Linear Algebra Note

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Contents

1 Chapter 1

1.1 § 1

1.1.1 Notes

A generic vector space V is not a field because there is no definition of v^{-1} for some $v \in V$, fulfilling not the definition of a field.

1. **Pg. 4 Proof of** $(-1)v = v$

$$\begin{aligned}(-1)v + v &= (-1)v + 1 \cdot v \\ &= (-1+1)v \\ &= v + (-v)\end{aligned}$$

   Thus, $(-1)v = -v$.

2. **Pg. 6 Proof of SP 3**

$$\begin{aligned}(xA) \cdot B &= \sum_{i=1}^{n} (xa_i)b_i \\ &= \sum_{i=1}^{n} x(a_i b_i) \\ &= x \sum_{i=1}^{n} a_i b_i \\ &= x(A \cdot B) \\ A \cdot (xB) &= \sum_{i=1}^{n} a_i(xb_i) \\ &= \sum_{i=1}^{n} x(a_i b_i) \\ &= x \sum_{i=1}^{n} a_i b_i \\ &= x(A \cdot B)\end{aligned}$$

3. **Pg. 7**

   Upper one:

$$\begin{aligned}(A+B)^2 &= (A+B) \cdot (A+B) \\ &= (A+B) \cdot A + (A+B) \cdot B && \text{Use SP 2} \\ &= A^2 + B \cdot A + A \cdot B + B^2 && \text{Use SP 1}\end{aligned}$$

   Bottom one: Since K is a field, all **VS** s regarding summation or product of functions are actually closed on K. By applying field axioms, V is then a vector space over K.

4. **Pg. 9**

   Let $a_1 = (u_1 + w_1), a_2 = (u_2 + w_2)$. Both of them $\in (U + W)$.
   Since U, W are subspaces of V, $U, W \in V$. Thus, $a_1, a_2 \in V$ as $u_1, w_1, u_2, w_2 \in V$, moreover, $(U + W) \subset V$.
   $a_1 + a_2 = (u_1 + u_2) + (w_1 + w_2) \in (U + W)$
   $ca_1 = c(u_1 + w_1) = (cu_1) + (cw_1) \in (U + W)$
   Since $O \in U$ and $O \in W$, $O = O + O \in (U + W)$. Thus, $(U + W)$ is a subspace of V.

1.1.2 Exercises

1. **Exercise 1** Let $v \in V$, $c[v + (-v)] = cv + c(-v) = cv + (-c)v = v \cdot 0 = v \cdot (1 - 1) = v + (-v) = O$

2. **Exercise 2** Since $c \neq 0$

$$\begin{aligned} O &= cv + [-(cv)] \\ cv &= cv + [-(cv)] \\ O &= -(cv) \\ \frac{-1}{c} \cdot O &= (-c)v \cdot \frac{-1}{c} \\ \frac{-1}{c} \cdot (v - v) &= v \\ \frac{-1}{c} \cdot v + \frac{1}{c} \cdot v &= v \\ v \cdot (1 - 1) &= v \\ v - v &= v \\ O &= v \end{aligned}$$

3. **Exercise 3**

   $\forall g \in V, (g + f)(x) = g(x) + f(x) = f(x) + g(x) = (f + g)(x) \Rightarrow g + f = f + g$.
   If $O + u = u$, $(O + u)(x) = O(x) + u(x) = u(x)$. Therefore, $O(x) = 0$.

4. **Exercise 4**

$$\begin{aligned} v + w &= O \\ v + w &= v + (-v) \\ w &= -v \end{aligned}$$

5. **Exercise 5**

$$\begin{aligned} v + w &= v \\ v + (-v) + w &= v + (-v) \\ O + w &= O \end{aligned}$$

   Since $\forall u, O + u = u$, we have $w = O$.

6. **Exercise 6**

   Let $W = \{B | B \cdot A_1 = O \text{ and } B \cdot A_2 = O\}$. Specifically, it is clear that $O \in W$ as $O \cdot A = \sum_{i=1}^{n} b_i a_i = \sum_{i=1}^{n} 0 \times a_i = 0$.
   Let $v_1, v_2 \in W$ such that $v_1 \cdot A_1 = 0$, $v_1 \cdot A_2 = 0$, $v_2 \cdot A_1 = 0$, $v_2 \cdot A_2 = 0$. Thus,

$$\begin{aligned} (v_1 + v_2) \cdot A_1 &= v_1 \cdot A_1 + v_2 \cdot A_1 \\ &= O + O \\ &= O \\ [c(v_1 + v_2)] \cdot A_1 &= (cv_1 + cv_2) \cdot A_1 \\ &= (cv_1) \cdot A_1 + (cv_2) \cdot A_1 \\ &= c(v_1 \cdot A_1 + v_2 \cdot A_1) \\ &= cO \\ &= O \end{aligned}$$

   . It is easy to show for A_2 then. Therefore, $(v_1 + v_2) \in W$.

7. **Exercise 7** Same to apply as Exercise 6.

8. **Exercise 8**

   Name the set as W.

(a) Proof
$v_1 + v_2 = (x_1 + x_2, y_1 + y_2), x_1 + x_2 = y_1 + y_2 \Rightarrow (v_1 + v_2) \in W$
$cv = (cx, cy), cx = cy \Rightarrow cv \in W$
$O = (0, 0) \in W$

(b) Proof See Part (a).

(c) Proof Same technique as in Part (a).

9. **Exercise 9** See Exercise 8.

10. **Exercise 10**

    For $U \cap W$, let $v_1, v_2 \in U \cap W$. Since $v_1, v_2 \in U$ and U is a subspace, $v_1 + v_2 \in U$. In same way, we can see that $v_1 + v_2 \in W$. Thus, $v_1 + v_2 \in U \cap W$.
    Since $v_1 \in U$, $cv_1 \in U$. Also, it shows $cv_1 \in W$ in the same way. Thus, $cv_1 \in U \cap W$. Because U, W are subspaces, $O \in U$ and $O \in W$. Thus, $O \in U \cap W$. Therefore, $U \cap W$ is a subspace.
    Refer to the note part for proof for $U + W$.

11. **Exercise 11** Since L is a field, **VS1, VS3, VS4, VS8** are established under field axioms, and multiplication and addition are closed in L. For **VS5, VS6, VS7**, they are all valid as $K \subset L$. O is simply 0, and $1 \cdot u = u$ is established in L.

12. **Exercise 12**

    For $x, y \in K$, we have
    $x + y = a_1 + b_1\sqrt{2} + a_2 + b_2\sqrt{2} = (a_1 + a_2) + (b_1 + b_2)\sqrt{2}$. Since $a_1, b_1, a_2, b_2 \in \mathbb{Q}$, $(a_1 + a_2), (b_1 + b_2) \in \mathbb{Q}$. Thus, $x + y \in K$.
    $xy = (a_1a_2 + 2b_1b_2) + (a_2b_1 + a_1b_2) \times \sqrt{2}$. Since $a_1, b_1, a_2, b_2 \in \mathbb{Q}$, $(a_1a_2 + 2b_1b_2), (a_2b_1 + a_1b_2) \in \mathbb{Q}$. Thus, $x + y \in K$.
    $-x = -a + -b\sqrt{2}$. Since $a, b \in \mathbb{Q}$, $-a, -b \in \mathbb{Q}$. Thus, $-x \in K$.
    If $a + b\sqrt{2} \neq 0$, $a, b \neq 0$, and $a - b\sqrt{2} \neq 0$. Thus, $x^{-1} = \frac{1}{a+b\sqrt{2}} = \frac{a-b\sqrt{2}}{a^2-2b^2} = \frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2}$. It is easy to see that **new** $a, b \in \mathbb{Q}$ as $a, b \in \mathbb{Q}$. Thus, $x^{-1} \in K$. Specifically, if $a = b = 0$, $0 \in \mathbb{Q}$. If $a = 1, b = 0$, $1 \in \mathbb{Q}$. Thus, K is a field.

13. **Exercise 13** Same technique as Exercise 12.

14. **Exercise 14** Same technique as Exercise 12.

1.2 § 2

1.2.1 Exercises

1. **Exercise 1** Using result from **Exercise 4**, easy to prove.

2. **Exercise 2**

    (a) $(1, -1)$
    (b) $(\frac{1}{2}, \frac{3}{2})$
    (c) $(1, 1)$
    (d) $(3, 2)$

3. **Exercise 3** Same technique as in **Exercise 2**.

4. **Exercise 4**

    Following set of equations is an equivalent of $x(a, b) + y(c, d) = O$,

$$ax + cy = 0 \quad (1)$$
$$bx + dy = 0 \quad (2)$$

$$(1) \times d - (2) \times c \Rightarrow (ad - cb)x + cdy - cdy = 0$$
$$(ad - cb)x = 0$$

For $ad - cb \neq 0$ part, clearly we shall see that $x = 0$ as $(ad - cb)x = 0$. Plugging x back to (1), we get $y = 0$. Thus, two vectors are linear independent.
For $ad - cb = 0$ part, we need to prove that $x(a, b) + y(c, d) = O$ has solution other than $x = y = 0$.
First, suppose $a, b, c, d \neq 0$. Since $ad - cb = 0$, $x \in \mathbb{R}$. By applying technique, we could also show $y \in \mathbb{R}$. Thus, (a, b), (c, d) are linear independent.
If $a, b, c, d \neq 0$ does **NOT** hold. Without lose of generality (for all the possibilities, a, d and $c.b$ are interchangeable), consider following scenarios in a xy -plane,

(a) $a = 0, c = 0$
If $a = c = 0$, $x, y \in \mathbb{R}$ in (1). Because the (2) is a line in the plane, there must exist some $x, y \neq 0$.

(b) $a = 0, b = 0, c = 0$
Same argument as above, despite the line represented by (2) is a little bit peculiar (it is $y = 0$).

(c) $a = 0, d = 0, c = 0$
Same argument as the first, despite the line represented by (2) is a little bit peculiar (it is $x = 0$).

(d) $a = 0, d = 0, b = 0, c = 0$
Both $(1), (2)$ represent the whole plane, thus, $x, y \in \mathbb{R}$.

5. **Exercise 9**

$$\sum_{i=1}^{r}[a_i \cdot (A_i \cdot \sum_{j=i+1}^{r} A_j)] = O \qquad \text{All vectors are mutually perpendicular}$$
$$= \sum_{i=1}^{r}[(a_i \cdot A_i) \cdot \sum_{j=i+1}^{r} A_j]$$

Since $\forall A \in \{A_i\}, A \neq O$, it is only possible that every a is 0. Thus, A_i are linearly independent.