

# Linear Algebra Notes

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# 1 Chapter 1

## 1.1 § 1

### 1.1.1 Notes

A generic vector space  $V$  is not a field because there is no definition of  $v^{-1}$  for some  $v \in V$ , fulfilling not the definition of a field.

#### 1. Pg. 4 Proof of $(-1)v = -v$

$$\begin{aligned}(-1)v + v &= (-1)v + 1 \cdot v \\&= (-1 + 1)v \\&= v + (-v)\end{aligned}$$

Thus,  $(-1)v = -v$ .

#### 2. Pg. 6 Proof of SP 3

$$\begin{aligned}(xA) \cdot B &= \sum_{i=1}^n (xa_i)b_i \\&= \sum_{i=1}^n x(a_ib_i) \\&= x \sum_{i=1}^n a_ib_i \\&= x(A \cdot B) \\A \cdot (xB) &= \sum_{i=1}^n a_i(xb_i) \\&= \sum_{i=1}^n x(a_ib_i) \\&= x \sum_{i=1}^n a_ib_i \\&= x(A \cdot B)\end{aligned}$$

#### 3. Pg. 7

Upper one:

$$\begin{aligned}(A+B)^2 &= (A+B) \cdot (A+B) \\&= (A+B) \cdot A + (A+B) \cdot B \quad \text{Use SP 2} \\&= A^2 + B \cdot A + A \cdot B + B^2 \quad \text{Use SP 1}\end{aligned}$$

Bottom one: Since  $K$  is a field, all **VS** s regarding summation or product of functions are actually closed on  $K$ . By applying field axioms,  $V$  is then a vector space over  $K$ .

#### 4. Pg. 9

Let  $a_1 = (u_1 + w_1), a_2 = (u_2 + w_2)$ . Both of them  $\in (U + W)$ .

Since  $U, W$  are subspaces of  $V$ ,  $U, W \in V$ . Thus,  $a_1, a_2 \in V$  as  $u_1, w_1, u_2, w_2 \in V$ , moreover,  $(U + W) \subset V$ .

$$a_1 + a_2 = (u_1 + u_2) + (w_1 + w_2) \in (U + W)$$

$$ca_1 = c(u_1 + w_1) = (cu_1) + (cw_1) \in (U + W)$$

Since  $O \in U$  and  $O \in W$ ,  $O = O + O \in (U + W)$ . Thus,  $(U + W)$  is a subspace of  $V$ .

### 1.1.2 Exercises

1. **Exercise 1** Let  $v \in V$ ,  $c[v + (-v)] = cv + c(-v) = cv + (-c)v = v \cdot 0 = v \cdot (1 - 1) = v + (-v) = O$

2. **Exercise 2** Since  $c \neq 0$

$$\begin{aligned} O &= cv + [-(cv)] \\ cv &= cv + [-(cv)] \\ O &= -(cv) \\ \frac{-1}{c} \cdot O &= (-c)v \cdot \frac{-1}{c} \\ \frac{-1}{c} \cdot (v - v) &= v \\ \frac{-1}{c} \cdot v + \frac{1}{c} \cdot v &= v \\ v \cdot (1 - 1) &= v \\ v - v &= v \\ O &= v \end{aligned}$$

3. **Exercise 3**

$\forall g \in V, (g + f)(x) = g(x) + f(x) = f(x) + g(x) = (f + g)(x) \Rightarrow g + f = f + g$ .  
If  $O + u = u$ ,  $(O + u)(x) = O(x) + u(x) = u(x)$ . Therefore,  $O(x) = 0$ .

4. **Exercise 4**

$$\begin{aligned} v + w &= O \\ v + w &= v + (-v) \\ w &= -v \end{aligned}$$

5. **Exercise 5**

$$\begin{aligned} v + w &= v \\ v + (-v) + w &= v + (-v) \\ O + w &= O \end{aligned}$$

Since  $\forall u, O + u = u$ , we have  $w = O$ .

6. **Exercise 6**

Let  $W = \{B \mid B \cdot A_1 = O \text{ and } B \cdot A_2 = O\}$ . Specifically, it is clear that  $O \in W$  as  $O \cdot A = \sum_{i=1}^n b_i a_i = \sum_{i=1}^n 0 \times a_i = 0$ .  
Let  $v_1, v_2 \in W$  such that  $v_1 \cdot A_1 = 0$ ,  $v_1 \cdot A_2 = 0$ ,  $v_2 \cdot A_1 = 0$ ,  $v_2 \cdot A_2 = 0$ . Thus,

$$\begin{aligned} (v_1 + v_2) \cdot A_1 &= v_1 \cdot A_1 + v_2 \cdot A_1 \\ &= O + O \\ &= O \\ [c(v_1 + v_2)] \cdot A_1 &= (cv_1 + cv_2) \cdot A_1 \\ &= (cv_1) \cdot A_1 + (cv_2) \cdot A_1 \\ &= c(v_1 \cdot A_1 + v_2 \cdot A_1) \\ &= cO \\ &= O \end{aligned}$$

. It is easy to show for  $A_2$  then. Therefore,  $(v_1 + v_2) \in W$ .

7. **Exercise 7** Same to apply as Exercise 6.

8. **Exercise 8**

Name the set as  $W$ .

(a) Proof

$$v_1 + v_2 = (x_1 + x_2, y_1 + y_2), x_1 + x_2 = y_1 + y_2 \Rightarrow (v_1 + v_2) \in W$$

$$cv = (cx, cy), cx = cy \Rightarrow cv \in W$$

$$O = (0, 0) \in W$$

(b) Proof See Part (a).

(c) Proof Same technique as in Part (a).

9. **Exercise 9** See Exercise 8.

10. **Exercise 10**

For  $U \cap W$ , let  $v_1, v_2 \in U \cap W$ . Since  $v_1, v_2 \in U$  and  $U$  is a subspace,  $v_1 + v_2 \in U$ . In same way, we can see that  $v_1 + v_2 \in W$ . Thus,  $v_1 + v_2 \in U \cap W$ .

Since  $v_1 \in U$ ,  $cv_1 \in U$ . Also, it shows  $cv_1 \in W$  in the same way. Thus,  $cv_1 \in U \cap W$ . Because  $U, W$  are subspaces,  $O \in U$  and  $O \in W$ . Thus,  $O \in U \cap W$ . Therefore,  $U \cap W$  is a subspace.

Refer to the [note part](#) for proof for  $U + W$ .

11. **Exercise 11** Since  $L$  is a field, **VS1**, **VS3**, **VS4**, **VS8** are established under field axioms, and multiplication and addition are closed in  $L$ . For **VS5**, **VS6**, **VS7**, they are all valid as  $K \subset L$ .  $O$  is simply 0, and  $1 \cdot u = u$  is established in  $L$ .

12. **Exercise 12**

For  $x, y \in K$ , we have

$$x + y = a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2} = (a_1 + a_2) + (b_1 + b_2)\sqrt{2}. \text{ Since } a_1, b_1, a_2, b_2 \in \mathbb{Q}, (a_1 + a_2), (b_1 + b_2) \in \mathbb{Q}. \text{ Thus, } x + y \in K.$$

$$xy = (a_1a_2 + 2b_1b_2) + (a_2b_1 + a_1b_2) \times \sqrt{2}. \text{ Since } a_1, b_1, a_2, b_2 \in \mathbb{Q}, (a_1a_2 + 2b_1b_2), (a_2b_1 + a_1b_2) \in \mathbb{Q}. \text{ Thus, } xy \in K.$$

$$-x = -a - b\sqrt{2}. \text{ Since } a, b \in \mathbb{Q}, -a, -b \in \mathbb{Q}. \text{ Thus, } -x \in K.$$

If  $a + b\sqrt{2} \neq 0$ ,  $a, b \neq 0$ , and  $a - b\sqrt{2} \neq 0$ . Thus,  $x^{-1} = \frac{1}{a+b\sqrt{2}} = \frac{a-b\sqrt{2}}{a^2-2b^2} = \frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2}$ . It is easy to see that **new**  $a, b \in \mathbb{Q}$  as  $a, b \in \mathbb{Q}$ . Thus,  $x^{-1} \in K$ . Specifically, if  $a = b = 0$ ,  $0 \in \mathbb{Q}$ . If  $a = 1, b = 0$ ,  $1 \in \mathbb{Q}$ . Thus,  $K$  is a field.

13. **Exercise 13** Same technique as Exercise 12.

14. **Exercise 14** Same technique as Exercise 12.

## 1.2 § 2

### 1.2.1 Notes

Another quite helpful equivalent of definition of linear independence is that (stated following without loss of generality)

$$\forall a_1 \neq 0, a_1 v_1 \neq \sum_{i=2}^n a_i v_i$$

Here is the *proof* of equivalence between above statement and definition of linear independence.

Since  $a_1 \neq 0$ ,

$$v_1 \neq \sum_{i=2}^n \frac{a_i}{a_1} v_i$$

$$O \neq -v_1 + \sum_{i=2}^n \frac{a_i}{a_1} v_i$$

$$\lambda O \neq (-\lambda)v_1 + \sum_{i=2}^n \frac{\lambda a_i}{a_1} v_i \quad \lambda \in K \text{ and } \lambda \neq 0$$

$$O \neq (-\lambda)v_1 + \sum_{i=2}^n \frac{\lambda a_i}{a_1} v_i \quad \lambda \in K \text{ and } \lambda \neq 0$$

$\lambda$  and  $a_i$  could be arbitrary, thus from above we could conclude that  $a'_1 v_1 \neq \sum_{i=2}^n a'_i v_i$  if and only if all  $a'_i = 0$ , which is the definition of linear independence.

Also, another point that worth paying attention to is that generators could be **linear dependent**. This is true because you could put arbitrary vectors at the end of a basis of a vector space and just set coefficients for these extraneous vectors when it is producing new linear combinations.

### 1.2.2 Exercises

1. **Exercise 1** Using result from [Exercise 4](#), easy to prove.

2. **Exercise 2**

- (a)  $(1, -1)$
- (b)  $(\frac{1}{2}, \frac{3}{2})$
- (c)  $(1, 1)$
- (d)  $(3, 2)$

3. **Exercise 3**

- (a)  $(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$
- (b)  $(1, 0, 1)$
- (c)  $(\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3})$

4. **Exercise 4**

Following set of equations is an equivalent of  $x(a, b) + y(c, d) = O$ ,

$$ax + cy = 0 \quad (1)$$

$$bx + dy = 0 \quad (2)$$

$$(1) \times d - (2) \times c \Rightarrow (ad - cb)x + cdy - cdy = 0$$

$$(ad - cb)x = 0$$

For  $ad - cb \neq 0$  part, clearly we shall see that  $x = 0$  as  $(ad - cb)x = 0$ . Plugging  $x$  back to (1), we get  $y = 0$ . Thus, two vectors are linear independent.

For  $ad - cb = 0$  part, we need to prove that  $x(a, b) + y(c, d) = O$  has solution other than  $x = y = 0$ .

First, suppose  $a, b, c, d \neq 0$ . Since  $ad - cb = 0$ ,  $x \in \mathbb{R}$ . By applying technique, we could also show  $y \in \mathbb{R}$ . Thus,  $(a, b), (c, d)$  are linear independent.

If  $a, b, c, d \neq 0$  does **NOT** hold. Without lose of generality (for all the possibilities,  $a, d$  and  $c, b$  are interchangeable), consider following scenarios in a  $xy$ -plane,

(a)  $a = 0, c = 0$

If  $a = c = 0$ ,  $x, y \in \mathbb{R}$  in (1). Because the (2) is a line in the plane, there must exist some  $x, y \neq 0$ .

(b)  $a = 0, b = 0, c = 0$

Same argument as above, despite the line represented by (2) is a little bit peculiar (it is  $y = 0$ ).

(c)  $a = 0, d = 0, c = 0$

Same argument as the first, despite the line represented by (2) is a little bit peculiar (it is  $x = 0$ ).

(d)  $a = 0, d = 0, b = 0, c = 0$

Both (1), (2) represent the whole plane, thus,  $x, y \in \mathbb{R}$ .

5. **Exercise 5,6**

To correctly understand how could functions be elements(vectors) in vector space, we need to understand that function  $f : S \rightarrow K$  is essentially a set of pairs  $(s, k), \forall s \in S$ . Functions have scalar multiplication and

addition defined.

$f + g$  is defined as  $\{(s, f(s) + g(s)) | s \in S\}$ , and  $cf, c \in K$  is defined as  $\{(s, c \cdot f(s)) | s \in S\}$ .

It is easy to verify that  $V$  of every  $f : S \rightarrow K$  is a vector space over  $K$ . Particularly,  $O$  for  $V$  is  $\{(s, 0) | s \in S\}$ .

So like other vector spaces, linear dependence is **about**

$$f_{sum} = \sum_{i=1}^n a_i f_i = O$$

Since right-hand-side of the equation is  $\{(s, 0) | s \in S\}$ , we can say that  $\forall v \in V, f_{sum}(s) = 0$ . This is useful in solving problems in **Exercise 5** and **Exercise 6**.

For example, we need to show that  $f(s) = 1$  and  $g(s) = t$  are linear independent. This means that we need to consider following equation,

$$af + bg = O$$

which is an equivalent of

$$\forall t, a + bt = 0$$

Above conversion is quite helpful since we could put in arbitrary  $t$  and the equation should holds. Thus, we could put in particular values of  $t$  to **construct** set of equations to show that  $a = b = 0$ . For example, here we plug in  $t = 0$ , then  $a = 0$ , and if we plug back  $a = 0$  into original equation with  $t = 0$  again,  $b = 0$ .

This method could be used throughout **Exercise 5,6**.

6. **Exercise 7** (3, 5)

7. **Exercise 8** *Calculus involved, not doing now.*

8. **Exercise 9**

$$\begin{aligned} \sum_{i=1}^r [a_i \cdot (A_i \cdot \sum_{j=i+1}^r A_j)] &= O & \text{All vectors are mutually perpendicular} \\ &= \sum_{i=1}^r [(a_i \cdot A_i) \cdot \sum_{j=i+1}^r A_j] \end{aligned}$$

Since  $\forall A \in \{A_i\}, A \neq O$ , it is only possible that every  $a$  is 0. Thus,  $A_i$  are linearly independent.

9. **Exercise 10**

Since  $v, w$  are linear dependent, for

$$nv + mw = O$$

at least one of  $n, m \neq 0$ . Consider following scenarios, we can see that there would be  $a = 0$  or  $a = -\frac{n}{m}$ .

$$(a) \quad n = 0, m \neq 0 \Rightarrow w = O$$

$$(b) \quad n \neq 0, m = 0 \Rightarrow v = O. \text{ This contradicts with } v \neq O \text{ in problem. Thus, this is impossible.}$$

$$(c) \quad n \neq 0, m \neq 0 \Rightarrow w = -\frac{n}{m}v$$

## 1.3 § 3

### 1.3.1 Notes

This subsection comprises a lot of concise proofs. But in conclusion, we need to know that

Basis  $\Leftrightarrow$  Maximal linear independent vector set proof at **Theorem3.1**

Basis  $\Leftrightarrow$  Maximal linear independent vector set  $\Rightarrow$  Generators proof at **Theorem2.2**

Generators  $\nRightarrow$  Basis Generators are not always linear independent.

Thus, all possible bases of a vector space  $V$  are of one and only one possible number of elements, which is equal to the one of maximal independent vector set.

## 1.4 § 4

### 1.4.1 Notes

*Proof* for

$$\dim(U \times W) = \dim U + \dim W$$

Because  $\forall u \in (U \times W), (O_u + O_w) + u = u + (O_u + O_w) = u$ . Thus, by definition,  $O = (O_u, O_w)$ .

Let  $A = \{u_i\}$  be a basis of  $U$  and  $B = \{w_i\}$  be a basis of  $W$ . Note the dimension of  $U, W$  as  $n, m$  respectively. Let

$$C = \{(u_i, 0) | u_i \in A\} \cup \{(0, w_i) | w_i \in B\}$$

Since there would be no intersection between two sets being union above, the number of elements in  $C$  is  $n + m$ .

If we could show that  $C$  is a basis of  $U \times W$ , then we could show the original statement.

First we need to show that all elements in  $C$  is linear independent. This means  $a_i \in K, c_i \in C$

$$\sum_{i=1}^{n+m} a_i c_i$$

if and only if all the  $a_i = 0$ .

Because multiplication by scalar and addition for  $U \times W$  is defined componentwise, we shall see that (if we keep the "order" of elements in  $C$  as  $A$  and  $B$  are merged)

$$\begin{aligned} \sum_{i=1}^n a_i u_i &= O_u \\ \sum_{i=n+1}^{n+m} a_i w_i &= O_w \end{aligned}$$

Since both  $A$  and  $B$  are basis of  $U$  and  $W$  respectively, all the  $a_i$  should be 0.

Now, we need to show that  $C$  generates  $U \times W$ . Since  $A$  and  $B$  are basis of  $U$  and  $W$  respectively,

$$\forall (a, b) \in (U \times W), \exists f_i, g_i \in K : \sum_{i=1}^n f_i u_i = a \text{ and } \sum_{i=1}^m g_i w_i = b$$

Thus, by setting set of scalar for "order"-kept  $C$  as  $\{f_i\} \cup \{g_i\}$ , it is easy to see that it generates  $U \times W$ .

Therefore, we see that

$$\dim(U \times W) = \dim U + \dim W$$

and

$$\{(u_i, 0) | u_i \in A\} \cup \{(0, w_i) | w_i \in B\}$$

is a basis for  $U \times W$ .

### 1.4.2 Exercises

#### 1. Exercise 1

For the first part, we need to show that  $\forall v \in V, \exists$  unique  $u \in U, w \in W : v = u + w$ . Since  $(2, 1)$  and  $(0, 1)$  are linear independent, they are a basis of  $V = \mathbb{R}^2$ . This means

$$\forall v \in V, \exists \text{ unique } a, b \in K : v = a \cdot (2, 1) + b \cdot (0, 1)$$

Thus, just set  $u = a \cdot (2, 1)$  and  $w = b \cdot (0, 1)$ , and we have proved it.

It is same for  $(2, 1)$  and  $(1, 1)$ .

2. **Exercise 2**

Since  $(1, 0, 0), (1, 1, 0), (0, 1, 1)$  are linear independent, we obtain that

$$\forall v \in V, \exists \text{ unique } a, b, c \in K : v = a \cdot (1, 0, 0) + b \cdot (1, 1, 0) + c \cdot (0, 1, 1)$$

Set  $u = a \cdot (1, 0, 0)$  and  $w = b \cdot (1, 1, 0) + c \cdot (0, 1, 1)$ , it would be proved.

3. **Exercise 3** According to argument provided [here](#),  $\forall c \in K, cA \neq B$  means that  $A, B$  are linear independent. Also, according to **Theorem 3.4**, they are a basis of  $\mathbb{R}^2$ .

Based on the similar argument in **Exercise 1**, second part could be proved.

4. **Exercise 4** See notes