Linear Algebra Note

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Contents

1 Chapter 1

1.1 § 1

1.1.1 Notes

A generic vector space V is not a field because there is no definition of v^{-1} for some $v \in V$, fulfilling not the definition of a field.

1. **Pg. 4 Proof of** $(-1)v = v$

$$\begin{aligned}(-1)v + v &= (-1)v + 1 \cdot v \\ &= (-1+1)v \\ &= v + (-v)\end{aligned}$$

   Thus, $(-1)v = -v$.

2. **Pg. 6 Proof of SP 3**

$$\begin{aligned}(xA) \cdot B &= \sum_{i=1}^{n} (xa_i)b_i \\ &= \sum_{i=1}^{n} x(a_i b_i) \\ &= x\sum_{i=1}^{n} a_i b_i \\ &= x(A \cdot B) \\ A \cdot (xB) &= \sum_{i=1}^{n} a_i(xb_i) \\ &= \sum_{i=1}^{n} x(a_i b_i) \\ &= x\sum_{i=1}^{n} a_i b_i \\ &= x(A \cdot B)\end{aligned}$$

3. **Pg. 7**

   Upper one:

$$\begin{aligned}(A+B)^2 &= (A+B) \cdot (A+B) \\ &= (A+B) \cdot A + (A+B) \cdot B && \text{Use SP 2} \\ &= A^2 + B \cdot A + A \cdot B + B^2 && \text{Use SP 1}\end{aligned}$$

   Bottom one: Since K is a field, all **VS** s regarding summation or product of functions are actually closed on K. By applying field axioms, V is then a vector space over K.

4. **Pg. 9**

   Let $a_1 = (u_1 + w_1), a_2 = (u_2 + w_2)$. Both of them $\in (U + W)$.
   Since U, W are subspaces of V, $U, W \in V$. Thus, $a_1, a_2 \in V$ as $u_1, w_1, u_2, w_2 \in V$, moreover, $(U + W) \subset V$.
   $a_1 + a_2 = (u_1 + u_2) + (w_1 + w_2) \in (U + W)$
   $ca_1 = c(u_1 + w_1) = (cu_1) + (cw_1) \in (U + W)$
   Since $O \in U$ and $O \in W$, $O = O + O \in (U + W)$. Thus, $(U + W)$ is a subspace of V.

1.1.2 Exercises

1. **Exercise 1** Let $v \in V$, $c[v + (-v)] = cv + c(-v) = cv + (-c)v = v \cdot 0 = v \cdot (1-1) = v + (-v) = O$

2. **Exercise 2** Since $c \neq 0$

$$
\begin{aligned}
O &= cv + [-(cv)] \\
cv &= cv + [-(cv)] \\
O &= -(cv) \\
\frac{-1}{c} \cdot O &= (-c)v \cdot \frac{-1}{c} \\
\frac{-1}{c} \cdot (v - v) &= v \\
\frac{-1}{c} \cdot v + \frac{1}{c} \cdot v &= v \\
v \cdot (1-1) &= v \\
v - v &= v \\
O &= v
\end{aligned}
$$

3. **Exercise 3**

   $\forall g \in V, (g+f)(x) = g(x) + f(x) = f(x) + g(x) = (f+g)(x) \Rightarrow g + f = f + g$.
   If $O + u = u$, $(O+u)(x) = O(x) + u(x) = u(x)$. Therefore, $O(x) = 0$.

4. **Exercise 4**

$$
\begin{aligned}
v + w &= O \\
v + w &= v + (-v) \\
w &= -v
\end{aligned}
$$

5. **Exercise 5**

$$
\begin{aligned}
v + w &= v \\
v + (-v) + w &= v + (-v) \\
O + w &= O
\end{aligned}
$$

   Since $\forall u, O + u = u$, we have $w = O$.

6. **Exercise 6**

   Let $W = \{B | B \cdot A_1 = O \text{ and } B \cdot A_2 = O\}$. Specifically, it is clear that $O \in W$ as $O \cdot A = \sum_{i=1}^{n} b_i a_i = \sum_{i=1}^{n} 0 \times a_i = 0$.
   Let $v_1, v_2 \in W$ such that $v_1 \cdot A_1 = 0$, $v_1 \cdot A_2 = 0$, $v_2 \cdot A_1 = 0$, $v_2 \cdot A_2 = 0$. Thus,

$$
\begin{aligned}
(v_1 + v_2) \cdot A_1 &= v_1 \cdot A_1 + v_2 \cdot A_1 \\
&= O + O \\
&= O \\
[c(v_1 + v_2)] \cdot A_1 &= (cv_1 + cv_2) \cdot A_1 \\
&= (cv_1) \cdot A_1 + (cv_2) \cdot A_1 \\
&= c(v_1 \cdot A_1 + v_2 \cdot A_1) \\
&= cO \\
&= O
\end{aligned}
$$

   . It is easy to show for A_2 then. Therefore, $(v_1 + v_2) \in W$.

7. **Exercise 7** Same to apply as Exercise 6.

8. **Exercise 8**

   Name the set as W.

(a) Proof
$v_1 + v_2 = (x_1 + x_2, y_1 + y_2), x_1 + x_2 = y_1 + y_2 \Rightarrow (v_1 + v_2) \in W$
$cv = (cx, cy), cx = cy \Rightarrow cv \in W$
$O = (0, 0) \in W$

(b) Proof See Part (a).

(c) Proof Same technique as in Part (a).

9. **Exercise 9** See Exercise 8.

10. **Exercise 10**

For $U \cap W$, let $v_1, v_2 \in U \cap W$. Since $v_1, v_2 \in U$ and U is a subspace, $v_1 + v_2 \in U$. In same way, we can see that $v_1 + v_2 \in W$. Thus, $v_1 + v_2 \in U \cap W$.
Since $v_1 \in U$, $cv_1 \in U$. Also, it shows $cv_1 \in W$ in the same way. Thus, $cv_1 \in U \cap W$. Because U, W are subspaces, $O \in U$ and $O \in W$. Thus, $O \in U \cap W$. Therefore, $U \cap W$ is a subspace.
Refer to the note part for proof for $U + W$.

11. **Exercise 11** Since L is a field, **VS1, VS3, VS4, VS8** are established under field axioms, and multiplication and addition are closed in L. For **VS5, VS6, VS7**, they are all valid as $K \subset L$. O is simply 0, and $1 \cdot u = u$ is established in L.

12. **Exercise 12**

For $x, y \in K$, we have
$x + y = a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2} = (a_1 + a_2) + (b_1 + b_2)\sqrt{2}$. Since $a_1, b_1, a_2, b_2 \in \mathbb{Q}$, $(a_1 + a_2), (b_1 + b_2) \in \mathbb{Q}$. Thus, $x + y \in K$.
$xy = (a_1a_2 + 2b_1b_2) + (a_2b_1 + a_1b_2) \times \sqrt{2}$. Since $a_1, b_1, a_2, b_2 \in \mathbb{Q}$, $(a_1a_2 + 2b_1b_2), (a_2b_1 + a_1b_2) \in \mathbb{Q}$. Thus, $x + y \in K$.
$-x = -a + -b\sqrt{2}$. Since $a, b \in \mathbb{Q}$, $-a, -b \in \mathbb{Q}$. Thus, $-x \in K$.
If $a + b\sqrt{2} \neq 0$, $a, b \neq 0$, and $a - b\sqrt{2} \neq 0$. Thus, $x^{-1} = \frac{1}{a+b\sqrt{2}} = \frac{a-b\sqrt{2}}{a^2-2b^2} = \frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2}$. It is easy to see that **new** $a, b \in \mathbb{Q}$ as $a, b \in \mathbb{Q}$. Thus, $x^{-1} \in K$. Specifically, if $a = b = 0$, $0 \in \mathbb{Q}$. If $a = 1, b = 0$, $1 \in \mathbb{Q}$. Thus, K is a field.

13. **Exercise 13** Same technique as Exercise 12.

14. **Exercise 14** Same technique as Exercise 12.

1.2 § 2

1.2.1 Exercises

1. **Exercise 1** Using result from **Exercise 4**, easy to prove.

2. **Exercise 2**

(a) $(1, -1)$
(b) $(\frac{1}{2}, \frac{3}{2})$
(c) $(1, 1)$
(d) $(3, 2)$

3. **Exercise 3** Same technique as in **Exercise 2**.

4. **Exercise 4**

Following set of equations is an equivalent of $x(a, b) + y(c, d) = O$,

$$ax + cy = 0 \quad (1)$$
$$bx + dy = 0 \quad (2)$$

$$
\begin{aligned}
(1)\times d-(2)\times c \Rightarrow (ad-cb)x+cdy-cdy&=0\\
(ad-cb)x&=0
\end{aligned}
$$

For $ad-cb\neq 0$ part, clearly we shall see that $x=0$ as $(ad-cb)x=0$. Plugging x back to (1), we get $y=0$. Thus, two vectors are linear independent.
For $ad-cb=0$ part, we need to prove that $x(a,b)+y(c,d)=O$ has solution other than $x=y=0$.
First, suppose $a,b,c,d\neq 0$. Since $ad-cb=0$, $x\in\mathbb{R}$. By applying technique, we could also show $y\in\mathbb{R}$. Thus, (a,b), (c,d) are linear independent.
If $a,b,c,d\neq 0$ does **NOT** hold. Without lose of generality (for all the possibilities, a,d and $c.b$ are interchangeable), consider following scenarios in a xy -plane,

(a) $a=0,c=0$
If $a=c=0$, $x,y\in\mathbb{R}$ in (1). Because the (2) is a line in the plane, there must exist some $x,y\neq 0$.

(b) $a=0,b=0,c=0$
Same argument as above, despite the line represented by (2) is a little bit peculiar (it is $y=0$).

(c) $a=0,d=0,c=0$
Same argument as the first, despite the line represented by (2) is a little bit peculiar (it is $x=0$).

(d) $a=0,d=0,b=0,c=0$
Both $(1),(2)$ represent the whole plane, thus, $x,y\in\mathbb{R}$.

5. **Exercise 9**

$$
\begin{aligned}
\sum_{i=1}^{r}[a_i\cdot(A_i\cdot\sum_{j=i+1}^{r}A_j)]&=O && \text{All vectors are mutually perpendicular}\\
&=\sum_{i=1}^{r}[(a_i\cdot A_i)\cdot\sum_{j=i+1}^{r}A_j]
\end{aligned}
$$

Since $\forall A\in\{A_i\}, A\neq O$, it is only possible that every a is 0. Thus, A_i are linearly independent.