

Linear Algebra Notes

Harry Ying

Contents

1	Chapter 1	2
1.1	§ 1	2
1.1.1	Notes	2
1.1.2	Exercises	3
1.2	§ 2	4
1.2.1	Notes	4
1.2.2	Exercises	5
1.3	§ 3	6
1.3.1	Notes	6
1.4	§ 4	7
1.4.1	Notes	7
1.4.2	Exercises	7
2	Chapter 2	8
2.1	§ 1	8
2.1.1	Exercises	8
2.2	§ 2	9
2.2.1	Notes	9
2.2.2	Exercises	10
2.3	§ 3	11
2.3.1	Exercises	11
3	Chapter 3	16
3.1	§ 1	16
3.1.1	Notes	16
3.1.2	Exercises	17
3.2	§ 2	17
3.2.1	Notes	17
3.2.2	Exercises	18

1 Chapter 1

1.1 § 1

1.1.1 Notes

A generic vector space V is not a field because there is no definition of v^{-1} for some $v \in V$, fulfilling not the definition of a field.

1. Pg. 4 Proof of $(-1)v = -v$

$$\begin{aligned}(-1)v + v &= (-1)v + 1 \cdot v \\&= (-1 + 1)v \\&= v + (-v)\end{aligned}$$

Thus, $(-1)v = -v$.

2. Pg. 6 Proof of SP 3

$$\begin{aligned}(xA) \cdot B &= \sum_{i=1}^n (xa_i)b_i \\&= \sum_{i=1}^n x(a_ib_i) \\&= x \sum_{i=1}^n a_ib_i \\&= x(A \cdot B) \\A \cdot (xB) &= \sum_{i=1}^n a_i(xb_i) \\&= \sum_{i=1}^n x(a_ib_i) \\&= x \sum_{i=1}^n a_ib_i \\&= x(A \cdot B)\end{aligned}$$

3. Pg. 7

Upper one:

$$\begin{aligned}(A+B)^2 &= (A+B) \cdot (A+B) \\&= (A+B) \cdot A + (A+B) \cdot B \quad \text{Use SP 2} \\&= A^2 + B \cdot A + A \cdot B + B^2 \quad \text{Use SP 1}\end{aligned}$$

Bottom one: Since K is a field, all **VS** s regarding summation or product of functions are actually closed on K . By applying field axioms, V is then a vector space over K .

4. Pg. 9

Let $a_1 = (u_1 + w_1), a_2 = (u_2 + w_2)$. Both of them $\in (U + W)$.

Since U, W are subspaces of V , $U, W \in V$. Thus, $a_1, a_2 \in V$ as $u_1, w_1, u_2, w_2 \in V$, moreover, $(U + W) \subset V$.

$$a_1 + a_2 = (u_1 + u_2) + (w_1 + w_2) \in (U + W)$$

$$ca_1 = c(u_1 + w_1) = (cu_1) + (cw_1) \in (U + W)$$

Since $O \in U$ and $O \in W$, $O = O + O \in (U + W)$. Thus, $(U + W)$ is a subspace of V .

1.1.2 Exercises

1. **Exercise 1** Let $v \in V$, $c[v + (-v)] = cv + c(-v) = cv + (-c)v = v \cdot 0 = v \cdot (1 - 1) = v + (-v) = O$

2. **Exercise 2** Since $c \neq 0$

$$\begin{aligned} O &= cv + [-(cv)] \\ cv &= cv + [-(cv)] \\ O &= -(cv) \\ \frac{-1}{c} \cdot O &= (-c)v \cdot \frac{-1}{c} \\ \frac{-1}{c} \cdot (v - v) &= v \\ \frac{-1}{c} \cdot v + \frac{1}{c} \cdot v &= v \\ v \cdot (1 - 1) &= v \\ v - v &= v \\ O &= v \end{aligned}$$

3. **Exercise 3**

$\forall g \in V, (g + f)(x) = g(x) + f(x) = f(x) + g(x) = (f + g)(x) \Rightarrow g + f = f + g$.
If $O + u = u$, $(O + u)(x) = O(x) + u(x) = u(x)$. Therefore, $O(x) = 0$.

4. **Exercise 4**

$$\begin{aligned} v + w &= O \\ v + w &= v + (-v) \\ w &= -v \end{aligned}$$

5. **Exercise 5**

$$\begin{aligned} v + w &= v \\ v + (-v) + w &= v + (-v) \\ O + w &= O \end{aligned}$$

Since $\forall u, O + u = u$, we have $w = O$.

6. **Exercise 6**

Let $W = \{B \mid B \cdot A_1 = O \text{ and } B \cdot A_2 = O\}$. Specifically, it is clear that $O \in W$ as $O \cdot A = \sum_{i=1}^n b_i a_i = \sum_{i=1}^n 0 \times a_i = 0$.
Let $v_1, v_2 \in W$ such that $v_1 \cdot A_1 = 0$, $v_1 \cdot A_2 = 0$, $v_2 \cdot A_1 = 0$, $v_2 \cdot A_2 = 0$. Thus,

$$\begin{aligned} (v_1 + v_2) \cdot A_1 &= v_1 \cdot A_1 + v_2 \cdot A_1 \\ &= O + O \\ &= O \\ [c(v_1 + v_2)] \cdot A_1 &= (cv_1 + cv_2) \cdot A_1 \\ &= (cv_1) \cdot A_1 + (cv_2) \cdot A_1 \\ &= c(v_1 \cdot A_1 + v_2 \cdot A_1) \\ &= cO \\ &= O \end{aligned}$$

. It is easy to show for A_2 then. Therefore, $(v_1 + v_2) \in W$.

7. **Exercise 7** Same to apply as Exercise 6.

8. **Exercise 8**

Name the set as W .

(a) Proof

$$v_1 + v_2 = (x_1 + x_2, y_1 + y_2), x_1 + x_2 = y_1 + y_2 \Rightarrow (v_1 + v_2) \in W$$

$$cv = (cx, cy), cx = cy \Rightarrow cv \in W$$

$$O = (0, 0) \in W$$

(b) Proof See Part (a).

(c) Proof Same technique as in Part (a).

9. **Exercise 9** See Exercise 8.

10. **Exercise 10**

For $U \cap W$, let $v_1, v_2 \in U \cap W$. Since $v_1, v_2 \in U$ and U is a subspace, $v_1 + v_2 \in U$. In same way, we can see that $v_1 + v_2 \in W$. Thus, $v_1 + v_2 \in U \cap W$.

Since $v_1 \in U$, $cv_1 \in U$. Also, it shows $cv_1 \in W$ in the same way. Thus, $cv_1 \in U \cap W$. Because U, W are subspaces, $O \in U$ and $O \in W$. Thus, $O \in U \cap W$. Therefore, $U \cap W$ is a subspace.

Refer to the [note part](#) for proof for $U + W$.

11. **Exercise 11** Since L is a field, **VS1**, **VS3**, **VS4**, **VS8** are established under field axioms, and multiplication and addition are closed in L . For **VS5**, **VS6**, **VS7**, they are all valid as $K \subset L$. O is simply 0, and $1 \cdot u = u$ is established in L .

12. **Exercise 12**

For $x, y \in K$, we have

$x + y = a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2} = (a_1 + a_2) + (b_1 + b_2)\sqrt{2}$. Since $a_1, b_1, a_2, b_2 \in \mathbb{Q}$, $(a_1 + a_2), (b_1 + b_2) \in \mathbb{Q}$. Thus, $x + y \in K$.

$xy = (a_1a_2 + 2b_1b_2) + (a_2b_1 + a_1b_2) \times \sqrt{2}$. Since $a_1, b_1, a_2, b_2 \in \mathbb{Q}$, $(a_1a_2 + 2b_1b_2), (a_2b_1 + a_1b_2) \in \mathbb{Q}$. Thus, $xy \in K$.

$-x = -a - b\sqrt{2}$. Since $a, b \in \mathbb{Q}$, $-a, -b \in \mathbb{Q}$. Thus, $-x \in K$.

If $a + b\sqrt{2} \neq 0$, $a, b \neq 0$, and $a - b\sqrt{2} \neq 0$. Thus, $x^{-1} = \frac{1}{a+b\sqrt{2}} = \frac{a-b\sqrt{2}}{a^2-2b^2} = \frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2}$. It is easy to see that **new** $a, b \in \mathbb{Q}$ as $a, b \in \mathbb{Q}$. Thus, $x^{-1} \in K$. Specifically, if $a = b = 0$, $0 \in \mathbb{Q}$. If $a = 1, b = 0$, $1 \in \mathbb{Q}$. Thus, K is a field.

13. **Exercise 13** Same technique as Exercise 12.

14. **Exercise 14** Same technique as Exercise 12.

1.2 § 2

1.2.1 Notes

Another quite helpful equivalent of definition of linear independence is that (stated following without loss of generality)

$$\forall a_1 \neq 0, a_1 v_1 \neq \sum_{i=2}^n a_i v_i$$

Here is the *proof* of equivalence between above statement and definition of linear independence.

Since $a_1 \neq 0$,

$$v_1 \neq \sum_{i=2}^n \frac{a_i}{a_1} v_i$$

$$O \neq -v_1 + \sum_{i=2}^n \frac{a_i}{a_1} v_i$$

$$\lambda O \neq (-\lambda)v_1 + \sum_{i=2}^n \frac{\lambda a_i}{a_1} v_i \quad \lambda \in K \text{ and } \lambda \neq 0$$

$$O \neq (-\lambda)v_1 + \sum_{i=2}^n \frac{\lambda a_i}{a_1} v_i \quad \lambda \in K \text{ and } \lambda \neq 0$$

λ and a_i could be arbitrary, thus from above we could conclude that $a'_1 v_1 \neq \sum_{i=2}^n a'_i v_i$ if and only if all $a'_i = 0$, which is the definition of linear independence.

Also, another point that worth paying attention to is that generators could be **linear dependent**. This is true because you could put arbitrary vectors at the end of a basis of a vector space and just set coefficients for these extraneous vectors when it is producing new linear combinations.

1.2.2 Exercises

1. **Exercise 1** Using result from [Exercise 4](#), easy to prove.

2. **Exercise 2**

- (a) $(1, -1)$
- (b) $(\frac{1}{2}, \frac{3}{2})$
- (c) $(1, 1)$
- (d) $(3, 2)$

3. **Exercise 3**

- (a) $(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$
- (b) $(1, 0, 1)$
- (c) $(\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3})$

4. **Exercise 4**

Following set of equations is an equivalent of $x(a, b) + y(c, d) = O$,

$$ax + cy = 0 \quad (1)$$

$$bx + dy = 0 \quad (2)$$

$$(1) \times d - (2) \times c \Rightarrow (ad - cb)x + cdy - cdy = 0$$

$$(ad - cb)x = 0$$

For $ad - cb \neq 0$ part, clearly we shall see that $x = 0$ as $(ad - cb)x = 0$. Plugging x back to (1), we get $y = 0$. Thus, two vectors are linear independent.

For $ad - cb = 0$ part, we need to prove that $x(a, b) + y(c, d) = O$ has solution other than $x = y = 0$.

First, suppose $a, b, c, d \neq 0$. Since $ad - cb = 0$, $x \in \mathbb{R}$. By applying technique, we could also show $y \in \mathbb{R}$. Thus, (a, b) , (c, d) are linear independent.

If $a, b, c, d \neq 0$ does **NOT** hold. Without lose of generality (for all the possibilities, a, d and c, b are interchangeable), consider following scenarios in a xy -plane,

(a) $a = 0, c = 0$

If $a = c = 0$, $x, y \in \mathbb{R}$ in (1). Because the (2) is a line in the plane, there must exist some $x, y \neq 0$.

(b) $a = 0, b = 0, c = 0$

Same argument as above, despite the line represented by (2) is a little bit peculiar (it is $y = 0$).

(c) $a = 0, d = 0, c = 0$

Same argument as the first, despite the line represented by (2) is a little bit peculiar (it is $x = 0$).

(d) $a = 0, d = 0, b = 0, c = 0$

Both (1), (2) represent the whole plane, thus, $x, y \in \mathbb{R}$.

5. **Exercise 5,6**

To correctly understand how could functions be elements(vectors) in vector space, we need to understand that function $f : S \rightarrow K$ is essentially a set of pairs $(s, k), \forall s \in S$. Functions have scalar multiplication and

addition defined.

$f + g$ is defined as $\{(s, f(s) + g(s)) | s \in S\}$, and $cf, c \in K$ is defined as $\{(s, c \cdot f(s)) | s \in S\}$.

It is easy to verify that V of every $f : S \rightarrow K$ is a vector space over K . Particularly, O for V is $\{(s, 0) | s \in S\}$.

So like other vector spaces, linear dependence is **about**

$$f_{sum} = \sum_{i=1}^n a_i f_i = O$$

Since right-hand-side of the equation is $\{(s, 0) | s \in S\}$, we can say that $\forall v \in V, f_{sum}(s) = 0$. This is useful in solving problems in **Exercise 5** and **Exercise 6**.

For example, we need to show that $f(s) = 1$ and $g(s) = t$ are linear independent. This means that we need to consider following equation,

$$af + bg = O$$

which is an equivalent of

$$\forall t, a + bt = 0$$

Above conversion is quite helpful since we could put in arbitrary t and the equation should hold. Thus, we could put in particular values of t to **construct** set of equations to show that $a = b = 0$. For example, here we plug in $t = 0$, then $a = 0$, and if we plug back $a = 0$ into original equation with $t = 0$ again, $b = 0$.

This method could be used throughout **Exercise 5,6**.

6. **Exercise 7** (3, 5)

7. **Exercise 8** *Calculus involved, not doing now.*

8. **Exercise 9**

$$\begin{aligned} \sum_{i=1}^r [a_i \cdot (A_i \cdot \sum_{j=i+1}^r A_j)] &= O && \text{All vectors are mutually perpendicular} \\ &= \sum_{i=1}^r [(a_i \cdot A_i) \cdot \sum_{j=i+1}^r A_j] \end{aligned}$$

Since $\forall A \in \{A_i\}, A \neq O$, it is only possible that every a is 0. Thus, A_i are linearly independent.

9. **Exercise 10**

Since v, w are linear dependent, for

$$nv + mw = O$$

at least one of $n, m \neq 0$. Consider following scenarios, we can see that there would be $a = 0$ or $a = -\frac{n}{m}$.

$$(a) \ n = 0, m \neq 0 \Rightarrow w = O$$

$$(b) \ n \neq 0, m = 0 \Rightarrow v = O. \text{ This contradicts with } v \neq O \text{ in problem. Thus, this is impossible.}$$

$$(c) \ n \neq 0, m \neq 0 \Rightarrow w = -\frac{n}{m}v$$

1.3 § 3

1.3.1 Notes

This subsection comprises a lot of concise proofs. But in conclusion, we need to know that

Basis \Leftrightarrow Maximal linear independent vector set proof at **Theorem3.1**

Basis \Leftrightarrow Maximal linear independent vector set \Rightarrow Generators proof at **Theorem2.2**

Generators \nRightarrow Basis Generators are not always linear independent.

Thus, all possible bases of a vector space V are of one and only one possible number of elements, which is equal to the one of maximal independent vector set.

1.4 § 4

1.4.1 Notes

Proof for

$$\dim(U \times W) = \dim U + \dim W$$

Because $\forall u \in (U \times W), (O_u + O_w) + u = u + (O_u + O_w) = u$. Thus, by definition, $O = (O_u, O_w)$.

Let $A = \{u_i\}$ be a basis of U and $B = \{w_i\}$ be a basis of W . Note the dimension of U, W as n, m respectively. Let

$$C = \{(u_i, 0) | u_i \in A\} \cup \{(0, w_i) | w_i \in B\}$$

Since there would be no intersection between two sets being union above, the number of elements in C is $n + m$. If we could show that C is a basis of $U \times W$, then we could show the original statement.

First we need to show that all elements in C is linear independent. This means $a_i \in K, c_i \in C$

$$\sum_{i=1}^{n+m} a_i c_i = O$$

if and only if all the $a_i = 0$.

Because multiplication by scalar and addition for $U \times W$ is defined componentwise, we shall see that (if we keep the "order" of elements in C as A and B are merged)

$$\begin{aligned} \sum_{i=1}^n a_i u_i &= O_u \\ \sum_{i=n+1}^{n+m} a_i w_i &= O_w \end{aligned}$$

Since both A and B are basis of U and W respectively, all the a_i should be 0.

Now, we need to show that C generates $U \times W$. Since A and B are basis of U and W respectively,

$$\forall (a, b) \in (U \times W), \exists f_i, g_i \in K : \sum_{i=1}^n f_i u_i = a \text{ and } \sum_{i=1}^m g_i w_i = b$$

Thus, by setting set of scalar for "order"-kept C as $\{f_i\} \cup \{g_i\}$, it is easy to see that it generates $U \times W$. Therefore, we see that

$$\dim(U \times W) = \dim U + \dim W$$

and

$$\{(u_i, 0) | u_i \in A\} \cup \{(0, w_i) | w_i \in B\}$$

is a basis for $U \times W$.

1.4.2 Exercises

1. Exercise 1

For the first part, we need to show that $\forall v \in V, \exists$ unique $u \in U, w \in W : v = u + w$. Since $(2, 1)$ and $(0, 1)$ are linear independent, they are a basis of $V = \mathbb{R}^2$. This means

$$\forall v \in V, \exists \text{ unique } a, b \in K : v = a \cdot (2, 1) + b \cdot (0, 1)$$

Thus, just set $u = a \cdot (2, 1)$ and $w = b \cdot (0, 1)$, and we have proved it.

It is same for $(2, 1)$ and $(1, 1)$.

2. Exercise 2

Since $(1, 0, 0), (1, 1, 0), (0, 1, 1)$ are linear independent, we obtain that

$$\forall v \in V, \exists \text{ unique } a, b, c \in K : v = a \cdot (1, 0, 0) + b \cdot (1, 1, 0) + c \cdot (0, 1, 1)$$

Set $u = a \cdot (1, 0, 0)$ and $w = b \cdot (1, 1, 0) + c \cdot (0, 1, 1)$, it would be proved.

3. **Exercise 3** According to argument provided [here](#), $\forall c \in K, cA \neq B$ means that A, B are linear independent. Also, according to **Theorem 3.4**, they are a basis of \mathbb{R}^2 .

Based on the similar argument in **Exercise 1**, second part could be proved.

4. **Exercise 4** See notes

2 Chapter 2

2.1 § 1

2.1.1 Exercises

1. **Exercise 1** Skip

2. **Exercise 2** Skip

3. **Exercise 3** Skip

4. **Exercise 4** Skip

5. **Exercise 5** Let $C = {}^t(A + B) = (c_{ij})$. Then, $c_{ij} = (a_{ij} + b_{ij})' = a_{ji} + b_{ji}$. Thus, $C = {}^tA + {}^tB$.

6. **Exercise 6** Let $B = {}^t(cA)$. Then, $b_{ij} = ca_{ji}$. Since ${}^tA = (a'_{ij}) = (a_{ji}) = A$, $B = c{}^tA$.

7. **Exercise 7** No difference.

8. **Exercise 8** Skip

9. **Exercise 9** Skip

10. **Exercise 10** Let $B = A + {}^tA = (b_{ij}) = (a_{ij} + a_{ji})$. Since, $b_{ij} = a_{ij} + a_{ji} = a_{ji} + b_{ij} = b_{ji}$, B is symmetric.

11. **Exercise 11** Skip

12. **Exercise 12** Skip

13. **Exercise 13**

For followings, we mean ones in *Exercises on Dimension* section.

Followings are linear independent.

$$U_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$U_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$U_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$U_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Apply $a \cdot U_1 + b \cdot U_2 + c \cdot U_3 + d \cdot U_4 = O$ to verify it. Because it generates the matrix vector space $Mat_{2 \times 2}K$ over K (For every $v \in Mat_{2 \times 2}K$, simply let a, b, c, d be v 's components) and $\{U_i\}$ are linear independent, $\{U_i\}$ is a basis of $Mat_{2 \times 2}K$.

Because the number of elements in a basis is the dimension of the vector space, we see that the dimension of it is 4.

14. **Exercise 14** Similar argument to **Exercise 13**. Dimension of it is mn .

15. **Exercise 15** Dimension of it is n . Simply build up a basis to see.

16. **Exercise 16** Similarly, dimension of it is $\frac{(n+1)n}{2}$.

17. **Exercise 17**

Basis is a set comprises

$$U_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$U_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$U_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Then, it is easy to see that dimension is 3.

18. **Exercise 18** Basis similar to the one in **Exercise 17** is linear independent and generates space. And, indeed, the number of elements in the basis is the same as one in **Exercise 16**. Thus, dimension of it is $\frac{n(n+1)}{2}$.

19. **Exercise 19** Same as **Exercise 15**.

20. **Exercise 20**

Let U be the subspace of V . There would be a maximal number m of linear independent vectors (**Theorem 3.1** in chapter 1). Suppose the number $m > \dim V$. Then it would contradicts **Theorem 3.1** in chapter 1 as any number of vectors more than $\dim V$ would be linear dependent, which means the basis of U would be linear dependent (remember U is a subspace of V). Thus, $m \leq \dim V$.

Dimension could be 0, 1, 2.

21. **Exercise 21**

According to the lemma we proved in **Exercise 20**, dimension of subspace of \mathbb{R}^3 could be 0, 1, 2, 3.

2.2 § 2

2.2.1 Notes

Lemma Let A be a set of linear dependent vectors that generates V . Then, for all $v \in V$, there exists infinite linear combinations of A that form v .

Proof Say that number of vectors in A is n . Since A generates V , $\forall v \in V, \exists \{a_i\} : v = \sum_{i=1}^n a_i A_i$. Let L be a set of linear combinations that form v (here L is a set of sets). We have

$$\begin{aligned} v &= \sum_{i=1}^n a_i A_i + O \\ &= \sum_{i=1}^n a_i A_i + \sum_{i=1}^n b_i A_i \\ &= \sum_{i=1}^n (a_i + b_i) A_i \end{aligned}$$

Since A is linear dependent, there exists $\{b_i\}$ where not every element is 0. Therefore, $\{a_i + b_i\} \in L$ and $\{a_i + b_i\} \neq \{a_i\}$ for some $\{b_i\}$.

This means that $\forall \ell \in L$, we can always form a new $\ell' \in L$. And since for all $v \in V$ we always have one linear combination, we can do it infinitely, which means number of elements in L is infinite. Therefore, we have shown what was to be shown. ■

Here we discuss the number of solutions for general linear equations. (A is a $m \times n$ matrix. X is a $n \times 1$ column matrix. B is a $m \times 1$ column matrix).

$$AX = B$$

If $n > m$, according to **Theorem 3.1 in chapter 1**, they must be linear dependent, resulting in infinite number of solutions because of **Lemma** above.

If $n = m$ and they are linear independent (it is then a basis because they are maximal independent vectors), there would only be one solution as **Theorem 2.1 in chapter 1** stated. If they are linear dependent and B is in the subspace generated by column vectors of A , there would be infinite number of solutions (**Lemma**), else the equations are not solvable (there exists no linear combination to represent B).

If $n < m$ and they are independent and B is in the subspace generated by column vectors of A , there would be only one solution. If they are linear independent but B is not in subspace, then it is unsolvable. If they are linear dependent and B is in subspace, infinite solutions occur. If they are linear dependent but B is not in subspace, equations are not solvable.

In general,

1. If B is in the vector space generated by column vectors of A and they are linear independent, there exists one unique solution.
2. If B is in the vector space generated by column vectors of A and they are linear dependent, there exists infinite solutions.
3. If B is not in the vector space generated by column vectors of A , there would be no solution.

2.2.2 Exercises

1. **Exercise 1** See notes and refer to the definition of linear independence.

2. Exercise 2

Let u be one set of solution and w be another.

We want to show that $u + w \in X$.

$$\sum_{i=1}^n (u_i + w_i) \cdot A^i = \sum_{i=1}^n u_i \cdot A^i + \sum_{i=1}^n w_i \cdot A^i = O + O = O$$

Thus, $u + w \in X$. Also, we need to show $cu \in X$ where $c \in K$.

$$c \sum_{i=1}^n u_i \cdot A^i = cO = O$$

Other **VS** s are easy to follow as we define the addition of vectors in X componentwise, O as a vector whose components are all zero, 1 as a vector whose components are all one.

3. **Exercise 3** We want to show following

$$\begin{aligned} \sum_{i=1}^n (a_i + b_i i) A^i &= O_{\mathbb{C}} \\ \sum_{i=1}^n a_i A^i + \sum_{i=1}^n b_i i \cdot A^i &= O_{\mathbb{C}} \\ O_{\mathbb{C}} + \sum_{i=1}^n b_i i \cdot A^i &= O_{\mathbb{C}} \\ \sum_{i=1}^n b_i \cdot A^i &= O_{\mathbb{C}} \end{aligned}$$

This means that $\{A^i\}$ should be linear independent over \mathbb{R} ($\sum_{i=1}^n b_i \cdot A^i = O_{\mathbb{C}}$ is equal to $\sum_{i=1}^n b_i \cdot A^i = O_{\mathbb{R}}$ as there is no imaginary part). Since it is known to us that $\{A^i\}$ is linear independent over \mathbb{R} , it has been proved as we do it reversely.

4. **Exercise 4** We know that

$$\sum_{i=1}^n (a_i + b_i i) A^i = O_{\mathbb{C}}$$

which means that $\sum_{i=1}^n a_i A^i = O_{\mathbb{C}}$ and/or $\sum_{i=1}^n b_i A^i = O_{\mathbb{C}}$. For either cases, we have shown it is linear dependent over \mathbb{R} ($a_i, b_i \in \mathbb{R}$).

2.3 § 3

2.3.1 Exercises

1. **Exercise 1** $AI = IA = A$

2. **Exercise 2** $AO = O$

3. **Exercise 3**

For every A and B , $(AB)C = A(BC)$.

(a) Case 1

$$\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$$

(b) Case 2

$$\begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

(c) Case 3

$$\begin{pmatrix} 33 & 37 \\ 11 & -18 \end{pmatrix}$$

4. **Exercise 4** This one could be proved as it is proved [here](#).

5. **Exercise 5**

$$AB = \begin{pmatrix} 4 & 2 \\ 5 & -1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix}$$

6. **Exercise 6**

$$CA = AC = \begin{pmatrix} 7 & 14 \\ 21 & -7 \end{pmatrix}$$

$$CB = BC = \begin{pmatrix} 14 & 0 \\ 7 & 7 \end{pmatrix}$$

General rule is that for symmetric one, we may have $AB = BA$? (I am not sure here).

7. **Exercise 7**

$$XA = \begin{pmatrix} 3 & 1 & 5 \end{pmatrix}$$

8. **Exercise 8**

$$X_1 A = A_2$$

$$X_2 A = A_3$$

Let X_i be a unit vector with only i -th component equal to 1. $X_i A = A_i$

9. **Exercise 9**

Skip the steps involving verifications. ${}^t(AB) = {}^t B {}^t A$ has already been proved in §2. Thus, ${}^t[(AB)C] = {}^t C \cdot {}^t(AB) = {}^t C \cdot {}^t B \cdot {}^t A$.

10. **Exercise 10**

Firstly, we know A is of $1 \times n$, M is of $n \times n$ and B is of $1 \times n$. This means that $\dim(\langle A, B \rangle) = 1$. Also, it implies that ${}^t(\langle A, B \rangle) = \langle A, B \rangle$. Thus, we have

$$\begin{aligned} \langle A, B \rangle &= {}^t(\langle A, B \rangle) \\ &= {}^t(AM^t B) \\ &= {}^t({}^t B) \cdot {}^t M \cdot {}^t A \quad \text{Exercise 9} \\ &= BM^t A \\ &= \langle B, A \rangle \end{aligned}$$

which is **SP 1**. Also, let

$$N = {}^t(B + C)$$

Then, $n_{ij} = n'_{ji} = b_{ji} + c_{ji}$. This implies also $N = {}^t A + {}^t B$. Therefore,

$$\langle A, B + C \rangle = AM^t(B + C) = AM({}^t B + {}^t C) = \langle A, B \rangle + \langle A, C \rangle$$

which is **SP 2**. Finally

$$\langle cA, B \rangle = cAM^t B = c\langle A, B \rangle$$

which is **SP 3**.

11. **Exercise 11**

For part (a), see **Exercise 35**.

Part (b)

$$A^2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 1 & 4 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

12. **Exercise 12**

$$(AX)_a = \begin{pmatrix} 4 \\ 7 \\ 5 \end{pmatrix} \quad (AX)_b = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$(AX)_c = \begin{pmatrix} x_2 \\ 0 \end{pmatrix} \quad (AX)_d = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}$$

13. **Exercise 13**

$$(AX)_a = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$(AX)_b = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$(AX)_c = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

14. **Exercise 14**

$$(AX)_a = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$(AX)_b = \begin{pmatrix} 12 \\ 3 \\ 9 \end{pmatrix}$$

$$(AX)_c = \begin{pmatrix} 5 \\ 4 \\ 8 \end{pmatrix}$$

15. **Exercise 15** $AX = A^2$ (second column of A).

16. **Exercise 16** $AX = A^i$

17. **Exercise 17**

Let U_i be a unit column vector which only has 1 on its i -th component. The proposed form of C^k could be written in the following way.

$$\begin{aligned} C^k &= \sum_{i=1}^n b_{ik} A^i \\ &= \sum_{i=1}^n b_{ik} \left[\sum_{j=1}^m (a_{ji} \cdot U_j) \right] \\ &= \sum_{i=1}^n \left[\sum_{j=1}^m a_{ji} b_{ik} \cdot U_j \right] \\ C^k &= \sum_{j=1}^m A_j \cdot B^k \cdot U_j \\ &= \sum_{j=1}^m \left[\sum_{i=1}^n a_{ji} b_{ik} \cdot U_j \right] \end{aligned}$$

Two forms are essentially the same if you expand them and compare. Thus, we have proved that the proposed formula is an equivalence of the original definition.

18. **Exercise 18**

(a) $A^{-1} = (I + A) \Rightarrow A \cdot A^{-1} = I^2 - A^2 = I$

(b) $A^{-1} = (I^2 + IA + A^2) \Rightarrow A \cdot A^{-1} = I^3 - A^3 = I$

(c) For real number I and A , we see that $I^n - A^n$ can be factored into $I - A$ and another polynomial, because according to remainder theorem, plugging in $I = A$ results in $I^n - A^n = 0$. Thus, we could follow the same pattern to construct always a A^{-1} .

(d) Set $A^{-1} = (-A - 2I)$

(e) Set $A^{-1} = (-A^2 - A)$

19. **Exercise 19**

$$AB = \begin{pmatrix} 1 & ab \\ 0 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}$$

Inductive step:

$$\begin{aligned} A^{n+1} &= A^n \cdot A \\ &= \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix} \cdot A \\ &= \begin{pmatrix} 1 & (n+1)a \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Thus, we have proved it.

20. **Exercise 20**

$$A^{-1} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$$

21. **Exercise 21** We now show that $B^{-1}A^{-1}$ would be an inverse of AB .

$$(AB)(B^{-1}A^{-1}) = A(B \cdot B^{-1})A^{-1} = A \cdot A^{-1} = I$$

And for the reverse, it is easy to verify either.

22. **Exercise 22** See the solution manual

23. **Exercise 23**

$$\begin{aligned} A^2 &= A \cdot A \\ &= \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \end{aligned}$$

Inductive step:

$$\begin{aligned} A^{n+1} &= \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} \cdot A \\ &= \begin{pmatrix} \cos n\theta \cos \theta - \sin n\theta \sin \theta & -(\sin n\theta \cos \theta + \sin \theta \cos n\theta) \\ \sin n\theta \cos \theta + \sin \theta \cos n\theta & -\sin n\theta + \cos n\theta \cos \theta \end{pmatrix} = \begin{pmatrix} \cos(n+1)\theta & -\sin(n+1)\theta \\ \sin(n+1)\theta & \cos(n+1)\theta \end{pmatrix} \end{aligned}$$

Thus, we have determined A^n

24. **Exercise 24**

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

25. **Exercise 25**

- (a) $\text{tr}(A) = 2$
- (b) $\text{tr}(A) = 4$
- (c) $\text{tr}(A) = 8$

26. **Exercise 26** See **Exercise 27**.

27. **Exercise 27**

$$\begin{aligned} \text{tr}(AB) &= \sum_{i=1}^n \left[\sum_{j=1}^n a_{ij} b_{ji} \right] \\ &= \sum_{i=1}^n \left[\sum_{j=1}^n b_{ji} a_{ij} \right] \\ &= \sum_{i=1}^n \left[\sum_{j=1}^n b_{ij} a_{ji} \right] \quad \text{They are the same if you expand} \\ &= \text{tr}(BA) \end{aligned}$$

28. **Exercise 28** As diagonal line keeps same after transpose, trace of the matrix would not change as well.

29. **Exercise 29** $A^n = ((a_{ij})^n)$

30. **Exercise 30**

$$A^2 = \begin{pmatrix} a_1^2 & 0 & \cdots & 0 \\ 0 & a_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^2 \end{pmatrix}$$

Inductive step

$$A^{k+1} = \begin{pmatrix} a_1^k & 0 & \cdots & 0 \\ 0 & a_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^k \end{pmatrix} \cdot A = \begin{pmatrix} a_1^{k+1} & 0 & \cdots & 0 \\ 0 & a_2^{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^{k+1} \end{pmatrix}$$

■

31. **Exercise 31** See **Exercise 35**

32. **Exercise 32** We want to show

$$\begin{aligned} {}^t(A^{-1}) &= ({}^tA)^{-1} \\ {}^t(A^{-1}) \cdot {}^t(A) &= ({}^tA)^{-1} \cdot ({}^tA) \\ {}^t(A^{-1}) \cdot {}^t(A) &= I_n \end{aligned}$$

Let $C = {}^t(A^{-1}) \cdot {}^t(A)$. We then know

$$\begin{aligned} c_{ij} &= \sum_{k=1}^n a'_{ik}{}^{-1} a'_{kj} \\ &= \sum_{k=1}^n a_{jk} a_{ki}^{-1} \\ &= A_j \cdot A^{-1} \cdot i \end{aligned}$$

Thus,

$$\begin{aligned} C &= {}^t(A \cdot A^{-1}) \\ &= {}^t(I_n) = I_n \end{aligned}$$

If we do it in the reverse way, then we can prove it.

33. **Exercise 33** Let $B = {}^t(\bar{A})$, then $b_{ij} = \bar{a}_{ji}$. Let $C = \overline{{}^tA}$, then $c_{ij} = \bar{a}'_{ij} = \bar{a}_{ji}$. Thus, $B = C$.

34. **Exercise 34** Its inverse is

$$\begin{pmatrix} \frac{1}{a_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_n} \end{pmatrix}$$

35. **Exercise 35** See solution manual. Here I would not like to introduce complex formal reasoning to simulate computation result.

36. **Exercise 36**

By result of **Exercise 35** we see that $N^{n+1} = O$ as $N = A - I_n$ is of the form being described in **Exercise 35**.

For inverse part, see **Exercise 18**.

37. **Exercise 37**

$$(I - N)(I + N + \cdots + N^r) = I^{r+1} - N^{r+1} = I^{r+1} = I$$

38. **Exercise 38** See solution manual for detail computation.

39. **Exercise 39** Since we know $AB = BA$ or A, B fulfills **SP 1**, we may say

$$(AB)^r = A^r B^r = O$$

For $(A + B)$, we discuss $(A + B)^{2r}$ where r is the larger r for A and B .

$$(A + B)^{2r} = \sum_{k=0}^{2r} \binom{2r}{k} A^{2r-k} B^k$$

If $1 \geq k \leq r$, then $2r - k \geq r$ and $A^{2r-k} = O$. If $r < k \leq 2r$, then $B^k = O$. Thus, essentially, $(A + B)^{2r} = O$.

3 Chapter 3

3.1 § 1

3.1.1 Notes

If we want to say that S is the image of A under F , we are essentially trying to say followings:

$$\begin{aligned} \forall z \in S, \exists x : F(x) = z. &\Rightarrow S \subset F(A) \\ \forall a \in A, F(a) \in S. &\Rightarrow F(A) \subset S \end{aligned}$$

Above are exactly what **Example 6** on Pg. 45 are saying.

Also, we shall work on the equality of two linear mappings. Two linear mappings $F : S_1 \rightarrow T_1, G : S_2 \rightarrow T_2$ are said to be equal if and only if followings are fulfilled:

$$\begin{aligned} S_1 &= S_2 \\ T_1 &= T_2 \\ \forall z \in S_1, F(z) &= G(z) \end{aligned}$$

Proofs left to readers on Pg. 49.

If u_1, u_2 are elements of V , then $T_{u_1+u_2} = T_{u_1} \circ T_{u_2}$.

$$\begin{aligned} \forall v \in V, T_{u_1+u_2} &= (u_1 + u_2) + v \\ &= u_1 + (u_2 + v) \\ &= T_{u_1}(u_2 + v) \\ &= T_{u_1}(T_{u_2}(v)) \\ &= T_{u_1} \circ T_{u_2}(v) \end{aligned}$$

Which means that $T_{u_1+u_2} = T_{u_1} \circ T_{u_2}$ according to our definition of linear mapping equality.

If u is an element of V , then $T_u : V \rightarrow V$ has an inverse mapping which is nothing but the translation T_{-u} .

First, it is easy to verify that T_{-u} is an inverse of T_u . Then, we say that there is an inverse T_u^{-1} . According to the definition of inverse of a linear mapping, we have, for every $v \in V$ that

$$\begin{aligned} T_u^{-1}(T_u(v)) &= I_V(v) = v \\ T_u^{-1}(v + u) &= v \\ T_u^{-1}(x) &= x - u \quad \text{Let } x = v + u \end{aligned}$$

which attests $T_u^{-1} = T_{-u}$.

3.1.2 Exercises

1. **Exercise 1** *Calculus involved, not doing now.*

2. **Exercise 2** Proved in notes.

3. **Exercise 3**

(a) $L(X) = 11$

(b) $L(X) = 13$

(c) $L(X) = 6$

4. **Exercise 4**

$$F(1) = (e, 1), F(0) = (1, 0), F(-1) = (e^{-1}, -1)$$

5. **Exercise 5**

$$(F + G)(1) = (e + 1, 3), (F + G)(2) = (e^2 + 2, 6), (F + G)(0) = (1, 0)$$

6. **Exercise 6**

$$(2F)(0) = (2, 0), (\pi F)(1) = (\pi e, \pi)$$

7. **Exercise 7** For (a), it is 1. For (b), it is 11.

8. **Exercise 8**

The image is a ellipse of the equation

$$\frac{u^2}{4} + \frac{w^2}{9} = 1$$

Proof is omitted.

9. **Exercise 9**

The image is a straight line

$$y = \frac{1}{2}x$$

Proof. $A = \{(2, y) | y \in \mathbb{R}\}$, $S = \{(2x, x) | x \in \mathbb{R}\}$. We contend that $\forall a \in A, F(a) \in S$.

$$\forall y \in \mathbb{R}, F(2, y) = (2y, y) \in S$$

Conversely, $\forall s = (x, \frac{1}{2}x) \in S$, let $a = (2, \frac{1}{2}x) \in A$, so that $F(a) = s$, which means $S \subset F(A)$.

10. **Exercise 10** It is a circle of center $(0, 0)$ and radius e^c . Proof is omitted.

11. **Exercise 11**

It is a cylinder of radius 1 and center $(0, 0)$. Proof is omitted.

12. **Exercise 12** $x^2 + y^2 = 1$. Proof is omitted.

3.2 § 2

3.2.1 Notes

Here we have an important **Lemma**

Let $F : V \rightarrow W$ is a linear mapping. If for some $v_i \in V$, we have $F(v_i)$ are linear independent, then v_i are linear

independent.

Proof. If $\sum_{i=1}^n t_i v_i = O$, then we have

$$\sum_{i=1}^n t_i v_i = O$$

$$F\left(\sum_{i=1}^n t_i v_i\right) = F(O) = O \quad (\text{This is ensured as "output" of a mapping is unique for same "input"})$$

$$\sum_{i=1}^n t_i F(v_i) = O$$

which means if $\sum_{i=1}^n t_i v_i = O$, we must have $\sum_{i=1}^n t_i F(v_i) = O$. Since $F(v_i)$ are linear independent, we obtain that t_i is always equal to 0, which is another word for v_i are linear independent.

It is noteworthy that reversal of this **Lemma** is **NOT** always true as $F(v) = O$ doesn't ensure that $v = O$.

3.2.2 Exercises

1. **Exercise 1** Only (a), (b), (d), (e), (f), (h) are linear mappings. For (h), it involves *Calculus*.

2. **Exercise 2** $T(O) = T[v + (-v)] = T(v) + T(-v) = T(v) - T(v) = O$

3. **Exercise 3** $T(u + v) = T(u) + T(v) = w + O = w$

4. **Exercise 4**

Let the set of elements $v \in V$ satisfying $T(v) = w$ be S . We contend that $\forall v \in S, \exists u \in U : v = u + v_0$.

Proof. let $u = v - v_0$. $F(u) = F(v - v_0) = F(v) - F(v_0) = O$. Thus, $v \in U$. This means $S \subset (v_0 + U)$.

Conversely, we contend that $\forall u \in U$, we have $(v_0 + u) \in S$.

Proof. $T(v_0 + u) = T(v_0) + T(u) = w + O = w$. This means $(v_0 + U) \subset S$.

Thus, $S = (v_0 + U)$.

5. **Exercise 5** As **Exercise 2** said, $T(O) = T(v - v) = T(v) + T(-v) = O \Rightarrow T(-v) = -T(v)$.

6. **Exercise 6**

Firstly, $F(v_1 + v_2) = (f(v_1) + f(v_2), g(v_1) + g(v_2)) = (f(v_1), g(v_1)) + (f(v_2), g(v_2)) = F(v_1) + F(v_2)$

Secondly, $F(cv) = (cf(v), cg(v)) = c(f(v), g(v)) = cF(v)$

7. **Exercise 7**