Linear Algebra Notes

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1 Chapter 1

1.1 § 1

1.1.1 Notes

A generic vector space V is not a field because there is no definition of v^{-1} for some $v \in V$, fulfilling not the definition of a field.

1. **Pg. 4 Proof of** $(-1)v = v$

$$\begin{aligned}(-1)v + v &= (-1)v + 1 \cdot v \\ &= (-1+1)v \\ &= v + (-v)\end{aligned}$$

   Thus, $(-1)v = -v$.

2. **Pg. 6 Proof of SP 3**

$$\begin{aligned}(xA) \cdot B &= \sum_{i=1}^{n} (xa_i)b_i \\ &= \sum_{i=1}^{n} x(a_i b_i) \\ &= x \sum_{i=1}^{n} a_i b_i \\ &= x(A \cdot B) \\ A \cdot (xB) &= \sum_{i=1}^{n} a_i(xb_i) \\ &= \sum_{i=1}^{n} x(a_i b_i) \\ &= x \sum_{i=1}^{n} a_i b_i \\ &= x(A \cdot B)\end{aligned}$$

3. **Pg. 7**

   Upper one:

$$\begin{aligned}(A+B)^2 &= (A+B) \cdot (A+B) \\ &= (A+B) \cdot A + (A+B) \cdot B && \text{Use SP 2} \\ &= A^2 + B \cdot A + A \cdot B + B^2 && \text{Use SP 1}\end{aligned}$$

   Bottom one: Since K is a field, all **VS** s regarding summation or product of functions are actually closed on K. By applying field axioms, V is then a vector space over K.

4. **Pg. 9**

   Let $a_1 = (u_1 + w_1), a_2 = (u_2 + w_2)$. Both of them $\in (U + W)$.
   Since U, W are subspaces of V, $U, W \in V$. Thus, $a_1, a_2 \in V$ as $u_1, w_1, u_2, w_2 \in V$, moreover, $(U + W) \subset V$.
   $a_1 + a_2 = (u_1 + u_2) + (w_1 + w_2) \in (U + W)$
   $ca_1 = c(u_1 + w_1) = (cu_1) + (cw_1) \in (U + W)$
   Since $O \in U$ and $O \in W$, $O = O + O \in (U + W)$. Thus, $(U + W)$ is a subspace of V.

1.1.2 Exercises

1. **Exercise 1** Let $v \in V$, $c[v+(-v)] = cv + c(-v) = cv + (-c)v = v \cdot 0 = v \cdot (1-1) = v + (-v) = O$

2. **Exercise 2** Since $c \neq 0$

$$\begin{aligned} O &= cv + [-(cv)] \\ cv &= cv + [-(cv)] \\ O &= -(cv) \\ \frac{-1}{c} \cdot O &= (-c)v \cdot \frac{-1}{c} \\ \frac{-1}{c} \cdot (v - v) &= v \\ \frac{-1}{c} \cdot v + \frac{1}{c} \cdot v &= v \\ v \cdot (1-1) &= v \\ v - v &= v \\ O &= v \end{aligned}$$

3. **Exercise 3**

   $\forall g \in V, (g+f)(x) = g(x) + f(x) = f(x) + g(x) = (f+g)(x) \Rightarrow g + f = f + g$.
   If $O + u = u$, $(O+u)(x) = O(x) + u(x) = u(x)$. Therefore, $O(x) = 0$.

4. **Exercise 4**

$$\begin{aligned} v + w &= O \\ v + w &= v + (-v) \\ w &= -v \end{aligned}$$

5. **Exercise 5**

$$\begin{aligned} v + w &= v \\ v + (-v) + w &= v + (-v) \\ O + w &= O \end{aligned}$$

   Since $\forall u, O + u = u$, we have $w = O$.

6. **Exercise 6**

   Let $W = \{B | B \cdot A_1 = O \text{ and } B \cdot A_2 = O\}$. Specifically, it is clear that $O \in W$ as $O \cdot A = \sum_{i=1}^{n} b_i a_i = \sum_{i=1}^{n} 0 \times a_i = 0$.
   Let $v_1, v_2 \in W$ such that $v_1 \cdot A_1 = 0$, $v_1 \cdot A_2 = 0$, $v_2 \cdot A_1 = 0$, $v_2 \cdot A_2 = 0$. Thus,

$$\begin{aligned} (v_1 + v_2) \cdot A_1 &= v_1 \cdot A_1 + v_2 \cdot A_1 \\ &= O + O \\ &= O \\ [c(v_1 + v_2)] \cdot A_1 &= (cv_1 + cv_2) \cdot A_1 \\ &= (cv_1) \cdot A_1 + (cv_2) \cdot A_1 \\ &= c(v_1 \cdot A_1 + v_2 \cdot A_1) \\ &= cO \\ &= O \end{aligned}$$

   . It is easy to show for A_2 then. Therefore, $(v_1 + v_2) \in W$.

7. **Exercise 7** Same to apply as Exercise 6.

8. **Exercise 8**

   Name the set as W.

(a) Proof
$v_1 + v_2 = (x_1 + x_2, y_1 + y_2), x_1 + x_2 = y_1 + y_2 \Rightarrow (v_1 + v_2) \in W$
$cv = (cx, cy), cx = cy \Rightarrow cv \in W$
$O = (0, 0) \in W$

(b) Proof See Part (a).

(c) Proof Same technique as in Part (a).

9. **Exercise 9** See Exercise 8.

10. **Exercise 10**

For $U \cap W$, let $v_1, v_2 \in U \cap W$. Since $v_1, v_2 \in U$ and U is a subspace, $v_1 + v_2 \in U$. In same way, we can see that $v_1 + v_2 \in W$. Thus, $v_1 + v_2 \in U \cap W$.
Since $v_1 \in U$, $cv_1 \in U$. Also, it shows $cv_1 \in W$ in the same way. Thus, $cv_1 \in U \cap W$. Because U, W are subspaces, $O \in U$ and $O \in W$. Thus, $O \in U \cap W$. Therefore, $U \cap W$ is a subspace.
Refer to the note part for proof for $U + W$.

11. **Exercise 11** Since L is a field, **VS1, VS3, VS4, VS8** are established under field axioms, and multiplication and addition are closed in L. For **VS5, VS6, VS7**, they are all valid as $K \subset L$. O is simply 0, and $1 \cdot u = u$ is established in L.

12. **Exercise 12**

For $x, y \in K$, we have
$x + y = a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2} = (a_1 + a_2) + (b_1 + b_2)\sqrt{2}$. Since $a_1, b_1, a_2, b_2 \in \mathbb{Q}$, $(a_1 + a_2), (b_1 + b_2) \in \mathbb{Q}$. Thus, $x + y \in K$.
$xy = (a_1a_2 + 2b_1b_2) + (a_2b_1 + a_1b_2) \times \sqrt{2}$. Since $a_1, b_1, a_2, b_2 \in \mathbb{Q}$, $(a_1a_2 + 2b_1b_2), (a_2b_1 + a_1b_2) \in \mathbb{Q}$. Thus, $x + y \in K$.
$-x = -a + -b\sqrt{2}$. Since $a, b \in \mathbb{Q}$, $-a, -b \in \mathbb{Q}$. Thus, $-x \in K$.
If $a + b\sqrt{2} \neq 0$, $a, b \neq 0$, and $a - b\sqrt{2} \neq 0$. Thus, $x^{-1} = \frac{1}{a+b\sqrt{2}} = \frac{a-b\sqrt{2}}{a^2-2b^2} = \frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2}$. It is easy to see that **new** $a, b \in \mathbb{Q}$ as $a, b \in \mathbb{Q}$. Thus, $x^{-1} \in K$. Specifically, if $a = b = 0$, $0 \in \mathbb{Q}$. If $a = 1, b = 0$, $1 \in \mathbb{Q}$. Thus, K is a field.

13. **Exercise 13** Same technique as Exercise 12.

14. **Exercise 14** Same technique as Exercise 12.

1.2 § 2

1.2.1 Notes

Another quite helpful equivalent of definition of linear independence is that (stated following without loss of generality)

$$\forall a_1 \neq 0, a_1v_1 \neq \sum_{i=2}^{n} a_i$$

Here is the *proof* of equivalence between above statement and definition of linear independence.
Since $a_1 \neq 0$,

$$v_1 \neq \sum_{i=2}^{n} \frac{a_i}{a_1} v_i$$

$$O \neq -v_1 + \sum_{i=2}^{n} \frac{a_i}{a_1} v_i$$

$$\lambda O \neq (-\lambda)v_1 + \sum_{i=2}^{n} \frac{\lambda a_i}{a_1} v_i \quad \lambda \in K \text{ and } \lambda \neq 0$$

$$O \neq (-\lambda)v_1 + \sum_{i=2}^{n} \frac{\lambda a_i}{a_1} v_i \quad \lambda \in K \text{ and } \lambda \neq 0$$

λ and a_i could be arbitrary, thus from above we could conclude that $a_1' v_1 \neq \sum_{i=2}^{n} a_i'$ if and only if all $a' = 0$, which is the definition of linear independence.
Also, another point that worth paying attention to is that generators could be **linear dependent**. This is true because you could put arbitrary vectors at the end of a basis of a vector space and just set coefficients for these extraneous vectors when it is producing new linear combinations.

1.2.2 Exercises

1. **Exercise 1** Using result from **Exercise 4**, easy to prove.
2. **Exercise 2**
   (a) $(1, -1)$
   (b) $(\frac{1}{2}, \frac{3}{2})$
   (c) $(1, 1)$
   (d) $(3, 2)$
3. **Exercise 3**
   (a) $(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$
   (b) $(1, 0, 1)$
   (c) $(\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3})$
4. **Exercise 4**

   Following set of equations is an equivalent of $x(a,b) + y(c,d) = O$,

$$
\begin{aligned}
ax + cy = 0 \quad (1)\\
bx + dy = 0 \quad (2)
\end{aligned}
$$

$$
\begin{aligned}
(1) \times d - (2) \times c \Rightarrow (ad - cb)x + cdy - cdy &= 0 \\
(ad - cb)x &= 0
\end{aligned}
$$

   For $ad - cb \neq 0$ part, clearly we shall see that $x = 0$ as $(ad - cb)x = 0$. Plugging x back to (1), we get $y = 0$. Thus, two vectors are linear independent.
   For $ad - cb = 0$ part, we need to prove that $x(a,b) + y(c,d) = O$ has solution other than $x = y = 0$.
   First, suppose $a, b, c, d \neq 0$. Since $ad - cb = 0$, $x \in \mathbb{R}$. By applying technique, we could also show $y \in \mathbb{R}$. Thus, (a,b), (c,d) are linear independent.
   If $a, b, c, d \neq 0$ does **NOT** hold. Without lose of generality (for all the possibilities, a, d and $c.b$ are interchangeable), consider following scenarios in a xy -plane,

   (a) $a = 0, c = 0$
   If $a = c = 0$, $x, y \in \mathbb{R}$ in (1). Because the (2) is a line in the plane, there must exist some $x, y \neq 0$.
   (b) $a = 0, b = 0, c = 0$
   Same argument as above, despite the line represented by (2) is a little bit peculiar (it is $y = 0$).
   (c) $a = 0, d = 0, c = 0$
   Same argument as the first, despite the line represented by (2) is a little bit peculiar (it is $x = 0$).
   (d) $a = 0, d = 0, b = 0, c = 0$
   Both (1), (2) represent the whole plane, thus, $x, y \in \mathbb{R}$.
5. **Exercise 5,6**

   To correctly understand how could functions be elements(vectors) in vector space, we need to understand that function $f : S \to K$ is essentially a set of pairs $(s, k), \forall s \in S$. Functions have scalar multiplication and

addition defined.
$f+g$ is defined as $\{(s, f(s)+g(s)) | s \in S\}$, and $cf, c \in K$ is defined as $\{(s, c \cdot f(s)) | s \in S\}$.
It is easy to verify that V of every $f : S \to K$ is a vector space over K. Particularly, O for V is $\{(s, 0) | s \in S\}$.
So like other vector spaces, linear dependence is **about**

$$f_{sum} = \sum_{i=1}^{n} a_i f_i = O$$

Since right-hand-side of the equation is $\{(s, 0) | s \in S\}$, we can say that $\forall v \in V, f_s um(s) = 0$. This is useful in solving problems in **Exercise 5** and **Exercise 6**.
For example, we need to show that $f(s) = 1$ and $g(s) = t$ are linear independent. This means that we need to consider following equation,

$$af + bg = O$$

which is an equivalent of

$$\forall t, a + bt = 0$$

Above conversion is quite helpful since we could put in arbitrary t and the equation should holds. Thus, we could put in particular values of t to **construct** set of equations to show that $a = b = 0$. For example, here we plug in $t = 0$, then $a = 0$, and if we plug back $a = 0$ into original equation with $t = 0$ again, $b = 0$.
This method could be used throughout **Exercise 5,6**.

6. **Exercise 7** $(3, 5)$

7. **Exercise 8** ***Calculus involved, not doing now.***

8. **Exercise 9**

$$\sum_{i=1}^{r} [a_i \cdot (A_i \cdot \sum_{j=i+1}^{r} A_j)] = O \qquad \text{All vectors are mutually perpendicular}$$

$$= \sum_{i=1}^{r} [(a_i \cdot A_i) \cdot \sum_{j=i+1}^{r} A_j]$$

   Since $\forall A \in \{A_i\}, A \neq O$, it is only possible that every a is 0. Thus, A_i are linearly independent.

9. **Exercise 10**

   Since v, w are linear dependent, for

$$nv + mw = O$$

   at least one of $n, m \neq 0$. Consider following scenarios, we can see that there would be $a = 0$ or $a = -\frac{n}{m}$.

   (a) $n = 0, m \neq 0 \Rightarrow w = O$
   (b) $n \neq 0, m = 0 \Rightarrow v = O$. This contradicts with $v \neq O$ in problem. Thus, this is impossible.
   (c) $n \neq 0, m \neq 0 \Rightarrow w = \frac{-n}{m} v$

1.3 § 3

1.3.1 Notes

This subsection comprises a lot of concise proofs. But in conclusion, we need to know that

Basis $\Leftrightarrow$ Maximal linear independent vector set	proof at **Theorem3.1**
Basis $\Leftrightarrow$ Maximal linear indpendent vector set $\Rightarrow$ Generators	proof at **Theorem2.2**
Generators $\nRightarrow$ Basis	Generators are not always linear independent.

Thus, all possible bases of a vector space V are of one and only one possible number of elements, which is equal to the one of maximal independent vector set.

1.4 § 4

1.4.1 Notes

Proof for

$$\dim(U \times W) = \dim U + \dim W$$

Because $\forall u \in (U \times W), (O_u + O_w) + u = u + (O_u + O_w) = u$. Thus, by definition, $O = (O_u, O_w)$.
Let $A = \{u_i\}$ be a basis of U and $B = \{w_i\}$ be a basis of W. Note the dimension of U, W as n, m respectively. Let

$$C = \{(u_i, 0)|u_i \in A\} \cup \{(0, w_i)|w_i \in B\}$$

Since there would be no intersection between two sets being union above, the number of elements in C is $n + m$.
If we could show that C is a basis of $U \times W$, then we could show the original statement.
First we need to show that all elements in C is linear independent. This means $a_i \in K, c_i \in C$

$$\sum_{i=1}^{n+m} a_i c_i = O$$

if and only if all the $a_i = 0$.
Because multiplication by scalar and addition for $U \times W$ is defined componentwise, we shall see that (if we keep the "order" of elements in C as A and B are merged)

$$\sum_{i=1}^{n} a_i u_i = O_u$$
$$\sum_{i=n+1}^{n+m} a_i w_i = O_w$$

Since both A and B are basis of U and W respectively, all the a_i should be 0.
Now, we need to show that C generates $U \times W$. Since A and B are basis of U and W respectively,

$$\forall (a, b) \in (U \times W), \exists f_i, g_i \in K : \sum_{i=1}^{n} f_i u_i = a \text{ and } \sum_{i=1}^{m} g_i w_i = b$$

Thus, by setting set of scalar for "order"-kept C as $\{f_i\} \cup \{g_i\}$, it is easy to see that it generates $U \times W$.
Therefore, we see that

$$\dim(U \times W) = \dim U + \dim W$$

and

$$\{(u_i, 0)|u_i \in A\} \cup \{(0, w_i)|w_i \in B\}$$

is a basis for $U \times W$.

1.4.2 Exercises

1. **Exercise 1**

   For the first part, we need to show that $\forall v \in V, \exists$ unique $u \in U, w \in W : v = u + w$. Since $(2, 1)$ and $(0, 1)$ are linear independent, they are a basis of $V = \mathbb{R}^2$. This means

   $$\forall v \in V, \exists \text{ unique } a, b \in K : v = a \cdot (2, 1) + b \cdot (0, 1)$$

   Thus, just set $u = a \cdot (2, 1)$ and $w = b \cdot (0, 1)$, and we have proved it.
   It is same for $(2, 1)$ and $(1, 1)$.

2. **Exercise 2**

   Since $(1,0,0),(1,1,0),(0,1,1)$ are linear independent, we obtain that

   $$\forall v \in V, \exists \text{ unique } a,b,c \in K : v = a \cdot (1,0,0) + b \cdot (1,1,0) + c \cdot (0,1,1)$$

   Set $u = a \cdot (1,0,0)$ and $w = b \cdot (1,1,0) + c \cdot (0,1,1)$, it would be proved.

3. **Exercise 3** According to argument provided here, $\forall c \in K, cA \neq B$ means that A, B are linear independent.
   Also, according to **Theorem 3.4**, they are a basis of $\mathbb{R}^2$.
   Based on the similar argument in **Exercise 1**, second part could be proved.

4. **Exercise 4** See notes