Linear Algebra Notes

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Contents

1 Chapter 1

1.1 § 1

1.1.1 Notes

A generic vector space V is not a field because there is no definition of v^{-1} for some $v \in V$, fulfilling not the definition of a field.

1. **Pg. 4 Proof of** $(-1)v = v$

$$\begin{aligned}(-1)v + v &= (-1)v + 1 \cdot v \\ &= (-1+1)v \\ &= v + (-v)\end{aligned}$$

   Thus, $(-1)v = -v$.

2. **Pg. 6 Proof of SP 3**

$$\begin{aligned}(xA) \cdot B &= \sum_{i=1}^{n} (xa_i) b_i \\ &= \sum_{i=1}^{n} x(a_i b_i) \\ &= x \sum_{i=1}^{n} a_i b_i \\ &= x(A \cdot B) \\ A \cdot (xB) &= \sum_{i=1}^{n} a_i (x b_i) \\ &= \sum_{i=1}^{n} x(a_i b_i) \\ &= x \sum_{i=1}^{n} a_i b_i \\ &= x(A \cdot B)\end{aligned}$$

3. **Pg. 7**

   Upper one:

$$\begin{aligned}(A+B)^2 &= (A+B) \cdot (A+B) \\ &= (A+B) \cdot A + (A+B) \cdot B && \text{Use SP 2} \\ &= A^2 + B \cdot A + A \cdot B + B^2 && \text{Use SP 1}\end{aligned}$$

   Bottom one: Since K is a field, all **VS** s regarding summation or product of functions are actually closed on K. By applying field axioms, V is then a vector space over K.

4. **Pg. 9**

   Let $a_1 = (u_1 + w_1), a_2 = (u_2 + w_2)$. Both of them $\in (U + W)$.
   Since U, W are subspaces of V, $U, W \in V$. Thus, $a_1, a_2 \in V$ as $u_1, w_1, u_2, w_2 \in V$, moreover, $(U + W) \subset V$.
   $a_1 + a_2 = (u_1 + u_2) + (w_1 + w_2) \in (U + W)$
   $ca_1 = c(u_1 + w_1) = (cu_1) + (cw_1) \in (U + W)$
   Since $O \in U$ and $O \in W$, $O = O + O \in (U + W)$. Thus, $(U + W)$ is a subspace of V.

1.1.2 Exercises

1. **Exercise 1** Let $v \in V$, $c[v + (-v)] = cv + c(-v) = cv + (-c)v = v \cdot 0 = v \cdot (1-1) = v + (-v) = O$

2. **Exercise 2** Since $c \neq 0$

$$\begin{aligned} O &= cv + [-(cv)] \\ cv &= cv + [-(cv)] \\ O &= -(cv) \\ \frac{-1}{c} \cdot O &= (-c)v \cdot \frac{-1}{c} \\ \frac{-1}{c} \cdot (v - v) &= v \\ \frac{-1}{c} \cdot v + \frac{1}{c} \cdot v &= v \\ v \cdot (1-1) &= v \\ v - v &= v \\ O &= v \end{aligned}$$

3. **Exercise 3**

   $\forall g \in V, (g+f)(x) = g(x) + f(x) = f(x) + g(x) = (f+g)(x) \Rightarrow g + f = f + g$.
   If $O + u = u$, $(O+u)(x) = O(x) + u(x) = u(x)$. Therefore, $O(x) = 0$.

4. **Exercise 4**

$$\begin{aligned} v + w &= O \\ v + w &= v + (-v) \\ w &= -v \end{aligned}$$

5. **Exercise 5**

$$\begin{aligned} v + w &= v \\ v + (-v) + w &= v + (-v) \\ O + w &= O \end{aligned}$$

   Since $\forall u, O + u = u$, we have $w = O$.

6. **Exercise 6**

   Let $W = \{B | B \cdot A_1 = O \text{ and } B \cdot A_2 = O\}$. Specifically, it is clear that $O \in W$ as $O \cdot A = \sum_{i=1}^{n} b_i a_i = \sum_{i=1}^{n} 0 \times a_i = 0$.
   Let $v_1, v_2 \in W$ such that $v_1 \cdot A_1 = 0$, $v_1 \cdot A_2 = 0$, $v_2 \cdot A_1 = 0$, $v_2 \cdot A_2 = 0$. Thus,

$$\begin{aligned} (v_1 + v_2) \cdot A_1 &= v_1 \cdot A_1 + v_2 \cdot A_1 \\ &= O + O \\ &= O \\ [c(v_1 + v_2)] \cdot A_1 &= (cv_1 + cv_2) \cdot A_1 \\ &= (cv_1) \cdot A_1 + (cv_2) \cdot A_1 \\ &= c(v_1 \cdot A_1 + v_2 \cdot A_1) \\ &= cO \\ &= O \end{aligned}$$

   . It is easy to show for A_2 then. Therefore, $(v_1 + v_2) \in W$.

7. **Exercise 7** Same to apply as Exercise 6.

8. **Exercise 8**

   Name the set as W.

(a) Proof
$v_1 + v_2 = (x_1 + x_2, y_1 + y_2), x_1 + x_2 = y_1 + y_2 \Rightarrow (v_1 + v_2) \in W$
$cv = (cx, cy), cx = cy \Rightarrow cv \in W$
$O = (0, 0) \in W$

(b) Proof See Part (a).

(c) Proof Same technique as in Part (a).

9. **Exercise 9** See Exercise 8.

10. **Exercise 10**

For $U \cap W$, let $v_1, v_2 \in U \cap W$. Since $v_1, v_2 \in U$ and U is a subspace, $v_1 + v_2 \in U$. In same way, we can see that $v_1 + v_2 \in W$. Thus, $v_1 + v_2 \in U \cap W$.
Since $v_1 \in U$, $cv_1 \in U$. Also, it shows $cv_1 \in W$ in the same way. Thus, $cv_1 \in U \cap W$. Because U, W are subspaces, $O \in U$ and $O \in W$. Thus, $O \in U \cap W$. Therefore, $U \cap W$ is a subspace.
Refer to the note part for proof for $U + W$.

11. **Exercise 11** Since L is a field, **VS1, VS3, VS4, VS8** are established under field axioms, and multiplication and addition are closed in L. For **VS5, VS6, VS7**, they are all valid as $K \subset L$. O is simply 0, and $1 \cdot u = u$ is established in L.

12. **Exercise 12**

For $x, y \in K$, we have
$x + y = a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2} = (a_1 + a_2) + (b_1 + b_2)\sqrt{2}$. Since $a_1, b_1, a_2, b_2 \in \mathbb{Q}$, $(a_1 + a_2), (b_1 + b_2) \in \mathbb{Q}$. Thus, $x + y \in K$.
$xy = (a_1a_2 + 2b_1b_2) + (a_2b_1 + a_1b_2) \times \sqrt{2}$. Since $a_1, b_1, a_2, b_2 \in \mathbb{Q}$, $(a_1a_2 + 2b_1b_2), (a_2b_1 + a_1b_2) \in \mathbb{Q}$. Thus, $x + y \in K$.
$-x = -a + -b\sqrt{2}$. Since $a, b \in \mathbb{Q}$, $-a, -b \in \mathbb{Q}$. Thus, $-x \in K$.
If $a + b\sqrt{2} \neq 0$, $a, b \neq 0$, and $a - b\sqrt{2} \neq 0$. Thus, $x^{-1} = \frac{1}{a+b\sqrt{2}} = \frac{a-b\sqrt{2}}{a^2-2b^2} = \frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2}$. It is easy to see that **new** $a, b \in \mathbb{Q}$ as $a, b \in \mathbb{Q}$. Thus, $x^{-1} \in K$. Specifically, if $a = b = 0$, $0 \in \mathbb{Q}$. If $a = 1, b = 0$, $1 \in \mathbb{Q}$. Thus, K is a field.

13. **Exercise 13** Same technique as Exercise 12.

14. **Exercise 14** Same technique as Exercise 12.

1.2 § 2

1.2.1 Notes

Another quite helpful equivalent of definition of linear independence is that (stated following without loss of generality)

$$\forall a_i \in K \text{ and some } a_i \neq 0, \text{ we have } a_1v_1 \neq \sum_{i=2}^{n} a_iv_i$$

Here is the *proof* of equivalence between above statement and definition of linear independence.

$$a_1v_1 \neq \sum_{i=2}^{n} a_iv_i$$
$$O \neq \sum_{i=1}^{n} a_iv_i$$

This means as long as **some** $a_i \neq 0$, $O \neq \sum_{i=1}^{n} a_iv_i$. In other words, only if all $a_i = 0$, $O = \sum_{i=1}^{n} a_iv_i$. This means any v_i fulfilling our statement are linear independent. Conversely, if v_i are linear independent, it is clear that as long

as **not all** $v_i = 0$, $a_1 v_1 \neq \sum\limits_{i=2}^{n} a_i v_i$, which is equal to our statement.
A simple but useful variation of this is

$$\forall v_i \in K, v_1 \neq \sum_{i=2}^{n} x_i v_i$$

Proof. We see that

$$O \neq -v_1 + \sum_{i=2}^{n} x_i v_i$$

$$O \neq (-\lambda) v_1 + \sum_{i=2}^{n} \lambda x_i v_i \quad \lambda \neq 0 \text{ (If } \lambda = 0 \text{ inequality holds not)}$$

Since λ and v_i can be arbitrary and they cannot be 0 all at once, we see it falls into the case of original statement. Also, another point that worth paying attention to is that generators could be **linear dependent**. This is true because you could put arbitrary vectors at the end of a basis of a vector space and just set coefficients for these extraneous vectors when it is producing new linear combinations.

1.2.2 Exercises

1. **Exercise 1** Using result from **Exercise 4**, easy to prove.
2. **Exercise 2**
   (a) $(1, -1)$
   (b) $(\frac{1}{2}, \frac{3}{2})$
   (c) $(1, 1)$
   (d) $(3, 2)$
3. **Exercise 3**
   (a) $(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$
   (b) $(1, 0, 1)$
   (c) $(\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3})$
4. **Exercise 4**

   Following set of equations is an equivalent of $x(a, b) + y(c, d) = O$,

$$ax + cy = 0 \quad (1)$$
$$bx + dy = 0 \quad (2)$$

$$(1) \times d - (2) \times c \Rightarrow (ad - cb)x + cdy - cdy = 0$$
$$(ad - cb)x = 0$$

   For $ad - cb \neq 0$ part, clearly we shall see that $x = 0$ as $(ad - cb)x = 0$. Plugging x back to (1), we get $y = 0$. Thus, two vectors are linear independent.
   For $ad - cb = 0$ part, we need to prove that $x(a, b) + y(c, d) = O$ has solution other than $x = y = 0$.
   First, suppose $a, b, c, d \neq 0$. Since $ad - cb = 0$, $x \in \mathbb{R}$. By applying technique, we could also show $y \in \mathbb{R}$. Thus, (a, b), (c, d) are linear independent.
   If $a, b, c, d \neq 0$ does **NOT** hold. Without lose of generality (for all the possibilities, a, d and $c.b$ are interchangeable), consider following scenarios in a xy -plane,

   (a) $a = 0, c = 0$
   If $a = c = 0$, $x, y \in \mathbb{R}$ in (1). Because the (2) is a line in the plane, there must exist some $x, y \neq 0$.

(b) $a = 0, b = 0, c = 0$

Same argument as above, despite the line represented by (2) is a little bit peculiar (it is $y = 0$).

(c) $a = 0, d = 0, c = 0$

Same argument as the first, despite the line represented by (2) is a little bit peculiar (it is $x = 0$).

(d) $a = 0, d = 0, b = 0, c = 0$

Both $(1), (2)$ represent the whole plane, thus, $x, y \in \mathbb{R}$.

5. **Exercise 5,6**

To correctly understand how could functions be elements(vectors) in vector space, we need to understand that function $f : S \to K$ is essentially a set of pairs $(s, k), \forall s \in S$. Functions have scalar multiplication and addition defined.
$f + g$ is defined as $\{(s, f(s) + g(s))|s \in S\}$, and $cf, c \in K$ is defined as $\{(s, c \cdot f(s))|s \in S\}$.
It is easy to verify that V of every $f : S \to K$ is a vector space over K. Particularly, O for V is $\{(s, 0)|s \in S\}$.
So like other vector spaces, linear dependence is **about**

$$f_{sum} = \sum_{i=1}^{n} a_i f_i = O$$

Since right-hand-side of the equation is $\{(s, 0)|s \in S\}$, we can say that $\forall v \in V, f_s um(s) = 0$. This is useful in solving problems in **Exercise 5** and **Exercise 6**.
For example, we need to show that $f(s) = 1$ and $g(s) = t$ are linear independent. This means that we need to consider following equation,

$$af + bg = O$$

which is an equivalent of

$$\forall t, a + bt = 0$$

Above conversion is quite helpful since we could put in arbitrary t and the equation should hold. Thus, we could put in particular values of t to **construct** set of equations to show that $a = b = 0$. For example, here we plug in $t = 0$, then $a = 0$, and if we plug back $a = 0$ into original equation with $t = 0$ again, $b = 0$.
This method could be used throughout **Exercise 5,6**.

6. **Exercise 7** $(3, 5)$

7. **Exercise 8** ***Calculus involved, not doing now.***

8. **Exercise 9**

$$\sum_{i=1}^{r}[a_i \cdot (A_i \cdot \sum_{j=i+1}^{r} A_j)] = O \qquad \text{All vectors are mutually perpendicular}$$

$$= \sum_{i=1}^{r}[(a_i \cdot A_i) \cdot \sum_{j=i+1}^{r} A_j]$$

Since $\forall A \in \{A_i\}, A \neq O$, it is only possible that every a is 0. Thus, A_i are linearly independent.

9. **Exercise 10**

Since v, w are linear dependent, for

$$nv + mw = O$$

at least one of $n, m \neq 0$. Consider following scenarios, we can see that there would be $a = 0$ or $a = -\frac{n}{m}$.

(a) $n = 0, m \neq 0 \Rightarrow w = O$

(b) $n \neq 0, m = 0 \Rightarrow v = O$. This contradicts with $v \neq O$ in problem. Thus, this is impossible.

(c) $n \neq 0, m \neq 0 \Rightarrow w = \frac{-n}{m} v$

1.3 § 3

1.3.1 Notes

This subsection comprises a lot of concise proofs. But in conclusion, we need to know that

Basis $\Leftrightarrow$ Maximal linear independent vector set	proof at **Theorem3.1**
Basis $\Leftrightarrow$ Maximal linear indpendent vector set $\Rightarrow$ Generators	proof at **Theorem2.2**
Generators $\nRightarrow$ Basis	Generators are not always linear independent.

Thus, all possible bases of a vector space V are of one and only one possible number of elements, which is equal to the one of maximal independent vector set.

1.4 § 4

1.4.1 Notes

Proof for

$$\dim(U \times W) = \dim U + \dim W$$

Because $\forall u \in (U \times W), (O_u + O_w) + u = u + (O_u + O_w) = u$. Thus, by definition, $O = (O_u, O_w)$.
Let $A = \{u_i\}$ be a basis of U and $B = \{w_i\}$ be a basis of W. Note the dimension of U, W as n, m respectively. Let

$$C = \{(u_i, 0)|u_i \in A\} \cup \{(0, w_i)|w_i \in B\}$$

Since there would be no intersection between two sets being union above, the number of elements in C is $n + m$.
If we could show that C is a basis of $U \times W$, then we could show the original statement.
First we need to show that all elements in C is linear independent. This means $a_i \in K, c_i \in C$

$$\sum_{i=1}^{n+m} a_i c_i = O$$

if and only if all the $a_i = 0$.
Because multiplication by scalar and addition for $U \times W$ is defined componentwise, we shall see that (if we keep the "order" of elements in C as A and B are merged)

$$\sum_{i=1}^{n} a_i u_i = O_u$$
$$\sum_{i=n+1}^{n+m} a_i w_i = O_w$$

Since both A and B are basis of U and W respectively, all the a_i should be 0.
Now, we need to show that C generates $U \times W$. Since A and B are basis of U and W respectively,

$$\forall (a, b) \in (U \times W), \exists f_i, g_i \in K : \sum_{i=1}^{n} f_i u_i = a \text{ and } \sum_{i=1}^{m} g_i w_i = b$$

Thus, by setting set of scalar for "order"-kept C as $\{f_i\} \cup \{g_i\}$, it is easy to see that it generates $U \times W$.
Therefore, we see that

$$\dim(U \times W) = \dim U + \dim W$$

and

$$\{(u_i, 0)|u_i \in A\} \cup \{(0, w_i)|w_i \in B\}$$

is a basis for $U \times W$.

1.4.2 Exercises

1. **Exercise 1**

   For the first part, we need to show that $\forall v \in V, \exists$ unique $u \in U, w \in W : v = u + w$. Since $(2,1)$ and $(0,1)$ are linear independent, they are a basis of $V = \mathbb{R}^2$. This means

   $$\forall v \in V, \exists \text{ unique } a, b \in K : v = a \cdot (2,1) + b \cdot (0,1)$$

   Thus, just set $u = a \cdot (2,1)$ and $w = b \cdot (0,1)$, and we have proved it.
   It is same for $(2,1)$ and $(1,1)$.

2. **Exercise 2**

   Since $(1,0,0), (1,1,0), (0,1,1)$ are linear independent, we obtain that

   $$\forall v \in V, \exists \text{ unique } a, b, c \in K : v = a \cdot (1,0,0) + b \cdot (1,1,0) + c \cdot (0,1,1)$$

   Set $u = a \cdot (1,0,0)$ and $w = b \cdot (1,1,0) + c \cdot (0,1,1)$, it would be proved.

3. **Exercise 3**

   $$\begin{aligned} &cA \neq B \\ &O \neq B - cA \\ &O \neq \lambda B - c\lambda A \quad \lambda \neq 0 \text{ (If } \lambda = 0 \text{ inequality holds not)} \end{aligned}$$

   Since λ, c are arbitrary and $\lambda \neq 0$, coefficients before A and B can be anything but not equal to 0 together. According to argument provided here, A, B are linear independent. Also, according to **Theorem 3.4**, they are a basis of $\mathbb{R}^2$.
   Based on the similar argument in **Exercise 1**, second part could be proved.

4. **Exercise 4** See notes

2 Chapter 2

2.1 § 1

2.1.1 Exercises

1. **Exercise 1** Skip

2. **Exercise 2** Skip

3. **Exercise 3** Skip

4. **Exercise 4** Skip

5. **Exercise 5** Let $C =^t (A + B) = (c_{ij})$. Then, $c_{ij} = (a_{ij} + b_{ij})' = a_{ji} + b_{ji}$. Thus, $C =^t A +^t B$.

6. **Exercise 6** Let $B =^t (cA)$. Then, $b_{ij} = ca_{ji}$. Since ${}^tA = (a'_{ij}) = (a_{ji}) = A$, $B = c^tA$.

7. **Exercise 7** No difference.

8. **Exercise 8** Skip

9. **Exercise 9** Skip

10. **Exercise 10** Let $B = A +^t A = (b_{ij}) = (a_{ij} + a_{ji})$. Since, $b_{ij} = a_{ij} + a_{ji} = a_{ji} + b_{ij} = b_{ji}$, B is symmetric.

11. **Exercise 11** Skip

12. **Exercise 12** Skip

13. **Exercise 13**

    For followings, we mean ones in *Exercises on Dimension* section.
    Followings are linear independent.

$$U_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$U_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$U_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$U_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

    Apply $a \cdot U_1 + b \cdot U_2 + c \cdot U_3 + d \cdot U_4 = O$ to verify it. Because it generates the matrix vector space $Mat_{2\times 2}K$ over K (For every $v \in Mat_{2\times 2}K$, simply let a, b, c, d be v's components) and $\{U_i\}$ are linear independent, $\{U_i\}$ is a basis of $Mat_{2\times 2}K$.
    Because the number of elements in a basis is the dimension of the vector space, we see that the dimension of it is 4.

14. **Exercise 14** Similar argument to **Exercise 13**. Dimension of it is mn.

15. **Exercise 15** Dimension of it is n. Simply build up a basis to see.

16. **Exercise 16** Similarly, dimension of it is $\frac{(n+1)n}{2}$.

17. **Exercise 17**

    Basis is a set comprises

$$U_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$U_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$U_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

    Then, it is easy to see that dimension is 3.

18. **Exercise 18** Basis similar to the one in **Exercise 17** is linear independent and generates space. And, indeed, the number of elements in the basis is the same as one in **Exercise 16**. Thus, dimension of it is $\frac{n(n+1)}{2}$.

19. **Exercise 19** Same as **Exercise 15**.

20. **Exercise 20**

    Let U be the subspace of V. There would be a maximal number m of linear independent vectors (**Theorem 3.1** in chapter 1). Suppose the number $m > \dim V$. Then it would contradicts **Theorem 3.1** in chapter 1 as any number of vectors more than $\dim V$ would be linear dependent, which means the basis of U would be linear dependent (remember U is a subspace of V). Thus, $m \leq \dim V$.
    Dimension could be $0, 1, 2$.

21. **Exercise 21**

    According to the lemma we proved in **Exercise 20**, dimension of subspace of $\mathbb{R}^3$ could be $0, 1, 2, 3$.

2.2 § 2

2.2.1 Notes

Lemma Let A be a set of linear dependent vectors that generates V. Then, for all $v \in V$, there exists infinite linear combinations of A that form v.
Proof Say that number of vectors in A is n. Since A generates V, $\forall v \in V, \exists \{a_i\} : v = \sum_{i=1}^{n} a_i A_i$. Let L be a set of linear combinations that form v (here L is a set of sets). We have

$$\begin{aligned} v &= \sum_{i=1}^{n} a_i A_i + O \\ &= \sum_{i=1}^{n} a_i A_i + \sum_{i=1}^{n} b_i A_i \\ &= \sum_{i=1}^{n} (a_i + b_i) A_i \end{aligned}$$

Since A is linear dependent, there exists $\{b_i\}$ where not every element is 0. Therefore, $\{a_i + b_i\} \in L$ and $\{a_i + b_i\} \neq \{a_i\}$ for some $\{b_i\}$.
This means that $\forall \ell \in L$, we can always form a new $\ell' \in L$. And since for all $v \in V$ we always have one linear combination, we can do it infinitely, which means number of elements in L is infinite. Therefore, we have shown what was to be shown. ■
Here we discuss the number of solutions for general linear equations. (A is a $m \times n$ matrix. X is a $n \times 1$ column matrix. B is a $m \times 1$ column matrix).

$$AX = B$$

If $n > m$, according to **Theorem 3.1 in chapter 1**, they must be linear dependent, resulting in infinite number of solutions because of **Lemma** above.
If $n = m$ and they are linear independent (it is then a basis because they are maximal independent vectors), there would only be one solution as **Theorem 2.1 in chapter 1** stated. If they are linear dependent and B is in the subspace generated by column vectors of A, there would be infinite number of solutions (**Lemma**), else the equations are not solvable (there exists no linear combination to represent B).
If $n < m$ and they are independent and B is in the subspace generated by column vectors of A, there would be only one solution. If they are linear independent but B is not in subspace, then it is unsolvable. If they are linear dependent and B is in subspace, infinite solutions occur. If they are linear dependent but B is not in subspace, equations are not solvable.
In general,

1. If B is in the vector space generated by column vectors of A and they are linear independent, there exists one unique solution.

2. If B is in the vector space generated by column vectors of A and they are linear dependent, there exists infinite solutions.

3. If B is not in the vector space generated by column vectors of A, there would be no solution.

2.2.2 Exercises

1. **Exercise 1** See notes and refer to the definition of linear independence.

2. **Exercise 2**

   Let u be one set of solution and w be another.
   We want to show that $u + w \in X$.

$$\sum_{i=1}^{n} (u_i + w_i) \cdot A^i = \sum_{i=1}^{n} u_i \cdot A^i + \sum_{i=1}^{n} w_i \cdot A^i = O + O = O$$

Thus, $u+w \in X$. Also, we need to show $cu \in X$ where $c \in K$.

$$c\sum_{i=1}^{n} u_i \cdot A^i = cO = O$$

Other **VS** s are easy to follow as we define the addition of vectors in X componentwise, O as a vector whose components are all zero, 1 as a vector whose components are all one.

3. **Exercise 3** We want to show following

$$\sum_{i=1}^{n}(a_i + b_i\mathrm{i})A^i = O_{\mathbb{C}}$$
$$\sum_{i=1}^{n} a_i A^i + \sum_{i=1}^{n} b_i\mathrm{i} \cdot A^i = O_{\mathbb{C}}$$
$$O_{\mathbb{C}} + \sum_{i=1}^{n} b_i\mathrm{i} \cdot A^i = O_{\mathbb{C}}$$
$$\sum_{i=1}^{n} b_i \cdot A^i = O_{\mathbb{C}}$$

This means that $\{A^i\}$ should be linear independent over $\mathbb{R}$ ($\sum_{i=1}^{n} b_i \cdot A^i = O_{\mathbb{C}}$ is equal to $\sum_{i=1}^{n} b_i \cdot A^i = O_{\mathbb{R}}$ as there is no imaginary part). Since it is known to us that $\{A^i\}$ is linear independent over $\mathbb{R}$, it has been proved as we do it reversely.

4. **Exercise 4** We know that

$$\sum_{i=1}^{n}(a_i + b_i\mathrm{i})A^i = O_{\mathbb{C}}$$

which means that $\sum_{i=1}^{n} a_i A^i = O_{\mathbb{C}}$ and/or $\sum_{i=1}^{n} b_i A^i = O_{\mathbb{C}}$. For either cases, we have shown it is linear dependent over $\mathbb{R}$ ($a_i, b_i \in \mathbb{R}$).

2.3 § 3

2.3.1 Exercises

1. **Exercise 1** $AI = IA = A$

2. **Exercise 2** $AO = O$

3. **Exercise 3**

   For every A and B, $(AB)C = A(BC)$.

   (a) Case 1

   $$\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$$

   (b) Case 2

   $$\begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

   (c) Case 3

   $$\begin{pmatrix} 33 & 37 \\ 11 & -18 \end{pmatrix}$$

4. **Exercise 4** This one could be proved as it is proved here.

5. **Exercise 5**

$$AB = \begin{pmatrix} 4 & 2 \\ 5 & -1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix}$$

6. **Exercise 6**

$$CA = AC = \begin{pmatrix} 7 & 14 \\ 21 & -7 \end{pmatrix}$$

$$CB = BC = \begin{pmatrix} 14 & 0 \\ 7 & 7 \end{pmatrix}$$

   General rule is that for symmetric one, we may have $AB = BA$? (I am not sure here).

7. **Exercise 7**

$$XA = \begin{pmatrix} 3 & 1 & 5 \end{pmatrix}$$

8. **Exercise 8**

$$X_1 A = A_2$$
$$X_2 A = A_3$$

   Let X_i be a unit vector with only i -th component equal to 1. $X_i A = A_i$

9. **Exercise 9**

   Skip the steps involving verifications. ${}^t(AB) =^t B^t A$ has already been proved in §2. Thus, ${}^t[(AB)C] =^t C \cdot^t (AB) =^t C \cdot^t B \cdot^t A$.

10. **Exercise 10**

    Firstly, we know A is of $1 \times n$, M is of $n \times n$ and B is of $1 \times n$. This means that $\dim(\langle A, B\rangle) = 1$. Also, it implies that ${}^t(\langle A, B\rangle) = \langle A, B\rangle$. Thus, we have

$$\begin{aligned} \langle A, B\rangle &=^t (\langle A, B\rangle) \\ &=^t (AM^t B) \\ &=^t ({}^t B) \cdot^t M \cdot^t A \quad \text{Exercise 9} \\ &= BM^t A \\ &= \langle B, A\rangle \end{aligned}$$

    which is **SP 1**. Also, let

$$N =^t (B + C)$$

    Then, $n_{ij} = n'_{ji} = b_{ji} + c_{ji}$. This implies also $N =^t A +^t B$. Therefore,

$$\langle A, B + C\rangle = AM^t(B + C) = AM({}^t B +^t C) = \langle A, B\rangle + \langle A, C\rangle$$

    which is **SP 2**. Finally

$$\langle cA, B\rangle = cAM^t B = c\langle A, B\rangle$$

    which is **SP 3**.

11. **Exercise 11**

    For part **(a)**, see **Exercise 35**.
    Part **(b)**

$$A^2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 1 & 4 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

12. **Exercise 12**

$$(AX)_a = \begin{pmatrix} 4 \\ 7 \\ 5 \end{pmatrix} (AX)_b = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$(AX)_c = \begin{pmatrix} x_2 \\ 0 \end{pmatrix} (AX)_d = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}$$

13. **Exercise 13**

$$(AX)_a = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$(AX)_b = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$(AX)_c = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

14. **Exercise 14**

$$(AX)_a = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$(AX)_b = \begin{pmatrix} 12 \\ 3 \\ 9 \end{pmatrix}$$

$$(AX)_c = \begin{pmatrix} 5 \\ 4 \\ 8 \end{pmatrix}$$

15. **Exercise 15** $AX = A^2$ (second column of A).

16. **Exercise 16** $AX = A^i$

17. **Exercise 17**

    Let U_i be a unit column vector which only has 1 on its i -th component. The proposed form of C^k could be written in the following way.

$$\begin{aligned} C^k &= \sum_{i=1}^{n} b_{ik} A^i \\ &= \sum_{i=1}^{n} b_{ik} [\sum_{j=1}^{m} (a_{ji} \cdot U_j)] \\ &= \sum_{i=1}^{n} [\sum_{j=1}^{m} a_{ji} b_{ik} \cdot U_j] \end{aligned}$$

$$\begin{aligned} C^k &= \sum_{j=1}^{m} A_j \cdot B^k \cdot U_j \\ &= \sum_{j=1}^{m} [\sum_{i=1}^{n} a_{ji} b_{ik} \cdot U_j] \end{aligned}$$

Two forms are essentially the same if you expand them and compare. Thus, we have proved that the proposed formula is an equivalence of the original definition.

18. **Exercise 18**

    (a) $A^{-1} = (I + A) \Rightarrow A \cdot A^{-1} = I^2 - A^2 = I$
    (b) $A^{-1} = (I^2 + IA + A^2) \Rightarrow A \cdot A^{-1} = I^3 - A^3 = I$
    (c) For real number I and A, we see that $I^n - A^n$ can be factored into $I - A$ and another polynomial, because according to remainder theorem, plugging in $I = A$ results in $I^n - A^n = 0$. Thus, we could follow the same pattern to construct always a A^{-1}.
    (d) Set $A^{-1} = (-A - 2I)$
    (e) Set $A^{-1} = (-A^2 - A)$

19. **Exercise 19**

$$AB = \begin{pmatrix} 1 & ab \\ 0 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}$$

Inductive step:

$$\begin{aligned} A^{n+1} &= A^n \cdot A \\ &= \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix} \cdot A \\ &= \begin{pmatrix} 1 & (n+1)a \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Thus, we have proved it.

20. **Exercise 20**

$$A^{-1} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$$

21. **Exercise 21** We now show that $B^{-1}A^{-1}$ would be an inverse of AB.

$$(AB)(B^{-1}A^{-1}) = A(B \cdot B^{-1})A^{-1} = A \cdot A^{-1} = I$$

And for the reverse, it is easy to verify either.

22. **Exercise 22** See the solution manual

23. **Exercise 23**

$$\begin{aligned} A^2 &= A \cdot A \\ &= \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \end{aligned}$$

Inductive step:

$$\begin{aligned} A^{n+1} &= \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} \cdot A \\ &= \begin{pmatrix} \cos n\theta \cos\theta - \sin n\theta \sin\theta & -(\sin n\theta \cos\theta + \sin\theta \cos n\theta) \\ \sin n\theta \cos\theta + \sin\theta \cos n\theta & -\sin n\theta + \cos n\theta \cos\theta \end{pmatrix} = \begin{pmatrix} \cos(n+1)\theta & -\sin(n+1)\theta \\ \sin(n+1)\theta & \cos(n+1)\theta \end{pmatrix} \end{aligned}$$

Thus, we have determined A^n

24. **Exercise 24**

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

25. **Exercise 25**

    (a) $\mathrm{tr}(A) = 2$

    (b) $\mathrm{tr}(A) = 4$

    (c) $\mathrm{tr}(A) = 8$

26. **Exercise 26** See **Exercise 27**.

27. **Exercise 27**

$$\begin{aligned}\mathrm{tr}(AB) &= \sum_{i=1}^{n}[\sum_{j=1}^{n} a_{ij}b_{ji}] \\ &= \sum_{i=1}^{n}[\sum_{j=1}^{n} b_{ji}a_{ij}] \\ &= \sum_{i=1}^{n}[\sum_{j=1}^{n} b_{ij}a_{ji}] \quad \text{They are the same if you expand} \\ &= \mathrm{tr}(BA)\end{aligned}$$

28. **Exercise 28** As diagonal line keeps same after transpose, trace of the matrix would not change as well.

29. **Exercise 29** $A^n = ((a_{ij})^n)$

30. **Exercise 30**

$$A^2 = \begin{pmatrix} a_1^2 & 0 & \cdots & 0 \\ 0 & a_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^2 \end{pmatrix}$$

Inductive step

$$A^{k+1} = \begin{pmatrix} a_1^k & 0 & \cdots & 0 \\ 0 & a_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^k \end{pmatrix} \cdot A = \begin{pmatrix} a_1^{k+1} & 0 & \cdots & 0 \\ 0 & a_2^{k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^{k+1} \end{pmatrix}$$

■

31. **Exercise 31** See **Exercise 35**

32. **Exercise 32** We want to show

$$\begin{aligned} {}^t(A^{-1}) &= ({}^tA)^{-1} \\ {}^t(A^{-1}) \cdot^t (A) &= ({}^tA)^{-1} \cdot ({}^tA) \\ {}^t(A^{-1}) \cdot^t (A) &= I_n \end{aligned}$$

Let $C =^t (A^{-1}) \cdot^t (A)$. We then know

$$\begin{aligned} c_{ij} &= \sum_{k=1}^{n} a'^{-1}_{ik} a'_{kj} \\ &= \sum_{k=1}^{n} a_{jk} a^{-1}_{ki} \\ &= A_j \cdot A^{-1\ i} \end{aligned}$$

Thus,

$$\begin{aligned} C &=^t (A \cdot A^{-1}) \\ &=^t (I_n) = I_n \end{aligned}$$

If we do it in the reverse way, then we can prove it.

33. **Exercise 33** Let $B =^t (\bar{A})$, then $b_{ij} = \bar{a}_{ji}$. Let $C = \overline{{}^tA}$, then $c_{ij} = \bar{a'}_{ij} = \bar{a}_{ji}$. Thus, $B = C$.

34. **Exercise 34** Its inverse is

$$\begin{pmatrix} \frac{1}{a_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_n} \end{pmatrix}$$

35. **Exercise 35** See solution manual. Here I would not like to introduce complex formal reasoning to simulate computation result.

36. **Exercise 36**

    By result of **Exercise 35** we see that $N^{n+1} = O$ as $N = A - I_n$ is of the form being described in **Exercise 35**.
    For inverse part, see **Exercise 18**.

37. **Exercise 37**

$$(I - N)(I + N + \cdot + N^r) = I^{r+1} - N^{r+1} = I^{r+1} = I$$

38. **Exercise 38** See solution manual for detail computation.

39. **Exercise 39** Since we know $AB = BA$ or A, B fulfills **SP 1**, we may say

$$(AB)^r = A^r B^r = O$$

    For $(A + B)$, we discuss $(A + B)^{2r}$ where r is the larger r for A and B.

$$(A + B)^{2r} = \sum_{k=0}^{2r} \binom{2r}{k} A^{2r-k} B^k$$

    If $1 \geq k \leq r$, then $2r - k \geq r$ and $A^{2r-k} = O$. If $r < k \leq 2r$, then $B^k = O$. Thus, essentially, $(A + B)^{2r} = O$.

3 Chapter 3

3.1 § 1

3.1.1 Notes

If we want to say that S is the image of A under F, we are essentially trying to say followings:

$$\forall z \in S, \exists x : F(x) = z. \Rightarrow S \subset F(A)$$
$$\forall a \in A, F(a) \in S. \Rightarrow F(A) \subset S$$

Above are exactly what **Example 6** on Pg. 45 are saying.
Also, we shall work on the equality of two linear mappings. Two linear mappings $F : S_1 \to T_1, G : S_2 \to T_2$ are said to be equal if and only if followings are fulfilled:

$$S_1 = S_2$$
$$T_1 = T_2$$
$$\forall z \in S_1, F(z) = G(z)$$

Proofs left to readers on Pg. 49.
If u_1, u_2 are elements of V, then $T_{u_1+u_2} = T_{u_1} \circ T_{u_2}$.

$$\begin{aligned} \forall v \in V, T_{u_1+u_2} &= (u_1 + u_2) + v \\ &= u_1 + (u_2 + v) \\ &= T_{u_1}(u_2 + v) \\ &= T_{u_1}(T_{u_2}(v)) \\ &= T_{u_1} \circ T_{u_2}(v) \end{aligned}$$

Which means that $T_{u_1+u_2} = T_{u_1} \circ T_{u_2}$ according to our definition of linear mapping equality.
If u is an element of V, then $T_u : V \to V$ has an inverse mapping which is nothing but the translation T_{-u}.
First, it is easy to verify that T_{-u} is an inverse of T_u. Then, we say that there is an inverse T_u^{-1}. According to the definition of inverse of a linear mapping, we have, for every $v \in V$ that

$$\begin{aligned} T_u^{-1}(T_u(v)) &= I_V(v) = v \\ T_u^{-1}(v+u) &= v \\ T_u^{-1}(x) &= x - u \qquad \text{Let } x = v + u \end{aligned}$$

which attests $T_u^{-1} = T_{-u}$.
Here comes words on bijectivity, inverse and function composition:

1. For two mappings $F : S_1 \to T_1$ and $F : S_2 \to T_2$, $F \circ G$ is only defined if $T_1 = S_2$.

2. A more clear proof for *If $F : S \to V$ has an inverse $G : V \to S$, then F is bijective.* *Proof.* If $F(x) = F(y)$ given $x, y \in S$, then $G(F(x)) = G(F(y))$. Also, since F, G are inverse of each other, we have

$$\forall s \in S, (G \circ F)(s) = G(F(s)) = I_s(s) = s$$
$$\forall v \in V, (F \circ G)(v) = F(G(v)) = I_v(v) = v$$

which means $x = G(F(x)) = G(F(y)) = y$. Also, we contend that $\forall v \in V, \exists x : F(x) = v$. Since we know $\forall v \in V, (F \circ G)(v) = F(G(v)) = I_v(v) = v$, we can simply let $x = G(v)$ so that $F(x) = v$. This proves the theorem.

3.1.2 Exercises

1. **Exercise 1** ***Calculus involved, not doing now.***

2. **Exercise 2** Proved in notes.

3. **Exercise 3**

   (a) $L(X) = 11$
   (b) $L(X) = 13$
   (c) $L(X) = 6$

4. **Exercise 4**

   $F(1) = (e, 1)$, $F(0) = (1, 0)$, $F(-1) = (e^{-1}, -1)$

5. **Exercise 5**

   $(F + G)(1) = (e + 1, 3)$, $(F + G)(2) = (e^2 + 2, 6)$, $(F + G)(0) = (1, 0)$

6. **Exercise 6**

   $(2F)(0) = (2, 0)$, $(\pi F)(1) = (\pi e, \pi)$

7. **Exercise 7** For (a), it is 1. For (b), it is 11.

8. **Exercise 8**

   The image is a ellipse of the equation

$$\frac{u^2}{4} + \frac{w^2}{9} = 1$$

   Proof is omitted.

9. **Exercise 9**

   The image is a straight line
   $$y = \frac{1}{2}x$$

   Proof. $A = \{(2, y)|y \in \mathbb{R}\}$, $S = \{(2x, x)|x \in \mathbb{R}\}$. We contend that $\forall a \in A, F(a) \in S$.
   $$\forall y \in \mathbb{R}, F(2, y) = (2y, y) \in S$$
   Conversely, $\forall s = (x, \frac{1}{2}x) \in S$, let $a = (2, \frac{1}{2}x) \in A$, so that $F(a) = s$, which means $S \subset F(A)$.

10. **Exercise 10** It is a circle of center $(0, 0)$ and radius e^c. Proof is omitted.

11. **Exercise 11**

    It is a cylinder of radius 1 and center $(0, 0)$. Proof is omitted.

12. **Exercise 12** $x^2 + y^2 = 1$. Proof is omitted.

3.2 § 2

3.2.1 Notes

Here we have an important **Lemma**
Let $F : V \to W$ is a linear mapping. If for some $v_i \in V$, we have $F(v_i)$ are linear independent, then v_i are linear independent.
Proof. If $\sum_{i=1}^{n} t_i v_i = O$, then we have

$$\sum_{i=1}^{n} t_i v_i = O$$
$$F(\sum_{i=1}^{n} t_i v_i) = F(O) = O \quad \text{(This is ensured as "output" of a mapping is unique for same "input")}$$
$$\sum_{i=1}^{n} t_i F(v_i) = O$$

which means if $\sum_{i=1}^{n} t_i v_i = O$, we must have $\sum_{i=1}^{n} t_i F(v_i) = O$. Since $F(v_i)$ are linear independent, we obtain that t_i is always equal to 0, which is another word for v_i are linear independent.
It is noteworthy that reversal of this **Lemma** is **NOT** always true as $F(v) = O$ doesn't ensure that $v = O$. In fact, in later subsections, we shall see that F is injective if and only if Ker $F = O$.

3.2.2 Exercises

1. **Exercise 1** Only (a), (b), (d), (e), (f), (h) are linear mappings. For (h), it involves ***Calculus***.

2. **Exercise 2** $T(O) = T[v + (-v)] = T(v) + T(-v) = T(v) - T(v) = O$

3. **Exercise 3** $T(u + v) = T(u) + T(v) = w + O = w$

4. **Exercise 4**

   Let the set of elements $v \in V$ satisfying $T(v) = w$ be S. We contend that $\forall v \in S, \exists u \in U : v = u + v_0$.
   Proof. let $u = v - v_0$. $F(u) = F(v - v_0) = F(v) - F(v_0) = O$. Thus, $v \in U$. This means $S \subset (v_0 + U)$.
   Conversely, we contend that $\forall u \in U$, we have $(v_0 + u) \in S$.
   Proof. $T(v_0 + u) = T(v_0) + T(u) = w + O = w$. This means $(v_0 + U) \subset S$.
   Thus, $S = (v_0 + U)$.

5. **Exercise 5** As **Exercise 2** said, $T(O) = T(v - v) = T(v) + T(-v) = O \Rightarrow T(-v) = -T(v)$.

6. **Exercise 6**

   Firstly, $F(v_1 + v_2) = (f(v_1) + f(v_2), g(v_1) + g(v_2)) = (f(v_1), g(v_1)) + (f(v_2), g(v_2)) = F(v_1) + F(v_2)$
   Secondly, $F(cv) = (cf(v), cg(v)) = c(f(v), g(v)) = cF(v)$

7. **Exercise 7**

   (a) Prove $(u_1 + u_2) \in U$. We have $F(u_1 + u_2) = F(u_1) + F(u_2) = O + O = O$.

   (b) Prove $cu \in U$. We have $F(cu) = cF(u) = O$.

8. **Exercise 8** Mapping 8 is linear, others are not.

9. **Exercise 9**

   By definition later introduced in §5, it is line segment between $F(v)$ and $F(v + w)$.
   If $F(w) \neq O$, then it is a line segment. If $F(w) = O$, then it is a point.

10. **Exercise 10** By definition, it is a parallelogram.

11. **Exercise 11** Note that E_1, E_2 are standard generators. Since S is a set of points that can be written in the form $t_1E_1 + t_2E_2$ where $0 \leq t_1 \leq 1$ and $0 \leq t_2 \leq 1$. Thus, $F(t_1E_1 + t_2E_2) = t_1F(E_1) + t_2F(E_2)$ where $0 \leq t_1 \leq 1$ and $0 \leq t_2 \leq 1$. Hence, prove the statement.

12. **Exercise 12** We know $3E_1$ and E_2 are also linear independent. So are $3F(E_1)$ and $F(E_2)$. Thus, adopting similar reasoning in **Exercise 11**, we prove statement.

13. **Exercise 13** It is a parallelogram generated by $5A$ and $2B$.

14. **Exercise 14** $T_u(v_1 + v_2) = v_1 + v_2 + u = T_u(v_1) + T_u(v_2) = v_1 + v_2 + 2u$. Thus, we have $2u = u$ and $u = O$.

15. **Exercise 15** It is shown in Lemma.

16. **Exercise 16**

    If $v \in W$, simply let $c = 0$ and $w = v$.
    If $v \notin W$, let $c = \frac{F(v)}{F(v_0)}$ and $w = v - cv_0$. We then contend that $w \in W$. Since $F(w) = F(v - cv_0) = F(v) - cF(v_0) = 0$, we conclude $w \in W$. Thus, concludes.

17. **Exercise 17**

    We see that $F(w_1 + w_2) = F(w_1) + F(w_2) = 0 + 0 = 0$. Also, $F(cw) = cF(w) = 0$. And since F is linear and as we proved before, $F(O) = O$ and $O \in W$. Thus, W is a subspace of V.
    We know by **Exercise 16** that $\{v_0, v_1, \cdots, v_n\}$ generates V. We contend then that they are linear independent. It is by definition that $\{v_1, \cdots, v_n\}$ are linear independent. Since $v_0 \notin W$, we see that v_0 cannot be expressed by linear combination of $\{v_0, v_1, \cdots, v_n\}$, thereby v_0 is linear independent from others. Thus, they are linear independent. Then, by definition, this set is a basis of V.

18. **Exercise 18**

    (a) $(-1, -1)$

    (b) $(-\frac{2}{3}, 1)$

    (c) $(-2, -1)$

19. **Exercise 19**

    (a) $(4, 5)$

    (b) $(\frac{11}{3}, -3)$

    (c) $(4, 2)$

3.3 § 3

3.3.1 Notes

Another part for **Theorem 3.3**. If Im $L = W$, then $\dim \mathrm{Im}\ L = \dim W$ and $\dim \mathrm{Ker}\ L = 0$. Thus, Ker $L = \{O\}$.

3.3.2 Exercises

1. **Exercise 1**

   We know $\dim \mathbb{R}^n = n$. According to rank-nullity law, we see that $\dim \mathbb{R}^2 = \dim \mathrm{Ker}\ F + \dim \mathrm{Im}\ F$. Thus, $2 = \dim \mathrm{Ker}\ F + n$ and $2 - n = \dim \mathrm{Ker}\ F$ Since $n, \dim \mathrm{Ker}\ F \geq 0$, we see that $0 \leq n \leq 2$.

   (a) $n = 2$, then $\dim \mathrm{Ker}\ F = 0$ and Ker $F = \{O\}$. This means that $t_1F(A)+t_2F(B) = O \Rightarrow F(t_1A+t_2B) = O \Rightarrow t_1A + t_2B = O \Rightarrow t_1 = t_2 = 0$.

   (b) $n = 1$, then $\dim \mathrm{Im}\ F = 1$.

   (c) $n = 0$, then $\dim \mathrm{Im}\ F = 0$ and Im $F = \{O\}$.

2. **Exercise 2**

   We know then that $\dim \mathrm{Ker}\ F \neq 0$. Since $2 - \dim \mathrm{Ker}\ F = \dim \mathrm{Im}\ F$, we see that $\dim \mathrm{Im}\ F = 0$ or 1. This concludes our prove.

3. **Exercise 3** Consider $L : \mathbb{R}^4 \to \mathbb{R}^2$ such that $L(x_1, x_2, x_3, x_4) = L(x_1 + 2x_2, x_3 - 15x_4)$. According to rank-nullity theorem and since Ker $L = W$, we see that $\dim \mathbb{R}^4 = \dim W + \dim \mathbb{R}^2$. Thus $\dim W = 2$.

4. **Exercise 4** We contend that there exists such a u. $\forall X$, $L(X - v_0) = L(X) - L(v_0) = O$. This means if we let $u = X - v_0$, then $u \in$ Ker L.

5. **Exercise 5-9** ***Calculus Involved, not done now.***

6. **Exercise 10**

   (a) Let such a subspace be W. Consider $L : \mathbb{R}^n \to \mathbb{R}$ such that $L(X) = \sum_{i=1}^{n} x_i$. According to rank-nullity theorem and since Ker $L = W$, we see that $\dim \mathbb{R}^n = \dim W + \dim \mathbb{R}$. Thus $\dim W = n - 1$.

   (b) Let such a subspace be W. Consider $tr : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathbb{R}$ such that $tr(A) = \sum_{i=1}^{n} a_i i$. According to rank-nullity theorem and since Ker $tr = W$, we see that $\dim \mathrm{Mat}_{n\times n}(\mathbb{R}) = \dim W + \dim \mathbb{R}$. Thus $\dim W = n^2 - 1$.

7. **Exercise 11**

   (a) We have $tr(A+B) = \sum_{i=1}^{n}(a_{ii} + b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = tr(A) + tr(B)$ and $tr(cA) = \sum_{i=1}^{n}(ca_{ii}) = c\sum_{i=1}^{n} a_{ii} = c \cdot tr(A)$. This concludes linearity for tr.

   (b)

$$\begin{aligned}
tr(AB) &= \sum_{i=1}^{n} A_i B^i \\
&= \sum_{i,j=1}^{n} a_{ij} b_{ji} \\
&= \sum_{i,j=1}^{n} b_{ji} a_{ij} \\
&= \sum_{i=1}^{n} B_i A^i \\
&= tr(BA)
\end{aligned}$$

(c) Since we know $tr(AB) = tr(BA)$, we have $tr[(B^{-1}A)B] = tr[B(B^{-1}A)] = tr[(BB^{-1})A] = tr(I_n A) = tr(A)$

(d) Firstly, $\langle A, B\rangle = tr(AB) = tr(BA) = \langle B, A\rangle$. Secondly, $\langle A, B + C\rangle = tr[A(B + C)] = tr[AB + AC] = tr(AB) + tr(AC) = \langle A, B\rangle + \langle A, C\rangle$. Thirdly, $c\langle A, B\rangle = c\sum_{i=1}^{n} A_i B^i = \sum_{i=1}^{n}(cA_i)B^i = \langle cA, B\rangle$.

(e) $tr(AB - BA) = tr(AB) - tr(BA) = 0$. Since $tr(I_n) = n$, this could not be possible.

8. **Exercise 12** $\dim S = \frac{n(n+1)}{2}$. Basis are trivial, skipped.

9. **Exercise 13** $tr(AA) = \sum_{i,j=1}^{n} a_{ij}a_{ji} = \sum_{i,j=1}^{n} (a_{ij})^2 \geq 0$