# Demand Response Management via Real-time Electricity Price Control in Smart Grids

Li Ping Qian, Ying Jun (Angela) Zhang, Jianwei Huang, and Yuan Wu

**Abstract**

This paper proposes a real-time pricing scheme that reduces the peak-to-average load ratio through demand response management in smart grid systems. The proposed scheme solves a two-stage optimization problem. On one hand, each user reacts to prices announced by the retailer and maximizes its payoff, which is the difference between its quality-of-usage and the payment to the retailer. On the other hand, the retailer designs the real-time prices in response to the forecasted user reactions to maximize its profit. In particular, each user computes its optimal energy consumption either in closed forms or through an efficient iterative algorithm as a function of the prices. At the retailer side, we develop a Simulated-Annealing-based Price Control (SAPC) algorithm to solve the non-convex price optimization problem. In terms of practical implementation, the users and the retailer interact with each other via a limited number of message exchanges to find the optimal prices. By doing so, the retailer can overcome the uncertainty of users' responses, and users can determine their energy usage based on the actual prices to be used. Our simulation results show that the proposed real-time pricing scheme can effectively shave the energy usage peaks, reduce the retailer's cost, and improve the payoffs of the users.

**Index Terms**

Real-time pricing, Demand response management, Payoff maximization, Profit maximization, Non-convex optimization.

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## I. INTRODUCTION

In today's electric power grid, we often observe substantial hourly variation in the wholesale electricity price, and the spikes usually happen during peak hours due to the high generation cost. However, almost all end users nowadays are charged some flat-rate retail electricity price [1], [2], which does not reflect the actual wholesale price. With the flat-rate pricing, users often consume much more electricity during peak hours, such as the time between late afternoon and bed time for residential users. This leads to a large fluctuation of electricity consumption between off-peak hours and peak hours. The high peak-hour demand not only induces high cost to the retailers due to the high wholesale prices in those hours, but also has a negative impact on the reliability of the power grid. Ideally, the retailer would like to have the electricity consumption evenly spread across different hours of the day through a proper demand response management.

For the demand response management, researchers have introduced real-time pricing schemes to encourage users to shift their usage to off-peak hours [1]–[7]. A real-time price charges each users based on not only "how much" electricity is consumed but also "when" it is consumed. A properly designed real-time pricing scheme may result in a "triple-win" solution: flattened load demand curves enhances the robustness and lowers the generation cost for the power grid; a lower generation cost leads to a lower wholesale price, which in turn increases the retailers' profit; users may reduce their electricity expenditures by responding to the time-varying price.

The existing research in the real-time pricing can be divided into three main threads. The first thread is concerned with how users respond to the real-time price, hopefully in an automated manner, to achieve their desired level of comfort with lower electricity bill payment (e.g., [2], [8], [9]). These work, however, does not mention how the real-time prices should be set. The second thread of work is concerned with setting the real-time price at the retailer side (e.g., [10]), without taking into account users' potential responses to the forecasted price. For example, the retailer may adjust the real-time retail electricity price through linking it closely to the wholesale electricity price in [10]. The last thread of work is concerned with setting the real-time retail electricity price based on the maximization of the aggregate surplus of users and retailers subject to the supply-demand matching (e.g., [11]–[15]). However, the price obtained in this way may not do good to the retailer's profit. In principle, the retailer should be able to design real-time prices that maximizes its own profit by taking into account the users' potential responses to the prices (e.g., responses that maximize users' own payoff). Ideally, the real-time pricing scheme should be able to achieve a “triple-win” solution that benefits the grid, the retailer, and the end users.

In this paper, we endeavor to design such a real-time pricing scheme to reduce the peak-to-average load ratio, and to maximize each user’s payoff and the retailer’s profit in the meantime. The key hurdle that prevented previous work from doing so lies in the asymmetry of information. For example, when the prices are announced before the energy scheduling horizon (i.e., ex-ante price), the retailer has to face the uncertainty of user response and reimburse the wholesale cost based on the actual electricity consumption by the users. On the other hand, if the prices are fixed after the energy is being consumed (i.e., ex-post price), the users have to bear the uncertainty, as they only adjust their demand according to a prediction of the actual price.

As mentioned in [16], smart grid is an electricity delivery system enhanced with communication facilities and information technologies. Roughly speaking, smart grid is a system integrating the traditional power grids and the communication networks (e.g., Local Area Network (LAN)). Suppose that each user is equipped with a smart meter that is capable of having two way communications with the retailer through a communication network. Based on it, we introduce a novel ex-ante real-time pricing scheme for the future smart grid, where prices are determined at the beginning of each energy scheduling horizon. The contributions of this paper can be summarized as follows:

- We formulate the real-time pricing scheme as a two-stage optimization problem. On one hand, each user reacts to the price and maximizes its payoff, which is the difference between the quality-of-usage and the payment. On the other hand, the retailer designs the real-time price in response to the forecasted user reactions to maximize its profit.
- The proposed algorithm allows each user to optimally schedule its energy consumption in closed forms or through an efficient iterative algorithm. Furthermore, the users and the retailer interact with each other through a limited number of message exchanges to find the optimal price[1], which facilitates the elimination of cost uncertainty at the retailer side.
- We propose a real-time pricing algorithm based on the idea of simulated annealing to reduce the peak-to-average load ratio in smart grid systems. For the practical implementation of the algorithm, we further study how to set the length of interaction period so that the retailer is guaranteed to

[1] In this paper, we assume that the users and the retailer declare their information truthfully. We will consider the incentive issue in a future work.

obtain the optimal price through communications with users.

The rest of this paper is organized as follows. Section II introduces the system model and the problem formulations. In Section III, we provide the closed-form expression of the electricity consumption scheduling with elaborate mathematical analysis, and propose an efficient iterative algorithm for electricity consumption scheduling. The simulated annealing based algorithm used to adjust the real-time retail electricity prices at the retailer side is proposed in Section IV. In Section V, we evaluate the performance of the proposed algorithm through several simulations. The paper is concluded in Section VI.

## II. System Model and Problem Formulation

We consider a microgrid (as shown in Fig. 1 [9]) with two types of participants: end users (i.e., customers), and a retailer from which end users purchase electricity. In this paper, we consider time-varying prices, with the hope to reduce peak-to-average load demand ratio, increase the retailer's profit, and maximize the users' utilities with minimum payment. In the following, we present the problems considered by users and retailer, respectively.

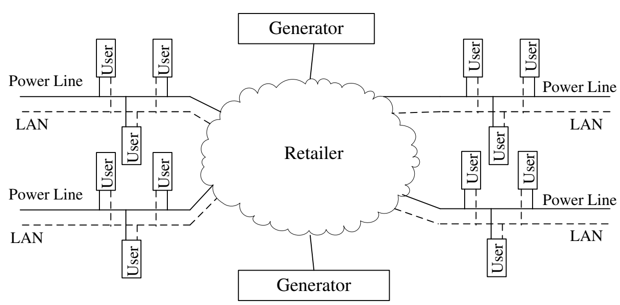


Fig. 1. A simplified illustration of the retail electricity market.

### A. Residential End Users

We assume that each user is equipped with a smart meter as shown in Fig. 2. The retailer sets the real-time retail electricity prices and informs to users via LAN. At the user side, the energy scheduler in the smart meter optimally computes and distributes energy consumption according to the prices for the upcoming scheduling horizon $\mathcal{H}$.

Let $\mathcal{U} = \{1, 2, \cdots, U\}$ denote the set of residential users. Assume that each user $u$ has three types of appliances, denoted by $\mathcal{A}_u$, $\mathcal{B}_u$ and $\mathcal{C}_u$. The first category $\mathcal{A}_u$ includes *background appliances*, which

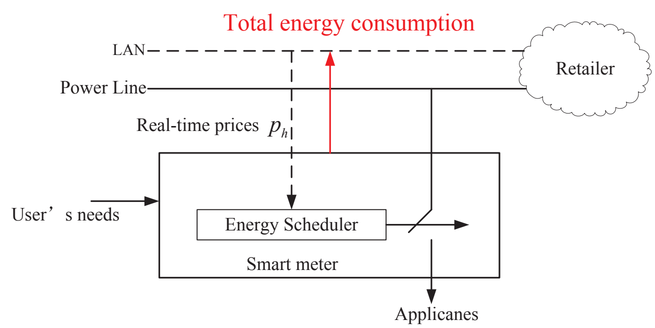


Fig. 2. The operation of smart meter and the price setting of retailer in our design.

consumes a fixed amount of energy per unit time during a fixed period of time. The background appliances are inelastic, in the sense that there is no flexibility to adjust the energy consumption across time. Examples of such appliances include lighting, refrigerator and electric kettle. The second category $\mathcal{B}_u$ includes elastic appliances, which have higher quality-of-usage (satisfaction) for more energy consumed per unit time (with a maximum consumption upper bound). The evaluation of quality-of-usage can be time-dependent for elastic appliances, as users may obtain a higher satisfaction to consume certain amount of energy during a certain time than in other time durations. Examples of such appliances include air conditioner, electric fan and iron. The last category $\mathcal{C}_u$ includes semi-inelastic appliances, which consume a fixed total energy within a preferred time period. This category is semi-inelastic in the sense that there is flexibility to choose when to consume the energy within the preferred time period, but no flexibility to adjust the total energy consumption. Examples include washer/dryer, dishwasher, plug-in hybrid electric vehicle (PHEV) and electric geyser.

For each appliance $a_u$, we express its energy consumption over the scheduling horizon $\mathcal{H}$ by a scheduling vector $\boldsymbol{e}_{a_u}$ as follows:

$$\boldsymbol{e}_{a_u} = (e_{a_u,1}, \cdots, e_{a_u,H}). \tag{1}$$

The retailer announces the price $p_h$ for time slots $h \in \mathcal{H} = \{1, ..., H\}$ at the beginning of the scheduling horizon, and each user $u$ computes its optimal $\boldsymbol{e}_{a_u}$ accordingly. A time slot can be, for example, one hour, and the scheduling horizon can be one day, i.e., $H = 24$ hours. In what follows, we will introduce the energy consumption constraints of the three categories of appliances.

As a background appliance, each $a_u \in \mathcal{A}_u$ works in a working period $H_{a_u} \in \mathcal{H}$, during which it consumes $r_{a_u}$ energy per time slot. This is mathematically described as

$$e_{a_u,h} = \begin{cases} r_{a_u,h}, & h \in \mathcal{H}_{a_u}, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

We note that the time slots in $\mathcal{H}_{a_u}$ are allowed to be intermittent. There is no flexibility to redistribute the load due to this type of appliances in response to the price. However, such appliances are ubiquitous in power systems and contribute a large percentage of the peak during high-demand hours. It is thus important to properly schedule categories $\mathcal{B}_u$ and $\mathcal{C}_u$ appliances to avoid further overloading the peaks.

For each appliance $a_u \in \mathcal{B}_u$, user $u$ obtains different levels of satisfaction for the same amount of energy consumed in different time slots. Suppose that the satisfaction is measured by a time-dependent quality-of-usage function $U_{a_u,h}(e_{a_u,h})$, which depends on who, when, and how much energy is consumed. For example, $U_{a_u,h}(e_{a_u,h})$ may be equal to zero during undesirable operation hours. It is reasonable to assume that $U_{a_u,h}(e_{a_u,h})$ is a non-decreasing concave function of $e_{a_u,h}$ at any time slot $h \in \mathcal{H}$. Besides, for each appliance $a_u \in \mathcal{B}_u$, the energy consumption per time slot is subject to

$$0 \leq e_{a_u,h} \leq r_{a_u}^{\max}, \forall h \in \mathcal{H}, \tag{3}$$

where $r_{a_u}^{\max}$ is the maximum energy that can be consumed in the time slot when appliance $a_u$ is working.

As a semi-inelastic appliance, each $a_u \in \mathcal{C}_u$ works in a working period $\mathcal{H}'_{a_u} \in \mathcal{H}$, during which it consumes $E_{a_u}$ energy in total. This is written as

$$\sum_{h \in \mathcal{H}'_{a_u}} e_{a_u,h} = E_{a_u} \tag{4}$$

for each appliance $a_u \in \mathcal{C}_u$. Noticeably, the period $\mathcal{H}'_{a_u}$ is consecutive with the beginning $\alpha_{a_u} \in \mathcal{H}$ and the end $\beta_{a_u} \in \mathcal{H}$. Thus, $\mathcal{H}'_{a_u}$ can be rewritten as $\mathcal{H}'_{a_u} = \{\alpha_{a_u}, \alpha_{a_u}+1, \cdots, \beta_{a_u}\}$. In practice, the choice of $\alpha_{a_u}$ and $\beta_{a_u}$ depends on the habit (or preference) of user $u$. Besides, the energy consumption per time slot is subject to the following constraint due to this type of appliance, i.e.,

$$0 \leq e_{a_u,h} \leq r_{a_u}^{\max}, \forall h \in \mathcal{H}'_{a_u}. \tag{5}$$

Furthermore, we have $e_{a_u,h} = 0$ for any $h \notin \mathcal{H}'_{a_u}$ as no operation (and hence energy consumption) is needed outside the working period $\mathcal{H}'_{a_u}$. For this type of appliance, there is flexibility to distribute the total load during the working period in response to the price.

For three categories of appliances, there is usually a limit on the total allowable energy consumption for each user $u$ at each time slot. This limit, denoted by $C_u^{max}$, can be set by the retailer to impose the following set of constraints on energy scheduling:

$$\sum_{a_u \in \mathcal{A}_u, \mathcal{B}_u, \mathcal{C}_u} e_{a_u,h} \leq C_u^{\max}, \forall h \in \mathcal{H}. \tag{6}$$

In practice, such constraints are used to protect the total energy consumption from exceeding the grid capacity. To summarize, all valid energy consumption scheduling vectors can be determined by constraints (2)-(6).

Noticeably, each user $u$ has two contradicting goals, given that all its demands (i.e., constraints (2)-(6)) are met. The first is to maximize its overall satisfaction, given by

$$\sum_{h \in \mathcal{H}} \sum_{a_u \in \mathcal{B}_u} U_{a_u,h}(e_{a_u,h}). \tag{7}$$

The second goal is to minimize its electricity bill payment, obtained as

$$\sum_{h \in \mathcal{H}} p_h \Bigg( \sum_{a_u \in \mathcal{A}_u, \mathcal{B}_u, \mathcal{C}_u} e_{a_u,h} \Bigg). \tag{8}$$

To balance the two objectives, each user formulates the energy scheduling problem as maximizing its payoff, i.e.,

$$\begin{aligned}
\textbf{P1:} \quad & \text{maximize} && \sum_{h \in \mathcal{H}} \sum_{a_u \in \mathcal{B}_u} U_{a_u,h}(e_{a_u,h}) - \sum_{h \in \mathcal{H}} p_h \Bigg( E_{u,h} + \sum_{a_u \in \mathcal{B}_u, \mathcal{C}_u} e_{a_u,h} \Bigg) && (9.1) \\
& \text{subject to} && 0 \leq e_{a_u,h} \leq r_{a_u}^{\max}, \forall a_u \in \mathcal{B}_u, \forall h \in \mathcal{H}, && (9.2) \\
& && 0 \leq e_{a_u,h} \leq r_{a_u}^{\max}, \forall a_u \in \mathcal{C}_u, \forall h \in \mathcal{H}'_{a_u}, && (9.3) \\
& && e_{a_u,h} = 0, \forall a_u \in \mathcal{C}_u, \forall h \notin \mathcal{H}'_{a_u}, && (9.4) \\
& && \sum_{h \in \mathcal{H}'_{a_u}} e_{a_u,h} = E_{a_u}, \forall a_u \in \mathcal{C}_u, && (9.5) \\
& && \sum_{a_u \in \mathcal{B}_u, \mathcal{C}_u} e_{a_u,h} \leq C_u^{\max} - E_{u,h}, \forall h \in \mathcal{H}, && (9.6) \\
& \text{variables} && \boldsymbol{e}_{a_u}, \forall a_u \in \mathcal{B}_u, \mathcal{C}_u,
\end{aligned}$$

where $E_{u,h}$ is the total energy consumed by background appliances at time slot $h$, satisfying

$$E_{u,h} = \sum_{a_u : h \in \mathcal{H}_{a_u}} r_{a_u,h}. \tag{10}$$

Note that through solving Problem 1, each user can independently determine its optimal energy usage based on the prices forecasted by the retailer.

### B. Retailer

When we solve Problem **P1**, the optimal total energy consumption from all users in each time slot depends on the price vector $\boldsymbol{p} = [p_1, \cdots, p_H]$. For notational convenience, let $S_{u,h}(\boldsymbol{p})$ denote the corresponding optimal total energy consumption of user $u$ at time slot $h$. As shown in the next section, given the price, each user can easily calculate $S_{u,h}(\boldsymbol{p})$'s from Problem **P1**.

Note that the retailer has one goal to maximize its profit, which is the difference between revenue and cost with which it buys energy from the generators. In particular, the revenue is given by

$$\sum_{h} \left( \sum_{u \in \mathcal{U}} S_{u,h}(\boldsymbol{p}) \right) p_h, \tag{11}$$

and the cost is denoted as an increasing convex function regarding the load demand at each time slot, i.e.,

$$\sum_{h \in \mathcal{H}} a \left( \sum_{u \in \mathcal{U}} S_{u,h}(\boldsymbol{p}) \right)^2 + b \left( \sum_{u \in \mathcal{U}} S_{u,h}(\boldsymbol{p}) \right)^3, \tag{12}$$

where $a$ and $b$ are positive factors [22]. Mathematically, the goal of the retailer is therefore written as

$$\begin{aligned}
\textbf{P2:}\ \text{maximize}\ & L(\boldsymbol{p}) = \sum_{h \in \mathcal{H}} \left( \sum_{u \in \mathcal{U}} S_{u,h}(\boldsymbol{p}) \right) p_h - w \left( \sum_{h \in \mathcal{H}} a \left( \sum_{u \in \mathcal{U}} S_{u,h}(\boldsymbol{p}) \right)^2 + b \left( \sum_{u \in \mathcal{U}} S_{u,h}(\boldsymbol{p}) \right)^3 \right) \\
\text{subject to}\quad & p^l \leq p_h \leq p^u,\ \forall h \in \mathcal{H}, \\
\text{variables}\quad & \boldsymbol{p},
\end{aligned} \tag{13}$$

where $p^l$ and $p^u$ denote the lower-bound price and the upper-bound price due to regulation, respectively, and the coefficient $w$ reflects the weight of cost in the net profit.

Noticeably, the optimal solution of Problem **P2** depends on the form of the total electricity consumption (i.e., $\sum_{u \in \mathcal{U}} S_{u,h}(\boldsymbol{p})$). The detailed solution to Problem **P2** will be discussed in Section IV.

## III. Implementation of Electricity Consumption Scheduling

In this section, we first derive $S_{u,h}(\boldsymbol{p})$'s at the user side. Each user can then inform the retailer of $S_{u,h}(\boldsymbol{p})$'s via the communication channels. With this information, the retailer can then efficiently calculate the demand response from each user for any price vector $\boldsymbol{p}$, thus removing the uncertainty of user response.

Due to the concavity of $U_{a_u,h}(e_{a_u,h})$, Problem **P1** is a convex optimization problem, and the optimal solution can be obtained by the primal-dual arguments [17]. That is, we can solve Problem **P1** through maximizing its Lagrangian and minimizing the corresponding dual function. Let $\boldsymbol{\eta}_u = (\eta_{u,1}, \cdots, \eta_{u,H})$ be the Lagrange multiplier vector corresponding to the constraint (10.5). We define the Lagrangian $L(\{\boldsymbol{e}_{a_u}\}_{a_u}, \boldsymbol{\eta}_u)$ associated with Problem **P1**, i.e.,

$$\begin{aligned} L(\{\boldsymbol{e}_{a_u}\}_{a_u}, \boldsymbol{\eta}_u) =& \sum_{h\in\mathcal{H}}\sum_{a_u\in\mathcal{B}_u} U_{a_u,h}(e_{a_u,h}) - \sum_{h\in\mathcal{H}} p_h\Bigg(\sum_{a_u\in\mathcal{B}_u,\mathcal{C}_u} e_{a_u,h}\Bigg) \\ &+ \sum_{h\in\mathcal{H}} \eta_{u,h}\Bigg(C_u^{\max} - E_{u,h} - \sum_{a_u\in\mathcal{B}_u,\mathcal{C}_u} e_{a_u,h}\Bigg). \end{aligned} \tag{14}$$

Mathematically, the optimizations of the Lagrangian and the dual problem are expressed as

$$\begin{aligned} g(\boldsymbol{\eta}_u) =& \text{maximize } L(\{\boldsymbol{e}_{a_u}\}_{a_u}, \boldsymbol{\eta}_u) \\ & \text{subject to } 0 \le e_{a_u,h} \le r_{a_u}^{\max}, \forall h \in \mathcal{H}, \forall a_u \in \mathcal{B}_u, \\ & \qquad \sum_{h\in\mathcal{H}'_{a_u}} e_{a_u,h} = E_{a_u}, \forall a_u \in \mathcal{C}_u, \\ & \qquad 0 \le e_{a_u,h} \le r_{a_u}^{\max}, \forall a_u \in \mathcal{C}_u, \forall h \in \mathcal{H}'_{a_u}, \\ & \qquad e_{a_u,h} = 0, \forall a_u \in \mathcal{C}_u, \forall h \notin \mathcal{H}'_{a_u}, \\ & \text{variables} \quad \boldsymbol{e}_{a_u}, \forall a_u \in \mathcal{B}_u, \mathcal{C}_u. \end{aligned} \tag{15}$$

and

$$\begin{aligned} \text{Dual Problem: minimize } & \; g(\boldsymbol{\eta}_u) \\ \text{subject to } & \; \eta_{u,h} \ge 0, \forall h \in \mathcal{H}, \\ \text{variables } & \; \boldsymbol{\eta}_u, \end{aligned} \tag{16}$$

respectively.

Let $U'_{a_u,h}(e_{a_u,h})$ denote $\frac{\partial U_{a_u,h}(e_{a_u,h})}{\partial e_{a_u,h}}$ and $U'^{-1}_{a_u,h}(\cdot)$ denote the inverse function of $U'_{a_u,h}(\cdot)$. Let $\hat{\mathcal{H}}_u$ be the set of time slots in which there might be semi-elastic appliances for user $u$. It is clear that $\hat{\mathcal{H}}_u = \bigcup\limits_{a_u\in\mathcal{C}_u} \mathcal{H}'_{a_u}$. Likewise, let the optimal solution to (15) be $\hat{e}_{a_u,h}(p_h, \eta_{u,h})$ for all $a_u \in \mathcal{B}_u$ and $\hat{e}_{a_u,h}(\boldsymbol{p}, (\eta_{u,h}, \forall h \in \hat{\mathcal{H}}_u))$ for all $a_u \in \mathcal{C}_u$ in each in each $h \in \mathcal{H}$, respectively. According to the theory of convex optimization and linear optimization [17], we have the following result as shown in Lemma 1[2].

[2]For simplicity, we assume that the smart grid can tolerate that every category $\mathcal{C}_u$ appliance consumes $r_{a_u}^{\max}$ at the same time slot, i.e., $\sum\limits_{a_u\in\mathcal{C}_u} r_{a_u}^{\max} \le C_u^{\max} - E_{u,h}, \forall h$.

**Lemma 1.** The optimal solution to (15) satisfies[3]

$$\hat{e}_{a_u,h}(p_h,\eta_{u,h}) = \left[U'^{-1}_{a_u,h}(p_h+\eta_{u,h})\right]_0^{r^{\max}_{a_u}}, \forall a_u \in \mathcal{B}_u, \forall h \in \mathcal{H}, \tag{17}$$

and for all $a_u \in \mathcal{C}_u$ and $h_i \in \mathcal{H}$

$$\hat{e}_{a_u,h_i}(\boldsymbol{p},(\eta_{u,h},\forall h \in \hat{\mathcal{H}}_u)) = \begin{cases} r^{\max}_{a_u} & \text{if } i \in \{1,2,\cdots,\lfloor \frac{E_{a_u}}{r^{\max}_{a_u}} \rfloor\} \text{ and } h_i \in \mathcal{H}'_{a_u} \\ E_{a_u} - r^{\max}_{a_u} \times \lfloor \frac{E_{a_u}}{r^{\max}_{a_u}} \rfloor, & \text{elseif } i = \lfloor \frac{E_{a_u}}{r^{\max}_{a_u}} \rfloor + 1 \text{ and } h_i \in \mathcal{H}'_{a_u} \\ 0, & \text{otherwise,} \end{cases} \tag{18}$$

where $h_i$ follows that $p_{h_i} + \eta_{u,h_i} \le p_{h_{i+1}} + \eta_{u,h_{i+1}}$.

The proof of Lemma 1 is deferred to Appendix A.

After obtaining the optimal solution to problem (15), we want to minimize the dual problem (16). In particular, problem (16) can be decomposed into (19) and (20).

$$\begin{aligned} \text{minimize} \quad & \sum_{h \notin \hat{\mathcal{H}}_u} \sum_{a_u \in \mathcal{B}_u} U_{a_u,h}(e_{a_u,h}) - \sum_{h \notin \hat{\mathcal{H}}_u} (p_h + \eta_{u,h}) \left( \sum_{a_u \in \mathcal{B}_u} e_{a_u,h} \right) + \sum_{h \notin \hat{\mathcal{H}}_u} \eta_{u,h} \left( C^{\max}_u - E_{u,h} \right) \\ \text{subject to} \quad & e_{a_u,h} = \left[U'^{-1}_{a_u,h}(p_h+\eta_{u,h})\right]_0^{r^{\max}_{a_u}}, \forall h \notin \hat{\mathcal{H}}_u, \forall a_u \in \mathcal{B}_u, \\ & \eta_{u,h} \ge 0, \forall h \notin \hat{\mathcal{H}}_u, \\ \text{variables} \quad & \eta_{u,h}, \forall h \notin \hat{\mathcal{H}}_u. \end{aligned} \tag{19}$$

$$\begin{aligned} \text{minimize} \quad & \sum_{h_i \in \hat{\mathcal{H}}_u} \sum_{a_u \in \mathcal{B}_u} U_{a_u,h_i}(e_{a_u,h_i}) - \sum_{h_i \in \hat{\mathcal{H}}_u} (p_{h_i} + \eta_{u,h_i}) \left( \sum_{a_u \in \mathcal{B}_u, \mathcal{C}_u} e_{a_u,h_i} \right) + \sum_{h_i \in \hat{\mathcal{H}}_u} \eta_{u,h_i} \left( C^{\max}_u - E_{u,h_i} \right) \\ \text{subject to} \quad & e_{a_u,h_i} = \left[U'^{-1}_{a_u,h_i}(p_{h_i}+\eta_{u,h_i})\right]_0^{r^{\max}_{a_u}}, \forall h_i \in \hat{\mathcal{H}}_u, \forall a_u \in \mathcal{B}_u, \\ & \forall a_u \in \mathcal{C}_u, e_{a_u,h_i} = \begin{cases} r^{\max}_{a_u} & \text{if } i \in \{1,2,\cdots,\lfloor \frac{E_{a_u}}{r^{\max}_{a_u}} \rfloor\} \text{ and } h_i \in \mathcal{H}'_{a_u} \\ E_{a_u} - r^{\max}_{a_u} \times \lfloor \frac{E_{a_u}}{r^{\max}_{a_u}} \rfloor, & \text{elseif } i = \lfloor \frac{E_{a_u}}{r^{\max}_{a_u}} \rfloor + 1 \text{ and } h_i \in \mathcal{H}'_{a_u} \\ 0, & \text{otherwise,} \end{cases} \\ & \eta_{u,h_i} \ge 0, \forall h_i \in \hat{\mathcal{H}}_u, \\ \text{variables} \quad & \eta_{u,h_i}, \forall h_i \in \hat{\mathcal{H}}_u. \end{aligned} \tag{20}$$

[3]Notation $[x]_a^b$ means $\max\{\min\{x,b\},a\}$, and notation $\lfloor x \rfloor$ returns the nearest integer that is less than or equal to $x$.

From problem (19), we have Lemma 2 as shown in the following.

**Lemma 2.** If

$$\sum_{a_u\in\mathcal{B}_u}\left[U'^{-1}_{a_u,h}(p_h)\right]_0^{r_{a_u}^{\max}} \le C_u^{\max} - E_{u,h}, h \notin \hat{\mathcal{H}}_u \tag{21}$$

the optimal solution to (19) is zero, and the optimal energy consumption of each appliance $a_u$ at time slot $h \notin \hat{\mathcal{H}}_u$ satisfies

$$e^*_{a_u,h}(p_h) = \left[U'^{-1}_{a_u,h}(p_h)\right]_0^{r_{a_u}^{\max}},\ \forall a_u \in \mathcal{B}_u, h \notin \hat{\mathcal{H}}_u. \tag{22}$$

Otherwise, the unique optimal solution to (19) is a function of $p_h$, denoted by $\eta^*_{u,h}(p_h)$, and the optimal energy consumption of each appliance $a_u$ at time slot $h \notin \hat{\mathcal{H}}_u$ satisfies

$$e^*_{a_u,h}(p_h) = \left[U'^{-1}_{a_u,h}(p_h + \eta^*_{u,h}(p_h))\right]_0^{r_{a_u}^{\max}},\ \forall a_u \in \mathcal{B}_u, h \notin \hat{\mathcal{H}}_u. \tag{23}$$

Moreover, the total optimal energy consumption at time slot $h \notin \hat{\mathcal{H}}_u$ is equal to $C_u^{\max}$. That is,

$$\sum_{a_u\in\mathcal{B}_u} e^*_{a_u,h}(p_h) = C_u^{\max} - E_{u,h}, h \notin \hat{\mathcal{H}}_u. \tag{24}$$

The proof of Lemma 2 is deferred to Appendix B.

Denote by $S_{u,h}(\boldsymbol{p})$ the total energy consumed by user $u$ in each time slot $h$. By Lemma 2, we have

$$S_{u,h}(\boldsymbol{p}) = \sum_{a_u\in\mathcal{B}_u} e^*_{a_u,h}(p_h) + E_{u,h} = \min\left\{C_u^{\max}, E_{u,h} + \sum_{a_u\in\mathcal{B}_u}\left[U'^{-1}_{a_u,h}(p_h)\right]_0^{r_{a_u}^{\max}}\right\}, \forall h \notin \hat{\mathcal{H}}_u. \tag{25}$$

**Remark 1.** The total energy consumption follows (25) as long as there is no energy consumption of semi-elastic appliances in time slot $h$.

Next, we focus on calculating the total energy consumption in time slot $h$ when there may exist semi-elastic appliances, i.e., Problem (20). A close look at Problem (20) reveals that to solve Problem (20), we first need to sort the time slots in $\hat{\mathcal{H}}_u$ as the order $\{h_1, \cdots, h_i, \cdots, h_L\}$ such that $p_{h_i} + \eta_{u,h_i} \le p_{h_{i+1}} + \eta_{u,h_{i+1}}$ for all $h_i \in \hat{\mathcal{H}}_u$, where $L = |\hat{\mathcal{H}}_u|$. Then, the energy consumption of each semi-elastic appliance is given according to the order of time slots. Since the order of time slots depends on the summation of $\eta_{u,h}$ and $p_h$ for all $h \in \hat{\mathcal{H}}_u$, the energy consumption has no closed-form expression for the semi-elastic appliances. Therefore, it is difficult to obtain the optimal solution $\eta^*_{u,h}(\boldsymbol{p})$'s from (20). Alternatively, for a given $\boldsymbol{p}$, we can adopt an iterative numerical algorithm to obtain $\eta^*_{u,h}(\boldsymbol{p})$'s. In particular, at each iteration $k$, the energy consumption is first updated by Lemma 2. Then, the multiplier variables are updated as $\eta_{u,h}^{(k)}$'s by the subgradient method [18], i.e.,

$$\eta_{u,h}^{(k)} = \left[\eta_{u,h}^{(k-1)} - \psi_{u,h}^{(k)}(C_u^{\max} - E_{u,h} - \sum_{a_u \in \mathcal{B}_u} \hat{e}_{a_u,h}(p_h + \eta_{u,h}^{(k-1)}) - \sum_{a_u \in \mathcal{C}_u} \hat{e}_{a_u,h}(\boldsymbol{p}, (\eta_{u,h}^{(k-1)}, \forall h \in \hat{\mathcal{H}}_u))\right]^+, \quad \forall h \in \hat{\mathcal{H}}_u. \tag{26}$$

Here, $\psi_{u,h}^{(k)}$'s are stepsizes at the $k$th iteration. Besides, $\hat{e}_{a_u,h}(p_h + \eta_{u,h}^{(k-1)})$'s and $\hat{e}_{a_u,h}(\boldsymbol{p}, (\eta_{u,h}^{(k-1)}, \forall h \in \hat{\mathcal{H}}_u))$'s are the optimal solution to (15) at the $(k-1)$th iteration.

Now, we present the implementation of electricity consumption in time slot $h$ when there may exist semi-elastic appliances in Algorithm 1.

**Remark 2.** At each iteration of Algorithm 1, the energy consumption and the multiplier variables are updated in closed forms. Therefore, the complexity of Algorithm 1 is $\mathrm{O}(|\mathcal{B}_u|H + |\mathcal{C}_u|H + H)$ at each iteration.

---

**Algorithm 1** Implementation of Electricity Consumption Scheduling for given $\boldsymbol{p}$ in $\hat{\mathcal{H}}_u$

1: **Initialization**; Randomly choose $\eta_{u,h}^{(0)} > 0, \forall h \in \hat{\mathcal{H}}_u$. Let $k = 1$.
2: **repeat**
3: Calculate $\hat{e}_{a_u,h}(p_h + \eta_{u,h}^{(k-1)})$'s and $\hat{e}_{a_u,h}(\boldsymbol{p}, (\eta_{u,h}^{(k-1)}, \forall h \in \hat{\mathcal{H}}_u))$'s by Lemma 1.
4: Update the multiplier variables $\eta_{u,h}^{(k)}$'s according to (26).
5: $k = k + 1$.
6: **until** $\sum_{h \in \hat{\mathcal{H}}_u} \left( \sum_{a_u \in \mathcal{B}_u} (\hat{e}_{a_u,h}(p_h + \eta_{u,h}^{(k-1)}) - \hat{e}_{a_u,h}(p_h + \eta_{u,h}^{(k-2)}))^2 + \sum_{a_u \in \mathcal{C}_u} (\hat{e}_{a_u,h}(\boldsymbol{p}, (\eta_{u,h}^{(k-1)}, \forall h \in \hat{\mathcal{H}}_u)) - \hat{e}_{a_u,h}(\boldsymbol{p}, (\eta_{u,h}^{(k-2)}, \forall h \in \hat{\mathcal{H}}_u)))^2 \right) \leq \epsilon$.

---

Note that the optimal solution to Problem P1 is a function of $\boldsymbol{p}$, and thus we denote it as $\boldsymbol{e}_{a_u}^*(\boldsymbol{p})$'s. The following theorem shows that when the stepsizes $\psi_{u,h}^{(k)}$'s in (26) are small, the optimal solution of (15), i.e., $\hat{e}_{a_u,h}(p_h + \eta_{u,h}^{(k)})$'s and $\hat{e}_{a_u,h}(\boldsymbol{p}, (\eta_{u,h}^{(k)}, \forall h \in \hat{\mathcal{H}}_u))$'s, will converge within a small neighborhood of the optimal solution $\boldsymbol{e}_{a_u}^*(\boldsymbol{p})$'s.

**Theorem 1.** (a) **Constant Stepsize**. Assume that $\psi_{u,h}^{(k)} = s\psi_{u,h}^{(0)}$, where $\psi_{u,h}^{(0)}$'s are arbitrary positive constants. For any $\epsilon > 0$, there exists some $s_0 > 0$ and a time $T_0$, such that for any $s \leq s_0$, any initial multiplier variables $\eta_{u,h}^{(0)}$'s and all $k \geq T_0$, we have

$$\sum_{h\in\hat{\mathcal{H}}_u}\left(\sum_{a_u\in\mathcal{B}_u}(\hat{e}_{a_u,h}(p_h+\eta_{u,h}^{(k)})-\boldsymbol{e}_{a_u}^*(\boldsymbol{p}))^2+\sum_{a_u\in\mathcal{C}_u}(\hat{e}_{a_u,h}(\boldsymbol{p},(\eta_{u,h}^{(k)},\forall h\in\hat{\mathcal{H}}_u))-\boldsymbol{e}_{a_u}^*(\boldsymbol{p}))^2\right)\leq\epsilon. \quad (27)$$

(b) **Adaptive Stepsize**. If the stepsizes are iteration-varying and they are chosen such that $\psi_{u,h}^{(k)} = s^{(k)}\psi_{u,h}^{(0)}$ with $\lim\limits_{k\to\infty} s^{(k)} = 0$ and $\sum\limits_{k=1}^{\infty} s^{(k)} = +\infty$, then (28) is satisfied.

$$\lim_{k\to\infty}\sum_{h\in\hat{\mathcal{H}}_u}\left(\sum_{a_u\in\mathcal{B}_u}(\hat{e}_{a_u,h}(p_h+\eta_{u,h}^{(k)})-\boldsymbol{e}_{a_u}^*(\boldsymbol{p}))^2+\sum_{a_u\in\mathcal{C}_u}(\hat{e}_{a_u,h}(\boldsymbol{p},(\eta_{u,h}^{(k)},\forall h\in\hat{\mathcal{H}}_u))-\boldsymbol{e}_{a_u}^*(\boldsymbol{p}))^2\right)=0. \quad (28)$$

The detailed proof of Theorem 1 is relegated to Appendix C.

With Algorithm 1, the smart meter can quickly calculate the energy consumption of each appliance (and hence $S_{u,h}(\boldsymbol{p})$) in each time $h \in \hat{\mathcal{H}}_u$ in response to the price $\boldsymbol{p}$ forecasted by the retailer. This implies that the total energy consumption can be obtained by the smart meter through Algorithm 1 or the closed-form expression (25) according to the forecasted price. The following example will illustrate both situations.

**Example 1** (User's response to the forecasted price $\boldsymbol{p}$): Assume that user $u$ has six appliances, including two background appliances $a_1$ and $a_2$, two elastic appliances $a_3$ and $a_4$, and two semi-elastic appliances $a_5$ and $a_6$. Let the scheduling horizon $\mathcal{H}$ be $\{1, 2, \cdots, 8\}$. Specifically, the working period of appliance $a_5$ is $\{3, 4, 5, 6\}$, and the working period of appliance $a_6$ is $\{4, 5, 6, 7\}$. The total energy consumption of background appliances is $[4.0, 3.0, 3.0, 3.5, 2.5, 3.5, 3.5, 3.0]$ kWh in the scheduling horizon. The maximum allowable energy consumption $C_u^{\max}$ of user $u$ is 40 kWh at each time slot. The maximum allowable energy consumptions of appliances $a_3$, $a_4$, $a_5$ and $a_6$ are 20 kWh, 20 kWh, 4 kWh and 6 kWh at each time slot, respectively. The total energy consumptions of semi-elastic appliances $a_5$ and $a_6$ are 10 kWh and 10 kWh in the working period, respectively. Assume appliance $a_3$ and appliance $a_4$ have a quality-of-usage of $U_{a_3,h}(e_{a_3,h}) = 1.5w_{1,h}\log(m_{1,h} + e_{a_3,h})$ and a quality-of-service of $U_{a_4,h}(e_{a_4,h}) = 1.5w_{2,h}\log(m_{2,h} + e_{a_4,h})$ at time slot $h$, respectively. The parameters in these two functions are given as follows,

$$\begin{pmatrix} w_{1,h} \\ w_{2,h} \end{pmatrix}_{h\in\mathcal{H}} = \begin{pmatrix} 6 & 8 & 6 & 8 & 6 & 10 & 8 & 6 \\ 6 & 8 & 10 & 8 & 10 & 6 & 10 & 8 \end{pmatrix}$$

and

$$\begin{pmatrix} m_{1,h} \\ m_{2,h} \end{pmatrix}_{h \in \mathcal{H}} = \begin{pmatrix} 1.0 & 3.0 & 1.5 & 3.5 & 3.0 & 3.5 & 0.5 & 3.0 \\ 3.0 & 1.0 & 1.5 & 3.0 & 1.5 & 3.5 & 2.0 & 1.0 \end{pmatrix}.$$

Consider a given price vector $\boldsymbol{p} = \$[1.1, 1.0, 1.2, 1.2, 1.9, 1.4, 1.9, 1.0]$. In this case, the amount of consumed energy is calculated with the closed-form expression (25) in time slots $\{1, 2, 8\}$, since there is no semi-elastic appliance in these time slots. For other time slots, the smart meter needs to calculate the energy consumption according to Algorithm 1.

Fig. 3 shows the scheduled energy consumption of each appliance in response to the forecasted price vector $\boldsymbol{p}$. In this example, Algorithm 1 takes 4 iterations to converge to the desirable energy consumptions in time slots $\{3, 4, 5, 6, 7\}$. Recall Remark 2 that the complexity of Algorithm 1 is O$(|\mathcal{B}_u|H + |\mathcal{C}_u|H + H)$ at each iteration. Therefore, it is of practical meaning for the smart meter because of its low computing capability. It can be further seen from Fig. 3 that the larger price the less energy consumption for semi-elastic appliances. This is because that the minimum payment is the target of semi-elastic appliances. However, it might not be satisfied for the elastic appliances, because the quality-of-service also needs to be considered except the payment.

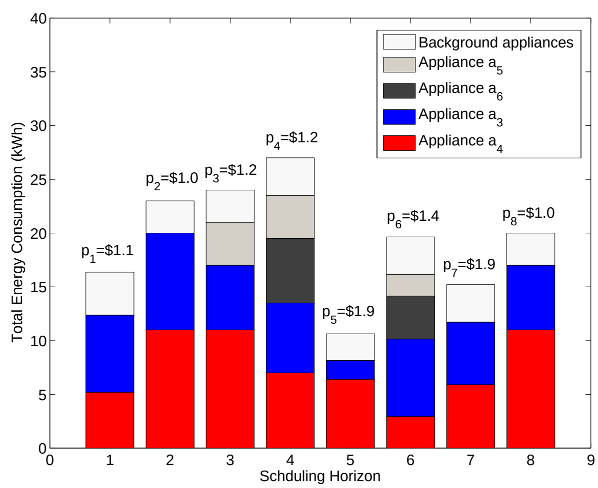


Fig. 3. The scheduled energy consumption of each appliance in response to the forecasted price vector $\boldsymbol{p}$.

## IV. Simulated Annealing based Algorithm for Price Control

In the section above, we have obtained the solution of electricity consumption in response to electricity prices. In this section, we consider how the retailer adjusts the electricity prices according to users' responses.

### A. Simulated Annealing based Price Control (SAPC) Algorithm

The retailer determines the optimal price vector for a certain time period (namely a scheduling horizon) right before the start of the time period. For example, the price used for a day can be calculated during the last few minutes of the previous day. Due to the non-convexity of $S_{u,h}(\boldsymbol{p})$'s, convex optimization methods cannot be used to solve the retailer side problem **P2**. The proposed price control algorithm here is based on the use of Simulated Annealing (SA) [19], and is referred to as SAPC.

Suppose that two-way communications between the retailer and users is possible through certain types of communication networks in future smart grids. The SAPC algorithm solves Problem **P2** in an iterative manner. In each iteration, the retailer broadcasts to all users a tentative price vector $\boldsymbol{p}$. Each user then responds with $S_{u,h}(\boldsymbol{p})$ for all $h$. If no semi-elastic appliance is present, $S_{u,h}(\boldsymbol{p})$ can be analytically calculated through the closed-form expression (25). Otherwise, the users compute $S_{u,h}(\boldsymbol{p})$ using Algorithm 1. Based on these responses, the retailer updates the price vector based on the concept of SA, which will be discussed shortly. The updated price is then broadcasted to probe the users' responses. When the price vector is finalized, the retailer sends it to all users, who will schedule energy consumption for the next scheduling horizon accordingly.

**Update of $\boldsymbol{p}$:** In each iteration of the SAPC algorithm, one entry of $\boldsymbol{p}$ is updated while the others keep fixed. Different entries $p_1$, $p_2$, $\cdots$, $p_H$ are updated in a round robin manner in consecutive iterations. When $p_h$ is to be updated, for example, the retailer randomly picks $p'_h$ in $[p^l, p^u]$ for time slot $h$, and then broadcasts the updated power vector $(\boldsymbol{p}_{-h}, p'_h)$ to users. Here, $\boldsymbol{p}_{-h}$ denotes the vector $(p_1, ..., p_{h-1}, p_{h+1}, ..., p_H)$. Having received the responses $S_{u,h}(\boldsymbol{p}_{-h}, p'_h)$'s from all users, the retailer calculates the value $L(\boldsymbol{p}_{-h}, p'_h)$, where $L(\cdot)$ is defined in (13). Then, the retailer compares this value with $L(\boldsymbol{p})$ calculated in the previous iteration. If $\Delta = L(\boldsymbol{p}_{-h}, p'_h) - L(\boldsymbol{p})$ is larger than 0, then $p'_h$ is accepted as the new $p_h$ with probability 1. Otherwise, it is accepted as the new $p_h$ with a probability $\exp(\frac{\Delta}{T})$, and the old $p_h$ keeps with a probability $1 - \exp(\frac{\Delta}{T})$, where $T$ is a control parameter (also referred to as temperature). For convergence, $T$ decreases with each iteration. Intuitively, the acceptance of uphill-moving becomes less and less likely with the decrease of $T$, implying convergence when $T$ is sufficiently small. When $p_h$ is updated as $p'_h$, the retailer updates the memory of $L(\boldsymbol{p})$ as $L(\boldsymbol{p}_{-h}, p'_h)$. After the SAPC algorithm is terminated, the retailer broadcasts the updated price vector encapsulated in a packet to the smart meters at the beginning of the following scheduling horizon. Such updated price vector is used for the electricity prices in the following scheduling horizon.

The following propositions discusses the convergence of the SA-based algorithms.

**Proposition 1** ( [20], [21])**.** The SAPC algorithm converges to the global optimal solution to Problem **P2**, as the control parameter $T$ approaches to zero with $T = \frac{T_0}{\log(k)}$.

For practical implementation, a solution very close to the global optimal solution is obtained when $T < \epsilon$, where $\epsilon$ is a very small number. The details of the SAPC algorithm is given in Algorithm 2.

### *B. Computational Complexity of SAPC*

Recall the SAPC algorithm, and the number of rounds needed for the SAPC algorithm is $\exp(\frac{T_0}{\epsilon})$. Since $H$ iterations are needed in one round, the total number of iterations needed for the SAPC algorithm is equal to $\exp(\frac{T_0}{\epsilon})H$, where $H$ is the number of time slots in the scheduling horizon. Therefore, we have the following Lemma 3.

**Lemma 3.** Given the initial temperature $T_0$ and the stopping criterion $\epsilon$, the SAPC algorithm needs $\exp(\frac{T_0}{\epsilon})H$ iterations.

Two-way communications between the retailer and users are needed when the price in any time slot is updated, as the retailer needs to broadcast the updated prices to users in a control packet, and each user needs to inform the retailer of its possible energy consumption in a data packet. Therefore, the time needed for one iteration consists of the transmit time of packets, the computational time in response to the updated price at each smart meter, and the computational time of updating the price at the retailer side. The transmit time depends on the underlying communication technology. In practice, the transmit time of packets with 32 bytes is at the order of $1 \sim 10^3$ ms per iteration over a broadband with a speed of 100 Mbps. One the other hand, the computational time totally depends on the processors of the retailer and smart meters, the number of users, and the number of appliances of each user. Let the transmit time and the computational time be $T_t$ time units and $T_c$ time units, respectively. Then, by Lemma 3, the SAPC algorithm takes $(T_t + T_c)H\exp(\frac{T_0}{\epsilon})$ time units. This implies that the retailer needs to update the prices for the following scheduling horizon $(T_t + T_c)H\exp(\frac{T_0}{\epsilon})$ time units before the end of the current scheduling horizon.

**Algorithm 2** The SAPC Algorithm

**Procedure at the retailer side:**

1: **Initialization**: Set $T = T_0$, $k = 1$, and $\boldsymbol{p}$ is initialized as the price vector used in the current scheduling horizon.
2: **repeat**
3:     **for all** $h$'s in the order of $\{1, 2, \cdots, H\}$ **do**
4:         The retailer randomly pick $p'_h \in [p^l, p^u]$, and broadcasts $p'_h$ and $p_{h-1}$ encapsulated in a packet to the smart meters via LAN.
5:         After receiving the packet from the users including the energy information $S_{u,h}(\boldsymbol{p}_{-h}, p'_h)$'s, the retailer calculates $L(\boldsymbol{p}_{-h}, p'_h)$ in Problem **P2** according to the received $S_{u,h}(\cdot)$'s.
6:         The retailer computes $\Delta = L(\boldsymbol{p}_{-h}, p'_h) - L(\boldsymbol{p})$, and let $p_h = p'_h$ with probability 1 if $\Delta_h \geq 0$. Otherwise, let $p_h = p'_h$ with probability $\exp(\frac{\Delta}{T})$, or $p_h = p_h$ with probability $(1 - \exp(\frac{\Delta}{T}))$.
7:         When $p_h$ is updated as $p'_h$, the retailer updates the memory of $L(\boldsymbol{p})$ as $L(\boldsymbol{p}_{-h}, p'_h)$.
8:     **end for**
9:     $k = k + 1$.
10:     $T = \frac{T_0}{\log(k)}$.
11: **until** $T < \epsilon$.
12: The retailer broadcasts the updated price vector $\boldsymbol{p}$ encapsulated in a packet to the smart meters at the beginning of the following scheduling horizon.

**Procedure at the user side:**

1: After receiving the packet from the retailer, the smart meter updates the tentative price vector as $(\boldsymbol{p}_{-h}, p'_h)$ in its memory.
2: The smart meter calculates the response to $(\boldsymbol{p}_{-h}, p'_h)$ by either (25) or Algorithm 1.
3: The smart meter informs the retailer of the total energy consumption in each time slot encapsulated in a packet.

## V. Simulation Results

In this section, we conduct simulations to illustrate the effectiveness of the proposed real-time pricing scheme.

**Example 2**: We consider a smart grid with 100 users, where each user $u$ has four elastic appliances (i.e., $u_1$, $u_2$, $u_3$ and $u_4$) and two semi-elastic appliances (i.e., $u_5$ and $u_6$). Assume each user $u$ has a quality-of-usage of $U_{u_i,h}(e_{u_i,h}) = -a_{u_i,h}(e_{u_i,h} + b_{u_i,h})^{-1}$ for each elastic appliance $u_i$ at each time slot $h$. Specifically, each parameter $a_{u_i,h}$ is chosen from the uniform distribution on $[10, 20]$, and each parameter $b_{u_i,h}$ is chosen from the uniform distribution on $[2, 5]$. The time scheduling horizon is $\mathcal{H} = \{1, 2, \cdots, 12\}$. The maximum allowable energy consumption of each user follows the uniform distribution on $[10, 15]$ kWh at each time slot. The maximum allowable energy consumptions of each appliance follows the uniform distribution on $[1.0, 2.0]$ kWh at each time slot. The total energy consumption of each semi-elastic appliance follows the uniform distribution on $[4, 6]$ kWh in the working period. Assume that the total energy consumption of background appliances follows the uniform distribution on [1,2] kWh at each time slot for each user. Let the time scheduling horizon be $\mathcal{H} = \{1, 2, \cdots, 12\}$. Let the working period of each semi-elastic appliance be consecutive with the beginning $\alpha_{u_i}$ and the end $\beta_{u_i}$, where $\alpha_{u_i}$ and $\beta_{u_i}$ are randomly chosen from $\mathcal{H}$. Finally, let $w = 1$, $a = 10^{-4}$, and $b = 2 \times 10^{-5}$ in (13).

We first compare the total energy consumption at each time slot under different settings of price in Fig. 4. Here, both the optimal flat-rate price and the optimal real-time price are computed by the proposed SAPC algorithm. When the proposed SAPC algorithm is used for computing the optimal flat-rate price, all elements in the price vector are simultaneously updated to the same value at each round. From Fig. 4, we can see that in any time slot, the increase of electricity price leads to the reduction of total energy consumption in the time slot, regardless of the prices setting in other time slots. For example, in time slot 5, the most energy consumption happens when $p_5$=\$0.58, while the least enegy consumption happens when $p_5$=\$1.50. For the flat-rate scheme, this implies that when the retailer increases the price in each time slot, all load demands are reduced in the scheduling horizon accordingly, which can be also found in Fig. 4 from the two flat-rate schemes. The reduction of load demand further leads to the reduction of peak demand. Furthermore, Fig. 4 shows that compared to the two flat-rate pricing schemes, the real-time pricing scheme flattens the load demand curve and reduces the peak-to-average load ratio. In particular, the real-time pricing scheme reduces the peak-to-average load ratio by about 20% compared to the flat-rate pricing schemes.

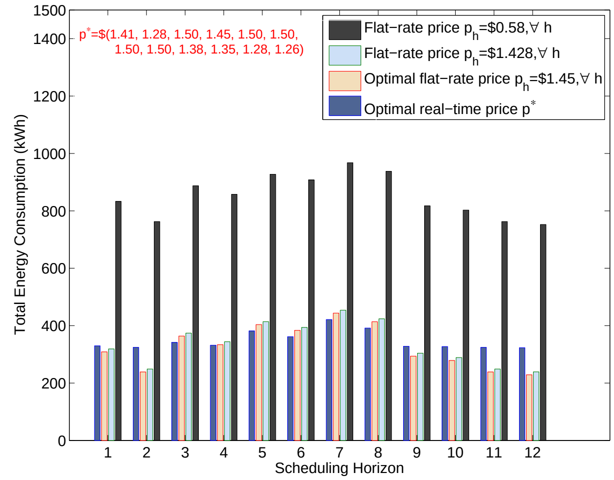


Fig. 4. The total energy consumption under different settings of price.

We then compare the revenue, the cost, and the profit under different settings of price in Table I. Specifically, these performances are evaluated at the retailer side. It can be seen from Table I that (i) to make ensure the same total payment from users (i.e.,$5923.6), the cost under the flat-rate pricing (i.e., $6930.7) is much higher due to the increase of peak demand (as shown in Fig.4); (ii) to make the same cost, the retailer has to set the flat rate to be high enough to compensate the peak cost; (iii) under the flat-rate pricing scheme, the retailer achieves the maximum profit with the price of $1.45, which is lower than that achieved by our real-time pricing scheme.

**Example 3:** In this example, we conduct an experiment to observe the total time needed for obtaining the optimal price vector if the two-way communication in the SAPC algorithm is done through general-purpose networks, such as Internet. Our experiment emulates the retailers and users by computers that are connected to the public network through sub-networks from different service providers. The retailer computer is located in The Chinese University of Hong Kong, and the user computers are scattered throughout the Hong Kong city. We consider an experiment situation with $N$ user computers, where $N$ is from 100 to 1000. Each user has six elastic appliances, four semi-elastic appliances, and several background appliances. Assume the retailer side procedure and the user side procedure in the SAPC

TABLE I

USERS' AND RETAILER'S BEHAVIORS UNDER DIFFERENT PRICE SETTING

| Flat-rate pricing | | | |
|---|---|---|---|
| Price setting | Total Payment/Revenue | Cost | Profit |
| $p_h = \$0.58, \forall h$ | \$5923.6 | \$6930.7 | \$-1007.1 |
| $p_h = \$1.428, \forall h$ | \$5787 | \$1191.2 | \$4595.8 |
| Optimal flat price $p_h = \$1.45, \forall h$ | \$5698.1 | \$1097.5 | \$4600.6 |
| Real-time pricing (Achieved by SAPC) | | | |
| Optimal price | Total Payment/Revenue | Cost | Profit |
| $\boldsymbol{p}^* = \$(1.41, 1.28, 1.50, 1.45,$ 1.50, 1.50, 1.50 1.50, 1.38, 1.35, 1.28, 1.26) | \$5923.6 | \$1191.2 | \$4732.4 |

* The total payment is the sum of all users' payment, equal to the retailer's revenue. The cost and the profit are evaluated by the retailer.

algorithm are implemented with MATLAB[4]. The scheduling horizon is $\mathcal{H} = \{1, 2, \cdots, 24\}$ hours. The initial price vector is randomly picked.

Fig. 5 shows the total time needed to the optimal price vector. It can be seen that both the total time and the time for two-way communications do not change much when the number of users increases. For example, it only takes in total 580 seconds for the retailer to communicate with the users for the purpose of probing $S_{u,h}(\boldsymbol{p})$'s, even when the number of users is 1000. The total time needed goes up to 600 seconds, i.e., 10 minutes, when the MATLAB computational time is also included. Note that the computational time in real-system deployment can be much shorter with, say, special-purpose FPGAs. This result is very encouraging, as it implies that the retailer only needs a few minutes before midnight to determine the optimal price vector for the next day (24 hours). More importantly, the result implies that the algorithm is rather scalable with the number of users, as the time cost does not increase much when the user number becomes large.

[4]In this paper, MATLAB with version R2010b is used on a HP Compaq dx7300 desktop with 3.6GHz processors and 1Gb of RAM.

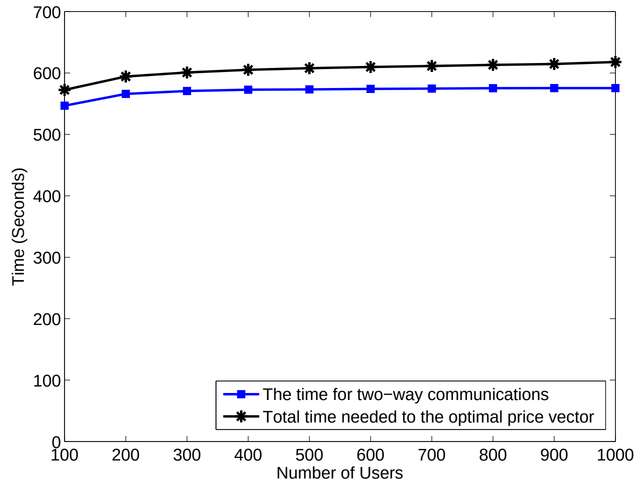


Fig. 5. Total time needed for obtaining the price vector VS number of users

## VI. Conclusions

In this paper, we proposed an optimal real-time pricing scheme for the reduction of peak-to-average load ratio in smart grid. The proposed scheme solves a two-stage optimization problem, as which the real-time pricing scheme is formulated. At the users' side, we can obtain the optimal energy consumption that maximizes the quality-of-usage with minimum electricity payment either in closed forms or through an efficient iterative algorithm. As the retailer side, we used a Simulated-Annealing-based Price Control (SAPC) algorithm to obtain the optimal real-time price that maximizes the profit. In terms of practical implementation, users and the retailer interact with each other through a limited number of message exchanges over a communication network to reach the optimal prices. Simulation results showed that our proposed algorithm can lead to performance improvement for both the retailer and users.

## Appendix A
## Proof of Lemma 1

Due to the decoupling of energy consumption among the $\mathcal{B}_u$ category appliances and the $\mathcal{C}_u$ category appliances, problem (15) is decomposed into

$$\begin{aligned}&\text{maximize } \sum_{h\in\mathcal{H}}\sum_{a_u\in\mathcal{B}_u} U_{a_u,h}(e_{a_u,h}) - \sum_{h\in\mathcal{H}}(p_h+\eta_{u,h})\Bigg(\sum_{a_u\in\mathcal{B}_u} e_{a_u,h}\Bigg)\\ &\text{subject to } 0\le e_{a_u,h}\le r_{a_u}^{\max}, \forall h\in\mathcal{H}, \forall a_u\in\mathcal{B}_u,\\ &\text{variables } \ \boldsymbol{e}_{a_u}, a_u\in\mathcal{B}_u,\end{aligned} \tag{29}$$

and

$$\begin{aligned}
\text{minimize} \quad & \sum_{h\in\hat{\mathcal{H}}_u}(p_h+\eta_{u,h})\Big(\sum_{a_u\in\mathcal{C}_u}e_{a_u,h}\Big)\\
\text{subject to} \quad & \sum_{h\in\mathcal{H}'_{a_u}}e_{a_u,h}=E_{a_u},\forall a_u\in\mathcal{C}_u,\\
& 0\le e_{a_u,h}\le r^{\max}_{a_u},\forall a_u\in\mathcal{C}_u,\forall h\in\mathcal{H}'_{a_u},\\
& e_{a_u,h}=0,\forall a_u\in\mathcal{C}_u,\forall h\notin\mathcal{H}'_{a_u},\\
\text{variables} \quad & \boldsymbol{e}_{a_u},a_u\in\mathcal{C}_u.
\end{aligned} \tag{30}$$

Due to the concave nature of $U_{e_{a_u}}(\cdot)$, the optimal solution of problem (29) can be obtained when the first derivative of the objective function over $e_{a_u}$'s is set to zero. In particular, such optimal solution satisfies (17). Based on the assumption of $\sum_{a_u\in\mathcal{C}_u} r^{\max}_{a_u}\le C^{\max}_u-E_{u,h}$ for all $h$, the optimal solution of the linear optimization problem (30) can be obtained by partitioning more energy to the working time slot with smaller price. In particular, such optimal solution satisfies (18). Therefore, Lemma 1 follows. ■

## APPENDIX B
## PROOF OF LEMMA 2

Due to the convex nature of the dual problem, we obtain the optimal solution $\eta^*_{u,h}(p_h)$'s by the Karush-Kuhn-Tucker (KKT) sufficient and necessary conditions of (19). Let $\upsilon_h$ is the multiplier variable regarding the constraint of $\eta_{u,h}\ge 0$. In particular, the KKT of (19) follows

$$\begin{aligned}
&\sum_{a_u\in\mathcal{B}_u}\Bigg(U'_{a_u,h}\Bigg(\Big[U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))\Big]_0^{r^{\max}_{a_u}}\Bigg)-(p_h+\eta^*_{u,h}(p_h))\Bigg)\frac{\partial\Big[U'^{-1}_{a_u,h}(p_h+\eta_{u,h})\Big]_0^{r^{\max}_{a_u}}}{\partial\eta_{u,h}}\Bigg|_{\eta_{u,h}=\eta^*_{u,h}(p_h)}\\
&\quad+(C^{\max}_u-E_{u,h}-\sum_{a_u\in\mathcal{B}_u}\Big[U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))\Big]_0^{r^{\max}_{a_u}})-\upsilon_h=0\\
&\upsilon_h\eta^*_{u,h}(p_h)=0\\
&\upsilon_h\ge 0,\forall h\notin\hat{\mathcal{H}}_u.
\end{aligned} \tag{31}$$

Due to the concavity of $U_{a_u,h}(\cdot)$, we have

$$U'_{a_u,h}\Bigg(\Big[U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))\Big]_0^{r^{\max}_{a_u}}\Bigg)=(p_h+\eta^*_{u,h}(p_h)),\ \text{if } 0\le U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))\le r^{\max}_{a_u}, \tag{32}$$

and

$$\frac{\partial \left[U'^{-1}_{a_u,h}(p_h+\eta_{u,h})\right]_0^{r_{a_u}^{\max}}}{\partial \eta_{u,h}} = 0, \text{ if } U'^{-1}_{a_u,h}(p_h+\eta_{u,h}) > r_{a_u}^{\max} \text{ or } U'^{-1}_{a_u,h}(p_h+\eta_{u,h}) < 0. \tag{33}$$

From (32) and (33), we get

$$\sum_{a_u\in\mathcal{B}_u}\left(U'_{a_u,h}\left(\left[U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))\right]_0^{r_{a_u}^{\max}}\right)-(p_h+\eta^*_{u,h}(p_h))\right)\frac{\partial \left[U'^{-1}_{a_u,h}(p_h+\eta_{u,h})\right]_0^{r_{a_u}^{\max}}}{\partial \eta_{u,h}}\Bigg|_{\eta_{u,h}=\eta^*_{u,h}(p_h)}=0. \tag{34}$$

Thus, (31) becomes

$$\begin{aligned} &C_u^{\max}-E_{u,h}-\sum_{a_u\in\mathcal{B}_u}\left[U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))\right]_0^{r_{a_u}^{\max}}-\upsilon_h=0\\ &\upsilon_h\eta^*_{u,h}(p_h)=0\\ &\upsilon_h\geq 0, \forall h\in\mathcal{H}. \end{aligned} \tag{35}$$

The concavity of $U_{a_u,h}(\cdot)$ implies that $U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))$ decreases with $\eta^*_{u,h}(p_h)$, and thus

$$\left[U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))\right]_0^{r_{a_u}^{\max}} \leq \left[U'^{-1}_{a_u,h}(p_h)\right]_0^{r_{a_u}^{\max}}, \forall \eta^*_{u,h}(p_h)>0. \tag{36}$$

This implies that if (21) is satisfied, then

$$\sum_{a_u\in\mathcal{B}_u}\left[U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))\right]_0^{r_{a_u}^{\max}} < C_u^{\max}-E_{u,h}, \forall \eta^*_{u,h}(p_h)>0. \tag{37}$$

It follows that if $\eta^*_{u,h}(p_h)$ is positive, then the first equation of (35) is satisfied only when $\upsilon_h$ is positive. Obviously, it contradicts with the second equation of (35). Therefore, if (21) is satisfied, then $\eta^*_{u,h}(p_h)=0$, and hence (22).

Next, we prove the latter part of Lemma 2. Since $U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))$ decreases with the increase of $\eta^*_{u,h}(p_h)$, there must exist a positive $\eta^*_{u,h}(p_h)$ such that[5]

$$\sum_{a_u\in\mathcal{B}_u}\left[U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))\right]_0^{r_{a_u}^{\max}} = C_u^{\max}-E_{u,h} \tag{38}$$

when

$$\sum_{a_u\in\mathcal{B}_u}\left[U'^{-1}_{a_u,h}(p_h)\right]_0^{r_{a_u}^{\max}} > C_u^{\max}-E_{u,h}. \tag{39}$$

Therefore, we have (23) and (24). ■

[5]Due to the monotonicity of $U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))$, we can obtain $\eta^*_{u,h}(p_h)$ satisfying (38) through the bisection searching.

## Appendix C
## Proof of Theorem 1

Theorem 1 is a consequence of Theorems 2.1, 2.2 and 2.3 in [23]. In the following, we first prove part (a). Let $\boldsymbol{\eta}_u^* = (\eta_{u,h}^*, \forall h \in \hat{\mathcal{H}}_u)$ be the optimal solution to (20). For notational convenience, let $\boldsymbol{\eta}_u$ denote the concatenation of variables $\eta_{u,h}$'s for all $h \in \hat{\mathcal{H}}_u$, and let $\hat{e}_{a_u,h}^{(k)}$ and $\check{e}_{a_u,h}^{(k)}$ denote $\hat{e}_{a_u,h}(p_h + \eta_{u,h}^{(k)})$ and $\hat{e}_{a_u,h}(\boldsymbol{p}, (\eta_{u,h}^{(k)}, \forall h \in \hat{\mathcal{H}}_u))$, respectively. Define

$$||\boldsymbol{\eta}_u||_\psi = \sum_{h \in \hat{\mathcal{H}}_u} \frac{(\eta_{u,h})^2}{\psi_{u,h}^{(0)}}. \tag{40}$$

Together with $\psi_{u,h}^{(k)} = s\psi_{u,h}^{(0)}$, by (26), we have

$$\begin{aligned} ||\boldsymbol{\eta}_u^{(k+1)} - \boldsymbol{\eta}_u^*||_\psi \leq ||\boldsymbol{\eta}_u^{(k)} - \boldsymbol{\eta}_u^*||_\psi - 2s \sum_{h \in \hat{\mathcal{H}}_u} (\eta_{u,h}^{(k)} - \eta_{u,h}^*)(C_u^{\max} - E_{u,h} - \sum_{a_u \in \mathcal{B}_u} \hat{e}_{a_u,h}^{(k)} - \sum_{a_u \in \mathcal{C}_u} \check{e}_{a_u,h}^{(k)}) \\ + \sum_{h \in \mathcal{H}} s^2 \psi_{u,h}^{(0)} (C_u^{\max} - E_{u,h} - \sum_{a_u \in \mathcal{B}_u} \hat{e}_{a_u,h}^{(k)} - \sum_{c_u \in \mathcal{C}_u} \check{e}_{a_u,h}^{(k)})^2. \end{aligned} \tag{41}$$

For simplicity, let

$$\begin{aligned} f^*(\boldsymbol{\eta}_u^*) = &\sum_{h \in \hat{\mathcal{H}}_u} \sum_{a_u \in \mathcal{B}_u} U_{a_u,h}(e_{a_u,h}^*(\boldsymbol{p})) - \sum_{h \in \hat{\mathcal{H}}_u} p_h \Bigg( \sum_{a_u \in \mathcal{B}_u, \mathcal{C}_u} e_{a_u,h}^*(\boldsymbol{p}) \Bigg) \\ &+ \sum_{h \in \hat{\mathcal{H}}_u} \eta_{u,h}^* \Bigg( C_u^{\max} - E_{u,h} - \sum_{a_u \in \mathcal{B}_u, \mathcal{C}_u} e_{a_u,h}^*(\boldsymbol{p}) \Bigg), \end{aligned} \tag{42}$$

$$\begin{aligned} f^{(k)}(\boldsymbol{\eta}_u^{(k)}) = &\sum_{h \in \hat{\mathcal{H}}_u} \sum_{a_u \in \mathcal{B}_u} U_{a_u,h}(\hat{e}_{a_u,h}^{(k)}) - \sum_{h \in \hat{\mathcal{H}}_u} p_h \Bigg( \sum_{a_u \in \mathcal{B}_u} \hat{e}_{a_u,h}^{(k)} + \sum_{c_u \in \mathcal{C}_u} \check{e}_{a_u,h}^{(k)} \Bigg) \\ &+ \sum_{h \in \hat{\mathcal{H}}_u} \eta_{u,h}^{(k)} \Bigg( C_u^{\max} - E_{u,h} - \sum_{a_u \in \mathcal{B}_u} \hat{e}_{a_u,h}^{(k)} - \sum_{c_u \in \mathcal{C}_u} \check{e}_{a_u,h}^{(k)} \Bigg), \end{aligned} \tag{43}$$

and

$$\begin{aligned} f^{(k)}(\boldsymbol{\eta}_u^*) = &\sum_{h \in \hat{\mathcal{H}}_u} \sum_{a_u \in \mathcal{B}_u} U_{a_u,h}(\hat{e}_{a_u,h}^{(k)}) - \sum_{h \in \hat{\mathcal{H}}_u} p_h \Bigg( \sum_{a_u \in \mathcal{B}_u} \hat{e}_{a_u,h}^{(k)} + \sum_{c_u \in \mathcal{C}_u} \check{e}_{a_u,h}^{(k)} \Bigg) \\ &+ \sum_{h \in \hat{\mathcal{H}}_u} \eta_{u,h}^* \Bigg( C_u^{\max} - E_{u,h} - \sum_{a_u \in \mathcal{B}_u} \hat{e}_{a_u,h}^{(k)} - \sum_{c_u \in \mathcal{C}_u} \check{e}_{a_u,h}^{(k)} \Bigg). \end{aligned} \tag{44}$$

Due to the strict convexity of Problem **P1**, we have

$$f^*(\boldsymbol{\eta}_u^*) \geq f^{(k)}(\boldsymbol{\eta}_u^*). \tag{45}$$

Thus, it follows from (45) that

$$\begin{aligned} f^*(\boldsymbol{\eta}_u^*) - f^{(k)}(\boldsymbol{\eta}_u^{(k)}) \geq & f^{(k)}(\boldsymbol{\eta}_u^*) - f^{(k)}(\boldsymbol{\eta}_u^{(k)}) \\ & = \sum_{h\in\hat{\mathcal{H}}_u} (\eta^*_{u,h} - \eta^{(k)}_{u,h})(C_u^{\max} - E_{u,h} - \sum_{a_u\in\mathcal{B}_u} \hat{e}^{(k)}_{a_u,h} - \sum_{c_u\in\mathcal{C}_u} \check{e}^{(k)}_{a_u,h})) \end{aligned} \tag{46}$$

Substituting (46) into (41), we get

$$\begin{aligned} ||\boldsymbol{\eta}_u^{(k+1)} - \boldsymbol{\eta}_u^*||_\psi \leq & ||\boldsymbol{\eta}_u^{(k)} - \boldsymbol{\eta}_u^*||_{k\psi} + 2s(f^*(\boldsymbol{\eta}_u^*) - f^{(k)}(\boldsymbol{\eta}_u^{(k)})) \\ & + \sum_{h\in\hat{\mathcal{H}}_u} s^2\psi^{(0)}_{u,h}(C_u^{\max} - E_{u,h} - \sum_{a_u\in\mathcal{B}_u} \hat{e}^{(k)}_{a_u,h} - \sum_{c_u\in\mathcal{C}_u} \check{e}^{(k)}_{a_u,h}))^2. \end{aligned} \tag{47}$$

Given $\lambda > 0$, let

$$\Phi(\lambda) = \left\{ \boldsymbol{\eta}_u^{(0)} | f^{(0)}(\boldsymbol{\eta}_u^{(0)}) \leq f^*(\boldsymbol{\eta}_u^*) + \lambda \right\} \tag{48}$$

Based on variables $e_{a_u,h}$'s, we can find an $M < \infty$ that is no smaller than the optimal value of (49),

$$\begin{aligned} \text{maximize} \ & \sum_{h\in\hat{\mathcal{H}}_u} s^2\psi^{(0)}_{u,h}(C_u^{\max} - E_{u,h} - \sum_{a_u\in\mathcal{B}_u,\mathcal{C}_u} e_{a_u,h})^2 \\ \text{subject to} \ & 0 \leq e_{a_u,h} \leq r^{\max}_{a_u}, \forall h \in \mathcal{H}, \forall a_u \in \mathcal{B}_u, \\ & \sum_{h\in\mathcal{H}'_{a_u}} e_{a_u,h} = E_{a_u}, \forall a_u \in \mathcal{C}_u, \\ & 0 \leq e_{a_u,h} \leq r^{\max}_{a_u}, \forall a_u \in \mathcal{C}_u, \forall h \in \mathcal{H}'_{a_u}, \\ \text{variables} \ \ & e_{a_u,h}, \forall h \in \hat{\mathcal{H}}_u, \forall a_u \in \mathcal{B}_u, \mathcal{C}_u. \end{aligned} \tag{49}$$

Therefore, if we pick

$$s \leq \lambda/M, \tag{50}$$

then as long as $\boldsymbol{\eta}_u^{(k)} \notin \Phi(\lambda)$, we have

$$||\boldsymbol{\eta}_u^{(k+1)} - \boldsymbol{\eta}_u^*||_\psi \leq ||\boldsymbol{\eta}_u^{(k)} - \boldsymbol{\eta}_u^*||_\psi - s\lambda \tag{51}$$

Thus, $\boldsymbol{\eta}_u^{(k)}$ will enter the set $\Phi(\lambda)$ eventually. Once $\boldsymbol{\eta}_u^{(k)} \in \Phi(\lambda)$, if we pick

$$s \leq \lambda/\sqrt{M}, \tag{52}$$

then we have

$$\sqrt{||\boldsymbol{\eta}_u^{(k+1)} - \boldsymbol{\eta}_u^*||_\psi} \leq \sqrt{||\boldsymbol{\eta}_u^{(k)} - \boldsymbol{\eta}_u^*||_\psi} + \sqrt{||\boldsymbol{\eta}_u^{(k+1)} - \boldsymbol{\eta}_u^{(k)}||_\psi} \leq \sqrt{||\boldsymbol{\eta}_u^{(k)} - \boldsymbol{\eta}_u^*||_\psi} + \lambda \tag{53}$$

Therefore, if

$$s \leq \min\{\lambda/M, \lambda/\sqrt{M}\}, \tag{54}$$

then there exists a number $T_0$ such that

$$\sqrt{||\boldsymbol{\eta}_u^{(k)} - \boldsymbol{\eta}_u^*||_\psi} \leq \xi(\lambda) = \max_{\boldsymbol{\eta}_u \in \Phi(\lambda)} \sqrt{||\boldsymbol{\eta}_u - \boldsymbol{\eta}_u^*||_\psi}, \forall k \geq T_0. \tag{55}$$

It is clear that, as $\lambda \to 0$, $\xi(\lambda) \to 0$. Therefore, for any $\epsilon > 0$, we can pick $\lambda$ (and hence $s$) sufficiently small such that $\xi(\lambda) < \epsilon$, i.e., there exists time $T_0$ such that

$$\sqrt{||\boldsymbol{\eta}_u^{(k)} - \boldsymbol{\eta}_u^*||_\psi} \leq \epsilon, \forall k \geq T_0. \tag{56}$$

Finally, since the mapping from $\boldsymbol{\eta}_u^{(k)}$ to $(\hat{e}_{a_u,h}(p_h + \eta_{u,h}^{(k)})$'s, $\hat{e}_{a_u,h}(\boldsymbol{p}, (\eta_{u,h}^{(k)}, \forall h \in \hat{\mathcal{H}}_u))$'s) is continuous, we can pick $\lambda$ (and hence $s$) sufficiently small such that (27) is satisfied.

The proof of part (b) is similar to part (a), and thus omitted here. Interested readers are referred to Theorem 3 in [23]. Consequently, the proof of Theorem 1 is finished. ■

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# Demand Response Management via Real-time Electricity Price Control in Smart Grids

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***Abstract*—This paper proposes a real-time pricing scheme that reduces the peak-to-average load ratio through demand response management in smart grid systems. The proposed scheme solves a two-stage optimization problem. On one hand, each user reacts to prices announced by the retailer and maximizes its payoff, which is the difference between its quality-of-usage and the payment to the retailer. On the other hand, the retailer designs the real-time prices in response to the forecasted user reactions to maximize its profit. In particular, each user computes its optimal energy consumption either in closed forms or through an efficient iterative algorithm as a function of the prices. At the retailer side, we develop a Simulated-Annealing-based Price Control (SAPC) algorithm to solve the non-convex price optimization problem. In terms of practical implementation, the users and the retailer interact with each other via a limited number of message exchanges to find the optimal prices. By doing so, the retailer can overcome the uncertainty of users' responses, and users can determine their energy usage based on the actual prices to be used. Our simulation results show that the proposed real-time pricing scheme can effectively shave the energy usage peaks, reduce the retailer's cost, and improve the payoffs of the users.**

***Index Terms*—Real-time pricing, Demand response management, Payoff maximization, Profit maximization, Non-convex optimization.**

## I. Introduction

In today's electric power grid, we often observe substantial hourly variation in the wholesale electricity price, and the spikes usually happen during peak hours due to the high generation cost. However, almost all end users nowadays are charged some flat-rate retail electricity price [1], [2], which does not reflect the actual wholesale price. With the flat-rate pricing, users often consume much more electricity during peak hours, such as the time between late afternoon and bed time for residential users. This leads to a large fluctuation of electricity consumption between off-peak hours and peak hours. The high peak-hour demand not only induces high cost to the retailers due to the high wholesale prices in those hours, but also has a negative impact on the reliability of the power grid. Ideally, the retailer would like to have the electricity consumption evenly spread across different hours of the day through a proper demand response management.

For the demand response management, researchers have introduced real-time pricing schemes to encourage users to shift their usage to off-peak hours [1]–[7]. A real-time price charges each users based on not only "how much" electricity is consumed but also "when" it is consumed. A properly designed real-time pricing scheme may result in a "triple-win" solution: flattened load demand curves enhances the robustness and lowers the generation cost for the power grid; a lower generation cost leads to a lower wholesale price, which in turn increases the retailers' profit; users may reduce their electricity expenditures by responding to the time-varying price.

The existing research in the real-time pricing can be divided into three main threads. The first thread is concerned with how users respond to the real-time price, hopefully in an automated manner, to achieve their desired level of comfort with lower electricity bill payment (e.g., [2], [8], [9]). These work, however, does not mention how the real-time prices should be set. The second thread of work is concerned with setting the real-time price at the retailer side (e.g., [10]), without taking into account users' potential responses to the forecasted price. For example, the retailer may adjust the real-time retail electricity price through linking it closely to the wholesale electricity price in [10]. The last thread of work is concerned with setting the real-time retail electricity price based on the maximization of the aggregate surplus of users and retailers subject to the supply-demand matching (e.g., [11]–[15]). However, the price obtained in this way may not do good to the retailer's profit. In principle, the retailer should be able to design real-time prices that maximizes its own profit by taking into account the users' potential responses to the prices (e.g., responses that maximize users' own payoff). Ideally, the real-time pricing scheme should be able to achieve a "triple-win" solution that benefits the grid, the retailer, and the end users.

In this paper, we endeavor to design such a real-time pricing scheme to reduce the peak-to-average load ratio, and to maximize each user's payoff and the retailer's profit in the meantime. The key hurdle that prevented previous work from doing so lies in the asymmetry of information. For example, when the prices are announced before the energy scheduling horizon (i.e., ex-ante price), the retailer has to face the uncertainty of user response and reimburse the wholesale cost based on the actual electricity consumption by the users. On the other hand, if the prices are fixed after the energy is being consumed (i.e., ex-post price), the users have to bear the uncertainty, as they only adjust their demand according to a prediction of the actual price.

As mentioned in [16], smart grid is an electricity delivery system enhanced with communication facilities and information technologies. Roughly speaking, smart grid is a system integrating the traditional power grids and the communication networks (e.g., Local Area Network (LAN)). Suppose that each user is equipped with a smart meter that is capable of having two way communications with the retailer through a communication network. Based on it, we introduce a novel ex-ante real-time pricing scheme for the future smart grid, where prices are determined at the beginning

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of each energy scheduling horizon. The contributions of this paper can be summarized as follows:

- We formulate the real-time pricing scheme as a two-stage optimization problem. On one hand, each user reacts to the price and maximizes its payoff, which is the difference between the quality-of-usage and the payment. On the other hand, the retailer designs the real-time price in response to the forecasted user reactions to maximize its profit.
- The proposed algorithm allows each user to optimally schedule its energy consumption in closed forms or through an efficient iterative algorithm. Furthermore, the users and the retailer interact with each other through a limited number of message exchanges to find the optimal price[1], which facilitates the elimination of cost uncertainty at the retailer side.
- We propose a real-time pricing algorithm based on the idea of simulated annealing to reduce the peak-to-average load ratio in smart grid systems. For the practical implementation of the algorithm, we further study how to set the length of interaction period so that the retailer is guaranteed to obtain the optimal price through communications with users.

The rest of this paper is organized as follows. Section II introduces the system model and the problem formulations. In Section III, we provide the closed-form expression of the electricity consumption scheduling with elaborate mathematical analysis, and propose an efficient iterative algorithm for electricity consumption scheduling. The simulated annealing based algorithm used to adjust the real-time retail electricity prices at the retailer side is proposed in Section IV. In Section V, we evaluate the performance of the proposed algorithm through several simulations. The paper is concluded in Section VI.

## II. System Model and Problem Formulation

We consider a microgrid (as shown in Fig. 1 [9]) with two types of participants: end users (i.e., customers), and a retailer from which end users purchase electricity. In this paper, we consider time-varying prices, with the hope to reduce peak-to-average load demand ratio, increase the retailer's profit, and maximize the users' utilities with minimum payment. In the following, we present the problems considered by users and retailer, respectively.

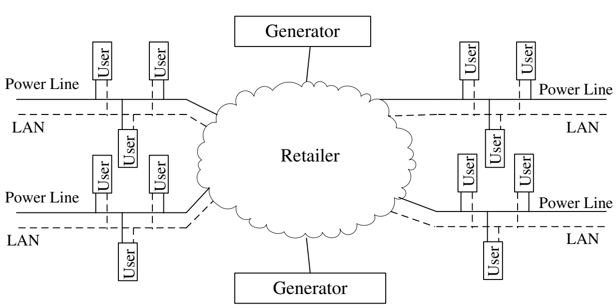


Fig. 1. A simplified illustration of the retail electricity market.

[1] In this paper, we assume that the users and the retailer declare their information truthfully. We will consider the incentive issue in a future work.

### A. Residential End Users

We assume that each user is equipped with a smart meter as shown in Fig. 2. The retailer sets the real-time retail electricity prices and informs to users via LAN. At the user side, the energy scheduler in the smart meter optimally computes and distributes energy consumption according to the prices for the upcoming scheduling horizon $\mathcal{H}$.

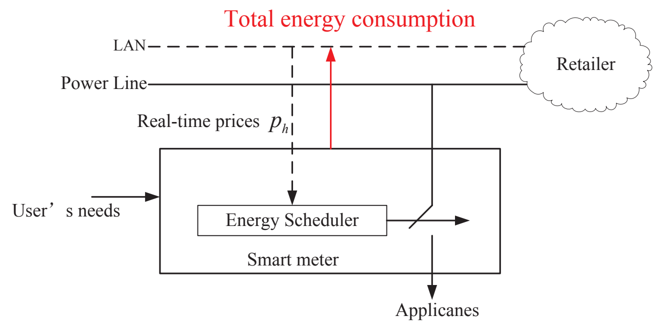


Fig. 2. The operation of smart meter and the price setting of retailer in our design.

Let $\mathcal{U} = \{1, 2, \cdots, U\}$ denote the set of residential users. Assume that each user $u$ has three types of appliances, denoted by $\mathcal{A}_u$, $\mathcal{B}_u$ and $\mathcal{C}_u$. The first category $\mathcal{A}_u$ includes *background appliances*, which consumes a fixed amount of energy per unit time during a fixed period of time. The background appliances are inelastic, in the sense that there is no flexibility to adjust the energy consumption across time. Examples of such appliances include lighting, refrigerator and electric kettle. The second category $\mathcal{B}_u$ includes elastic appliances, which have higher quality-of-usage (satisfaction) for more energy consumed per unit time (with a maximum consumption upper bound). The evaluation of quality-of-usage can be time-dependent for elastic appliances, as users may obtain a higher satisfaction to consume certain amount of energy during a certain time than in other time durations. Examples of such appliances include air conditioner, electric fan and iron. The last category $\mathcal{C}_u$ includes semi-inelastic appliances, which consume a fixed total energy within a preferred time period. This category is semi-inelastic in the sense that there is flexibility to choose when to consume the energy within the preferred time period, but no flexibility to adjust the total energy consumption. Examples include washer/dryer, dishwasher, plug-in hybrid electric vehicle (PHEV) and electric geyser.

For each appliance $a_u$, we express its energy consumption over the scheduling horizon $\mathcal{H}$ by a scheduling vector $\boldsymbol{e}_{a_u}$ as follows:

$$\boldsymbol{e}_{a_u} = (e_{a_u,1}, \cdots, e_{a_u,H}). \tag{1}$$

The retailer announces the price $p_h$ for time slots $h \in \mathcal{H} = \{1, ..., H\}$ at the beginning of the scheduling horizon, and each user $u$ computes its optimal $\boldsymbol{e}_{a_u}$ accordingly. A time slot can be, for example, one hour, and the scheduling horizon can be one day, i.e., $H = 24$ hours. In what follows, we will introduce the energy consumption constraints of the three categories of appliances.

As a background appliance, each $a_u \in \mathcal{A}_u$ works in a working period $H_{a_u} \in \mathcal{H}$, during which it consumes $r_{a_u}$ energy per time slot. This is mathematically described as

$$e_{a_u,h} = \begin{cases} r_{a_u,h}, & h \in \mathcal{H}_{a_u}, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

We note that the time slots in $\mathcal{H}_{a_u}$ are allowed to be intermittent. There is no flexibility to redistribute the load due to this type of appliances in response to the price. However, such appliances are ubiquitous in power systems and contribute a large percentage of the peak during high-demand hours. It is thus important to properly schedule categories $\mathcal{B}_u$ and $\mathcal{C}_u$ appliances to avoid further overloading the peaks.

For each appliance $a_u \in \mathcal{B}_u$, user $u$ obtains different levels of satisfaction for the same amount of energy consumed in different time slots. Suppose that the satisfaction is measured by a time-dependent quality-of-usage function $U_{a_u,h}(e_{a_u,h})$, which depends on who, when, and how much energy is consumed. For example, $U_{a_u,h}(e_{a_u,h})$ may be equal to zero during undesirable operation hours. It is reasonable to assume that $U_{a_u,h}(e_{a_u,h})$ is a non-decreasing concave function of $e_{a_u,h}$ at any time slot $h \in \mathcal{H}$. Besides, for each appliance $a_u \in \mathcal{B}_u$, the energy consumption per time slot is subject to

$$0 \le e_{a_u,h} \le r_{a_u}^{\max}, \forall h \in \mathcal{H}, \tag{3}$$

where $r_{a_u}^{\max}$ is the maximum energy that can be consumed in the time slot when appliance $a_u$ is working.

As a semi-inelastic appliance, each $a_u \in \mathcal{C}_u$ works in a working period $\mathcal{H}'_{a_u} \in \mathcal{H}$, during which it consumes $E_{a_u}$ energy in total. This is written as

$$\sum_{h \in \mathcal{H}'_{a_u}} e_{a_u,h} = E_{a_u} \tag{4}$$

for each appliance $a_u \in \mathcal{C}_u$. Noticeably, the period $\mathcal{H}'_{a_u}$ is consecutive with the beginning $\alpha_{a_u} \in \mathcal{H}$ and the end $\beta_{a_u} \in \mathcal{H}$. Thus, $\mathcal{H}'_{a_u}$ can be rewritten as $\mathcal{H}'_{a_u} = \{\alpha_{a_u}, \alpha_{a_u}+1, \cdots, \beta_{a_u}\}$. In practice, the choice of $\alpha_{a_u}$ and $\beta_{a_u}$ depends on the habit (or preference) of user $u$. Besides, the energy consumption per time slot is subject to the following constraint due to this type of appliance, i.e.,

$$0 \le e_{a_u,h} \le r_{a_u}^{\max}, \forall h \in \mathcal{H}'_{a_u}. \tag{5}$$

Furthermore, we have $e_{a_u,h} = 0$ for any $h \notin \mathcal{H}'_{a_u}$ as no operation (and hence energy consumption) is needed outside the working period $\mathcal{H}'_{a_u}$. For this type of appliance, there is flexibility to distribute the total load during the working period in response to the price.

For three categories of appliances, there is usually a limit on the total allowable energy consumption for each user $u$ at each time slot. This limit, denoted by $C_u^{max}$, can be set by the retailer to impose the following set of constraints on energy scheduling:

$$\sum_{a_u \in \mathcal{A}_u, \mathcal{B}_u, \mathcal{C}_u} e_{a_u,h} \le C_u^{\max}, \forall h \in \mathcal{H}. \tag{6}$$

In practice, such constraints are used to protect the total energy consumption from exceeding the grid capacity. To summarize, all valid energy consumption scheduling vectors can be determined by constraints (2)-(6).

Noticeably, each user $u$ has two contradicting goals, given that all its demands (i.e., constraints (2)-(6)) are met. The first is to maximize its overall satisfaction, given by

$$\sum_{h \in \mathcal{H}} \sum_{a_u \in \mathcal{B}_u} U_{a_u,h}(e_{a_u,h}). \tag{7}$$

The second goal is to minimize its electricity bill payment, obtained as

$$\sum_{h \in \mathcal{H}} p_h \Bigg( \sum_{a_u \in \mathcal{A}_u, \mathcal{B}_u, \mathcal{C}_u} e_{a_u,h} \Bigg). \tag{8}$$

To balance the two objectives, each user formulates the energy scheduling problem as maximizing its payoff, i.e.,

$$\begin{aligned}
\textbf{P1:}\quad & \text{maximize} && \sum_{h \in \mathcal{H}} \sum_{a_u \in \mathcal{B}_u} U_{a_u,h}(e_{a_u,h}) \\
& && - \sum_{h \in \mathcal{H}} p_h \Bigg( E_{u,h} + \sum_{a_u \in \mathcal{B}_u, \mathcal{C}_u} e_{a_u,h} \Bigg) && (9.1) \\
& \text{subject to} && 0 \le e_{a_u,h} \le r_{a_u}^{\max}, \forall a_u \in \mathcal{B}_u, \forall h \in \mathcal{H}, && (9.2) \\
& && 0 \le e_{a_u,h} \le r_{a_u}^{\max}, \forall a_u \in \mathcal{C}_u, \forall h \in \mathcal{H}'_{a_u}, && (9.3) \\
& && e_{a_u,h} = 0, \forall a_u \in \mathcal{C}_u, \forall h \notin \mathcal{H}'_{a_u}, && (9.4) \\
& && \sum_{h \in \mathcal{H}'_{a_u}} e_{a_u,h} = E_{a_u}, \forall a_u \in \mathcal{C}_u, && (9.5) \\
& && \sum_{a_u \in \mathcal{B}_u, \mathcal{C}_u} e_{a_u,h} \le C_u^{\max} - E_{u,h}, \forall h \in \mathcal{H}, && (9.6) \\
& \text{variables} && \boldsymbol{e}_{a_u}, \forall a_u \in \mathcal{B}_u, \mathcal{C}_u,
\end{aligned}$$

where $E_{u,h}$ is the total energy consumed by background appliances at time slot $h$, satisfying

$$E_{u,h} = \sum_{a_u : h \in \mathcal{H}_{a_u}} r_{a_u,h}. \tag{10}$$

Note that through solving Problem 1, each user can independently determine its optimal energy usage based on the prices forecasted by the retailer.

### *B. Retailer*

When we solve Problem **P1**, the optimal total energy consumption from all users in each time slot depends on the price vector $\boldsymbol{p} = [p_1, \cdots, p_H]$. For notational convenience, let $S_{u,h}(\boldsymbol{p})$ denote the corresponding optimal total energy consumption of user $u$ at time slot $h$. As shown in the next section, given the price, each user can easily calculate $S_{u,h}(\boldsymbol{p})$'s from Problem **P1**.

Note that the retailer has one goal to maximize its profit, which is the difference between revenue and cost with which it buys energy from the generators. In particular, the revenue is given by

$$\sum_h \left( \sum_{u \in \mathcal{U}} S_{u,h}(\boldsymbol{p}) \right) p_h, \tag{11}$$

and the cost is denoted as an increasing convex function regarding the load demand at each time slot, i.e.,

$$\sum_{h \in \mathcal{H}} a \left( \sum_{u \in \mathcal{U}} S_{u,h}(\boldsymbol{p}) \right)^2 + b \left( \sum_{u \in \mathcal{U}} S_{u,h}(\boldsymbol{p}) \right)^3, \tag{12}$$

where $a$ and $b$ are positive factors [22]. Mathematically, the goal of the retailer is therefore written as

$$\begin{aligned}\textbf{P2:}\ \text{maximize}\ L(\boldsymbol{p}) &= \sum_{h\in\mathcal{H}}\left(\sum_{u\in\mathcal{U}}S_{u,h}(\boldsymbol{p})\right)p_h \\ &- w\left(\sum_{h\in\mathcal{H}} a\left(\sum_{u\in\mathcal{U}}S_{u,h}(\boldsymbol{p})\right)^2 + b\left(\sum_{u\in\mathcal{U}}S_{u,h}(\boldsymbol{p})\right)^3\right) \\ \text{subject to}\quad & p^l \le p_h \le p^u,\ \forall h\in\mathcal{H}, \\ \text{variables}\quad & \boldsymbol{p},\end{aligned} \tag{13}$$

where $p^l$ and $p^u$ denote the lower-bound price and the upper-bound price due to regulation, respectively, and the coefficient $w$ reflects the weight of cost in the net profit.

Noticeably, the optimal solution of Problem **P2** depends on the form of the total electricity consumption (i.e., $\sum_{u\in\mathcal{U}} S_{u,h}(\boldsymbol{p})$). The detailed solution to Problem **P2** will be discussed in Section IV.

## III. Implementation of Electricity Consumption Scheduling

In this section, we first derive $S_{u,h}(\boldsymbol{p})$'s at the user side. Each user can then inform the retailer of $S_{u,h}(\boldsymbol{p})$'s via the communication channels. With this information, the retailer can then efficiently calculate the demand response from each user for any price vector $\boldsymbol{p}$, thus removing the uncertainty of user response.

Due to the concavity of $U_{a_u,h}(e_{a_u,h})$, Problem **P1** is a convex optimization problem, and the optimal solution can be obtained by the primal-dual arguments [17]. That is, we can solve Problem **P1** through maximizing its Lagrangian and minimizing the corresponding dual function. Let $\boldsymbol{\eta}_u = (\eta_{u,1}, \cdots, \eta_{u,H})$ be the Lagrange multiplier vector corresponding to the constraint (10.5). We define the Lagrangian $L(\{\boldsymbol{e}_{a_u}\}_{a_u}, \boldsymbol{\eta}_u)$ associated with Problem **P1**, i.e.,

$$\begin{aligned}&L(\{\boldsymbol{e}_{a_u}\}_{a_u}, \boldsymbol{\eta}_u) \\ &= \sum_{h\in\mathcal{H}}\sum_{a_u\in\mathcal{B}_u} U_{a_u,h}(e_{a_u,h}) - \sum_{h\in\mathcal{H}} p_h\left(\sum_{a_u\in\mathcal{B}_u,\mathcal{C}_u} e_{a_u,h}\right) \\ &\quad + \sum_{h\in\mathcal{H}}\eta_{u,h}\left(C_u^{\max} - E_{u,h} - \sum_{a_u\in\mathcal{B}_u,\mathcal{C}_u} e_{a_u,h}\right).\end{aligned} \tag{14}$$

Mathematically, the optimizations of the Lagrangian and the dual problem are expressed as

$$\begin{aligned}g(\boldsymbol{\eta}_u) = &\text{maximize}\ L(\{\boldsymbol{e}_{a_u}\}_{a_u}, \boldsymbol{\eta}_u) \\ &\text{subject to}\ 0 \le e_{a_u,h} \le r_{a_u}^{\max}, \forall h\in\mathcal{H}, \forall a_u\in\mathcal{B}_u, \\ &\qquad \sum_{h\in\mathcal{H}'_{a_u}} e_{a_u,h} = E_{a_u}, \forall a_u\in\mathcal{C}_u, \\ &\qquad 0 \le e_{a_u,h} \le r_{a_u}^{\max}, \forall a_u\in\mathcal{C}_u, \forall h\in\mathcal{H}'_{a_u}, \\ &\qquad e_{a_u,h} = 0, \forall a_u\in\mathcal{C}_u, \forall h\notin\mathcal{H}'_{a_u}, \\ &\text{variables}\quad \boldsymbol{e}_{a_u}, \forall a_u\in\mathcal{B}_u, \mathcal{C}_u.\end{aligned} \tag{15}$$

and

$$\begin{aligned}\text{Dual Problem: minimize}\quad & g(\boldsymbol{\eta}_u) \\ \text{subject to}\ & \eta_{u,h} \ge 0, \forall h\in\mathcal{H}, \\ \text{variables}\quad & \boldsymbol{\eta}_u,\end{aligned} \tag{16}$$

respectively.

Let $U'_{a_u,h}(e_{a_u,h})$ denote $\frac{\partial U_{a_u,h}(e_{a_u,h})}{\partial e_{a_u,h}}$ and $U'^{-1}_{a_u,h}(\cdot)$ denote the inverse function of $U'_{a_u,h}(\cdot)$. Let $\hat{\mathcal{H}}_u$ be the set of time slots in which there might be semi-elastic appliances for user $u$. It is clear that $\hat{\mathcal{H}}_u = \bigcup_{a_u\in\mathcal{C}_u} \mathcal{H}'_{a_u}$. Likewise, let the optimal solution to (15) be $\hat{e}_{a_u,h}(p_h, \eta_{u,h})$ for all $a_u\in\mathcal{B}_u$ and $\hat{e}_{a_u,h}(\boldsymbol{p}, (\eta_{u,h}, \forall h\in\hat{\mathcal{H}}_u))$ for all $a_u\in\mathcal{C}_u$ in each in each $h\in\mathcal{H}$, respectively. According to the theory of convex optimization and linear optimization [17], we have the following result as shown in Lemma 1[2].

**Lemma 1.** The optimal solution to (15) satisfies[3]

$$\begin{aligned}&\hat{e}_{a_u,h}(p_h, \eta_{u,h}) \\ &= \left[U'^{-1}_{a_u,h}(p_h + \eta_{u,h})\right]_0^{r_{a_u}^{\max}}, \forall a_u\in\mathcal{B}_u, \forall h\in\mathcal{H},\end{aligned} \tag{17}$$

and for all $a_u\in\mathcal{C}_u$ and $h_i\in\mathcal{H}$

$$\begin{aligned}&\hat{e}_{a_u,h_i}(\boldsymbol{p}, (\eta_{u,h}, \forall h\in\hat{\mathcal{H}}_u)) \\ &= \begin{cases} r_{a_u}^{\max},\ \text{if}\ i\in\{1,2,\cdots,\lfloor\frac{E_{a_u}}{r_{a_u}^{\max}}\rfloor\}\ \text{and}\ h_i\in\mathcal{H}'_{a_u} \\ E_{a_u} - r_{a_u}^{\max}\times\lfloor\frac{E_{a_u}}{r_{a_u}^{\max}}\rfloor,\ \text{elseif}\ i = \lfloor\frac{E_{a_u}}{r_{a_u}^{\max}}\rfloor + 1\ \text{and}\ h_i\in\mathcal{H}'_{a_u} \\ 0,\ \text{otherwise},\end{cases}\end{aligned} \tag{18}$$

where $h_i$ follows that $p_{h_i} + \eta_{u,h_i} \le p_{h_{i+1}} + \eta_{u,h_{i+1}}$.

The proof of Lemma 1 is deferred to Appendix A.

After obtaining the optimal solution to problem (15), we want to minimize the dual problem (16). In particular, problem (16) can be decomposed into (19) and (20).

From problem (19), we have Lemma 2 as shown in the following.

**Lemma 2.** If

$$\sum_{a_u\in\mathcal{B}_u}\left[U'^{-1}_{a_u,h}(p_h)\right]_0^{r_{a_u}^{\max}} \le C_u^{\max} - E_{u,h}, h\notin\hat{\mathcal{H}}_u \tag{21}$$

the optimal solution to (19) is zero, and the optimal energy consumption of each appliance $a_u$ at time slot $h\notin\hat{\mathcal{H}}_u$ satisfies

$$e^*_{a_u,h}(p_h) = \left[U'^{-1}_{a_u,h}(p_h)\right]_0^{r_{a_u}^{\max}},\ \forall a_u\in\mathcal{B}_u, h\notin\hat{\mathcal{H}}_u. \tag{22}$$

Otherwise, the unique optimal solution to (19) is a function of $p_h$, denoted by $\eta^*_{u,h}(p_h)$, and the optimal energy consumption of each appliance $a_u$ at time slot $h\notin\hat{\mathcal{H}}_u$ satisfies

$$e^*_{a_u,h}(p_h) = \left[U'^{-1}_{a_u,h}(p_h + \eta^*_{u,h}(p_h))\right]_0^{r_{a_u}^{\max}},\ \forall a_u\in\mathcal{B}_u, h\notin\hat{\mathcal{H}}_u. \tag{23}$$

Moreover, the total optimal energy consumption at time slot $h\notin\hat{\mathcal{H}}_u$ is equal to $C_u^{\max}$. That is,

$$\sum_{a_u\in\mathcal{B}_u} e^*_{a_u,h}(p_h) = C_u^{\max} - E_{u,h}, h\notin\hat{\mathcal{H}}_u. \tag{24}$$

[2]For simplicity, we assume that the smart grid can tolerate that every category $\mathcal{C}_u$ appliance consumes $r_{a_u}^{\max}$ at the same time slot, i.e., $\sum_{a_u\in\mathcal{C}_u} r_{a_u}^{\max} \le C_u^{\max} - E_{u,h}, \forall h$.

[3]Notation $[x]_a^b$ means $\max\{\min\{x,b\},a\}$, and notation $\lfloor x\rfloor$ returns the nearest integer that is less than or equal to $x$.

$$\begin{aligned}
\text{minimize} \quad & \sum_{h\notin\hat{\mathcal{H}}_u}\sum_{a_u\in\mathcal{B}_u} U_{a_u,h}(e_{a_u,h}) - \sum_{h\notin\hat{\mathcal{H}}_u}(p_h+\eta_{u,h})\Bigg(\sum_{a_u\in\mathcal{B}_u} e_{a_u,h}\Bigg) + \sum_{h\notin\hat{\mathcal{H}}_u}\eta_{u,h}\Bigg(C_u^{\max}-E_{u,h}\Bigg)\\
\text{subject to} \quad & e_{a_u,h} = \Bigg[U'^{-1}_{a_u,h}(p_h+\eta_{u,h})\Bigg]_0^{r_{a_u}^{\max}}, \forall h\notin\hat{\mathcal{H}}_u, \forall a_u\in\mathcal{B}_u, \qquad (19)\\
& \eta_{u,h}\ge 0, \forall h\notin\hat{\mathcal{H}}_u,\\
\text{variables} \quad & \eta_{u,h}, \forall h\notin\hat{\mathcal{H}}_u.
\end{aligned}$$

$$\begin{aligned}
\text{minimize} \quad & \sum_{h_i\in\hat{\mathcal{H}}_u}\sum_{a_u\in\mathcal{B}_u} U_{a_u,h_i}(e_{a_u,h_i}) - \sum_{h_i\in\hat{\mathcal{H}}_u}(p_{h_i}+\eta_{u,h_i})\Bigg(\sum_{a_u\in\mathcal{B}_u,\mathcal{C}_u} e_{a_u,h_i}\Bigg) + \sum_{h_i\in\hat{\mathcal{H}}_u}\eta_{u,h_i}\Bigg(C_u^{\max}-E_{u,h_i}\Bigg)\\
\text{subject to} \quad & e_{a_u,h_i} = \Bigg[U'^{-1}_{a_u,h_i}(p_{h_i}+\eta_{u,h_i})\Bigg]_0^{r_{a_u}^{\max}}, \forall h_i\in\hat{\mathcal{H}}_u, \forall a_u\in\mathcal{B}_u,\\
& \forall a_u\in\mathcal{C}_u, e_{a_u,h_i} = \begin{cases} r_{a_u}^{\max} & \text{if } i\in\{1,2,\cdots,\lfloor\frac{E_{a_u}}{r_{a_u}^{\max}}\rfloor\} \text{ and } h_i\in\mathcal{H}'_{a_u}\\ E_{a_u}-r_{a_u}^{\max}\times\lfloor\frac{E_{a_u}}{r_{a_u}^{\max}}\rfloor, & \text{elseif } i=\lfloor\frac{E_{a_u}}{r_{a_u}^{\max}}\rfloor+1 \text{ and } h_i\in\mathcal{H}'_{a_u}\\ 0, & \text{otherwise,}\end{cases} \qquad (20)\\
& \eta_{u,h_i}\ge 0, \forall h_i\in\hat{\mathcal{H}}_u,\\
\text{variables} \quad & \eta_{u,h_i}, \forall h_i\in\hat{\mathcal{H}}_u.
\end{aligned}$$

The proof of Lemma 2 is deferred to Appendix B.

Denote by $S_{u,h}(\boldsymbol{p})$ the total energy consumed by user $u$ in each time slot $h$. By Lemma 2, we have

$$\begin{aligned}
S_{u,h}(\boldsymbol{p}) &= \sum_{a_u\in\mathcal{B}_u} e^*_{a_u,h}(p_h) + E_{u,h}\\
&= \min\Bigg\{C_u^{\max}, E_{u,h} + \sum_{a_u\in\mathcal{B}_u}\Bigg[U'^{-1}_{a_u,h}(p_h)\Bigg]_0^{r_{a_u}^{\max}}\Bigg\}, \forall h\notin\hat{\mathcal{H}}_u. \qquad (25)
\end{aligned}$$

**Remark 1.** The total energy consumption follows (25) as long as there is no energy consumption of semi-elastic appliances in time slot $h$.

Next, we focus on calculating the total energy consumption in time slot $h$ when there may exist semi-elastic appliances, i.e., Problem (20). A close look at Problem (20) reveals that to solve Problem (20), we first need to sort the time slots in $\hat{\mathcal{H}}_u$ as the order $\{h_1,\cdots,h_i,\cdots,h_L\}$ such that $p_{h_i}+\eta_{u,h_i}\le p_{h_{i+1}}+\eta_{u,h_{i+1}}$ for all $h_i\in\hat{\mathcal{H}}_u$, where $L=|\hat{\mathcal{H}}_u|$. Then, the energy consumption of each semi-elastic appliance is given according to the order of time slots. Since the order of time slots depends on the summation of $\eta_{u,h}$ and $p_h$ for all $h\in\hat{\mathcal{H}}_u$, the energy consumption has no closed-form expression for the semi-elastic appliances. Therefore, it is difficult to obtain the optimal solution $\eta^*_{u,h}(\boldsymbol{p})$'s from (20). Alternatively, for a given $\boldsymbol{p}$, we can adopt an iterative numerical algorithm to obtain $\eta^*_{u,h}(\boldsymbol{p})$'s. In particular, at each iteration $k$, the energy consumption is first updated by Lemma 2. Then, the multiplier variables are updated as $\eta^{(k)}_{u,h}$'s by the subgradient method [18], i.e.,

$$\begin{aligned}
\eta^{(k)}_{u,h} = \Bigg[&\eta^{(k-1)}_{u,h} - \psi^{(k)}_{u,h}(C_u^{\max} - E_{u,h} - \sum_{a_u\in\mathcal{B}_u}\hat{e}_{a_u,h}(p_h+\eta^{(k-1)}_{u,h})\\
&- \sum_{a_u\in\mathcal{C}_u}\hat{e}_{a_u,h}(\boldsymbol{p},(\eta^{(k-1)}_{u,h},\forall h\in\hat{\mathcal{H}}_u))\Bigg]^+, \forall h\in\hat{\mathcal{H}}_u. \qquad (26)
\end{aligned}$$

Here, $\psi^{(k)}_{u,h}$'s are stepsizes at the $k$th iteration. Besides, $\hat{e}_{a_u,h}(p_h+\eta^{(k-1)}_{u,h})$'s and $\hat{e}_{a_u,h}(\boldsymbol{p},(\eta^{(k-1)}_{u,h},\forall h\in\hat{\mathcal{H}}_u))$'s are the optimal solution to (15) at the $(k-1)$th iteration.

Now, we present the implementation of electricity consumption in time slot $h$ when there may exist semi-elastic appliances in Algorithm 1.

**Remark 2.** At each iteration of Algorithm 1, the energy consumption and the multiplier variables are updated in closed forms. Therefore, the complexity of Algorithm 1 is $\mathrm{O}(|\mathcal{B}_u|H+|\mathcal{C}_u|H+H)$ at each iteration.

Note that the optimal solution to Problem P1 is a function of $\boldsymbol{p}$, and thus we denote it as $\boldsymbol{e}^*_{a_u}(\boldsymbol{p})$'s. The following theorem shows that when the stepsizes $\psi^{(k)}_{u,h}$'s in (26) are small, the optimal solution of (15), i.e., $\hat{e}_{a_u,h}(p_h+\eta^{(k)}_{u,h})$'s and $\hat{e}_{a_u,h}(\boldsymbol{p},(\eta^{(k)}_{u,h},\forall h\in\hat{\mathcal{H}}_u))$'s, will converge within a small neighborhood of the optimal solution $\boldsymbol{e}^*_{a_u}(\boldsymbol{p})$'s.

**Theorem 1.** (a) **Constant Stepsize**. Assume that $\psi^{(k)}_{u,h} = s\psi^{(0)}_{u,h}$, where $\psi^{(0)}_{u,h}$'s are arbitrary positive constants. For any $\epsilon>0$, there exists some $s_0>0$ and a time $T_0$, such that for any $s\le s_0$, any initial multiplier variables $\eta^{(0)}_{u,h}$'s

**Algorithm 1** Implementation of Electricity Consumption Scheduling for given $\boldsymbol{p}$ in $\hat{\mathcal{H}}_u$

**Initialization**; Randomly choose $\eta_{u,h}^{(0)} > 0, \forall h \in \hat{\mathcal{H}}_u$. Let $k = 1$.
**repeat**
  Calculate $\hat{e}_{a_u,h}(p_h + \eta_{u,h}^{(k-1)})$'s and $\hat{e}_{a_u,h}(\boldsymbol{p}, (\eta_{u,h}^{(k-1)}, \forall h \in \hat{\mathcal{H}}_u))$'s by Lemma 1.
  Update the multiplier variables $\eta_{u,h}^{(k)}$'s according to (26).
  $k = k + 1$.
**until** $\sum\limits_{h\in\hat{\mathcal{H}}_u}\Bigg(\sum\limits_{a_u\in\mathcal{B}_u}(\hat{e}_{a_u,h}(p_h+\eta_{u,h}^{(k-1)})-\hat{e}_{a_u,h}(p_h+\eta_{u,h}^{(k-2)}))^2 + \sum\limits_{a_u\in\mathcal{C}_u}(\hat{e}_{a_u,h}(\boldsymbol{p},(\eta_{u,h}^{(k-1)},\forall h \in \hat{\mathcal{H}}_u)) - \hat{e}_{a_u,h}(\boldsymbol{p},(\eta_{u,h}^{(k-2)},\forall h \in \hat{\mathcal{H}}_u)))^2\Bigg) \leq \epsilon.$

and all $k \geq T_0$, we have

$$\sum_{h\in\hat{\mathcal{H}}_u}\Bigg(\sum_{a_u\in\mathcal{B}_u}(\hat{e}_{a_u,h}(p_h + \eta_{u,h}^{(k)}) - \boldsymbol{e}^*_{a_u}(\boldsymbol{p}))^2 + \sum_{a_u\in\mathcal{C}_u}(\hat{e}_{a_u,h}(\boldsymbol{p},(\eta_{u,h}^{(k)},\forall h\in\hat{\mathcal{H}}_u)) - \boldsymbol{e}^*_{a_u}(\boldsymbol{p}))^2\Bigg) \leq \epsilon. \tag{27}$$

(b) **Adaptive Stepsize**. If the stepsizes are iteration-varying and they are chosen such that $\psi_{u,h}^{(k)} = s^{(k)}\psi_{u,h}^{(0)}$ with $\lim\limits_{k\to\infty} s^{(k)} = 0$ and $\sum\limits_{k=1}^{\infty} s^{(k)} = +\infty$, then (28) is satisfied.

$$\lim_{k\to\infty}\sum_{h\in\hat{\mathcal{H}}_u}\Bigg(\sum_{a_u\in\mathcal{B}_u}(\hat{e}_{a_u,h}(p_h + \eta_{u,h}^{(k)}) - \boldsymbol{e}^*_{a_u}(\boldsymbol{p}))^2 + \sum_{a_u\in\mathcal{C}_u}(\hat{e}_{a_u,h}(\boldsymbol{p},(\eta_{u,h}^{(k)},\forall h\in\hat{\mathcal{H}}_u)) - \boldsymbol{e}^*_{a_u}(\boldsymbol{p}))^2\Bigg) = 0. \tag{28}$$

The detailed proof of Theorem 1 is relegated to Appendix C.

With Algorithm 1, the smart meter can quickly calculate the energy consumption of each appliance (and hence $S_{u,h}(\boldsymbol{p})$) in each time $h \in \hat{\mathcal{H}}_u$ in response to the price $\boldsymbol{p}$ forecasted by the retailer. This implies that the total energy consumption can be obtained by the smart meter through Algorithm 1 or the closed-form expression (25) according to the forecasted price. The following example will illustrate both situations.

**Example 1** (User's response to the forecasted price $\boldsymbol{p}$): Assume that user $u$ has six appliances, including two background appliances $a_1$ and $a_2$, two elastic appliances $a_3$ and $a_4$, and two semi-elastic appliances $a_5$ and $a_6$. Let the scheduling horizon $\mathcal{H}$ be $\{1, 2, \cdots, 8\}$. Specifically, the working period of appliance $a_5$ is $\{3, 4, 5, 6\}$, and the working period of appliance $a_6$ is $\{4, 5, 6, 7\}$. The total energy consumption of background appliances is $[4.0, 3.0, 3.0, 3.5, 2.5, 3.5, 3.5, 3.0]$ kWh in the scheduling horizon. The maximum allowable energy consumption $C_u^{\max}$ of user $u$ is 40 kWh at each time slot. The maximum allowable energy consumptions of appliances $a_3$, $a_4$, $a_5$ and $a_6$ are 20 kWh, 20 kWh, 4 kWh and 6 kWh at each time slot, respectively. The total energy consumptions of semi-elastic appliances $a_5$ and $a_6$ are 10 kWh and 10 kWh in the working period, respectively. Assume appliance $a_3$ and appliance $a_4$ have a quality-of-usage of $U_{a_3,h}(e_{a_3,h}) = 1.5w_{1,h}\log(m_{1,h} + e_{a_3,h})$ and a quality-of-service of $U_{a_4,h}(e_{a_4,h}) = 1.5w_{2,h}\log(m_{2,h} + e_{a_4,h})$ at time slot $h$, respectively. The parameters in these two functions are given as follows,

$$\begin{pmatrix} w_{1,h} \\ w_{2,h} \end{pmatrix}_{h\in\mathcal{H}} = \begin{pmatrix} 6 & 8 & 6 & 8 & 6 & 10 & 8 & 6 \\ 6 & 8 & 10 & 8 & 10 & 6 & 10 & 8 \end{pmatrix}$$

and

$$\begin{pmatrix} m_{1,h} \\ m_{2,h} \end{pmatrix}_{h\in\mathcal{H}} = \begin{pmatrix} 1.0 & 3.0 & 1.5 & 3.5 & 3.0 & 3.5 & 0.5 & 3.0 \\ 3.0 & 1.0 & 1.5 & 3.0 & 1.5 & 3.5 & 2.0 & 1.0 \end{pmatrix}.$$

Consider a given price vector $\boldsymbol{p} = \$[1.1, 1.0, 1.2, 1.2, 1.9, 1.4, 1.9, 1.0]$. In this case, the amount of consumed energy is calculated with the closed-form expression (25) in time slots $\{1, 2, 8\}$, since there is no semi-elastic appliance in these time slots. For other time slots, the smart meter needs to calculate the energy consumption according to Algorithm 1.

Fig. 3 shows the scheduled energy consumption of each appliance in response to the forecasted price vector $\boldsymbol{p}$. In this example, Algorithm 1 takes 4 iterations to converge to the desirable energy consumptions in time slots $\{3, 4, 5, 6, 7\}$. Recall Remark 2 that the complexity of Algorithm 1 is $\mathrm{O}(|\mathcal{B}_u|H + |\mathcal{C}_u|H + H)$ at each iteration. Therefore, it is of practical meaning for the smart meter because of its low computing capability. It can be further seen from Fig. 3 that the larger price the less energy consumption for semi-elastic appliances. This is because that the minimum payment is the target of semi-elastic appliances. However, it might not be satisfied for the elastic appliances, because the quality-of-service also needs to be considered except the payment.

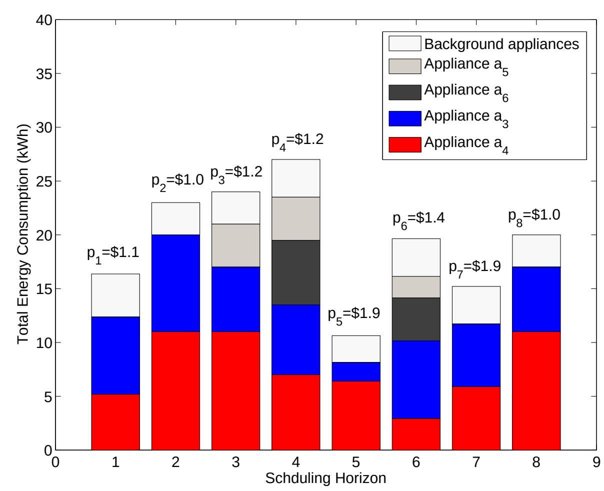


Fig. 3. The scheduled energy consumption of each appliance in response to the forecasted price vector $\boldsymbol{p}$.

## IV. Simulated Annealing based Algorithm for Price Control

In the section above, we have obtained the solution of electricity consumption in response to electricity prices. In this section, we consider how the retailer adjusts the electricity prices according to users' responses.

### A. Simulated Annealing based Price Control (SAPC) Algorithm

The retailer determines the optimal price vector for a certain time period (namely a scheduling horizon) right before the start of the time period. For example, the price used for a day can be calculated during the last few minutes of the previous day. Due to the non-convexity of $S_{u,h}(\boldsymbol{p})$'s, convex optimization methods cannot be used to solve the retailer side problem **P2**. The proposed price control algorithm here is based on the use of Simulated Annealing (SA) [19], and is referred to as SAPC.

Suppose that two-way communications between the retailer and users is possible through certain types of communication networks in future smart grids. The SAPC algorithm solves Problem **P2** in an iterative manner. In each iteration, the retailer broadcasts to all users a tentative price vector $\boldsymbol{p}$. Each user then responds with $S_{u,h}(\boldsymbol{p})$ for all $h$. If no semi-elastic appliance is present, $S_{u,h}(\boldsymbol{p})$ can be analytically calculated through the closed-form expression (25). Otherwise, the users compute $S_{u,h}(\boldsymbol{p})$ using Algorithm 1. Based on these responses, the retailer updates the price vector based on the concept of SA, which will be discussed shortly. The updated price is then broadcasted to probe the users' responses. When the price vector is finalized, the retailer sends it to all users, who will schedule energy consumption for the next scheduling horizon accordingly.

**Update of $\boldsymbol{p}$:** In each iteration of the SAPC algorithm, one entry of $\boldsymbol{p}$ is updated while the others keep fixed. Different entries $p_1$, $p_2$, $\cdots$, $p_H$ are updated in a round robin manner in consecutive iterations. When $p_h$ is to be updated, for example, the retailer randomly picks $p'_h$ in $[p^l, p^u]$ for time slot $h$, and then broadcasts the updated power vector $(\boldsymbol{p}_{-h}, p'_h)$ to users. Here, $\boldsymbol{p}_{-h}$ denotes the vector $(p_1, ..., p_{h-1}, p_{h+1}, ..., p_H)$. Having received the responses $S_{u,h}(\boldsymbol{p}_{-h}, p'_h)$'s from all users, the retailer calculates the value $L(\boldsymbol{p}_{-h}, p'_h)$, where $L(\cdot)$ is defined in (13). Then, the retailer compares this value with $L(\boldsymbol{p})$ calculated in the previous iteration. If $\Delta = L(\boldsymbol{p}_{-h}, p'_h) - L(\boldsymbol{p})$ is larger than 0, then $p'_h$ is accepted as the new $p_h$ with probability 1. Otherwise, it is accepted as the new $p_h$ with a probability $\exp(\frac{\Delta}{T})$, and the old $p_h$ keeps with a probability $1 - \exp(\frac{\Delta}{T})$, where $T$ is a control parameter (also referred to as temperature). For convergence, $T$ decreases with each iteration. Intuitively, the acceptance of uphill-moving becomes less and less likely with the decrease of $T$, implying convergence when $T$ is sufficiently small. When $p_h$ is updated as $p'_h$, the retailer updates the memory of $L(\boldsymbol{p})$ as $L(\boldsymbol{p}_{-h}, p'_h)$. After the SAPC algorithm is terminated, the retailer broadcasts the updated price vector encapsulated in a packet to the smart meters at the beginning of the following scheduling horizon. Such updated price vector is used for the electricity prices in the following scheduling horizon.

The following propositions discusses the convergence of the SA-based algorithms.

**Proposition 1** ( [20], [21])**.** The SAPC algorithm converges to the global optimal solution to Problem **P2**, as the control parameter $T$ approaches to zero with $T = \frac{T_0}{\log(k)}$.

For practical implementation, a solution very close to the global optimal solution is obtained when $T < \epsilon$, where $\epsilon$ is a very small number. The details of the SAPC algorithm is given in Algorithm 2.

**Algorithm 2** The SAPC Algorithm

**Procedure at the retailer side:**

**Initialization**: Set $T = T_0$, $k = 1$, and $\boldsymbol{p}$ is initialized as the price vector used in the current scheduling horizon.
**repeat**
  **for all** $h$'s in the order of $\{1, 2, \cdots, H\}$ **do**
    The retailer randomly pick $p'_h \in [p^l, p^u]$, and broadcasts $p'_h$ and $p_{h-1}$ encapsulated in a packet to the smart meters via LAN.
    After receiving the packet from the users including the energy information $S_{u,h}(\boldsymbol{p}_{-h}, p'_h)$'s, the retailer calculates $L(\boldsymbol{p}_{-h}, p'_h)$ in Problem **P2** according to the received $S_{u,h}(\cdot)$'s.
    The retailer computes $\Delta = L(\boldsymbol{p}_{-h}, p'_h) - L(\boldsymbol{p})$, and let $p_h = p'_h$ with probability 1 if $\Delta_h \geq 0$. Otherwise, let $p_h = p'_h$ with probability $\exp(\frac{\Delta}{T})$, or $p_h = p_h$ with probability $(1 - \exp(\frac{\Delta}{T}))$.
    When $p_h$ is updated as $p'_h$, the retailer updates the memory of $L(\boldsymbol{p})$ as $L(\boldsymbol{p}_{-h}, p'_h)$.
  **end for**
  $k = k + 1$.
  $T = \frac{T_0}{\log(k)}$.
**until** $T < \epsilon$.
The retailer broadcasts the updated price vector $\boldsymbol{p}$ encapsulated in a packet to the smart meters at the beginning of the following scheduling horizon.

**Procedure at the user side:**

After receiving the packet from the retailer, the smart meter updates the tentative price vector as $(\boldsymbol{p}_{-h}, p'_h)$ in its memory.
The smart meter calculates the response to $(\boldsymbol{p}_{-h}, p'_h)$ by either (25) or Algorithm 1.
The smart meter informs the retailer of the total energy consumption in each time slot encapsulated in a packet.

### B. Computational Complexity of SAPC

Recall the SAPC algorithm, and the number of rounds needed for the SAPC algorithm is $\exp(\frac{T_0}{\epsilon})$. Since $H$ iterations are needed in one round, the total number of iterations needed for the SAPC algorithm is equal to $\exp(\frac{T_0}{\epsilon})H$, where $H$ is the number of time slots in the scheduling horizon. Therefore, we have the following Lemma 3.

**Lemma 3.** Given the initial temperature $T_0$ and the stopping criterion $\epsilon$, the SAPC algorithm needs $\exp(\frac{T_0}{\epsilon})H$ iterations.

Two-way communications between the retailer and users are needed when the price in any time slot is updated, as the retailer needs to broadcast the updated prices to users in a control packet, and each user needs to inform the retailer of its possible energy consumption in a data packet. Therefore, the time needed for one iteration consists of the transmit time of packets, the computational time in response to the updated price at each smart meter, and the computational time of updating the price at the retailer side. The transmit time depends on the underlying communication technology. In practice, the transmit time of packets with 32 bytes is at the order of $1 \sim 10^3$ ms per iteration over a broadband with a speed of 100 Mbps. One the other hand, the computational time totally depends on the processors of the retailer and smart meters, the number of users, and the number of appliances of each user. Let the transmit time and the computational time be $T_t$ time units and $T_c$ time units, respectively. Then, by Lemma 3, the SAPC algorithm takes $(T_t + T_c)H \exp(\frac{T_0}{\epsilon})$ time units. This implies that the retailer needs to update the prices for the following scheduling horizon $(T_t+T_c)H \exp(\frac{T_0}{\epsilon})$ time units before the end of the current scheduling horizon.

## V. Simulation Results

In this section, we conduct simulations to illustrate the effectiveness of the proposed real-time pricing scheme.

**Example 2**: We consider a smart grid with 100 users, where each user $u$ has four elastic appliances (i.e., $u_1$, $u_2$, $u_3$ and $u_4$) and two semi-elastic appliances (i.e., $u_5$ and $u_6$). Assume each user $u$ has a quality-of-usage of $U_{u_i,h}(e_{u_i,h}) = -a_{u_i,h}(e_{u_i,h}+b_{u_i,h})^{-1}$ for each elastic appliance $u_i$ at each time slot $h$. Specifically, each parameter $a_{u_i,h}$ is chosen from the uniform distribution on $[10, 20]$, and each parameter $b_{u_i,h}$ is chosen from the uniform distribution on $[2, 5]$. The time scheduling horizon is $\mathcal{H} = \{1, 2, \cdots, 12\}$. The maximum allowable energy consumption of each user follows the uniform distribution on $[10, 15]$ kWh at each time slot. The maximum allowable energy consumptions of each appliance follows the uniform distribution on $[1.0, 2.0]$ kWh at each time slot. The total energy consumption of each semi-elastic appliance follows the uniform distribution on $[4, 6]$ kWh in the working period. Assume that the total energy consumption of background appliances follows the uniform distribution on [1,2] kWh at each time slot for each user. Let the time scheduling horizon be $\mathcal{H} = \{1, 2, \cdots, 12\}$. Let the working period of each semi-elastic appliance be consecutive with the beginning $\alpha_{u_i}$ and the end $\beta_{u_i}$, where $\alpha_{u_i}$ and $\beta_{u_i}$ are randomly chosen from $\mathcal{H}$. Finally, let $w = 1$, $a = 10^{-4}$, and $b = 2 \times 10^{-5}$ in (13).

We first compare the total energy consumption at each time slot under different settings of price in Fig. 4. Here, both the optimal flat-rate price and the optimal real-time price are computed by the proposed SAPC algorithm. When the proposed SAPC algorithm is used for computing the optimal flat-rate price, all elements in the price vector are simultaneously updated to the same value at each round. From Fig. 4, we can see that in any time slot, the increase of electricity price leads to the reduction of total energy consumption in the time slot, regardless of the prices setting in other time slots. For example, in time slot 5, the most energy consumption happens when $p_5$=\$0.58, while the least enegy consumption happens when $p_5$=\$1.50. For the flat-rate scheme, this implies that when the retailer increases the price in each time slot, all load demands are reduced in the scheduling horizon accordingly, which can be also found in Fig. 4 from the two flat-rate schemes. The reduction of load demand further leads to the reduction of peak demand. Furthermore, Fig. 4 shows that compared to the two flat-rate pricing schemes, the real-time pricing scheme flattens the load demand curve and reduces the peak-to-average load ratio. In particular, the real-time pricing scheme reduces the peak-to-average load ratio by about 20% compared to the flat-rate pricing schemes.

We then compare the revenue, the cost, and the profit under different settings of price in Table I. Specifically, these performances are evaluated at the retailer side. It can be seen from Table I that (i) to make ensure the same total payment from users (i.e.,\$5923.6), the cost under the flat-rate pricing (i.e., \$6930.7) is much higher due to the increase of peak demand (as shown in Fig.4); (ii) to make the same cost, the retailer has to set the flat rate to be high enough to compensate the peak cost; (iii) under the flat-rate pricing scheme, the retailer achieves the maximum profit with the price of \$1.45, which is lower than that achieved by our real-time pricing scheme.

**Example 3:** In this example, we conduct an experiment to observe the total time needed for obtaining the optimal price vector if the two-way communication in the SAPC algorithm is done through general-purpose networks, such as Internet. Our experiment emulates the retailers and users by computers that are connected to the public network through sub-networks from different service providers. The retailer computer is located in The Chinese University of Hong Kong, and the user computers are scattered throughout the Hong Kong city. We consider an experiment situation with $N$ user computers, where $N$ is from 100 to 1000. Each user has six elastic appliances, four semi-elastic appliances, and several background appliances. Assume

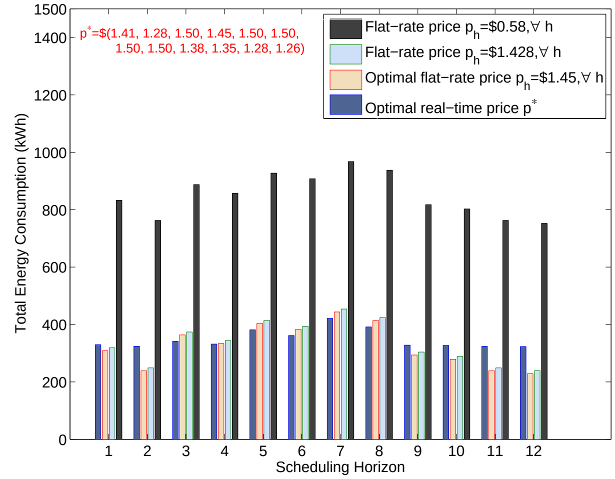


Fig. 4. The total energy consumption under different settings of price.

TABLE I
USERS' AND RETAILER'S BEHAVIORS UNDER DIFFERENT PRICE SETTING

| Flat-rate pricing | | | |
|---|---|---|---|
| Price setting | Total Payment/Revenue | Cost | Profit |
| $p_h = \$0.58, \forall h$ | \$5923.6 | \$6930.7 | \$-1007.1 |
| $p_h = \$1.428, \forall h$ | \$5787 | \$1191.2 | \$4595.8 |
| Optimal flat price $p_h = \$1.45, \forall h$ | \$5698.1 | \$1097.5 | \$4600.6 |
| Real-time pricing (Achieved by SAPC) | | | |
| Optimal price | Total Payment/Revenue | Cost | Profit |
| $\boldsymbol{p}^* =$ \$(1.41, 1.28, 1.50, 1.45, 1.50, 1.50, 1.50 1.50, 1.38, 1.35, 1.28, 1.26) | \$5923.6 | \$1191.2 | \$4732.4 |

* The total payment is the sum of all users' payment, equal to the retailer's revenue. The cost and the profit are evaluated by the retailer.

the retailer side procedure and the user side procedure in the SAPC algorithm are implemented with MATLAB[4]. The scheduling horizon is $\mathcal{H} = \{1, 2, \cdots, 24\}$ hours. The initial price vector is randomly picked.

Fig. 5 shows the total time needed to the optimal price vector. It can be seen that both the total time and the time for two-way communications do not change much when the number of users increases. For example, it only takes in total 580 seconds for the retailer to communicate with the users for the purpose of probing $S_{u,h}(\boldsymbol{p})$'s, even when the number of users is 1000. The total time needed goes up to 600 seconds, i.e., 10 minutes, when the MATLAB computational time is also included. Note that the computational time in real-system deployment can be much shorter with, say, special-purpose FPGAs. This result is very encouraging, as it implies that the retailer only needs a few minutes before midnight to determine the optimal price vector for the next day (24 hours). More importantly, the result implies that the algorithm is rather scalable with the number of users, as the time cost does not increase much when the user number becomes large.

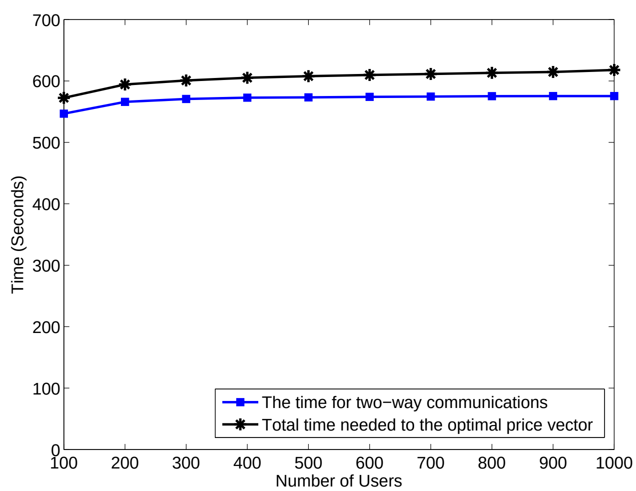


Fig. 5. Total time needed for obtaining the price vector VS number of users

## VI. Conclusions

In this paper, we proposed an optimal real-time pricing scheme for the reduction of peak-to-average load ratio in smart grid. The proposed scheme solves a two-stage optimization problem, as which the real-time pricing scheme is formulated. At the users' side, we can obtain the optimal energy consumption that maximizes the quality-of-usage with minimum electricity payment either in closed forms or through an efficient iterative algorithm. As the retailer side, we used a Simulated-Annealing-based Price Control (SAPC) algorithm to obtain the optimal real-time price that maximizes the profit. In terms of practical implementation, users and the retailer interact with each other through a limited number of message exchanges over a communication network to reach the optimal prices. Simulation results showed that our proposed algorithm can lead to performance improvement for both the retailer and users.

[4]In this paper, MATLAB with version R2010b is used on a HP Compaq dx7300 desktop with 3.6GHz processors and 1Gb of RAM.

## Appendix A
## Proof of Lemma 1

Due to the decoupling of energy consumption among the $\mathcal{B}_u$ category appliances and the $\mathcal{C}_u$ category appliances, problem (15) is decomposed into

$$\begin{aligned} \text{maximize } & \sum_{h\in\mathcal{H}} \sum_{a_u\in\mathcal{B}_u} U_{a_u,h}(e_{a_u,h}) \\ & - \sum_{h\in\mathcal{H}} (p_h + \eta_{u,h}) \Big( \sum_{a_u\in\mathcal{B}_u} e_{a_u,h} \Big) \\ \text{subject to } & 0 \le e_{a_u,h} \le r_{a_u}^{\max}, \forall h \in \mathcal{H}, \forall a_u \in \mathcal{B}_u, \\ \text{variables } & \boldsymbol{e}_{a_u}, a_u \in \mathcal{B}_u, \end{aligned} \tag{29}$$

and

$$\begin{aligned} \text{minimize } & \sum_{h\in\hat{\mathcal{H}}_u} (p_h + \eta_{u,h}) \Big( \sum_{a_u\in\mathcal{C}_u} e_{a_u,h} \Big) \\ \text{subject to } & \sum_{h\in\mathcal{H}'_{a_u}} e_{a_u,h} = E_{a_u}, \forall a_u \in \mathcal{C}_u, \\ & 0 \le e_{a_u,h} \le r_{a_u}^{\max}, \forall a_u \in \mathcal{C}_u, \forall h \in \mathcal{H}'_{a_u}, \\ & e_{a_u,h} = 0, \forall a_u \in \mathcal{C}_u, \forall h \notin \mathcal{H}'_{a_u}, \\ \text{variables } & \boldsymbol{e}_{a_u}, a_u \in \mathcal{C}_u. \end{aligned} \tag{30}$$

Due to the concave nature of $U_{e_{a_u}}(\cdot)$, the optimal solution of problem (29) can be obtained when the first derivative of the objective function over $e_{a_u}$'s is set to zero. In particular, such optimal solution satisfies (17). Based on the assumption of $\sum_{a_u\in\mathcal{C}_u} r_{a_u}^{\max} \le C_u^{\max} - E_{u,h}$ for all $h$, the optimal solution of the linear optimization problem (30) can be obtained by partitioning more energy to the working time slot with smaller price. In particular, such optimal solution satisfies (18). Therefore, Lemma 1 follows. ■

## Appendix B
## Proof of Lemma 2

Due to the convex nature of the dual problem, we obtain the optimal solution $\eta^*_{u,h}(p_h)$'s by the Karush-Kuhn-Tucker (KKT) sufficient and necessary conditions of (19). Let $\upsilon_h$ is the multiplier variable regarding the constraint of $\eta_{u,h} \ge 0$. In particular, the KKT of (19) follows

$$\begin{aligned} & \sum_{a_u\in\mathcal{B}_u} \Bigg( U'_{a_u,h}\Big( \Big[ U'^{-1}_{a_u,h}(p_h + \eta^*_{u,h}(p_h)) \Big]_0^{r_{a_u}^{\max}} \Big) \\ & - (p_h + \eta^*_{u,h}(p_h)) \Bigg) \frac{\partial \Big[ U'^{-1}_{a_u,h}(p_h + \eta_{u,h}) \Big]_0^{r_{a_u}^{\max}}}{\partial \eta_{u,h}} \Bigg|_{\eta_{u,h}=\eta^*_{u,h}(p_h)} \\ & + (C_u^{\max} - E_{u,h} - \sum_{a_u\in\mathcal{B}_u} \Big[ U'^{-1}_{a_u,h}(p_h + \eta^*_{u,h}(p_h)) \Big]_0^{r_{a_u}^{\max}} ) = \upsilon_h \\ & \upsilon_h \eta^*_{u,h}(p_h) = 0 \\ & \upsilon_h \ge 0, \forall h \notin \hat{\mathcal{H}}_u. \end{aligned} \tag{31}$$

Due to the concavity of $U_{a_u,h}(\cdot)$, we have

$$U'_{a_u,h}\left(\left[U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))\right]_0^{r^{\max}_{a_u}}\right) = (p_h+\eta^*_{u,h}(p_h)),\ \text{if } 0\le U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))\le r^{\max}_{a_u}, \tag{32}$$

and

$$\frac{\partial\left[U'^{-1}_{a_u,h}(p_h+\eta_{u,h})\right]_0^{r^{\max}_{a_u}}}{\partial \eta_{u,h}} = 0,\ \text{if } U'^{-1}_{a_u,h}(p_h+\eta_{u,h})>r^{\max}_{a_u} \text{ or } U'^{-1}_{a_u,h}(p_h+\eta_{u,h})<0. \tag{33}$$

From (32) and (33), we get (34). Thus, (31) becomes

$$\begin{aligned} &C^{\max}_u - E_{u,h} - \sum_{a_u\in\mathcal{B}_u}\left[U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))\right]_0^{r^{\max}_{a_u}} - \upsilon_h = 0\\ &\upsilon_h\eta^*_{u,h}(p_h)=0\\ &\upsilon_h\ge 0, \forall h\in\mathcal{H}. \end{aligned} \tag{35}$$

The concavity of $U_{a_u,h}(\cdot)$ implies that $U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))$ decreases with $\eta^*_{u,h}(p_h)$, and thus

$$\left[U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))\right]_0^{r^{\max}_{a_u}} \le \left[U'^{-1}_{a_u,h}(p_h)\right]_0^{r^{\max}_{a_u}}, \forall \eta^*_{u,h}(p_h)>0. \tag{36}$$

This implies that if (21) is satisfied, then

$$\sum_{a_u\in\mathcal{B}_u}\left[U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))\right]_0^{r^{\max}_{a_u}} < C^{\max}_u - E_{u,h}, \forall \eta^*_{u,h}(p_h)>0. \tag{37}$$

It follows that if $\eta^*_{u,h}(p_h)$ is positive, then the first equation of (35) is satisfied only when $\upsilon_h$ is positive. Obviously, it contradicts with the second equation of (35). Therefore, if (21) is satisfied, then $\eta^*_{u,h}(p_h)=0$, and hence (22).

Next, we prove the latter part of Lemma 2. Since $U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))$ decreases with the increase of $\eta^*_{u,h}(p_h)$, there must exist a positive $\eta^*_{u,h}(p_h)$ such that[5]

$$\sum_{a_u\in\mathcal{B}_u}\left[U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))\right]_0^{r^{\max}_{a_u}} = C^{\max}_u - E_{u,h} \tag{38}$$

when

$$\sum_{a_u\in\mathcal{B}_u}\left[U'^{-1}_{a_u,h}(p_h)\right]_0^{r^{\max}_{a_u}} > C^{\max}_u - E_{u,h}. \tag{39}$$

Therefore, we have (23) and (24). ■

## APPENDIX C
## PROOF OF THEOREM 1

Theorem 1 is a consequence of Theorems 2.1, 2.2 and 2.3 in [23]. In the following, we first prove part (a). Let $\boldsymbol{\eta}^*_u = (\eta^*_{u,h}, \forall h\in\hat{\mathcal{H}}_u)$ be the optimal solution to (20). For notational convenience, let $\boldsymbol{\eta}_u$ denote the concatenation of variables $\eta_{u,h}$'s for all $h\in\hat{\mathcal{H}}_u$, and let $\hat{e}^{(k)}_{a_u,h}$ and $\check{e}^{(k)}_{a_u,h}$ denote $\hat{e}_{a_u,h}(p_h+\eta^{(k)}_{u,h})$ and $\hat{e}_{a_u,h}(\boldsymbol{p},(\eta^{(k)}_{u,h},\forall h\in\hat{\mathcal{H}}_u))$, respectively. Define

$$||\boldsymbol{\eta}_u||_\psi = \sum_{h\in\hat{\mathcal{H}}_u}\frac{(\eta_{u,h})^2}{\psi^{(0)}_{u,h}}. \tag{40}$$

Together with $\psi^{(k)}_{u,h}=s\psi^{(0)}_{u,h}$, by (26), we have (41). For simplicity, let

$$\begin{aligned} &f^*(\boldsymbol{\eta}^*_u)\\ &=\sum_{h\in\hat{\mathcal{H}}_u}\sum_{a_u\in\mathcal{B}_u}U_{a_u,h}(e^*_{a_u,h}(\boldsymbol{p}))\\ &-\sum_{h\in\hat{\mathcal{H}}_u}p_h\Big(\sum_{a_u\in\mathcal{B}_u,\mathcal{C}_u}e^*_{a_u,h}(\boldsymbol{p})\Big)\\ &+\sum_{h\in\hat{\mathcal{H}}_u}\eta^*_{u,h}\Big(C^{\max}_u-E_{u,h}-\sum_{a_u\in\mathcal{B}_u,\mathcal{C}_u}e^*_{a_u,h}(\boldsymbol{p})\Big), \end{aligned} \tag{42}$$

$$\begin{aligned} &f^{(k)}(\boldsymbol{\eta}^{(k)}_u)\\ &=\sum_{h\in\hat{\mathcal{H}}_u}\sum_{a_u\in\mathcal{B}_u}U_{a_u,h}(\hat{e}^{(k)}_{a_u,h})\\ &-\sum_{h\in\hat{\mathcal{H}}_u}p_h\Big(\sum_{a_u\in\mathcal{B}_u}\hat{e}^{(k)}_{a_u,h}+\sum_{c_u\in\mathcal{C}_u}\check{e}^{(k)}_{a_u,h}\Big)\\ &+\sum_{h\in\hat{\mathcal{H}}_u}\eta^{(k)}_{u,h}\Big(C^{\max}_u-E_{u,h}-\sum_{a_u\in\mathcal{B}_u}\hat{e}^{(k)}_{a_u,h}-\sum_{c_u\in\mathcal{C}_u}\check{e}^{(k)}_{a_u,h}\Big), \end{aligned} \tag{43}$$

and

$$\begin{aligned} &f^{(k)}(\boldsymbol{\eta}^*_u)\\ &=\sum_{h\in\hat{\mathcal{H}}_u}\sum_{a_u\in\mathcal{B}_u}U_{a_u,h}(\hat{e}^{(k)}_{a_u,h})\\ &-\sum_{h\in\hat{\mathcal{H}}_u}p_h\Big(\sum_{a_u\in\mathcal{B}_u}\hat{e}^{(k)}_{a_u,h}+\sum_{c_u\in\mathcal{C}_u}\check{e}^{(k)}_{a_u,h}\Big)\\ &+\sum_{h\in\hat{\mathcal{H}}_u}\eta^*_{u,h}\Big(C^{\max}_u-E_{u,h}-\sum_{a_u\in\mathcal{B}_u}\hat{e}^{(k)}_{a_u,h}-\sum_{c_u\in\mathcal{C}_u}\check{e}^{(k)}_{a_u,h}\Big). \end{aligned} \tag{44}$$

Due to the strict convexity of Problem **P1**, we have

$$f^*(\boldsymbol{\eta}^*_u)\ge f^{(k)}(\boldsymbol{\eta}^*_u). \tag{45}$$

Thus, it follows from (45) that

$$\begin{aligned} &f^*(\boldsymbol{\eta}^*_u)-f^{(k)}(\boldsymbol{\eta}^{(k)}_u)\\ &\ge f^{(k)}(\boldsymbol{\eta}^*_u)-f^{(k)}(\boldsymbol{\eta}^{(k)}_u)\\ &=\sum_{h\in\hat{\mathcal{H}}_u}(\eta^*_{u,h}-\eta^{(k)}_{u,h})(C^{\max}_u-E_{u,h}\\ &\qquad -\sum_{a_u\in\mathcal{B}_u}\hat{e}^{(k)}_{a_u,h}-\sum_{c_u\in\mathcal{C}_u}\check{e}^{(k)}_{a_u,h})) \end{aligned} \tag{46}$$

Substituting (46) into (41), we get

$$\begin{aligned} &||\boldsymbol{\eta}^{(k+1)}_u-\boldsymbol{\eta}^*_u||_\psi\\ &\le||\boldsymbol{\eta}^{(k)}_u-\boldsymbol{\eta}^*_u||_{k\psi}+2s(f^*(\boldsymbol{\eta}^*_u)-f^{(k)}(\boldsymbol{\eta}^{(k)}_u))\\ &+\sum_{h\in\hat{\mathcal{H}}_u}s^2\psi^{(0)}_{u,h}(C^{\max}_u-E_{u,h}-\sum_{a_u\in\mathcal{B}_u}\hat{e}^{(k)}_{a_u,h}-\sum_{c_u\in\mathcal{C}_u}\check{e}^{(k)}_{a_u,h}))^2. \end{aligned} \tag{47}$$

[5]Due to the monotonicity of $U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))$, we can obtain $\eta^*_{u,h}(p_h)$ satisfying (38) through the bisection searching.

$$\sum_{a_u\in\mathcal{B}_u}\left(U'_{a_u,h}\left(\left[U'^{-1}_{a_u,h}(p_h+\eta^*_{u,h}(p_h))\right]_0^{r^{\max}_{a_u}}\right)-(p_h+\eta^*_{u,h}(p_h))\right)\frac{\partial\left[U'^{-1}_{a_u,h}(p_h+\eta_{u,h})\right]_0^{r^{\max}_{a_u}}}{\partial\eta_{u,h}}\Bigg|_{\eta_{u,h}=\eta^*_{u,h}(p_h)}=0. \tag{34}$$

$$\begin{aligned}||\boldsymbol{\eta}_u^{(k+1)}-\boldsymbol{\eta}_u^*||_\psi\le||\boldsymbol{\eta}_u^{(k)}-\boldsymbol{\eta}_u^*||_\psi-2s\sum_{h\in\hat{\mathcal{H}}_u}(\eta^{(k)}_{u,h}-\eta^*_{u,h})(C_u^{\max}-E_{u,h}-\sum_{a_u\in\mathcal{B}_u}\hat{e}^{(k)}_{a_u,h}-\sum_{a_u\in\mathcal{C}_u}\check{e}^{(k)}_{a_u,h})\\+\sum_{h\in\mathcal{H}}s^2\psi^{(0)}_{u,h}(C_u^{\max}-E_{u,h}-\sum_{a_u\in\mathcal{B}_u}\hat{e}^{(k)}_{a_u,h}-\sum_{c_u\in\mathcal{C}_u}\check{e}^{(k)}_{a_u,h})^2.\end{aligned} \tag{41}$$

Given $\lambda>0$, let

$$\Phi(\lambda)=\left\{\boldsymbol{\eta}_u^{(0)}|f^{(0)}(\boldsymbol{\eta}_u^{(0)})\le f^*(\boldsymbol{\eta}_u^*)+\lambda\right\} \tag{48}$$

Based on variables $e_{a_u,h}$'s, we can find an $M<\infty$ that is no smaller than the optimal value of (49),

$$\begin{aligned}\text{maximize}\quad&\sum_{h\in\hat{\mathcal{H}}_u}s^2\psi^{(0)}_{u,h}(C_u^{\max}-E_{u,h}-\sum_{a_u\in\mathcal{B}_u,\mathcal{C}_u}e_{a_u,h})^2\\\text{subject to}\quad&0\le e_{a_u,h}\le r^{\max}_{a_u},\forall h\in\mathcal{H},\forall a_u\in\mathcal{B}_u,\\&\sum_{h\in\mathcal{H}'_{a_u}}e_{a_u,h}=E_{a_u},\forall a_u\in\mathcal{C}_u,\\&0\le e_{a_u,h}\le r^{\max}_{a_u},\forall a_u\in\mathcal{C}_u,\forall h\in\mathcal{H}'_{a_u},\\\text{variables}\quad&e_{a_u,h},\forall h\in\hat{\mathcal{H}}_u,\forall a_u\in\mathcal{B}_u,\mathcal{C}_u.\end{aligned} \tag{49}$$

Therefore, if we pick

$$s\le\lambda/M, \tag{50}$$

then as long as $\boldsymbol{\eta}_u^{(k)}\notin\Phi(\lambda)$, we have

$$||\boldsymbol{\eta}_u^{(k+1)}-\boldsymbol{\eta}_u^*||_\psi\le||\boldsymbol{\eta}_u^{(k)}-\boldsymbol{\eta}_u^*||_\psi-s\lambda \tag{51}$$

Thus, $\boldsymbol{\eta}_u^{(k)}$ will enter the set $\Phi(\lambda)$ eventually. Once $\boldsymbol{\eta}_u^{(k)}\in\Phi(\lambda)$, if we pick

$$s\le\lambda/\sqrt{M}, \tag{52}$$

then we have

$$\begin{aligned}\sqrt{||\boldsymbol{\eta}_u^{(k+1)}-\boldsymbol{\eta}_u^*||_\psi}&\le\sqrt{||\boldsymbol{\eta}_u^{(k)}-\boldsymbol{\eta}_u^*||_\psi}+\sqrt{||\boldsymbol{\eta}_u^{(k+1)}-\boldsymbol{\eta}_u^{(k)}||_\psi}\\&\le\sqrt{||\boldsymbol{\eta}_u^{(k)}-\boldsymbol{\eta}_u^*||_\psi}+\lambda\end{aligned} \tag{53}$$

Therefore, if

$$s\le\min\{\lambda/M,\lambda/\sqrt{M}\}, \tag{54}$$

then there exists a number $T_0$ such that

$$\sqrt{||\boldsymbol{\eta}_u^{(k)}-\boldsymbol{\eta}_u^*||_\psi}\le\xi(\lambda)=\max_{\boldsymbol{\eta}_u\in\Phi(\lambda)}\sqrt{||\boldsymbol{\eta}_u-\boldsymbol{\eta}_u^*||_\psi},\forall k\ge T_0. \tag{55}$$

It is clear that, as $\lambda\to0$, $\xi(\lambda)\to0$. Therefore, for any $\epsilon>0$, we can pick $\lambda$ (and hence $s$) sufficiently small such that $\xi(\lambda)<\epsilon$, i.e., there exists time $T_0$ such that

$$\sqrt{||\boldsymbol{\eta}_u^{(k)}-\boldsymbol{\eta}_u^*||_\psi}\le\epsilon,\forall k\ge T_0. \tag{56}$$

Finally, since the mapping from $\boldsymbol{\eta}_u^{(k)}$ to $(\hat{e}_{a_u,h}(p_h+\eta^{(k)}_{u,h})$'s, $\hat{e}_{a_u,h}(\boldsymbol{p},(\eta^{(k)}_{u,h},\forall h\in\hat{\mathcal{H}}_u))$'s) is continuous, we can pick $\lambda$ (and hence $s$) sufficiently small such that (27) is satisfied.

The proof of part (b) is similar to part (a), and thus omitted here. Interested readers are referred to Theorem 3 in [23]. Consequently, the proof of Theorem 1 is finished. ■

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