

# Linear Algebra Notes

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## Contents

<b>1</b>	<b>Chapter 1</b>	<b>2</b>
1.1	§ 1	2
1.1.1	Notes	2
1.1.2	Exercises	3
1.2	§ 2	4
1.2.1	Notes	4
1.2.2	Exercises	5
1.3	§ 3	6
1.3.1	Notes	6
1.4	§ 4	7
1.4.1	Notes	7
1.4.2	Exercises	7
<b>2</b>	<b>Chapter 2</b>	<b>8</b>
2.1	§ 1	8
2.1.1	Exercises	8
2.2	§ 2	9
2.2.1	Notes	9
2.2.2	Exercises	10
2.3	§ 3	11
2.3.1	Exercises	11
<b>3</b>	<b>Chapter 3</b>	<b>16</b>
3.1	§ 1	16
3.1.1	Notes	16
3.1.2	Exercises	17
3.2	§ 2	18
3.2.1	Notes	18
3.2.2	Exercises	18

# 1 Chapter 1

## 1.1 § 1

### 1.1.1 Notes

A generic vector space  $V$  is not a field because there is no definition of  $v^{-1}$  for some  $v \in V$ , fulfilling not the definition of a field.

#### 1. Pg. 4 Proof of $(-1)v = -v$

$$\begin{aligned}(-1)v + v &= (-1)v + 1 \cdot v \\&= (-1 + 1)v \\&= v + (-v)\end{aligned}$$

Thus,  $(-1)v = -v$ .

#### 2. Pg. 6 Proof of SP 3

$$\begin{aligned}(xA) \cdot B &= \sum_{i=1}^n (xa_i)b_i \\&= \sum_{i=1}^n x(a_ib_i) \\&= x \sum_{i=1}^n a_ib_i \\&= x(A \cdot B) \\A \cdot (xB) &= \sum_{i=1}^n a_i(xb_i) \\&= \sum_{i=1}^n x(a_ib_i) \\&= x \sum_{i=1}^n a_ib_i \\&= x(A \cdot B)\end{aligned}$$

#### 3. Pg. 7

Upper one:

$$\begin{aligned}(A+B)^2 &= (A+B) \cdot (A+B) \\&= (A+B) \cdot A + (A+B) \cdot B \quad \text{Use SP 2} \\&= A^2 + B \cdot A + A \cdot B + B^2 \quad \text{Use SP 1}\end{aligned}$$

Bottom one: Since  $K$  is a field, all **VS** s regarding summation or product of functions are actually closed on  $K$ . By applying field axioms,  $V$  is then a vector space over  $K$ .

#### 4. Pg. 9

Let  $a_1 = (u_1 + w_1), a_2 = (u_2 + w_2)$ . Both of them  $\in (U + W)$ .

Since  $U, W$  are subspaces of  $V$ ,  $U, W \in V$ . Thus,  $a_1, a_2 \in V$  as  $u_1, w_1, u_2, w_2 \in V$ , moreover,  $(U + W) \subset V$ .

$$a_1 + a_2 = (u_1 + u_2) + (w_1 + w_2) \in (U + W)$$

$$ca_1 = c(u_1 + w_1) = (cu_1) + (cw_1) \in (U + W)$$

Since  $O \in U$  and  $O \in W$ ,  $O = O + O \in (U + W)$ . Thus,  $(U + W)$  is a subspace of  $V$ .

### 1.1.2 Exercises

1. **Exercise 1** Let  $v \in V$ ,  $c[v + (-v)] = cv + c(-v) = cv + (-c)v = v \cdot 0 = v \cdot (1 - 1) = v + (-v) = O$

2. **Exercise 2** Since  $c \neq 0$

$$\begin{aligned} O &= cv + [-(cv)] \\ cv &= cv + [-(cv)] \\ O &= -(cv) \\ \frac{-1}{c} \cdot O &= (-c)v \cdot \frac{-1}{c} \\ \frac{-1}{c} \cdot (v - v) &= v \\ \frac{-1}{c} \cdot v + \frac{1}{c} \cdot v &= v \\ v \cdot (1 - 1) &= v \\ v - v &= v \\ O &= v \end{aligned}$$

3. **Exercise 3**

$\forall g \in V, (g + f)(x) = g(x) + f(x) = f(x) + g(x) = (f + g)(x) \Rightarrow g + f = f + g$ .  
If  $O + u = u$ ,  $(O + u)(x) = O(x) + u(x) = u(x)$ . Therefore,  $O(x) = 0$ .

4. **Exercise 4**

$$\begin{aligned} v + w &= O \\ v + w &= v + (-v) \\ w &= -v \end{aligned}$$

5. **Exercise 5**

$$\begin{aligned} v + w &= v \\ v + (-v) + w &= v + (-v) \\ O + w &= O \end{aligned}$$

Since  $\forall u, O + u = u$ , we have  $w = O$ .

6. **Exercise 6**

Let  $W = \{B \mid B \cdot A_1 = O \text{ and } B \cdot A_2 = O\}$ . Specifically, it is clear that  $O \in W$  as  $O \cdot A = \sum_{i=1}^n b_i a_i = \sum_{i=1}^n 0 \times a_i = 0$ .  
Let  $v_1, v_2 \in W$  such that  $v_1 \cdot A_1 = 0$ ,  $v_1 \cdot A_2 = 0$ ,  $v_2 \cdot A_1 = 0$ ,  $v_2 \cdot A_2 = 0$ . Thus,

$$\begin{aligned} (v_1 + v_2) \cdot A_1 &= v_1 \cdot A_1 + v_2 \cdot A_1 \\ &= O + O \\ &= O \\ [c(v_1 + v_2)] \cdot A_1 &= (cv_1 + cv_2) \cdot A_1 \\ &= (cv_1) \cdot A_1 + (cv_2) \cdot A_1 \\ &= c(v_1 \cdot A_1 + v_2 \cdot A_1) \\ &= cO \\ &= O \end{aligned}$$

. It is easy to show for  $A_2$  then. Therefore,  $(v_1 + v_2) \in W$ .

7. **Exercise 7** Same to apply as Exercise 6.

8. **Exercise 8**

Name the set as  $W$ .

(a) Proof

$$v_1 + v_2 = (x_1 + x_2, y_1 + y_2), x_1 + x_2 = y_1 + y_2 \Rightarrow (v_1 + v_2) \in W$$

$$cv = (cx, cy), cx = cy \Rightarrow cv \in W$$

$$O = (0, 0) \in W$$

(b) Proof See Part (a).

(c) Proof Same technique as in Part (a).

9. **Exercise 9** See Exercise 8.

10. **Exercise 10**

For  $U \cap W$ , let  $v_1, v_2 \in U \cap W$ . Since  $v_1, v_2 \in U$  and  $U$  is a subspace,  $v_1 + v_2 \in U$ . In same way, we can see that  $v_1 + v_2 \in W$ . Thus,  $v_1 + v_2 \in U \cap W$ .

Since  $v_1 \in U$ ,  $cv_1 \in U$ . Also, it shows  $cv_1 \in W$  in the same way. Thus,  $cv_1 \in U \cap W$ . Because  $U, W$  are subspaces,  $O \in U$  and  $O \in W$ . Thus,  $O \in U \cap W$ . Therefore,  $U \cap W$  is a subspace.

Refer to the [note part](#) for proof for  $U + W$ .

11. **Exercise 11** Since  $L$  is a field, **VS1**, **VS3**, **VS4**, **VS8** are established under field axioms, and multiplication and addition are closed in  $L$ . For **VS5**, **VS6**, **VS7**, they are all valid as  $K \subset L$ .  $O$  is simply 0, and  $1 \cdot u = u$  is established in  $L$ .

12. **Exercise 12**

For  $x, y \in K$ , we have

$$x + y = a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2} = (a_1 + a_2) + (b_1 + b_2)\sqrt{2}. \text{ Since } a_1, b_1, a_2, b_2 \in \mathbb{Q}, (a_1 + a_2), (b_1 + b_2) \in \mathbb{Q}. \text{ Thus, } x + y \in K.$$

$$xy = (a_1a_2 + 2b_1b_2) + (a_2b_1 + a_1b_2) \times \sqrt{2}. \text{ Since } a_1, b_1, a_2, b_2 \in \mathbb{Q}, (a_1a_2 + 2b_1b_2), (a_2b_1 + a_1b_2) \in \mathbb{Q}. \text{ Thus, } xy \in K.$$

$$-x = -a - b\sqrt{2}. \text{ Since } a, b \in \mathbb{Q}, -a, -b \in \mathbb{Q}. \text{ Thus, } -x \in K.$$

If  $a + b\sqrt{2} \neq 0$ ,  $a, b \neq 0$ , and  $a - b\sqrt{2} \neq 0$ . Thus,  $x^{-1} = \frac{1}{a+b\sqrt{2}} = \frac{a-b\sqrt{2}}{a^2-2b^2} = \frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2}$ . It is easy to see that **new**  $a, b \in \mathbb{Q}$  as  $a, b \in \mathbb{Q}$ . Thus,  $x^{-1} \in K$ . Specifically, if  $a = b = 0$ ,  $0 \in \mathbb{Q}$ . If  $a = 1, b = 0$ ,  $1 \in \mathbb{Q}$ . Thus,  $K$  is a field.

13. **Exercise 13** Same technique as Exercise 12.

14. **Exercise 14** Same technique as Exercise 12.

## 1.2 § 2

### 1.2.1 Notes

Another quite helpful equivalent of definition of linear independence is that (stated following without loss of generality)

$$\forall a_1 \neq 0, a_1 v_1 \neq \sum_{i=2}^n a_i v_i$$

Here is the *proof* of equivalence between above statement and definition of linear independence.

Since  $a_1 \neq 0$ ,

$$v_1 \neq \sum_{i=2}^n \frac{a_i}{a_1} v_i$$

$$O \neq -v_1 + \sum_{i=2}^n \frac{a_i}{a_1} v_i$$

$$\lambda O \neq (-\lambda)v_1 + \sum_{i=2}^n \frac{\lambda a_i}{a_1} v_i \quad \lambda \in K \text{ and } \lambda \neq 0$$

$$O \neq (-\lambda)v_1 + \sum_{i=2}^n \frac{\lambda a_i}{a_1} v_i \quad \lambda \in K \text{ and } \lambda \neq 0$$

$\lambda$  and  $a_i$  could be arbitrary, thus from above we could conclude that  $a'_1 v_1 \neq \sum_{i=2}^n a'_i v_i$  if and only if all  $a'_i = 0$ , which is the definition of linear independence.

Also, another point that worth paying attention to is that generators could be **linear dependent**. This is true because you could put arbitrary vectors at the end of a basis of a vector space and just set coefficients for these extraneous vectors when it is producing new linear combinations.

### 1.2.2 Exercises

1. **Exercise 1** Using result from [Exercise 4](#), easy to prove.

2. **Exercise 2**

- (a)  $(1, -1)$
- (b)  $(\frac{1}{2}, \frac{3}{2})$
- (c)  $(1, 1)$
- (d)  $(3, 2)$

3. **Exercise 3**

- (a)  $(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$
- (b)  $(1, 0, 1)$
- (c)  $(\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3})$

4. **Exercise 4**

Following set of equations is an equivalent of  $x(a, b) + y(c, d) = O$ ,

$$ax + cy = 0 \quad (1)$$

$$bx + dy = 0 \quad (2)$$

$$(1) \times d - (2) \times c \Rightarrow (ad - cb)x + cdy - cdy = 0$$

$$(ad - cb)x = 0$$

For  $ad - cb \neq 0$  part, clearly we shall see that  $x = 0$  as  $(ad - cb)x = 0$ . Plugging  $x$  back to (1), we get  $y = 0$ . Thus, two vectors are linear independent.

For  $ad - cb = 0$  part, we need to prove that  $x(a, b) + y(c, d) = O$  has solution other than  $x = y = 0$ .

First, suppose  $a, b, c, d \neq 0$ . Since  $ad - cb = 0$ ,  $x \in \mathbb{R}$ . By applying technique, we could also show  $y \in \mathbb{R}$ . Thus,  $(a, b), (c, d)$  are linear independent.

If  $a, b, c, d \neq 0$  does **NOT** hold. Without lose of generality (for all the possibilities,  $a, d$  and  $c, b$  are interchangeable), consider following scenarios in a  $xy$ -plane,

(a)  $a = 0, c = 0$

If  $a = c = 0$ ,  $x, y \in \mathbb{R}$  in (1). Because the (2) is a line in the plane, there must exist some  $x, y \neq 0$ .

(b)  $a = 0, b = 0, c = 0$

Same argument as above, despite the line represented by (2) is a little bit peculiar (it is  $y = 0$ ).

(c)  $a = 0, d = 0, c = 0$

Same argument as the first, despite the line represented by (2) is a little bit peculiar (it is  $x = 0$ ).

(d)  $a = 0, d = 0, b = 0, c = 0$

Both (1), (2) represent the whole plane, thus,  $x, y \in \mathbb{R}$ .

5. **Exercise 5,6**

To correctly understand how could functions be elements(vectors) in vector space, we need to understand that function  $f : S \rightarrow K$  is essentially a set of pairs  $(s, k), \forall s \in S$ . Functions have scalar multiplication and

addition defined.

$f + g$  is defined as  $\{(s, f(s) + g(s)) | s \in S\}$ , and  $cf, c \in K$  is defined as  $\{(s, c \cdot f(s)) | s \in S\}$ .

It is easy to verify that  $V$  of every  $f : S \rightarrow K$  is a vector space over  $K$ . Particularly,  $O$  for  $V$  is  $\{(s, 0) | s \in S\}$ .

So like other vector spaces, linear dependence is **about**

$$f_{sum} = \sum_{i=1}^n a_i f_i = O$$

Since right-hand-side of the equation is  $\{(s, 0) | s \in S\}$ , we can say that  $\forall v \in V, f_{sum}(s) = 0$ . This is useful in solving problems in **Exercise 5** and **Exercise 6**.

For example, we need to show that  $f(s) = 1$  and  $g(s) = t$  are linear independent. This means that we need to consider following equation,

$$af + bg = O$$

which is an equivalent of

$$\forall t, a + bt = 0$$

Above conversion is quite helpful since we could put in arbitrary  $t$  and the equation should hold. Thus, we could put in particular values of  $t$  to **construct** set of equations to show that  $a = b = 0$ . For example, here we plug in  $t = 0$ , then  $a = 0$ , and if we plug back  $a = 0$  into original equation with  $t = 0$  again,  $b = 0$ .

This method could be used throughout **Exercise 5,6**.

6. **Exercise 7** (3, 5)

7. **Exercise 8** *Calculus involved, not doing now.*

8. **Exercise 9**

$$\sum_{i=1}^r [a_i \cdot (A_i \cdot \sum_{j=i+1}^r A_j)] = O$$

All vectors are mutually perpendicular

$$= \sum_{i=1}^r [(a_i \cdot A_i) \cdot \sum_{j=i+1}^r A_j]$$

Since  $\forall A \in \{A_i\}, A \neq O$ , it is only possible that every  $a$  is 0. Thus,  $A_i$  are linearly independent.

9. **Exercise 10**

Since  $v, w$  are linear dependent, for

$$nv + mw = O$$

at least one of  $n, m \neq 0$ . Consider following scenarios, we can see that there would be  $a = 0$  or  $a = -\frac{n}{m}$ .

(a)  $n = 0, m \neq 0 \Rightarrow w = O$

(b)  $n \neq 0, m = 0 \Rightarrow v = O$ . This contradicts with  $v \neq O$  in problem. Thus, this is impossible.

(c)  $n \neq 0, m \neq 0 \Rightarrow w = -\frac{n}{m}v$

## 1.3 § 3

### 1.3.1 Notes

This subsection comprises a lot of concise proofs. But in conclusion, we need to know that

Basis  $\Leftrightarrow$  Maximal linear independent vector set

proof at **Theorem3.1**

Basis  $\Leftrightarrow$  Maximal linear independent vector set  $\Rightarrow$  Generators

proof at **Theorem2.2**

Generators  $\nRightarrow$  Basis

Generators are not always linear independent.

Thus, all possible bases of a vector space  $V$  are of one and only one possible number of elements, which is equal to the one of maximal independent vector set.

## 1.4 § 4

### 1.4.1 Notes

*Proof* for

$$\dim(U \times W) = \dim U + \dim W$$

Because  $\forall u \in (U \times W), (O_u + O_w) + u = u + (O_u + O_w) = u$ . Thus, by definition,  $O = (O_u, O_w)$ .

Let  $A = \{u_i\}$  be a basis of  $U$  and  $B = \{w_i\}$  be a basis of  $W$ . Note the dimension of  $U, W$  as  $n, m$  respectively. Let

$$C = \{(u_i, 0) | u_i \in A\} \cup \{(0, w_i) | w_i \in B\}$$

Since there would be no intersection between two sets being union above, the number of elements in  $C$  is  $n + m$ .

If we could show that  $C$  is a basis of  $U \times W$ , then we could show the original statement.

First we need to show that all elements in  $C$  is linear independent. This means  $a_i \in K, c_i \in C$

$$\sum_{i=1}^{n+m} a_i c_i = O$$

if and only if all the  $a_i = 0$ .

Because multiplication by scalar and addition for  $U \times W$  is defined componentwise, we shall see that (if we keep the "order" of elements in  $C$  as  $A$  and  $B$  are merged)

$$\begin{aligned} \sum_{i=1}^n a_i u_i &= O_u \\ \sum_{i=n+1}^{n+m} a_i w_i &= O_w \end{aligned}$$

Since both  $A$  and  $B$  are basis of  $U$  and  $W$  respectively, all the  $a_i$  should be 0.

Now, we need to show that  $C$  generates  $U \times W$ . Since  $A$  and  $B$  are basis of  $U$  and  $W$  respectively,

$$\forall (a, b) \in (U \times W), \exists f_i, g_i \in K : \sum_{i=1}^n f_i u_i = a \text{ and } \sum_{i=1}^m g_i w_i = b$$

Thus, by setting set of scalar for "order"-kept  $C$  as  $\{f_i\} \cup \{g_i\}$ , it is easy to see that it generates  $U \times W$ .

Therefore, we see that

$$\dim(U \times W) = \dim U + \dim W$$

and

$$\{(u_i, 0) | u_i \in A\} \cup \{(0, w_i) | w_i \in B\}$$

is a basis for  $U \times W$ .

### 1.4.2 Exercises

#### 1. Exercise 1

For the first part, we need to show that  $\forall v \in V, \exists$  unique  $u \in U, w \in W : v = u + w$ . Since  $(2, 1)$  and  $(0, 1)$  are linear independent, they are a basis of  $V = \mathbb{R}^2$ . This means

$$\forall v \in V, \exists \text{ unique } a, b \in K : v = a \cdot (2, 1) + b \cdot (0, 1)$$

Thus, just set  $u = a \cdot (2, 1)$  and  $w = b \cdot (0, 1)$ , and we have proved it.

It is same for  $(2, 1)$  and  $(1, 1)$ .

## 2. Exercise 2

Since  $(1, 0, 0), (1, 1, 0), (0, 1, 1)$  are linear independent, we obtain that

$$\forall v \in V, \exists \text{ unique } a, b, c \in K : v = a \cdot (1, 0, 0) + b \cdot (1, 1, 0) + c \cdot (0, 1, 1)$$

Set  $u = a \cdot (1, 0, 0)$  and  $w = b \cdot (1, 1, 0) + c \cdot (0, 1, 1)$ , it would be proved.

3. **Exercise 3** According to argument provided [here](#),  $\forall c \in K, cA \neq B$  means that  $A, B$  are linear independent. Also, according to **Theorem 3.4**, they are a basis of  $\mathbb{R}^2$ .

Based on the similar argument in **Exercise 1**, second part could be proved.

4. **Exercise 4** See notes

## 2 Chapter 2

### 2.1 § 1

#### 2.1.1 Exercises

1. **Exercise 1** Skip

2. **Exercise 2** Skip

3. **Exercise 3** Skip

4. **Exercise 4** Skip

5. **Exercise 5** Let  $C = {}^t(A + B) = (c_{ij})$ . Then,  $c_{ij} = (a_{ij} + b_{ij})' = a_{ji} + b_{ji}$ . Thus,  $C = {}^tA + {}^tB$ .

6. **Exercise 6** Let  $B = {}^t(cA)$ . Then,  $b_{ij} = ca_{ji}$ . Since  ${}^tA = (a'_{ij}) = (a_{ji}) = A$ ,  $B = c{}^tA$ .

7. **Exercise 7** No difference.

8. **Exercise 8** Skip

9. **Exercise 9** Skip

10. **Exercise 10** Let  $B = A + {}^tA = (b_{ij}) = (a_{ij} + a_{ji})$ . Since,  $b_{ij} = a_{ij} + a_{ji} = a_{ji} + b_{ij} = b_{ji}$ ,  $B$  is symmetric.

11. **Exercise 11** Skip

12. **Exercise 12** Skip

13. **Exercise 13**

For followings, we mean ones in *Exercises on Dimension* section.

Followings are linear independent.

$$U_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$U_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$U_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$U_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Apply  $a \cdot U_1 + b \cdot U_2 + c \cdot U_3 + d \cdot U_4 = O$  to verify it. Because it generates the matrix vector space  $Mat_{2 \times 2}K$  over  $K$  (For every  $v \in Mat_{2 \times 2}K$ , simply let  $a, b, c, d$  be  $v$ 's components) and  $\{U_i\}$  are linear independent,  $\{U_i\}$  is a basis of  $Mat_{2 \times 2}K$ .

Because the number of elements in a basis is the dimension of the vector space, we see that the dimension of it is 4.



14. **Exercise 14** Similar argument to **Exercise 13**. Dimension of it is  $mn$ .

15. **Exercise 15** Dimension of it is  $n$ . Simply build up a basis to see.

16. **Exercise 16** Similarly, dimension of it is  $\frac{(n+1)n}{2}$ .

17. **Exercise 17**

Basis is a set comprises

$$U_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$U_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$U_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Then, it is easy to see that dimension is 3.

18. **Exercise 18** Basis similar to the one in **Exercise 17** is linear independent and generates space. And, indeed, the number of elements in the basis is the same as one in **Exercise 16**. Thus, dimension of it is  $\frac{n(n+1)}{2}$ .

19. **Exercise 19** Same as **Exercise 15**.

20. **Exercise 20**

Let  $U$  be the subspace of  $V$ . There would be a maximal number  $m$  of linear independent vectors (**Theorem 3.1** in chapter 1). Suppose the number  $m > \dim V$ . Then it would contradicts **Theorem 3.1** in chapter 1 as any number of vectors more than  $\dim V$  would be linear dependent, which means the basis of  $U$  would be linear dependent (remember  $U$  is a subspace of  $V$ ). Thus,  $m \leq \dim V$ .

Dimension could be 0, 1, 2.

21. **Exercise 21**

According to the lemma we proved in **Exercise 20**, dimension of subspace of  $\mathbb{R}^3$  could be 0, 1, 2, 3.

## 2.2 § 2

### 2.2.1 Notes

**Lemma** Let  $A$  be a set of linear dependent vectors that generates  $V$ . Then, for all  $v \in V$ , there exists infinite linear combinations of  $A$  that form  $v$ .

*Proof* Say that number of vectors in  $A$  is  $n$ . Since  $A$  generates  $V$ ,  $\forall v \in V, \exists \{a_i\} : v = \sum_{i=1}^n a_i A_i$ . Let  $L$  be a set of linear combinations that form  $v$  (here  $L$  is a set of sets). We have

$$\begin{aligned} v &= \sum_{i=1}^n a_i A_i + O \\ &= \sum_{i=1}^n a_i A_i + \sum_{i=1}^n b_i A_i \\ &= \sum_{i=1}^n (a_i + b_i) A_i \end{aligned}$$

Since  $A$  is linear dependent, there exists  $\{b_i\}$  where not every element is 0. Therefore,  $\{a_i + b_i\} \in L$  and  $\{a_i + b_i\} \neq \{a_i\}$  for some  $\{b_i\}$ .

This means that  $\forall \ell \in L$ , we can always form a new  $\ell' \in L$ . And since for all  $v \in V$  we always have one linear combination, we can do it infinitely, which means number of elements in  $L$  is infinite. Therefore, we have shown what was to be shown. ■

Here we discuss the number of solutions for general linear equations. ( $A$  is a  $m \times n$  matrix.  $X$  is a  $n \times 1$  column matrix.  $B$  is a  $m \times 1$  column matrix).

$$AX = B$$

If  $n > m$ , according to **Theorem 3.1 in chapter 1**, they must be linear dependent, resulting in infinite number of solutions because of **Lemma** above.

If  $n = m$  and they are linear independent (it is then a basis because they are maximal independent vectors), there would only be one solution as **Theorem 2.1 in chapter 1** stated. If they are linear dependent and  $B$  is in the subspace generated by column vectors of  $A$ , there would be infinite number of solutions (**Lemma**), else the equations are not solvable (there exists no linear combination to represent  $B$ ).

If  $n < m$  and they are independent and  $B$  is in the subspace generated by column vectors of  $A$ , there would be only one solution. If they are linear independent but  $B$  is not in subspace, then it is unsolvable. If they are linear dependent and  $B$  is in subspace, infinite solutions occur. If they are linear dependent but  $B$  is not in subspace, equations are not solvable.

In general,

1. If  $B$  is in the vector space generated by column vectors of  $A$  and they are linear independent, there exists one unique solution.
2. If  $B$  is in the vector space generated by column vectors of  $A$  and they are linear dependent, there exists infinite solutions.
3. If  $B$  is not in the vector space generated by column vectors of  $A$ , there would be no solution.

### 2.2.2 Exercises

1. **Exercise 1** See notes and refer to the definition of linear independence.

#### 2. Exercise 2

Let  $u$  be one set of solution and  $w$  be another.

We want to show that  $u + w \in X$ .

$$\sum_{i=1}^n (u_i + w_i) \cdot A^i = \sum_{i=1}^n u_i \cdot A^i + \sum_{i=1}^n w_i \cdot A^i = O + O = O$$

Thus,  $u + w \in X$ . Also, we need to show  $cu \in X$  where  $c \in K$ .

$$c \sum_{i=1}^n u_i \cdot A^i = cO = O$$

Other **VS** s are easy to follow as we define the addition of vectors in  $X$  componentwise,  $O$  as a vector whose components are all zero,  $1$  as a vector whose components are all one.

3. **Exercise 3** We want to show following

$$\begin{aligned} \sum_{i=1}^n (a_i + b_i i) A^i &= O_{\mathbb{C}} \\ \sum_{i=1}^n a_i A^i + \sum_{i=1}^n b_i i \cdot A^i &= O_{\mathbb{C}} \\ O_{\mathbb{C}} + \sum_{i=1}^n b_i i \cdot A^i &= O_{\mathbb{C}} \\ \sum_{i=1}^n b_i \cdot A^i &= O_{\mathbb{C}} \end{aligned}$$

This means that  $\{A^i\}$  should be linear independent over  $\mathbb{R}$  ( $\sum_{i=1}^n b_i \cdot A^i = O_{\mathbb{C}}$  is equal to  $\sum_{i=1}^n b_i \cdot A^i = O_{\mathbb{R}}$  as there is no imaginary part). Since it is known to us that  $\{A^i\}$  is linear independent over  $\mathbb{R}$ , it has been proved as we do it reversely.

4. **Exercise 4** We know that

$$\sum_{i=1}^n (a_i + b_i i) A^i = O_{\mathbb{C}}$$

which means that  $\sum_{i=1}^n a_i A^i = O_{\mathbb{C}}$  and/or  $\sum_{i=1}^n b_i A^i = O_{\mathbb{C}}$ . For either cases, we have shown it is linear dependent over  $\mathbb{R}$  ( $a_i, b_i \in \mathbb{R}$ ).

## 2.3 § 3

### 2.3.1 Exercises

1. **Exercise 1**  $AI = IA = A$

2. **Exercise 2**  $AO = O$

3. **Exercise 3**

For every  $A$  and  $B$ ,  $(AB)C = A(BC)$ .

(a) Case 1

$$\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$$

(b) Case 2

$$\begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

(c) Case 3

$$\begin{pmatrix} 33 & 37 \\ 11 & -18 \end{pmatrix}$$

4. **Exercise 4** This one could be proved as it is proved [here](#).

5. **Exercise 5**

$$AB = \begin{pmatrix} 4 & 2 \\ 5 & -1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix}$$

6. **Exercise 6**

$$CA = AC = \begin{pmatrix} 7 & 14 \\ 21 & -7 \end{pmatrix}$$

$$CB = BC = \begin{pmatrix} 14 & 0 \\ 7 & 7 \end{pmatrix}$$

General rule is that for symmetric one, we may have  $AB = BA$ ? (I am not sure here).

7. **Exercise 7**

$$XA = \begin{pmatrix} 3 & 1 & 5 \end{pmatrix}$$

8. **Exercise 8**

$$X_1 A = A_2$$

$$X_2 A = A_3$$

Let  $X_i$  be a unit vector with only  $i$ -th component equal to 1.  $X_i A = A_i$

9. **Exercise 9**

Skip the steps involving verifications.  ${}^t(AB) = {}^t B {}^t A$  has already been proved in §2. Thus,  ${}^t[(AB)C] = {}^t C \cdot {}^t(AB) = {}^t C \cdot {}^t B \cdot {}^t A$ .

10. **Exercise 10**

Firstly, we know  $A$  is of  $1 \times n$ ,  $M$  is of  $n \times n$  and  $B$  is of  $1 \times n$ . This means that  $\dim(\langle A, B \rangle) = 1$ . Also, it implies that  ${}^t(\langle A, B \rangle) = \langle A, B \rangle$ . Thus, we have

$$\begin{aligned} \langle A, B \rangle &= {}^t(\langle A, B \rangle) \\ &= {}^t(AM^t B) \\ &= {}^t({}^t B) \cdot {}^t M \cdot {}^t A \quad \text{Exercise 9} \\ &= BM^t A \\ &= \langle B, A \rangle \end{aligned}$$

which is **SP 1**. Also, let

$$N = {}^t(B + C)$$

Then,  $n_{ij} = n'_{ji} = b_{ji} + c_{ji}$ . This implies also  $N = {}^t A + {}^t B$ . Therefore,

$$\langle A, B + C \rangle = AM^t(B + C) = AM({}^t B + {}^t C) = \langle A, B \rangle + \langle A, C \rangle$$

which is **SP 2**. Finally

$$\langle cA, B \rangle = cAM^t B = c\langle A, B \rangle$$

which is **SP 3**.

11. **Exercise 11**

For part (a), see **Exercise 35**.

Part (b)

$$A^2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 1 & 4 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

12. **Exercise 12**

$$(AX)_a = \begin{pmatrix} 4 \\ 7 \\ 5 \end{pmatrix} \quad (AX)_b = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$(AX)_c = \begin{pmatrix} x_2 \\ 0 \end{pmatrix} \quad (AX)_d = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}$$

13. **Exercise 13**

$$(AX)_a = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$(AX)_b = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$(AX)_c = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

14. **Exercise 14**

$$(AX)_a = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$(AX)_b = \begin{pmatrix} 12 \\ 3 \\ 9 \end{pmatrix}$$

$$(AX)_c = \begin{pmatrix} 5 \\ 4 \\ 8 \end{pmatrix}$$

15. **Exercise 15**  $AX = A^2$  (second column of  $A$ ).

16. **Exercise 16**  $AX = A^i$

17. **Exercise 17**

Let  $U_i$  be a unit column vector which only has 1 on its  $i$ -th component. The proposed form of  $C^k$  could be written in the following way.

$$\begin{aligned} C^k &= \sum_{i=1}^n b_{ik} A^i \\ &= \sum_{i=1}^n b_{ik} \left[ \sum_{j=1}^m (a_{ji} \cdot U_j) \right] \\ &= \sum_{i=1}^n \left[ \sum_{j=1}^m a_{ji} b_{ik} \cdot U_j \right] \\ C^k &= \sum_{j=1}^m A_j \cdot B^k \cdot U_j \\ &= \sum_{j=1}^m \left[ \sum_{i=1}^n a_{ji} b_{ik} \cdot U_j \right] \end{aligned}$$

Two forms are essentially the same if you expand them and compare. Thus, we have proved that the proposed formula is an equivalence of the original definition.

18. **Exercise 18**

(a)  $A^{-1} = (I + A) \Rightarrow A \cdot A^{-1} = I^2 - A^2 = I$

(b)  $A^{-1} = (I^2 + IA + A^2) \Rightarrow A \cdot A^{-1} = I^3 - A^3 = I$

(c) For real number  $I$  and  $A$ , we see that  $I^n - A^n$  can be factored into  $I - A$  and another polynomial, because according to remainder theorem, plugging in  $I = A$  results in  $I^n - A^n = 0$ . Thus, we could follow the same pattern to construct always a  $A^{-1}$ .

(d) Set  $A^{-1} = (-A - 2I)$

(e) Set  $A^{-1} = (-A^2 - A)$

19. **Exercise 19**

$$AB = \begin{pmatrix} 1 & ab \\ 0 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}$$

Inductive step:

$$\begin{aligned} A^{n+1} &= A^n \cdot A \\ &= \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix} \cdot A \\ &= \begin{pmatrix} 1 & (n+1)a \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Thus, we have proved it.

20. **Exercise 20**

$$A^{-1} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$$

21. **Exercise 21** We now show that  $B^{-1}A^{-1}$  would be an inverse of  $AB$ .

$$(AB)(B^{-1}A^{-1}) = A(B \cdot B^{-1})A^{-1} = A \cdot A^{-1} = I$$

And for the reverse, it is easy to verify either.

22. **Exercise 22** See the solution manual

23. **Exercise 23**

$$\begin{aligned} A^2 &= A \cdot A \\ &= \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \end{aligned}$$

Inductive step:

$$\begin{aligned} A^{n+1} &= \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} \cdot A \\ &= \begin{pmatrix} \cos n\theta \cos \theta - \sin n\theta \sin \theta & -(\sin n\theta \cos \theta + \sin \theta \cos n\theta) \\ \sin n\theta \cos \theta + \sin \theta \cos n\theta & -\sin n\theta + \cos n\theta \cos \theta \end{pmatrix} = \begin{pmatrix} \cos(n+1)\theta & -\sin(n+1)\theta \\ \sin(n+1)\theta & \cos(n+1)\theta \end{pmatrix} \end{aligned}$$

Thus, we have determined  $A^n$

24. **Exercise 24**

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

25. **Exercise 25**

- (a)  $\text{tr}(A) = 2$
- (b)  $\text{tr}(A) = 4$
- (c)  $\text{tr}(A) = 8$

26. **Exercise 26** See **Exercise 27**.

27. **Exercise 27**

$$\begin{aligned} \text{tr}(AB) &= \sum_{i=1}^n \left[ \sum_{j=1}^n a_{ij} b_{ji} \right] \\ &= \sum_{i=1}^n \left[ \sum_{j=1}^n b_{ji} a_{ij} \right] \\ &= \sum_{i=1}^n \left[ \sum_{j=1}^n b_{ij} a_{ji} \right] \quad \text{They are the same if you expand} \\ &= \text{tr}(BA) \end{aligned}$$

28. **Exercise 28** As diagonal line keeps same after transpose, trace of the matrix would not change as well.

29. **Exercise 29**  $A^n = ((a_{ij})^n)$

30. **Exercise 30**

$$A^2 = \begin{pmatrix} a_1^2 & 0 & \cdots & 0 \\ 0 & a_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^2 \end{pmatrix}$$

Inductive step

$$A^{k+1} = \begin{pmatrix} a_1^k & 0 & \cdots & 0 \\ 0 & a_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^k \end{pmatrix} \cdot A = \begin{pmatrix} a_1^{k+1} & 0 & \cdots & 0 \\ 0 & a_2^{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^{k+1} \end{pmatrix}$$

■

31. **Exercise 31** See **Exercise 35**

32. **Exercise 32** We want to show

$$\begin{aligned} {}^t(A^{-1}) &= ({}^tA)^{-1} \\ {}^t(A^{-1}) \cdot {}^t(A) &= ({}^tA)^{-1} \cdot ({}^tA) \\ {}^t(A^{-1}) \cdot {}^t(A) &= I_n \end{aligned}$$

Let  $C = {}^t(A^{-1}) \cdot {}^t(A)$ . We then know

$$\begin{aligned} c_{ij} &= \sum_{k=1}^n a'_{ik}{}^{-1} a'_{kj} \\ &= \sum_{k=1}^n a_{jk} a_{ki}^{-1} \\ &= A_j \cdot A^{-1} \cdot i \end{aligned}$$

Thus,

$$\begin{aligned} C &= {}^t(A \cdot A^{-1}) \\ &= {}^t(I_n) = I_n \end{aligned}$$

If we do it in the reverse way, then we can prove it.

33. **Exercise 33** Let  $B = {}^t(\bar{A})$ , then  $b_{ij} = \bar{a}_{ji}$ . Let  $C = \overline{{}^tA}$ , then  $c_{ij} = \bar{a}'_{ij} = \bar{a}_{ji}$ . Thus,  $B = C$ .

34. **Exercise 34** Its inverse is

$$\begin{pmatrix} \frac{1}{a_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_n} \end{pmatrix}$$

35. **Exercise 35** See solution manual. Here I would not like to introduce complex formal reasoning to simulate computation result.

36. **Exercise 36**

By result of **Exercise 35** we see that  $N^{n+1} = O$  as  $N = A - I_n$  is of the form being described in **Exercise 35**.

For inverse part, see **Exercise 18**.

37. **Exercise 37**

$$(I - N)(I + N + \cdots + N^r) = I^{r+1} - N^{r+1} = I^{r+1} = I$$

38. **Exercise 38** See solution manual for detail computation.

39. **Exercise 39** Since we know  $AB = BA$  or  $A, B$  fulfills **SP 1**, we may say

$$(AB)^r = A^r B^r = O$$

For  $(A + B)$ , we discuss  $(A + B)^{2r}$  where  $r$  is the larger  $r$  for  $A$  and  $B$ .

$$(A + B)^{2r} = \sum_{k=0}^{2r} \binom{2r}{k} A^{2r-k} B^k$$

If  $1 \geq k \leq r$ , then  $2r - k \geq r$  and  $A^{2r-k} = O$ . If  $r < k \leq 2r$ , then  $B^k = O$ . Thus, essentially,  $(A + B)^{2r} = O$ .

### 3 Chapter 3

#### 3.1 § 1

##### 3.1.1 Notes

If we want to say that  $S$  is the image of  $A$  under  $F$ , we are essentially trying to say followings:

$$\forall z \in S, \exists x : F(x) = z. \Rightarrow S \subset F(A)$$

$$\forall a \in A, F(a) \in S. \Rightarrow F(A) \subset S$$

Above are exactly what **Example 6** on Pg. 45 are saying.

Also, we shall work on the equality of two linear mappings. Two linear mappings  $F : S_1 \rightarrow T_1, G : S_2 \rightarrow T_2$  are said to be equal if and only if followings are fulfilled:

$$S_1 = S_2$$

$$T_1 = T_2$$

$$\forall z \in S_1, F(z) = G(z)$$

Proofs left to readers on Pg. 49.

If  $u_1, u_2$  are elements of  $V$ , then  $T_{u_1+u_2} = T_{u_1} \circ T_{u_2}$ .

$$\begin{aligned} \forall v \in V, T_{u_1+u_2} &= (u_1 + u_2) + v \\ &= u_1 + (u_2 + v) \\ &= T_{u_1}(u_2 + v) \\ &= T_{u_1}(T_{u_2}(v)) \\ &= T_{u_1} \circ T_{u_2}(v) \end{aligned}$$

Which means that  $T_{u_1+u_2} = T_{u_1} \circ T_{u_2}$  according to our definition of linear mapping equality.

If  $u$  is an element of  $V$ , then  $T_u : V \rightarrow V$  has an inverse mapping which is nothing but the translation  $T_{-u}$ .

First, it is easy to verify that  $T_{-u}$  is an inverse of  $T_u$ . Then, we say that there is an inverse  $T_u^{-1}$ . According to the definition of inverse of a linear mapping, we have, for every  $v \in V$  that

$$T_u^{-1}(T_u(v)) = I_V(v) = v$$

$$T_u^{-1}(v + u) = v$$

$$T_u^{-1}(x) = x - u \quad \text{Let } x = v + u$$

which attests  $T_u^{-1} = T_{-u}$ .

Here comes words on bijectivity, inverse and function composition:

1. For two mappings  $F : S_1 \rightarrow T_1$  and  $F : S_2 \rightarrow T_2$ ,  $F \circ G$  is only defined if  $T_1 = S_2$ .



2. A more clear proof for *If  $F : S \rightarrow V$  has an inverse  $G : V \rightarrow S$ , then  $F$  is bijective.* *Proof.* If  $F(x) = F(y)$  given  $x, y \in S$ , then  $G(F(x)) = G(F(y))$ . Also, since  $F, G$  are inverse of each other, we have

$$\begin{aligned}\forall s \in S, (G \circ F)(s) &= G(F(s)) = I_s(s) = s \\ \forall v \in V, (F \circ G)(v) &= F(G(v)) = I_v(v) = v\end{aligned}$$

which means  $x = G(F(x)) = G(F(y)) = y$ . Also, we contend that  $\forall v \in V, \exists x : F(x) = v$ . Since we know  $\forall v \in V, (F \circ G)(v) = F(G(v)) = I_v(v) = v$ , we can simply let  $x = G(v)$  so that  $F(x) = v$ . This proves the theorem.

### 3.1.2 Exercises

1. **Exercise 1** *Calculus involved, not doing now.*

2. **Exercise 2** Proved in notes.

3. **Exercise 3**

- (a)  $L(X) = 11$
- (b)  $L(X) = 13$
- (c)  $L(X) = 6$

4. **Exercise 4**

$$F(1) = (e, 1), F(0) = (1, 0), F(-1) = (e^{-1}, -1)$$

5. **Exercise 5**

$$(F + G)(1) = (e + 1, 3), (F + G)(2) = (e^2 + 2, 6), (F + G)(0) = (1, 0)$$

6. **Exercise 6**

$$(2F)(0) = (2, 0), (\pi F)(1) = (\pi e, \pi)$$

7. **Exercise 7** For (a), it is 1. For (b), it is 11.

8. **Exercise 8**

The image is a ellipse of the equation

$$\frac{u^2}{4} + \frac{w^2}{9} = 1$$

Proof is omitted.

9. **Exercise 9**

The image is a straight line

$$y = \frac{1}{2}x$$

*Proof.*  $A = \{(2, y) | y \in \mathbb{R}\}$ ,  $S = \{(2x, x) | x \in \mathbb{R}\}$ . We contend that  $\forall a \in A, F(a) \in S$ .

$$\forall y \in \mathbb{R}, F(2, y) = (2y, y) \in S$$

Conversely,  $\forall s = (x, \frac{1}{2}x) \in S$ , let  $a = (2, \frac{1}{2}) \in A$ , so that  $F(a) = s$ , which means  $S \subset F(A)$ .

10. **Exercise 10** It is a circle of center  $(0, 0)$  and radius  $e^c$ . Proof is omitted.

11. **Exercise 11**

It is a cylinder of radius 1 and center  $(0, 0)$ . Proof is omitted.

12. **Exercise 12**  $x^2 + y^2 = 1$ . Proof is omitted.

## 3.2 § 2

### 3.2.1 Notes

Here we have an important **Lemma**

Let  $F : V \rightarrow W$  is a linear mapping. If for some  $v_i \in V$ , we have  $F(v_i)$  are linear independent, then  $v_i$  are linear independent.

*Proof.* If  $\sum_{i=1}^n t_i v_i = O$ , then we have

$$\sum_{i=1}^n t_i v_i = O$$
$$F\left(\sum_{i=1}^n t_i v_i\right) = F(O) = O \quad (\text{This is ensured as "output" of a mapping is unique for same "input"})$$

$$\sum_{i=1}^n t_i F(v_i) = O$$

which means if  $\sum_{i=1}^n t_i v_i = O$ , we must have  $\sum_{i=1}^n t_i F(v_i) = O$ . Since  $F(v_i)$  are linear independent, we obtain that  $t_i$  is always equal to 0, which is another word for  $v_i$  are linear independent.

It is noteworthy that reversal of this **Lemma** is **NOT** always true as  $F(v) = O$  doesn't ensure that  $v = O$ . In fact, in later subsections, we shall see that  $F$  is injective if and only if  $\text{Ker } F = O$ .

### 3.2.2 Exercises

1. **Exercise 1** Only (a), (b), (d), (e), (f), (h) are linear mappings. For (h), it involves **Calculus**.

2. **Exercise 2**  $T(O) = T[v + (-v)] = T(v) + T(-v) = T(v) - T(v) = O$

3. **Exercise 3**  $T(u + v) = T(u) + T(v) = w + O = w$

4. **Exercise 4**

Let the set of elements  $v \in V$  satisfying  $T(v) = w$  be  $S$ . We contend that  $\forall v \in S, \exists u \in U : v = u + v_0$ .

*Proof.* let  $u = v - v_0$ .  $F(u) = F(v - v_0) = F(v) - F(v_0) = O$ . Thus,  $v \in U$ . This means  $S \subset (v_0 + U)$ .

Conversely, we contend that  $\forall u \in U$ , we have  $(v_0 + u) \in S$ .

*Proof.*  $T(v_0 + u) = T(v_0) + T(u) = w + O = w$ . This means  $(v_0 + U) \subset S$ .

Thus,  $S = (v_0 + U)$ .

5. **Exercise 5** As **Exercise 2** said,  $T(O) = T(v - v) = T(v) + T(-v) = O \Rightarrow T(-v) = -T(v)$ .

6. **Exercise 6**

Firstly,  $F(v_1 + v_2) = (f(v_1) + f(v_2), g(v_1) + g(v_2)) = (f(v_1), g(v_1)) + (f(v_2), g(v_2)) = F(v_1) + F(v_2)$

Secondly,  $F(cv) = (cf(v), cg(v)) = c(f(v), g(v)) = cF(v)$

7. **Exercise 7**

(a) Prove  $(u_1 + u_2) \in U$ . We have  $F(u_1 + u_2) = F(u_1) + F(u_2) = O + O = O$ .

(b) Prove  $cu \in U$ . We have  $F(cu) = cF(u) = O$ .

8. **Exercise 8** Mapping 8 is linear, others are not.

9. **Exercise 9**

By definition later introduced in §5, it is line segment between  $F(v)$  and  $F(v + w)$ .

If  $F(w) \neq O$ , then it is a line segment. If  $F(w) = O$ , then it is a point.

10. **Exercise 10** By definition, it is a parallelogram.

11. **Exercise 11** Note that  $E_1, E_2$  are standard generators. Since  $S$  is a set of points that can be written in the form  $t_1E_1 + t_2E_2$  where  $0 \leq t_1 \leq 1$  and  $0 \leq t_2 \leq 1$ . Thus,  $F(t_1E_1 + t_2E_2) = t_1F(E_1) + t_2F(E_2)$  where  $0 \leq t_1 \leq 1$  and  $0 \leq t_2 \leq 1$ . Hence, prove the statement.
12. **Exercise 12** We know  $3E_1$  and  $E_2$  are also linear independent. So are  $3F(E_1)$  and  $F(E_2)$ . Thus, adopting similar reasoning in **Exercise 11**, we prove statement.
13. **Exercise 13** It is a parallelogram generated by  $5A$  and  $2B$ .
14. **Exercise 14**  $T_u(v_1 + v_2) = v_1 + v_2 + u = T_u(v_1) + T_u(v_2) = v_1 + v_2 + 2u$ . Thus, we have  $2u = u$  and  $u = O$ .