5}$$

and

$$\tan(\alpha_i/2) = \frac{1 - \cos\alpha_i}{\sin\alpha_i}, \tag{2.6}$$

equation (2.4) reduces to equation (2.3). ■

Not only are these coordinates positive, but we can bound them away from zero. This might be useful when considering the conditioning of the linear systems used in [**2,5,6**]. If $L^\star = \max_i ||v_i - v_0||$, $L_\star = \min_i ||v_i - v_0||$ and $\alpha^\star = \max_i \alpha_i$, $\alpha_\star = \min_i \alpha_i$, then from (2.1) we have

$$\frac{2\tan(\alpha_\star/2)}{L^\star} \le w_i \le \frac{2\tan(\alpha^\star/2)}{L_\star},$$

and

$$\frac{1}{Ck} \le \lambda_i \le \frac{C}{k}, \tag{2.7}$$

where

$$C = \frac{L^\star \tan(\alpha^\star/2)}{L_\star \tan(\alpha_\star/2)} \ge 1.$$

The inequality (2.7) becomes an equality when $C = 1$ which occurs when $v_1, \ldots, v_k$ is a regular polygon and v_0 is its centre.

3. Motivation

The motivation behind the coordinates was an attempt to approximate harmonic maps by piecewise linear maps over triangulations, in such a way that injectivity is preserved. Recall that a C^2 function u defined over a planar region Ω is *harmonic* if it satisfies the Laplace equation

$$u_{xx} + u_{yy} = 0.$$

Suppose we want to approximate the solution u with respect to Dirichlet boundary conditions, $u|_{\partial\Omega} = u_0$, by a piecewise linear function $u_{\mathcal{T}}$ over some triangulation $\mathcal{T}$ of Ω. Thus $u_{\mathcal{T}}$ will be an element of the spline space $S_1^0(\mathcal{T})$. The finite element approach to this problem is to take $u_{\mathcal{T}}$ to be the unique element of $S_1^0(\mathcal{T})$ which minimizes

$$\int_\Omega |\nabla u_{\mathcal{T}}|^2 dx$$

subject to the boundary conditions. As is well known, this leads to a sparse linear system in the values of $u_{\mathcal{T}}$ at the interior vertices of the triangulation $\mathcal{T}$. To be precise, suppose

v_0 in Figure 1 is an interior vertex of the triangulation. As described in several books on numerical analysis, the equation associated with v_0 can be expressed as

$$u_{\mathcal{T}}(v_0) = \sum_{i=1}^{k} \lambda_i u_{\mathcal{T}}(v_i), \tag{3.1}$$

where λ_i is given by equation (1.4).

Suppose now that we use this method component-wise to approximate a *harmonic map* $\phi : \Omega \to \mathbb{R}^2$, that is a map $\phi = (\phi_1, \phi_2)$ for which both ϕ_1 and ϕ_2 are harmonic functions, by a piecewise linear map $\phi_{\mathcal{T}} : \Omega \to \mathbb{R}^2$. We would like the map $\phi_{\mathcal{T}}$ to be injective and a sufficient condition has been derived in [**13**] and [**4**]: if $\phi_{\mathcal{T}}$ is a *convex combination map*, i.e. at every interior vertex v_0 of $\mathcal{T}$, we have

$$\phi_{\mathcal{T}}(v_0) = \sum_{i=1}^{k} \lambda_i \phi_{\mathcal{T}}(v_i), \tag{3.2}$$

for some *positive* weights $\lambda_1, \ldots, \lambda_k$ which sum to one, and if $\phi_{\mathcal{T}}$ maps the boundary $\partial\mathcal{T}$ homeomorphically to the boundary of a convex region, then $\phi_{\mathcal{T}}$ is one-to-one.

From this point of view, the standard finite element approximation $\phi_{\mathcal{T}}$ has a drawback: the theory of [**13**] and [**4**] cannot be applied. The map $\phi_{\mathcal{T}}$ will not in general be a convex combination map because the weights λ_i in (1.4) can be negative which can lead to "foldover" in the map (an example is given in [**3**]).

This motivates, if possible, approximating a harmonic map ϕ by a convex combination map $\phi_{\mathcal{T}}$, i.e., one with positive weights in (3.2). Consider the following alternative discretization of a harmonic function u. Recall that harmonic functions satisfy the mean value theorem. The mean value theorem comes in two forms.

Circumferential Mean Value Theorem. *For a disc $B = B(v_0, r) \subset \Omega$ with boundary Γ,*

$$u(v_0) = \frac{1}{2\pi r} \int_{\Gamma} u(v)\, ds.$$

Solid Mean Value Theorem.

$$u(v_0) = \frac{1}{\pi r^2} \int_{B} u(v)\, dx\, dy.$$

This suggests finding the element $u_{\mathcal{T}}$ of $S_1^0(\mathcal{T})$ which satisfies one of the two mean value theorems locally at every interior vertex v_0 of $\mathcal{T}$. We will concentrate on the first version and demand that

$$u_{\mathcal{T}}(v_0) = \frac{1}{2\pi r} \int_{\Gamma} u_{\mathcal{T}}(v)\, ds, \tag{3.3}$$

for r sufficiently small that the disc $B(v_0, r)$ is entirely contained in the union of the triangles containing v_0; see Figure 2. It turns out that this equation can be expressed in the form of (3.1) where the weights λ_i are those of (2.1), *independent* of the choice of r.

To see this, consider the triangle $T_i = [v_0, v_i, v_{i+1}]$ in Figure 2 and let Γ_i be the part of Γ contained in T_i.

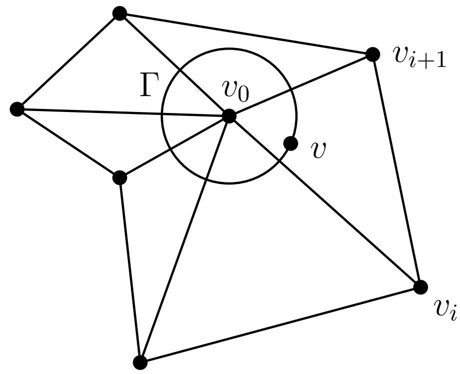

Figure 2. Circle for the Mean Value Theorem.

Lemma 1. *If $f : T_i \to \mathbb{R}$ is any linear function then*

$$\int_{\Gamma_i} f(v)\,ds = r\alpha_i f(v_0) + r^2 \tan(\alpha_i/2)\left(\frac{f(v_i)-f(v_0)}{||v_i - v_0||} + \frac{f(v_{i+1})-f(v_0)}{||v_{i+1}-v_0||}\right). \tag{3.4}$$

Proof: We represent any point $v \in \Gamma_i$ in polar coordinates with respect to v_0, i.e.,

$$v = v_0 + r(\cos\theta, \sin\theta),$$

and we let

$$v_j = v_0 + r_j(\cos\theta_j, \sin\theta_j), \qquad j = i, i+1.$$

Then

$$\int_{\Gamma_i} f(v)\,ds = r\int_{\theta_i}^{\theta_{i+1}} f(v)\,d\theta. \tag{3.5}$$

Since f is linear, and using barycentric coordinates, we have

$$f(v) = f(v_0) + \lambda_1(f(v_i) - f(v_0)) + \lambda_2(f(v_{i+1}) - f(v_0)), \tag{3.6}$$

where $\lambda_1 = A_1/A$ and $\lambda_2 = A_2/A$ and A_1 and A_2 are the areas of the two triangles $[v_0, v, v_{i+1}]$, $[v_0, v_i, v]$ and A the area of the the whole triangle T_i. Using trigonometry we have

$$A = \frac{1}{2}r_i r_{i+1}\sin\alpha_i, \qquad A_1 = \frac{1}{2}r r_{i+1}\sin(\theta_{i+1}-\theta), \qquad A_2 = \frac{1}{2}r r_i \sin(\theta - \theta_i),$$

and

$$\lambda_1 = \frac{r\sin(\theta_{i+1}-\theta)}{r_i\sin\alpha_i}, \qquad \lambda_2 = \frac{r\sin(\theta-\theta_i)}{r_{i+1}\sin\alpha_i}. \tag{3.7}$$

It follows that the substitution of the expression for $f(v)$ in (3.6) into equation (3.5) and applying the identities (2.5–2.6) leads to equation (3.4). ■

It is now a simple matter to show that equation (3.3) reduces to the linear combination (3.1) with the coordinates λ_i of (2.1).

Proposition 2. *Suppose the piecewise linear function $u_\mathcal{T} : \Omega \to \mathbb{R}$ satisfies the local mean value theorem, i.e., for each interior vertex v_0, it satisfies equation (3.3) for some $r > 0$ suitably small. Then $u_\mathcal{T}(v_0)$ is given by the convex combination in (3.1) with the weights λ_i of (2.1).*

Proof: Equation (3.3) can be written as

$$u_\mathcal{T}(v_0) = \frac{1}{2\pi r} \sum_{i=1}^{k} \int_{\Gamma_i} u_\mathcal{T}(v)\, ds,$$

which, after applying Lemma 1 to $u_\mathcal{T}$, reduces to

$$0 = \sum_{i=1}^{k} \tan(\alpha_i/2) \left(\frac{u_\mathcal{T}(v_i) - u_\mathcal{T}(v_0)}{||v_i - v_0||} + \frac{u_\mathcal{T}(v_{i+1}) - u_\mathcal{T}(v_0)}{||v_{i+1} - v_0||} \right),$$

which is equivalent to equation (3.1) with the weights of (2.1). ■

We now notice that Proposition 1 follows from Proposition 2 due to the simple observation that linear bivariate functions are trivially harmonic.

It is not difficult to show that Proposition 2 remains true when the solid mean value theorem is used instead of the circumferential one, the main point being that λ_1 and λ_2 in equation (3.7) are linear in r.

We remark finally that it is well known that the standard finite element approximation $u_\mathcal{T}$ converges to u in various norms as the mesh size of the triangulation $\mathcal{T}$ tends to zero, under certain conditions on the angles of the triangles. Initial numerical tests suggest that the mean value 'approximation' does not converge to u in general. An interesting question for future research is whether it is in fact possible to approximate a harmonic map by a convex combination map over an arbitrary triangulation with sufficiently small mesh size.

4. Applications

In the parameterization method of [**2**], the triangulation is a spatial one, so that the vertices $v_0, \ldots, v_k$ are points in $\mathbb{R}^3$. However, the mean value coordinates can be applied directly to the triangulation; we simply compute the coordinates λ_i of equation (2.1) directly from the vertices $v_0, \ldots, v_k \in \mathbb{R}^3$. The numerical example of Figure 3 shows the result of parameterizing a triangle mesh (3a) and mapping a rectangular grid in the parameter plane back onto the mesh. The three parameterizations used are uniform (3b) (i.e. Tutte's embedding), shape-preserving (3c), and mean value (3d). In this example the mean value parameterization looks at least as good as the 'shape-preserving' one of [**2**]. In addition, the mean value coordinates are faster to compute than the shape-preserving coordinates, and have the theoretical advantage that the resulting parameterization will depend smoothly on the vertices of the triangulation.

Mean value coordinates can also be used to morph pairs of compatible planar triangulations, adapting the method of [**5**]. Such a morph will depend smoothly on the vertices of the two triangulations. Surazhsky and Gotsman found recently that mean value morphs are visually smoother than previous morphs; see [**12**] for details.

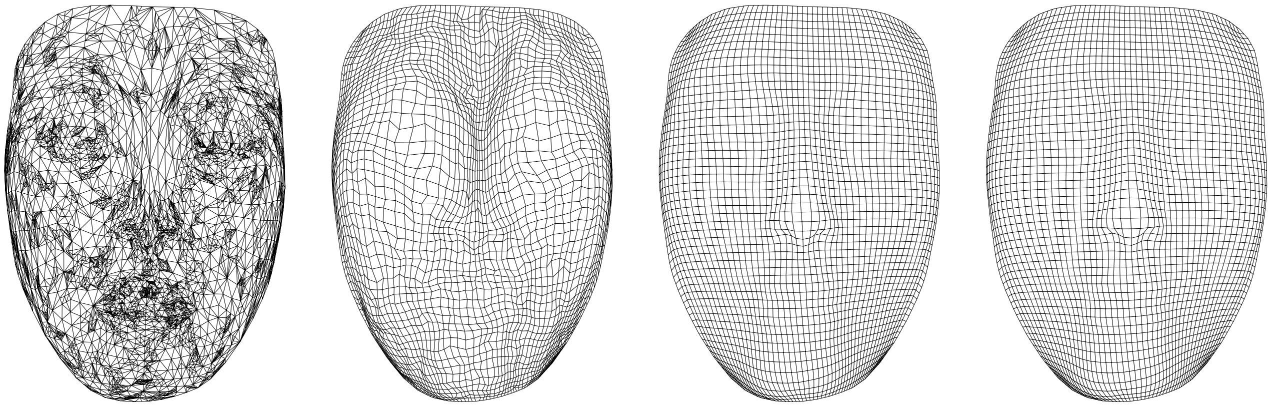

Figure 3. Comparisons from left to right:
(3a) Triangulation, (3b) Tutte, (3c) shape-preserving, (3d) mean value

5. Final remarks

When $k = 3$ the mean value coordinates, like Wachspress's coordinates, are equal to the three barycentric coordinates, due to uniqueness. When $k = 4$ the mean value coordinates and Wachspress coordinates are different in general. For example, when the points v_1, v_2, v_3, v_4 form a rectangle, the Wachspress coordinates are bilinear, while the mean value coordinates are not.

If the points $v_1, \ldots, v_k$ form a *convex* polygon, the mean value coordinates are defined for all points v_0 inside the polygon, but due to the use of the angles α_i in formula (2.1), it is not obvious whether the coordinates can be extended to the polygon itself. Though this paper was not intended to deal with this issue, it would be important if these coordinates were to be used for generalizing Bezier surfaces, as in [**8**]. For Wachspress's coordinates (1.3), this is not a problem because by multiplying by the product of all areas $A(v_j, v_{j+1}, v_0)$, they have the well-known equivalent expression

$$\lambda_i = \frac{w_i}{\sum_{j=1}^k w_j}, \qquad w_i = A(v_{i-1}, v_i, v_{i+1}) \prod_{j \neq i-1, i} A(v_j, v_{j+1}, v_0).$$

It turns out that the mean value coordinates can also be continuously extended to the polygon itself. Moreover, like Wachspress's coordinates, they are linear along each edge of the polygon and have the Lagrange property at the vertices: if $v_0 = v_i$ then $\lambda_i = 1$ and $\lambda_j = 0$ for $j \neq i$. A proof of this as well as other properties of these coordinates will appear in a forthcoming paper.

Acknowledgement. I wish to thank Mathieu Desbrun for supplying the data set for the numerical example and the referees for their comments. This work was partly supported by the European Union project MINGLE, contract num. HPRN-CT-1999-00117.

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