# Linear Algebra Notes

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## Contents

# 1 Chapter 1

## 1.1 § 1

### 1.1.1 Notes

A generic vector space $V$ is not a field because there is no definition of $v^{-1}$ for some $v \in V$, fulfilling not the definition of a field.

1. **Pg. 4 Proof of** $(-1)v = v$
$$\begin{aligned}(-1)v + v &= (-1)v + 1 \cdot v \\ &= (-1+1)v \\ &= v + (-v)\end{aligned}$$
Thus, $(-1)v = -v$.

2. **Pg. 6 Proof of SP 3**
$$\begin{aligned}(xA) \cdot B &= \sum_{i=1}^{n} (xa_i)b_i \\ &= \sum_{i=1}^{n} x(a_i b_i) \\ &= x \sum_{i=1}^{n} a_i b_i \\ &= x(A \cdot B) \\ A \cdot (xB) &= \sum_{i=1}^{n} a_i(xb_i) \\ &= \sum_{i=1}^{n} x(a_i b_i) \\ &= x \sum_{i=1}^{n} a_i b_i \\ &= x(A \cdot B)\end{aligned}$$

3. **Pg. 7**
Upper one:
$$\begin{aligned}(A+B)^2 &= (A+B) \cdot (A+B) \\ &= (A+B) \cdot A + (A+B) \cdot B && \text{Use SP 2} \\ &= A^2 + B \cdot A + A \cdot B + B^2 && \text{Use SP 1}\end{aligned}$$
Bottom one: Since $K$ is a field, all **VS** s regarding summation or product of functions are actually closed on $K$. By applying field axioms, $V$ is then a vector space over $K$.

4. **Pg. 9**
Let $a_1 = (u_1 + w_1), a_2 = (u_2 + w_2)$. Both of them $\in (U + W)$.
Since $U, W$ are subspaces of $V$, $U, W \in V$. Thus, $a_1, a_2 \in V$ as $u_1, w_1, u_2, w_2 \in V$, moreover, $(U + W) \subset V$.
$a_1 + a_2 = (u_1 + u_2) + (w_1 + w_2) \in (U + W)$
$ca_1 = c(u_1 + w_1) = (cu_1) + (cw_1) \in (U + W)$
Since $O \in U$ and $O \in W$, $O = O + O \in (U + W)$. Thus, $(U + W)$ is a subspace of $V$.

### 1.1.2 Exercises

1. **Exercise 1** Let $v \in V$, $c[v+(-v)] = cv + c(-v) = cv + (-c)v = v \cdot 0 = v \cdot (1-1) = v + (-v) = O$

2. **Exercise 2** Since $c \neq 0$

$$\begin{aligned} O &= cv + [-(cv)] \\ cv &= cv + [-(cv)] \\ O &= -(cv) \\ \frac{-1}{c} \cdot O &= (-c)v \cdot \frac{-1}{c} \\ \frac{-1}{c} \cdot (v-v) &= v \\ \frac{-1}{c} \cdot v + \frac{1}{c} \cdot v &= v \\ v \cdot (1-1) &= v \\ v - v &= v \\ O &= v \end{aligned}$$

3. **Exercise 3**

   $\forall g \in V, (g+f)(x) = g(x) + f(x) = f(x) + g(x) = (f+g)(x) \Rightarrow g + f = f + g$.
   If $O + u = u$, $(O+u)(x) = O(x) + u(x) = u(x)$. Therefore, $O(x) = 0$.

4. **Exercise 4**

$$\begin{aligned} v + w &= O \\ v + w &= v + (-v) \\ w &= -v \end{aligned}$$

5. **Exercise 5**

$$\begin{aligned} v + w &= v \\ v + (-v) + w &= v + (-v) \\ O + w &= O \end{aligned}$$

   Since $\forall u, O + u = u$, we have $w = O$.

6. **Exercise 6**

   Let $W = \{B | B \cdot A_1 = O \text{ and } B \cdot A_2 = O\}$. Specifically, it is clear that $O \in W$ as $O \cdot A = \sum_{i=1}^{n} b_i a_i = \sum_{i=1}^{n} 0 \times a_i = 0$.
   Let $v_1, v_2 \in W$ such that $v_1 \cdot A_1 = 0$, $v_1 \cdot A_2 = 0$, $v_2 \cdot A_1 = 0$, $v_2 \cdot A_2 = 0$. Thus,

$$\begin{aligned} (v_1 + v_2) \cdot A_1 &= v_1 \cdot A_1 + v_2 \cdot A_1 \\ &= O + O \\ &= O \\ [c(v_1 + v_2)] \cdot A_1 &= (cv_1 + cv_2) \cdot A_1 \\ &= (cv_1) \cdot A_1 + (cv_2) \cdot A_1 \\ &= c(v_1 \cdot A_1 + v_2 \cdot A_1) \\ &= cO \\ &= O \end{aligned}$$

   . It is easy to show for $A_2$ then. Therefore, $(v_1 + v_2) \in W$.

7. **Exercise 7** Same to apply as Exercise 6.

8. **Exercise 8**

   Name the set as $W$.

(a) Proof
$v_1 + v_2 = (x_1 + x_2, y_1 + y_2), x_1 + x_2 = y_1 + y_2 \Rightarrow (v_1 + v_2) \in W$
$cv = (cx, cy), cx = cy \Rightarrow cv \in W$
$O = (0, 0) \in W$
(b) Proof See Part (a).
(c) Proof Same technique as in Part (a).

9. **Exercise 9** See Exercise 8.

10. **Exercise 10**

For $U \cap W$, let $v_1, v_2 \in U \cap W$. Since $v_1, v_2 \in U$ and $U$ is a subspace, $v_1 + v_2 \in U$. In same way, we can see that $v_1 + v_2 \in W$. Thus, $v_1 + v_2 \in U \cap W$.
Since $v_1 \in U$, $cv_1 \in U$. Also, it shows $cv_1 \in W$ in the same way. Thus, $cv_1 \in U \cap W$. Because $U, W$ are subspaces, $O \in U$ and $O \in W$. Thus, $O \in U \cap W$. Therefore, $U \cap W$ is a subspace.
Refer to the note part for proof for $U + W$.

11. **Exercise 11** Since $L$ is a field, **VS1, VS3, VS4, VS8** are established under field axioms, and multiplication and addition are closed in $L$. For **VS5, VS6, VS7**, they are all valid as $K \subset L$. $O$ is simply 0, and $1 \cdot u = u$ is established in $L$.

12. **Exercise 12**

For $x, y \in K$, we have
$x + y = a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2} = (a_1 + a_2) + (b_1 + b_2)\sqrt{2}$. Since $a_1, b_1, a_2, b_2 \in \mathbb{Q}$, $(a_1 + a_2), (b_1 + b_2) \in \mathbb{Q}$. Thus, $x + y \in K$.
$xy = (a_1a_2 + 2b_1b_2) + (a_2b_1 + a_1b_2) \times \sqrt{2}$. Since $a_1, b_1, a_2, b_2 \in \mathbb{Q}$, $(a_1a_2 + 2b_1b_2), (a_2b_1 + a_1b_2) \in \mathbb{Q}$. Thus, $x + y \in K$.
$-x = -a + -b\sqrt{2}$. Since $a, b \in \mathbb{Q}$, $-a, -b \in \mathbb{Q}$. Thus, $-x \in K$.
If $a + b\sqrt{2} \neq 0$, $a, b \neq 0$, and $a - b\sqrt{2} \neq 0$. Thus, $x^{-1} = \frac{1}{a+b\sqrt{2}} = \frac{a-b\sqrt{2}}{a^2-2b^2} = \frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2}$. It is easy to see that **new** $a, b \in \mathbb{Q}$ as $a, b \in \mathbb{Q}$. Thus, $x^{-1} \in K$. Specifically, if $a = b = 0$, $0 \in \mathbb{Q}$. If $a = 1, b = 0$, $1 \in \mathbb{Q}$. Thus, $K$ is a field.

13. **Exercise 13** Same technique as Exercise 12.

14. **Exercise 14** Same technique as Exercise 12.

## 1.2 § 2

### 1.2.1 Notes

Another quite helpful equivalent of definition of linear independence is that (stated following without loss of generality)

$$\forall a_1 \neq 0, a_1 v_1 \neq \sum_{i=2}^{n} a_i$$

Here is the *proof* of equivalence between above statement and definition of linear independence.
Since $a_1 \neq 0$,

$$v_1 \neq \sum_{i=2}^{n} \frac{a_i}{a_1} v_i$$

$$O \neq -v_1 + \sum_{i=2}^{n} \frac{a_i}{a_1} v_i$$

$$\lambda O \neq (-\lambda)v_1 + \sum_{i=2}^{n} \frac{\lambda a_i}{a_1} v_i \quad \lambda \in K \text{ and } \lambda \neq 0$$

$$O \neq (-\lambda)v_1 + \sum_{i=2}^{n} \frac{\lambda a_i}{a_1} v_i \quad \lambda \in K \text{ and } \lambda \neq 0$$

$\lambda$ and $a_i$ could be arbitrary, thus from above we could conclude that $a'_1 v_1 \neq \sum_{i=2}^{n} a'_i$ if and only if all $a' = 0$, which is the definition of linear independence.
Also, another point that worth paying attention to is that generators could be **linear dependent**. This is true because you could put arbitrary vectors at the end of a basis of a vector space and just set coefficients for these extraneous vectors when it is producing new linear combinations.

### 1.2.2 Exercises

1. **Exercise 1** Using result from **Exercise 4**, easy to prove.
2. **Exercise 2**
   (a) $(1, -1)$
   (b) $(\frac{1}{2}, \frac{3}{2})$
   (c) $(1, 1)$
   (d) $(3, 2)$
3. **Exercise 3**
   (a) $(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$
   (b) $(1, 0, 1)$
   (c) $(\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3})$
4. **Exercise 4**

   Following set of equations is an equivalent of $x(a, b) + y(c, d) = O$,

$$\begin{aligned} ax + cy &= 0 \quad (1) \\ bx + dy &= 0 \quad (2) \end{aligned}$$

$$\begin{aligned} (1) \times d - (2) \times c \Rightarrow (ad - cb)x + cdy - cdy &= 0 \\ (ad - cb)x &= 0 \end{aligned}$$

   For $ad - cb \neq 0$ part, clearly we shall see that $x = 0$ as $(ad - cb)x = 0$. Plugging $x$ back to (1), we get $y = 0$. Thus, two vectors are linear independent.
   For $ad - cb = 0$ part, we need to prove that $x(a, b) + y(c, d) = O$ has solution other than $x = y = 0$.
   First, suppose $a, b, c, d \neq 0$. Since $ad - cb = 0$, $x \in \mathbb{R}$. By applying technique, we could also show $y \in \mathbb{R}$. Thus, $(a, b)$, $(c, d)$ are linear independent.
   If $a, b, c, d \neq 0$ does **NOT** hold. Without lose of generality (for all the possibilities, $a, d$ and $c.b$ are interchangeable), consider following scenarios in a $xy$ -plane,

   (a) $a = 0, c = 0$
   If $a = c = 0$, $x, y \in \mathbb{R}$ in (1). Because the (2) is a line in the plane, there must exist some $x, y \neq 0$.
   (b) $a = 0, b = 0, c = 0$
   Same argument as above, despite the line represented by (2) is a little bit peculiar (it is $y = 0$).
   (c) $a = 0, d = 0, c = 0$
   Same argument as the first, despite the line represented by (2) is a little bit peculiar (it is $x = 0$).
   (d) $a = 0, d = 0, b = 0, c = 0$
   Both $(1), (2)$ represent the whole plane, thus, $x, y \in \mathbb{R}$.
5. **Exercise 5,6**

   To correctly understand how could functions be elements(vectors) in vector space, we need to understand that function $f : S \to K$ is essentially a set of pairs $(s, k), \forall s \in S$. Functions have scalar multiplication and

addition defined.
$f+g$ is defined as $\{(s, f(s)+g(s)) | s \in S\}$, and $cf, c \in K$ is defined as $\{(s, c \cdot f(s)) | s \in S\}$.
It is easy to verify that $V$ of every $f : S \to K$ is a vector space over $K$. Particularly, $O$ for $V$ is $\{(s, 0) | s \in S\}$.
So like other vector spaces, linear dependence is **about**

$$f_{sum} = \sum_{i=1}^{n} a_i f_i = O$$

Since right-hand-side of the equation is $\{(s, 0) | s \in S\}$, we can say that $\forall v \in V, f_s um(s) = 0$. This is useful in solving problems in **Exercise 5** and **Exercise 6**.
For example, we need to show that $f(s) = 1$ and $g(s) = t$ are linear independent. This means that we need to consider following equation,

$$af + bg = O$$

which is an equivalent of

$$\forall t, a + bt = 0$$

Above conversion is quite helpful since we could put in arbitrary $t$ and the equation should hold. Thus, we could put in particular values of $t$ to **construct** set of equations to show that $a = b = 0$. For example, here we plug in $t = 0$, then $a = 0$, and if we plug back $a = 0$ into original equation with $t = 0$ again, $b = 0$.
This method could be used throughout **Exercise 5,6**.

6. **Exercise 7** $(3, 5)$

7. **Exercise 8** ***Calculus involved, not doing now.***

8. **Exercise 9**

$$\sum_{i=1}^{r} [a_i \cdot (A_i \cdot \sum_{j=i+1}^{r} A_j)] = O \qquad \text{All vectors are mutually perpendicular}$$
$$= \sum_{i=1}^{r} [(a_i \cdot A_i) \cdot \sum_{j=i+1}^{r} A_j]$$

   Since $\forall A \in \{A_i\}, A \neq O$, it is only possible that every $a$ is 0. Thus, $A_i$ are linearly independent.

9. **Exercise 10**

   Since $v, w$ are linear dependent, for

$$nv + mw = O$$

   at least one of $n, m \neq 0$. Consider following scenarios, we can see that there would be $a = 0$ or $a = -\frac{n}{m}$.

   (a) $n = 0, m \neq 0 \Rightarrow w = O$
   (b) $n \neq 0, m = 0 \Rightarrow v = O$. This contradicts with $v \neq O$ in problem. Thus, this is impossible.
   (c) $n \neq 0, m \neq 0 \Rightarrow w = \frac{-n}{m} v$

## 1.3 § 3

### 1.3.1 Notes

This subsection comprises a lot of concise proofs. But in conclusion, we need to know that

| | |
|---|---|
| Basis $\Leftrightarrow$ Maximal linear independent vector set | proof at **Theorem3.1** |
| Basis $\Leftrightarrow$ Maximal linear indpendent vector set $\Rightarrow$ Generators | proof at **Theorem2.2** |
| Generators $\nRightarrow$ Basis | Generators are not always linear independent. |

Thus, all possible bases of a vector space $V$ are of one and only one possible number of elements, which is equal to the one of maximal independent vector set.

### 1.4 § 4

#### 1.4.1 Notes

*Proof* for
$$\dim(U \times W) = \dim U + \dim W$$
Because $\forall u \in (U \times W), (O_u + O_w) + u = u + (O_u + O_w) = u$. Thus, by definition, $O = (O_u, O_w)$.
Let $A = \{u_i\}$ be a basis of $U$ and $B = \{w_i\}$ be a basis of $W$. Note the dimension of $U, W$ as $n, m$ respectively. Let
$$C = \{(u_i, 0)|u_i \in A\} \cup \{(0, w_i)|w_i \in B\}$$
Since there would be no intersection between two sets being union above, the number of elements in $C$ is $n + m$.
If we could show that $C$ is a basis of $U \times W$, then we could show the original statement.
First we need to show that all elements in $C$ is linear independent. This means $a_i \in K, c_i \in C$
$$\sum_{i=1}^{n+m} a_i c_i = O$$
if and only if all the $a_i = 0$.
Because multiplication by scalar and addition for $U \times W$ is defined componentwise, we shall see that (if we keep the "order" of elements in $C$ as $A$ and $B$ are merged)
$$\sum_{i=1}^{n} a_i u_i = O_u$$
$$\sum_{i=n+1}^{n+m} a_i w_i = O_w$$
Since both $A$ and $B$ are basis of $U$ and $W$ respectively, all the $a_i$ should be 0.
Now, we need to show that $C$ generates $U \times W$. Since $A$ and $B$ are basis of $U$ and $W$ respectively,
$$\forall (a, b) \in (U \times W), \exists f_i, g_i \in K : \sum_{i=1}^{n} f_i u_i = a \text{ and } \sum_{i=1}^{m} g_i w_i = b$$
Thus, by setting set of scalar for "order"-kept $C$ as $\{f_i\} \cup \{g_i\}$, it is easy to see that it generates $U \times W$.
Therefore, we see that
$$\dim(U \times W) = \dim U + \dim W$$
and
$$\{(u_i, 0)|u_i \in A\} \cup \{(0, w_i)|w_i \in B\}$$
is a basis for $U \times W$.

#### 1.4.2 Exercises

1. **Exercise 1**

   For the first part, we need to show that $\forall v \in V, \exists$ unique $u \in U, w \in W : v = u + w$. Since $(2, 1)$ and $(0, 1)$ are linear independent, they are a basis of $V = \mathbb{R}^2$. This means
   $$\forall v \in V, \exists \text{ unique } a, b \in K : v = a \cdot (2, 1) + b \cdot (0, 1)$$
   Thus, just set $u = a \cdot (2, 1)$ and $w = b \cdot (0, 1)$, and we have proved it.
   It is same for $(2, 1)$ and $(1, 1)$.

2. **Exercise 2**

   Since $(1,0,0),(1,1,0),(0,1,1)$ are linear independent, we obtain that

   $$\forall v \in V, \exists \text{ unique } a,b,c \in K : v = a\cdot(1,0,0)+b\cdot(1,1,0)+c\cdot(0,1,1)$$

   Set $u = a\cdot(1,0,0)$ and $w = b\cdot(1,1,0)+c\cdot(0,1,1)$, it would be proved.

3. **Exercise 3** According to argument provided here, $\forall c \in K, cA \neq B$ means that $A, B$ are linear independent. Also, according to **Theorem 3.4**, they are a basis of $\mathbb{R}^2$.
   Based on the similar argument in **Exercise 1**, second part could be proved.

4. **Exercise 4** See notes

# 2 Chapter 2

## 2.1 § 1

### 2.1.1 Exercises

1. **Exercise 1** Skip

2. **Exercise 2** Skip

3. **Exercise 3** Skip

4. **Exercise 4** Skip

5. **Exercise 5** Let $C =^t (A+B) = (c_{ij})$. Then, $c_{ij} = (a_{ij}+b_{ij})' = a_{ji}+b_{ji}$. Thus, $C =^t A +^t B$.

6. **Exercise 6** Let $B =^t (cA)$. Then, $b_{ij} = ca_{ji}$. Since ${}^tA = (a'_{ij}) = (a_{ji}) = A$, $B = c^tA$.

7. **Exercise 7** No difference.

8. **Exercise 8** Skip

9. **Exercise 9** Skip

10. **Exercise 10** Let $B = A +^t A = (b_{ij}) = (a_{ij}+a_{ji})$. Since, $b_{ij} = a_{ij}+a_{ji} = a_{ji}+b_{ij} = b_{ji}$, $B$ is symmetric.

11. **Exercise 11** Skip

12. **Exercise 12** Skip

13. **Exercise 13**

    **For followings, we mean ones in *Exercises on Dimension* section.**
    Followings are linear independent.

    $$U_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

    $$U_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

    $$U_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

    $$U_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

    Apply $a\cdot U_1 + b\cdot U_2 + c\cdot U_3 + d\cdot U_4 = O$ to verify it. Because it generates the matrix vector space $Mat_{2\times 2}K$ over K (For every $v \in Mat_{2\times 2}K$, simply let $a,b,c,d$ be $v$'s components) and $\{U_i\}$ are linear independent, $\{U_i\}$ is a basis of $Mat_{2\times 2}K$.
    Because the number of elements in a basis is the dimension of the vector space, we see that the dimension of it is 4.

14. **Exercise 14** Similar argument to **Exercise 13**. Dimension of it is $mn$.

15. **Exercise 15** Dimension of it is $n$. Simply build up a basis to see.

16. **Exercise 16** Similarly, dimension of it is $\frac{(n+1)n}{2}$.

17. **Exercise 17**

    Basis is a set comprises

$$U_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$U_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$U_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

    Then, it is easy to see that dimension is 3.

18. **Exercise 18** Basis similar to the one in **Exercise 17** is linear independent and generates space. And, indeed, the number of elements in the basis is the same as one in **Exercise 16**. Thus, dimension of it is $\frac{n(n+1)}{2}$.

19. **Exercise 19** Same as **Exercise 15**.

20. **Exercise 20**

    Let $U$ be the subspace of $V$. There would be a maximal number $m$ of linear independent vectors (**Theorem 3.1** in chapter 1). Suppose the number $m > \dim V$. Then it would contradicts **Theorem 3.1** in chapter 1 as any number of vectors more than $\dim V$ would be linear dependent, which means the basis of $U$ would be linear dependent (remember $U$ is a subspace of $V$). Thus, $m \leq \dim V$.
    Dimension could be $0, 1, 2$.

21. **Exercise 21**

    According to the lemma we proved in **Exercise 20**, dimension of subspace of $\mathbb{R}^3$ could be $0, 1, 2, 3$.

## 2.2 § 2

### 2.2.1 Notes

**Lemma** Let $A$ be a set of linear dependent vectors that generates $V$. Then, for all $v \in V$, there exists infinite linear combinations of $A$ that form $v$.
*Proof* Say that number of vectors in $A$ is $n$. Since $A$ generates $V$, $\forall v \in V, \exists \{a_i\} : v = \sum_{i=1}^{n} a_i A_i$. Let $L$ be a set of linear combinations that form $v$ (here $L$ is a set of sets). We have

$$\begin{aligned} v &= \sum_{i=1}^{n} a_i A_i + O \\ &= \sum_{i=1}^{n} a_i A_i + \sum_{i=1}^{n} b_i A_i \\ &= \sum_{i=1}^{n} (a_i + b_i) A_i \end{aligned}$$

Since $A$ is linear dependent, there exists $\{b_i\}$ where not every element is 0. Therefore, $\{a_i + b_i\} \in L$ and $\{a_i + b_i\} \neq \{a_i\}$ for some $\{b_i\}$.
This means that $\forall \ell \in L$, we can always form a new $\ell' \in L$. And since for all $v \in V$ we always have one linear combination, we can do it infinitely, which means number of elements in $L$ is infinite. Therefore, we have shown what was to be shown. ■

Here we discuss the number of solutions for general linear equations. ($A$ is a $m \times n$ matrix. $X$ is a $n \times 1$ column matrix. $B$ is a $m \times 1$ column matrix).

$$AX = B$$

If $n > m$, according to **Theorem 3.1 in chapter 1**, they must be linear dependent, resulting in infinite number of solutions because of **Lemma** above.
If $n = m$ and they are linear independent (it is then a basis because they are maximal independent vectors), there would only be one solution as **Theorem 2.1 in chapter 1** stated. If they are linear dependent and $B$ is in the subspace generated by column vectors of $A$, there would be infinite number of solutions (**Lemma**), else the equations are not solvable (there exists no linear combination to represent $B$).
If $n < m$ and they are independent and $B$ is in the subspace generated by column vectors of $A$, there would be only one solution. If they are linear independent but $B$ is not in subspace, then it is unsolvable. If they are linear dependent and $B$ is in subspace, infinite solutions occur. If they are linear dependent but $B$ is not in subspace, equations are not solvable.
In general,

1. If $B$ is in the vector space generated by column vectors of $A$ and they are linear independent, there exists one unique solution.

2. If $B$ is in the vector space generated by column vectors of $A$ and they are linear dependent, there exists infinite solutions.

3. If $B$ is not in the vector space generated by column vectors of $A$, there would be no solution.

### 2.2.2 Exercises

1. **Exercise 1** See notes and refer to the definition of linear independence.

2. **Exercise 2**

   Let $u$ be one set of solution and $w$ be another.
   We want to show that $u + w \in X$.

$$\sum_{i=1}^{n}(u_i + w_i) \cdot A^i = \sum_{i=1}^{n} u_i \cdot A^i + \sum_{i=1}^{n} w_i \cdot A^i = O + O = O$$

   Thus, $u + w \in X$. Also, we need to show $cu \in X$ where $c \in K$.

$$c\sum_{i=1}^{n} u_i \cdot A^i = cO = O$$

   Other **VS** s are easy to follow as we define the addition of vectors in $X$ componentwise, $O$ as a vector whose components are all zero, 1 as a vector whose components are all one.

3. **Exercise 3** We want to show following

$$\sum_{i=1}^{n}(a_i + b_i \mathrm{i})A^i = O_{\mathbb{C}}$$

$$\sum_{i=1}^{n} a_i A^i + \sum_{i=1}^{n} b_i \mathrm{i} \cdot A^i = O_{\mathbb{C}}$$

$$O_{\mathbb{C}} + \sum_{i=1}^{n} b_i \mathrm{i} \cdot A^i = O_{\mathbb{C}}$$

$$\sum_{i=1}^{n} b_i \cdot A^i = O_{\mathbb{C}}$$

This means that $\{A^i\}$ should be linear independent over $\mathbb{R}$ ($\sum_{i=1}^{n} b_i \cdot A^i = O_{\mathbb{C}}$ is equal to $\sum_{i=1}^{n} b_i \cdot A^i = O_{\mathbb{R}}$ as there is no imaginary part). Since it is known to us that $\{A^i\}$ is linear independent over $\mathbb{R}$, it has been proved as we do it reversely.

4. **Exercise 4** We know that

$$\sum_{i=1}^{n}(a_i + b_i\mathrm{i})A^i = O_{\mathbb{C}}$$

which means that $\sum_{i=1}^{n} a_i A^i = O_{\mathbb{C}}$ and/or $\sum_{i=1}^{n} b_i A^i = O_{\mathbb{C}}$. For either cases, we have shown it is linear dependent over $\mathbb{R}$ ($a_i, b_i \in \mathbb{R}$).

## 2.3 § 3

### 2.3.1 Exercises

1. **Exercise 1** $AI = IA = A$

2. **Exercise 2** $AO = O$

3. **Exercise 3**

   For every $A$ and $B$, $(AB)C = A(BC)$.

   (a) Case 1

   $$\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$$

   (b) Case 2

   $$\begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

   (c) Case 3

   $$\begin{pmatrix} 33 & 37 \\ 11 & -18 \end{pmatrix}$$

4. **Exercise 4** This one could be proved as it is proved here.

5. **Exercise 5**

   $$AB = \begin{pmatrix} 4 & 2 \\ 5 & -1 \end{pmatrix}$$

   $$BA = \begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix}$$

6. **Exercise 6**

   $$CA = AC = \begin{pmatrix} 7 & 14 \\ 21 & -7 \end{pmatrix}$$

   $$CB = BC = \begin{pmatrix} 14 & 0 \\ 7 & 7 \end{pmatrix}$$

   General rule is that for symmetric one, we may have $AB = BA$? (I am not sure here).

7. **Exercise 7**

   $$XA = \begin{pmatrix} 3 & 1 & 5 \end{pmatrix}$$

8. **Exercise 8**

   $$X_1 A = A_2$$
   $$X_2 A = A_3$$

   Let $X_i$ be a unit vector with only $i$ -th component equal to 1. $X_i A = A_i$

9. **Exercise 9**

   Skip the steps involving verifications. ${}^t(AB) =^t B^t A$ has already been proved in §2. Thus, ${}^t[(AB)C] =^t C \cdot^t (AB) =^t C \cdot^t B \cdot^t A$.

10. **Exercise 10**

    Firstly, we know $A$ is of $1 \times n$, $M$ is of $n \times n$ and $B$ is of $1 \times n$. This means that $\dim(\langle A, B\rangle) = 1$. Also, it implies that ${}^t(\langle A, B\rangle) = \langle A, B\rangle$. Thus, we have

$$\begin{aligned}\langle A, B\rangle &=^t (\langle A, B\rangle)\\ &=^t (AM^tB)\\ &=^t ({}^tB) \cdot^t M \cdot^t A \quad \text{Exercise 9}\\ &= BM^tA\\ &= \langle B, A\rangle\end{aligned}$$

    which is **SP 1**. Also, let

$$N =^t (B + C)$$

    Then, $n_{ij} = n'_{ji} = b_{ji} + c_{ji}$. This implies also $N =^t A +^t B$. Therefore,

$$\langle A, B + C\rangle = AM^t(B + C) = AM({}^tB +^t C) = \langle A, B\rangle + \langle A, C\rangle$$

    which is **SP 2**. Finally

$$\langle cA, B\rangle = cAM^tB = c\langle A, B\rangle$$

    which is **SP 3**.

11. **Exercise 11**

    For part **(a)**, see **Exercise 35**.
    Part **(b)**

$$A^2 = \begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}$$

$$A^3 = \begin{pmatrix}1 & 3 & 6\\ 0 & 1 & 3\\ 0 & 0 & 1\end{pmatrix}$$

$$A^4 = \begin{pmatrix}1 & 4 & 10\\ 0 & 1 & 4\\ 0 & 0 & 1\end{pmatrix}$$

12. **Exercise 12**

$$(AX)_a = \begin{pmatrix}4\\ 7\\ 5\end{pmatrix} (AX)_b = \begin{pmatrix}3\\ 1\end{pmatrix}$$

$$(AX)_c = \begin{pmatrix}x_2\\ 0\end{pmatrix} (AX)_d = \begin{pmatrix}0\\ x_1\end{pmatrix}$$

13. **Exercise 13**

$$(AX)_a = \begin{pmatrix}2\\ 4\end{pmatrix}$$

$$(AX)_b = \begin{pmatrix}4\\ 6\end{pmatrix}$$

$$(AX)_c = \begin{pmatrix}3\\ 5\end{pmatrix}$$

14. **Exercise 14**

$$(AX)_a = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$(AX)_b = \begin{pmatrix} 12 \\ 3 \\ 9 \end{pmatrix}$$

$$(AX)_c = \begin{pmatrix} 5 \\ 4 \\ 8 \end{pmatrix}$$

15. **Exercise 15** $AX = A^2$ (second column of $A$).

16. **Exercise 16** $AX = A^i$

17. **Exercise 17**

    Let $U_i$ be a unit column vector which only has 1 on its $i$ -th component. The proposed form of $C^k$ could be written in the following way.

$$\begin{aligned} C^k &= \sum_{i=1}^{n} b_{ik} A^i \\ &= \sum_{i=1}^{n} b_{ik} [\sum_{j=1}^{m} (a_{ji} \cdot U_j)] \\ &= \sum_{i=1}^{n} [\sum_{j=1}^{m} a_{ji} b_{ik} \cdot U_j] \end{aligned}$$

$$\begin{aligned} C^k &= \sum_{j=1}^{m} A_j \cdot B^k \cdot U_j \\ &= \sum_{j=1}^{m} [\sum_{i=1}^{n} a_{ji} b_{ik} \cdot U_j] \end{aligned}$$

    Two forms are essentially the same if you expand them and compare. Thus, we have proved that the proposed formula is an equivalence of the original definition.

18. **Exercise 18**

    (a) $A^{-1} = (I + A) \Rightarrow A \cdot A^{-1} = I^2 - A^2 = I$
    (b) $A^{-1} = (I^2 + IA + A^2) \Rightarrow A \cdot A^{-1} = I^3 - A^3 = I$
    (c) For real number $I$ and $A$, we see that $I^n - A^n$ can be factored into $I - A$ and another polynomial, because according to remainder theorem, plugging in $I = A$ results in $I^n - A^n = 0$. Thus, we could follow the same pattern to construct always a $A^{-1}$.
    (d) Set $A^{-1} = (-A - 2I)$
    (e) Set $A^{-1} = (-A^2 - A)$

19. **Exercise 19**

$$AB = \begin{pmatrix} 1 & ab \\ 0 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}$$

Inductive step:

$$\begin{aligned} A^{n+1} &= A^n \cdot A \\ &= \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix} \cdot A \\ &= \begin{pmatrix} 1 & (n+1)a \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Thus, we have proved it.

20. **Exercise 20**

$$A^{-1} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$$

21. **Exercise 21** We now show that $B^{-1}A^{-1}$ would be an inverse of $AB$.

$$(AB)(B^{-1}A^{-1}) = A(B \cdot B^{-1})A^{-1} = A \cdot A^{-1} = I$$

And for the reverse, it is easy to verify either.

22. **Exercise 22** See the solution manual

23. **Exercise 23**

$$\begin{aligned} A^2 &= A \cdot A \\ &= \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \end{aligned}$$

Inductive step:

$$\begin{aligned} A^{n+1} &= \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} \cdot A \\ &= \begin{pmatrix} \cos n\theta \cos\theta - \sin n\theta \sin\theta & -(\sin n\theta \cos\theta + \sin\theta \cos n\theta) \\ \sin n\theta \cos\theta + \sin\theta \cos n\theta & -\sin n\theta + \cos n\theta \cos\theta \end{pmatrix} \quad = \begin{pmatrix} \cos(n+1)\theta & -\sin(n+1)\theta \\ \sin(n+1)\theta & \cos(n+1)\theta \end{pmatrix} \end{aligned}$$

Thus, we have determined $A^n$

24. **Exercise 24**

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

25. **Exercise 25**

    (a) $\mathrm{tr}(A) = 2$

    (b) $\mathrm{tr}(A) = 4$

    (c) $\mathrm{tr}(A) = 8$

26. **Exercise 26** See **Exercise 27**.

27. **Exercise 27**

$$\begin{aligned} \mathrm{tr}(AB) &= \sum_{i=1}^{n}[\sum_{j=1}^{n} a_{ij}b_{ji}] \\ &= \sum_{i=1}^{n}[\sum_{j=1}^{n} b_{ji}a_{ij}] \\ &= \sum_{i=1}^{n}[\sum_{j=1}^{n} b_{ij}a_{ji}] \quad \text{They are the same if you expand} \\ &= \mathrm{tr}(BA) \end{aligned}$$

28. **Exercise 28** As diagonal line keeps same after transpose, trace of the matrix would not change as well.

29. **Exercise 29** $A^n = ((a_{ij})^n)$

30. **Exercise 30**

$$A^2 = \begin{pmatrix} a_1^2 & 0 & \cdots & 0 \\ 0 & a_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^2 \end{pmatrix}$$

Inductive step

$$A^{k+1} = \begin{pmatrix} a_1^k & 0 & \cdots & 0 \\ 0 & a_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^k \end{pmatrix} \cdot A = \begin{pmatrix} a_1^{k+1} & 0 & \cdots & 0 \\ 0 & a_2^{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^{k+1} \end{pmatrix}$$

■

31. **Exercise 31** See **Exercise 35**

32. **Exercise 32** We want to show

$$\begin{aligned} {}^t(A^{-1}) &= ({}^tA)^{-1} \\ {}^t(A^{-1}) \cdot^t (A) &= ({}^tA)^{-1} \cdot ({}^tA) \\ {}^t(A^{-1}) \cdot^t (A) &= I_n \end{aligned}$$

Let $C =^t (A^{-1}) \cdot^t (A)$. We then know

$$\begin{aligned} c_{ij} &= \sum_{k=1}^{n} a'^{-1}_{ik} a'_{kj} \\ &= \sum_{k=1}^{n} a_{jk} a^{-1}_{ki} \\ &= A_j \cdot A^{-1\ i} \end{aligned}$$

Thus,

$$\begin{aligned} C &=^t (A \cdot A^{-1}) \\ &=^t (I_n) = I_n \end{aligned}$$

If we do it in the reverse way, then we can prove it.

33. **Exercise 33** Let $B =^t (\bar{A})$, then $b_{ij} = \bar{a}_{ji}$. Let $C = \overline{{}^tA}$, then $c_{ij} = \bar{a'}_{ij} = \bar{a}_{ji}$. Thus, $B = C$.

34. **Exercise 34** Its inverse is

$$\begin{pmatrix} \frac{1}{a_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_n} \end{pmatrix}$$

35. **Exercise 35** See solution manual. Here I would not like to introduce complex formal reasoning to simulate computation result.

36. **Exercise 36**

By result of **Exercise 35** we see that $N^{n+1} = O$ as $N = A - I_n$ is of the form being described in **Exercise 35**.
For inverse part, see **Exercise 18**.

37. **Exercise 37**

$$(I - N)(I + N + \cdot + N^r) = I^{r+1} - N^{r+1} = I^{r+1} = I$$

38. **Exercise 38** See solution manual for detail computation.

39. **Exercise 39** Since we know $AB = BA$ or $A, B$ fulfills **SP 1**, we may say

$$(AB)^r = A^r B^r = O$$

For $(A + B)$, **I don't know how to prove by now because I don't know binomial formula well**.