

Math Word Problems

FOR

DUMMIES®

by Mary Jane Sterling



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About the Author

Mary Jane Sterling is also the author of *Algebra For Dummies*, *Trigonometry For Dummies*, *Algebra II For Dummies*, *CliffsStudySolver Algebra I*, and *CliffsStudySolver Algebra II*. She taught junior high and high school math for many years before beginning her current tenure at Bradley University in Peoria, Illinois. Mary Jane especially enjoys working with future teachers, doing volunteer work with her college students and fellow Kiwanians, and sitting down with a glass of lemonade and a good murder mystery.

Dedication

I dedicate this book to my children, Jon, Jim, and Jane. Each is truly an individual — and none seems to have any hesitation about facing the challenges and adventures that the world has to offer. Each of them makes my husband, Ted, and me so very proud.

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Introduction

Math word problems (or *story problems*, depending on where and when you went to school). What topic has caused more hair to be pulled from tender heads, more tears and anguish, and, at the same time, more feeling of satisfaction and accomplishment? When I told friends that I was writing this book, their responses were varied, but none was mild or without a strong opinion one way or the other. Oh, the *stories* (pardon the pun) I heard. And, lucky you, I've taken some of the accounts and incorporated the better stories in this book. Everyone has his *favorite* word problem, most of them starting with, "If Jim is twice as old as Ted was. . . ."

I was never crazy about math word problems until I got to teach them. It's all a matter of perspective. I've taken years (and years and years) of experience of trying to convey the beauty and structure of math word problems to others and put the best of my efforts in this book. I hope that you enjoy the problems and explanations as much as I've enjoyed writing them.

About This Book

Math word problems are really a part of life. Pretty much everything is a word problem until you change it into an arithmetic problem or algebra problem or logic problem and then solve it. In this book, you first find the basic steps or processes that you use to solve any math word problem. I list the steps, illustrated by examples, and then later incorporate those steps into the different types of word problems throughout the rest of the book. The same basic techniques and processes work whether you're doing a third-grade arithmetic problem or a college geometry problem.

You'll see that I use the processes and steps over and over in the examples — reinforcing the importance of using such steps. Because the steps are carried throughout, you can start anywhere you want in this book and be able to either backtrack or jump forward and still find a familiar friend in a similar step.

The different types of word problems are divided into categories, in case you're only looking for help with age problems or in case you're only interested in interest problems. Most of the examples have a firm basis in reality, but a few are off the wall, just because you need to have a good sense of humor when dealing with math word problems.

Conventions Used in This Book

For the most part, when I use a specific math word or expression, I define it right then and there. For example, if you read a math word problem about a *regular* hexagon, you immediately find the definition of *regular* (all sides and all angles are the same measure), so you don't have to hunt around to understand what's being asked.

You'll find lots of cross-referencing in the chapters. If a problem requires the use of the quadratic formula, I send you to the chapter or section where I introduce that formula. Each section and each chapter stands by itself — you don't really need to go through the chapters sequentially. You're more than welcome to go back and forth as much as you want. This isn't a murder mystery where the whole plot will be exposed if you go to the end first. When reading this book, do it *your way!*

What You're Not to Read

Math can get pretty technical — whether you want it to or not. So you'll find this book to be pretty self-contained. All you need to get you through the technical formulas and complicated algebraic manipulations is found right here in this book. You won't need a table of values or computer manual to understand what I present here.

You'll find the material in this book peppered with sidebars. What are sidebars? They're the text you see in gray boxes throughout the book. Most of the sidebars in this book are brainteasers. You have your mental juices flowing as you're reading this book, so you're probably in the mood to tackle a little twist of logic or a sassy question. The answers to the brainteasers follow immediately, so you won't have to wait or be frustrated at not having the answer. And if you're not in the mood to have your brain teased, just skip on over them. (In fact, you can skip any sidebar, whether it's a brainteaser or not.)

Foolish Assumptions

The math word problems in this book span some basic problems (using arithmetic) to the more complex (requiring algebraic skills). Even though I like to make example problems come out with whole-number answers, sometimes

fractions or decimals are just unavoidable. So I'm assuming that you know your way around adding, subtracting, multiplying, and dividing fractions and that you can reduce fractions to the lowest terms.

Another assumption I make is that you have access to a calculator. A scientific calculator works best, because you can raise numbers to powers and take roots. But you can always make do with a nonscientific calculator. Graphing calculators are a bit of overkill, but they come in handy for making tables and programming different processes.

For the math word problems requiring algebra, all you need to know is how to solve some basic linear equations, such as solving for x in $4x + 7 = 9$. For the problems ending up with the need to solve a quadratic equation, you may want to review factoring techniques and the quadratic formula. *Algebra For Dummies*, written by yours truly (and published by Wiley) is a great reference for many of the basic algebraic skills. Other great sources for math review are *Everyday Math For Dummies*, by Charles Seiter (Wiley), and *Basic Math & Pre-Algebra For Dummies*, by Mark T. Zegarelli (Wiley).

If you're reading this book, I'm making the not-so-foolish assumption that you know your way around basic arithmetic and algebra. With the rest, I'm here to help you!

How This Book Is Organized

This book is broken into five different parts, each with a common thread or theme. You can start anywhere — you don't have to go from Part I to Part II, and so on. But the logical arrangement of topics helps you find your way through the material.

Part I: Lining Up the Basic Strategies

The four chapters in this part contain general plans of attack — how you approach a word problem and what you do with all those words. I introduce the basic vocabulary of math in word problems, and I outline the steps you use for solving any kind of word problem. You see how to work your way through the various units: linear, area, volume, rate. And finally, I use a grand example of handling a math word problem to demonstrate the various techniques you use to solve the rest of the problems in the book.

Part II: Taking Charge of the Math

The main emphasis of the chapters in this part is on using the correct operations and formulas. You get to use probability and proportions, money and mixtures, formulas and figuring. One of the first hurdles to overcome when doing math word problems is choosing the correct process, operation, or rule. Money plays a big part in these chapters — as it plays a big part in most people's lives.

Part III: Tackling Word Problems from Algebra

These chapters and problems may be the ones that you've really been looking forward to all along. Here you see how to take foreign-sounding, confounding, baffling, challenging word combinations and change them into mathematical problems that you can perform. Or, on the other hand, maybe these chapters present a new experience for you — math word problems that aren't based on simple, practical applications. Enjoy this journey into the word problems that so many people remember with such delight (or a shudder).

Part IV: Taking the Shape of Geometric Word Problems

The problems in this part are solid or geometric in nature. Most people are very visual, too, finding that a picture clears up the confusing and gives direction to the perplexing. The geometric word problems in this section almost always have a rule or formula attached. You'll use perimeter, area, and volume formulas, and you'll find Pythagoras very useful when approaching these problems.

Part V: The Part of Tens

The chapters in this part are short, sweet, and to the point. The first chapter contains classic brainteasers and their solutions. The second chapter contains very brief descriptions of mathematicians — or pseudo-mathematicians. You'll find a president and world conqueror among the ten listed.

Icons Used in This Book

Following along in a book like this is easier if you go from topic to topic or icon to icon. Here are the icons you find in this book:



If you're the kind of person who loves puzzles, challenges, or just needs a break from all-word-problems-all-the-time, you'll love the brainteasers marked by this icon. If you find teasing to be annoying or mean, don't worry: I provide instant relief with the answers (upside-down and in small print).



The main emphasis of this book is how to handle math word problems in their raw form. After I introduce a problem, this icon tells you that the solution and basic steps are available for your perusal.



I reintroduce those long-forgotten or ignored tidbits and facts with this icon. You don't have to look up that old formula or math rule — you'll find it here.



Many people get caught or stuck with some particular math process. If it's tricky and you need to avoid the pitfall, the warning icon is here to alert you.

Where to Go from Here

Oh, where to start? You have so many choices.

I'd just pick a topic — your favorite, if possible. Or, maybe you have a problem that needs to be solved tonight. Go for it! Find the information you need and conquer your challenge. Then you can take the time to wander through the rest of the sections and chapters to find out what the other good stuff is.

This book isn't meant to be read from beginning to end. I'd never do that to you! Go to the chapter or section that interests you today. And go to another part that interests you tomorrow. This book has a topic for every occasion, right at your fingertips.

Part I

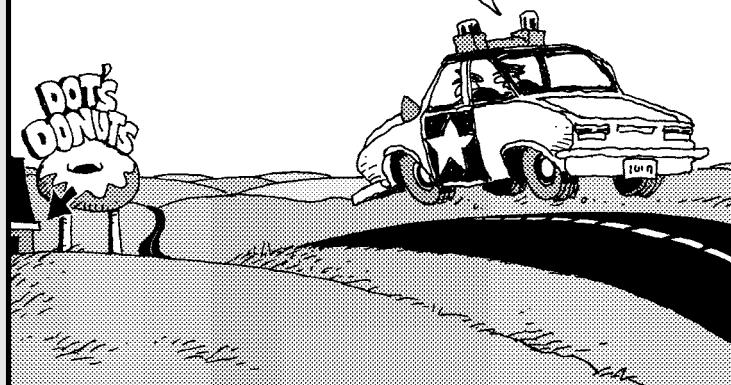
Lining Up the Basic Strategies

The 5th Wave

By Rich Tennant

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The fugitives left 200 miles from here traveling 70 miles per hour. We're traveling in the opposite direction going 60 miles per hour. At what point... hey, donut shop, donut shop! Pull over!



In this part . . .

You find how to deal with problems that include words such as *sum*, *twice*, *ratio*, and *difference*. Throw in units such as *inches* and *quarts* and rates such as *miles per hour*. If you mix it all up in a mathematical container, such as a box, you have the ingredients for a math word problem. You find the basic strategies and procedures for doing word problems. The methods I present in this part follow you throughout the entire book.

Chapter 1

Getting Comfortable with Math Speak

In This Chapter

- ▶ Introducing terminology and mathematical conventions
 - ▶ Comparing sentence and equation structure for more clarity
 - ▶ Using pictures for understanding
 - ▶ Looking to tables and charts for organization of information
-

Mathematicians decided long ago to conserve on words and explanations and replace them with symbols and single letters. The only problem is that a completely different language was created, and you need to know how to translate from the cryptic language of symbols into the language of words. The operations have designations such as $+$, $-$, \times , and \div . Algebraic equations use letters and arrangements of those letters and numbers to express relationships between different symbols.

In this chapter, you get a refresher of the math speak you've seen in the past. I review the vocabulary of algebra and geometry and give examples using the appropriate symbols and operations.

Latching onto the Lingo

Words used in mathematics are very precise. The words have the same meaning no matter who's doing the reading of a problem or when it's being done. These precise designations may seem restrictive, but being strict is necessary — you want to be able to count on a mathematical equation or expression meaning the same thing each time you use it.

For example, in mathematics, the word *rational* refers to a type of number or function. A person is *rational* if he acts in a controlled, logical way. A number is *rational* if it acts in a controlled, structured way. If you use the word *rational* to describe a number, and if the person you're talking to also knows what a rational number is, then you don't have to go into a long, drawn-out explanation about what you mean. You're both talking in the same language, so to speak.

Defining types of numbers

Numbers are classified by their characteristics. One number can have more than one classification. For example, the number 2 is a *whole* number, an *even* number, and a *prime* number. Knowing which numbers belong in which classification will help you when you're trying to solve problems in which the answer has to be of a certain type of number.

Naming numbers

Numbers have names that you speak. For example, when you write down a phone number that someone is reciting, you hear *two, one, six, nine, three, two, seven*, and you write down 216-9327. Some other names associated with numbers refer to how the numbers are classified.

- ✓ **Natural (counting):** The numbers starting with 1 and going up by ones forever: 1, 2, 3, 4, 5, ...
- ✓ **Whole:** The numbers starting with 0 and going up by ones forever. Whole numbers are different from the natural numbers by just the number 0: 0, 1, 2, 3, 4, ...
- ✓ **Integer:** The positive and negative whole numbers and 0: ..., -3, -2, -1, 0, 1, 2, 3, 4, ...
- ✓ **Rational:** Numbers that can be written as $\frac{p}{q}$ where both p and q are integers, but q is never 0: $\frac{3}{4}, \frac{19}{8}, -\frac{5}{21}, \frac{24}{6}$, and so on
- ✓ **Even:** Numbers evenly divisible by 2: ..., -4, -2, 0, 2, 4, 6, ...
- ✓ **Odd:** Numbers not evenly divisible by 2: ..., -3, -1, 1, 3, 5, 7, ...
- ✓ **Prime** Numbers divisible evenly only by 1 and themselves: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, ...
- ✓ **Composite:** Numbers that are not prime; numbers that are evenly divisible by some number other than just 1 and themselves: 4, 6, 8, 9, 10, 12, 14, 15, ...

Relating numbers

Numbers of the same or even different classifications are often related in another way that makes them usable in problems. For example, if you want only *multiples of five*, you draw from evens, odds, and integers — several types to create a new relationship.

- ✓ **Consecutive:** A listing of numbers, in order, from smallest to largest, that have the same difference between them: 22, 33, 44, 55, . . . are consecutive multiples of 11 starting with 22.
- ✓ **Multiples:** Numbers that all have a common multiplier: 21, 28, and 63 are multiples of 7.

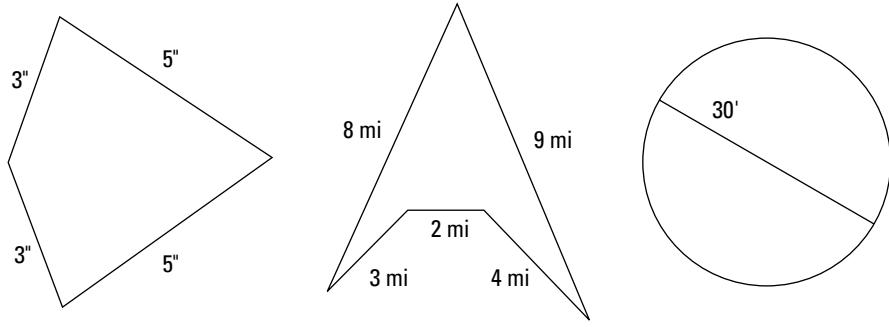
Gauging the geometric

Geometric figures appear frequently in mathematical applications — and in life. Geometric figures have names, classifications, and characteristics. The figures are also measured in two or more ways. Flat figures have the lengths of their sides, their whole perimeter, or their area measured. Solid figures have their surface area and volume measured. You can find all the formulas you need on the Cheat Sheet and in Chapters 18, 19, and 20. What you find here is a description of what the measures mean.

Plying perimeter

The *perimeter* is a linear measure: inches, feet, centimeters, miles, kilometers, and so on. Perimeter is a measure of distance — the distance around the outside of a flat figure. The perimeter of a figure made up of line segments is equal to the sum of the length of all the segments. The perimeter of a circle is also called its *circumference* and is always slightly more than three times the circle's diameter. In Figure 1-1, you see several sketches and their respective perimeters.

Figure 1-1:
Add up
the lengths
of the
segments
to get the
perimeter.



$$P = 3 + 5 + 5 + 3 = 16"$$

$$P = 8 + 9 + 4 + 2 + 3 = 26 \text{ mi.}$$

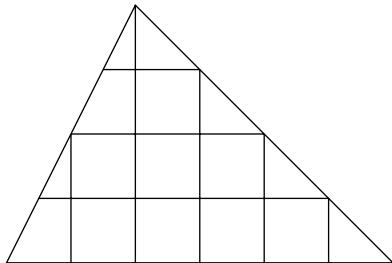
$$C = 30\pi \approx 94.2'$$

Assembling the area

The *area* of a figure is a two-dimensional measure. The area is a measure of how many squares you can fit into the figure. If the figure doesn't have 90-degree or squared-off angles, then you have to count up pieces of squares — break them up and put them back together — to get the whole area. Think about putting square tiles in a room — you have to cut some of them to go around cabinets or fit along a wall. The formulas that you use to compute areas help you with the piecing-together of squares.

In Figure 1-2, you see a triangle with an area of exactly 12 square units. See if you can figure out how the pieces go together to form a total of 12 squares. If that doesn't work, you can compute the area by just looking up the formula for the area of a triangle.

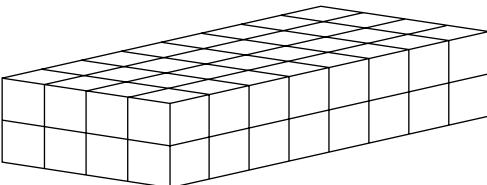
Figure 1-2:
How many
squares are
in the
triangle?



Coming to the surface with surface area

The *surface area* of a solid figure is the sum of the areas of all the sides. A four-sided figure has a triangle on each side, so you add up the areas of each of the triangles to get the total surface area. How do you get the area of each triangle? You go back to the formula for finding the area of triangles of that particular size — or just count how many squares! Figure 1-3 shows three of the six sides of a right rectangular prism and how each side has its area determined by all the squares it can fit on that side.

Figure 1-3:
How much
paper will
you need to
wrap the
package?

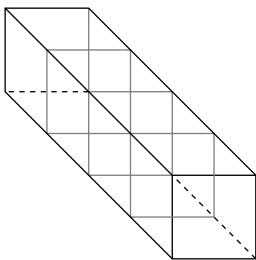


The prism in Figure 1-3 has a surface area of 112 square units. That's how many squares cover the six surfaces of this solid figure. Formulas are much easier to use than actually trying to count squares.

Vanquishing volume

The *volume* of a solid figure is a three-dimensional type of unit. When you compute the volume of something, you're determining how many cubes (like sugar cubes or dice) will fit inside the figure. When the sides slant, of course, you have to slice, trim, and fit to make all the cubes go inside — or you can use a handy-dandy formula. Figure 1-4 shows how you can set cubes next to one another and then stack them to determine the volume of a solid.

Figure 1-4:
Cubes all in
a row.



Formulating financials

Most people are interested in money, in one way or another. Money is the way people keep count of whether they can trade for what they want or need. Financial formulas aid with the computation of money-type situations.

The financial formulas here are divided into two different types: interest formulas and revenue formulas. The interest formulas both involve a percentage that needs to be changed into a decimal before being inserted into the formula. To change a percent into a decimal, you move the decimal point two places to the left. So 3.4 percent becomes 0.034 and 67 percent becomes 0.67.

The interest formulas are of two types: simple interest and compound interest. The simple-interest formula is $I = Prt$. The I indicates how much interest your money has earned — or how much interest you owe. The P is the principal — how much money you invested or are borrowing. The r represents the interest rate — the percentage that gets changed to a decimal. And the t stands for time, which is usually a number of years.



"Tomorrow, tomorrow, I love ya, tomorrow . . ."

If yesterday had been Wednesday's tomorrow,
and if tomorrow is Sunday's yesterday, then
what day is it today?

Answer: It's Friday.

Compounding interest means that you split up the rate of interest into a designated number of subintervals (every three months, twice a year, daily, and so on), figure the interest earned during that subinterval, add the interest to the principal, and then figure the next interval's interest on the sum of the original principal plus the interest you've added. As you may expect, you'll have more money in the end if you deposit it where you can earn compound interest rather than just a flat amount. The formula for compound interest is $A = P \left(1 + \frac{r}{n}\right)^t$. The A represents the total amount of money — all the principal plus the interest earned. The r and t are the same as in simple interest. The n represents the number of times each year that the interest is compounded. Most banks compound quarterly, so the value of n is 4 in those cases.

Interpreting the Operations

What would mathematics be without its operations? The basic operations are addition, subtraction, multiplication, and division. You then add raising to powers and finding roots. Many more operations exist, but these six basic operations are the ones you'll find in this book. Also listed here are some of the special names for multiplying by two or three.

Naming the results

Each operation has a result, and just naming that result is sometimes more convenient than going into a big explanation as to what you want done. You can economize with words, space, time, and ink. The following are results of operations most commonly used.

✓ **Sum:** The result of adding

✓ **Difference:** The result of subtracting

- ✓ **Product:** The result of multiplying
- ✓ **Twice or double:** The result of multiplying by two
- ✓ **Thrice or triple:** The result of multiplying by three
- ✓ **Quotient:** The result of dividing
- ✓ **Half:** The result of dividing by two

Assigning the variables

A variable is something that changes. In mathematics, a variable is represented by a letter — usually one from the end of the alphabet — and it *always* represents a number (usually an unknown number). For example, if you’re doing a problem involving Jake and Jim and their ages, you can let x represent Jake’s age, but you can’t let x represent Jake.



As you work on a problem, it’s a good idea to make a notation as to what you’re letting the variable or variables represent, so you don’t forget or get confused when constructing an equation to solve the problem.

Aligning symbols and word forms

One of the things that people see as a challenge in word problems is that they’re full of *words!* After you’ve changed the words to symbols and equations, it’s smooth sailing. But you have to get from there to here. Table 1-1 lists some typical translations of words into symbols and an example of their use.

Table 1-1 Translating into Math Shorthand

| Words | Symbols | Examples |
|-------------|---------|--|
| Is, are | = | Jack is twice as old as Jill: $x = 2y$, where x represents Jack’s age, and y represents Jill’s age. |
| And, total | + | Six dimes and 5 quarters: $10x + 25q$, where x represents the number of dimes and q represents the number of quarters; each is multiplied by its monetary value. |
| Less, fewer | - | Jon worked 3 fewer hours than Jim: $x = y - 3$, where x represents the number of hours that Jon worked, and y represents the number of hours that Jim worked. |

(continued)

Table 1-1 (continued)

| Words | Symbols | Examples |
|---------------|----------------|---|
| Of, times | \times | One-half of Clare's money: $\frac{1}{2}x$, where x represents Clare's money. |
| Ratio | \div | The ratio of pennies to quarters: $\frac{x}{y}$, where x represents the number of pennies, and y represents the number of quarters |
| Approximately | \approx | The fraction $\frac{22}{7}$ is about 3.14: $\frac{22}{7} \approx 3.14$ |

Drawing a Picture

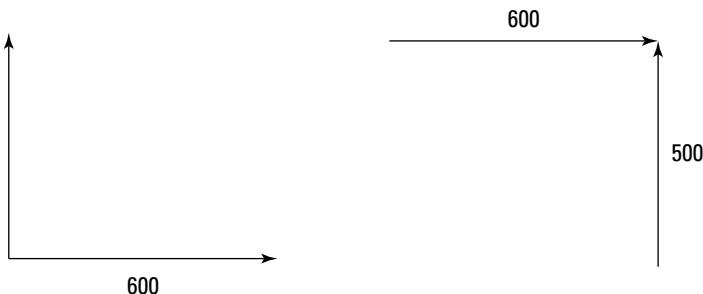
One of the most powerful tools you can use when working on word problems is drawing a picture. Most people are very visual — they understand relationships between things when they write something down and/or draw a picture illustrating the situation.

Visualizing relationships

The words in a math problem suggest how different parts of the situation are connected — or not connected. Drawing a picture helps to make the connections and, often, suggests how to proceed with a solution.

For example, consider a word problem starting out with: “A plane is flying east at 600 mph while another plane is flying north at 500 mph. . . .” You need more information than this to determine what the question and answer are, but a picture suggests what process to use. Look at Figure 1-5, where two possible scenarios for the statement are illustrated.

Figure 1-5:
The planes are leaving or approaching one another.



The precise relationship between the planes has to be given, but both sketches suggest that a triangle can be formed by connecting the ends of the arrows. Right triangles suggest the Pythagorean theorem, and other triangles come with their respective perimeter and area formulas. In any case, the picture solidifies the situation and makes interpretations possible.

Another example where a picture is helpful involves a situation where you're cutting a piece of paper. The word problem starts out with: "A rectangular piece of paper has equal squares cut out of its corners. . . ." You draw a rectangle, and you show what it looks like to remove squares that are all the same size. Figure 1-6 illustrates one interpretation.

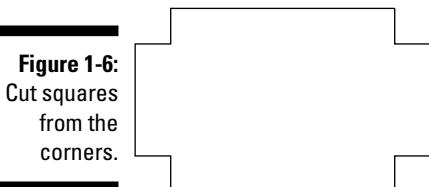


Figure 1-6:
Cut squares
from the
corners.

With the figure in view, you see that the lengths of the outer edges are reduced by two times some unknown amount. The picture helps you write expressions about the relationships between the original piece of paper and the cut-up one.

Labeling accurately

Pictures are great for clarifying the words in a problem, but equally important are the labels that you put on the picture. By labeling the different parts — especially with their units in feet, miles per hour, and so on — you improve your chances of writing an expression or equation that represents the situation.

You're told "A trapezoidal piece of land has 300 feet between the two parallel sides, and the other two sides are 400 feet and 500 feet in length, while the two parallel bases are 600 feet and 1,200 feet." This statement has five different numbers in it, and you need to sort them out. Figure 1-7 shows how the different measures sort out from the statement.

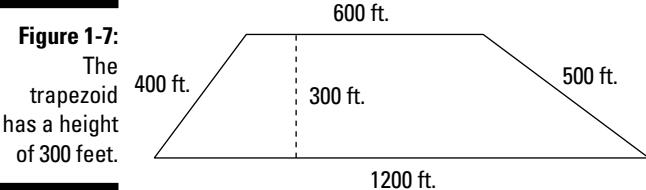


Figure 1-7:
The
trapezoid
has a height
of 300 feet.



Watch out for the black cat

A completely black cat was ambling down the street during a total blackout — electricity was down throughout the entire town. Not a single streetlight had been on for hours. Just as the cat was crossing the middle of the street, a Jeep with two broken headlights came racing toward

where the cat was. Just as the Jeep got to where the cat was crossing, it swerved out of the way to avoid hitting the cat. How could the driver of the Jeep have seen the cat in time to swerve and avoid hitting it?

lights were on — everyone could see just fine.

Answer: This happened in the middle of the day. It didn't matter that no street

Constructing a Table or Chart

A really nice way to determine what's going on with a word problem is to make a list of different possibilities and see what fits in the list or what pattern forms. Patterns often suggest a formula or equation; the values in the listing sometimes even provide the exact answer. Just as with pictures, making a chart is a way of visualizing what's going on.

Finding the values

When creating a table or chart, designate a variable to represent a part of the problem, and see what the results are as you systematically change that variable. For example, if you're trying to find two numbers the product of which is 60 and the sum of which is as small as possible, let the first number be x .

Then the other number is $\frac{60}{x}$. Add the two numbers together to see what you get. Table 1-2 shows the different values for the two numbers and the sum — if you stick to whole numbers.

Table 1-2

Finding the Smallest Possible Sum

| <i>x (The First Number)</i> | <i>$\frac{60}{x}$ (The Second Number)</i> | <i>The Sum of the Numbers</i> |
|-----------------------------|--|-------------------------------|
| 1 | 60 | $1 + 60 = 61$ |
| 2 | 30 | $2 + 30 = 32$ |
| 3 | 20 | $3 + 20 = 23$ |

| <i>x (The First Number)</i> | $\frac{60}{x}$ (<i>The Second Number</i>) | <i>The Sum of the Numbers</i> |
|-----------------------------|---|-------------------------------|
| 4 | 15 | $4 + 15 = 19$ |
| 5 | 12 | $5 + 12 = 17$ |
| 6 | 10 | $6 + 10 = 16$ |
| 10 | 6 | $10 + 6 = 16$ |

I stop here, because I'm just going to get the same pairs of numbers in the opposite order. If the rule is that the numbers can't be fractions, then the two numbers with a product of 60 and with the smallest possible sum are 6 and 10.

Increasing in steps

When making a table or chart, you want to be as systematic as possible so you don't miss anything – especially if that *anything* is the correct answer. After you've determined a variable to represent a quantity in the problem, you need to go up in logical steps — by ones or twos or halves or whatever is appropriate. In Table 1-2, in the preceding section, you can see that I went up in steps of 1 until I got to the 6. One more than 6 is 7, but 7 doesn't divide into 60 evenly, so I skipped it. Even though the work isn't shown here, I mentally tried 7, 8, and 9 and discarded them, because they didn't work in the problem. When you're working with more complicated situations, you don't want to skip any steps — show them all.

Chapter 2

Planning Your Attack on a Word Problem

In This Chapter

- ▶ Deciphering between fact and fiction
 - ▶ Getting organized and planning an approach
 - ▶ Turning guessing into a science
-

Getting ready to solve a math word problem involves more than sharpening your pencil, aligning your sheets of paper, and taking a deep breath. Math word problems have the reputation of being obscure, difficult, confusing — you name it. The only way to overcome this bad rep is to be ready, able, and willing to take them on. Start with the right attitude and preparation so that you can approach math word problems in a confident, organized fashion.

In this chapter, I review the importance of isolating the question, determining just what information is needed, and ignoring the fluff. Math word problems often contain information that makes the wording of the problem more interesting but adds nothing to what's needed for the solution. Also in this chapter, you see the importance of doing a reality check — does the answer really make sense? Is it what you expected? If not, why not?

Singling Out the Question

A math word problem is full of words — big surprise! Word problems really represent the real world. When you have a problem to solve at the office involving ordering new file cabinets, you don't sit down to write out your times tables, and you aren't handed a piece of paper with an algebra problem asking you to solve $2x + 3 = 27$. To be successful with a word problem, you

have to translate from words to symbols, formulate the equation or problem needed, and then perform the operations and arithmetic correctly to find the answer.

Wading through the swamp of information

One of my favorite sayings is: “When you’re knee deep in alligators, it’s hard to remember that your mission is to drain the swamp.” This certainly applies to math word problems and sorting out what the question is from all the other stuff. Consider the following problem. See if you can find the question in all the verbiage.

The Problem: The 17 office workers and 4 managers at Super Mart all need new file cabinets before the end of this month, which has 31 days. The file cabinets cost \$300 each, and the supplier of file cabinets will haul away the old cabinets for \$5 each. How much will it cost to replace the file cabinets if each of the office workers takes his old file cabinet home and each of the managers elects to have them hauled away?



There’s a lot of information in this problem, but you need to first look for the question — what is it that you want to find? Look for *what, how many, how much, when, find*, or other questioning or seeking words. Ignore the rest for now until you determine what you’re looking for. Is it a number of file cabinets? Is it an amount of money? Is it a number of people? Get your question nailed down, and then worry about how to put it all together. In this problem, the question is basically: “How much money?”

If you can’t stand to let a problem go unanswered and you want to solve this one, the answer is \$6,320. You figure out how much money it costs by determining how much was spent on each office worker and then adding the total to how much the managers spent. Each of the 17 office workers spent just \$300 (just the cost of the cabinets), so $17 \times \$300 = \$5,100$. The managers had their file cabinets hauled away, which added \$5 on to the cost. The total for each manager is \$305, so $4 \times \$305 = \$1,220$. Add the amounts together, and the cost of the file cabinets for the entire office is $\$5,100 + \$1,220 = \$6,320$.



How many?

A farmer has 40 sheep, 20 pigs, 10 cows, and 5 chickens. If he decides to call the pigs cows, how many cows does he have?

Answer: The farmer has 10 cows. He can call the pigs whatever he wants, but they’re still pigs.

Going to the end

In 95 percent of all math word problems, the question is at the end of the description. No, I didn't do a scientific survey. This is just my best guess based on years of experience and reading way too many problems. Just trust me. Most of the questions in word problems are at the end. That's just the way word problems are most efficiently constructed.

Reading the last sentence first isn't a bad idea. When you find the question that you need to answer, you can go back and wade through all the information and sort out what's needed to find the answer. The following questions are examples of how word problems are constructed and how the question seems to come more naturally at the end.

The Problem: In a particular parking garage, you have spaces for regular-size cars, compact cars, and large vans. The regular-size cars can park in the spaces designed for a van, and the compact cars can fit in the spaces designed for either a regular-size car or a van. If there are 200 spaces for regular-size cars, 40 spaces for compacts, and 30 spaces for vans, *how many* regular-sized cars can park in the garage if vans are not allowed on a particular day?

The Problem: A magazine salesman gets a 5 percent commission on all one-year subscriptions, a 10 percent commission on all two-year subscriptions, a 20 percent commission on all three-year subscriptions, and a flat fee of \$5 for every subscription he sells in excess of 100 subscriptions in any one week. On Monday, he sold 14 one-year subscriptions, 23 two-year subscriptions, and 6 three-year subscriptions. On Tuesday, Wednesday, and Thursday, he sold 12 of each type of subscription. On Friday, he sold 60 two-year subscriptions. One-year subscriptions cost \$20, two-year subscriptions cost \$35, and three-year subscriptions cost \$52. On *which day* did he earn the most?

The Problem: If Tom is twice as old as Dick was ten years ago, and if the sum of their ages five years ago was 90, then *how old* is Tom now?

You want the answers to these questions? Okay, you'll get them, but you'll have to read on in this chapter where I cover eliminating the unwanted and doing the operations in order.

Organizing the Facts, Ma'am, Just the Facts

Some writers of word problems seem to need to wax poetic — they go on and on with unnecessary facts just to make the problem seem more interesting. You know that it isn't necessary to make these more interesting — they're fascinating enough already, right? Okay, don't answer that.

Eliminating the unneeded

After you've isolated the question, you go back to the problem to sort out the needed information from the extra fluff. Look at the three problems from the previous section again. I've drawn lines through the interesting-but-unnecessary.

The Problem: ~~In a particular parking garage, you have spaces for regular size cars, compact cars, and large vans.~~ The regular-size cars can park in the spaces designed for a van, and compact cars can fit in the spaces designed for either a regular-size car or a van. If there are 200 spaces for regular-size cars, ~~40 spaces for compacts~~ and 30 spaces for vans, how many regular-size cars can park in the garage if vans are not allowed on a particular day?

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The Problem: If Tom is twice as old as Dick was ten years ago, and if the sum of their ages five years ago was 90, then how old is Tom now?

You see that quite a bit is eliminated in the first problem, just a bit is eliminated in the second problem, and nothing is eliminated in the third problem. The information that's been eliminated may be useful to answer some other question, but it isn't needed for the question at hand.

Doing the chores in order

You've isolated the question and eliminated the riff-raff. Now it's time to set up the arithmetic problems or equations needed to solve the problems. The order in which you perform the operations is pretty much dictated by what the question is. The last operation performed is what gives you the final answer. I'll take the problems one at a time.

The Problem: ~~In a particular parking garage, you have spaces for regular size cars, compact cars, and large vans.~~ The regular-size cars can park in the spaces designed for a van, and ~~compact cars can fit in the space designed for either a regular size car or a van.~~ If there are 200 spaces for regular-size cars,

40 spaces for compacts and 30 spaces for vans, *how many* regular-sized cars can park in the garage if vans are not allowed on a particular day?



To solve this problem, you need to answer the question *how many*, which is a total of two types of parking spaces. The total is the sum of 200 spaces plus 30 spaces, which is $200 + 30 = 230$ spaces. For more problems of this type, turn to Chapter 5.

The Problem: A magazine salesman gets a 5 percent commission on all one-year subscriptions, 10 percent commission on all two-year subscriptions, a 20 percent commission on all three-year subscriptions, ~~and a flat fee of \$5 for every subscription he sells in excess of 100 subscriptions in any one week.~~ On Monday he sold 14 one-year subscriptions, 23 two-year subscriptions, and 6 three-year subscriptions. On Tuesday, Wednesday, and Thursday, he sold 12 of each type of subscription. On Friday, he sold 60 two-year subscriptions. One-year subscriptions cost \$20, two-year subscriptions cost \$35, and three-year subscriptions cost \$52. On which day did he earn the most?



Solving this problem requires comparing the commissions made on three different days. Only three days are necessary, because the commissions are the same for Tuesday, Wednesday, and Thursday. Comparing the commissions requires finding the total commission for each day by multiplying the number of each type times the rate for that type times the price; then the commissions from the different types are all added together. A table or chart is helpful in this case to keep everything in order (see Table 2-1). You can do the commission amount computations ahead of time. For one-year subscriptions, 5 percent of \$20 is \$1. I got that by multiplying $0.05 \times \$20$. Two-year subscriptions earn the salesman 10 percent of \$35, or \$3.50; and three-year subscriptions are worth 20 percent of \$52, which equals \$10.40. Refer to Chapter 6 for more on percentage problems involving decimals and percents.

Table 2-1 Determining the Daily Commission

| Day | Number of One-Year Subscriptions $\times \$1$ | Number of Two-Year Subscriptions $\times \$3.50$ | Number of Three-Year Subscriptions $\times \$10.40$ | Total |
|------------------------------------|--|---|--|----------|
| Monday | $14 \times 1 = \$14$ | $23 \times \$3.50 = \80.50 | $6 \times \$10.40 = \62.40 | \$156.90 |
| Tuesday, Wednesday, Thursday | $12 \times 1 = \$12$ | $12 \times \$3.50 = \42 | $12 \times \$10.40 = \124.80 | \$178.80 |
| Friday | $0 \times 1 = \$0$ | $60 \times \$3.50 = \210 | $0 \times \$10.40 = \0 | \$210 |

He had his highest commission on Friday. In Chapter 14, you see many problems involving *quality* \times *quantity* — more like this type of problem.

The Problem: If Tom is twice as old as Dick was ten years ago, and if the sum of their ages five years ago was 90, then how old is Tom now?



This problem is more like the word problems that most people remember from their high school years. The problem almost seems like double-speak when you first read it. I hate to give too much away right here, because I cover age problems in great detail in Chapter 15, but let this be a *tickler* — something to get you all enthused about doing more age problems.

You want to answer the question “How old is Tom?” So let Tom’s age be represented by t . You’re comparing Tom’s age to Dick’s age. So let Dick’s age be represented by d . But Tom’s age is compared to Dick’s age *ten years ago*. How old was Dick ten years ago? You get that by subtracting 10, so Dick was $d - 10$ ten years ago, and Tom is twice that or $2(d - 10)$. So you can say that $t = 2(d - 10)$. Well, there are lots of numbers that satisfy the equation $t = 2(d - 10)$, so you need a bit more information. You’re told that *five years ago* . . . Their ages five years ago were $t - 5$ and $d - 5$. The sum of their ages at that time was 90, so $t - 5 + d - 5 = 90$, which simplifies down to $t + d = 100$. So their ages today must add up to 100. The algebraic way of solving the system of equations $t = 2(d - 10)$ and $t + d = 100$ is found in Chapter 17. For now, I’ll let you off the hook and tell you that the solution is that Tom is 60 and Dick is 40.

Estimating an Answer to Check for Sense

No matter how easy or complicated a math word problem, you should always have some guess or inkling or idea as to what the answer may be before you even get started on solving it. Even if you’re way off with your guess, this exercise is very useful. You’re more apt to check the work if you think that the answer is *way out there*. Many times, you just find that you didn’t do a very good job of guessing. Other times, you find that you made a mistake in the arithmetic and you can go back and correct your error.

Guessing an answer

One of a student’s favorite instructions on a test is: *Must show work*. And here I am, encouraging you to guess the answer. I’m a firm advocate of showing work, too. But guessing is a great skill to have, and it’s useful when doing these word problems. Showing the work and all the steps involved is good practice, too, so you can apply the same steps and techniques on problems that aren’t quite as easy to guess for the answers.

The Problem: You have \$10,000 to invest over the next year and will put some of the money in an account earning 12 percent (risky) and three times as much as that in an account earning 4 percent (safer). How much interest will you earn?



You're going to multiply some money by 12 percent and the rest by 4 percent. (Refer to Chapter 10 on using formulas and Chapters 6 and 14 for more problems like this.) You make an estimate of the total amount of interest you'll earn by picking a percentage somewhere between 12 percent and 4 percent and multiplying the \$10,000 by that percent. Because more money is going to be invested at the lower rate, you'll probably use either 5 percent or 6 percent. Let your guess be the 6 percent, making your estimate on the interest earned be 6 percent of \$10,000, or \$600. Now, when you actually do the problem, if you get some out-of-the-world answer like \$8,000, you'll know that you've put a decimal point in a wrong place or done some other silly computation. What's the actual answer, you ask? It really is \$600. Sometimes I even surprise myself. I multiplied \$2,500 by 12 percent and \$7,500 (three times as much as \$2,500) by 4 percent and got a total of \$600.

Doing a reality check

Some math word problems have answers that just don't seem right. These answers are the ones that definitely need checking. Other answers are just obviously wrong, meaning that the problem needs reworking.

Being struck by the obvious

Math problems sometimes require solving algebraic equations. The algebra is wonderful, but you run the risk of creating some *extraneous roots*. Extraneous roots or solutions are answers that satisfy the algebraic equation but don't really mean a thing in the situation.

The Problem: If Folabo leaves home and drives 400 miles north while her sister Fatima leaves home at the same time and drives 300 miles east, how far apart are they?



A line segment drawn 400 miles upward connected to a line segment drawn 300 miles eastward forms two sides of a right triangle. You find out how far apart they are by finding the hypotenuse of a right triangle. (In Chapter 18, Pythagoras and his famous triangle are discussed at length. Pardon the pun.) Anyway, to solve this you solve $400^2 + 300^2 = c^2$ for c and get $c = \pm 500$. Obviously, the distance can't be negative, so you just use the positive value and determine that they're 500 miles apart.

Getting a firm grip on reality

Sometimes a weird answer looks good at first, and it takes careful checking to show that it doesn't work.

The Problem: I'm thinking of a number. If you subtract 1 from the number and find the square root of the difference, you get the same answer as when you subtract the number from 7. What is my number?



This problem requires a nice algebraic equation. (Chapter 11 has lots of problems like this, if these are up *your* alley.) Letting the number be represented by x , the equation to use is $\sqrt{x - 1} = 7 - x$. Squaring both sides of the equation and then solving the quadratic equation by factoring, you get

$$\begin{aligned}(\sqrt{x - 1})^2 &= (7 - x)^2 \\x - 1 &= 49 - 14x + x^2 \\0 &= 50 - 15x + x^2 \\0 &= (10 - x)(5 - x)\end{aligned}$$

The two answers to this equation are $x = 10$ and $x = 5$. They both look perfectly wonderful, don't they? *Not!* If you put 10 back in for x in the equation, you get that $3 = -3$. Doesn't work. But putting the 5 in for x , you get that $2 = 2$. The 5 is a solution, and the 10 is *extraneous* — it satisfies the quadratic equation, but it makes no sense in the answer. (If you like this type of number problem, you'll find lots of them in Chapter 11.)

Chapter 3

Coordinating the Units

In This Chapter

- ▶ Adjusting units for ease in computations
 - ▶ Changing from English to metric units and back again
 - ▶ Squaring off with square and cubic units
-

Many mathematical problems involve units of length, weight, volume, or money. You incorporate the units into your computations and then report them in the answers so the solution makes sense and is useful. Sometimes you're confronted with problems that have two or more different units — such as feet and inches or pounds and ounces — and you have to make a decision as to which unit to use.

In this chapter, I offer suggestions on how to choose the unit or units and then how to work with the unit or units you've chosen. This chapter also covers the tricky conversions of square feet to square inches or cubic yards to cubic feet. And, of course, no discussion of units is complete without introducing meters and kilograms, so you get conversions involving metric and English measures.

Choosing the Best Measure

When a problem involves two different measures, you choose one or the other measure to work with and convert the unchosen measure to the unit you want so that they're all the same. You may even decide to change measures when they're already all the same — just because you think that another measure may work better.

Using miles instead of inches

A mile is much longer than an inch. In fact, there are $12 \times 5,280 = 63,360$ inches in 1 mile. If you're measuring how far it is from one side of the desk to another, you'll use inches. If you're measuring how far it is from your home to your workplace, you'll usually measure in miles.

The Problem: Train tracks are made of metal, and metal expands when it gets hot and shrinks when the weather is cold. When the tracks are put in place, a gap should be left between the adjacent tracks to allow for expansion. A metal track that's 1 mile long expands with the heat and increases in length by 12 inches. There was no gap between the tracks, however, so it buckled in the middle and formed a V shape. (See Figure 3-1 for a picture of what the track looks like.) How high up did the track rise?

Figure 3-1
The train
track
expanded
with the
heat.



Two different measures are given: 1 mile and 12 inches. You can work with inches, and change the 1 mile to 63,360 inches, or you can work with miles and write the 12 inches (1 foot) as $\frac{1}{5,280}$ mile. The choice here is between using really large numbers and using fractions or decimals.

I choose to go with the fractions — to work with pieces of a mile. To find out how high the rise in the track is, I use the Pythagorean theorem (Chapter 18 is completely devoted to that theorem of Pythagoras) and one right triangle going halfway down the track. The bottom segment of the triangle is $\frac{1}{2}$ mile long, and the *hypotenuse* (longest side) is $\frac{1}{2}$ mile plus 6 inches or $\frac{1}{10,650}$ mile long. To solve for the rise, which I'll represent with x , I solve the equation for x .

$$\left(\frac{1}{2}\right)^2 + x^2 = \left(\frac{1}{2} + \frac{1}{10,650}\right)^2$$

This looks pretty nasty, but a scientific calculator makes short work of the problem, and you get

$$0.25 + x^2 \approx 0.2500947$$

$$x^2 = 0.0000947$$

$$x = 0.009732$$

The height or rise of 0.009732 doesn't seem like much, but, remember, this is in miles. Multiply by 5,280 feet and you get over 51 feet. Whoa! That's quite a rise!

Working with square feet instead of square yards

When you buy carpeting, you usually buy it in square yards — 3 feet by 3 feet. But you probably bought your last tile floor in terms of a number of square feet.

The Problem: Using your yardstick, you measure the length of a room to be 6 yards and its width to be 5 yards. You plan on putting in 1-foot-square tiles. How many tiles will you need?



Before determining the area of the room, first change the yards to feet using 1 yard = 3 feet. So 6 yards is $6 \times 3 = 18$ feet. Five yards is $5 \times 3 = 15$ feet. A room that's 18 feet by 15 feet is $18 \times 15 = 270$ square feet.

But what if you preferred finding the area in square yards, first, and then changing the area to square feet? The area of the room is 6×5 yards or 30 square yards. A square yard is equal to 9 square feet ($3 \text{ feet} \times 3 \text{ feet}$). So multiply 30×9 to get 270 square feet.

Converting from One Measure to Another

When a problem contains more than one measure, you change everything to the same measure before doing the computing on the problem or solving the equation. You can't add 6 inches to 4 feet and get 10 — you have to change the inches to feet or feet to inches. Knowing when to multiply and when to divide sometimes gets confusing, so your best bet is to write down the equivalence or change of units and then work from the equation.

Changing linear measures

First, here's a list of some common equivalences used when working with lengths. I cover the English and metric equivalences later, in "Mixing It Up with Measures."



$$1 \text{ foot} = 12 \text{ inches}$$

$$1 \text{ yard} = 3 \text{ feet}$$

$$1 \text{ mile} = 5,280 \text{ feet}$$

The measure equivalences are used to convert from one measure to another. You may need to do more than one computation if there isn't a direct equivalence between units — such as changing inches to yards or yards to miles.

The Problem: Cheryl has 48 rolls of satin ribbon, each containing 15 yards of ribbon. She plans to wrap packages to send overseas as gifts, and each package requires 30 inches of ribbon. How many packages can she wrap?



First, determine how many yards of ribbon are in those 48 rolls. Then change the yards to feet using $1 \text{ yard} = 3 \text{ feet}$ and the feet to inches using $1 \text{ foot} = 12 \text{ inches}$. After you have the total number of inches, you can divide by 30 to get the number of packages that can be wrapped.

Multiplying $48 \text{ rolls} \times 15 \text{ yards}$ you get 720 yards. Start with the equivalence involving yards and feet. To change 720 yards to feet, you multiply each side of the equation $1 \text{ yard} = 3 \text{ feet}$ by 720.

$$\begin{array}{rcl} 1 \text{ yard} & = & 3 \text{ feet} \\ \times 720 & & \times 720 \\ \hline 720 \text{ yards} & = & 2,160 \text{ feet} \end{array}$$

You have 2,160 feet of ribbon. Change this to inches by using the equivalence involving feet and inches, $1 \text{ foot} = 12 \text{ inches}$.

$$\begin{array}{rcl} 1 \text{ foot} & = & 12 \text{ inches} \\ \times 2,160 & & \times 2,160 \\ \hline 2,160 \text{ feet} & = & 25,920 \text{ inches} \end{array}$$

That's 25,920 inches of ribbon. Divide 25,920 by 30 to get 864 packages that Cheryl can wrap.

Adjusting area and volume

Area is a two-dimensional measure. You're counting up how many squares — all the same size — fit into some flat region. You use area measures for floors in buildings and spaces in parking lots, as well as when you want to find out how much room there is in a backyard.

Volume is a three-dimensional measure and tells you how many cubes of a particular size fit into an object. Volume measures tell you about the inside of a refrigerator or the size of a cardboard carton.



$$1 \text{ square foot} = 144 \text{ square inches} (12 \text{ inches} \times 12 \text{ inches})$$

$$1 \text{ square yard} = 9 \text{ square feet} (3 \text{ feet} \times 3 \text{ feet})$$

$$1 \text{ square mile} = 640 \text{ acres}$$

$$1 \text{ cubic foot} = 1,728 \text{ cubic inches} (12 \text{ inches} \times 12 \text{ inches} \times 12 \text{ inches})$$

$$1 \text{ cubic yard} = 27 \text{ cubic feet} (3 \text{ feet} \times 3 \text{ feet} \times 3 \text{ feet})$$

The Problem: Jimmy is going to play a prank on his dad and fill the refrigerator with ice cubes that are 1 inch on a side. The refrigerator can hold 6 cubic feet. How many ice cubes will Jimmy need?



Determine the number of cubic inches in 6 cubic feet by using the equivalence $1 \text{ cubic foot} = 1,728 \text{ cubic inches}$ and multiplying each side of the equation by 6.

$$1 \text{ cubic foot} = 1,728 \text{ cubic inches}$$

$$\begin{array}{r} \times 6 \\ \hline 6 \end{array} \qquad \qquad \begin{array}{r} \times 6 \\ \hline 6 \end{array}$$

$$6 \text{ cubic feet} = 10,368 \text{ cubic inches}$$

That's over 10,000 ice cubes. Jimmy had better rethink his plan. He'll never get the ice cubes all stacked inside the refrigerator before they start melting — and he gets frostbite.

The Problem: Timothy bought 3,200 acres of land and intends to plant seedling trees on it. If each seedling requires an area of 9 square yards to grow properly, how many seedlings can he plant on his new acreage?



First, change the acres to square miles using $1 \text{ square mile} = 640 \text{ acres}$ and then the square miles to square feet. Determine how many square feet are in 9 square yards using $1 \text{ square yard} = 9 \text{ square feet}$ and dividing the result into the number of square feet in the acreage.

Changing the acres to square miles:

$$1 \text{ square mile} = 640 \text{ acres}$$

$$x \text{ square miles} = 3,200 \text{ acres}$$

Make a proportion of the equivalences, lining up the numbers and the x exactly as they appear — opposite one another. Solve for x .

$$\begin{aligned}\frac{1}{x} &= \frac{640}{3,200} \\ 3,200 &= 640x \\ \frac{3,200}{640} &= \frac{640}{640}x \\ 5 &= x\end{aligned}$$

So 3,200 acres is equivalent to 5 square miles. Change the square miles to square feet by multiplying each side of the equation 1 square mile = $5,280 \times 5,280$ square feet by 5. You get that 5 square miles = $5,280 \times 5,280 \times 5 = 139,392,000$ square feet.

Now find the number of square feet that each tree needs. If each seedling needs 9 square yards, use the equivalence that 1 square yard = 9 square feet and multiply each side of the equation by 9 to get 9 square yards = 81 square feet.

Now divide 139,392,000 square feet by 81 square feet to get the number of trees that will fit on the acreage. 139,392,000 divided by 81 = 1,720,888.89 trees. That's a lot of seedlings.

Keeping It All in English Units

Many countries, including the United States, use primarily the English units of measurement. Pressure to change to metric hasn't been strong enough, even though advocates have proposed changing to metric for over 40 years. The awkwardness of the English units is that they have all sorts of different numbers in their equivalences — as compared to the metric system where all the numbers are multiples of 10.

Comparing measures with unlikely equivalences



As disjointed as the English measurement system seems to be, it has a long tradition and some interesting and charming equivalences. Here are some more uncommon but historic measures, plus, to finish it off, a rate.

1 rod = $16\frac{1}{2}$ feet

1 fathom = 6 feet

1 furlong = 220 yards

1 hand = 4 inches

1 league = 3 miles

1 pica = 12 points

1 mile per hour = 88 feet per minute = $1\frac{7}{15}$ feet per second

The Problem: The Preakness, one of the horse races in the Triple Crown, has a distance of 9.5 furlongs. How many miles is that?



Change the furlongs to yards using 1 furlong = 220 yards and the yards to feet using 1 yard = 3 feet. Then change the feet to miles using 1 mile = 5,280 feet. The race is 9.5 furlongs, so multiply 9.5×220 to get 2,090 yards. Multiply the number of yards by 3 to get the number of feet: $2,090 \times 3 = 6,270$. Now divide the 6,270 feet by 5,280 to get the number of miles: $\frac{6,270}{5,280} = 1.1875$ miles. The decimal 0.1875 is equal to $\frac{3}{16}$, so the race is $1\frac{3}{16}$.



To change a terminating decimal to its fractional equivalent, create a fraction that has all the digits to the right of the decimal point in the numerator and, in the denominator, a power of 10 that has as many zeros as there are digits in the numerator. Then reduce the fraction. For the decimal 0.1875, you write 1875 in the numerator and a 1 followed by four zeroes in the denominator.

$$0.1875 = \frac{1875}{10,000}$$

Now reduce the fraction. You can first divide both numerator and denominator by 25 and then divide the resulting numerator and denominator by 25.

$$\frac{1875^{75}}{10,000^{400}} = \frac{75^3}{400^{16}} = \frac{3}{16}$$

The Problem: A popular method for determining how far away a bolt of lightning has struck is to count the number of seconds between the lightning flash and the sound of the thunder. If sound travels at about 1,100 feet per second, and if it's 6 seconds between the flash of lightning and the roar of the thunder, then how far away was the lightning strike in miles? And what is the speed of the sound in miles per hour?



First, determine how many feet the sound traveled by multiplying 1,100 feet by 6 to get 6,600 feet. Determine the number of miles using 1 mile = 5,280 feet. You divide 6,600 feet by 5,280 and you get 1.25 miles. According to what I was told, the number of seconds you count is the number of miles away. I don't think I was told right.

Now, to the speed of sound in miles per hour, use the equivalence that 1 mile per hour is equal to $1\frac{7}{15}$ feet per second. The speed of sound is about 1,100 feet per second. Write a proportion using these figures, letting the speed of sound in miles per hour be represented by x . Then solve for x .

$$\frac{1 \text{ mile per hour}}{x \text{ miles per hour}} = \frac{1\frac{7}{15} \text{ feet per second}}{1,100 \text{ feet per second}}$$

$$\frac{1}{x} = \frac{\frac{22}{15}}{1,100} = \frac{22^2}{15(1,100)} = \frac{2}{1,500} = \frac{1}{750}$$

$$\frac{1}{x} = \frac{1}{750}$$

The speed of sound comes out to be about 750 miles per hour. This is a bit over the speed you usually see quoted. The textbooks say that the speed of sound is actually 1,088 feet per second at 32° F. I rounded the number up to 1,100 feet for ease in computation and assumed that the temperature during a thunderstorm would be a bit warmer than 32° F.



Working for a bar of gold

Television's favorite billionaire is interviewing yet another group of potential employees. To keep from being told, "You're fired!" the finalists have the following problem posed to them, and the first person to come up with the solution will not hear the dreaded words. The problem: You've hired someone to work for you for the

next seven days. You must pay him $\frac{1}{7}$ of a bar of gold per day, but he requires a daily payment of $\frac{1}{7}$ of that bar of gold — no credit. It's expensive to cut through a bar of gold, so what are the fewest number of cuts necessary to meet his requirements?

the $\frac{1}{7}$ piece and give him the $\frac{6}{7}$ piece. On Day 7, give him the last $\frac{1}{7}$ piece. the $\frac{1}{7}$ piece, so he'll have the $\frac{6}{7}$ piece and the $\frac{1}{7}$ piece. On Day 6, take back the $\frac{1}{7}$ piece and give back the $\frac{6}{7}$ piece and $\frac{1}{7}$ pieces. On Day 4, give the worker the remaining $\frac{1}{7}$ added to the $\frac{1}{7}$ he already has. On Day 3, give the worker the original $\frac{1}{7}$ to be and take the first $\frac{1}{7}$ back. On Day 2, cut off $\frac{1}{7}$ of the bar, give that to the worker and give it to the worker. On Day 1, cut the bar at the $\frac{1}{7}$ mark

Answer: You need make only two cuts. On Day 1, cut the bar at the $\frac{1}{7}$ mark

Loving you a bushel and a peck

Volumes and weights take on some historically interesting values when working with equivalences. Do you buy your apples by the bushel, or is a peck all you need?



- 1 quart = 2 pints = 4 cups
- 1 gallon = 4 quarts = 32 gills
- 1 bushel = 4 pecks
- 1 pound = 16 ounces
- 1 ton = 2,000 pounds = 20 hundredweights

The Problem: According to a recent census, the typical American ate 13.8 pounds of turkey in a recent year. If this represents the total amount of turkey eaten at 12 different meals, then how many ounces of turkey were consumed at each meal?



Change the number of pounds to ounces using $1 \text{ pound} = 16 \text{ ounces}$ by multiplying 16×13.8 . Then divide the total number of ounces by 12. The product of $16 \times 13.8 = 220.8$ ounces. Divide $220.8 \div 12$ and you get 18.4 ounces per meal. That's over a pound of turkey at a sitting!

The next problem involves a phenomenon of agriculture and requires that fruit growers adopt a good balance in their orchards. Consider an orchard where a certain number of apples are produced by each tree in an average year. If the number of trees in the orchard is increased, there'll be more trees producing apples, but the crowding causes a reduction in the number of apples per tree. The balancing act for growers amounts to adding enough trees to increase production but not too many to decrease the production per tree by too much.

The Problem: An orchard contains 240 apple trees, which produce, on average, 2 bushels of apples per year. If the orchard manager increases the number of trees in the orchard by 60, she calculates that the amount of apples will be reduced by 1 peck per tree. If she goes ahead and plants those 60 additional trees, will the total crop be greater or smaller than the crop obtained from the original 240 trees?



First, determine how many bushels of apples are produced by 240 apple trees by multiplying $240 \times 2 = 480$ bushels. Increasing the number of trees by 60 results in 300 apple trees. The production of 300 trees will be 2 bushels less 1 peck. Change pecks to bushels with $1 \text{ bushel} = 4 \text{ pecks}$, giving you that 1 bushel is $\frac{1}{4}$ peck. Subtract $\frac{1}{4}$ bushel from 2 bushels, and the yield per tree will be $1\frac{3}{4}$ bushels. Multiply the number of trees by the new yield, and you get a total of $300 \times 1\frac{3}{4} = 300 \times 1.75 = 525$ bushels. There's an increase of $525 - 480 = 45$ bushels of apples.

Mixing It Up with Measures

Most of the problems in this book use the English measures of length, volume, and weight. But metric measures are very important to know, because of the great incidence of foreign travel and trade with other countries that use metrics.

Matching metric with metric

The metric measurement system is extremely easy to use, because all the units and equivalents are powers of 10. A kilogram is 1,000 times as big as a gram, and a centimeter is 0.01 as big as a meter. The multiplication and division problems using metric measures are really a piece of cake. When you learn what the different prefixes stand for, you can navigate your way through the metric measurement system.



In the metric system, *kilo* means 1,000 times as much, *hecto* means 100 times as much, *deca* means 10 times as much, *deci* means $\frac{1}{10}$ as much, *centi* means $\frac{1}{100}$ as much, and *milli* means $\frac{1}{1,000}$ as much.

The Problem: Stephen is the track manager at a race-car competition that's 16 kilometers long. If he wants to put a spotter every 25 meters for the length of the race, then how many spotters will he need?



Change kilometers to meters using $1 \text{ kilometer} = 1,000 \text{ meters}$ and then divide the total number of meters by 25. The 16-kilometer race is $16 \times 1,000 = 16,000$ meters long. Divide $16,000 \div 25$ to get 640 spotters. That's a lot of people!

The Problem: Stephanie works for a candy company and got permission to produce a piece of licorice that's 12 meters long. She's going to take the licorice to a party (just for the effect) and then divide it into individual pieces that are each 3 centimeters long. How many pieces of licorice will she have?



A centimeter is $\frac{1}{100}$ of a meter, so there are 100 centimeters in a meter. Multiply the 12 meters by 100, and you get 1,200 centimeters of licorice. Divide that total by 3, and Stephanie will have 400 pieces of candy.

Changing from metric to English

You've decided to go to Europe and you want to be sure that you order the right size beverage, know how far you'll be traveling by car, and dress appropriately for the weather on any particular day. All these functions relate to changing from English units of measure to metric measure. Here are some of the more useful conversion equivalences you'll need for your travels. For help with the temperatures, refer to Chapter 10 for conversions from Celsius to Fahrenheit and back again.



$$1 \text{ meter} = 39.37 \text{ inches}$$

$$1 \text{ kilometer} = 0.621 \text{ mile}$$

$$1 \text{ liter} = 1.057 \text{ quarts}$$

$$1 \text{ kilogram} = 2.205 \text{ pounds}$$

The Problem: You're in Europe and about to take a day trip with a rented car and trusty map. You're going to drive from your hotel to a famous cathedral. According to the map, the distance is 500 kilometers. How far is that in miles?



Use the equivalence 1 kilometer = 0.621 mile and multiply each side of the equation by 500. You get that 500 kilometers = 310.5 miles. That's a pretty long trip, depending on what kinds of roads you're going to find. You may want to check on an overnight stop.

The Problem: You're driving along and notice that you'll be needing fuel very soon. You spot a service station and pull over to buy fuel. The price on the sign is \$2.25. You gulp after you realize that the price is for 1 liter of fuel. What is the price per gallon?



First, use the equivalence 1 liter = 1.057 quarts in a proportion with the price of 1 liter = \$2.25 to determine how much the fuel costs per quart. Then you multiply that price by 4 because 1 gallon = 4 quarts.

$$\begin{aligned}\frac{1 \text{ liter}}{\$2.25} &= \frac{1.057 \text{ quarts}}{x \text{ dollars}} \\ \frac{1}{2.25} &= \frac{1.057}{x} \\ x &= 2.25(1.057) = 2.37825\end{aligned}$$

The fuel is about \$2.38 per quart. Multiply by four, and $2.37825 \times 4 = 9.513$ or about \$9.51 per gallon. And you thought gas prices were bad in the United States!

Changing from English to metric

You're on your European tour and you've brought some fabric samples to make curtains for your hostess and a recipe so you can cook up a thank-you dinner. Now you have the challenge of converting some of your measures into the measures of the country you're visiting.



$$1 \text{ yard} = 0.9144 \text{ meter}$$

$$1 \text{ pound} = 0.454 \text{ kilogram}$$

$$1 \text{ cup} = 0.2365 \text{ liter}$$

Part of the challenge of cooking in another country is trying to find the ingredients that you're used to working with at home. The other challenge comes when you need to measure those ingredients.

The Problem: Your recipe for lasagna calls for a 16-ounce jar (2 cups) of tomato sauce. You find a can of tomato puree (which you'll have to spice up a bit), and the can contains $\frac{3}{4}$ liter of puree. How many cans of the puree will you need to buy?



Create a proportion using $1 \text{ cup} = 0.2365 \text{ liter}$ and $2 \text{ cups} = x \text{ liters}$. Solve for x , which will be the amount of tomato sauce you need in terms of liters. Then compare that amount to $\frac{3}{4}$ or 0.75 liter.

$$\begin{aligned} \frac{1 \text{ cup}}{2 \text{ cups}} &= \frac{0.2365 \text{ liter}}{x \text{ liters}} \\ \frac{1}{2} &= \frac{0.2365}{x} \\ x &= 2(0.2365) = 0.473 \end{aligned}$$

The can contains 0.473 liter of tomato sauce. You need 0.75 liter. You'll have to buy 2 cans, giving you 0.946 liter, and just save the extra for the next project. Good luck with the measuring part!

You've had way too good of a time on your trip to Europe. You've been avoiding getting on the scale to see if you've gained any weight, but you finally decide, the day before leaving for home, to get on the scale to see what the damage is. Omigosh! You've lost weight! You've lost a *lot* of weight! Then you realize that the scale is in kilograms.

The Problem: You weigh yourself on a metric scale and it says 68. How many pounds do you really weigh?



Use the equivalence $1 \text{ kilogram} = 2.205 \text{ pounds}$ and multiply each side of the equation by 68. You get that $68 \text{ kilograms} = 149.9$ or 150 pounds.

Chapter 4

Stepping through the Problem

In This Chapter

- ▶ Going step-by-step through a problem using suggested techniques
- ▶ Using both sketch and table to illustrate the problem
- ▶ Solving a problem using Pythagoras's theorem and a quadratic equation

Solving a math word problem may seem daunting at first, but it doesn't have to be if you have a plan and the proper mindset. You don't have to use all the steps and procedures in this chapter for every math problem, but here you see how different techniques are useful when attempting to solve a problem.

In this chapter, I tell you what to look for in terms of the question and information needed. I fill you in on all the steps used to solve an equation — and the proper order to do them in. And I show you the need for checking your answer at the end.

Laying Out the Steps to a Solution

A math word problem presents challenges in understanding, organization, and launching the mathematical problem to be solved. To illustrate all these steps (and more), consider a problem involving two friends and their walking adventure. They both leave the same place at the same time; one walks north and the other walks east. One walks faster than the other. And, for some reason known only to them, they can determine how far apart they are after a period of time.

The Problem: Shelly and Shirley leave their dorm at 8 a.m. and start walking in different directions. Shelly starts walking due north, and Shirley walks due east. Shirley takes her time to smell the flowers and Shelly walks at a pretty steady pace. So, at noon, when they stop walking, Shelly has gone 1 mile less than twice as far as Shirley. At noon, they are 17 miles apart. How far did Shelly walk?



How old was the wife?

At the time a man and woman married, the man determined that his wife's age was $\frac{3}{4}$ of his age. After 12 years of marriage, he found that his

wife's age was now $\frac{5}{4}$ of his age. How old was his wife when they were married?

Answer: The wife was 18 years old. If she was $\frac{3}{4}$ his age, then he was 24 and she was 18. After 12 years of marriage, she was 36 and he was 36, and her age was $\frac{5}{4}$ his age.



Words, words, words. Somewhere in the description of the problem, you find the question that needs to be answered, information that needs to be organized, other information that can be ignored, an appropriate drawing that can be made, and an algebra problem begging to be written. In the following sections, I walk you through the various steps — steps you can apply to any math word problem.

Step 1: Determine the question

The main thing that empowers you when you're attempting a math word problem is determining the question. You wade through all that information and wonder what in the world you're going to do with everything. So, find the question, and you'll have more direction. The question is usually at the end of the problem description.

In this problem, the question is, indeed, at the end. *How far did Shelly walk?* The only unit mentioned is *miles* — because they're 17 miles apart — so your answer should be something like: "Shelly walked ____ miles."

Step 2: Organize the information

After you've determined what the question is (see the preceding section), you can go back to the problem and decide what information is needed and in what order you want to do the various steps. If you have an answer guess or goal in mind, it'll help you determine how the steps get arranged.

Eliminating the unneeded

The problem mentions the distance that Shelly and Shirley walked and the distance they're apart. Also, in the problem description you see times listed. You don't need the time they started and finished, because there's no mention of rates — requiring you to use $d = rt$. So just eliminate or ignore the mention of the times. It's just fluff. It also doesn't matter why Shirley walked slower; it could have been that her legs are shorter.

Shelly and Shirley leave their dorm at 8 a.m. and start walking in different directions. Shelly starts walking due north, and Shirley walks due east. Shirley takes her time to smell the flowers, and Shelly walks at a pretty steady pace. So, at noon, when they stop walking, Shelly has gone 1 mile less than twice as far as Shirley. At noon, they are 17 miles apart. How far did Shelly walk?

Making an educated guess

The problem involves two people walking in different directions and ending up 17 miles apart. If they walked in opposite directions — one north and one south — you'd expect the sum of the distances they walked to be 17. Neither could have walked more than 17 miles. In this case, they walked at right angles from one another, so the sum of the distances that they walked has to be less than 17. Keeping in mind a *reasonable* answer, you're more likely to set up the process (equations and operations) properly. So some possible answers are that Shelly walked 10 miles and Shirley walked 5 miles. The numbers don't really need to fit or be the right answer. You just have a relationship in mind — that Shelly walked farther and the sum of their distances is less than 17.

Getting organized

The question asks for the distance that Shelly walked. So you want some algebraic expression involving that distance. You could let the distance be represented by x . But look back at the problem to see what other distances are involved. You have the number 17 representing the distance between them, so you don't need a symbol for that distance. The only other distance is the distance that Shirley walked. You could let the distance that Shirley walked be represented by y , but then you'd have two different variables — x and y — to worry about. You want to keep to one variable, if possible.

The last bit of information that hasn't been used is that *Shelly has gone 1 mile less than twice as far as Shirley*. The two distances are compared to one another. Letting the distance that Shirley traveled be represented by y , you can let Shelly's distance be written in terms of y . Shelly went $2y - 1$ miles: twice Shirley's less 1. Now you have y for Shirley's distance and $2y - 1$ for Shelly's distance.

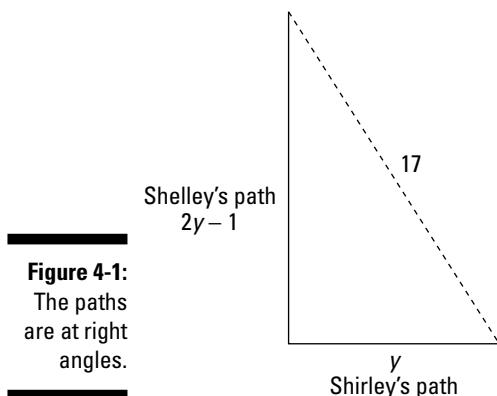
Step 3: Draw a picture or make a chart

This problem just begs to have a picture drawn. But a chart or table may be helpful, too. In the following sections, I show you both options and let you decide which works best.

Providing some artwork

Shelly walks north and Shirley walks east. You have to assume that it's possible for them to stay exactly on track and walk due north and due east, respectively, and not have to veer off. Math problems are usually about optimal situations, not the reality of how roads are laid out.

Figure 4-1 shows you a sketch of the paths taken by Shelly and Shirley. The representations of their distances — y for Shirley and $2y - 1$ for Shelley — are shown, as is their distance apart.



The picture of their paths seems to suggest a right triangle. And a right triangle comes with that most famous formula, the Pythagorean theorem. If you hadn't thought of that theorem before you saw the picture, you probably did after seeing it.

Figuring out possible values with a table

This problem has Shelly traveling one less than twice as far as Shirley. You could make a table of possible values for the distances they traveled. You'd probably be interested only in whole-number values, which won't solve the problem if the answer is a fraction, but you may get fairly close to the answer. Table 4-1 has some possible numbers or distances, starting with Shirley going 1 mile and ending to keep the total distance from getting larger than 17.



Counting cars

On a warm, sunny day, Clark went to the car races to watch an automobile race. As the cars sped around the track and Clark tried to watch them, he got dizzy. So he decided to keep his eyes on one particular car — the bright green one.

Clark then decided to count how many cars were in the race. He noticed that the total number of cars was equal to one-third of the cars in front of the green car plus three-quarters of the cars behind the green car plus the green car. How many cars were in the race?

Answer: The total number of cars was 13. In the problem, you're given information about one-third and three-quarters of the cars in front of and behind the green car. This total has to be divisible by 12, so the number of cars must be 12, 24, 36, and so on. Using the 12, you get a winner. Add the green car, and you get 13.

Table 4-1

Shirley and Shelly's Walk

| Miles Shirley Walked (y) | Miles Shelly Walked ($2y - 1$) | Total Distance They Walked |
|--|--|---------------------------------------|
| 1 | 1 | 2 |
| 2 | 3 | 5 |
| 3 | 5 | 8 |
| 4 | 7 | 11 |
| 5 | 9 | 14 |
| 6 | 11 | 17 |

This table doesn't really get you any closer to the answer, but it shows you how the distances are related. You know that the total distance has to be less than 17, so several of the entries in the table give possible solutions to the problem.

Step 4: Align the units

Many problems come with two or more different units. You may have information on time in terms of minutes and seconds, information on distance in terms of feet and yards, or some combination of all these. You can't really

change minutes and feet into the same unit, but feet and yards can be changed to feet, and minutes and seconds can be changed into minutes (or seconds). (Turn to Chapter 3 for a full coverage of how to deal with units.)

In this problem, because the only unit mentioned (after the time was eliminated) is miles, you just leave the unit as is. You're finished in terms of the units.

Step 5: Set up the operations or tasks

This is where all the previous steps should be coming together. You have the question in mind and a picture of what's happening. You have an estimate or guess of how the answer should come out. You have variable expressions representing the distances traveled. Now you need to set up a process or equation. Sometimes the process is no more than multiplying a number by 2 or 3. That's the best-case scenario. But it's much more fun when you set up an equation to be solved.

The previous work on this problem suggests an equation. You have two different distances represented by expressions involving a y . Write an equation and solve for y . What equation? Why Pythagoras's, of course!

In a right triangle whose shorter sides are a and b in length and whose longest side is c long, the following is always true: $a^2 + b^2 = c^2$.

The distances that Shirley and Shelly walked are the a and b of a right triangle. The distance that they're apart, 17 miles, is the c value. Substituting into the Pythagorean theorem, the equation $a^2 + b^2 = c^2$ becomes $(y)^2 + (2y - 1)^2 = 17^2$.



Finding a concrete solution

Mike had just finished pouring and smoothing out a new concrete sidewalk when he noticed that someone had written in the numbers 777 111 999 in the walk. He should have been angry or disgusted, but instead, he made a puzzle

of the numbers. His puzzle: *Cross out six of these nine numbers, leaving three, such that the sum of the numbers remaining is 20.* How did he get 20?

ANSWER: Mike crossed out the three 7s, the first 1, and two of the 9s. That left him with 11 + 9 = 20.

Solving the Problem

A math word problem is different from other arithmetic and algebra problems, because you first have to translate from the words to the symbols before you do the operations or solve the equation for the answer.

Step 6: Perform the operations or solving the equation

The problem involving Shelly and Shirley boils down to an equation to solve. The equation is quadratic — requiring either factoring (coupled with the multiplication property of zero) or the quadratic formula. Some operations are performed on the equation first — squaring the binomial and simplifying terms, and the result of the simplifying is a quadratic equation set equal to 0. The equation $(y)^2 + (2y - 1)^2 = 17^2$ can actually be solved by factoring, but I'll show it both that way and with the quadratic formula, because the factoring may not be entirely obvious to you.

Solving the equation using factoring

The equation $(y)^2 + (2y - 1)^2 = 17^2$ is quadratic. You first square each of the terms, including the binomial, and then simplify the terms by combining what you can. Then move all the terms to the left to set it equal to 0.

$$\begin{aligned}(y)^2 + (2y - 1)^2 &= 17^2 \\ y^2 - 4y^2 - 4y + 1 &= 289 \\ 5y^2 - 4y - 288 &= 0\end{aligned}$$

The factorization of this quadratic is the product of two binomials. The first terms in the binomials have to be $5y$ and y . There's no other choice. It's the second numbers that will be the challenge. You have to find two numbers whose product is 288 — that's challenge enough. But then you have to figure out how to arrange the factors so that the difference between the outer and inner products is $4y$.

$$\begin{aligned}5y^2 - 4y - 288 &= 0 \\ (5y \quad)(y \quad) &= 0\end{aligned}$$

The two factors that work are 36 and 8. Their product is 288, and, when you multiply the 8 by 5 and the 36 by 1 you get 40 and 36, respectively. The difference between the two products is 4. The product $40y$ has to be negative for

the difference $4y$ to come out negative. So the factor 8 is negative and the factor 36 is positive in the factorization.

$$(5y + 36)(y - 8) = 0$$

Setting the two factors equal to 0, you get $y = -\frac{36}{5}$ or $y = 8$. The negative fraction doesn't make any sense if this is supposed to represent distance, so you go with the solution $y = 8$, only.

Solving the equation using the quadratic formula

Not all quadratic equations can be solved by factoring. And sometimes those that can be solved by factoring are more easily solved using the quadratic formula. On the other hand, *all* quadratic equations can be solved using the formula. It's just that factoring is usually quicker, easier, and more accurate (not as many opportunities for error).



The quadratic formula says that if a quadratic equation is written in the form $ax^2 + bx + c = 0$, then its solutions are found with $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Solving the equation $5y^2 - 4y - 288 = 0$ with the quadratic formula, you get

$$\begin{aligned} y &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(5)(-288)}}{2(5)} \\ &= \frac{4 \pm \sqrt{16 + 5,760}}{10} \\ &= \frac{4 \pm \sqrt{5,776}}{10} \\ &= \frac{4 \pm 76}{10} = \frac{80}{10} \text{ or } \frac{-72}{10} \end{aligned}$$

This gives you the same two answers that you get with the factoring method: 8 and $-\frac{36}{5}$.

Step 7: Answer the question

After working hard at solving the quadratic equation, it's tempting to just sit back, relax, and think that your work is done. Not so. The solution of the equation that you think will work is that $y = 8$. What is y ? Does its value answer the question?

First, y represents the distance that Shirley walked. The question asks how far Shelly walked, so the number 8 is not the answer to the question. To determine how far Shelly walked, use $2y - 1$ and replace the y with 8. You get that $2(8) - 1 = 16 - 1 = 15$. So Shelly walked 15 miles.

Step 8: Check for accuracy and common sense

Accuracy and common sense go hand in hand. You can't have one without the other. The accuracy part goes to the arithmetic or algebra involved in solving the problem and whether the relationships hold. The common sense refers to whether the answer — even though it's the solution of an equation — really fits the problem and the real world.

In the case of Shirley and Shelly who walked away from each other in directions that are right angles from one another, you find that Shirley walked 8 miles and Shelly walked 15 miles. They're only 17 miles apart — even though they walked a total of 23 miles — because their journeys and the distance between them formed a right triangle. They didn't walk in opposite directions. As far as the accuracy of the solution, check out the Pythagorean theorem with the distances:

$$\begin{aligned} 8^2 + 15^2 &= 17^2 \\ 64 + 225 &= 289 \\ 289 &= 289 \end{aligned}$$

The equation checks, the answer makes sense, and the problem is solved with the question answered.

Part II

Taking Charge of the Math

The 5th Wave

By Rich Tennant

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"Okay – let's play the statistical probabilities of this situation. There are 4 of us and 1 of him. Phillip will probably start screaming, Nora will probably faint, you'll probably yell at me for leaving the truck open, and there's a good probability I'll run like a weenie if he comes toward us."

In this part . . .

Change from percents to decimals, and you have a problem that can be solved using arithmetic processes. The problems that I introduce in this part all involve basic arithmetic principles — no big surprises. Use the formulas. Count the money. Combine the ingredients. Measure the temperatures. The formulas and math are here.

Chapter 5

Deciding On the Operation

In This Chapter

- ▶ Deciding between addition and multiplication
 - ▶ Determining whether it's a difference or ratio
 - ▶ Mixing it up with the different operations
-

If you're like most people, you're leery of word problems because the tasks involved in solving them aren't always immediately apparent. The task or operation doesn't cooperate by standing up and shouting, "Here I am!" Sure, everyone is more comfortable with a simple addition problem or maybe a combination multiply-then-subtract problem. Give me a + or × any day, and I'm in hog heaven. (Well, that may be stretching it a bit.) The challenge of doing a word problem comes when you have to decide which operations to use and in what order.

In this chapter, you get some down-to-earth tips — so you can change a wordy description or situation into a standard arithmetic problem. Here I show you how to spot the clues that lie along the path toward a word problem's solution.

Does It All Add Up?

The easiest arithmetic operation you encounter is addition. The first things that kids master in school are addition rules and addition tables. So it's always comforting to find a word problem that involves the operation of addition. A big clue that you're probably dealing with an addition problem is the word *and*. (See Chapter 1 for more on the mathematical meanings of everyday words.)

Determining when the sum is needed

The word *and* indicates that you want a sum, but you can't assume that everything on one side of an *and* and everything on the other side of an *and* gets added up. For example, if you're talking about Jim and Jon going to a

hockey game, you don't add Jim and Jon together. You can add their ages or their weights or the number of hamburgers that they eat, but you don't add people or colors. You just add numbers.

The Problem: Michael and Owen went shopping. Michael bought new jeans that cost \$49.95 *and* a T-shirt that cost \$12.50. What was the total cost of his purchase?



The last sentence tells you that you need a total; the word *and* between the prices of the clothing suggests that you add. So, finding the sum of \$49.95 + \$12.50, you get \$62.45. (**Note:** This sum doesn't include any sales tax; I cover finding and adding in sales tax in Chapter 6.)

Adding up two or more

You probably remember your second-grade teacher (mine was Mrs. Dopke — haven't thought about her in years) and those splendid columns of numbers that you added for days and days and days. For example, you got to find the sum of 299 *and* 401 *and* 650 *and* 850 *and* 76 *and* 24. Notice the repeated use of the word *and*? That's your signal to add and add and add.

As you do the addition problem, you want to take advantage of any grouping of numbers that makes the sum more convenient. Notice in the listing in the preceding paragraph that the 99 in 299 and the 1 in 401 add up to an even 100. The 50s in the 650 and 850 add up to an even 100. Nice combinations like these don't happen all the time, but you can take advantage of the nice sums if you catch them in time.

The Problem: Liam and Cassidy spent the day picking strawberries — a hot and backbreaking job, but someone has to do it. Every time they filled a flat with fresh fruit, they took it back to the weighing shed and picked up a new, empty box to fill. In all, they filled 12 flats of strawberries weighing: 9.3 pounds, 10.4 pounds, 8.7 pounds, 11.2 pounds, 9.5 pounds, 10.5 pounds, 11.0 pounds, 8.9 pounds, 9.8 pounds, 8.6 pounds, 10.1 pounds, and 9.1 pounds. What is the total weight of all the strawberries that they picked that day?



It's probably pretty obvious that you want to add all the numbers together. Multiplying 12 numbers at a time or in a row doesn't sound like a good idea — or like a lot of fun. The *and* tells you to add, and the last digits in the weights are just begging for you to take advantage of grouping numbers together before adding. Specifically, group the numbers whose decimal values add up to 10. (Actually, they add up to 1: 0.3 and 0.7 = 1.0 or 1.) Rewriting the 12 numbers in groups of nicely-combining numbers, you get: (9.3 + 8.7) and (10.4 + 8.6) and (11.2 + 9.8) and (9.5 + 10.5) and (8.9 + 10.1) and (11.0 + 9.1). Actually, the



An old counting trick

Old Mother Hubbard had more problems than a bare cupboard. She also had to figure out where to put all her children when they went to bed. She had 11 children to put in 10 beds, and she had promised them that each would have his or her own bed. She accomplished this great feat by putting two children in the first bed — with the promise that the second would have a bed to himself later, after all the others were settled in.

Then she put the third child in the second bed, the fourth child in the third bed, the fifth in the fourth bed, the sixth in the fifth bed, the seventh in the sixth bed, the eighth in the seventh bed, the ninth in the eighth bed, and the tenth in the ninth bed. She then put the eleventh child, who was temporarily located in the first bed, in the bed that was left. What an amazing woman! How did she do it?

ANSWER: Poor old Mother Hubbard must have been extra tired, not extra clever. She assumed that the eleventh child was the extra child in the first bed. She actually had ten children in the first nine beds, with the eleventh child still left standing. She had counted the second child and the eleventh child as being the same.

last two don't have decimals that add up to 1, they're just left over, and it seemed nice to let them form a group, too. Adding up the numbers in the parentheses and then summing the sums, you get $18.0 + 19.0 + 21.0 + 20.0 + 19.0 + 20.1 = 37.0 + 41.0 + 39.1 = 117.1$ pounds.

Did you notice how I grouped the groups to make the addition easier? I wish Mrs. Dopke had shown me that trick earlier, to avoid all the pain and agony of those huge columns of numbers.

What's the Difference — When You Subtract?

Subtraction is an operation that's just a little harder than addition. Subtraction is still a basic operation, though, and shouldn't pose too much of a problem for you. The main challenge in subtraction is getting the order of the numbers right. Do you subtract 90 minus 47 or 47 minus 90? It's pretty obvious, when doing word problems about practical matters, such as weights and money, which number you subtract from which — it's more in the algebra word problems, where you can end up with negative numbers, that you have to be especially careful about the order of subtraction.



How many apples?

If I took 12 apples from a pile of 23 apples, how many apples would I have?

Answer: Is your answer 11 apples? Wrong! If I took 12 apples from the pile, then I have 12 apples!

Deciphering the subtraction lingo

Here are some words that tell you to use the operation of subtraction: *less*, *lower*, *fewer*, *difference*, *minus*, and, of course, *subtract*. Someone can weigh 10 pounds *less* than another person (fie on them). Your colleague can make a *lower* salary than you (but you deserve it). To make comparisons of things such as weights or money or temperatures, you subtract.

The Problem: Avery lives 47 miles from Grandma's house, and Reid lives 63 miles from Grandma's house. How many fewer miles from Grandma does Avery live than Reid?



The problem suggests subtraction, because there's a comparison of numbers. You subtract 47 miles from 63 miles to get: $63 - 47 = 16$ miles, and the answer is that Avery lives 16 miles *fewer* from Grandma than Reid.

Actually, the question could have been: "How many *more* miles away does Reid live from Grandma than Avery?" Notice that the word *fewer* changed to *more* and that the names got reversed, too. It's sort of like negating things twice. The answer is the same, it's just stated differently.



When combining signed (positive and negative) numbers in algebra, two negatives or reverses can make a positive. For instance, the opposite of negative 4, written $-(-4)$, is equal to positive 4.

Subtracting for the answer

Sometimes a clue or key word isn't really apparent in a word problem; it may be there and you don't recognize it, or it just may not be there at all. You have to determine which operation or operations are needed without much help. In the case of problems that finally end up using subtraction, you'll find that the situation has to do with finding the difference between two values — how much bigger one is than the other.



Using four 4s

Can you create the numbers 1, 2, 3, 4, and 5 using four 4s and the operations of addition, subtraction, multiplication, and division, only?

Answer: To write the number 1, use $4 \div 4 = 1$. To write the number 2, how about $4 \times 4 \div (4 + 4) = 2$? And then $3 = (4 + 4 + 4) \div 4$. The number 4 should be the easiest, but it isn't. Here's the trick: $4 + 4 - (4 - 4) = 4$. And the number 5 is found with $((4 \times 4) + 4) \div 4$. These are the only ways to write the number 5 through 5 using four 4s. See if you can find other ways to do the task.



The Problem: Ruby is taking a mathematics class where 4 tests, 10 quizzes, and 20 homework assignments determine her grade. She has one test left to take and needs 920 points to get an A in the course. She currently has 847 points, and the test is worth 100 points. How many points does she have to get on the test to get an A? Is it possible for her to get an A?

This problem is an example of a situation where you have to wade through all the information to figure out what's needed to answer the question. The fact that there are 4 tests, 10 quizzes, and 20 homework assignments has nothing to do with the question. Also, there aren't any clue words such as *difference* or *less*. To answer the question, Ruby needs to subtract 847 from 920 to see how many points she needs. Then she can compare that answer to the number of points possible. Doing the subtraction, $920 - 847 = 73$. It looks like, if Ruby gets 73 percent of the points on the test, she'll get her A.

How Many Times Do I Have to Tell You?

Multiplication is really just repeated addition. If you're adding the same number to itself over and over again, then you're really doing multiplication, and the multiplication facts are very helpful. Some of the key or clue words for multiplication are: *times*, *multiplied*, *twice*, and *thrice*. (Okay, so nobody says "thrice" anymore, but if they did, it would be a clue that multiplication was involved.)

Doing multiplication instead of repeated addition

Addition is a comfortable operation, but it can get tedious after a while, if you have to repeat the same task over and over again. Word problems involving

multiplication may be spelled out pretty clearly, or they may be masked with some math jargon such as *thrice* or *quadruple*.

Dealing with clear-cut multiplications

You already know that you aren't interested in repeated additions when multiplication is so much easier to handle. The multiplication problems can be fairly simple or bordering on the challenging. No matter what the situation, you can handle it.

The Problem: Sydney has a new job and wants to draw up a timetable for every day of the week so she can get everything done at home as well as at her job. She wants to know, "How many hours are in the month of January?"



With 24 hours in a day, and 31 days in the month of January, Sidney can add 24 plus 24 plus 24 a total of 31 times. Of course, it makes much more sense to multiply 31×24 to get 744 hours.

The Problem: Have you ever been amazed that your heart keeps beating and beating without your thinking about it or doing anything to help? Just how hard does your heart work? An average adult's heart beats 72 times each minute; a child's heart beats faster than that — more like 90 times each minute. Use the heart rate of an average adult — 72 times each minute — to answer the question: How many times does the heart of an average adult beat in one day?



You want to multiply to get this answer, and there are actually two different multiplication problems to deal with. First, you need to know how many minutes there are in one day; you determine that by multiplying 60 minutes times 24 hours. Then you can multiply the number of minutes times the number of heart beats. For the number of minutes in one day, $60 \times 24 = 1,440$ minutes. Multiplying that by 72, you get $1,440 \times 72 = 103,680$ beats of the heart. That's just in one day!

The problem involving minutes and hours and heart beats involved several ratios of units. There are minutes per hour, hours per day, and beats per minute. And your answer comes out to be beats per day. How does the answer work out to be in the correct number of units? Look at the preceding problem, written as fractions multiplied together.

$$\begin{aligned} & \frac{60 \text{ min.}}{\text{hour}} \times \frac{24 \text{ hours}}{\text{day}} \times \frac{72 \text{ beats}}{\text{min.}} \\ &= \frac{60 \text{ min.}}{\cancel{\text{hour}}} \times \frac{24 \cancel{\text{hours}}}{\text{day}} \times \frac{72 \text{ beats}}{\cancel{\text{min.}}} \\ &= \frac{60 \times 24 \times 72}{\text{day}} = 103,680 \text{ beats/day} \end{aligned}$$

Notice how the minutes and hours cancel out, leaving beats and day.

Doubling or tripling your pleasure

The special math words that deal with multiplying something a number of times are somewhat recognizable. Here are the more commonly used multipliers:

- ✓ **Double:** Two times
- ✓ **Twice:** Two times
- ✓ **Triple:** Three times
- ✓ **Thrice:** Three times, in Shakespeare's day
- ✓ **Quadruple:** Four times
- ✓ **Quintuple:** Five times

The Problem: Juan was told that he had to quadruple the amount of his sales in the next six months in order to win that trip to Tahiti. His sales are currently at \$230,000, so what level of sales does he have to reach in the next six months to get that vacation?



The word *quadruple* means to multiply by 4, so $\$230,000 \times 4 = \$920,000$. Aw, Juan can do it!

Taking charge of the number of times

Multiplying numbers can result in quite large results. This isn't a bad thing — especially if you're talking about your bank account. It's just that large numbers can get unwieldy — to write and to deal with in a calculator. Here are two options for handling large numbers: Write the numbers in scientific notation, or change the units.

Writing numbers using scientific notation

Scientific notation was developed to allow scientists, mathematicians, and people like you and me to write really, really large numbers or teeny, tiny small numbers without using up a full page to express the numbers for the reader.



The format for a number written in scientific notation is a number between 1 and 10 times a power of 10: $n \times 10^p$. For example, the number 230,000,000,000,000,000,000,000 is written as 2.3×10^{26} in scientific notation. The number 2.3 is between 0 and 10, and it multiplies a power of 10. See how much shorter the number in scientific notation is than the number written out the long way? Also, comparing two large numbers is easier when they're written in scientific notation. You look at the power of 10, first, and then compare the multiplier.



Rolling out the barrel

A man wants to fill a barrel exactly half full of oil, but there are no markings on the barrel to tell where the halfway mark is, and there are no measuring instruments to do any measurements.

The man is very clever, though (he's read this book), and he is able to fill the barrel exactly halfway. How did he do it?

ANSWER: As the oil flowed into the barrel, the men had to move the barrel around the bottom of the barrel. The oil was then exactly half the volume of the barrel. The barrel just reached the lip of the barrel and, at the same time, just covered the side with some help — it got pretty heavy! He watched until the oil in the barrel just reached the lip of the barrel and, at the same time, just covered the bottom of the barrel. The oil was then exactly half the volume of the barrel.

To change a number into scientific notation, you move the decimal point from the right end of the number until there's just one digit to the left of the decimal point. With just one digit in front of the decimal point, you automatically have a number between 1 and 10. The power that you put on the 10 is the number of places that you had to move the decimal point. Moving the decimal point to the left gives you a positive exponent, and moving the decimal point to the right requires a negative exponent.



Many scientific calculators write scientific notation using the letter *E* instead of a power of 10. On your calculator screen, you'll see 2.3E26 instead of 2.3×10^{26} or 1.23E-39 instead of 1.23×10^{-39} . This isn't a big problem — you just want to be aware of this cryptic notation so that you know what it means and can write it correctly.

Changing units to make numbers smaller

Keep in mind how units are related when working with a large number — that happens to be of one type of unit. For example, because there are 5,280 feet in a mile, you can say 45 miles instead of 23,760 feet. With 16 ounces in a pound and 2,000 pounds in a ton, you can say 9 tons instead of 288,000 ounces.

The Problem: Don has 150 drafting rulers, each of which is 18 inches long. If he lays the rulers end to end, how long a line of rulers can he form?



Multiply 150×18 to get 2,700 inches. That's a perfectly good answer. But Don can also report that the line is 225 feet or 75 yards. How do you get the other units? There are 12 inches in a foot and 36 inches in a yard. So you can divide 2,700 either by 12 or by 36 to get the feet or yards, respectively.

Another way to look at the problem is to change the 18 inches to feet or yards first, before you multiply by 150. Eighteen inches is 1.5 feet. Multiplying 1.5×150 , you get 225 feet. Eighteen inches is also half a yard. Multiplying $\frac{1}{2} \times 150$ you get 75 yards. The answers are the same, of course. You just have to choose how you want to deal with the numbers — and when to change them to smaller units.

Dividing and Conquering

Division is usually the last of the four basic operations that kids study in school. Why? Because many of the results are not whole numbers. When you add, subtract, or multiply whole numbers together, you always get a whole number as a result (or an integer — in the case of subtracting a larger number from a smaller number). Not so with division. Not every division problem comes out evenly, and dealing with a remainder can be a bit unsettling or even daunting.

Using division instead of subtraction

Just as multiplication is used instead of repeated addition, you can say that division is used instead of repeated subtraction. For example, if Keisha wants to hand out 4 pieces of candy to each of her friends and she has 38 pieces of candy, she can give 4 pieces to the first friend and 4 pieces to the second friend and so on until 9 friends have 4 pieces of candy each — and Keisha has 2 left over. Of course, you would do the division problem $38 \div 4 = 9$ with a remainder of 2. The remainder can also be written as a fraction ($\frac{2}{4}$, which can be simplified to $\frac{1}{2}$).

The Problem: Clara ordered a barrel of M&M's online and was billed for 11,000 ounces of candy. How many pounds of M&M's did Clara order?



The world's smallest computer

This morning, I bought a word processor small enough to fit in my pocket. It can write in any language that I want and use any alphabet I need. It can also add, subtract, multiply, and

divide. It has a delete capability that will correct all errors. No battery or power outlet is needed. And, believe it or not, it only cost me a quarter. How is this possible?

Answer: My word processor is a pencil.



One pound is equivalent to 16 ounces, so divide 11,000 by 16. This doesn't come out even: $11,000 \div 16 = 687$ with a remainder of 8. Clara ordered $687\frac{1}{2}$ (or $687\frac{1}{2}$) pounds of M&M's.

Making use of pesky remainders

When a division problem doesn't come out even — when the number you're dividing into isn't a multiple of the number you're dividing by — you have several options:

- ✓ You can report the remainder as a number, which is the actual value of the remainder.
- ✓ You can report the remainder as a fraction.
- ✓ You can round the number up to the next higher number.
- ✓ You can just lop off the remainder and ignore it.

The following problems give examples of when and how to use the different options.

Reporting the amount remaining

You report the amount of a remainder if the amount left over is to be treated in a different way — that is, if it doesn't figure into the answer to the problem. An example involves a doughnut shop and a worker putting a certain number of doughnuts into boxes.

The Problem: Pete works at a doughnut shop and is packing the morning's supply of doughnuts into boxes containing one dozen doughnuts each. The owner of the doughnut shop tells Pete that he can keep any that are left after filling as many boxes as possible. Pete started with 4,000 doughnuts. How many doughnuts does he get to keep?



Pete has to put one dozen doughnuts in each box. It takes 12 donuts to make a dozen. Divide 4,000 by 12 and determine the remainder — if there is one: $4,000 \div 12 = 333$ with 4 left over. The owner of the shop has 333 boxes of donuts to sell, and Pete gets the last 4. This is a case where it didn't make sense to change the remainder to a fraction or decimal. The last four doughnuts weren't going to be sold — but the number of doughnuts remaining was pretty important to Pete!

Creating a fraction or decimal of the remainder

When you're dividing one number by another, you often have the situation where the division doesn't come out even. This happens when the number

being divided isn't a multiple of the number doing the dividing. The remainder may or may not have importance. Examples of cases where the remainder *does* have importance are when vendors sell fractions of pounds of candy or meat or some other product. In these cases, the vendors include the remainder in the computations — they don't want to lose even a small fraction of the amount.

The Problem: Chuck, the candy-store owner, has a large box of jelly beans that weighs 480 pounds. He wants to divide this large box into 100 smaller boxes and sell each of the smaller packages for \$2.50 per pound. How much does each of the smaller boxes weigh?



The 480 pounds of candy divided into 100 smaller units doesn't come out to a whole number of pounds of candy in smaller boxes. Chuck changes the remainder into a fraction and uses it in determining the cost of each small box. To answer the question, you don't need to know the cost per pound — that's just extra information needed by Chuck. Just do the division: $480 \div 100 = 4$ with a remainder of 80. Change the 80 to a fraction or decimal, and the result of the division is $4 + \frac{80}{100} = 4\frac{4}{5}$ or 4.8 pounds per box.

Rounding up — numbers, not cattle

Rounding the answer of a division problem to a whole number results in a new answer that's an approximation of the real, exact answer. Rounding answers is practical and makes sense in many situations.



The standard rule for rounding numbers is to round *down* when the fractional or decimal part is less than half (when the remainder is less than half of the divisor), round *up* when the fractional part is more than half, and round to the closer even number when the fractional part is exactly half.

Some businesses can't (or don't choose to) use the standard rule of rounding. They round all fractions up to the next higher whole number. For example, with some phone companies, you pay for a whole number of minutes, and any fraction of a minute is considered to be the whole thing.

The Problem: Stefanie made some homemade jelly and wants to divide it among ten of her best friends. She has a total of 64 ounces of jelly. She has some jars that hold 6 ounces of jelly, other jars that hold 7 ounces of jelly, and still other jars that hold 8 ounces of jelly. What size jars will she use if she wants to distribute all the jelly evenly, and not have any left over?



If Stefanie has 64 ounces of jelly and divides that among ten friends, then you divide $64 \div 10 = 6.4$ ounces per friend. You can't force 6.4 ounces of jelly into a 6 ounce jar, so you round up to 7 and leave a little room at the top of each of those jars. The 8-ounce jars are overkill. Rounding up does have its practical applications.

Pocketing the difference

The story goes (and it may just be one of those urban legends) that an enterprising bank employee figured out how to program the bank's computers so that any fractions of cents that occurred when the interest was computed on a savings account got deposited in his own personal account. So, if a person was supposed to earn \$37.43255 in interest, the \$0.00255 was

opped off the interest amount and placed in the employee's account. This fraction of a dollar doesn't seem like much, but, if you add up the fractions of dollars from every account, the amount grows and grows. Again, according to the story, he got away with it for a few years. But, as happens to those who go astray, he finally got caught.

Rounding down or truncating the remainder

Most scientific calculators round up or down automatically when an answer has more decimal places than are available in the display. Back in the good old days of early, handheld calculators, you were more apt to see long decimal answers being *truncated* — meaning that the extra decimal values were just dropped off with no consideration given for rounding one way or the other.

The Problem: Ebenezer announced to his employees that he would be doing some profit-sharing with them. The nine employees would each get one-ninth of the \$75 profit each month — and be paid in silver dollars. So the employees each got \$8 per month, for a total of \$96 for the year. How could Ebenezer have been just a bit more fair to his employees?



First, dividing 75 by 9, you get 8 with a remainder of 3. In decimals, this is $8.\bar{3}$. (That line over the 3 indicates that the number repeats forever.) It looks like Ebenezer just lopped off the remainder and kept it himself — because he paid in silver dollars and no other coins, he couldn't give his employees the fraction of the total. But if Ebenezer really wanted to be fair, he would multiply the 75 times the 12 months of the year to get 900. Dividing that by 9, each employee would get 100 dollars for the whole year instead of 96 dollars (8×12). He could give the extra as a holiday bonus.

Mixing Up the Operations

Most of the interesting problems — the ones that you find in real life — are made up of words describing a situation or question that is solved using one or more operations. The *best* problems are those that involve more than one operation. Okay, that's from my perspective, but I'm going to try to win you over to my side with some scintillating examples.

When more than one operation is involved, you need to be cautious about which operations to use and in what order to use them. Making decisions about the operations and the order requires a mixture of good old common sense and a willingness to tackle a problem, slightly seasoned with some good old math rules.

Doing the operations in the correct order

One challenge of dealing with two or more operations is determining the order in which you do these operations. Sometimes the path or order is clear cut. Sometimes the problem is loaded with pitfalls or opportunities to flub. You need to keep in mind the *order of operations* from algebra when doing multiple operations in story problems.



The *order of operations* says that, when you have more than one operation to do, first perform any powers or roots, then do any multiplication or division, and end by doing any addition or subtraction. The order of operations can be interrupted, though, by parentheses or brackets or other grouping symbols.

Here are some examples of using the order of operations:

$$4 + 12 \div 3 - 1 = 7$$

First you do the division, $12 \div 3 = 4$. Then you have the problem $4 + 4 - 1$. Do these operations in order, moving from left to right: $8 - 1 = 7$.

$$(6^2 - 10) \div 13 + 5 = 7$$

First you have to simplify the terms in the parentheses. Square the 6 to get 36. Subtract 10 from 36 to get 26. The problem now reads: $26 \div 13 + 5$. Do the division and then add. $2 + 5 = 7$.

This is just a quick review of the order of operations. If you need more of an explanation, refer to my book *Algebra For Dummies* (Wiley), where I give a more thorough explanation.

Now I'll show you how the order of operations works in a practical situation. Consider a problem where you have to figure out the cost of lunch, and see how it works.

The Problem: You have a buy-one-get-one-free coupon from a local restaurant. The coupon is good for the cheaper of the two lunches. The amount of that meal is deducted after the tax is added. You and a friend go to lunch. You have the \$6.95 chef salad, and she has the \$6.50 fruit plate (you're both eating so healthy). The tax comes to \$1.08, and you want to leave a tip of \$2.80. You're going to split the total cost of the lunch equally, so what will lunch cost you?



Whew! It's going to take several operations to solve this problem. You have to *add* up all the costs, *subtract* the amount for the coupon, and then *divide* the resulting amount by 2. You'll use parentheses and brackets so that the correct amount is divided by 2. Here's what the problem looks like using the operations and grouping symbols:

$$[(6.95 + 6.50 + 1.08 + 2.80) - 6.50] \div 2$$

Even though parentheses are around all the addition, you can drop the parentheses and include the subtraction of 6.50 with that group. You want to do this, because you get a 0 by cleverly adding and subtracting the same number ($+ 6.50 - 6.50$). The new version of the problem is:

$$[6.95 + 1.08 + 2.80] \div 2 = [10.83] \div 2 = 5.415$$

You can't split a penny in half, so your friend very generously offers to pay the extra penny because you brought the coupon. You pay \$5.41 and she pays \$5.42. (It doesn't make sense to round up or down with that $\frac{1}{2}$ of a cent, because you'd be paying too little or too much if you both paid the same amount after rounding.)

Determining which of the many operations to use

Using the operations of addition, subtraction, multiplication, and division isn't all that taxing. It's deciding *when* to use *which* operation — and how often. You pick up on the word clues and try out something that makes sense. Check to see if your answer is what you expected or is something very surprising. Surprising isn't bad, but it should make you go back to see if you've made some sort of calculation error or if it's just your estimate that was wrong.



Operating on the digits

I'm thinking of a number in which the second digit is smaller than the first digit by 4. When you

divide the number by the sum of the digits, the quotient is 7. What is the number I'm thinking of?

ANSWER To do this, think of all the two-digit numbers where the second digit is smaller than the first by 4. Those numbers are 51, 62, 73, 84, and 95. You use subtraction to figure out the first digit: 5, 6, 7, 8, and 9. You sums of the digits in those numbers: 6, 8, 10, 12, and 14, respectively. Only one of the two-digit numbers is evenly divisible by the sum of its digits — the number 84 divided by 12 is 7. Now you know that my number is 84.

The following examples incorporate several operations in each, for your perusal. The first problem uses multiplication and division.

The Problem: Ted can use a disposable razor six times before it gets too dull to do a good job. He shaves twice on Saturday — to look his very best on Saturday night. He buys the razors in packs of 10. Each pack of razors costs \$4.95. How much does Ted spend on razors every year?



First, you want to wade through all the information to get to the pertinent facts. And, for a problem like this, make an estimate in your head. If he buys a pack of razors every month (which is more than he needs) and spends \$5 per pack (rounding up to make the math easier), then that's $12 \times \$5 = \60 . You know that the estimate is high, but it gives you a ballpark figure to aim at. Now you need to determine how many *shaves* Ted has per year, divide that number by 6 to get the number of razors, divide that number by 10 to get the number of packs, and multiply the result by \$4.95. Yes, you could divide the number of shaves by 60 instead of doing two separate divisions, but I want to show every step and not make big leaps.

Start with the number of shaves. You take the number of weeks in a year, 52, and multiply that by 8. (Why 8? Because Ted shaves twice on Saturday, and you add 1 to the number of days in a week.) So $52 \times 8 = 416$ shaves per year. Divide the number of shaves by 6 (the number of shaves per razor), and you get $416 \div 6 = 69$ and a remainder of 2, which is the number of razors needed. How do you handle the remainder in this case? (If you need help, see “Making use of pesky remainders,” earlier in this chapter.) Because this is a number of razors needed, and you can’t break apart a razor, you choose to round up to 70 razors. Now divide 70 by 10, the number of razors in a pack. Ted needs $70 \div 10 = 7$ packs. At \$4.95 per pack, that’s $\$4.95 \times 7 = \34.65 per year for razors. That’s much cheaper than going to the barber every day — and it’s much lower than the estimate of \$60. This doesn’t necessarily fit in the *surprising* category. The estimate in these types of problems are especially useful if you make an error in the decimal point. For instance, if you came out with an answer of \$346.50, you’d know that something was very wrong with your answer.

The next problem uses addition — with a tad of multiplication — and subtraction. You need to be careful about the grouping — keeping the additions separate before subtracting.

The Problem: Twins Jake and Jill are always competing with each another. They’re in the same math class and want to compare their quiz scores to see who has done the best on quizzes. Jake’s scores are 19, 18, 17, 16, 17, 18, 19, 20, 20, and 20; Jill’s scores are 20, 20, 20, 20, 14, 13, 14, 15, 16, 20. Who has the higher total, and by how much?



Tricky arithmetic

Divide 30 by $\frac{1}{2}$ and add 10. What is the answer?

Answer: Did you get 25? Gotcha! If that was your answer, then you divided 30 by $\frac{1}{2}$ instead of $\frac{1}{2}$. When you divide 30 by $\frac{1}{2}$, you get 60. Adding 10 gives you a final answer of 70.



You first want to total each person's quizzes and take advantage of repeated scores to cut down on the number of computations. By grouping and multiplying, Jake's scores look like the following:

$$([2 \times 19] + [2 \times 18] + [2 \times 17] + 16 + [3 \times 20]) = (38 + 36 + 34 + 16 + 60) = 184$$

Now, grouping and multiplying Jill's scores you get:

$$([5 \times 20] + [2 \times 14] + 13 + 15 + 16) = (100 + 28 + 13 + 15 + 16) = 172$$

It looks like Jake has the better total. You subtract $184 - 172 = 12$, which means that Jake's score is better than Jill's by 12 points.

This last problem uses addition, multiplication and division. Again, you have to group the correct values and then do the division at the end.

The Problem: Five friends — Adam, Ben, Charlie, Duncan, and Eduardo — have decided to work together on a project and then split up the proceeds evenly. A neighbor has offered to pay them \$6.50 per hour to paint the fence around his pasture. (If you've never painted a fence — with all the surfaces and nooks and crannies — you've never lived. Oh, yeah.) The boys were to come when they could and report the number of hours that they spent painting. They'd all be paid at the end. Adam worked for $8\frac{1}{2}$ hours, Ben worked for 6 hours, Charlie worked for $9\frac{1}{4}$ hours, Duncan worked for 7 hours, and Eduardo worked for 9 hours. How much did each boy make?



First, you need to total all the hours by adding them up. Then multiply the number of hours by \$6.50. Last, divide the total amount paid by 5 to get each person's share. Adding, $8\frac{1}{2} + 6 + 9\frac{1}{4} + 7 + 9 = 39\frac{3}{4}$ hours. Then multiply $39\frac{3}{4} \times \$6.50$ and you get \$258.375. Now, dividing the total by 5, you get $\$258.375 \div 5 = \51.675 , which rounds up to \$51.68. That seemed like good money to the boys, but I think that the neighbor made out pretty well, too.

Chapter 6

Improving Your Percentages

In This Chapter

- ▶ Switching from fractions to decimals to percents and back again
 - ▶ Investigating both the practical and impractical with percents
 - ▶ Using percentages to your advantage — in your best interest
-

Decimals and percents are really just fractions — in a more manageable format. Doing problems that involve percents of things involves changing the percents to decimals and then doing the indicated operations. That's not a big deal, if you handle the decimals correctly. And the computations are much easier than with fractions, which can have very uncooperative denominators.

In this chapter, you see how to figure percent increase and percent decrease and determine whether what you see advertised is a good deal. Everyone is affected by interest on money — whether you're borrowing or saving — so I include problems dealing with computing interest as well.

Relating Fractions, Decimals, and Percents

The usual move from fractions to percents is through decimals — the decimal format is the middleman in the process. You probably already know some of the more common equivalences of percents and fractions. You know that 50 percent is equivalent to $\frac{1}{2}$ and 25 percent is equivalent to $\frac{1}{4}$. Well, I'm assuming that you know this, but, just in case, here are some properties and techniques that you can use to make the transitions easier. You don't have to memorize these properties, but having some of them in mind as you're working on percentage problems is helpful.

Changing from fractions to decimals to percents

A decimal is a fraction and vice versa. To change a fraction to a percent, you first determine the decimal value and then fiddle with the decimal point. That's all there's to it — really.



To write a fraction as a decimal, you divide the numerator by the denominator, inserting the decimal point where needed. To change a decimal to a percent, you move the decimal point two places to the right and use a percent sign (%).

For instance, changing the fraction $\frac{7}{16}$ to a percent, you first find the decimal by dividing the numerator, 7, by the denominator, 16. The decimal that you get is 0.4375. Here's what the division looks like:

$$\begin{array}{r} 0.4375 \\ 16 \overline{)7.0000} \\ 64 \\ \hline 60 \\ 48 \\ \hline 120 \\ 112 \\ \hline 80 \\ 80 \\ \hline \end{array}$$

To change the decimal to a percent, you move the decimal point two places to the right to get 43.75 percent. This makes sense, because 7 is not quite half of 16, and 43.75 percent is just short of half of 100 percent.

Finding terminating decimal values

Fractions all have decimal values, but some of these decimals *terminate* (come to an end) and some *repeat* (never end). As long as the denominator of the fraction is the product of 2s and 5s and nothing else, then the decimal equivalent of the fraction will terminate. To find this terminating decimal, you divide the denominator (bottom) of the fraction into the numerator (top) and keep dividing until there's no remainder. You may have to keep adding 0s in the divisor for a while, but the division will end.

For example, to find the decimal equivalent for the fraction $\frac{14}{25}$, you divide 25 into 14. To begin the division, you put a decimal point to the right of the 4 and add a 0 to the right of the decimal point. You keep adding 0s as needed in the division. This is what I get when I do the division.

$$\begin{array}{r} 0.56 \\ 25 \overline{)14.00} \\ \underline{125} \\ 150 \\ \underline{150} \\ 0 \end{array}$$

So $\frac{14}{25} = 0.56$. As you see, this decimal *terminates*. The division stopped because I eventually ended up with no remainder. You can predict that this will happen (the termination of the decimal), because the denominator, the 25, is equal to 5 times 5. The only factors of 25 are 5s.

Next, you see the division required to find the decimal equivalent of the fraction $\frac{9}{160}$. This decimal will terminate, too, because the denominator is equal to $2 \times 2 \times 2 \times 2 \times 2 \times 5$. Only 2s and 5s are factors of the denominator. It doesn't matter what the numerator is; the decimal will terminate. Finding the decimal equivalent:

$$\begin{array}{r} 0.050625 \\ 160 \overline{)9.000000} \\ \underline{800} \\ 100 \\ \underline{0} \\ 1000 \\ \underline{960} \\ 400 \\ \underline{320} \\ 800 \\ \underline{800} \\ 0 \end{array}$$

It took a while, but you can now say that $\frac{9}{160} = 0.050625$.

Computing decimals that keep repeating themselves

Terminating decimals are just dandy, but they're in no way the only type of decimal value out there. Repeating decimals occur when you change a fraction to a decimal and the denominator of the fraction has some factor other than 2 or 5. It only takes one such factor to create the repeating situation. For

example, the fraction $\frac{5}{12}$ repeats when you divide the numerator by the denominator, because the denominator has a factor of 3.

$$\begin{array}{r} 0.4166\dots \\ 12 \overline{)5.0000} \\ 48 \\ \hline 20 \\ 12 \\ \hline 80 \\ 72 \\ \hline 80 \\ 72 \\ \hline 8 \end{array}$$

As you can see, the remainder will now forever be 8, and the corresponding number in the quotient (answer of a division problem) will be 6. The three dots following the last 6 shown indicates that the 6 keeps repeating forever and ever.



To indicate a repeating decimal, you can either write three dots (ellipsis) after some repeated digits, or you can draw a horizontal bar across the digits that repeat. For instance, 0.41666 . . . can also be written $0.\overline{416}$. Note that the bar is over the 6 only. The first two digits don't repeat.

When you're using repeating decimals in a problem, you decide how many decimal places you want and then round the number to that *approximate* value. Rounded decimals aren't exactly the same value as the repeating decimals, but you can make them pretty accurate in an application by using enough digits in the decimal equivalent.

Making the switch from fractions to percents

The middle step in changing a fraction to a percent is finding the decimal equivalent (or, in the case of a repeating decimal, the approximate). Table 6-1 shows you some fractions, their decimal value, and then the percent that you get by moving the decimal point two places to the right.

Table 6-1 Fraction, Decimal, and Percent Equivalences

| <i>Fraction</i> | <i>Decimal</i> | <i>Percent</i> |
|------------------------|-----------------------|-----------------------|
| $\frac{3}{4}$ | 0.75 | 75 |
| $\frac{9}{16}$ | 0.5625 | 56.25 |
| $\frac{1}{3,125}$ | 0.00032 | 0.032 |

| Fraction | Decimal | Percent |
|-----------------|--|----------------|
| $\frac{9}{160}$ | 0.050625 | 5.0625 |
| $\frac{1}{3}$ | 0.333 . . . (0.3333 when rounded to 4 places) | 33.33 |
| $\frac{47}{66}$ | 0.71212 . . . (0.7121 when rounded to 4 places) | 71.21 |
| $\frac{5}{12}$ | 0.41666 . . . (0.4167 when rounded to 4 places) | 41.67 |
| $\frac{1}{999}$ | 0.001001 . . . (0.0010 when rounded to 4 places) | 0.1 |

Changing from percents back to fractions

Percents are very descriptive — they tell you how many you have out of 100 things. The only problem with percents is that you can't use the percent format when doing computations. You need to change a percent back to a decimal or a fraction, if you want to multiply or divide using the percent.



To change a percent to a decimal number, move the decimal place in the percent two places to the left, adding 0s if necessary. To change that decimal number to a fraction, write the digits in the decimal over a power of 10 that has as many zeros in it as there are digits in the decimal — then reduce the fraction if you can.

For example, 45 percent has a decimal value of 0.45, and the fraction equivalent is $\frac{45}{100} = \frac{9}{20}$. Another example is 0.032 percent, which has a decimal value of 0.00032 and a fraction of $\frac{32}{100,000} = \frac{1}{3125}$.

Tackling Basic Percentage Problems

The nice thing about percents is that their values are easy to relate to. If you're 75 percent finished with a project, then you know that you're well on your way. You compare the percentage to 100 — a nice, round number — and have a good idea of what the value is in the comparison. To be more exact with an answer, though, you need to convert percentages to decimals and create a more exact value to use in computations. Using the decimal equivalents, you can solve for the percent of a value and get the answer in items, and you can also solve for how many items are needed to reach a certain percent.

Finding the percent amount

When you're told that you have 60 percent of the work done or 85 percent of the problems correct, you multiply the total number of hours needed to do the work or the total number of problems on the test by the percent to get the numerical value of what you're discussing. Percents are convenient amounts for comparison. You convert percents to decimals to use them in problems.

The Problem: You sign up for Weight Watchers, and you're told that you need to lose 10 percent of your current weight. If you weigh 160 pounds, how much do you need to lose?



Changing 10 percent to a decimal, you get 0.10. Multiply 160×0.10 and you get 16 pounds. That should be a piece of cake. Oops! Not on Weight Watchers — make that a carrot stick.

The Problem: You're told that 95 out of 100 of the people who buy a Honda motorcycle will buy another Honda when they need to buy another motorcycle. Last year, the number of Honda motorcycles sold in North America was 570,000. How many of these owners will buy a Honda when they make their next motorcycle purchase?



First, change the fraction $\frac{95}{100}$ to a decimal. The value of 95 out of 100 is 95 percent, but you can't use the percentage in a computation — you want the decimal value. This fraction is equal to 0.95, so $570,000 \times 0.95 = 541,500$ people who will buy a Honda motorcycle the next time they make a purchase.

The Problem: To get an A in your math class, you need to have an average of 92 percent of all the points available. You currently have 540 out of a possible 600 points, and you still have the 200-point final to take. What do you need to score on the final to get an A?



This problem involves several operations: addition, multiplication, and subtraction. First, determine the total number of points needed. Add $600 + 200$ for a total of 800 points. You need 92 percent of 800 points to get an A, so multiply $0.92 \times 800 = 736$ points. You currently have 540 points. Subtract $736 - 540 = 196$. You need a score of 196 out of 200 points, which is $\frac{196}{200} = \frac{98}{100} = 0.98 = 98$ percent. You have your work cut out for you.

Finding the whole when given the percent

Working backward to find out where you get a certain percent amount requires division instead of multiplication. This process of using division makes sense, because multiplication and division are inverse operations — one undoes the other.

The Problem: Forty-five percent of the class is boys. And you're told that the class has 18 boys in it. How many students are in the class?



Falling golf balls

Two large barrels contain the same amount of water. The water temperature in the first barrel is 49°F, and the temperature in the other is 29°F. Two identical golf balls, positioned at the same

height above the two barrels, are dropped at exactly the same time. Which ball will touch the bottom of the barrel first?

Answer: Did you catch the temperature in the second barrel? Yes, the water is frozen. The golf ball in the second barrel won't touch bottom until spring, when the ice melts. So the golf ball dropped into the barrel with 49°F water will reach the bottom first.



Divide 18 by 45 percent — $18 \div 0.45 = 40$. There are 40 students in the class. You don't believe this works so easily? Then check the work by finding 45 percent of 40 to see if you get 18 boys. Multiplying $0.45 \times 40 = 18$ boys. By golly, it works!

The Problem: In a large bag of Skittles, 20 percent of the candy is colored yellow and 18 percent is colored red. If you counted 100 yellow candies, then how many are red?



First, determine how many pieces of candy there are altogether, and then determine what 18 percent of that number is to find out how many are red. You know that 20 percent of the candy corresponds to 100 pieces, so divide 100 by 20 percent — $100 \div 0.20 = 500$ pieces of candy. Now multiply 500 times 18 percent, or $500 \times 0.18 = 90$ pieces of candy that are red in color.

Looking At Percent Increase and Percent Decrease

You've been drawn to a store when it advertises "All Prices Slashed 20 Percent" or "Take 15 Percent Off the Reduced Price." Who wouldn't be tempted when you're offered such deals? And what about that meeting with the boss when she says that you need to increase productivity by 25 percent? Where does that put you as far as output? Can you do it? Percent decrease and percent increase are both based on changing the amount from 100 percent, or the full amount. You use the difference from 100 percent to help you when doing the problems.

Decreasing by percents

Figuring a percent decrease — or the new value of an item after the decrease is applied — just takes a deep breath and a little common sense. You can get messed up with the arithmetic if you don't think about what the answer should be ahead of time. In general, you multiply the total amount by the percent to get the decrease in the amount. You can then subtract that decrease from the original amount to get the new result. Another way of finding the resulting amount is to subtract the percent decrease from 100 percent and multiply this difference times the original amount.



To determine the net result or amount after applying a percent decrease, you use one of the following methods (either one works):

- ✓ Total amount \times percent decrease = decrease in amount
Total amount – decrease in amount = net result
- ✓ 100 percent – percent decrease = decreased percent
Total amount \times decreased percent = net result

The Problem: A local store is going out of business and has advertised that all items are 60 percent off the original price. You buy a toaster oven that's currently marked \$49.95. What will you pay for the toaster oven after the store applies the discount?



Using the first method, earlier, you first multiply $\$49.95 \times 0.60 = \29.97 . This is the amount of the decrease. Next, subtract $\$49.95 - \$29.97 = \$19.98$. Wow! Such a deal!

Now, using the second method, you subtract 100 percent – 60 percent = 40 percent. The item will cost 40 percent of the original cost. Taking $\$49.95 \times 0.40 = \19.98 . You get the same answer, of course.

You're pretty happy with your purchase of the toaster oven until, the next day, you see that the same store is offering to reduce the *previously reduced price* by another 20 percent. Does that mean that the new reduction is 80 percent, or is the reduction 20 percent of the previous 60 percent? Is there even a difference? Oh, yes, there is.

The Problem: What is the difference between an 80 percent decrease and a 60 percent decrease followed by a 20 percent decrease? For the sake of comparison, use an item that has an original cost of \$100.



To see if there is a difference at all, do the computations two different ways: Find the price after an 80 percent decrease. Then go back to the original price and figure a 60 percent decrease followed by a 20 percent decrease.

Determining the cost of the item if there's an 80 percent decrease in the price, you subtract 100 percent – 80 percent, giving you a final cost of 20 percent of \$100. Multiplying $100 \times 0.20 = \$20$. That's the cost of the item with a straight 80 percent decrease.

Now, find the cost when there's first a 60 percent decrease in cost followed by a 20 percent decrease in that result — 100 percent – 60 percent = 40 percent. Multiplying $100 \times 0.40 = \$40$. A 20 percent decrease becomes 100 percent – 20 percent = 80 percent. And, multiplying, $\$40 \times 0.80 = \32 . That's quite a bit different from the \$20 when figured the other way.

Not that I'm beating up on this poor store that's going out of business, but here's another scenario to consider when dealing with percent decreases. Consider a less-than-scrupulous manager who advertises that prices are going to be decreased by 60 percent off the original amount. What you don't know is that he changes all the prices the night before the big sale so that the 60 percent decreases result in the cost of the items all being the same as they were the previous day. How does he do the necessary math?

The Problem: A store advertises a 60 percent decrease in cost on all items. What does the price tag have to read so that an item costing \$100 originally will still cost \$100 after the 60 percent decrease?



Go back to the same process of subtracting the 60 percent from 100 percent to get a net cost of 40 percent. Then divide instead of multiplying to get the required price. Dividing $\$100 \div 0.40 = \250 . The price needed on the item is \$250. When you apply a 60 percent decrease to \$250, you get back to the original price of \$100. Somehow, I think that the shoppers just may notice the price hike.



Matching socks

You have 40 socks in a drawer in your bedroom. The power is out, though, and you can't see what color the socks are. You know that 35 percent of the socks are black and 65 percent of the socks

are blue. How many socks do you have to take out of the drawer to be absolutely sure that you have a matching pair of socks? You don't care which color, as long as they match.

ANSWER: You need to take just three socks out of the drawer. At least two of them have to be the same color. Did the percentages throw you off? I was hoping so.

Making the discount count

You're going to take advantage of the end-of-year discounts being offered at the local hardware store. You get a 25 percent discount on all items, but you have to figure in the 8 percent sales tax.

The Problem: You have purchases totaling \$75.45 and get to the checkout counter. Your coupon says 25 percent off, but there's an 8 percent sales tax, so the clerk tells you that she'll just subtract the 8 percent from the 25 percent, leaving a 17 percent discount, which means that you pay 83 percent of the total before all this magic arithmetic. Is this right?



You sense a shell game going on. The numbers are flashing through your head like those visions of sugar plums. Take a deep breath, and do the computations yourself. First, find the reduced cost, and then add the sales tax. The discount is 25 percent off, so you'll only be paying 75 percent of the original price. (Refer to "Decreasing by Percents," earlier in this chapter, for more on figuring the discounted price.) Computing 75 percent of \$75.45, you get $0.75 \times \$75.45 = \56.5875 or \$56.59. Now add on the 8 percent tax by multiplying \$56.59 times 108 percent. (This is the same as finding 8 percent and adding that amount on to the price. See "Determining an increase with percents," later in this chapter, for more information on this process.) The computation is $56.59 \times 1.08 = \$61.12$. The tax was 8 percent of \$56.59 or \$4.53.

How does this compare to the clerk's suggestion of just giving you a 17 percent discount? Computing 83 percent of \$75.45, you multiply $0.83 \times \$75.45 = \62.6235 . You would pay \$62.62, which is \$1.50 more than you should.

Determining an increase with percents

A percent increase may involve your goal in productivity or the amount of rainfall one summer or the amount that a price is increased due to sales tax. In general, to find the new amount after a percent increase, you either determine the increase in the amount that results from the percent increase and add it to the original, or you add the percent increase to 100 percent and multiply by the new percentage.



To determine the net result or amount after applying a percent increase, you use one of the following methods:

- ✓ Total amount × percent increase = increase in amount
Total amount + increase in amount = net result
- ✓ 100 percent + percent increase = increased percent
Total amount × increased percent = net result

The Problem: A sheep shearer figures that he can improve upon his average per day shearing of 100 sheep. He's set a goal of increasing the number of sheep by 15 percent. How many sheep per day will he have to shear to make that goal?



Using the first method, he first determines that 15 percent of 100 is $0.15 \times 100 = 15$. Add 15 to the usual 100 sheep, and he sees that he needs to shear 115 sheep per day.

Using the second method, if the shearer adds 15 percent + 100 percent, he gets 115 percent. Multiply 100×1.15 , and he gets 115 sheep.



Percentages of more than 100 percent have decimal equivalents that are greater than 1. Be careful when moving the decimal point. For instance, 250 percent = 2.5, 800 percent = 8 and 1,000 percent = 10.

Problems involving sales tax are pretty much just percent increase problems. They get even more interesting when you figure in both a percent decrease because of a sale price and also sales tax.

The Problem: You purchase new shoes that were advertised as being 25 percent off the original price. You look at the sales receipt and don't agree with the total price. You suspect that the sales tax was computed on the original price of the shoes, not the sale price. The shoes were originally \$120, and the sales tax is $8\frac{1}{4}$ percent. The amount you're being asked to pay is \$97.43. Is this the correct amount?



First, figure out the new price due to the decrease. Then figure the increase in price due to the sales tax. For the decrease in price, multiply the original cost times 75 percent. If you need help in determining where the 75 percent came from, go back to "Decreasing By Percents," earlier in this chapter. After you get the new, lower cost of the shoes, multiply that amount by 108.25 percent. This percentage is the result of applying a percent increase and adding the sales tax percentage to 100 percent.

Doubling up on amoebas

A jar contains seven amoebas. These particular amoebas multiply so fast (split into two) that they double in volume every minute. If it takes

40 minutes for the amoebas to fill the jar (for it to be 100 percent full), how long did it take to fill half the jar (50 percent full)?

Answer: Did you say 20 minutes? Note: Higher; it would take 39 minutes for the amoebas to be half full and then, because they double in 1 minute, they'd fill the jar in the next minute.

The new price of the shoes is $\$120 \times 0.75 = \90 . Now, to figure the total cost with tax, take $\$90 \times 1.0825 = \97.425 . The amount on the bill is correct, when you round up to the higher penny. No error was made here. If the tax had been figured on the original amount, the total would have been $\$99.90$. I get this amount by figuring the tax on $\$120$ to be $\$120 \times 0.0825 = \9.90 and adding that tax to the reduced price of the shoes. It doesn't hurt to check — errors easily can be made when figuring discounted prices and taxes.

Tipping the Waitress without Tipping Your Hand

Did you know that the word *tip* is an acronym for *To Insure Promptness*? The word *tip* may not have started out to stand for those words, but it does seem to fit the situation. Waiters and waitresses are at your mercy when it comes to tipping them properly, so you want to be able to compute their payment with a minimum of hassle and struggle — and do it accurately. Sometimes the restaurant makes it easy for you and adds on a 15 percent tip. Also, if you charge the meal, you're given a nice slip of paper to do your addition on — if you know how much you want to tip. A bit of a problem arises when you use a discount coupon. All these things have to be taken into consideration — and I walk you through them all in this section.

Figuring the tip on your bill

Even when the service is questionable, most people leave a tip. The amount can serve to indicate to the waiter just what you think of the service. The most commonly used tip percentages are 10 percent, 15 percent, and 20 percent. The first and last are fairly easy to compute. But 15 percent, is a bit of a challenge.

Adding on 10 percent or 20 percent

To figure a tip of 10 percent — which means that you aren't particularly impressed with the waiter's work — all you need do is take the total and move the decimal point one place.

Multiplying by 10 percent is accomplished by moving the decimal point one place to the left. If you multiply 48×10 percent this is $48 \times 0.10 = 4.80$ or 4.8.



Figuring the tip in your head

When you're in a situation where you're going to give only a 10 percent tip (and I hope that the situation doesn't come up very often), you can do a quick, neat calculation in your head to find the total amount — the tip plus the charge for the meal. This little method works because it's a quick computation used to multiply any number by 11.

The method you use is to picture a 0 in front of the meal cost and add the adjacent digits together. If any of the pairs add up to more than

9, you'll have to carry 1 to the next sum. For example, if your bill is \$45.32, then picture in your mind 045.32. Start on the right end and add $2 + 3 = 5$, $3 + 5 = 8$, $5 + 4 = 9$ and $4 + 0 = 4$. Write these sums in reverse order, with the decimal point in the correct place, and you get 49.85 — the sum of the meal plus a 10 percent tip. This calculation is going to be off by a penny if the last digit in the meal cost is bigger than 5 — because of the rounding. But, for a quick computation, it works pretty well.

The Problem: How much tip do you leave if you're going to pay 10 percent on a bill of \$18.80? And what is the total payment after adding the tip?



Move the decimal in \$18.80 one place to the left to get \$1.88. Add \$18.80 + \$1.88 to get a total of \$20.68.

To figure a 20 percent tip, all you do is figure the 10 percent tip and double it. You can do this by either doubling the cost and then moving the decimal point, or you can figure the 10 percent amount by moving the decimal and then doubling that amount for the tip.

The Problem: How much tip will you give the hairdresser if the charge for a haircut and perm is \$85 and you want to give a 20 percent tip?



When you're moving the decimal point in 85 one place to the left, you have to remember that there *is* a decimal point in the number. The decimal point is always assumed to be at the far right of the number, if it isn't showing. So one place to the left gives you 8.5 — which is \$8.50. Double that to get \$17. Add that tip to the bill for a total of $\$85 + \$17 = \$102$.

Computing a 15 percent tip

Multiplying a number by 15 isn't the hardest thing to do, but it isn't quite as sweet as multiplying by 10 or 20. So it won't come as a shock to find that figuring a 15 percent tip is a bit more involved than figuring a 10 percent or 20 percent tip.

The simplest approach is just to multiply the amount by 15 percent and be done with it.

The Problem: How much do you tip the waiter if the bill is \$164, and you're going to tip 15 percent? What is the total bill after the tip is added?



Multiply $164 \times 15\% = 164 \times 0.15 = \24.60 . So, adding $\$164 + \24.60 , you get a total of \$188.60 for the bill.

If you like to figure out the tip in her head, there's a trick to computing the amount of the tip when you want to leave a tip of 15 percent. First figure the 10 percent tip by moving the decimal point one place to the left. Then take half of that tip and add it on to the whole tip. There's your 15 percent tip — 10 percent and half of 10 percent.

The Problem: You've gone to lunch with nine of your friends and you've all agreed to just split the whole bill equally — ten ways — and leave a 15 percent tip. You're in charge of collecting everyone's money and then settling up at the cash register. The charge for the ten lunches is \$188. How much will each person give you for her share?



You first have to figure the tip, add it on to the cost of the meal, and then divide the total by 10. You're going to do this in your head — and hope you can do it correctly, or you're going to get stuck with any shortfall. A 15 percent tip is 10 percent plus half of 10 percent, so it's $\$18.80 + \$9.40 = \$28.20$. Add the tip to the bill to get $\$188 + \$28.20 = \$216.20$. To divide that by 10, you just have to move the decimal point again and get that everyone's share is \$21.62. Of course, what are the chances that everyone will have the correct change? Oh, sure.

Taking into account the discount

Everyone just loves those buy-one-get-one-free coupons or the percentage off the total discounts. These promotions get you in the door and are good for everyone involved. Sometimes you have to pay tax on the amount before the discount, and sometimes you pay tax on the lesser amount. It depends on what the product is — and if the merchant knows how to figure it correctly. That's why you need to be aware of what's going on so you can check the computations.

The Problem: Your favorite restaurant is offering a free second entree, as long as it costs less than the entree you're paying for. This is how most of the buy-one-get-one-free promotions work. You'll pay tax on the reduced price, but you need to tip the waitress based on the total cost before the discount. Your entree costs \$19.95, and your friend's entree costs \$21.95. You've ordered beverages totaling \$16 and shared an appetizer that costs \$6.95. The tax (sales plus restaurant tax) comes to 10.5 percent, and you want to give a 20 percent tip. You and your friend will split the bill. How much will each of you pay?



You're going to pay tax on all the items except the less-pricey meal, so add up all the items — food and beverages — except the \$19.95 and compute the tax on that. You're going to figure the tip on all the items except the tax, so you'll need a different sum to do that computation. Last, you'll add up the cost of the items you're paying for, the tax, and the tip. Divide that total by 2, and you'll have the amount that each of you owes.

First, computing the tax, add $\$21.95 + \$16 + \$6.95 = \44.90 . The tax on that is $\$44.90 \times 10.5\text{ percent} = 44.90 \times 0.105 = \4.7145 , making the tax \$4.71.

Next, to compute the tip, add up the cost of all the items, $\$19.95 + \$21.95 + \$16 + \$6.95 = \$64.85$. Multiplying $\$64.85 \times 20\text{ percent} = 64.85 \times 0.20 = \12.97 .

Your total cost is the cost of the meals plus the tax plus the tip. Add $\$44.90 + \$4.71 + \$12.97 = \62.58 . Divide that by 2, and each of you owes \$31.29. Now, I know that in practice, most people would round the tip up to \$13, but that doesn't really change the amount by much.

KISS: Keeping It Simple, Silly — with Simple Interest

Simple interest is the interest computed when *compounding* doesn't occur. The interest in a savings account *compounds*, because, if you don't withdraw any of the money you've invested, your interest earns interest. With compound interest, the amount of interest earned is added to the account total, and then the new interest is figured on the new total. Simple interest is computed only on the beginning amount.



The formula for *simple interest* is $I = prt$, where I is the amount of interest earned, p is the principal or amount of money involved, r is the interest rate (a percent changed to a decimal for the computation), and t is the amount of time involved — usually a number of years.

Determining how much interest you've earned

Problems involving *interest* are two types: interest earned, and interest you have to pay. The interest earned is the more-fun type. You get to add money to your savings account without even working at it.

The Problem: How much simple interest is earned on \$10,000 if this money is deposited for 6 years in an account that earns $4\frac{1}{4}$ percent interest?



Using the formula for simple interest and replacing the letters with their corresponding values, you get $I = \$10,000 \times 0.0425 \times 6 = 2,550$. You earn \$2,550 in interest, so now the total in the account is \$12,550.

The Problem: How much simple interest is earned on \$10,000 if you have it in one account for $6\frac{1}{2}$ years at 4 percent interest and then move that money and the interest it's earned to another account earning 6 percent interest for another $3\frac{1}{2}$ years?



To determine the total amount of interest earned, you have to add the two different interest amounts. One amount comes from the interest earned at 4 percent, and the other amount comes from the interest earned at 6 percent.

Apply the simple-interest formula on \$10,000 at 4 percent for $6\frac{1}{2}$ years with $I = 10,000 \times 0.04 \times 6.5 = \$2,600$.

Next, apply the simple-interest formula on \$12,600 at 6 percent for $3\frac{1}{2}$ years. Why is the amount of money different? Because the total amount from the first $6\frac{1}{2}$ years, the principal plus interest, is all deposited in the new account. You get $I = \$12,600 \times 0.06 \times 3.5 = \$2,646$.

Now, to answer the question, “How much interest is earned?”, you add the two interest values together — $\$2,600 + \$2,646 = \$5,246$.

Figuring out how much you need to invest

The simple-interest formula, $I = prt$, gives you the amount that your money has earned for you over a particular period of time. You have a set number of dollars and invest it for a chosen number of years. Of course, the longer you invest the money, the more interest you'll earn. Another situation is that you may have a target amount of interest or a target total of money, and you need to know how much to invest right now to reach that target. For example, you may want to set up an account (usually called an *endowment*) where only the interest is spent each year while the amount in the account stays the same and is never withdrawn. Or you may want to have a particular total amount of money to buy a boat in ten years and need to make a deposit today so that the money in the account will grow to that target amount over the years.

Spending only the interest

A benefactor wants to donate money to a local charity but doesn't want the charity to spend it. How does this help the charity? The arrangement is that the charity gets to spend only the interest, every year, and the amount in the account stays there to earn interest the next year and the next and so on.



Inheriting a fortune

A sheikh had two sons who were equally likely to inherit his position and his fortune. He devised a contest to see which son would be his successor. In this contest, the two sons would race their camels to a distant city. The winner would be the one whose camel was *slower!*

The sons started the race and wandered about aimlessly for several days; neither wanted to be

the first to the city, because each wanted his camel to be the slower one. They happened upon a wise man and asked him for advice on how they could finish this contest. The wise man gave it his best. After hearing what he had to say, the two sons jumped on the camels and raced as fast as they could to the city. What did the wise man tell them?

Answer: The wise man told each son to ride the other son's camel.



The Problem: How much money must be invested in an account that earns 5 percent simple interest per year if the interest must come out to be \$6,000?

Use the simple-interest formula, $I = prt$, replacing the letters with the corresponding values in the problem. You'll be solving for p , the principal. The equation becomes: $\$6,000 = p \times 0.05 \times 1$ or $\$6,000 = 0.05p$. Divide each side of the equation by 0.05 to get $p = \$120,000$. The person donating the money needs to put \$120,000 in the account for there to be \$6,000 of spendable money (or interest) each year.

Aiming at a future purchase

You have your eye on a new powerboat — one that seats up to 12 people and moves fast enough to pull a skier. You figure that the boat you want will cost close to \$100,000 when you're ready to buy it. You're going to put away a lump sum of money today, and let it grow in value for ten years — at which time you'll take all the money and the interest to buy the boat.

The Problem: How much money do you have to deposit in an account earning 8 percent simple interest if you want to have a total of \$100,000 in principal and interest in ten years?



If you use the simple-interest formula, $I = prt$, you'll get varying amounts of interest, depending on what the principal is. What you want is for the principal, p , and the interest, prt , to have a total of \$100,000. Your equation would look like: $p + prt = \$100,000$. You can solve for the principal by dividing the \$100,000 by $1 + rt$. Just add 1 to the product of the rate times the time and divide that into the \$100,000. You'll be dividing \$100,000 by $1 + (0.08 \times 10)$, which is: $\$100,000 \div 1.8 = \$55,555.56$. Actually, the money grows even faster if you use compound interest, but this is pretty impressive just as it is here.

Working out the payments

Many people use credit cards to pay for large purchases, but some stores still offer convenient short-term payment plans on their merchandise. If you want to buy an item, the store figures out the total cost of the item plus the interest over a period of time, divides the total into equal payments, and then lets you purchase the item paying back the same amount for a certain number of months or years.

The Problem: You want to buy an all-leather sectional sofa that costs \$3,500. The store will let you pay for it over the next 36 months at 9 percent interest. How much will your monthly payment be if you're going to be paying for the sofa and three years' simple interest in equal monthly payments?



Figure out the interest on \$3,500 at 9 percent for 3 years. Add the interest to the cost of the sofa and divide the total by 36. The interest is $\$3,500 \times 0.09 \times 3 = \945 . Add $\$3,500 + \945 to get $\$4,445$. Divide $\$4,445 \div 36 = \$123.47222\ldots$. You can't just lop off the remainder. The $0.00222\ldots$ represents a remainder of 8 cents in the division, so you'll pay \$123.48 for 8 months and \$123.47 for the other 28 months.

How do you get the 8¢ remainder? You multiply \$123.47 by 36, and you get \$4444.92, which is 8¢ short of the total. Long division gives you a remainder of 8¢. Calculators give you the decimal. And, if you're so lucky, you have a graphing calculator that changes decimals into fractions automatically.

Chapter 7

Making Things Proportional

In This Chapter

- Using proportions to figure fair shares
- Working with proportions effectively
- Weighing all the choices

Proportions are nothing more than two ratios or fractions set equal to one another. Proportions have several very handy properties that make working with them much easier to manage. In this chapter, you see how to set up the proportions correctly and how to solve the problems you've created with the proportions. You see how to apply the properties of proportions to make the solutions easier.

Working with the Math of Proportions

A proportion is an equation involving two fractions. The ratio of the numerators and denominators of the fractions must be equal. A proportion is a statement saying that two fractions are equivalent or equal in value. The fractions can be reduced in the normal way — the way you've seen since third grade — and they can be reduced in some rather unique ways, too. You use this property and several others to solve for unknown parts of a proportion.



Given the proportion $\frac{a}{b} = \frac{c}{d}$, the following also are true:

$$a \times d = b \times c$$

The cross products are equal.

$$\frac{b}{a} = \frac{d}{c}$$

The reciprocals are equal.

You can reduce (eliminate common divisors):

$$\frac{a}{b} = \frac{e \times f}{e \times g}, \frac{a}{b} = \frac{\not{e} \times f}{\not{e} \times g}, \frac{a}{b} = \frac{f}{g}$$

You can reduce the fractions vertically.

$$\frac{a}{e \times f} = \frac{c}{e \times g}, \frac{a}{\not{e} \times f} = \frac{c}{\not{e} \times g}, \frac{a}{f} = \frac{c}{g}$$

You can reduce the fractions horizontally.

Solving proportions by multiplying or flipping

Equations involving proportions are solved using the properties of proportions. When you cross-multiply, you get rid of the fraction format, which gives you an equation that is usually simpler to deal with. Also, if you flip the proportion, you make the problem more to your liking — easier to solve.

Cross-multiplying in a proportion

You can solve for the value of x in the proportion $\frac{2x+3}{x-3} = \frac{5}{7}$ by cross-multiplying and getting rid of the fractional format.

$$\begin{aligned}\frac{2x+3}{x-3} &= \frac{5}{7} \\ (2x+3) \times 7 &= (x-3) \times 5 \\ 14x + 21 &= 5x - 15 \\ 9x &= -36 \\ x &= -4\end{aligned}$$

Flipping your lid over a proportion

Even though cross-multiplying is a great tool to use when solving proportions, you can often take an easier route: Flip the fractions (set the reciprocals equal to one another) and then multiply each side by the same number to solve the equation. For example, in the following equation, I flip the proportion and then just have to multiply each side by the number under the x , reduce, and get the answer.

$$\begin{aligned}\frac{70}{x} &= \frac{35}{21} \\ \frac{x}{70} &= \frac{21}{35} \\ 70 \cdot \frac{x}{70} &= \frac{21}{35} \cdot 70^2 \\ x &= 21 \times 2 = 42\end{aligned}$$

Going every which way with reducing

Reducing fractions in proportions is a blast! You can reduce across the tops, across the bottoms, up and down on the left, or up and down on the right. You just can't reduce going diagonally — the crisscross motion is for multiplying only. By reducing the proportion, first, before cross-multiplying, you get to work with smaller numbers. That's a good idea not only for the ease of the problem, but also because it helps prevent errors.

Reducing vertically

Proportions are created from fractions, and the traditional way of reducing fractions is to find a number that divides the numerator and denominator evenly and divide each part of the fraction by that number. Reducing fractions is sometimes referred to as *cancelling*. In the following proportion, the numbers in the fraction on the left are each divisible by 5. After reducing the fraction on the left, you can cross-multiply and solve the equation for x .

$$\begin{aligned}\frac{25}{40} &= \frac{x}{16} \\ \frac{25^5}{40^8} &= \frac{x}{16} \\ \frac{5}{8} &= \frac{x}{16} \\ 5 \times 16 &= 8x \\ 80 &= 8x \\ 10 &= x\end{aligned}$$

You could also have reduced horizontally in this equation, because 8 and 16 are both divisible by 8. The next section shows you how that works.

Reducing horizontally

The property of proportions that allows you to cross-multiply and have a true statement is the same property that allows for reducing across the proportion horizontally. Both of these clever tools are due to the commutative property of multiplication — the fact that reversing the order of the numbers in a multiplication property doesn't change the answer. It's just neat that this property comes in so handy when working with proportions. In the following proportion, the two numerators are each divisible by 11. Then you see that the two numbers in the right fraction are divisible by 7. Reduce that fraction so that, when you cross-multiply, you don't have to multiply by 63.

$$\begin{aligned}\frac{22^2}{x} &= \frac{77^7}{63} \\ \frac{2}{x} &= \frac{7}{63} \\ \frac{2}{x} &= \frac{7^1}{63^9} \\ 2 \times 9 &= x \times 1 \\ 18 &= x\end{aligned}$$



Dividing \$17

Sneaky Pete thought that he could pull one over on his buddies and keep all the money that was paid to all of them for taking care of the neighbor's dog. The neighbor gave Pete \$17 and told him to divvy it up according to how much time each friend spent taking care of the dog. Pete took the money and posed this problem: "In my hand, I have \$17. Tom can have one-ninth of the money, because he worked one-ninth of the

time. Tim can have one-third of the money, because he worked one-third of the time. And Ted can have one-half of the money, because he worked half of the time. You aren't allowed to tear any of the dollar bills, and there's no change. If you can't figure out how to divide up the \$17, then I'll just keep it all." Pete's buddies were just too smart for him and divided up the money equitably. How did they do it?

Answer: Tom had a dollar in his pocket and added it to the \$17, making the amount \$18. One-ninth of eighteen is 2, one-third of eighteen is 6, and one-half of eighteen is 9. The sum of $2 + 6 + 9 = 17$. Tom took \$2, gave Tim \$6, and Ted \$9, then Tom then took the extra dollar he'd put in the pot. Sorry, Pete.

Dividing Things Up Equitably

Everyone should play fair. Whether it's in basketball or a card game, you expect to be treated fairly. The same can be said for sharing candy or money or time. If everyone gets an equal share (all the same amount), then the computation is easy — you just divide by the number of people who are involved. It gets a little more complicated when the shares are to be unequally divided, like someone getting twice as much as another because she did twice the work.

Splitting things between two people unevenly

If two children are to share equally in an inheritance, you just divide the total amount by two. It gets a bit stickier when one person gets more than the other. It strains the family relationship *and* the mathematics.

The Problem: Henry and Hilda are to share their father's \$2.5 million estate. Their father said that Henry is to get 65 percent of the estate and Hilda is to get the other 35 percent. How much does Henry get?



Using proportions, think of Henry's 65 percent as being 65 out of 100, where 100 is the total amount. Write the proportion with x being the unknown amount out of \$2.5 million. The proportion is $\frac{65}{100} = \frac{x}{2,500,000}$. The proportion says that 65 out of 100 is equal to some unknown value out of 2,500,000 (2.5 million).



When writing proportions, put *related* amounts either horizontally or vertically from one another. Units that are alike should be either across from one another or above and below one another.

In the problem, the units that are *related* are the \$2,500,000, which is *all* of the inheritance and the 100 which represents 100 percent or *all*. The x and the 65 each represent a *part* of the whole thing. To solve the proportion and determine Henry's share, first reduce across the bottom of the proportion, and then cross-multiply and solve for x .

$$\begin{aligned}\frac{65}{100} &= \frac{x}{2,500,000} \\ \frac{65}{1} &= \frac{x}{25,000} \\ 65 \times 25,000 &= x \times 1 \\ 1,625,000 &= x\end{aligned}$$

Henry gets \$1,625,000 of the \$2.5 million in their father's estate.

Figuring each person's share

Another possible scenario when dividing things up is that three or more people are involved, and they each get a different share or fraction of the total amount. A situation like this occurs when people do different amounts of the total work or when they are different ages or different weights or whatever makes them different from one another.

The proportion or proportions used to solve problems where three or more people get differing shares all have a common theme. You're always concerned with the total amount — and all the parts must add up to the total amount.

If a pie is to be divided among four people, and the shares are one-twelfth, one-sixth, one-fourth, and one-half of the pie, you have to be sure that these fractions all add up to 1 — which is the whole pie.

$$\begin{aligned}\frac{1}{12} + \frac{1}{6} + \frac{1}{4} + \frac{1}{2} &= \frac{1}{12} + \frac{2}{12} + \frac{3}{12} + \frac{6}{12} \\ &= \frac{1+2+3+6}{12} = \frac{12}{12} = 1\end{aligned}$$

Consider a situation where contestants share in the total prize depending on the number of points that they've scored.

The Problem: In a fishing tournament, a sports-equipment company has offered a prize of \$100,000 to be divided among the top five winners in proportion to their scores. The points earned by the top five winners are: 60, 40, 30, 20, and 10. How much money does each get?



Determine the total number of points, and then figure out the proportion or part of the prize that each gets. The total number of points is: $60 + 40 + 30 + 20 + 10 = 160$ points.

Writing a proportion with the 60 points for the top scorer, $\frac{60}{160} = \frac{x}{100,000}$. Reduce across the bottom, cross-multiply and solve for x .

$$\begin{aligned}\frac{60}{160} &= \frac{x}{100,000} \\ 60 \times 625 &= 1 \times x \\ 37,500 &= x\end{aligned}$$

The top scorer gets \$37,500. The rest of the winners get amounts as shown here:

The second-place winner gets \$25,000, determined from

$$\frac{40}{160} = \frac{x}{100,000}, \quad \frac{40}{160} = \frac{x}{100,000}, \quad 4x = 100,000.$$

The third-place winner gets \$18,750, from

$$\frac{30}{160} = \frac{x}{100,000}, \quad \frac{30}{160} = \frac{x}{100,000}, \quad 16x = 300,000.$$

The fourth-place winner gets \$12,500, using

$$\frac{20}{160} = \frac{x}{100,000}, \quad \frac{20}{160} = \frac{x}{100,000}, \quad 8x = 100,000.$$

The fifth-place winner gets \$6,250, which you determine by just subtracting all the other prizes from \$100,000 and seeing what's left.

Comparing the proportions for differing amounts of money

Some very interesting problems occur when an estate is divided up between all the heirs. And then there's the executor's share to be considered in the mix. Proportions are very handy when you're determining who gets how much money.



Diophantus's life

Diophantus was a Greek mathematician who had an impact on algebra back in the day. A puzzle that's attributed to his life goes: "Diophantus's boyhood lasted for one-sixth of his life. He married after one-seventh more. His

beard grew after one-twelfth more of his life, and his son was born 5 years later. The son lived to half his father's age, and Diophantus died four years after the son. How long did Diophantus live?

Answer: Diophantus lived for 84 years. His boyhood lasted for 14 years, his beard grew for 7 more years, when he was 33, and his son was born 5 years later, after he was 38. His son died at 42, when Diophantus was 80. And Diophantus died four years later, at 84. The simplest way to do this problem is to find a number divisible by 6, 7, and 12 because fractions with those denominators are used in the problem. The smallest number divisible by all three numbers is 84.



The Problem: A woman's estate totals \$5 million. She leaves $\frac{1}{20}$ of her estate to her butler, $\frac{3}{20}$ to her chauffeur, $\frac{1}{10}$ to her daughter, $\frac{1}{4}$ to her son, and $\frac{9}{20}$ is to be put in a trust to take care of her dog, Puddles. But all these bequests aren't to be made until *after* the executor of her will gets 10 percent of the original amount. The others get their share of the net amount. How much does each person get?

You first deduct the amount that the executor gets. Because she gets 10 percent of the estate, you multiply $\$5,000,000 \times 0.10 = \$500,000$. Subtract that from the estate, and it leaves \$4,500,000 to be divided among the others. The other shares are determined using proportions, letting the fractional share be one side of the proportion and an x divided by \$4,500,000 be the other side of the proportion. The value of x in each case is that person's share.

$$\text{Butler: } \frac{1}{20} = \frac{x}{4,500,000}, \frac{1}{20_1} = \frac{x}{\cancel{4,500,000}_{225,000}}, x = 225,000$$

$$\text{Chauffer: } \frac{3}{20} = \frac{x}{4,500,000}, \frac{3}{20_1} = \frac{x}{\cancel{4,500,000}_{225,000}}, x = 3 \cdot 225,000 = 675,000$$

$$\text{Daughter: } \frac{1}{10} = \frac{x}{4,500,000}, \frac{1}{10_1} = \frac{x}{\cancel{4,500,000}_{450,000}}, x = 450,000$$

$$\text{Son: } \frac{1}{4} = \frac{x}{4,500,000}, \frac{1}{4_1} = \frac{x}{\cancel{4,500,000}_{1,125,000}}, x = 1,125,000$$

$$\text{Puddles: } \frac{9}{20} = \frac{x}{4,500,000}, \frac{9}{20_1} = \frac{x}{\cancel{4,500,000}_{225,000}}, x = 9 \cdot 225,000 = 2,025,000$$

Comparing Apples and Oranges

One of the nicest things about proportions is that they can be used to solve problems involving items that don't seem to have anything in common except for their ratios to one another. For instance, if you're told that three apples can be traded for four oranges, then you can figure out how many apples you can get for 28 oranges by using a proportion.

The Problem: How many apples can you trade for your 28 oranges if the current trading rate is that three apples are worth four oranges?



Write the apples over the oranges in one fraction, and then put the 28 oranges on the bottom of the other fraction — opposite the oranges in the first fraction. Let the unknown number of apples be represented by x .

$$\frac{3 \text{ apples}}{4 \text{ oranges}} = \frac{x \text{ apples}}{28 \text{ oranges}}, \frac{3}{4} = \frac{x}{28}$$

This proportion is solved by reducing across the bottom and cross-multiplying.

$$\frac{3}{4} = \frac{x}{28}, 21 = x$$

It takes 21 apples to get 28 oranges. As long as you have apples across from apples and oranges across from oranges, the proportion will work. Another format for this is to have apples over apples and oranges over oranges, with the equivalence of 3 and 4 across from one another. (Refer to “Working with the Math of Proportions,” earlier in this chapter, if you need help with the manipulations of proportions.)

Determining the amounts in recipes

If you're a cook — or even if you don't have much interest in the culinary arts — you're apt to come across a situation where you need to double a recipe for a bigger crowd, halve a recipe that makes too much food, or compute some such multiple or part of a recipe. Even if the recipe is for cement, getting the amounts correct is important. Otherwise, the chili will taste too salty or the cake won't rise or the cement will never harden. Proportions are a huge help with these recipe challenges.

The Problem: Your favorite chili recipe calls for 2 pounds of hamburger and 3 onions. You're going to make enough chili for your whole fraternity and plan to use 28 pounds of hamburger. How many onions do you need?



Set up a proportion with pounds of hamburger divided by number of onions in one fraction and pounds of hamburger divided by onions in the other fraction. Place an x for the unknown number of onions. Be sure to put the 28 pounds of hamburger in the same fraction as the x number of onions.

$$\frac{2 \text{ pounds}}{3 \text{ onions}} = \frac{28 \text{ pounds}}{x \text{ onions}}, \frac{2}{3} = \frac{28}{x}$$

Solving this proportion is easier if you reduce across the top by dividing each numerator by 2. Then flip the proportion before cross-multiplying. (Refer to “Working with the Math of Proportions,” earlier in this chapter, if you need a refresher on these techniques.)

$$\frac{\cancel{2}^1}{\cancel{3}^1} = \frac{\cancel{28}^{14}}{\cancel{x}^1}, \frac{3}{1} = \frac{x}{14}, 3 \cdot 14 = x, 42 = x$$

You will need 42 onions. I feel a crying session coming on.

I inherited my grandmother’s recipe box, and there are some wonderful, old recipes in it. One of my favorites (just because I’m trying to imagine my very proper grandmother, Marion Jones Roby Ingersoll, making this much food) is for sausage. The recipe calls for 100 pounds of pork, $\frac{1}{4}$ pound of sage, $\frac{1}{4}$ pound of pepper, 2 pounds of salt, 1 tablespoon of mustard and a “little” summer savory. The recipe says to put the stuff in a crock by layers and “weigh it down.”

The Problem: I want to try my grandmother’s recipe, but not in that huge quantity. If I start with 5 pounds of pork instead of 100, how much sage and mustard will I need?



Even though the quantities are in pounds and tablespoons, I can still use proportions to solve for the amounts needed. Write a proportion with the original number of pounds of pork and sage in the numerator and denominator of the first fraction, and then write the 5 pounds of pork in the other fraction across from the 100 pounds. Solve for the reduced pounds of sage. Do the same thing with the original measures to solve for the amount of mustard.

$$\frac{100 \text{ pounds}}{\frac{1}{4} \text{ pound}} = \frac{5 \text{ pounds}}{x \text{ pound}}, \frac{100}{\frac{1}{4}} = \frac{\cancel{5}^1}{x}, 20x = \frac{1}{4}, x = \frac{1}{80}$$

When I divided each side by 20, I got $\frac{1}{80}$ pound of sage. It makes more sense to change the fraction of a pound to ounces. If there are 16 ounces in a pound, then multiply $16 \times \frac{1}{80}$ pound to get $\frac{1}{5}$ ounce. And now, for the mustard, write a new proportion and solve it.

$$\frac{100 \text{ pounds}}{\frac{1}{4} \text{ pound}} = \frac{1 \text{ T}}{x \text{ T}}, \frac{100}{\frac{1}{4}} = \frac{1}{x}, 100x = \frac{1}{4}, x = \frac{1}{400}$$

The measure of $\frac{1}{400}$ of a tablespoon is probably better known as a “dash.”



Truck on a bridge

A large truck is crossing a bridge that measures 1 mile in length. The bridge can only hold 14,000 pounds, which happens to be the exact weight

of the truck and all its cargo. The truck makes it halfway across the bridge and stops. A bird lands on the truck. Does the bridge collapse?

Answer: No, not to worry. The truck has driven half a mile and has used up a goodly amount of fuel. You can subtract the weight of the fuel, and the bird can land safely.

Figuring out weighted averages

Weighted averages are used to give more importance or emphasis to one thing than another. A prime example of weighted averages is when they're used to determine your grade in a college course. The weighting can go something like this: Tests count three times as much as papers, the final exam counts twice as much as a test, and attendance counts one-fourth as much as a paper. In general, to find a weighted average, you set up a proportion and multiply the weights times their respective amounts. Look at these next two problems, and you'll see what I mean.

The Problem: An Astrodollars Coffee shop sells several different types of whole coffee beans. Last Monday they sold 100 pounds of Honduran coffee beans, 70 pounds of Guatemalan coffee beans, 40 pounds of Nicaraguan coffee beans, and 40 pounds of Chilean coffee beans. The Honduran beans cost \$8 per pound, the Guatemalan beans cost \$9 per pound, the Nicaraguan beans cost \$10.50 per pound, and the Chilean beans cost \$13.50 per pound. What was the average cost per pound of the coffee beans sold on Monday?



Set up a proportion where each poundage multiplies its respective price. Put the sum of all the products you get in the numerator of a fraction, and divide by the total number of pounds. That's one side of the proportion. Set that fraction equal to another fraction with a 1 in the denominator (opposite the total number of pounds) and an x in the numerator.

$$\frac{(100 \times 8) + (70 \times 9.00) + (40 \times 10.50) + (40 \times 13.50)}{100 \text{ pounds} + 70 \text{ pounds} + 40 \text{ pounds} + 40 \text{ pounds}} = \frac{x}{1 \text{ pound}}$$

Now simplify the proportion by doing the multiplications and additions; then reduce the fraction on the left. Cross-multiply and solve for x , and you get \$9.56 for the average cost of the coffee beans sold on Monday.

$$\frac{800 + 630 + 420 + 540}{100 + 70 + 40 + 40} = \frac{x}{1 \text{ pound}}$$

$$\frac{2,390}{250} = \frac{x}{1}$$

$$\frac{\cancel{2,390}^{239}}{\cancel{250}^{25}} = \frac{x}{1}$$

$$239 = 25x$$

$$x = \frac{239}{25} = 9.56$$

A college grade point average (GPA) is usually a weighted average. Different courses are a different number of hours or quarters or other units, which serve to determine their relative worth. Consider a student who attends a college that measures courses in semester hours and uses grades of A, B, C, D, and F. The corresponding point values for the grades are: 4, 3, 2, 1, and 0 points.

The Problem: Nick took five courses last semester and needs to determine his GPA. He got an A in his 4-semester-hour calculus course, an A in his 1-semester-hour computer course, a B in his 5-semester-hour biology course, a C in his 3-semester-hour English course, and a B in his 3-semester-hour Spanish course. What is his GPA for the semester?



Multiply each number of semester hours by its worth in terms of points for that grade, and add up all the points; divide by the total number of semester hours. Set that fraction equal to x divided by 1.

$$\frac{(A \times 4) + (A \times 1) + (B \times 5) + (C \times 3) + (B \times 3)}{4 + 1 + 5 + 3 + 3} = \frac{x}{1}$$

$$\frac{(4 \times 4) + (4 \times 1) + (3 \times 5) + (2 \times 3) + (3 \times 3)}{16} = \frac{x}{1}$$

$$\frac{16 + 4 + 15 + 6 + 9}{16} = \frac{x}{1}$$

$$\frac{50}{16} = \frac{x}{1}$$

$$\frac{\cancel{50}^{25}}{\cancel{16}_8} = \frac{x}{1}$$

$$25 = 8x$$

$$\frac{25}{8} = x$$

$$3.125 = x$$

After simplifying the proportion, reducing the fraction on the left, cross-multiplying, and solving for x , you get that Nick's GPA is 3.125.

Computing Medicinal Doses Using Proportions

Modern medicine offers many wonderful options to reduce pain and suffering. The amount of the medication for a particular person has to be correct, though. The amount of medicine prescribed may depend on a person's weight, age, or current health status — or, often, a mixture of all these things. Proportions are used to determine dosages of many medications and the number of tablets needed per dose.

Figuring the tablets for doses

A *scored tablet* is a medicine tablet that is designed so that it can be broken into halves or quarters, making it possible to administer a dose that is less than the amount in the tablet. For purposes of these problems, assume that the tablets are scored into quarters. (You can break the tablet into four equal pieces.)

The Problem: A doctor prescribes 0.375 mg of Digoxin, and the scored tablets that are available contain 0.25 mg each. How many tablets should be administered?



Set up a proportion with the amount per tablet in one fraction and the needed dosage and number of tablets in the other fraction.

$$\frac{0.25 \text{ mg}}{1 \text{ tablet}} = \frac{0.375 \text{ mg}}{x \text{ tablets}}$$

Reduce the fractions across the top by dividing each numerator by 0.125. Then solve for x .

$$\begin{aligned}\frac{0.25^2}{1} &= \frac{0.375^3}{x} \\ 2x &= 3 \\ x &= 1.5\end{aligned}$$

The patient needs 1½ tablets.

The Problem: A patient is to take 0.5 g of Ampicillin, and the capsules available are 250 mg. How many capsules are needed?

From the metric system: 1 gram = 1,000 milligrams or 1 g = 1,000 mg.





Change the 0.5 grams to milligrams using a proportion with 1 gram over 1,000 milligrams in one fraction and 0.5 grams over x milligrams in the other fraction. Solve for x .

$$\frac{1 \text{ g}}{1,000 \text{ mg}} = \frac{0.5 \text{ g}}{x \text{ mg}}$$

$$\frac{1}{1,000} = \frac{0.5}{x}$$

$$x = 0.5 \times 1000 = 500$$

So $0.5 \text{ g} = 500 \text{ mg}$. If the patient is to take 500 mg of Ampicillin, and the capsules are 250 mg, then the patient will need two capsules.

Making the weight count

A person's weight can affect the dosage of the medication they're given. You use a proportion to determine the amount of medication based on that weight.

The Problem: Robert has been taking 80 mg of a medication every day. When it was prescribed, he weighed 170 pounds. He's lost 30 pounds, so how much should the new dosage be?



Create a proportion with 80 mg and 170 pounds in one fraction and 140 pounds in the other fraction. The 140 is found by subtracting $170 - 30$ to determine Robert's new weight. Be sure to put the 140 pounds opposite the 170 pounds in the proportion.

$$\frac{80 \text{ mg}}{170 \text{ lb}} = \frac{x \text{ mg}}{140 \text{ lb}}$$

Reduce either vertically in the left fraction or horizontally across the bottom. Then cross-multiply and solve for x .



You can "drop" the zeros in only one direction. Reducing by dividing by 10 in the proportion can be done either vertically or horizontally, but not both.

$$\frac{80}{170} = \frac{x}{140}$$

$$\frac{8}{17} = \frac{x}{14}$$

$$\frac{1,120}{17} = x$$

The dosage is about 65.88 mg of the medication. The doctor will have to round up or down to find a suitable tablet to use.

Chapter 8

Figuring the Probability and Odds

In This Chapter

- ▶ Covering the mathematics of probability computations
 - ▶ Computing the probabilities of desired outcomes
 - ▶ Figuring the odds
-

The concept of probability is all around us. You can't turn on the TV or radio or pick up a newspaper or magazine without hearing or reading about the probability of some event. From predicting the probability of a hurricane hitting land to declaring the odds that a horse will win the Kentucky Derby, predictions are rampant. The predictions themselves may not be correct, but probability isn't a *for sure*, it's just a *probably*.

This chapter starts out by showing you how to do the computations needed for solving probability problems. (If you need a more thorough review of the relationships between fractions, decimals, and percents, and how to change from one to another, refer to Chapter 5.) In this chapter, you use the percentages to make the predictions. And you see what the odds are that you'll just *love* this topic.

Defining and Computing Probability

The *probability* of an event is the likelihood that it'll happen. The most common way to express a probability is with a percent, such as 60 percent probability of rain or 70 percent likelihood that he'll hit the ball. Probability is also expressed in terms of fractions — in fact, a fraction supplies one of the nicest ways of defining how you get the probability of something happening.

When the probability of something happening is 95 percent, you can be pretty sure that the event will happen — 95 out of 100 times it does. A probability of 15 percent is pretty low. That sounds like the chance that I'll make a free throw in basketball.

Counting up parts of things for probability

Following is a standard version of a probability formula. This formula allows for you to count up how many different ways something can be done and then determine the probability that just a few of those things will happen.

The probability, P , that an event, e , will happen is found with:

$$P(e) = \frac{\text{number of ways event } e \text{ can happen}}{\text{total number of ways all events can happen}}$$

For example, consider a jar of marbles and the probability that you'll pick a red one. Or you look at the seats in an airplane and determine the probability of your getting a particular seat. Or how about the number of boys in a family? How does a number of boys become a probability problem? (Well, boys can be a *problem* — I don't know about the probability part.)

Dealing with a jar that contains 10 red, 20 yellow, and 50 green marbles

Picture a jar containing marbles and a task of drawing a marble out of the jar. You have your eyes blindfolded, and you can't tell the difference between the marbles by feeling them — they're all the same size.

The Problem: What is the probability that you get a green marble when you choose a marble at random out of the jar that contains 10 red, 20 yellow, and 50 green marbles?



Make a fraction that has the number of green marbles in the numerator and the total number of marbles in the denominator. Then reduce the fraction and write your final answer as a percent. (Go to Chapter 5 if you need help changing a fraction to a percent.)

$$\begin{aligned} P(\text{green}) &= \frac{\text{number of green marbles}}{\text{total number of marbles}} \\ &= \frac{50}{80} = \frac{5}{8} = 0.625 = 62.5 \text{ percent} \end{aligned}$$

A probability of 62.5 percent isn't particularly high, but it's better than half. If you had to guess which color marble you might draw at random, you'd pick the green marble, because it has the highest probability (more greens than any other color).

The Problem: What is the probability that you'll choose a marble at random and that it's *not* green if the jar contains 10 red, 20 yellow, and 50 green marbles?



You have two ways of doing this. The first is to find the number of marbles that are *not* green and divide by the total number of marbles.

$$\begin{aligned} P(\text{not green}) &= \frac{\text{number of red or yellow marbles}}{\text{total number of marbles}} \\ &= \frac{30}{80} = \frac{3}{8} = 0.375 = 37.5 \text{ percent} \end{aligned}$$

Even adding the red and yellow marbles together, you still have a better chance of drawing a green marble. The other way of solving a probability problem that has a *not* in it is to subtract from 100 percent.



The probability than an event, e , will *not* occur is 100 percent minus the probability that it *will* occur. $P(\text{not } e) = 100 \text{ percent} - P(e)$.

So, to solve for the probability that the marble is *not* green, you take the probability that it *is* green and subtract that from 100 percent. Subtracting, you get 100 percent – 62.5 percent = 37.5 percent.

The Problem: You have a jar that contains 10 red, 20 yellow, and 50 green marbles, and you take a marble out of the jar and put it in your right pocket without looking at the color. Now you draw another marble out of the same jar and put it in your left pocket. What is the probability that both marbles are red?



Right away, you should be thinking about how unlikely the probability is that both marbles are red. The red marbles are the fewest in the jar. The probability should be pretty low. To do the problem, first find the probability of drawing a red marble out of the 80 marbles, and then find the probability of drawing another red marble out of the 79 remaining marbles.



The probability that an event, e , and another event, f , will *both* occur is equal to the product of the two probabilities. $P(e \text{ and } f) = P(e) \times P(f)$.

The probability of drawing a red marble is $P(\text{red}) = \frac{10}{80} = \frac{1}{8} = .125 = 12.5$

percent. If that first marble is red, then there are only 9 red marbles left in the jar of 79 marbles. So the probability of a second marble being red is

$P(\text{red}) = \frac{9}{79} \approx .114 = 11.4$ percent. Multiply the two probabilities together.

$12.5 \text{ percent} \times 11.4 \text{ percent} = 0.125 \times 0.114 = 0.01425$ or about a 1.4 percent chance that both marbles are red.

Looking at 40 rows of seats on an airplane, with 7 seats per row

Nowadays, you can make your plane reservations online and, in some cases, even pick your seats. The following problems involve the case where you're on standby and don't get a seat assignment choice — you'll be happy just to

get on the airplane. So assume that the seat assignment is selected at random. Imagine that all 40 rows in the airplane have 7 seats across: 2 seats on either side along the windows, and 3 seats in the middle. So there are $40 \times 7 = 280$ seats possible.

The Problem: What is the probability that your seat will be in row 10 if there are 40 rows with 7 seats in a row?



Take the number of seats in row 10, which is 7, and divide by the total number of seats, 280. $P(\text{row 10}) = \frac{7}{280} = \frac{1}{40} = 0.025 = 2.5$ percent. The probability is only 2.5 percent. Sorry — you'll be disappointed if 10 is your lucky number and you were hoping for that row.

The Problem: What is the probability that you'll get a window seat if there are 40 rows with 7 seats in a row?



Determine the number of window seats. With 40 rows and a window seat at each end of each row, that gives you $40 \times 2 = 80$ window seats. Divide 80 by the total number of seats. $P(\text{window}) = \frac{80}{280} = \frac{8}{28} = \frac{2}{7} \approx 0.2857 = 28.57$ percent. Notice that you'd get the same answer if you just did the computation on a single row. Two of the seven seats are window seats. The fraction $\frac{2}{7}$ applies to figuring the probability of a window seat either in one row or the whole plane.

The Problem: What is the probability that you'll get a seat either in the last 4 rows or a window seat, if there are 40 rows and 7 seats in a row?



When counting the number of seats for this problem, you have to be careful not to count the same seats more than once. Some of the seats in the last four rows are also window seats.



When counting a number of items in two sets or categories, you first add up the number of items in one category plus the number of items in the second category, and then you subtract the number of items that they share. Let n represent “the number in”:

$$n(\text{Set 1 or Set 2}) = n(\text{Set 1}) + n(\text{Set 2}) - n(\text{Set 1 and Set 2})$$

Applying this rule to the seats on the plane, there are $4 \times 7 = 28$ seats in the last four rows and $2 \times 40 = 80$ window seats. The last four rows have $4 \times 2 = 8$ window seats. So the number of seats you want is: $28 + 80 - 8 = 100$ seats that are either in the last four rows or a window seat (or both). The probability that you'll get one of those seats is found with

$$P(\text{last 4 or window}) = \frac{100}{280} = \frac{10}{28} = \frac{5}{14} \approx 0.3571 = 35.71\text{ percent.}$$

Seeing whether a family of three children has a certain number of boys

Some classic probability problems have to do with children in a family and determining the likelihood that all three are boys or, maybe, that there are two boys and a girl, and so on. A wonderful way to handle problems like this (when it's possible) is listing all the possibilities and counting the number of possibilities you want. In a family of three children, the eight possible arrangements of boys (B) and girls (G), listing the children in order, are: BBB, BBG, BGB, GBB, BGG, GBG, GGB, and GGG. Notice that two of the arrangements are all one sex, three of the arrangements have two boys and one girl, and three of the arrangements have one boy and two girls. None of the arrangements is the same.

The Problem: When choosing a three-child family at random, what is the probability that the children are all the same sex?



Count the number of arrangements with all the same sex and divide by the total number of arrangements. Three children in a family can be arranged in the following eight ways: BBB, BBG, BGB, GBB, BGG, GBG, GGB, and GGG.

$P(3 \text{ same}) = \frac{2}{8} = \frac{1}{4} = 0.25 = 25\%$ percent. In 25 percent of three-child families, the children are all the same sex.

The Problem: What is the probability that a three-child family has at least one girl?



You can count how many of the arrangements of BBB, BBG, BGB, GBB, BGG, GBG, GGB, and GGG have one or more girls and find that there are seven of them. $P(\text{at least one girl}) = \frac{7}{8} = 0.875 = 87.5\%$ percent. Another approach, instead of counting how many families have a certain characteristic, is to use the *not* idea and subtract from 100 percent. (Refer to the “Dealing with a jar that contains 10 red, 20 yellow, and 50 green marbles” section, earlier in this chapter, for the rule.) Applying the *not* rule to this problem, determine how many arrangements have *no* girls — there’s only 1. Find the percentage, $P(\text{no girl}) = \frac{1}{8} = 0.125 = 12.5\%$ percent. Then subtract 100 percent – 12.5 percent = 87.5 percent.

Using probability to determine sums and numbers

When you have a percent or fraction that represents a probability, you can use the decimal equivalent of the percent or the fraction to determine a



Sons and daughters

In the Jefferson family, each daughter has the same number of brothers as she has sisters, and each son has twice as many sisters as he

has brothers. How many sons and daughters are there in this family?

brothers, making the family four girls and three boys — a match. (making the family two girls and two boys) or possibly four sisters and two brothers as brothers, with a possible scenario of two sisters and one brother (making it three girls and four boys), and so on. But each son has twice as many sisters as brothers, making the family two girls and one boy, two brothers and two sisters (making the family three girls and one boy), three brothers and three sisters (making it four girls and two boys), and so on.

ANSWER: There are three sons and four daughters. She has one brother and one sister

number of times that something may happen. The number answer may represent how many coins in a pouch are gold or how many moves you may be making while playing a board game. The probability value always assumes that you're using fair dice (each face equally likely to appear) or a fair coin (not weighted on one side or the other) or aren't peeking or aren't in some way altering the numerical value.

Rolling a die with faces of 1, 2, 3, 4, 5, and 6

When you roll a die, one of the faces lies on the table, and the other five faces are showing. The face that counts, in most games, is the face on the top of the die, after it's been rolled.

The Problem: You're nearing the end of a game of Trivial Pursuit and are just three spaces away from trying for the last pie piece (for non-Pursuit people, it's the last of six categories). You shake and shake and shake your die before rolling it. What is the probability that you'll roll either a 3 or a 5 (a 5 will allow you to roll again)?



Make a listing of all the possible results of the roll and a listing of all the numbers that you want to get. With six faces on the die, there are six different possibilities: 1, 2, 3, 4, 5, or 6. You want one of two different rolls: 3 or 5 (although you'd prefer the number 3), so you divide 2 by 6 to get about 0.3333 or 33.33 percent.

Rolling a die and flipping a coin at the same time

When you roll a standard die, you get 1, 2, 3, 4, 5, or 6. When you flip a coin, you get either heads or tails. Do the two actions at the same time (which takes some coordination not to have at least one fall on the floor) and you have the following results possible (let H represent *heads* and T represent *tails*): 1H, 2H, 3H, 4H, 5H, 6H, 1T, 2T, 3T, 4T, 5T, or 6T.

The Problem: When you roll a die and flip a coin at the same time, what is the probability that you get an even number on the die with tails?



Count up how many of the results have an even number (2, 4, or 6) and tails, at the same time. The results you want are: 2T, 4T, or 6T. That's 3 out of the 12 possible results. $P(\text{even and tails}) = \frac{3}{12} = \frac{1}{4} = 0.25 = 25\text{ percent.}$

The Problem: When you roll a die and flip a coin at the same time, what is the probability that you get *either* an even number *or* tails?



Note that this problem is different from the previous problem where you wanted *both* an even number and tails at the same time. In this case, you'll take either an even number or tails, or both. You can't just add up the number of even numbers and the number of tails. That sum is $6 + 6 = 12$, which is the same number as all the choices. You can look through the list and find all the choices that have either an even number or tails (or both) and get: 2H, 4H, 6H, 1T, 2T, 3T, 4T, 5T, or 6T, which is 9 items. The other method to use is the counting rule (refer to "Looking at 40 rows of seats on an airplane" for more on the counting rule) and get $n(\text{even or tails}) = n(\text{even}) + n(\text{tails}) - n(\text{both even and tails}) = 6 + 6 - 3 = 9$. In any case, the probability is computed $P(\text{even or tails}) = \frac{9}{12} = \frac{3}{4} = 0.75 = 75\text{ percent.}$

Rolling two dice at the same time

Board games like Monopoly involve moves based on the results of rolling two dice at the same time and adding up the face values. The sums that you get from adding the two numbers on the dice are any of the 11 numbers from 2 through 12, but there are 36 different ways to get those 11 numbers. The sums are best shown with a table of values. In Table 8-1, the face values of the two dice are shown along the top and down the left side. The sums are shown in the table.

Table 8-1 **Sums Obtained from Rolling Two Dice**

| 1 | 2 | 3 | 4 | 5 | 6 |
|----------------|----------------|----------------|-----------------|-----------------|-----------------|
| 1 1 + 1 = 2 | 2 1 + 2 = 3 | 3 1 + 3 = 4 | 4 1 + 4 = 5 | 5 1 + 5 = 6 | 6 1 + 6 = 7 |
| 2 2 + 1 = 3 | 2 2 + 2 = 4 | 3 2 + 3 = 5 | 4 2 + 4 = 6 | 5 2 + 5 = 7 | 6 2 + 6 = 8 |
| 3 3 + 1 = 4 | 3 3 + 2 = 5 | 3 3 + 3 = 6 | 4 3 + 4 = 7 | 5 3 + 5 = 8 | 6 3 + 6 = 9 |
| 4 4 + 1 = 5 | 4 4 + 2 = 6 | 4 4 + 3 = 7 | 4 4 + 4 = 8 | 5 4 + 5 = 9 | 6 4 + 6 = 10 |
| 5 5 + 1 = 6 | 5 5 + 2 = 7 | 5 5 + 3 = 8 | 5 5 + 4 = 9 | 5 5 + 5 = 10 | 6 5 + 6 = 11 |
| 6 6 + 1 = 7 | 6 6 + 2 = 8 | 6 6 + 3 = 9 | 6 6 + 4 = 10 | 6 6 + 5 = 11 | 6 6 + 6 = 12 |

You probably notice, in the table, that the numbers increase as you move down a column or across a row. Did you also notice that the sums in the diagonals are the same? The longest diagonal is the one with the sums of 7 — it has six entries that add up to 7.

The Problem: What is the probability that, when you roll two dice, you'll get a sum of 6?



Referring to the table, you see that there are five different ways to get a sum of 6; the sums are all found on a diagonal starting under the 5s column and moving downward to the left. Make a fraction putting 5 in the numerator and 36 in the denominator. $P(\text{sum of } 6) = \frac{5}{36} \approx 0.1389 = 13.89\%$ percent.

The Problem: You're playing Monopoly and sitting on the railroad right before reaching Go. You're also looking at Park Place and Boardwalk with your opponent's hotels, waiting for you to land on them. What is the probability that you'll roll a sum large enough to miss those hotels completely (you need a sum of 5 or more)?



You can refer to Table 8-1 and count up all the sums that are 5 or greater. Then put the number of sums you count over 36 and divide. An even easier task would be to count up those sums that are *less than* 5, find the probability, and then subtract that probability from 100 percent. (Refer to the “Counting up parts of things for probability” section for more on how the rule for *not* works.) You see six different entries in the table that are sums of 4 or less.

$$P(\text{sum 4 or less}) = \frac{6}{36} = \frac{1}{6} \approx 0.1667 = 16.67\% \text{ Subtracting } 100\% - 16.67\% = 83.33\%$$

The Problem: When you roll two dice, what is the probability that the sum will be even or that the roll will be a *doubles* (both faces the same)?



Count up the number of rolls that have an even sum, and then go along the diagonal to count up all the doubles. Subtract the rolls that the two categories have in common. $n(\text{even sums}) + n(\text{doubles}) - n(\text{even sums that are doubles}) = 18 + 6 - 6 = 18$. All of the doubles are even sums, of course. Twice a number (*any* number) is an even number. So the probability is found by dividing $18 \div 36$, which is half, and the probability is 50 percent.

Making words from words — or not

If you’re like me, you’ve had at least one teacher who sought to entertain you and your classmates on the day before a school holiday by having you make words from the letters in the name of the holiday, whatever the holiday was. Being a word junky, myself, I always liked these exercises. Those who aren’t that much into words will just have to indulge me a bit.

The rules for making words in statistical or probability problems is that the arrangement of letters doesn't have to actually form a real word. A better term than *word* in this process is *arrangement*. You just list all the different *arrangements* of letters possible. For example, you can make a list of 12 different 2-letter words (arrangements) formed from the letters in MATH. These words are: MA, MT, MH, AM, TM, HM, AT, AH, TA, HA, TH, HT. Of those 12 arrangements, the only words that are actually found in the dictionary are: MA, AM, AT, AH, and HA. (No, you can't use TA; it's an abbreviation for one of the elements in the periodic table, for teaching assistant, and for numerous other things as well.)

The Problem: You have the four Scrabble tiles M, A, T, and H in a bag. Drawing out two of the letters, what is the probability that you can form an English word with the two letters?



Of the 12 different arrangements possible with the four letters in the word MATH, five of them form words, so the probability computed is

$$P(\text{a word}) = \frac{5}{12} \approx 0.4167 = 41.67 \text{ percent.}$$

The Problem: You have the six Scrabble tiles S, T, A, R, E, and D in a bag. Drawing out two of the letters, what is the probability that you'll draw one vowel and one consonant?



This time, you don't need to worry about the order of the letters, just the type of letter that they are. List all the different possibilities, but don't list the same two letters in different orders (don't list both S and T and T and S). The 15 different possibilities are: ST, SA, SR, SE, SD, TA, TR, TE, TD, AR, AE, AD, RE, RD, ED. Of the 15 possibilities, 8 have 1 vowel and 1 consonant. Doing the math, $P(\text{one vowel, one consonant}) = \frac{8}{15} \approx 53.33 \text{ percent.}$

Predicting the Outcomes

Some probabilities are *empirical* (determined by observations or experiments) and other probabilities are *theoretical* (determined using mathematical formulas). For instance, the probability that a person will have type-O blood is empirical — determined over the years from testing the blood of millions of people. The probability that a person will win the lottery is theoretical — it can be computed using mathematical formulas. But these are just probabilities. Neither guarantees or forbids anything. Each determines a likelihood of something occurring. No matter how likely something will happen, there's still the chance that it won't happen.

Predicting using empirical probabilities

The weather is one of the most frequently talked-about probability situations. You get predictions on temperature, precipitation, humidity, and long-term trends.

Seeing that the probability of rain is 70 percent

You hear on the radio that for the month of April, the probability of rain will be 70 percent. This means that, 70 percent of the time, it'll be raining on a day in April. This is just a prediction, of course, but it's computed using past trends and observations of what's going on around the world.

The Problem: You're planning on an April wedding. If the prediction is that there's a 70 percent chance of rain on any day in April, how many days does that leave you to try to plan a dry-weather wedding?



You're going to assume that the dry periods will occur on full days — just to make the math simpler. (And you certainly don't want it to rain for 70 percent of the time *every day*!) If it's going to rain on 70 percent of the days, then it should be dry for 30 percent of them. Multiply the number of days in April, 30, by 30 percent to get $30 \times 0.30 = 9$. You have nine days to choose from for your wedding. Lots of luck!

Using the probability of being struck by lightning

The probability of being struck by lightning is a pretty low number, even if that probability is applied to every year of your life. Of course, your lifestyle can change that probability somewhat — depending on how much you go golfing or sailing.

The Problem: If the population of the United States is 300 million, and the probability of a person being hit by lightning is 0.00003 percent, then how many people do you expect to be hit by lightning this year?



You will multiply the number of people by the probability of being hit by lightning. Watch out for the two numerical challenges. The 300 million has to have zeros added, and the percentage has to have its decimal place changed before multiplying. The computation: $300,000,000 \times 0.0000003 = 90$ people in the United States who are expected to be hit by lightning.

Working with a probability of having type-O blood

If you have type-O blood, then you're very popular with the local hospitals and blood banks, because you're a universal donor. Approximately 45 percent of the population has type-O blood, 40 percent has type-A blood, 11 percent has type-B blood, and 4 percent has type-AB blood. Along with the four blood types in terms of letters, you are either RH+ or RH-. So, technically, there are eight different blood types: O+, O-, A+, A-, B+, B-, AB+, and AB-.



The Problem: You have type B+ blood and need a blood transfusion. You can use either type-O or type-B blood and can use either a + or – RH factor. What is the probability that a person selected at random can be a donor for you?

Determine the total percentage of eligible donors by adding the probabilities together: 45 percent + 11 percent = 56 percent.

The Problem: You work for the Red Cross blood services and need to secure some O– blood to have on hand for the imminent birth of quadruplets at the local hospital. Of the 45 percent of the population that has type-O blood, about 16 percent of the type-O people are O–. About how many people out of 100 has O– blood (what is the probability that a person selected at random out of 100 has O– blood)?



Percentages are out of 100. So, if 45 percent of the population has type-O blood, then chances are that 45 out of 100 have type-O blood. If 16 percent of the type O people have O– blood, then find 16 percent of 45 with $0.16 \times 45 = 7.2$. About 7 people in 100 have O– blood. The probability of a person at random having O– blood is about 7.2 percent.

Counting on Jimmy batting .300

When you say that a baseball player is *batting .300*, this means that his percentage is 300 out 1,000 or 30 percent. So the probability that he'll get a hit when he goes to bat is 30 percent.

The Problem: Jimmy is first in the lineup and typically gets to bat four times each game. His team is playing Chicago this weekend and they'll play four games in all. Jimmy is batting .300. How many hits do his fans expect him to get?



In the four-game series, Jimmy expects to bat $4 \times 4 = 16$ times. The probability of a hit at any time at bat is 30 percent, so multiply $16 \times 0.30 = 4.8$. Jimmy is expected to get about five hits.

Using theoretical probabilities

Theoretical probabilities are determined using mathematical counting techniques and mathematical formulas. These probabilities are still just that: predictions or likelihoods of events happening.

Betting on heads or tails

When flipping a fair coin, the probability is 50 percent that it'll be heads and 50 percent that it'll be tails.

The Problem: What is the probability of flipping three heads in a row?



This problem can be done one of two ways: Multiply the probability three times or make a list of possibilities and write a fraction. The probability of getting heads is 50 percent. The probability of heads the first time *and* the second time is 50 percent \times 50 percent. Carry that one more step for three flips, and you get 50 percent \times 50 percent \times 50 percent = $0.50 \times 0.50 \times 0.50 = 0.125 = 12.5$ percent. The chance of three heads in a row isn't very good. The other method, making a list instead of multiplying, has you write down all the possibilities of flipping three coins: HHH, HHT, HTH, THH, HTT, THT, TTH, TTT. You see 8 different arrangements, and 1 out of 8 is $P(\text{three heads}) = \frac{1}{8} = 0.125 = 12.5$ percent.

Working with the probability of winning the lottery

You have a better chance of being struck by lightning than of winning the lottery! Earlier in this chapter, I tell you that the probability of a person being hit by lightning is 0.00003 percent. And, as low as that probability is, it's better than your chance of winning the lottery. The probability of winning the lottery is mathematically generated; the lightning probability comes from observations (and not all strikes are reported).

A certain state has a lottery in which you have to guess which 6 of 54 numbers will be drawn on Saturday night. Your favorite numbers are: 10, 16, 17, 19, 26, and 28.

The Problem: The probability of winning a lottery, where you have to pick six numbers from the numbers 1 through 54, is 0.00000387 percent. How many tickets do you have to buy (and combinations of numbers do you have to choose) to be sure that you have a ticket with all the winning numbers on it?



The percentage is the probability that one ticket will be the winning ticket. Divide the number 1 by the percentage to get the number of tickets needed (the probability is 1 out of that number of tickets). $1 \div 0.000000387 \approx 25,830,000$ tickets. This is a rounded number, because the probability was rounded and most scientific calculators won't give the exact value. In any case, there isn't enough time in a week to buy all the tickets, even if you had the money to do it.

Dealing with the probability of drawing an ace

A standard deck of playing cards has four different suits (spades, hearts, diamonds, clubs) and 13 different cards in each suit. Whether you're playing poker or bridge or hearts or euchre, the probability of a particular card or a particular suit plays a big part in the result of the game. The rest of the game depends on your skill and your partner.

The birthday problem

Even though there are 366 different birthdays possible, you only need to have about 24 people in a room to have a better than 50 percent chance that two people will have the same birthday (not the year, necessarily, just the

date). In a room of 40 people, the probability is almost 90 percent that two people will have the same birthday. Next time you're in a boring meeting, try it by surveying the gathering.

The Problem: You're playing hearts with three other players. You're trying to decide what card to play. What are the chances that the person to your left has the queen of spades?



First, assume that you do *not* have the queen of spades. (Otherwise, why would you need to find the probability?) The cards remaining to be played are divided among four players. It doesn't matter how many cards have yet to be played — and you're only worried about three-fourths of them, because you know what cards you have. One out of the other three players has the queen of spades. Back to the question: what is the probability that the person to your left has the queen of spades? After wading through all the extra information, you determine that the probability is $1 \div 3 = 0.3333$ or about 33.33 percent chance that it's the person to your left with the queen of spades.

The Problem: You're drawing a card from a standard deck of playing cards. What is the probability that you'll draw either an ace or a club?



Use the counting method from the “Counting up parts of things for probability” section to get the number of cards in the category that you want: an ace or a club. $n(\text{aces}) + n(\text{clubs}) - n(\text{ace of clubs}) = 4 + 13 - 1 = 16$ cards. A full deck has 52 cards, so divide $16 \div 52$. $P(\text{an ace or a club}) = \frac{16}{52} = \frac{4}{13} \approx 0.3077 = 30.77\%$ percent.

Figuring Out the Odds

The probability that something will happen and the odds of it happening are closely tied by the mathematics involved. If the probability of an event is 50 percent, then the odds of the event are 1 to 1 (one that it will, and one that it won’t).

Changing from probability to odds and back again

The probability of an event is a percentage between 0 percent and 100 percent, which is computed by dividing the number of ways an event can happen by the total number of ways that all the events can happen. For example, in a family of two children, the probability of there being at least one girl is 75 percent, because in the listing of all the possibilities: BB, BG, GB, GG, you see that three of the choices have at least one girl. Divide $3 \div 4$ to get 0.75 or 75 percent.

The odds of there being at least one girl in a two-child family are 3 to 1 (also written 3:1). You read the odds as being three ways for the event to happen and one way for it not to happen.

Computing odds given the probability

The easiest way to compute the odds when you're given the probability is let the probability be represented by p percent and write: p to $(100 - p)$ and reduce the two numbers as if they were fractions. For example, if the probability is 75 percent, you write: 75 to $(100 - 75)$ which becomes 75 to 25. Then divide each number by 25 to get 3 to 1.

Computing probability given the odds

When you're given the odds of an event, you need to write a fraction to compute the probability. If the odds are a to b , write the fraction $\frac{a}{a+b}$ and then determine the decimal and the percentage. So, for example, if the odds are 4 to 1 for an event, the probability is $\frac{4}{4+1} = \frac{4}{5} = 0.80 = 80$ percent.

Making the odds work for you

What are the odds that a person between the ages of 18 and 29 does *not* read the newspaper regularly? Did you know that someone out there has determined those odds?

The Problem: The odds that a person's first marriage will survive without separation or divorce for 15 years are 1.3 to 1. What *percentage* of first marriages survive for 15 years?



Write the fraction $\frac{1.3}{1.3 + 1}$ and solve for the decimal and percent.

$$\frac{1.3}{1.3 + 1} = \frac{1.3}{2.3} \approx 0.5652 = 56.52\text{ percent. That's pretty grim.}$$

The Problem: The odds that a Pokey Joe will *lose* the Kentucky Derby are quoted as being 23 to 2. What is the probability that Pokey Joe will *win*?



First, rewrite the odds of winning as 2 to 23. Then write the fraction and determine the percentage. $\frac{2}{2 + 23} = \frac{2}{25} = 0.08 = 8\text{ percent.}$ Pokey Joe has an 8 percent chance of winning. Notice what would have happened if I had computed his probability of *losing* using the 23 to 2 odds. $\frac{23}{23 + 2} = \frac{23}{25} = 0.92 = 92\text{ percent.}$ You then determine the probability of *winning* by subtracting the 92 percent from 100 percent.

The Problem: You had 49¢ (in American coins) in your pocket, but your pocket has a hole in it, and one coin fell out. If you had exactly seven coins in your pocket, what are the *odds* that it was a dime that fell out?



First, determine what the coins are. You know that the sum of the two numbers in the odds a to b will have to be seven, but you need more information. To have exactly 49 cents in American coins, and for there to be only 7 coins, you have to have had 1 quarter, 2 dimes, and 4 pennies (this is the only combination of seven coins that equals 49 cents). So the odds that a dime fell out are 2 to 5. Of course, this doesn't take into account the fact that a dime is smaller than the other coins and more likely to slip through the hole.

Chapter 9

Counting Your Coins

In This Chapter

- ▶ Working with denominations and number of coins
 - ▶ Figuring out the total amount of money in coins or bills
 - ▶ Working with money from around the world
-

Money is a common denominator. We deal with money just about every day — from the spending end, the earning end, or both. Counting coins is one of the earliest number exercises for a small child. And collecting coins is a passion of many historians and numismatists.

This chapter deals with computing totals of money and figuring out the number of coins from the totals. You even get a short primer on monetary units from several different countries and see how the basic properties are the same for just about any monetary system.

Determining the Total Count

Counting up the coins in your pocket or from your piggy bank involves more than just knowing that you have 80 coins. The number of coins doesn't tell you how much money you have. You usually have different denominations, so you need to count each type of coin differently. Each type of coin gets multiplied by its monetary value.

Equating different money amounts

Each country has its own monetary system with its set of coins and other currency. Often, the money is imprinted with pictures of historic figures and places. In the United States, for example, each state is honored (or soon will

be) with its own quarter. What is similar to the coinage of different countries is that different coins take on different values. In the United States, it takes more nickels to make a dollar than it does quarters, but it sure feels like you have more money when you have a dollar's worth of nickels rather than a dollar's worth of quarters.

The Problem: You want to change your \$5 bill into nickels or quarters. How many nickels and how many quarters are needed to equal \$5?



A nickel is worth 5¢, and a quarter is worth 25¢. Five dollars is equal to 500¢. To determine the number of nickels in \$5, divide 500¢ by 5¢: $500 \div 5 = 100$ nickels. The number of quarters is determined by dividing 500 cents by 25 cents: $500 \div 25 = 20$ quarters.

The Problem: A roll of quarters from the bank equals \$10, a roll of dimes equals \$5, a roll of nickels equals \$2, and a roll of pennies equals 50¢. How many rolls of dimes is worth the same as 20 rolls of nickels?



Sort through the information, first. You only need to know about the dimes and nickels. Determine the total of 20 rolls of nickels. Then figure out how many rolls of dimes you can get for that amount of money. Twenty rolls of nickels is $20 \times \$2 = \40 . Divide 40 by 5¢: $\$40 \div \$5 = 8$ rolls of dimes.

The Problem: A roll of dimes is worth \$5, a roll of nickels is worth \$2, and a roll of pennies is worth 50¢. Which is worth more: 10 rolls of nickels, 4 rolls of dimes, or 50 rolls of pennies?



Multiply the number of rolls of nickels by \$2, the number of rolls of dimes by \$5, and the number of rolls of pennies by half a dollar, or 50¢. (An alternative would be to change the \$2 and \$5 to cents; in any case, the units all have to be the same to make a comparison.) Doing the calculations: 10 rolls of nickels $\times \$2 = \20 ; 4 rolls of dimes $\times \$5 = \20 ; 50 rolls of pennies $\times \$0.50 = \25 . The pennies are worth more than the nickels or dimes by \$5.

Adding it all up

To find the total amount of money you have in coins or bills, you multiply the number of each by their monetary value and add up all the products for the total.

The Problem: You open up your piggy bank and find 177 pennies, 123 nickels, 59 dimes, 33 quarters, and 5 half-dollars. What is the total amount of money in your piggy bank?



Multiply the number of pennies by 1, nickels by 5, dimes by 10, quarters by 25, and half-dollars by 50. This gives you the number of cents. It's easier to multiply by whole numbers, at first, and then change to dollars by moving the decimal point two places to the left. Doing the computation:

$$177 \times 1\text{¢} = 177\text{¢}$$

$$123 \times 5\text{¢} = 615\text{¢}$$

$$59 \times 10\text{¢} = 590\text{¢}$$

$$33 \times 25\text{¢} = 825\text{¢}$$

$$5 \times 50\text{¢} = 250\text{¢}$$

The sum of the products is: $177 + 615 + 590 + 825 + 250 = 2,457\text{¢}$. Moving the decimal point two places (or dividing by 100), you get that the total in the piggy bank is \$24.57.

The Problem: You are in charge of the concession stand at Friday's football game. You need to bring enough coins and bills to make change for the purchases of the customers. You're given a check for \$200 to pay for the change and decide to get 50 \$1 bills, 20 \$5 bills, and the rest of the change in quarters and nickels. Quarters come in rolls worth \$10, and nickels come in rolls worth \$2. How many rolls of quarters and nickels can you get?



You first determine how much money in coins you'll be getting after taking care of the \$1 bills and \$5 bills. Add up the total for the \$1 bills and \$5 bills. Then subtract the total of the bills from \$200. Many different combinations of rolls of quarters and nickels add up to the amount you'll need in coins. Make a chart, putting the number of rolls of quarters in one column, its worth in the next column, the value remaining in the third column (after spending this much on the quarters), and then the number of rolls of nickels you can buy in the last column.



Ancient coin

An archaeologist claims that he found an ancient coin dated 46 B.C. He put it on eBay and hoped to sell it for a small fortune. Fortunately,

all people who use eBay are much too smart for him. Why didn't anyone bid on the coin?

Answer In the year 46 B.C., the break in the years between B.C. and A.D. had been determined, but it was not known if the coin was really from 46 B.C. and would have been put on a calendar based on whatever calendar he was using at the time.

Determining the amount in bills: $50 \times \$1 = \50 and $20 \times \$5 = \100 . The total of \$150 leaves \$50 in coins ($\$200 - \$150 = \$50$). In Table 9-1, I filled in the number of rolls of quarters first. Then I computed the value of the quarters, leaving the amount for nickels. Finally, I computed the number of rolls of nickels.

Table 9-1**Determining the Rolls of Quarters and Nickels**

| <i>Rolls of Quarters</i> | <i>Value of Quarters</i> | <i>Value Left for Nickels</i> | <i>Rolls of Nickels</i> |
|---------------------------------|---------------------------------|--------------------------------------|--------------------------------|
| 0 | \$0 | $\$50 - 0 = \50 | $\$50 \div \$2 = 25$ |
| 1 | \$10 | $\$50 - 10 = \40 | $\$40 \div \$2 = 20$ |
| 2 | \$20 | $\$50 - 20 = \30 | $\$30 \div \$2 = 15$ |
| 3 | \$30 | $\$50 - 30 = \20 | $\$20 \div \$2 = 10$ |
| 4 | \$40 | $\$50 - 40 = \10 | $\$10 \div \$2 = 5$ |
| 5 | \$50 | $\$50 - 50 = 0$ | $\$0 \div \$2 = 0$ |

You'll need no quarters with 25 rolls of nickels, 1 roll of quarters and 20 rolls of nickels, 2 rolls of quarters with 15 rolls of nickels, 3 rolls of quarters and 10 rolls of nickels, 4 rolls of quarters and 5 rolls of nickels, or 5 rolls of quarters with no nickels.

Working Out the Denominations of Coins

If you have enough different kinds of coins, you can create any number of money amounts. Some money amounts are possible with several different combinations of coins. You want to be creative. The standard U.S. coins are used in the problems in this section: penny (1¢), nickel (5¢), dime (10¢), quarter (25¢), half-dollar (50¢), and dollar (100¢ or \$1).

Having the total and figuring out the coins

Some people like to have lots of coins in their pockets. It gives them a feeling of wealth. Others don't like to have their pockets bulge too much, so they prefer not to have a lot of extra coins around. There are problems in this section for both types of people — depending on which type you are.

The Problem: Stefanie wants to give each of her children 42¢. How can she do this with the least number of coins for each child?



Two strange coins

A certain country has just two kinds of coins. One coin is called a *sevna* and is worth 7 units. The other coin is a *levna*, and it's worth 11 units. Because of these strange units, some prices can't be paid for exactly using these coins.

Residents can pay for items costing 7 units or 14 units or 18 units, and so on. What is the highest priced item that *can't* be paid for with any combination of these coins?

for with sevens and levnas.
than 7s and 11s is the number 59. Any number bigger than 59 can be paid
in units. The highest priced item that isn't a multiple of 7 or 11 or a combina-
tion of 7s and 11s.

Answer: You may think that it's a pretty high price that can't be paid for, but



Make a chart of the different denominations of coins and the number of each type of coin that is needed to add up to 42¢. The table won't have to contain 50-cent pieces or dollars, because they're both worth more than the total needed. See the chart I created in Table 9-2.

Table 9-2 **Adding Up to 42¢**

| Quarters | Dimes | Nickels | Pennies | Total Coins |
|-----------------|--------------|----------------|----------------|--------------------|
| 1 × 25¢ | 1 × 10¢ | 1 × 5¢ | 2 × 1¢ | 5 |
| 1 × 25¢ | | 3 × 5¢ | 2 × 1¢ | 6 |
| | 4 × 10¢ | | 2 × 1¢ | 6 |
| | 3 × 10¢ | 2 × 5¢ | 2 × 1¢ | 7 |

I'll stop here with the table. The numbers aren't getting any better. It looks like one quarter, one dime, one nickel, and two pennies are the fewest number of coins needed to add up to 42¢.

The Problem: How many different ways can you make change for a quarter?



Haul out another chart. As you fill in the different amounts, do this in an organized fashion, using dimes as many ways as you can, nickels as many ways as you can, and so on, just to make it less likely that you'll miss something. Table 9-3 has just the totals for each coin, not the number of coins.

Table 9-3**Making Change for a Quarter**

| <i>Dimes</i> | <i>Nickels</i> | <i>Pennies</i> |
|--------------|----------------|----------------|
| 20¢ | 5¢ | |
| 20¢ | | 5¢ |
| 10¢ | 15¢ | |
| 10¢ | 10¢ | 5¢ |
| 10¢ | 5¢ | 10¢ |
| 10¢ | | 15¢ |
| | 25¢ | |
| 20¢ | | 5¢ |
| 15¢ | | 10¢ |
| 10¢ | | 15¢ |
| 5¢ | | 20¢ |
| | | 25¢ |

As you see, you can make change for a quarter in 12 different ways, using standard U.S. coins.

The Problem: How can you have a total of \$1.40 using exactly 24 coins?



You can do this problem with a chart — a *big* chart. But another, not-too-glamorous way, is to try to work it out through trial and error and reasoning. For example, you could try using five quarters (the greatest number of quarters possible), leaving $140\text{¢} - 125\text{¢} = 15$ cents. Even if you used all pennies for the rest of the coins, that's only a total of $5 + 15 = 20$ coins, which isn't enough, if you're aiming for 24 coins.

Next, you could try using four quarters, leaving $140 - 100 = 40$ ¢. You now need to use 20 more coins to add up to 40¢. One dime and 30 pennies is too many coins. Two dimes and 20 pennies is still too many coins. Three dimes and ten pennies is too few coins. Are you ready to go to a chart yet? No, I'm not giving up that easily.

Think about the pennies. If you're going to use pennies, they have to be a multiple of 5 for the total to come out to an even 140¢. You need to use 5, 10, 15, or 20 pennies so that the other coins will add up with them and come out right. Start with 20 pennies (the most you can use without exceeding 24

coins) and work downward. Using 20 pennies, you now need 4 more coins to add up to \$1.20. You can do this in two different ways: a dollar coin plus a dime and two nickels, or two 50-cent pieces plus two dimes. Whew! Two answers! And there are more. For example, 2 quarters plus 8 dimes plus 9 nickels plus 5 pennies are 24 coins that add up to \$1.40. Can you find any more? If so, e-mail me at sterling@bradley.edu, and I'll post them on my Web site (<http://hilltop.bradley.edu/~sterling/facinfo.html>)!

Going with choices of coins and bills

The coins used in the United States have several things in common with the coins used in other countries. The biggest commonality is having 100¢ in \$1. Most countries have monetary systems with 100 of some coin being equal to the main monetary unit.

In the metric system, each unit is ten times or one-tenth of another unit. Not so in our money system. Even with five nickels in a quarter and ten dimes in a dollar, there is still no way to divide a quarter into an equal number of dimes. Our paper currency is a bit more forgiving, but there are still challenges with the dividing and multiplying.

The Problem: How many nickels, dimes, quarters, and half-dollars are equal in value to \$5?



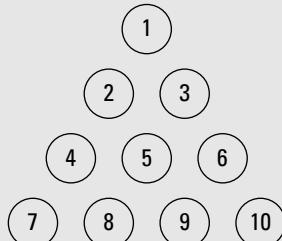
Divide 500¢ ($5 \times 100\text{¢}$) by the cents value of each of the coins. For nickels, $500\text{¢} \div 5\text{¢} = 100$ nickels. With dimes, $500\text{¢} \div 10\text{¢} = 50$ dimes. Quarters give you $500\text{¢} \div 25\text{¢} = 20$. And half-dollars yield $500\text{¢} \div 50\text{¢} = 10$ half-dollars.



The Problem: How many quarters are equivalent to \$1, \$2, \$5, \$10, \$20, \$50, and \$100?

Sliding the pennies

Look at the triangle of pennies shown in the figure. Can you reverse the triangle so that it points downward instead of upward — and do it by moving only three pennies?



Answer: Move penny 1 down, below pennies 8 and 9. Then move penny 7 up next to penny 2 and penny 10 up next to penny 3.



If you divide \$1 by 25¢, that's $100 \div 25 = 4$ quarters. Two dollars divided by 25¢ is $200 \div 25 = 8$ quarters. You can do the divisions for the rest of the bills, but an easier way than dividing by 25 is multiplying by 4. Because \$1 is equal to 4 quarters, \$20 is equal to $20 \times 4 = 80$ quarters, \$50 is equal to $50 \times 4 = 200$ quarters, and so on.

Figuring Coins from around the World

Most of the countries in the world have coin systems in which one of the coins is worth 100 of another. Other coins are then multiples of that smaller coin — usually 10 times, 20 times, or 50 times. The exceptions are few. The coin names are usually pretty interesting, though, and probably have some historical significance.

Making change in another country

In India, the monetary unit is the rupee. You find banknotes of 5, 10, 20, 50, and 100 rupees. There are also 1-rupee, 2-rupee, and 5-rupee coins. The smaller coins in India are paise. One rupee is equal to 100 paise. The coins that are multiples of the paise are 10-paise coins, 20-paise coins, 25-paise coins, and 50-paise coins.

The Problem: Is there a single coin in India that is equal to: three rupees, one 50-paise coin, three 20-paise coins, three 25-paise coins, and 15 paise? (In other words, if you have all that in your pocket, can you exchange it for a single coin?)



Determine the total amount of money all these coins are worth. Then look at the equivalence between rupees and paise and see if a coin in the list is equal to the sum. Just concentrate on the paise, at first. When you get a total, you can see if you'll need to change the rupees to paise or paise to rupees. Multiplying each of the paise coins by how many of each you have: $(1 \times 50) + (3 \times 20) + (3 \times 25) + (15 \times 1) = 50 + 60 + 75 + 15 = 200$ paise. Because 100 paise equals 1 rupee, the 200 paise are worth 2 rupees. Add these two rupees to the three rupees you already have, and you can exchange the five 1-rupee coins for a 5-rupee coin.

The Problem: Some interesting Indian coins are no longer used. In the late 1940s, you got 16 annas for 1 rupee, 4 pice for 1 anna, and 3 pies for 1 pice. How many pies could you get for 12 rupees?



Pushing a coin through paper

Can you push a quarter through a dime-sized hole in a piece of paper? Take a dime and trace around it on a piece of paper. Then carefully cut out along the circle you've drawn, leaving a hole

in the paper. Now push a quarter through the hole without tearing the paper. How can you do this?

Answer: Fold the paper so that the fold goes through the hole. Then take the ends of the fold and bend them (and the paper) down, toward each other.



Multiply the number of rupees by 16 to get the number of annas. Multiply that product by 4 to get the number of pies. Then multiply that product by 3 to get the number of pies.

$$\begin{aligned} 12 \text{ rupees} &\times \frac{16 \text{ annas}}{1 \text{ rupee}} \times \frac{4 \text{ pice}}{1 \text{ anna}} \times \frac{3 \text{ pies}}{1 \text{ pice}} \\ &= 12 \cancel{\text{ rupees}} \times \frac{16 \cancel{\text{ annas}}}{\cancel{1 \text{ rupee}}} \times \frac{4 \cancel{\text{ pice}}}{\cancel{1 \text{ anna}}} \times \frac{3 \cancel{\text{ pies}}}{\cancel{1 \text{ pice}}} \\ &= 12 \times 16 \times 4 \times 3 \text{ pies} = 2,304 \text{ pies} \end{aligned}$$

The Problem: In China, the monetary unit is yuan. One yuan is equal to 10 jiao, and 1 jiao is equal to 10 fen. The multiples and powers of ten are at work here, making the computation much easier. What is the fewest number of coins (or paper bills) you need if you have 6,348 fen?



First determine how many fen in a yuan. Then divide 6,348 by that number to get as many yuan as possible. Next, divide the remainder by 10 to convert them to jiao. The rest will have to be in fen, and there should be fewer than 10 fen. If there are 10 fen in 1 jiao and 10 jiao in 1 yuan, then there are 10×10 fen in 1 yuan. Divide 6,348 by 100 to get 63 yuan with a remainder of 48. Divide 48 by 10 to get 4 jiao with 8 left over. 63 yuan + 4 jiao + 8 fen is equal to 75 coins or bills.

Converting other currency to U.S. dollars

A common challenge for people traveling in foreign countries is converting their money to the currency of that country and back again. If you're unfamiliar with a particular monetary unit, you need to get acquainted with the relative

value (relative to your money) so that you know what the worth is of what you're buying. The exchange rates change daily, so the problems in this section use approximate values to give you an idea of the different conversions.

The Problem: If one U.S. dollar is equal to 8.28 Chinese yuan, then what is the approximate cost, in dollars, of a silk robe that's selling for 3,000 yuan?



The biggest challenge to doing these problems is in deciding whether to multiply or divide — and by what. A good approach to doing money conversions is to use proportions. Chapter 7 goes into more detail on the properties of proportions. In the case of dollars and yuans, write a fraction with \$1 divided by 8.28 yuan. Then set that fraction equal to x dollars divided by 3,000 yuan. Note that the dollars are opposite dollars, in the numerator, and the yuans are opposite yuans in the denominator.

$$\frac{1 \text{ dollar}}{8.28 \text{ yuan}} = \frac{x \text{ dollars}}{3,000 \text{ yuan}}$$

Now cross-multiply and solve for x .

$$\begin{aligned} 3000 &= 8.28x \\ \frac{3000}{8.28} &= \frac{8.28x}{8.28} \\ 362.3188 &\approx x \end{aligned}$$

The silk robe costs about \$362.

The Problem: In Vietnam, the monetary unit is the dong. One dong is equal to 10 hao, and 1 hao is equal to 10 xu. About 15,300 dong are equivalent to \$1. You have been traveling in Vietnam and have spent the day with a wonderful tour guide. You want to tip him accordingly and figure that an extra \$40 (American) will do that to your satisfaction. How many dong will you add to the bill for his tip?



Set up a proportion with dong and dollars. Solve for the needed number of dong.

$$\begin{aligned} \frac{1 \text{ dollar}}{15,300 \text{ dong}} &= \frac{40 \text{ dollars}}{x \text{ dong}} \\ x &= 15,300 \times 40 \\ &= 612,000 \end{aligned}$$

You will be tipping your guide 612,000 dong.

Chapter 10

Formulating a Plan with Formulas

In This Chapter

- ▶ Working with formula output and finding formula input
- ▶ Trying out formulas involving distance and temperature
- ▶ Applying the order of operations to formulas

A mathematical formula is a rule, comprised of different operations. A formula works and is true no matter how many times you use it. A formula is created either by observing something that repeats and writing down the pattern, or by doing some creative mathematical manipulations. For instance, Pythagoras observed that there is a relationship between the sides of a right triangle. Hence, the famous formula called the *Pythagorean theorem*. (Chapter 18 is completely devoted to Pythagoras’s contribution.)

In this chapter, you see how to handle, manipulate, and come to terms with different types of formulas. The mathematical computations must be done correctly, so I cover that here. Plugging values into a formula nets the formula output or result, but you’ll also see how to determine what input gave you a particular output.

Solving for the Formula Amount

Working with a formula requires that you have the formula written accurately, that you know what the symbols in the formula represent (so you can put the right numbers in the right places), and that you do the mathematics correctly. You should also make a prediction or plan ahead so that you have a fairly good idea of what the answer will be. This helps you spot an obvious error in computations, if it occurs.

Inserting the values correctly for area and perimeter formulas

The formula for the area of a rectangle is $A = lw$. Interpreting the symbols and operation, this formula says that the area of a rectangle, A , is equal to the product of the length of the rectangle, l , and the width of the rectangle, w .

The Problem: Find the area of a rectangle with a length of 14 inches and a width of 7 inches.

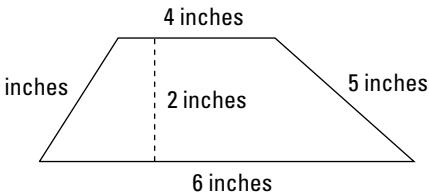


The information in this problem is pretty straightforward. Let $l = 14$ and $w = 7$, and you get that the area is $A = 14 \times 7 = 98$ square inches.

It's nice when the values for a formula are laid out for you clearly. But sometimes you have to work out the details from pictures or charts. A *trapezoid* is a four-sided polygon in which one pair of opposite sides is parallel. The formula for the area of a trapezoid is $A = \frac{1}{2} h(b_1 + b_2)$. Figure 10-1 shows you the sketch of a trapezoid with segments labeled by size. The formula for the area of a trapezoid says that the area is equal to $\frac{1}{2}$ times the height of the trapezoid times the sum of the two bases (the parallel sides). The height of a trapezoid is the distance between the two bases.

Figure 10-1:

A trapezoid is a special type of quadrilateral.



The Problem: Find the area of the trapezoid pictured in Figure 10-1 using the dimensions shown in the figure.



First, identify the values that correspond to the different letters in the formula. Then replace the letters with the numbers and simplify the expression. In the trapezoid in Figure 10-1, the height of the trapezoid is 2 inches. The two bases are 4 inches and 6 inches in length. The other two lengths — the sides — aren't needed to find the area of the trapezoid. Filling in the numbers, the area problem becomes $A = \frac{1}{2}(2)(4 + 6) = 1(10) = 10$ square inches.

If you were unhappy about not being able to use the lengths of the sides of the trapezoid in Figure 10-1, you need be unhappy no longer. The perimeter (distance around the outside) of a trapezoid is found with $P = s_1 + s_2 + s_3 + s_4$. The formula for the perimeter of a trapezoid isn't very elegant, because the lengths of the sides are often all different. You don't see pairs of lengths as in rectangles and parallelograms. But the formula for the perimeter of a trapezoid really just defines what the perimeter of any figure is: the distance around the outside.



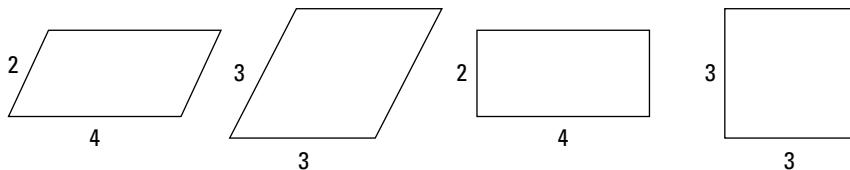
The Problem: Find the perimeter of the trapezoid in Figure 10-1.

Replace the side symbols in the formula with the measures of the sides. Do not use the height of the trapezoid. So $P = 3 + 4 + 5 + 6 = 18$ inches.

When a polygon has some regularity to it — when opposite sides are parallel or the adjacent sides are congruent or some such situation — then the perimeter formula is more elegant than just adding up the measures of the sides. The perimeter formula can contain time-saving, computation-easing operations and groupings. The perimeter of a *parallelogram* (a four-sided polygon with opposite sides that are parallel to one another) is found with $P = 2(l + w)$. The perimeter is equal to twice the sum of the length and width. Figure 10-2 shows four parallelograms, all with the same perimeter. The first (left-most) parallelogram is just that — your standard, run-of-the-mill parallelogram. The second parallelogram is also called a *rhombus*, a parallelogram with all the sides equal in measure. The next is a *rectangle*, a parallelogram with 90-degree angles, and the last is the most special of all parallelograms, a *square*.

Figure 10-2:

Four parallelograms all in a row.



The Problem: Find the perimeters of the parallelogram, rhombus, rectangle, and square from Figure 10-2 using $P = 2(l + w)$ for the parallelogram and rectangle and $P = 4s$ for the rhombus and square.



First, identify the values of l , w , and s in the figures. Then replace the letters with the correct number values in the formulas and simplify. The l and w represent the length and width of the parallelogram and rectangle. So the perimeter of each is $P = 2(4 + 2) = 2(6) = 12$ units. The letter s represents the length of any side of the rhombus or square. So the perimeter of these two figures is $P = 4(3) = 12$ units.



The horse eats the hay

A horse is tied to a 20-foot rope, and there is a bail of hay that's 35 feet away from him. But the horse still is able to eat the bail of hay. How is this possible?

Answer: Sure, the horse is tied to the rope, but the other end of the rope isn't tied to anything. So the horse can go anywhere it wants and just drag the rope behind it.

Using the correct order of operations when simplifying formulas

Mathematical formulas are made up of symbols and operations. The operations need to be performed in the correct order so that you get the correct answer.



The *order of operations* states that, when two or more operations are to be performed in a computation, you do the operations in the following order:

1. Powers or roots
2. Multiplication or division
3. Addition or subtraction

The *order of operations* is interrupted if you find grouping symbols. Operations inside parentheses or brackets or braces are performed before the results inside those grouping symbols are combined with other operations. Fraction lines also act as grouping symbols.

The Problem: The formula for finding the total amount of money in an account that earns compound interest is $A = P \left(1 + \frac{r}{n}\right)^t$ where A is the total amount of money that's accumulated, P is the principal or amount initially deposited, r is the interest rate written as a decimal, n is the number of times each year that the money in the account is compounded, and t is the number of years that all this is occurring. Find the total amount of money in an account after 10 years if \$15,000 is deposited and it earns 4.5 percent interest compounded quarterly.



Replace the letters in the formula with the corresponding numbers. The value of the expression inside the parentheses has to be computed first. But there's both addition and division inside the parentheses, so do the division, and then add the results to the 1.

$$\begin{aligned} A &= 15,000 \left(1 + \frac{.045}{4} \right)^{4(10)} \\ &= 15,000 (1 + .01125)^{4(10)} \\ &= 15,000 (1.01125)^{4(10)} \end{aligned}$$

Now the problem has multiplication, a power, and the power has a multiplication in it. Multiply the two numbers in the power. You don't see a grouping symbol around those two numbers, but it's implied — you have to combine them into one number before raising what's in the parentheses to the power. (Use a calculator — unless you want to drive yourself crazy.) Then, for the last step, multiply by the number in front.

$$\begin{aligned} A &= 15,000 (1.01125)^{40} \\ &\approx 15,000 (1.564377) \\ &= 23,465.65 \end{aligned}$$

The initial investment of \$15,000 grew to over \$23,000 in ten years.

The Problem: Heron's formula is used to find the area of a triangle when all you have to work with are the measures of the three sides of that triangle.

Heron's formula is $A = \sqrt{s(s-a)(s-b)(s-c)}$ where s is the value of the semi-perimeter (half the perimeter) and a , b , and c represent the lengths of the sides. Find the area of a triangle with sides that measure 7 yards, 18 feet, and 108 inches.



First, notice that the measures are all different units. Change all the measures to the same unit. The best choice is to change everything to yards, because inches will result in some really big numbers. Also, the sides all come out to being a whole number of yards. The second step is to find the perimeter of the triangle. $P = a + b + c$. The semi-perimeter, s , is half the perimeter. Then, finally, put all the values in their correct places in the formula and simplify.

Changing the measures, 18 feet is equal to $18 \div 3 = 6$ yards, because there are 3 feet in a yard. And 108 inches is equal to $108 \div 36 = 3$ yards, because there are 36 inches in a yard. The perimeter is: $7 + 6 + 3 = 16$ yards. So the semi-perimeter is 8 yards (half the perimeter).

$A = \sqrt{8(8-7)(8-6)(8-3)}$
 Using the formula, $= \sqrt{8(1)(2)(5)} = \sqrt{80} \approx 8.944$ square yards. Note that,

using the order of operations, the subtractions inside the parentheses are done first, because the parentheses are grouping symbols. Then the results are multiplied. And, finally, the square root of the product is found.

One more named formula — one that all algebra students will recognize quickly — is the *quadratic formula*. The quadratic formula is used to find the value of the unknown variable in a quadratic equation that's written in the form $ax^2 + bx + c = 0$. The quadratic formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where a is the coefficient (multiplier) of the squared term in a quadratic equation, b is the coefficient of the variable term raised to the first power, and c is the value of the constant.

The Problem: Find the values of x that make the equation $12x^2 - 31x + 20 = 0$ a true statement.



Use the quadratic formula, letting $a = 12$, $b = -31$, and $c = 20$. First, simplify what's under the radical, which acts as a grouping symbol. Do the power first, multiply the three terms together, then subtract the product from the power.

$$\begin{aligned} x &= \frac{-(-31) \pm \sqrt{(-31)^2 - 4(12)(20)}}{2(12)} \\ &= \frac{31 \pm \sqrt{961 - 960}}{24} = \frac{31 \pm \sqrt{1}}{24} \\ &= \frac{31 \pm 1}{24} \end{aligned}$$

A quadratic equation can result in two different answers. Using the quadratic formula, the two answers are found by doing the addition and subtraction in the numerator and dividing by the denominator.

$$\begin{aligned} \frac{31 \pm 1}{24} &= \\ \frac{31 + 1}{24} &= \frac{32}{24} = \frac{4}{3} \\ \text{or } \frac{31 - 1}{24} &= \frac{30}{24} = \frac{5}{4} \end{aligned}$$

Delving into a Formula and Its Input

A formula is used to find an answer to a problem that's computable. You use one of several formulas to find the area of a triangle. The area is *predictable*, because the area of a triangle can be computed exactly using one of those

formulas. The choice of a formula depends on what information or *input* you have available. The *input* into a formula consists of numbers that are particular to that problem or question. For the area of a triangle, the input may be the length of the base and the height of the triangle, or the input may be the measures of the three sides. It all depends on the formula being applied that time.

Taking an answer and finding the question

Using the formula for the area of a square, you can find the area when you know how long the sides are. The formula for the area of a square is: $A = s^2$ where A is the area and s is the length of a side. This area formula also allows you to find the question if you're given the answer. This is sort of like the game show *Jeopardy!*, where you give the question.

The Problem: If the area of a square is 144 square inches, then what is the length of the sides of that square?



Replace the A in the formula with 144 and solve for the value of s . You get that $144 = s^2$. Two numbers have squares that are 144, 12 and -12. Because it doesn't make any sense to have a side of a square be a negative number, then the answer must be $s = 12$.

The Problem: The area of a rectangle is found with the formula $A = lw$, where l is the length and w is the width. If the area of a rectangle is 70 square feet, and the length of the rectangle is 10 feet, then what is the perimeter of that rectangle?



This problem has two parts. First, you solve for the width of the rectangle. Then you use the length and width in the perimeter formula $P = 2(l + w)$ to find the perimeter of that rectangle.

Since $A = lw$, replace the A with 70 and the l with 10. You get $70 = 10w$. Dividing each side of the equation by 10, you get that $w = 7$. The width of the rectangle is 7 feet. Use this information to find the perimeter, replacing the l with 10 and the w with 7. The perimeter is found: $P = 2(10 + 7) = 2(17) = 34$ feet.

Consider a traveler who rents a car for \$30 per day plus mileage. He returns the car after a few days and is shocked by the bill, so he asks about the charge for the mileage.

The Problem: Clay rented a car for three days. The contract reads that the charge is \$30 per day plus mileage. The number where the mileage rate is given is all blurry. His total bill is \$310, and he drove 400 miles, so what is the charge per mile?



The total charge was based on the formula: $C = (\text{rate} \times m) + (\$30 \times d)$. This formula says that the charge, C , is equal to the mileage rate times the number of miles, m , plus \$30 times the number of days, d . Clay has all of the information he needs except for the rate per mile. Filling in the information that he has, Clay knows that $\$310 = (\text{rate} \times 400 \text{ miles}) + (\$30 \times 3 \text{ days})$. Letting the mileage rate be represented by R , the formula simplifies to: $310 = 400R + 90$. Subtract 90 from each side to get that $220 = 400R$. Divide each side by 400, and you get that $R = 0.55$. So it appears that the mileage rate was 55¢ per mile.

Comparing several inputs resulting in the same output

You can reconstruct exactly what input resulted in a certain output as long as the formula contains just one input variable. For example, if the perimeter of a square is found with $P = 4s$, and you're told that the perimeter is 48 yards, then solving $48 = 4s$, you get that s , the length of the side of the square, is 12 yards. Reconstructing the input gets a bit more challenging when the formula contains two or more input variables.

The Problem: The area of a triangle is 20 square inches. If the lengths of the base and height are whole numbers, then what are the base and height of this triangle?



The area of a triangle is found with the formula $A = \frac{1}{2}bh$ where b represents the length of the base of the triangle, and h represents the height of the triangle. (The height is drawn from the base up to the vertex of the triangle and is perpendicular to the base.) You find several possibilities for b and h that result in an area of 20 using the formula. Because the product of b and h is being multiplied by $\frac{1}{2}$, the product must be 40 before it's multiplied. Table 10-1 shows the possibilities resulting in an area of 20. **Remember:** The base and height are whole numbers.

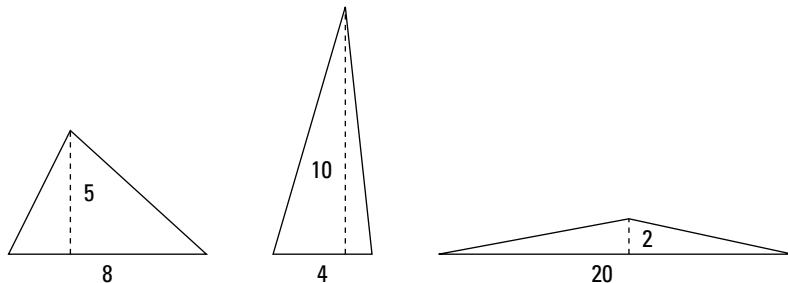
Table 10-1**The Area of a Triangle Is 20 Square Inches**

| Base of Triangle | Height of Triangle | Area ($\frac{1}{2}bh = 20$) |
|-------------------------|---------------------------|---|
| 1 | 40 | $\frac{1}{2}(1)(40) = \frac{1}{2}(40) = 20$ |
| 2 | 20 | $\frac{1}{2}(2)(20) = \frac{1}{2}(40) = 20$ |
| 4 | 10 | $\frac{1}{2}(4)(10) = \frac{1}{2}(40) = 20$ |

| <i>Base of Triangle</i> | <i>Height of Triangle</i> | <i>Area ($\frac{1}{2} bh = 20$)</i> |
|-------------------------|---------------------------|--|
| 5 | 8 | $\frac{1}{2}(5)(8) = \frac{1}{2}(40) = 20$ |
| 8 | 5 | $\frac{1}{2}(8)(5) = \frac{1}{2}(40) = 20$ |
| 10 | 4 | $\frac{1}{2}(10)(4) = \frac{1}{2}(40) = 20$ |
| 20 | 2 | $\frac{1}{2}(20)(2) = \frac{1}{2}(40) = 20$ |
| 40 | 1 | $\frac{1}{2}(40)(1) = \frac{1}{2}(40) = 20$ |

Each of the triangles given in the table has an area of 20 square inches. Figure 10-3 shows you three of the triangles as examples. Notice how different the shapes of the triangles are.

Figure 10-3:
Three triangles with an area of 20 square inches.



If you need to find out exactly which triangle the original 20 square inches came from, then you need more information about the triangle. For example, if you were told that the base was 6 inches longer than the height, then you know that the base of 10 inches and the height of 4 inches were the original input. In Chapter 13, you see how to write algebraic equations using this kind of additional information to solve problems.

The Pythagorean theorem is a formula saying that $a^2 + b^2 = c^2$. The Pythagorean theorem gives the relationship between the three sides of a right triangle. The letters a and b represent the lengths of the two shorter sides of a right triangle, and the letter c represents the length of the *hypotenuse* (the longest side, opposite the 90-degree angle) of a right triangle.

The Problem: What are the lengths of the two shorter sides of a right triangle if the hypotenuse is 25 cm? (You want only whole-number answers, no decimal approximations.)



Using the Pythagorean theorem and replacing the c with 25, you get that $a^2 + b^2 = 25^2 = 625$. You need to find two numbers such that the sum of their squares is 625. A table works well here (see Table 10-2). You solve for b in the original equation and put in numbers for a , starting with 1 and working your way up. Solving for b , you get $b = \sqrt{625 - a^2}$. As it turns out, there are four values of a that result in a perfect square under the radical and a nice value for b . The table just shows a few of the numbers that don't work and then the numbers that do give a nice answer. **Note:** In the table, I refer to a *Pythagorean triple*, which is a listing of three numbers that work in the Pythagorean theorem. They're always whole numbers.

Table 10-2 Solving for the Value of b According to Pythagoras

| <i>a</i> | <i>Solving for b</i> | <i>Comments</i> |
|----------|---|--|
| 1 | $b = \sqrt{625 - 1^2} = \sqrt{625 - 1} = \sqrt{625} \approx 24.979992$ | The result isn't a whole number. |
| 5 | $b = \sqrt{625 - 5^2} = \sqrt{625 - 25} = \sqrt{600} \approx 24.494897$ | This doesn't work, either. |
| 7 | $b = \sqrt{625 - 7^2} = \sqrt{625 - 49} = \sqrt{576} = 24$ | The Pythagorean triple is 7, 24, 25. |
| 15 | $b = \sqrt{625 - 15^2} = \sqrt{625 - 225} = \sqrt{400} = 20$ | The sides are 15, 20, 25. |
| 20 | $b = \sqrt{625 - 20^2} = \sqrt{625 - 400} = \sqrt{225} = 15$ | This is a rearrangement of the numbers you get using 15. |
| 24 | $b = \sqrt{625 - 24^2} = \sqrt{625 - 576} = \sqrt{49} = \sqrt{7}$ | This is another repeat of sorts. |

There are, of course, lots of not-so-nice answers that use decimal approximations. The answers listed here are the only ones that give whole-number measures for the sides of the triangles.

Going the Distance with Formulas

Distance is actually computed using several different formulas. The formula $d = rt$ is used to find distance when you have the rate at which an object is moving and you know how long that object has moved. Another distance formula is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, which is used to find the distance

between two points graphed in a coordinate system. A third distance formula, $s = -16t^2 + v_0t + s_0$, is actually a height formula, telling you how high something is after it travels for a period of time. You get to experience all three of these formulas here.

Distance, rate, and time are related to one another with a formula that says $distance = rate \times time$. The distance is usually given in terms of miles or feet or inches. The rate is often in miles per hour, feet per second, or inches per year. And the time is in hours, minutes, seconds, or years. In any case, the $d = rt$ formula is used to answer the questions: “How far have we gone?”, “Can you step on it?”, or, my all-time favorite from the kids sitting in the backseat, “Are we there yet?”

Solving for distance traveled

Depending on which distance formula you’re using, the input is quite different. One distance formula inputs the rate and time. The second distance formula inputs coordinates of points. And the third formula inputs just time. In all three cases, the output is a distance measure.

Distance is rate times time

When using the $d = rt$ formula, you need to watch out for the units. If the rate is miles per hour, then you need to multiply by hours.

The Problem: How far has a car traveled if it averaged 40 mph for $2\frac{1}{2}$ hours and then sped up to 55 mph for the next 45 minutes?



This problem involves two different distances that are added together — the first distance that’s traveled at 40 mph and the second distance that’s traveled at 55 mph. The 45 minutes is changed to $\frac{45}{60} = \frac{3}{4}$ hour. Computing the first distance, $d = 40 \text{ mph} \times 2.5 \text{ hours} = 100 \text{ miles}$. And the second distance is $d = 55 \text{ miles per hour} \times 0.75 \text{ hours} = 41.25 \text{ miles}$. Adding distances, $100 + 41.25 = 141.25 \text{ miles}$.

Multiplying miles per hour times hours or feet per second times seconds results in the distance because the fractions reduce, cancelling out units, as shown here.

$$d = \frac{m \text{ miles}}{\cancel{\text{hour}}} \times \frac{\cancel{h \text{ hours}}}{1} = m \times h \text{ miles}$$

$$d = \frac{f \text{ feet}}{\cancel{\text{second}}} \times \frac{\cancel{s \text{ seconds}}}{1} = f \times s \text{ feet}$$

$$d = \frac{n \text{ inches}}{\cancel{\text{year}}} \times \frac{\cancel{y \text{ years}}}{1} = n \times y \text{ inches}$$



Crossing the bridge

During a war, there was a bridge between two countries that was heavily guarded on the east side. A sentry tower on the east side was manned 24 hours a day. Every three minutes, a guard would come out of the tower to check on the bridge. His orders were to turn back anyone who was trying to cross the bridge to come to

the east and to shoot anyone who was trying to get out of the country on the east and go to the west. A woman desperately wanted to get into the western country. She knew that it would take her five minutes to cross the bridge, but she managed to do it. How did she cross from the east to the west? (No, she didn't swim.)

Answer: The lady started crossing the bridge to the west as soon as the sentry went back into his tower. After 2 minutes and 50 seconds, she turned around and started walking back to the east. The guard thought she was coming from the West and ordered her to turn around and go back.



The Problem: A glacier moves at the rate of 1.5 inches per month. How far does it travel in 20 years?

You can either change the inches per month to inches per year, using a proportion, or change 20 years to months and then use the formula. Instead of changing to a number of months, it's probably preferable to stay with years. Changing 1.5 inches per month into inches per year, use the following proportion:

$$\frac{1.5 \text{ inches}}{1 \text{ month}} = \frac{x \text{ inches}}{12 \text{ months}}$$

$$1.5(12) = x$$

$$18 = x$$

The glacier moves 18 inches in one year. Then find the distance with $d = 18$ inches per year \times 20 years = 360 inches (which is 30 feet).

Finding the distance between two points

The formula for the distance between two points on a graph uses the x and the y coordinates of the points. You find the difference between the pairs of coordinates, square the differences, and add them together. Then take the square root of the sum. The distance is in terms of *units* on the graph.

The Problem: A circle is sketched on graph paper, and the endpoints of the diameter of the circle are at the points $(-4, 7)$ and $(2, -1)$. What is the length of the diameter of this circle?



Use the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, to find the distance between the points. Put the first point's coordinates in where the subscripts are 1s and the second point where the subscripts are 2s.

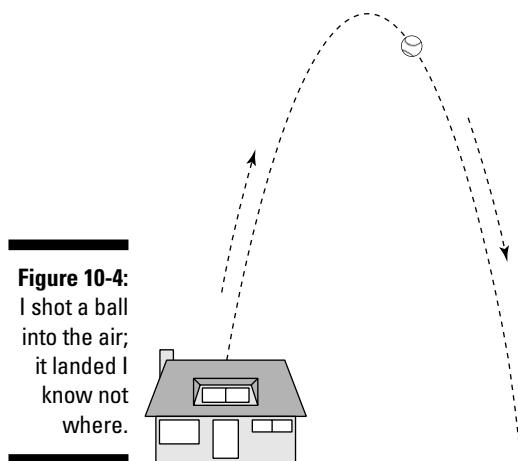
$$\begin{aligned} d &= \sqrt{(2 - (-4))^2 + (-1 - 7)^2} = \sqrt{6^2 + (-8)^2} \\ &= \sqrt{36 + 64} = \sqrt{100} = 10 \end{aligned}$$

The diameter is 10 units long.

Finding the height of the ball

When a ball is tossed into the air, it has an *initial velocity* (the speed it's traveling at the very beginning) that slows down after time because of the pull of gravity. The formula for the height of an object after a certain amount of time, t , is $s = -16t^2 + v_o t + s_o$ where s is the height of the object, $-16t^2$ represents the effects of the pull of gravity after t amount of time, v_o is the initial velocity, and s_o is the initial height of the object.

The Problem: How high is a baseball 5 seconds after it's hurled straight up into the air by a pitching machine if the ball is, at first, traveling at 100 mph and the pitching machine is sitting on the top of a building 30 feet up in the air?



The units in this problem are all over the place. The question involves seconds; the rate is in miles and hours; and one of the dimensions is in feet. First, change everything to seconds and feet. Then use the formula for the height. To change the miles per hour to feet per second, multiply miles per hour \times hour per seconds \times feet per mile. Notice how the units cancel to give you the rate you want.



1 mile = 5,280 feet and 1 hour = $60 \times 60 = 3,600$ seconds

$$\begin{aligned} & \frac{100 \text{ miles}}{1 \text{ hour}} \times \frac{1 \text{ hour}}{3,600 \text{ seconds}} \times \frac{5,280 \text{ feet}}{1 \text{ mile}} \\ &= \frac{100 \times 1 \times 5,280 \text{ feet}}{1 \times 3,600 \times 1 \text{ second}} = \frac{528,000 \text{ feet}}{3,600 \text{ seconds}} \\ &= \frac{528,000}{3,600} = \frac{5,280}{36} = 146 \frac{2}{3} \text{ feet/second} \end{aligned}$$

The rate is about 146.67 feet per second. Substituting all the information into the formula for the height, $s = -16(5)^2 + 146.67(5) + 30 = -400 + 733.35 + 30 = 363.35$ feet. After 5 seconds, the ball is 363.35 feet in the air. In case you've ever considered taking calculus, here's some further encouragement. In this problem, calculus tells you that this ball is on its way back down after reaching its highest point when t was equal to about 4.5 seconds. Now, don't you want to know more — and what else calculus can do for you?

Solving for rate or time

The distance formula, $d = rt$ gives you the distance traveled when you know how fast you're traveling and how long it's been since you started. With some simple algebraic manipulations, you change the formula into a new formula that gives you the speed when you have the distance and time, or you change the formula to one that gives you time when you're given the distance and rate.



The following formulas are all derived from one another:

$$d = rt, r = \frac{d}{t}, t = \frac{d}{r}$$

The Problem: Your friend shows up at your doorstep at 2 a.m. with baggage in hand. He left Boston, 1,200 miles away, at 6 a.m. yesterday morning. How fast was he traveling (what was his average rate of speed)?



Solve for the rate using $r = \frac{d}{t}$. Determine how many hours there are between 6 a.m. and 2 a.m. the next day and divide the distance by that number of hours. The easiest way to figure the number of hours is to just say that the amount of time was 4 hours short of a whole day. Subtract 4 from 24, and you get that he

traveled for 20 hours. Computing the rate, $r = \frac{1,200 \text{ miles}}{20 \text{ hours}} = \frac{1,200}{20} = 60$ miles per hour.

The Problem: During that fabled race between the tortoise and the hare, a ladybug was clinging to the back of the tortoise for the first 0.2 mile. The tortoise lumbered along at 3 feet per minute. As the hare passed by the tortoise, the ladybug jumped onto the neck of the hare and traveled for the next 0.3 mile of the race at a rate of 50 feet per minute, until the hare decided to take a nap. How long did the ladybug get to travel on these two racers before nap time took over?



Find the two different times and add them together. The time that the ladybug spent on the tortoise is found by dividing 0.2 mile by 3 feet per minute. Of course, first, the miles have to be changed to feet. Then find the amount of time on the hare by dividing 0.3 mile by 50 feet per minute. Add the two times together.

Changing 0.2 mile and 0.3 mile to feet, use the fact that 1 mile = 5,280 feet.

$$\begin{aligned}\frac{0.2 \text{ mile}}{1 \text{ mile}} &= \frac{x \text{ feet}}{5,280 \text{ feet}} & \frac{0.3 \text{ mile}}{1 \text{ mile}} &= \frac{x \text{ feet}}{5,280 \text{ feet}} \\ 0.2(5,280) &= x & 0.2(5,280) &= x \\ 1,056 &= x & 1,584 &= x\end{aligned}$$

Now use the formula $t = \frac{d}{r}$ to solve for the two times.

$$t = \frac{1,056 \text{ feet}}{3 \text{ feet/sec.}} = 352 \text{ sec.}; t = \frac{1,584 \text{ feet}}{50 \text{ feet/sec.}} = 31.68 \text{ sec.}$$

Adding the times together, $352 + 31.68 = 383.68$ seconds. This is about 6.39 minutes. What a ride!

The distance formula $s = -16t^2 + v_o t + s_o$ gives you the height of an object when you know how much time has elapsed. You determine the input, the amount of time, by solving the quadratic equation for t .

The Problem: A toy rocket is shot into the air at 144 feet per second from the top of a platform that's 2 feet high. How long does it take for the rocket to reach a height of 130 feet?



Divided by the mighty Mississippi

The Mississippi River is the dividing line between Tennessee and Arkansas. If an airplane crashed exactly in the middle of the

Mississippi River, where the distance between the two banks is exactly the same, then where would the survivors be buried?

Answer: This is an oldie but goodie. Goodness, you wouldn't bury survivors!



Using the formula, and filling in 130 for s , 144 for v_0 , and 2 for s_o , the equation becomes $130 = -16t^2 + 144t + 8$. Subtract 130 from each side of the equation, and you get the quadratic equation $0 = -16t^2 + 144t - 128$. This equation factors into $0 = -16(t^2 - 9t + 8) = -16(t-1)(t-8)$. Setting the two binomial factors equal to 0, you get that $t = 1$ or $t = 8$. The question is: How long does it take for the rocket to reach 130 feet? The answer is that it only takes 1 second. It then goes higher until it turns back down toward the ground. At 8 seconds after launching, the rocket is back down to 130 feet and eventually hits the ground (assuming there's no parachute attached).

Testing the Temperature of Your Surroundings

Temperature is an important measure to everyone. When the snow is falling and the wind blowing, it's much too cold. When the sun is blazing and the air not stirring, it's too hot. If you want your pie to bake to a perfect golden brown, you need the right setting on your oven. And if that isn't enough, more challenges arise when the thermometer is scaled in degrees Celsius instead of Fahrenheit — or vice versa, depending on your choice of measure.

Changing from Fahrenheit to Celsius

The Fahrenheit temperature scale registers freezing at 32°F and boiling at 212°F , a difference of 180° . The Celsius scale (also called *centigrade*) has freezing at 0°C and boiling at 100°C , a difference of 100° . The ratios of $180:100$ and $100:180$ are what make up a major part of the formulas for changing from one temperature to another. The formula for converting Fahrenheit temperatures to Celsius temperatures is ${}^\circ\text{C} = \frac{5}{9}({}^\circ\text{F} - 32^\circ)$. To figure out what the Celsius temperature is, you subtract 32° from the Fahrenheit temperature and multiply by $\frac{5}{9}$. The fraction comes from reducing $\frac{100}{180}$ — one of the ratios.

The Problem: A good friend of yours lives in England, and, when you told her that it was 104° outside yesterday, she gasped and said that it just couldn't be. You remembered that she uses the Celsius system, so you converted the temperature to one that she is more accustomed to. What new temperature did you give her?



Use the formula for changing from Fahrenheit to Celsius. ${}^\circ\text{C} = \frac{5}{9}(104^\circ - 32^\circ) = \frac{5}{9}(72^\circ) = 40^\circ$. You changed the temperature to 40° , and she was satisfied.

Changing from Celsius to Fahrenheit

The formula for changing a Celsius temperature to a Fahrenheit temperature is $^{\circ}F = \frac{9}{5}(^{\circ}C) + 32$. Notice that you first multiply the Celsius measure by the fraction and then add 32°.

The Problem: You're planning a trip to Germany and read that the expected temperature for the week of your visit is 25°. You pack a heavy coat and woolen mittens and lots of sweaters. Is this a mistake?



Change the Celsius temperature to Fahrenheit. $^{\circ}F = \frac{9}{5}(25) + 32 = 45 + 32 = 77$. It's going to be a balmy 77°. You're way overdressed and you're too warm in your winter clothes.

The Problem: What temperature in degrees Fahrenheit is the same temperature in degrees Celsius?



You want $^{\circ}C = ^{\circ}F$, so replace one of the sides of the equation with its conversion formula. Replacing the $^{\circ}F$ with its conversion equivalence, you get an equation that's solved by subtracting $\frac{9}{5}C$ from both sides and then multiplying both sides by the reciprocal of $-\frac{4}{5}$.

$$\begin{aligned} ^{\circ}C &= ^{\circ}F \\ C &= \frac{9}{5}C + 32 \\ -\frac{4}{5}C &= 32 \\ C &= -40 \end{aligned}$$

A temperature of $-40^{\circ}\text{C} = -40^{\circ}\text{F}$. This temperature is the only one that's the same in both scales.

Cooling off with Newton's Law

Newton's Law of Cooling says that the greater the difference in temperature of an object is from its surroundings, the more quickly the object's temperature will change toward the surrounding temperature. For instance, when you pour a cup of hot coffee and set it on the counter in a room that's 72°F, the coffee cools off much faster during the first few minutes than it does as it gets closer to the temperature of the room. This is the same with heating (opposite of cooling). Have you ever waited impatiently for that turkey to finish cooking on Thanksgiving day? The last couple of degrees on the thermometer always seem to take the longest — and they do!

The formula for Newton's Law of Cooling is: $T(t) = T_a + (T_o - T_a)e^{-kt}$ where T is the temperature of the object after the amount of time t , T_a is the temperature of the environment or room, T_o is the original temperature of the object, and the multiplier e^{-kt} has the number e which is about 2.71828 and the constant k which changes, depending on the density of the object.

The Problem: You've just cooked your pizza in a 450°F oven. You take it out and set it in a room that's 72°F. For the density of this pizza, the constant, k , in Newton's Law of Cooling is 0.0843. What will the temperature of your pizza be after five minutes? Will it be cool enough to eat?



Using Newton's Law of Cooling, let the temperature, t , be 5, the initial temperature of the pizza be 450°F, the temperature of the environment be 72°F, and the value of k = 0.0843. Replacing all the letters in the formula with these numbers and simplifying, you get:

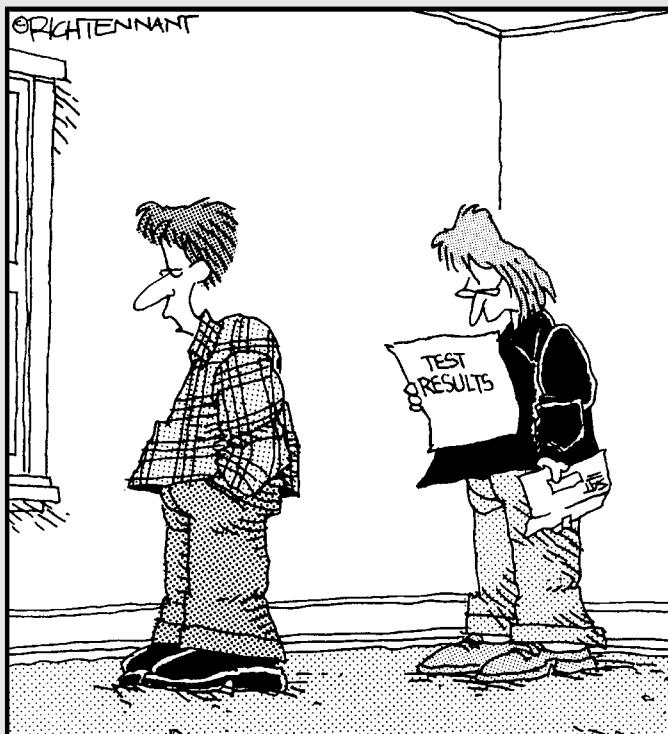
$$\begin{aligned}T(5) &= 72 + (450 - 72)e^{-(0.0843)(5)} \\&= 72 + 378e^{-0.4215} = 72 + 247.991 = 319.991\end{aligned}$$

At about 320°F, this is still too hot to eat.

Part III

Tackling Word Problems from Algebra

The 5th Wave By Rich Tennant



"If it's okay for them to ask experimental questions, I figure it should be okay for me to give some experimental answers."

In this part . . .

The problems in this part are probably where math word problems get their biggest bad rap. Introduce a math word problem that starts with: “If Matt drives 10 miles per hour faster than Michelle . . .” and you see eyes glaze over, a shudder pass through the body, and even a sweat break out on the forehead.

The chapters in this part help you find your way through the maze of mythical mathematical word problems. No more anxiety attacks — in this part, I show you how to move through the problems with confidence.

Chapter 11

Solving Basic Number Problems

In This Chapter

- ▶ Writing word problems as algebraic equations
- ▶ Making comparisons of numbers using mathematical operations
- ▶ Finding answers requiring two numbers and choosing which of two numbers is the answer

Word problems in algebra are problems that seem to take on a life of their own. You start out with a simple sentence in English and end up with an equation that, you hope, will answer the question that's been posed.

In this chapter, you see how to use word clues to convert from English sentences to algebraic statements that can be solved and checked. The problems in this chapter aren't really *themed* — they're all over the place — but they show how to put algebra to its best use.

Writing Equations Using Number Manipulations

Basic number problems are like the riddles or brainteasers that your fifth-grade teacher gave you to get you interested in doing numbers in your head. Your teacher may have said: "I'm thinking of a number. If I double it and subtract five, then I get a number that's two more than the original number." The first member of your class to guess the number got a prize, like a new, spiffy-doodle pencil. In this section, you find number problems like this. (Before starting, you may want to refer back to Chapter 5 for a review of operations and their corresponding words.)

Changing from words to math expressions

The word *and* is equivalent to addition; *difference* refers to subtraction. It's the variations on and the subtle use of these words that affect how the mathematical expression is written. Look at the following interpretations.

| | |
|--|---------------|
| The sum of three times x and y | $3x + y$ |
| Three times the sum of x and y | $3(x + y)$ |
| The product of x and four more than y | $x(y + 4)$ |
| Four more than the product of x and y | $xy + 4$ |
| Five less than the sum of x and y | $(x + y) - 5$ |
| Five minus the sum of x and y | $5 - (x + y)$ |
| The sum of four and the square of x | $4 + x^2$ |
| The square of the sum of x and 4 | $(x + 4)^2$ |
| Twice the square of the sum of six and y | $2(6 + y)^2$ |
| The square of the sum of six and twice y | $(6 + 2y)^2$ |

The subtleties in the wording are important in the interpretation. The implied grouping symbols need to be written when translated into mathematics. The English language can be misinterpreted (in word relations, too, of course), so it's usually a good idea to use as many descriptive words as necessary to be sure that your mathematical meaning is understood.

Solving equations involving one number

Many algebra word problems involve a single number and some operations performed on it. The problems of this sort are good for practice in writing algebraic equations, because you only need to worry about one unknown, or variable. For simplicity, I'll use x for the unknown.

The translations from word problems to algebraic problems have two standard rules:

- | ↗ The variable always represents a number.
- | ↗ The sentence is translated directly (in order) letting the verb be replaced by an equal sign.

The Problem: The sum of a number and 12 is 18. What is the number?



Let the number be represented by x . The word *sum* implies addition, so add the x and 12. Put an equal sign after the sum and before the 18. The equation you want reads: $x + 12 = 18$. Solving this linear equation, you subtract 12 from each side to get $x = 6$. Now, check your answer. Replace *a number* in the problem with 6, and it now reads: The sum of 6 and 12 is 18. You got it!

The Problem: The sum of a number and 12 is twice the number. What is that number?



This problem is different from the one earlier only in the result, or the value on the right side of the equation sign. You write *twice the number* using $2x$. So this equation is written: $x + 12 = 2x$. Subtracting x from each side, you get that $12 = x$. So, checking, "The sum of 12 and 12 is twice 12." Well, sure! It just sounded more complicated in math speak.

The Problem: The sum of a number and three more than twice the number is 36. What is the number?



The operation that's suggested here is addition. You want the sum of a number and *some stuff*. You now tackle the expression *three more than twice the number*. The words *more than* suggests addition, too. So you write *three more than twice a number* as $3 + 2x$. Now, back to the original sentence, you write $x + (3 + 2x)$ to represent the sum of a number and three more than twice the number. The parentheses aren't really necessary. I just put them there to emphasize the two different numbers in the sum. Writing the complete equation to replace the complete sentence: $x + (3 + 2x) = 36$. To solve this equation, you first combine the terms on the left and get $3x + 3 = 36$. Now subtract 3 from each side, giving you $3x = 33$. Dividing by 3, you get $x = 11$. Going back to the original problem, you check to see if the sum of 11 and three more than twice 11 is 36. Three more than twice 11 is 25. And the sum of 11 and 25 is, indeed, 36.

The Problem: When five is subtracted from twice a number, the result is two more than the number. What is that number?



This equation has operations on each side. Letting the unknown number be x , you write *five subtracted from twice the number* as $2x - 5$. Notice the order of the subtraction. You can't reverse subtraction and get the same result. Now, for the other side of the equation, you write *two more than the number* as $2 + x$. You could also have written $x + 2$, because addition is commutative and can be written in either order. Putting the two parts together, you get the equation $2x - 5 = 2 + x$. Add 5 to each side and subtract x from each side to get $x = 7$. Checking this with the original problem, you see that when 5 is subtracted from twice 7, you subtract 5 from 14 and get 9. Two more than 7 is also equal to 9. It checks. Do you recognize this problem? It's the example I accused your fifth-grade teacher of giving you (flip back to the first page of this chapter).



How many goats?

Avery and Reid both are raising goats for a 4-H project. Avery says to Reid, "If you let me have one of your goats, then I'll have twice as many goats as what you have left." Reid then counters

with, "If you let me have one of *your* goats, then we'll have the same number of goats." How many goats does each have right now?

Answer: Avery has seven goats, and Reid has five goats. The quickest way to do this problem is by trial and error: If the answer involved hundreds of goats, then you may have wanted to use algebraic equations. Start out by letting Reid have two goats and Avery have one goat. If Reid gives Avery one goat, then Avery will have two goats and Reid will have one. That takes care of the first part of the problem, but it doesn't work for the second. Keep trying numbers — increasing the number of goats for one, then the other, until you stumble on the answer. Just keep track of which numbers you've tried so you don't repeat yourself. Use a system, be methodical.

Comparing Two Numbers in a Problem

Number problems can contain situations involving one, two, or even more different integers, whole numbers, or fractions. The more numbers you have to solve for, the more interesting the problem becomes. Usually, when more than one number is involved in one of these problems, there's some sort of relationship between the numbers — some mathematical comparison. Chapter 12 is completely devoted to *consecutive integers*, so you'll find other types of problems here. Other problems requiring that you find two or more solutions are solved with systems of equations. (You'll find systems in Chapter 17.)

Looking at the bigger, the smaller, and the multiple

Children compare their bicycles or cookies or even their fathers by saying that theirs is bigger or faster or smarter or whatever comes to mind. Mathematical expressions take over where childish comparisons leave off, using operations to combine and compare different numbers.

The Problem: I'm thinking of two numbers. One of the numbers is four less than the other, and their sum is 100. What are the numbers?



Select a variable to represent one of the numbers, such as x . Now write an expression, using the x , to describe the other number. Even though it may be tempting to let the other number be represented by a new variable, y , you stay with just one unknown or variable. Because the other number is four less than the first, you can write the other number as $x - 4$. The sum of the two numbers is 100, so write the sum of x and $x - 4$ as being equal to 100. Your equation is $x + (x - 4) = 100$. Remove the parentheses and simplify the expression on the left. $2x - 4 = 100$. Now add 4 to each side, and you have $2x = 104$. Divide by 2, and $x = 52$. The other number is four less than 52. So $x - 4$ becomes $52 - 4 = 48$. The two numbers are 52 and 48.



What if you had decided, in the preceding problem, that the x should be the smaller number, so the larger number is written as $x + 4$? Note that x is four less than $x + 4$. How will that affect the answer? Try adding the numbers together and solving for x . You get $x + (x + 4) = 100$, $2x + 4 = 100$, $2x = 96$, and $x = 48$. The smaller number is 48, and the number four bigger than that is 52. You get the same two numbers! As long as you write the relationship between the numbers correctly, it really doesn't matter which of the two you solve for first.

The Problem: Separate the number 20 into two parts so that five times the smaller part plus eight is equal to the larger part. What are the two numbers?



One way to do this problem is to hope that the numbers are whole numbers and make a list or chart, guessing what they may be. The number 20 is small enough that a chart is reasonable. A more efficient way, though, is to use an algebraic equation. You'd definitely choose the equation route if the two numbers added up to 200 or 2,000 instead of a nice, civilized 20.

Let x be one of the parts. The sum of x and some other number is supposed to be 20. So you can write the other number as $20 - x$. (If the sum of two numbers is 20, and one of the numbers is 7, then the other number is $20 - 7 = 13$.)

Now you have the two parts of 20, x , and $20 - x$. Back to the problem. The problem says to multiply 5 times the smaller part. You choose the smaller part to be x . Yes, you *can* do that! Writing *five times the smaller part plus eight*, you get $5x + 8$. This sum is equal to the larger part (the other number), so your equation becomes $5x + 8 = 20 - x$. To solve the equation, add x to each side and subtract 8 from each side to get $6x = 12$. Now divide each side, and you get that $x = 2$. If the number 2 is one *part* of 20, then the other is $20 - 2 = 18$. Does this check with the original problem? Is $5 \times 2 + 8$ equal to 18? Doing the math, $5(2) + 8 = 10 + 8 = 18$. By golly, it works!

The Problem: One number is three less than two times another number, and their sum is 21. What are the numbers?



By special arrangement

What is so special about the following sequence of numbers? 8, 5, 4, 9, 1, 7, 6, 10, 3, 2, 0

Answer: They're in alphabetical order!



Choose x to be one of the numbers, and then write the other number using mathematical operations to express how they compare. If the second number is *three less than two times* x , you write that as $2x - 3$. Notice that the 3 is subtracted from the $2x$. So now you have the two numbers, x and $2x - 3$. The sum of the two numbers is 21. The equation expressing this is $x + (2x - 3) = 21$. Combine the terms on the left to get $3x - 3 = 21$. Add 3 to each side, and the equation becomes $3x = 24$. Divide by 3, and $x = 8$. The number 8 and another number are supposed to add up to 21. That other number has to be 13. Is the number 13 equal to three less than twice 8? Yes, $2(8) - 3 = 16 - 3 = 13$.

Varying the problems with variation

The word *variation* has a special definition in mathematics. When two numbers *vary directly*, it means that one of the numbers is a *direct* multiple of the other. The numbers 7 and 21 vary directly, because 21 is three times 7. When two numbers *vary inversely*, it means that one number is some multiple of the reciprocal of the other.



If y varies directly with x , then $y = kx$, where k is some constant.

If y varies inversely with x , then $y = \frac{k}{x}$, where k is some constant.

The usual situation with variation problems is that you'll be told that two numbers vary either directly or inversely with a specific result. Then you have to figure out what that particular variation is and how it applies to another number.

Varying directly

When two values vary directly, then one of the numbers is a multiple of the other. You need to determine what the multiplier (the value of the constant, k) is. Formulas and applications from the sciences often use direct variation.

The Problem: If y varies directly with x , and y is equal to 20 when x is equal to 5, then what is the value of y when x is 2?



First solve the direct variation equation $y = kx$ for the value of k , the constant that is particular to this problem. Substituting the values of y and x that are given, you get $20 = k5$. Dividing each side of the equation by 5, you find that $k = 4$. Now rewrite the relationship equation as $y = 4x$. When x is 2, you solve for y by putting 2 into the formula. $y = 4 \times 2 = 8$. When x is 2, y is 8.

The Problem: If the square of y varies directly with the cube of x , and if y is equal to 3 when x is equal to 2, then what is the value of x when y is 24?



The variation equation is written $y^2 = kx^3$, and you solve for k by solving the equation $(3)^2 = k(2)^3$. You get that $9 = 8k$. Dividing each side of the equation by 8, you see that $k = \frac{9}{8}$. Now use the relationship $y^2 = \frac{9}{8}x^3$, replacing the y with 24, to solve for x . $(24)^2 = \frac{9}{8}x^3$, $576 = \frac{9}{8}x^3$. Multiply each side of the equation by $\frac{8}{9}$, and the equation becomes $512 = x^3$. Taking the cube root of each side, you get that $x = 8$.

The Problem: The volume of a sphere varies directly with the cube of its radius. If a sphere with a radius of 3 yards has a volume of about 113.10 cubic yards, then what is the volume of a sphere that has a radius of 5 yards?



The variation equation is $V = kr^3$, where V represents the volume of the sphere and r represents the radius. Using the values given, rewrite the equation as $113.10 = k(3)^3$ or $113.10 = 27k$. Dividing each side by 27, k is about 4.19. Now the relationship between the volume of a sphere and its radius is written $V = 4.19r^3$. Replacing the radius, r , with 5, you get $V = 4.19(5)^3 = 4.19(125) = 523.75$ cubic yards.



The standard formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$. Letting π be approximately 3.1416, then $\frac{4}{3}\pi = \frac{4}{3}(3.1416)$ or about 4.19.

Varying inversely

When one number varies *inversely* with another, then the reciprocal of the one is some multiple of the other number.



Carnival game

A boy was at a carnival and went to a booth where the attendant offered the following: "If I write your exact weight on this piece of paper, then you have to give me 25¢. If I can't do that, then I'll give you \$1." The boy was pretty

confident, so he said, "You're on!" The attendant took a slip of paper, wrote something on it, and gave it to the boy. The boy paid him the 25¢. What was on the paper?

Answer: The attendant had written, "Your exact weight!"

The Problem: If y varies inversely with the square root of x , and if y is 9 when x is 4, then what is the value of y when x is 36?



Start with the inverse variation equation, $y = \frac{k}{\sqrt{x}}$. Replace the y with 9 and the x with 4 and solve for k . $9 = \frac{k}{\sqrt{4}}$, $9 = \frac{k}{2}$. Multiplying each side by 2, you get that $k = 18$. Now replace the k with 18 and the x with 36 in the new inverse variation equation and solve for y . $y = \frac{18}{\sqrt{36}} = \frac{18}{6} = 3$.

The Problem: The rate of vibration of a taut string varies inversely with the length of the string. If a particular string is 50 inches long, it vibrates 250 times per second. How long is a string that vibrates 100 times per second?



Using the inversely varying equation $V = \frac{k}{L}$ with V representing vibrations and L representing length, you solve for k $250 = \frac{k}{50}$, $5 = 250 \cdot 50 = 12,500$.

The value of k holds for this particular string at different lengths. Now, letting $k = 12,500$ and $V = 100$, solve for L . $100 = \frac{12,500}{L}$, $L = \frac{12,500}{100} = 125$.

The length of the string is 125 inches.

Squaring Off Using Quadratic Equations

A quadratic equation has a squared term in it. For instance, $x^2 + 2x - 3 = 0$ is a quadratic equation. An interesting feature of these equations is that many quadratic equations have two completely different solutions. In the equation $x^2 + 2x - 3 = 0$, x is equal to either +1 or -3. Either number works. Number problems using quadratic equations in their solutions may or may not use both answers. After solving the problem, you go back and look at the original question to see if both answers make sense.



Painting the number 9

Stefanie was hired to paint the numbers on the doors of the offices in a new academic building. There are 100 offices, numbered 1 through 100.

How many times did Stefanie have to paint the number 9?

Answer: Did you remember to include the 9s in the nineties? She had to paint the number 9 a total of 20 times: 9, 19, 29, 39, 49, 59, 69, 79, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, and 99.

Doubling your pleasure, doubling your fun

Number problems that need two answers or end up with two solutions have to be checked carefully. The equations used when solving these number problems frequently involve expressions relating one of the numbers to the other number using mathematical operations.

The Problem: The sum of a number and eight times its reciprocal is 6. What are the two numbers?



First write expressions for the number and its reciprocal. Letting x be the number, then its reciprocal is $\frac{1}{x}$. Multiplying that reciprocal by 8, $8\left(\frac{1}{x}\right) = \frac{8}{x}$. Now write an equation that represents the sentence exactly. The word *sum* indicates that the number and its reciprocal are added. The word *is* indicates where the equal sign goes. The equation you write is $x + \frac{8}{x} = 6$. Multiply each term in the equation by x , and the quadratic equation $x^2 + 8 = 6x$ emerges. Subtract $6x$ from each side to set the equation equal to 0.



When solving a quadratic equation of the form $ax^2 + bx + c = 0$, you either factor the expression and set the factors equal to 0 to solve for x , or you use the quadratic formula. (You'll find the quadratic formula in the Cheat Sheet.)

The quadratic equation $x^2 - 6x + 8 = 0$ factors into $(x - 2)(x - 4) = 0$. Setting the factor $x - 2 = 0$, you get that $x = 2$. Setting the factor $x - 4 = 0$, you get the answer $x = 4$. Each answer needs to be checked. Is the sum of 2 and eight times its reciprocal equal to 6? Checking this out, $2 + 8\left(\frac{1}{2}\right) = 2 + \frac{8}{2} = 2 + 4 = 6$. It's true, so 2 works. You'll find that the 4 works, also.
 $4 + 8\left(\frac{1}{4}\right) = 4 + \frac{8}{4} = 4 + 2 = 6$.

The Problem: The sum of two numbers is 7 and the sum of the squares of the two numbers is 29. What are the numbers?



You first write the two numbers in terms of the same unknown or variable. If the first number is x , then the other number is $7 - x$. How did I pull the $7 - x$ out of my hat? Think about two numbers having a sum of 7. If one of them is 5, then the other is $7 - 5$, or 2. If one of them is 3, then the other number is $7 - 3$, or 4. Sometimes, when you do easy problems in your head, it's hard to figure out how to write what you're doing in math speak. So, if the two numbers are x and $7 - x$, then you have to square each of them, add them together, and set the sum equal to 29. The equation to use is $x^2 + (7 - x)^2 = 29$. To solve this equation, you square the binomial, combine like terms, subtract 29 from each side, factor the quadratic equation, and then set each of the factors equal to 0.

$$\begin{aligned}
 x^2 + (7 - x)^2 &= 29 \\
 x^2 + 49 - 14x + x^2 &= 29 \\
 2x^2 - 14x + 20 &= 0 \\
 2(x^2 - 7x + 10) &= 0 \\
 2(x - 2)(x - 5) &= 0
 \end{aligned}$$

When $x - 2 = 0$, you get that $x = 2$. Subtracting $7 - 2$ for the other number, you get 5. The square of 2 is 4, and the square of 5 is 25. The sum of 4 and 25 is 29, so the 2 works. You don't have to check the other solution from the quadratic equation, because $x - 5 = 0$ gives you the number 5. Both solutions work and are just repeats of one another. But sometimes the numbers that appear as the two solutions aren't the two numbers that answer the problem.

The Problem: The product of two integers is 48, and one of the integers is two less than the other. What are the two numbers?



If one of the numbers in the product is x , then the other number is $\frac{48}{x}$.

Writing that one number is two less than the other, you can either write an equation where one of the numbers equals the other minus 2 or where one is equal to the other plus 2. It doesn't really matter, because you'd be writing a statement making one side two more than the other. One equation you may use to solve this problem is $x = \frac{48}{x} + 2$. Multiply each term by x and solve the equation.

$$\begin{aligned}
 x &= \frac{48}{x} + 2 \\
 x^2 &= 48 + 2x \\
 x^2 - 2x - 48 &= 0 \\
 (x - 8)(x + 6) &= 0
 \end{aligned}$$

If you let $x - 8 = 0$, you get that $x = 8$.

The other number is found by dividing 48 by 8, so you get that the other number is 6. And the number 6 is 2 less than 8. The other factor of the equation gives you that $x + 6 = 0$. The value of x this time is -6 . The -6 doesn't work with the 8, because the product of -6 and 8 is -48 , and the two numbers are different by 14, not 2. But what about the -6 ? If it's one of the numbers, then dividing 48 by -6 you get -8 . The product of -6 and -8 is 48, and the -8 is two less than -6 . So there are two different sets of answers, each with the two numbers required.



Guessing the money in the box

A new television game show starts out by eliminating contestants from the audience by having them guess the amount of money in four different boxes. One hundred contestants are shown the four boxes and told to guess which box has \$1, which has \$5, which has \$1,000, and which has \$10,000. Those who guess all four boxes correctly get to come on stage for round two.

When the entries are tallied, the game-show host announces that 50 people have none correct, 20 people have one correct guess, and 15 people have two correct. Those who got three correct would be given \$100 each, and those who got all four correct would get to come up on stage to compete again. How many people got to come up on stage?

Answer: No one could have gotten three correct, if they guessed a different amount of money for each box. If they missed one, then they had to miss two (switch the boxes). So 15 people went up on stage for round two ($100 - 50 - 20 = 15$).

Disposing of the nonanswers

Quadratic equations may yield two different answers. Sometimes both answers work in a word problem. But in some instances, only one works or neither of the answers actually answers the question. Just because you can solve a quadratic equation for a correct answer to the equation, it doesn't mean that what you get will solve the original question. The question may be unsolvable.

The Problem: The square of a positive number is six more than five times that number. What is the number?



Let the number be represented by x . The square of x is written x^2 . The expression *six more than five times the number* is written $6 + 5x$. Put an equal sign between the two expressions, to represent the *is*, and you have $x^2 = 6 + 5x$. Subtracting 6 and $5x$ from each side, $x^2 - 5x - 6 = 0$ factors into $(x - 6)(x + 1) = 0$. When $x - 6 = 0$, you get the answer 6. The number 6 squared is 36, and six more than five times 6 is $6 + 30 = 36$. The other factor gives you that $x + 1 = 0$, or $x = -1$. You don't consider this answer, because the problem asks for a positive number. This problem has just the one answer.

The Problem: One positive integer is three more than twice another positive integer, and the difference of their squares is nine. What are they?



The two numbers are represented by x and $3 + 2x$. Their squares are x^2 and $(3 + 2x)^2$. Their difference can be written as either $x^2 - (3 + 2x)^2$ or $(3 + 2x)^2 - x^2$. Either works. I'm going to use the second version, because I don't want to have to distribute the negative sign over the three terms in the square of the binomial. Writing that the difference is equal to 9, $(3 + 2x)^2 - x^2 = 9$.

$$(3 + 2x)^2 - x^2 = 9$$

$$9 + 12x + 4x^2 - x^2 = 9$$

$$3x^2 + 12x = 0$$

$$3x(x + 4) = 0$$

When you set $3x = 0$, you get that $x = 0$. When $x + 4 = 0$, you get that $x = -4$. Neither of these numbers works in the original problem, because you need a *positive* integer, and neither is positive. This problem has no answer. Even though the quadratic equation has solutions, neither works with the question.

Chapter 12

Charting Consecutive Integers

In This Chapter

- ▶ Creating lists of consecutive integers from descriptions
- ▶ Focusing on consecutive odds and evens
- ▶ Solving for one of several in a list
- ▶ Applying consecutive integers to practical problems

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You became familiar with your first list of consecutive integers when you were taught to count. You may not remember when you held up fingers and counted, “One, two, three. . . .” But we all did it — except maybe Einstein, who probably counted by threes.

In this chapter, you find consecutive integers, consecutive even integers, consecutive multiples of fives, and so on. The word problems come in as puzzles to find the first, the middle, or the last in a list of consecutive integers. After an introduction on ways to find the sum of a large number of consecutive numbers, you’ll see some interesting applications from seating charts to orchards.

Adding Up Sets of Consecutives

A list of consecutive integers may be one of the following:

- ✓ 3, 4, 5, 6, 7: Five consecutive integers starting with 3
- ✓ -12, -10, -8, -6: Four consecutive even integers ending with -6
- ✓ 15, 20, 25, 30, 35: Five consecutive multiples of 5 starting with 15
- ✓ 0, 4, 8, 12, 16: Five consecutive multiples of 4, with middle 8

Any list of consecutive integers can be described in many different ways. You may give the overall pattern that describes how far apart the numbers in the list are, and then you give one of the numbers in the list and its position. Or

you may give the rule on how far apart the numbers are and then some characteristic of the list. You may give the first and last numbers and tell how many are in the list. All you need is enough information to distinguish your list from any others.

Writing the list algebraically

A list of consecutive integers consists of numbers that are all the same distance apart. You structure the distance apart by adding some constant value repeatedly, until you have the desired number of integers in the list. For example, if you want a list of the first six multiples of 3, starting with the number 0, you find the numbers by starting with 0, then adding 3 to 0 to get 3, then adding another 3 to 3 to get 6, and so on. Here's how to create that list: $0, 0 + 3, 0 + 3 + 3, 0 + 3 + 3 + 3, 0 + 3 + 3 + 3 + 3, 0 + 3 + 3 + 3 + 3 + 3$. Simplifying these terms, I write them: $0, 0 + 3, 0 + 6, 0 + 9, 0 + 12, 0 + 15$. You're probably wondering why I can't take that final leap and add the zeros in. You'll see my point shortly.

When writing a list of consecutive integers, you let the first integer in the list be signified with a variable. Some people like to use x , because that's the universal unknown or variable. In this chapter, you see a consistent use of another variable, n . Using n for consecutive integers is pretty standard notation, too.

So, if the first integer in the list is n , then the next integer in the list is $n +$ something. The *something* is that common difference between the terms in the list. Writing a list of any six multiples of 3, I start with n , then $n + 3, n + 6, n + 9, n + 12$, and finally $n + 15$. You see that the list hinges on two things:

- ✓ That the terms are all the same distance apart
- ✓ That the first integer is what you say it is

I declared, here, that n is a multiple of 3. I can do that! *You* can do that!

Following are more examples of the algebraic representation of lists of consecutive integers. You choose the number for n , and the rest fall in line. Of course, if you want even integers, you have to pick an even integer for n :

- ✓ $n, n + 2, n + 4, n + 6$: Four consecutive even integers
- ✓ $n, n + 2, n + 4, n + 6$: Four consecutive odd integers (Yes, this list can be even or odd. It all depends on what that first n is.)
- ✓ $n, n + 5, n + 10, n + 15$: Four consecutive multiples of 5
- ✓ $n, n + 14, n + 28$: Three consecutive multiples of 14
- ✓ $n, n + 1, n + 2, n + 3, n + 4$: Five consecutive integers

Reconstructing a list

When solving consecutive integer problems, you solve for n and use that value to answer some question about the list or some number on that list. For example, assume that you've solved an equation and found that $n = 4$. Using that value of n , you can then reconstruct a list of numbers by substituting in the 4 for the n 's. Or you can answer a question about the third number in the list or some such thing.

If $n = 4$ and you're trying to find *three consecutive integers*, then your three integers are: 4, 5, and 6.

If $n = 4$ and your goal is to find *five consecutive even integers*, then your five integers are: 4, 6, 8, 10, and 12.

If $n = 4$ is what you got, and you want to find *three consecutive multiples of 4*, then your three integers are: 4, 8, and 12.

If $n = 4$ is the solution of your equation, and you want the second number in the list of consecutive multiples of 4, then your answer is: 8.

And if the answer $n = 4$ is what you get when you solve an equation to find four consecutive odd integers, then you've done something wrong — or the problem was written incorrectly. The number 4 isn't odd. When the equations are set up and solved correctly, you get a number belonging to the category you want.

Writing up sums and solving them

Consecutive integer problems have a common theme: Take a list of consecutive integers, perform a certain operation, and you get a numerical result. When writing the equations needed to solve these problems, you use a fairly common pattern: Let n represent the first number in the list, let $n + d$ (where d is the common difference) be the second number, let $n + 2d$ be the third number, and so on. The most common operation performed in these problems is addition — so add 'em up.

The Problem: The sum of three consecutive integers is 45. What are the integers?



Write the three integers symbolically as: n , $n + 1$, and $n + 2$. Now write the three numbers all added together and set them equal to 45: $n + (n + 1) + (n + 2) = 45$. Drop the parentheses, and combine the terms on the left to get $3n + 3 = 45$. Subtract 3 from each side, and $3n = 42$. Divide each side by 3, and you get $n = 14$. The three consecutive integers with a sum of 45 are 14, 15, and 16.

The Problem: The sum of four consecutive integers is 38. What is the largest number?



Write the four integers symbolically as: n , $n + 1$, $n + 2$, and $n + 3$. Add them up, and let the sum be equal to 38. $n + (n + 1) + (n + 2) + (n + 3) = 38$. Simplifying the terms on the left, the equation becomes $4n + 6 = 38$. Subtract 6 from each side to get $4n = 32$. Dividing each side by 4, you get $n = 8$. So the four terms are 8, 9, 10, and 11. The largest of the four consecutive integers is 11.

The Problem: The sum of eight consecutive integers is 4. What is their product?



You may think that there's been some error here. How can eight consecutive integers add up to a number smaller than the number of integers? The answer to that question is: "Negative numbers." Keep that in mind as you answer the question posed in this problem.

Write the eight consecutive integers as n , $n + 1$, $n + 2$, $n + 3$, $n + 4$, $n + 5$, $n + 6$, and $n + 7$. Adding them all up, you get $n + (n + 1) + (n + 2) + (n + 3) + (n + 4) + (n + 5) + (n + 6) + (n + 7) = 4$. Simplifying, the equation becomes $8n + 28 = 4$. Subtract 28 from each side, and the equation $8n = -24$ is ready to have each side divided by 8. With $n = -3$, the eight terms are: -3, -2, -1, 0, 1, 2, 3, and 4. Multiplying all the terms together, you get that the product is 0, because one of the terms is 0.

Looking At Consecutive Multiples

Consecutive integers are lists of integers that have a common difference between the terms. Odd and even numbers have a common difference of two between each term. Multiples of three have a common difference of three between the consecutive terms, and so on.

Working with evens and odds

The hardest part of doing problems involving consecutive even and consecutive odd integers is accepting the fact that the expressions for consecutive even and consecutive terms are the same. If you say that n , $n + 2$, $n + 4$, and $n + 6$ are four consecutive even numbers, and that $n = 10$, then you just add 2, 4, and 6 to the 10 and get 12, 14, and 16. Now, take the same four numbers, n , $n + 2$, $n + 4$, and $n + 6$, and say that they're all odd numbers. Letting $n = 11$ this time, you see that when you add 2 to 11 you get 13. Adding 4 gives you 15, and adding 6 gives you 17. It's the value of n that sets the pattern.

The Problem: The sum of four consecutive odd numbers is 64. What are the numbers?



First write the four numbers symbolically as n , $n + 2$, $n + 4$, and $n + 6$. Adding them together and setting the sum equal to 64, $n + (n + 2) + (n + 4) + (n + 6) = 64$. Simplifying the equation, you get $4n + 12 = 64$. Subtract 12 from each side, giving you $4n = 52$. Dividing each side of the equation by 4, $n = 13$. The other odd numbers are 15, 17, and 19.

The Problem: The sum of five consecutive even integers is 100. What is the difference between the largest and the smallest of these integers?



The five consecutive even integers are written as n , $n + 2$, $n + 4$, $n + 6$, and $n + 8$. Adding them together and setting the sum equal to 100, $n + (n + 2) + (n + 4) + (n + 6) + (n + 8) = 100$. Simplifying the equation, $5n + 20 = 100$. Subtract 20 from each side to get $5n = 80$. Dividing each side by 5, you get that $n = 16$. Rather than list the five integers, just find the largest even integer in the list by using $n + 8$ where $n = 16$. Since $16 + 8 = 24$, you know that the largest integer is 24 and the smallest is 16. The difference between the two integers is 8.

In the previous problem, you really didn't need to find the even numbers in the list. If you have five consecutive even integers, they're always n , $n + 2$, $n + 4$, $n + 6$, and $n + 8$, no matter what n is. The difference between the largest and the smallest is $(n + 8) - n = 8$.

Expanding to larger multiples

Pick a number, and you can make a list of integers that are multiples of that number. You can also make a symbolic list (using math symbols) of integers that are multiples of a number by starting with n and adding the number repeatedly. For example, a list of six consecutive multiples of 7, starting with 63 is: 63, 70, 77, 84, 91, 98. Symbolically, a list of any six multiples of 7 is: n , $n + 7$, $n + 14$, $n + 21$, $n + 28$, $n + 35$.

The Problem: The sum of four consecutive multiples of 5 is 210. What is the largest of the four multiples?



Write the four integers as n , $n + 5$, $n + 10$, and $n + 15$. Add them up and set the sum equal to 210. The equation is: $n + (n + 5) + (n + 10) + (n + 15) = 210$. Simplifying on the left, the equation becomes $4n + 30 = 210$. Subtract 30 from each side, and the equation $4n = 180$ is solved by dividing both sides by 4. The value of $n = 45$. The original question asks for the largest of these consecutive integers. From the list of integers, the largest is $n + 15$. So, if $n = 45$, then $n + 15 = 45 + 15 = 60$. Or, another approach is to just write the four multiples of 5, starting with 45: 45, 50, 55, 60. That's a nice way to check your work.

The Problem: The sum of five consecutive multiples of 3 is 15. What is the product of the largest two of the numbers?



Write the five integers as n , $n + 3$, $n + 6$, $n + 9$, and $n + 12$. Adding them together and setting the sum equal to 15, you get $5n + 30 = 15$. Subtract 30 from each side and divide by 5 to get $5n = -15$ and $n = -3$. The two largest numbers in the list of consecutive integers are found with $n + 9$ and $n + 12$, making them equal to 6 and 9. The product of 6 and 9 is 54.

Operating on consecutive integers

Consecutive integers can be added, subtracted, multiplied, and divided. Many interesting problems come from performing various operations on the integers and not just limiting yourself to addition. Also, consecutive integers can be squared and cubed and more, before being combined in some fashion — just to mix things up a bit.

Doing more than addition to consecutive integers

One nice thing about just adding consecutive integers is that you always have a linear equation to solve. The equations formed from adding consecutive integers all seem to boil down to something in the form $ax + b = c$, which is solved by subtracting and dividing. But, on the other hand, isn't it nice that some other types of equations, such as quadratic equations, are created as soon as other operations arrive on the scene?

The Problem: The product of two consecutive integers is 20. What are the integers?



Write the two consecutive integers as n and $n + 1$. Multiply them together and set the equation equal to 20. You get a quadratic equation that's solved by setting the equation equal to 0, factoring, and solving for the numbers.

$$\begin{aligned} n(n+1) &= 20 \\ n^2 + n &= 20 \\ n^2 + n - 20 &= 0 \\ (n-4)(n+5) &= 0 \\ n-4 = 0, n &= 4 \\ n+5 = 0, n &= -5 \end{aligned}$$

The quadratic equation gives you two different answers. When $n = 4$, you get the two consecutive integers 4 and 5. The product of 4 and 5 is, indeed, 20. But what about the solution $n = -5$? If $n = -5$, then $n + 1 = -5 + 1 = -4$. The product of -5 and -4 is also 20. This problem has two different solutions. As long as you're happy with negative integers, too, then you accept both sets of answers.

The Problem: The product of two *positive* consecutive odd integers is 143. What are the integers?



The product of the two consecutive odd integers is written as $n(n + 2)$. Set that product equal to 143, set the equation equal to 0, and solve the quadratic equation that results.

$$n(n + 2) = 143$$

$$n^2 + 2n = 143$$

$$n^2 + 2n - 143 = 0$$

$$(n - 11)(n + 13) = 0$$

$$n - 11 = 0, n = 11$$

$$n + 13 = 0, n = -13$$

The solution $n = 11$ is used to find the other odd integer by adding 2. The two consecutive integers are 11 and 13. You don't bother with the solution $n = -13$, because the problem specifies that you're to find positive integers.

Working with consecutive squares and cubes

The numbers 1, 2, 3, and 4 are four consecutive integers. The numbers 1, 4, 9, and 16 are four consecutive *squares* of those first four numbers. And the *cubes* of the same four numbers are 1, 8, 27, and 64. Squares and cubes of consecutive numbers are squared and cubed, respectively, *before* being added together or having some other operation performed upon them.

The Problem: The sum of the squares of two consecutive integers is 145. What are the numbers?



First write the two consecutive integers as n and $n + 1$. The squares of those two numbers are n^2 and $(n + 1)^2$. To write the equation, add the two squares together and set them equal to 145. Then do the squaring on the left, combine like terms, subtract 145 from each side to set the equation equal to 0, and solve the quadratic equation.

$$n^2 + (n + 1)^2 = 145$$

$$n^2 + n^2 + 2n + 1 = 145$$

$$2n^2 + 2n - 144 = 0$$

$$2(n^2 + n - 72) = 0$$

$$2(n - 8)(n + 9) = 0$$

$$n - 8 = 0, n = 8$$

$$n + 9 = 0, n = -9$$

When $n = 8$, then $n + 1 = 9$. The sum of their squares is $64 + 81 = 145$. The other solutions works as well. When $n = -9$, then $n + 1 = -8$. Their squares are 81 and 64 and also have a sum of 145.



The Problem: The difference between the cubes of two consecutive numbers is 127. What are the numbers?

First write the cubes of the numbers as n^3 and $(n + 1)^3$. Subtract the smaller number from the larger number and set the difference equal to 127. The equation you write is $(n + 1)^3 - n^3 = 127$. Now cube the binomial and simplify the terms on the left. The resulting equation is quadratic, which factors and yields two solutions.

$$\begin{aligned}
 (n + 1)^3 - n^3 &= 127 \\
 n^3 + 3n^2 + 3n + 1 - n^3 &= 127 \\
 3n^2 + 3n + 1 &= 127 \\
 3n^2 + 3n - 126 &= 0 \\
 3(n^2 + n - 42) &= 0 \\
 2(n - 6)(n + 7) &= 0 \\
 n - 6 = 0, n &= 6 \\
 n + 7 = 0, n &= -7
 \end{aligned}$$

When $n = 6$, you get 6 and 7 whose cubes are 216 and 343, respectively. The difference between their cubes is 127. When $n = -7$, you get -7 and -6 whose cubes are -343 and -216. Subtracting $-216 - (-343)$ you also get 127.

The expansion of the binomial $(n + 1)^3$ results in a polynomial where the coefficients of the terms have a distinctive, symmetric pattern. Refer to the Cheat Sheet for some of the other powers of a binomial. You'll see that the cube of $(n + 1)$ has coefficients in the 1-3-3-1 pattern.

Finding Sums of Sequences of Integers

A *sequence* of numbers is a list of numbers created by a particular pattern or mathematical rule. An *arithmetic sequence* is a list of numbers in which there is a common difference between the consecutive numbers in the sequence. So consecutive integers are a special type of arithmetic sequence. The rule that allows you to add up any number of terms in an arithmetic sequence also lets you solve some problems involving the sums of consecutive integers.

Setting the stage for the sums

Before applying the rule for the sum of a certain number of terms in an arithmetic sequence, you need to be able to find the n th term in a sequence when

given the first term or find the n when given the first and last terms. Then you can apply the general rule for the sum of the terms.

Finding the n th term or finding the difference

The n th term of an arithmetic sequence may be the fourth term or the tenth term or any number of term. You don't want to have to list the first 99 terms in order to find the hundredth term, so the following rule comes in handy.



The n th term of an arithmetic sequence is a_n , which is equal to $a_1 + d(n - 1)$, where a_1 is the first term and d is the difference between each of the terms in the sequence.

You use the formula for the n th term of an arithmetic sequence, $a_n = a_1 + d(n - 1)$, to find a particular term in the sequence.

The Problem: In the sequence of terms beginning with 7 and with a common difference of 5, what is the tenth term in that sequence?



Using the formula and replacing d with 5 and n with 10, $a_{10} = 7 + 5(10 - 1) = 7 + 5(9) = 7 + 45 = 52$. You can check this answer, because you only have ten terms to worry about. Here are the first ten terms of the sequence: 7, 12, 17, 22, 27, 32, 37, 42, 47, 52. Yes, the tenth term is 52. You want to be sure that the formula works when the sequence has 100 or 1,000 terms.

The Problem: If the first term of an arithmetic sequence is 3 and the 50th term is 297, then what is the common difference between the terms?



Using the formula for finding the n th term, you fill in values for everything except the difference and then solve for that difference. The equation is written $297 = 3 + d(50 - 1)$. Simplifying on the right, the equation becomes $297 = 3 + 49d$. Subtract 3 from each side for $294 = 49d$. Dividing each side of the equation by 49, you get that $d = 6$.

Summing up the terms

The generalized formula for the sum of any number of terms of an arithmetic sequence allows you to add up the terms no matter where you start and where you stop in the sequence.



The sum of n terms of an arithmetic sequence is equal to half n times the sum of the first and last terms of the sequence.

$$S_n = \frac{n}{2}(a_1 + a_n)$$

The Problem: Find the sum of the 16 terms in the sequence that start with the number 15 and end with the number 60.



Using the formula, and substituting in the values, $S_{16} = \frac{16}{2}(15 + 60) = 8(75) = 600$. Do you wonder what those terms are? You find them by using the formula for the n th term. Using $60 = 15 + d(16 - 1)$ which becomes $60 = 15 + 15d$, $45 = 15d$, or $d = 3$. So the numbers are $15, 18, 21, 24, \dots, 57, 60$. (Okay, I was curious about the numbers but not interested enough to write out all 16 of them.)

The Problem: If the sum of the ten terms in an arithmetic sequence is 1,135 and if the difference between the terms is 3, then what are the first and last terms?



First, write the first and last terms in an expression that relates one to the other. The tenth term is $a_{10} = a_1 + 3(10 - 1) = a_1 + 27$, using the formula found in *Finding the nth term or finding the difference*. Now use the sum formula, replacing each letter in the formula with its equivalent.

$$\begin{aligned} S_n &= \frac{n}{2}(a_1 + a_n) \\ 1,135 &= \frac{10}{2}(a_1 + a_1 + 27) \\ 1,135 &= 5(2a_1 + 27) \end{aligned}$$

Divide each side by 5 and solve for a_1 . When you find the first term, you add 27 to find the tenth term.

$$\begin{aligned} \frac{1,135}{5} &= \frac{\cancel{5}(2a_1 + 27)}{\cancel{5}} \\ 227 &= 2a_1 + 27 \\ 200 &= 2a_1 \\ 100 &= a_1 \\ 127 &= a_{10} \end{aligned}$$

Finding the sums of consecutive integers

In most consecutive integer problems, you're given the sum of a certain number of integers and told to figure out what those integers were that gave you a particular sum. For a change of pace, in the problems in this section, it's the sums that you compute. And, because the problems involve lots and lots of consecutive integers, you find the formulas to be very nice.

The Problem: Find the sum of the 20 consecutive multiples of 4 that begin with the number 60.



Strange book

There's a book in the local library where the forward comes after the epilogue, the end is in the first half of the book, the index comes before the

introduction, and the glossary comes before the title. What is the name of this book?

Answer: It's called a dictionary.



The formula for finding the sum of a list of consecutive integers requires that you have the first and last terms in the list. The multiples of 4 are all four units apart. You have an arithmetic sequence with terms the difference of which is 4. (You can find more on arithmetic sequences in "Setting the stage for the sums," earlier in this chapter.) So, to find the 20th term in the list of multiples of 4 that start with 60, use the formula $a_n = a_1 + d(n - 1)$, giving you that $a_{20} = 60 + 4(20 - 1) = 60 + 4(19) = 60 + 76 = 136$. Now, using the formula for the sum of the terms, and letting 60 be the first term and 136 be the 20th term,

$$\begin{aligned} S_n &= \frac{n}{2}(a_1 + a_n) \\ S_{20} &= \frac{20}{2}(60 + 136) \\ &= 10(196) = 1,960 \end{aligned}$$

Applying Consecutive Integers

Adding up lists of numbers is always a huge amount of fun — or not. It depends on what you like to do with your leisure time, I suppose. When formulas are available to make arithmetic computations easier and more accurate, you jump at the chance to use those formulas. Here I give you some applications of sums of consecutive integers.

Adding up building blocks

A child's set of building blocks usually consists of many wooden cubes — all the same size and decorated with letters of the alphabet, numbers, animals, or different colors. A typical stacking exercise involves making a row of blocks, with each block touching the one next to it, and then making a row on

top of that first row where each block in the second row straddles two blocks beneath it (it sits on the crack between the two blocks). In this way, as more rows are added, each row has one less block than the row it's sitting on.

The Problem: Little Jimmy is stacking his cube-shaped blocks. His first row has 20 blocks, and each subsequent row has one block less than the one below. If there are 250 blocks in Jimmy's set, will he have enough blocks to build a structure all the way up to one block on the top?



Find the sum of the consecutive integers 1, 2, 3, 4, 5, . . . 19, 20 by using the formula for the sum of consecutive integers. The first term is 1; the last term is 20; and the number of terms, n , is 20. Compare that sum with the number of blocks to see if Jimmy will have enough blocks.

$$\begin{aligned} S_n &= \frac{n}{2}(a_1 + a_n) \\ S_{20} &= \frac{20}{2}(1 + 20) \\ &= 10(21) = 210 \end{aligned}$$

Jimmy has more than enough blocks. In fact, he'll probably throw those extra blocks at his brother.

Finding enough seats

In most theaters, you have more seats in the rows toward the back and fewer seats in the rows up front as the room narrows toward the stage. Imagine having to clean up after a performance attended by a rowdy bunch of children who were all given huge bags of popcorn. You need to plan the amount of time needed to do the job.

The Problem: The theater at a local civic center has 13 seats in the first 3 rows, 15 seats in the next 3 rows, and the number of seats increases by 2 seats every third row until there are a total of 45 rows in the theater. How many seats are there?



Because the number of seats in a row is repeated three times, just figure out how many seats in one of each of those different rows and multiply the result by three. With 45 rows total, then there are 15 different numbers of seats in the rows ($45 \div 3 = 15$). The first row has 13 seats, and the number of seats per row increases by 2, so the number of seats per row is: 13, 15, 17, 19. . . . The 15th different row has $a_1 + d(n - 1) = 13 + 2(15 - 1) = 41$ seats. The sum of the 15 numbers starting with 13 and ending with 41 is:

$$\begin{aligned} S_{15} &= \frac{15}{2}(13 + 41) \\ &= \frac{15}{2}(54^2) = 15(27) = 405 \end{aligned}$$

If there were just one of each of these 15 different rows, you'd have 405 seats. But there are 45 rows — 3 of each of those 15 different ones. Multiplying 405 by 3, you get $405 \times 3 = 1,215$ seats all together.

Laying bricks for a stairway

Stairways to monuments get a lot of wear and tear, so paving bricks are used, because they can take all the punishment of many feet over many years.

The Problem: A brick stairway to a monument has 40 steps. The first step has 250 bricks, and each successive step has 3 fewer bricks. How many bricks are in the last step, and how many total bricks are there in the stairway?



You use the formula for the n th term of a sequence of numbers letting the difference be -3 . So the 40th step has $a_1 + d(n - 1) = 250 - 3(40 - 1) = 133$ bricks. The sum of the bricks in the 40 steps, going from 250 bricks down to 133 bricks, is $\frac{40}{2}(250 + 133) = 20(383) = 7,660$ bricks.

Chapter 13

Writing Equations Using Algebraic Language

In This Chapter

- ▶ Translating from the written language to math-speak
 - ▶ Putting the operations in their places in the equations
 - ▶ Choosing which types of equations and inequalities
 - ▶ Doing a reality check with solutions
-

This chapter contains lots of rules, hints, tidbits, and procedures for you to use when changing from the written language to letters — variables that represent the unknown quantity — and operations in the mathematical language. The equations you write are usually quite easy to solve — after you manage to write them.

Changing from words to equations involves identifying what the variables (the x 's or y 's or t 's) represent and how to arrange them in an equation. Solving an equation requires algebraic know-how, but, if your equation is nonsense or doesn't fit the problem, then the answer to the equation will get you no closer to the answer to the problem than you were before you started.

You'll see how sometimes more than one option exists for writing an equation. Sometimes one way is better than another. Other times it makes no difference which format you use. And, unfortunately, you'll see that some problems are just unsolvable — no method or means will ever answer the question.

This chapter also allows me to cover some word-problem topics that just don't seem to fit anywhere else. You can call this the miscellaneous chapter — it contains word problems that you're likely to come across but that don't have any particular place with all the others. These problems are great for illustrating some more of the techniques that are helpful when solving math word problems.

Assigning the Variable

When translating from the written language to an algebraic equation, one rule that absolutely must be followed is that the variable (letter) that you choose to solve for in the equation has to represent a number. The letter x might represent Jack's age or the number of chickens in the yard, but the x can't represent Jack or the chickens themselves. Getting past this hurdle (of assigning the variable to be a number) sets you off on the right track toward solving an equation and answering a question.

After you've solved an equation for the value of the variable, then you either have the answer directly or you may have to do some additional steps to completely answer the question that's been posed.

Getting the answer directly from the variable

It's always nicest to have the answer just plopped in your lap as soon as you come up with the solution to an equation. It isn't always possible to easily create equations that behave this way, but, when you can, take advantage of the situation.

The Problem: Ken, Kyle, and Keith worked on a project and kept track of how much time each spent. They determined that the total was 48 hours. Keith worked 3 hours longer than Ken, and Kyle worked 9 hours less than Keith. How many hours did Ken work?



To have this problem work out so that the solution of the equation is the answer of the question, you let the variable represent the number of hours that Ken worked. Then you write the number of hours that Kyle and Keith worked in terms of Ken's hours. If you let x represent the number of hours that Ken worked, then $x + 3$ represents the number of hours that Keith worked. Using Keith's hours, you write that the number of hours that Kyle worked is $x + 3 - 9 = x - 6$ hours. The total number of hours is 48, so $x + (x + 3) + (x - 6) = 48$. Simplifying on the left, you get $3x - 3 = 48$. Add 3 to each side, and the equation becomes $3x = 51$. Dividing by 3, $x = 17$. So Ken worked for 17 hours. To check the answer, just figure out how many hours Keith and Kyle worked, and see if the sum is 48. Keith worked three hours longer than Ken, so Keith worked for 20 hours. Kyle worked nine hours less than Keith, so he worked for 11 hours. The men worked for $17 + 20 + 11$ hours which is, indeed, 48 hours.

The Problem: It cost \$12 for a man to drive his van to work every day, but he had passengers, so he divided this cost equally among the passengers and himself. After a few weeks, two more passengers were added, which reduced everyone's cost by \$1 per day. How many people are now riding in the van?



Look at the last sentence and the final question: "How many people . . . ?" To have the solution of the equation come out to be the final answer, you let the variable, x , be equal to the number of passengers now riding in the van. The cost per person now is \$1 less than it used to be, so the equation you write will be something to the effect that \$12 divided by the number of riders now is equal to \$1 less than \$12 divided by the number of riders that there used to be. Math speak is much easier to write:

$$\frac{\$12.00}{\text{number now in van}} = \frac{\$12.00}{\text{number used to be in van}} - \$1.00 \text{ becomes}$$

$$\frac{\$12.00}{x} = \frac{\$12.00}{x-2} - \$1.00 \text{ when you let } x \text{ be the number in the van now and}$$

let $x - 2$ be the number who used to ride to work. To solve this equation, multiply each term in the equation (even the 1) by the common denominator of the fractions, $x(x - 2)$, and then solve the quadratic equation that results.

$$\begin{aligned}\frac{12}{x} &= \frac{12}{x-2} - 1 \\ \frac{12}{x} \times x(x-2) &= \frac{12}{x-2} \times x(x-2) - 1 \times x(x-2) \\ 12(x-2) &= 12x - x(x-2) \\ 12x - 24 &= 12x - x^2 + 2x \\ 12x - 24 &= 14x - x^2 \\ x^2 - 2x - 24 &= 0\end{aligned}$$

Factoring the quadratic, you get that $x = 6$ or $x = -4$.

$$\begin{aligned}x^2 - 2x - 24 &= 0 \\ (x - 6)(x + 4) &= 0 \\ x = 6 \text{ or } x &= -4\end{aligned}$$



Fly away fly

I'm sitting at a table, and 12 flies land in front of me. I use a fly swatter and smash 4 flies. How many remain on the table?

ANSWER Well, if I smashed four flies, those four stay on the table and the rest fly away.

Only the positive value makes sense — you can't have a negative number of people. If $x = 6$, then that's the number of people now in the van. Dividing the \$12 by 6, each person is paying \$2. There used to be only $x - 2$ or 4 people in the van. Each paid \$3 then, or \$1 more than now.

Adding a step to get the answer

It's not always convenient or easy to write your equation so that the variable is equal to the answer. Sometimes you have to write an equation that makes sense, and then figure out the answer to the question from the solution of the equation that you construct and solve.

The Problem: A landlord wants to have a total of \$42,000 income from the monthly rents in his apartment complex. Right now, he gets \$400 per month for each of his 100 apartments, giving him a total of \$40,000. For every \$20 he raises the rent, he loses three tenants who don't want to pay the higher rent. He will get more income, to offset the vacant apartments, but he can't raise the rent too high, or he'll lose too much money. What should he charge per unit to get to that goal of \$42,000?



You compute the total amount of the money collected by multiplying the rent charged per apartment times the number of apartments. The amount of rent and the number of apartments keep changing, depending on how many times he raises the rent by \$20. So let x represent the number of \$20 increases he adds to the rent of \$400. This way, the rent per unit is written as $400 + 20x$. For every \$20 increase, the number of tenants is going to decrease by three. Write the number of apartments that will be paying rent as $100 - 3x$. You write the income from rents as the money charged times the number of units, so now you can write an equation setting the total rent money goal of \$42,000 equal to the new rent amount, $400 + 20x$, times the new number of apartments being rented, $100 - 3x$.

$$42,000 = (400 + 20x)(100 - 3x)$$

Multiplying the binomials on the right and combining terms, you get a quadratic equation that is solved by setting everything equal to 0 and factoring.

$$42,000 = (400 + 20x)(100 - 3x)$$

$$42,000 = 40,000 + 800x - 60x^2$$

$$60x^2 - 800x + 2000 = 0$$

$$20(3x^2 - 40 + 100) = 0$$

$$20(3x - 10)(x - 10) = 0$$



Making an interesting purchase

A man went to a hardware store and found out, from the clerk, that it costs \$3 for 600, which meant that each part cost \$1. How much does it cost for 601?

Answer: It also costs \$3 for 601. He's buying individual numbers to put on his house to identify his address.

The solution of the quadratic equation results in two different numbers, $x = \frac{10}{3}$ and $x = 10$. You check the solutions to see if either one or both work to answer the question.

If $x = \frac{10}{3}$, then the rent charged is $(400 + 20 \times \frac{10}{3}) = 400 + \frac{200}{3} = 400 + 66\frac{2}{3} = 466\frac{2}{3}$, and the number of apartments to be rented is $(100 - 3 \times \frac{10}{3}) = 100 - 10 = 90$. Multiplying $466\frac{2}{3} \times 90 = (466 \times 90) + \left(\frac{2}{3} \times 90\right) = 41,940 + 60 = 42,000$, which is how much money the landlord wants. The main problem with this answer is that the amount charged per apartment is a fraction that can't be changed into an exact number of cents. Rounding the rent amount to \$467 or even \$470 would make sense, resulting in just a little more total income than the \$42,000.

Trying the other solution of the quadratic equation, when $x = 10$, then the rent charged is $(400 + 20 \times 10) = 400 + 200 = 600$, and the number of apartments that will be rented is $(100 - 3 \times 10) = 100 - 30 = 70$ apartments. Multiplying the rent of \$600 times 70 you get exactly \$42,000. The rent is higher, but fewer apartments will be occupied. That would probably cut down on the cost of maintenance and upkeep, but there are a lot of vacant apartments. Now it's up to the landlord (and his conscience).

Writing Operations and Using Sentence Structure

Many hints, clues, and downright instructions are available from the wording of math problems. When the problem talks about a *total* or *increase* or uses the word *altogether*, the operation of addition is suggested. You look for a

subtraction problem when *decrease* or *less* or *how many left* are part of the problem description. Multiplication is suggested with *twice* or *times* or the obvious *multiply*. And the word *half* can create a division problem or a multiplication problem, depending on how you want to handle the computations. In general, you try to replace the verbs *is* and *are* with an equal sign, and you align the different expressions created for the equation on either side of the equal sign, in the same positions as the words they came from on either side of the verb.

Making the most of addition

Problems involving addition are nice in several respects:

- ✓ They're fairly easy to spot, and the expressions to algebra translations are usually straightforward.
- ✓ The terms in addition problems are reversible, because addition is *commutative* (you can add two terms in either order and get the same answer).
- ✓ The equations that you create are relatively easy to solve.

The Problem: A service club is selling Super Bowl coffee mugs for a fundraiser. The setup charge for creating the design that goes on the mugs is a flat \$75. The mugs will cost the club 90¢ each if they order no more than 200 mugs, 85¢ each if they order between 201 and 500 mugs, and 80¢ each if they order more than 500 mugs. What is the total cost for 100 mugs, 300 mugs, and 1,000 mugs?



A *cost function* or equation consists of the *fixed* part and the *variable* part. Each of the three cost schedules in this problem has a fixed cost of \$75, but the variable part changes, depending on the number of mugs ordered. An efficient way of writing the equations in this problem is to use a *piecewise* function, which lists each rule and the input associated with each. The C in this function represents the total cost, and the x represents the number of mugs.

$$C(x) = \begin{cases} 75 + 0.90x & \text{if } x \leq 200 \\ 75 + 0.85x & \text{if } 201 \leq x \leq 500 \\ 75 + 0.80x & \text{if } x > 500 \end{cases}$$

Use the top rule to determine the total cost of 100 mugs. $C(100) = 75 + 0.90(100) = 75 + 90 = 165$. Use the middle rule for 300 mugs. $C(300) = 75 + 0.85(300) = 75 + 255 = 330$. And the bottom rule works for finding the total cost of 1,000 mugs. $C(1,000) = 75 + 0.80(1,000) = 75 + 800 = 875$.



Oranges in a crate

There are 24 oranges in a crate, and there are 24 people in the room. You need to distribute the oranges so that everyone gets one orange, but

you have to still have an orange left in the crate. How can you do this?

Answer: You give one person the crate with an orange in it.

Subtracting and multiplying solutions

The word *less* implies subtraction. You can even make a case for *more* indicating subtraction, although addition is usually an easier way to go with those problems. In any case, be careful when you write the algebra indicating subtraction. The terms need to be in the correct order. Subtraction is not *commutative*. The order that you subtract numbers makes a huge difference. When multiplying shows up in a word problem, it's often an outright instruction to multiply by two or three or one-half. Otherwise, it may be disguised as *twice* or *thrice*. The next problem has a little of each of these operations.

The Problem: The number of boys in a high school senior class is ten less than twice the number of girls. Also, six fewer than half the students in the class have cars. If 202 students own cars, then how many boys are in the class?



Write an equation letting x represent the number of girls. You can solve for the number of boys (to answer the question) after solving the equation for x . The number of boys is represented by $2x - 10$, which is ten less than twice the number of girls. Note the order of the subtraction. Next, write the total number of students as girls plus boys: $x + 2x - 10$. Multiply the total by $\frac{1}{2}$ and subtract 6. Set that expression equal to 202. Then simplify the equation and solve for x .

$$\frac{1}{2}[x + 2x - 10] - 6 = 202$$

$$\frac{1}{2}[x + 2x - 10] = 208$$

$$x + 2x - 10 = 416$$

$$3x - 10 = 416$$

$$3x = 426$$

$$x = 142$$

The solution $x = 142$ tells you that there are 142 girls in the class. The number of boys is ten less than twice that, so $2(142) - 10 = 284 - 10 = 274$ boys.

Dividing and conquering

You use division in problems when you want to break some total amount of stuff into equal shares. After setting up the division problem, though, the best course of action for solving the equation is usually to get rid of the division operation. Do this by multiplying each term in the equation by some common denominator; then solve the simpler equation that's been created.

The Problem: A man bought two apartment buildings for \$210,000 apiece. If there are a total of ten apartments in the two buildings, and if the apartments in one of the buildings cost \$40,000 more than the apartments in the other building, then what did he pay for the apartments in the two buildings?



Assume that each apartment in a building costs the same amount. You determine the cost of one apartment in the building by dividing \$210,000 by the number of apartments. Let x represent the number of apartments in one of the buildings. Then, since the total number of apartments in the two buildings is 10, you write the number of apartments in that building as $10 - x$. Divide 210,000 by x on one side of the equation. On the other side of the equation, divide 210,000 by $10 - x$ and add 40,000 to the result to show that the first apartments are \$40,000 more.

$$\frac{210,000}{x} = \frac{210,000}{10 - x} + 40,000$$

Multiply each term by the common denominator, $x(10 - x)$.

$$\begin{aligned}\frac{210,000}{x} \times x(10 - x) &= \frac{210,000}{10 - x} \times x(10 - x) + 40,000 \times x(10 - x) \\ \cancel{x} \frac{210,000}{\cancel{x}} \times (10 - x) &= \frac{210,000}{\cancel{10 - x}} \times \cancel{x}(10 - \cancel{x}) + 40,000 \times x(10 - x) \\ 210,000(10 - x) &= 210,000x + 40,000x(10 - x)\end{aligned}$$

Before multiplying out and simplifying the terms, make life easier for yourself by dividing each term by 10,000. You don't want to work with such huge numbers. Then set the equation equal to 0, factor, and solve for x .

$$\begin{aligned}21(10 - x) &= 21x + 4x(10 - x) \\ 210 - 21x &= 21x + 40x - 4x^2 \\ 4x^2 - 82x + 210 &= 0 \\ 2(2x^2 - 41x + 105) &= 0 \\ 2(2x - 35)(x - 3) &= 0\end{aligned}$$

The two solutions of the quadratic equation are $x = \frac{35}{2}$ and $x = 3$. The first solution is equal to 17.5 which makes no sense — there are only ten apartments. So discard that solution. When $x = 3$, though, you get that one

apartment building has three apartments costing $\$210,000 \div 3 = \$70,000$ each. The other apartment building has $\$210,000 \div 7 = \$30,000$ each. The more expensive apartments are $\$40,000$ more than the less expensive apartments.

Tackling an earlier problem

In Chapter 10, I suggest that you can solve for the base and height of a triangle if you're given enough information. Just having the area of a triangle doesn't tell you the length of the base or the height, you need to know one or the other measure, or you need to know something about how the measures relate to one another.

The Problem: The area of a triangle is 20 square inches, and the base is 6 inches longer than the height. What are the measures of the base and height?



Use the formula for the area of a triangle, and write the base and height, b and h , in terms of one or the other of the variables. If you let h represent the height, then the base is written as $6+h$. (If you had chosen to use b instead of h , you'd represent the height with $b-6$.) Now let the area, A , be equal to 20 and solve for h .

$$\begin{aligned} A &= \frac{1}{2}bh \\ 20 &= \frac{1}{2}(6+h)h \\ 40 &= (6+h)h \\ 40 &= 6h + h^2 \\ 0 &= h^2 + 6h - 40 \\ 0 &= (h+10)(h-4) \end{aligned}$$

The solutions of the quadratic equation are $h = -10$ and $h = 4$. The negative answer doesn't really make sense in the measure of a triangle, so you go with the height being 4 inches. The base is 6 inches longer than the height, so it measures $4 + 6 = 10$ inches.

Solving for Answers from Algebraic Solutions

When algebraic equations or inequalities are used to find the answer to a question or problem, you write the equation or inequality carefully, you solve the algebraic equation or inequality using all the correct rules, and then you check to be sure that the solution from the algebra really answers the question that's been posed.

Comparing the types of algebraic expressions

Many different types of equations and inequalities are used to solve math word problems. Each type has its own methods for solution, and many have some quirks to watch out for when solving. Here are some more frequently used equations and inequalities:

- ✓ **Linear equations:** $ax + b = c$, with one possible solution
- ✓ **Quadratic equations:** $ax^2 + bx + c = 0$, with two solutions
- ✓ **Rational (proportions) equations:** $\frac{x}{a} = \frac{b}{c}$, with one solution
- ✓ **Absolute value linear equations:** $|ax + b| = c$, with two solutions
- ✓ **Linear inequalities:** $ax + b > c$ or $ax + b < c$, with infinite solutions
- ✓ **Systems of linear equations:** $ax + by = c$ and $dx + ey = f$, with one possible solution (see Chapter 17)

Using linear equations $ax + b = c$ to find answers

The variable, x , in a linear equation has an exponent of 1. That's why you'll only get one solution from these equations. You solve the equation by isolating the variable on one side of the equation.

The Problem: A realty manager gets a base salary of \$4,000 per month plus 3 percent of whatever his agents earn in commissions. In February, the manager had total earnings of \$11,050. What did his agents earn in commissions?



The equation used to determine the manager's salary any month is $T(x) = 4,000 + 0.03x$, where T is the total earnings by the manager, and x is the amount of commissions earned by his agents. Substituting in the values you know, the equation becomes $11,050 = 4000 + 0.03x$. Subtract 4,000 from each side to get $7,050 = 0.03x$. Now divide each side by 0.03, and $x = \$235,000$.

Quibbling over quadratic solutions to answer questions

Quadratic equations can, and often do, provide two completely different solutions. Sometimes both solutions work in the problem. Other times only one or even neither works. You need to check to determine which situation applies in each case. (Refer to "Checking to see if a solution is an answer" later in this section for more on the checking.)

The Problem: If you add 5 to a number and multiply that by what you get if you subtract 4 from the same number, then the answer is 36. What is the number?



Let the number be represented by x . Adding 5 gives you $x + 5$, and subtracting 4 is written $x - 4$. Multiplying $(x + 5)(x - 4)$ and setting the product equal to 36, you get a quadratic equation that's solved by setting everything equal to 0 and solved by factoring.

$$\begin{aligned}(x + 5)(x - 4) &= 36 \\ x^2 + x - 20 &= 36 \\ x^2 + x - 56 &= 0 \\ (x + 8)(x - 7) &= 0\end{aligned}$$

The two solutions you get are $x = -8$ and $x = 7$. Letting $x = -8$, the number you get when you add 5 is -3 . Subtracting 4 from -8 , you get -12 . The product $(-3)(-12) = 36$. So -8 is an answer. Checking on $x = 7$, $(7 + 5)(7 - 4) = (12)(3) = 36$. Both answers work.

Rationalizing with rational equations

A rational equation has one or more fractions in it — usually with the variable appearing in more than one numerator or denominator. In the “Dividing and conquering” section, earlier in this chapter, you see how to clear the equation of fractions by multiplying everything by the common denominator. Another type of rational equation is one in which you have two fractions set equal to one another. This is called a *proportion*. (Refer to Chapter 7 for a full description of what you can do with proportions.) One of the nicest features of proportions is that their cross products are always equal.

The Problem: In an orchard, it requires 3 workers for every 16 trees to keep them pruned and cared for during the summer. How many trees could be tended by 40 workers?



Write a proportion in which one fraction is 3 divided by 16 and the other fraction is 40 divided by x . This keeps the number of workers across from a number of workers, in the numerators, and trees across from trees, in the denominators. Then cross-multiply and solve for x .

$$\begin{aligned}\frac{3}{16} &= \frac{40}{x} \\ 3x &= 640 \\ x &= \frac{640}{3} = 213\frac{1}{3}\end{aligned}$$

A worker isn't going to take care of a fraction of a tree, so the best answer is probably 213 trees.

Being absolutely sure with absolute value

The absolute value operation $|a|$ takes the number a and measures its distance from 0. Another way of putting this is that the absolute value of any number is positive. So $|3| = +3$ and $|-3| = +3$. When solving absolute-value equations, you have to take into account the fact that what's inside the operation could be either positive or negative, so there could be more than one answer.

The Problem: Mike's salary and Matt's salary differ by \$10,000. If Mike makes \$165,000, then what does Matt make?



You aren't told if Matt makes more or less than Mike. Letting Matt's salary be represented by x , you write the absolute value equation $|x - 165,000| = 10,000$. Solve two different linear equations. One equation assumes that the result in the absolute value operation is positive, and the other assumes that it's negative. The first equation is written $x - 165,000 = 10,000$. Solving for x , you get $x = 175,000$. The other equation is formed by negating the expression inside the absolute value. You get $-(x - 165,000) = 10,000$ which becomes $-x + 165,000 = 10,000$. Subtract 165,000 from each side and divide by -1 , and you get $x = 155,000$. So Matt makes either \$175,000 or \$155,000.

Lingering a while with linear inequalities

Linear inequalities are solved using almost all the same rules as those you use with linear equations. The one big exception is that when you multiply or divide each side of the inequality by a negative number, you have to switch the sign around from $>$ to $<$ or vice versa.

The Problem: Jennifer has grades of 87, 95, 99, 83, and 100 on her first five exams. What does she have to get on her next exam to end up with an average of at least 92 on all six exams?

**Fickle friend**

It's found in Minnesota, but Wisconsin it will avoid. It's always in the timber, but not in woods, so they're annoyed. It's never into you, but it's

always into me. With anything so fickle, what, oh what then can it be?

Answer: Try the letter m.



An average of 92 on all six exams means that the total number of points has to be at least $92 \times 6 = 552$ points. Let x represent the next test grade and solve the inequality: $87 + 95 + 99 + 83 + 100 + x \geq 552$. Simplify on the left by adding up the tests to get $464 + x \geq 552$. Subtract 464 from each side, and you get $x \geq 88$. Jennifer has to get a score of 88 or better to have a minimum average of 92.

Checking to see if a solution is an answer

You get that feeling of satisfaction when an algebraic equation or inequality works out and you get one or more solutions. The next step in word problems is then, of course, to see if the solution of the equation or inequality is an answer to the problem. If the solution doesn't work, then you go back to see if you've done some miscomputation. But sometimes, no amount of good mathematics is going to get you an answer. It could be that the question just doesn't have an answer.

Several problems in this chapter illustrate how there can be two answers that work or just one of two solutions that works for an answer. How can it possibly be that there is no answer?

The Problem: Ted and Jeff were comparing how many Christmas gifts they got. If Ted got 10 less than 3 times as many as Jeff, and if the total number of gifts they got is equal to the number of Jeff's gifts minus 13, then how many gifts did they each get?



Let the number of gifts that Jeff got be x . Then Ted got $3x - 10$ gifts. Add the two numbers of gifts together and set it equal to $x - 13$. So the equation to solve is $x + 3x - 10 = x - 13$. Simplifying, you get $4x - 10 = x - 13$. Subtract x from each side and add 10 to each side, and the equation becomes $3x = -3$. Dividing each side by 3, you get $x = -1$. Oops! How can Jeff get a negative number of gifts? The equation accurately represents the problem that's presented. It's just that the problem posed is impossible. There's no answer to the problem, even though the equation has a solution.

Chapter 14

Improving the Quality and Quantity of Mixture Problems

In This Chapter

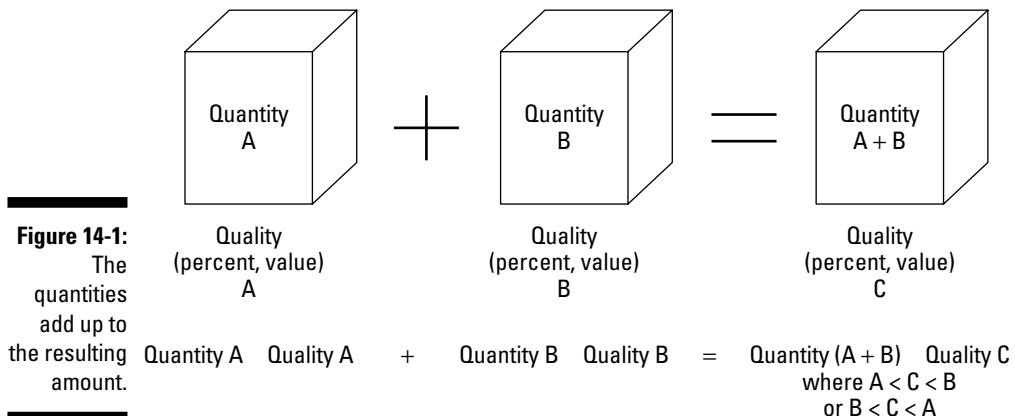
- ▶ Concentrating on the concentrations of mixtures
- ▶ Using quality times quantity as the standard
- ▶ Taking an interest in interest problems

A classic type of mathematical word problem is the mixture problem. But mixture problems go beyond just mixing up antifreeze with water or chocolate syrup with milk. Mixture problems involve all sorts of situations where you combine so much of one thing that has a certain amount of worth or density with so much of something else that has more worth or more density.

In this chapter, you find problems that multiply numbers of coins times their monetary value, pints of coolant with their concentrations and cartons of goodies with their product count. All these make for some interesting and, yes, even useful problems. So you can put your spoon away and find out what mixture problems are all about.

Standardizing Quality Times Quantity

The common theme in all mixture problems is that you take two or more different *amounts* (quantities) of two different *concentrations* (or qualities) and mix them together to get an amount (quantity) that's the sum of the two ingredient quantities and a concentration (quality) that's somewhere between the concentration of the two ingredients that you started with. Figure 14-1 illustrates this property or theme with containers.



The figure shows that when you add two quantities together, you get an amount or new quantity that's the sum of the two quantities that are put together. The concentration or quality of the resulting mixture is a blend of the two starting substances — its concentration is somewhere between the two that are being combined.

Mixing It Up with Mixtures

Mixture problems involving actual substances occur when you take two or more different solutions or granular compounds or anything that will combine or *mix* and create a new combination that's no longer purely one or the other. When you pour chocolate syrup into milk, you add volume to the liquid in the glass, and the color of the milk mixture isn't as dark as the chocolate or as white as the milk. The more chocolate, the darker the mixture. Yum!

Improving the concentration of antifreeze

The radiators of cars serve to cool down the engine with liquid that circulates around and pulls the heat away. The concentration of the fluid in the radiator can be changed to reflect the temperatures of the particular season. The concentration of antifreeze should be greater in the winter or cold months and less in the warmer months.

The Problem: A service station owner wants to mix up some 35 percent antifreeze. He wants to use up his current supply of 100 gallons of 20 percent antifreeze and add enough 40 percent antifreeze to bring the mixture up to 35 percent. How many gallons of 40 percent antifreeze should he add?



Keep in mind *quality* \times *quantity* for the two solutions being added together and the *quality* and *quantity* of the resulting solution. Let x represent the number of gallons of 40 percent antifreeze that needs to be added. The qualities are the percentages of the solutions, and the quantities are the gallons of the respective solutions.

Starting with $[20 \text{ percent} \times 100 \text{ gallons}] + [40 \text{ percent} \times x \text{ gallons}] = [35 \text{ percent} \times (100 + x) \text{ gallons}]$, you see that the desired concentration, 35 percent, is between 20 percent and 40 percent. Also, the resulting quantity is the sum of the two quantities x and 100. Now rewrite the equation so it can be solved, replacing percentages with the decimal equivalents. Solving for x ,

$$\begin{aligned} 0.20(100) + 0.40x &= 0.35(100 + x) \\ 20 + 0.40x &= 35 + 0.35x \\ 0.05x &= 15 \\ x &= \frac{15}{0.05} = 300 \end{aligned}$$

It'll take 300 gallons of the 40 percent antifreeze to bring the concentration up to what he wants. Hope he has a large enough container!

The Problem: The same service station owner from the previous problem still wants to make his 100 gallons of 20 percent antifreeze into 35 percent antifreeze. How many gallons of *pure* antifreeze will it take to raise the concentration?



You can use the same basic equation of *quality* \times *quantity* and multiply the unknown amount, x , by 100 percent. The equation takes the form of $[20 \text{ percent} \times 100 \text{ gallons}] + [100 \text{ percent} \times x \text{ gallons}] = [35 \text{ percent} \times (100 + x) \text{ gallons}]$. Writing this in a form to be solved,

$$\begin{aligned} 0.20(100) + 1.00x &= 0.35(100 + x) \\ 20 + 1.00x &= 35 + 0.35x \\ 0.65x &= 15 \\ x &= \frac{15}{0.65} \approx 23.077 \end{aligned}$$

It'll take just a little more than 23 gallons of pure antifreeze to raise the concentration to 35 percent.

If you want to add pure antifreeze to what's in your radiator right now, to increase the concentration, you have to take some of the mixture out of the radiator, first. The radiator holds only so much fluid. So, an even more interesting problem involves removing a certain amount of what's in the radiator now and replacing it with pure antifreeze to achieve the level of concentration that you want.

The Problem: A man has a 16-quart radiator that now contains 16 quarts of 20 percent antifreeze. How much of the current mixture has to be removed and replaced with pure antifreeze to raise the level of concentration to 35 percent?



The basic structure for the mixture problem will have one more term in it. You'll have a *quality* \times *quantity* term that's subtracted from the original amount. Let x represent the number of quarts taken out and put back into the radiator. The end quantity will have to be the original 16 quarts.

$$[20 \text{ percent} \times 16 \text{ quarts}] - [20 \text{ percent} \times x \text{ quarts}] + [100 \text{ percent} \times x \text{ quarts}] = [35 \text{ percent} \times 16 \text{ quarts}]$$

Solving this equation,

$$\begin{aligned} 0.20(16) - 0.20x + 1.00x &= 0.35(16) \\ 3.2 + 0.80x &= 5.6 \\ 0.80x &= 2.4 \\ x &= \frac{2.4}{0.80} = 3 \end{aligned}$$

Take out 3 quarts of the 20 percent solution and replace it with pure antifreeze.



Coffee, tea, or not

Two friends were at a restaurant and ordered coffee to have with their dessert. One of the friends found a fly in his coffee and called for the waiter to take it away and bring a fresh cup of

coffee. The waiter came back with another cup. But the friend became very upset and returned it again, saying that the second cup was the same as the first cup. How did he know?

Answer: The friend had put sugar in the first cup of coffee. When he tasted the new cup, he could taste the sugar in it and knew it was the same cup with the fly removed.

Watering down the wine

Just as concentrations of solutions are increased by adding a pure substance, the concentrations can also be decreased by adding pure water. You perform this type of operation all the time, such as when mixing up orange juice from frozen concentrate or when fixing mixed drinks such as bourbon and water.

The Problem: How much water must be added to 8 ounces of 40 percent alcohol to produce a mixture that's 7 percent alcohol?



Water is 0 percent alcohol, so use the *quality* \times *quantity* setup to solve for x , the number of ounces of water needed.

$$[40 \text{ percent} \times 8 \text{ ounces}] + [0 \text{ percent} \times x \text{ ounces}] = [7 \text{ percent} \times (8 + x) \text{ ounces}]$$

Writing an equation and solving for x ,

$$\begin{aligned} 0.40(8) + 0.00x &= 0.07(8 + x) \\ 3.2 + 0 &= 0.56 + 0.07x \\ 2.64 &= 0.07x \\ \frac{2.64}{0.07} &= x \\ x &\approx 37.71 \end{aligned}$$

It'll take about 38 ounces of water to get the alcohol down to the 7 percent level.

Mixing up insecticide

Farmers know how important it is to keep destructive insects out of their crop-producing fields. They apply enough insecticide to keep the bugs away, but they don't want to overdo it and poison the field for the future. Mixing the insecticides and applying them where necessary is a complicated problem.

The Problem: To get rid of a nasty strain of root worm, an insecticide mixture should have a concentration that's 9 percent insecticide. A farmer has two mixtures on hand — one with 5 percent insecticide and the other with 15 percent insecticide. What is the ratio of the 5 percent to 15 percent mixtures that should be combined to get a 9 percent mixture? If the farmer needs 40 gallons of mixture, how many gallons of each should she use?



Filling the pitcher

What can you fill a pitcher with so that it's lighter than if it's full of air?

Answer: You can fill it with holes.



First solve for the ratio of the different mixtures and then apply it to the quantity of 40 gallons. Use the *quality* \times *quantity* setup. Let x represent the fraction of 5 percent solution and $1 - x$ represent the fraction of 15 percent solution. Adding $x + 1 - x$ you get 1, which is *all* of the final mixture.

$$[5 \text{ percent} \times x] + [15 \text{ percent} \times (1 - x)] = [9 \text{ percent} \times 1]$$

Solving for x in the related equation,

$$\begin{aligned} 0.05(x) + 0.15(1 - x) &= 0.09(1) \\ 0.05x + 0.15 - 0.15x &= 0.09 \\ 0.15 - 0.10x &= 0.09 \\ -0.10x &= -0.06 \\ x &= \frac{-0.06}{-0.10} = \frac{6}{10} = \frac{3}{5} \end{aligned}$$

The farmer needs to let $\frac{3}{5}$, or 60 percent, of the insecticide be made up of the 5 percent solution and $\frac{2}{5}$, or 40 percent, of the mixture be 15 percent solution. In a 40-gallon situations, 60 percent is $0.60 \times 40 = 24$ gallons and the other 40 percent is $0.40 \times 40 = 16$ gallons.

Counting on the Money

Problems concerning money — no, I'm talking about *math* problems, not just any problem over money — involve the value of the currency or coin or vending unit along with the number of each of these monetary values. The theme of the *quality* \times *quantity* applies very well here, where the *quality* is the coin value, bill value, or commodity value in the problem.



Adding up the money

How can you make 30¢ with only two coins if one of the coins is not a nickel?

Answer: You let the other coin be the nickel and use that nickel and a quarter.

Determining how many of each denomination

Coins and paper money come in many different denominations, allowing people to carry a lot of money with just a few bills or to make change for purchases using smaller bills and coins. What's consistent for all money in all countries is that you determine the total amount that you have by multiplying the value of the coin or bill times the number of them that you're carrying.

The Problem: Cassie has a total of \$4.10 in dimes and quarters. If she has twenty coins in all, how many of them are quarters?



Multiply the value (quality) of a quarter, \$0.25, times the number of quarters and the value of a dime, \$0.10, times the number of dimes. The sum of the two results is set equal to \$4.10. You know that the number of quarters and dimes equals 20. Also, you want to solve for quarters. So let q represent the number of quarters, leaving $20 - q$ to represent the number of dimes. Setting up the equation and solving,

$$\begin{aligned} 0.25(q) + 0.10(20 - q) &= 4.10 \\ 0.25q + 2 - 0.10q &= 4.10 \\ 0.15q + 2 &= 4.10 \\ 0.15q &= 2.10 \\ q &= \frac{2.10}{0.15} = 14 \end{aligned}$$

Cassie has 14 quarters. You check your answer by determining the number of dimes, $20 - q = 20 - 14 = 6$. Multiply $14 \times 0.25 = 3.50$ and $6 \times 0.10 = 0.60$. The sum $3.50 + 0.60 = 4.10$. So the solution checks.

The Problem: Grace has twice as many nickels as quarters and five less than three times as many dimes as quarters. If she has a total of \$6, then how many of each coin does she have?



Because both the nickels and dimes are compared to quarters, let q represent the number of quarters and write expressions for the number of nickels and dimes using the relationships. The number of nickels is written as $2q$, and the number of dimes is $3q - 5$. Multiply each quantity of coin by its quality and add up the products, setting the sum equal to 6.

$$\begin{aligned} 0.25(q) + 0.05(2q) + 0.10(3q - 5) &= 6.00 \\ 0.25q + 0.10q + 0.30q - 0.50 &= 6.00 \\ 0.65q - 0.50 &= 6.00 \\ 0.65q &= 6.50 \\ q &= \frac{6.50}{0.65} = 10 \end{aligned}$$

Grace has 10 quarters. If she has twice as many nickels as quarters, that makes it 20 nickels. And 5 less than 3 times 10 is 30 minus 5 or 25 dimes. Checking on the total, $0.25(10) + 0.05(20) + 0.10(25) = 2.50 + 1 + 2.50 = 6$.

It isn't always clear-cut which coin or piece of currency you want to use to have their amount represent the variable. You almost always have more than one choice for x . Your goal is to pick the money amount that makes writing the problem as easy as possible.

The Problem: A cash drawer contains singles, fives, tens, and twenties. There are two more tens than fives, eight less than twice as many twenties as tens, and ten more than twice as many singles than twenties. If the total amount of money in the cash drawer is \$650, then how many of each bill is there?



Several different comparisons are going on here, each with the number of bills of a particular denomination being compared to the number of bills of another denomination. You need to pick a variable to represent how many you have of one type of bill and work from that amount.

One suggestion is to start by letting the number of fives be represented by f . Then the number of tens is represented by $f + 2$. Comparing twenties to tens, you write *eight less than twice as many twenties as tens* by subtracting 8 from 2 times the number of tens, or $2(f + 2) - 8$. Now take that number of twenties and write the number of singles (*ten more than twice as many singles as twenties*) as $2[2(f + 2) - 8] + 10$.

You then write an equation taking the number of each type of bill times its monetary value, adding up all these products, and setting the sum equal to \$650. But first, it's a good idea to simplify the expressions for the numbers of

each bill. The number of fives is f , which is fine. The number of tens, $f + 2$, is also as simple as it can get. But the number of twenties can be simplified.
 $2(f + 2) - 8 = 2f + 4 - 8 = 2f - 4$. And the number of singles has a long way to go, because $2[2(f + 2) - 8] + 10 = 2[2f + 4 - 8] + 10 = 2[2f - 4] + 10 = 4f - 8 + 10 = 4f + 2$. Now you're set to go. Write the equation.

$$\begin{aligned} 5(f) + 10(f + 2) + 20(2f - 4) + 1(4f + 2) &= 650 \\ 5f + 10f + 20 + 40f - 80 + 4f + 2 &= 650 \\ 59f - 58 &= 650 \\ 59f &= 708 \\ f &= \frac{708}{59} = 12 \end{aligned}$$

There are 12 five-dollar bills, two more tens than fives (or 14 ten-dollar bills), eight less than twice as many twenties as tens (or 20 twenties), and ten more than twice as many singles as twenties (or 50 singles). Checking this out for the total: $5(12) + 10(14) + 20(20) + 1(50) = 60 + 140 + 400 + 50 = 650$.

Making a marketable mixture of candy

Some people prefer dark chocolate. Others go for the gooey chocolate-covered cherries. Many don't care for coconut in their sweets. And most people are happy with any mixture of any kind. Different kinds of candies have different prices, depending on the ingredients. When you combine different candies in packages, the *quality* or price of each type is multiplied by the *quantity* or weight to determine the price of the mixture.

The Problem: Malted milk balls sell for \$3 per pound, and chocolate-covered peanuts cost \$4 per pound. How many pounds of each type of candy should be used to create a 5-pound box of candy that costs \$3.60 per pound?



Let m represent the number of pounds of malted milk balls. Because the total amount of candy is to be 5 pounds, then $5 - m$ can represent the number of pounds of chocolate-covered peanuts. Multiply \$3 times m and add it to the product of \$4 and $5 - m$. Set that sum equal to \$3.60 times 5. Each term has the price times the weight. Solve for m .

$$\begin{aligned} 3(m) + 4(5 - m) &= 3.60(5) \\ 3m + 20 - 4m &= 18 \\ 20 - m &= 18 \\ 2 &= m \end{aligned}$$

If you use 2 pounds of malted milk balls, you'll need 3 pounds of the peanuts to make 5 pounds. Checking, $3(2) + 4(3) = 6 + 12 = 18$. If you divide the total price of \$18 by 5 pounds, you get \$3.60, the cost per pound.

The Problem: A box of candy is to contain chocolate covered cherries that cost \$5 per pound, nougats that cost \$3 per pound, assorted crèmes that cost \$2 per pound, and caramels that cost \$6 per pound. There should be an equal amount (by weight) of the caramels and cherries. You want twice as many nougats as caramels and twice as many crèmes as nougats. How much of each type of candy should be used if the box is to cost \$4.25?



First, you have to stop drooling and think about the mathematics, not the candy. Because the caramels and cherries will have equal weight, it makes sense to assign a variable to represent the weights of those two candies. Let c be the number of pounds of caramels and c be the number of pounds of chocolate-covered cherries. For the nougat, you want twice as much as there are caramels, so that's represented by $2c$. And twice as much of the nougat is $2(2c) = 4c$, for the weight of the crèmes. Now multiply each weight of candy by the respective cost, add them together, and set the sum equal to 4.25.

$$\begin{aligned} 5(c) + 6(c) + 3(2c) + 2(4c) &= 4.25 \\ 5c + 6c + 6c + 8c &= 4.25 \\ 25c &= 4.25 \\ c &= \frac{4.25}{25} = 0.17 \end{aligned}$$

So the mixture will contain 0.17 pound of chocolate covered cherries, 0.17 pound of caramels, 0.34 pound of nougat, and 0.68 pound of assorted crèmes.

Running a concession stand

Picture this: It's game night, and you've made a dash to the food vendor to get a quick snack. So did half the people in the stadium. Now you have time to stand in line and wait and ponder the problem of combining different foods and drinks available in the concession stand so that you can spend every penny of the money in your pocket — no more, no less.

The Problem: Stan had exactly \$20.40 in his pocket and managed to spend it all at the concession stand. He bought three hot dogs, two servings of cheese fries, and one large drink for that amount of money. One hot dog costs \$1.80 more than one serving of cheese fries. And a drink costs \$1.20 less than a hot dog. How much did each item cost?



The number of items have to be multiplied by their respective prices, and the costs totaled. Set that total equal to \$20.40 to solve for the individual prices. Let the cost of cheese fries be represented by c . Then a hot dog costs $c + 1.80$. A drink is \$1.20 less than a hot dog, so it costs $c + 1.80 - 1.20 = c + 0.60$.

$$\begin{aligned}
 2(c) + 3(c + 1.80) + 1(c + 0.60) &= 20.40 \\
 2c + 3c + 5.40 + c + 0.60 &= 20.40 \\
 6c + 6.00 &= 20.40 \\
 6c &= 14.40 \\
 c &= \frac{14.40}{6} = 2.40
 \end{aligned}$$

So cheese fries cost \$2.40, the hot dogs are \$1.80 more than that (or \$4.20), and the drink is \$0.60 more than the fries (or \$3).

The Problem: You're buying supplies for the concession stand Friday night and need to purchase candy and pretzels and gum in bulk. Twix candy bars are 35¢ each if you buy a case of 36. M&M's bags are 36¢ each if you buy a case of 48. Pretzels are 19¢ a bag if you buy a case of 30. Cookies are 24¢ a package if you buy a case of 33. And gum costs 18¢ a pack if you buy a case of 40. You buy the same number of cases of cookies and gum. You buy twice as many cases of M&M's as cookies. You buy two less than twice as many cases of pretzels as cookies. And you buy five fewer cases of Twix than M&M's. If you spend \$270.72 on all these supplies, then how many individual items do you have on hand to sell Friday night?



This problem just begs to be organized. A spreadsheet would be a big help here. But the next best thing is a table showing the item name, the number of items in a case, the cost per case, and a representation of the number of cases ordered. Because the number of cases of cookies and gum is the same, let the number of cases of these items be represented by g . The number of cases of M&M's is $2g$. The number of cases of pretzels is $2g - 2$; and the number of cases of Twix is $2g - 5$. Table 14-1 shows all the entries.

Table 14-1 Concession-Stand Order

| Item | Number Per Case | Case Cost | Number of Cases |
|----------|-----------------|------------------------------|-----------------|
| Cookies | 33 | $33 \times \$0.24 = \7.92 | g |
| Gum | 40 | $40 \times \$0.18 = \7.20 | g |
| M&M's | 48 | $48 \times \$0.36 = \17.28 | $2g$ |
| Pretzels | 30 | $30 \times \$0.19 = \5.70 | $2g - 2$ |
| Twix | 36 | $36 \times \$0.35 = \12.60 | $2g - 5$ |

The equation consists of the sum of the products of the costs and the number of cases set equal to the total of \$270.72.

$$\begin{aligned}
 7.92(g) + 7.20(g) + 17.28(2g) + 5.70(2g - 2) + 12.60(2g - 5) &= 270.72 \\
 7.92g + 7.20g + 34.56g + 11.40g - 11.40 + 25.50g - 63 &= 270.72 \\
 86.28g - 74.40 &= 270.72 \\
 86.28g &= 345.12 \\
 g &= \frac{345.12}{86.28} \\
 g &= 4
 \end{aligned}$$

You bought 4 cases of cookies and 4 cases of gum. You bought 8 cases of M&M's, 6 cases of pretzels and 3 cases of Twix. Multiplying each number of cases by the number of items in that case, you have $4(33) + 4(40) + 8(48) + 6(30) + 3(36) = 964$ items. Have fun!

Being Interested in Earning Interest

Earning interest on money invested is of utmost importance to the wise investor. Some funds pay a higher rate of interest but may be a bit risky. To offset the risk, the shrewd investor puts some money in a high-risk fund and the rest in a fund that doesn't pay as well but one that can be trusted to give a return and not lose any of the investment.

Making your investment work for you

The problems in this section assume the use of the *simple-interest formula*, compounded annually. In actuality, financial institutions use compound interest and computer programs to figure out these problems. But you get a good idea of how it works — and a pretty good estimate of the actual answer using the less complex simple-interest formula.



The simple-interest formula says that the interest earned, I , is equal to the amount invested (principal), p , times the percentage rate, r , written as a decimal, times the number of years (time), t . The formula is: $I = prt$.

The Problem: Robert has \$50,000 to invest. He wants to put some of this money in an account that earns 8 percent interest and the rest in a riskier account that promises to earn at the rate of 12 percent. He needs yearly earnings of a total of \$4,500. How much of his \$50,000 should he invest in each account?



Write an equation in which the *first interest* plus the *second interest* is equal to \$4,500. The *first interest* and *second interest* are expressions that you write using the interest formula. This is another *quality times quantity* situation. Let x represent the amount of money invested in the first account and $50,000 - x$ be the amount of money in the second account. Assume that the time is one year. Then solve for x .

$$\begin{aligned}
 (prt)_{\text{first}} + (prt)_{\text{second}} &= 4500 \\
 x(0.08)(1) + (50,000 - x)(0.12)(1) &= 4500 \\
 0.08x + 6,000 - 0.12x &= 4500 \\
 6000 - 0.04x &= 4500 \\
 1500 &= 0.04x \\
 x = \frac{1500}{0.04} &= 37,500
 \end{aligned}$$

Robert needs to invest \$37,500 at 8 percent and the rest, $\$50,000 - \$37,500 = \$12,500$, at 12 percent.

The Problem: Stella has been watching several investment funds and has decided that she'll deposit some of her \$100,000 in each. She will put twice as much in the fund earning 5 percent as that earning 4 percent, \$5,000 more in the fund earning 10 percent than in the fund earning 4 percent, and \$5,000 less in the fund earning 6 percent as in the fund earning 4 percent. How much will she get in total interest earnings at the end of the year? How much did she invest in each fund?



First, determine how much she's investing in each fund by letting x represent the amount in the 4 percent fund, let $2x$ represent the amount in the 5 percent fund, let $x + 5,000$ represent the amount in the 10 percent fund, and let $x - 5,000$ represent the amount in the 6 percent fund. Add up all the fund amounts and set them equal to 100,000 to solve for x .

$$\begin{aligned}
 x + 2x + x + 5000 + x - 500 &= 100,000 \\
 5x &= 100,000 \\
 x = \frac{100,000}{5} &= 20,000
 \end{aligned}$$

If $x = 20,000$, then the amounts in the 4 percent, 5 percent, 10 percent, and 6 percent funds are \$20,000, \$40,000, \$25,000 and \$15,000, respectively. Multiply each fund amount and percentage together to get the total interest.

$\$20,000(0.04) + \$40,000(0.05) + \$25,000(0.10) + \$15,000(0.06) = \$800 + \$2,000 + \$2,500 + \$900 = \$6,200$. Stella should earn \$6,200 in interest.

Determining how much is needed for the future

It's all well and good if you have \$100,000 or some such amount of money to invest. Wouldn't we all just love that? Frequently, though, the question is more like, "How much do I need to invest in order to get that kind of interest?" Nonprofit organizations like to have *endowment funds* (money put aside that's never touched, with only the interest used to pay expenses). They want to know how much is needed in the endowment fund in order to have a particular income every year from the interest.

The Problem: The local Kiwanis Club needs \$20,000 in interest annually in order to adequately fund its benevolence programs. It currently has \$120,000 invested — one-third of it in a mutual fund earning 5 percent interest and the rest in a money-market fund earning 10 percent interest. This division of the funds is mandated by its endowment agreement. How much more money does the Kiwanis Club need, and how much does it have to have in each fund in order to earn that \$20,000 annual interest?



First determine the total amount of money needed to generate the \$20,000 interest each year when one-third goes into the 5 percent fund and two-thirds into the 10 percent fund. Then, after you find out how much is needed, you can see what the difference is between that and what the Kiwanis Club already has.

Think of the division of money as being divided into three parts — one part in the 5 percent fund and two-thirds in the 10 percent fund. Rather than use fractions, let the whole three-thirds be represented by $3x$. Then let x represent the amount of money invested at 5 percent and $2x$ represent the amount invested at 10 percent. Write the two interest terms, add them up, and set them equal to 20,000.

$$\begin{aligned}x(0.05) + 2x(0.10) &= 20,000 \\0.05x + 0.20x &= 20,000 \\0.25x &= 20,000 \\x &= \frac{20,000}{0.25} = 80,000\end{aligned}$$

It needs to have \$80,000 invested at 5 percent and twice that, \$160,000 invested at 10 percent. So a total of \$240,000 is needed in the two funds. Subtracting what it already has, $\$240,000 - \$120,000 = \$120,000$. The Kiwanis Club is halfway to its goal. Looks like it'll have to host a few more pancake breakfasts to get there!

Chapter 15

Feeling Your Age with Age Problems

In This Chapter

- ▶ Making sense of problems involving comparisons of ages
 - ▶ Adding and subtracting the years to solve problems
 - ▶ Dealing with complications of real puzzlers in age problems
-

One of my favorite Funky Winkerbean cartoons starts out with something like, “When Henry was twice as old as Hank was two years after half of Hank’s age. . . .” And the punch line: “How many apples is an orange worth?” Some people think this is hilarious. Others don’t find a bit of humor in these very involved, contrived, convoluted problems.

In this chapter, I acquaint you with techniques and procedures to use when tackling word problems involving the ages of one, two, or more people. When broken down into their component parts, age problems seem much less intimidating. Here you see how to deal with aging — well, ages of people after a certain amount of time. I hope you’ll see the humor in the cartoon, too.

Doing Age Comparisons

Age comparisons are done almost as soon as a person can talk. Children take advantage of being older than one another to get first in line or get to stay up later or get whatever the prize. Doing age comparisons makes for interesting math problems, too. The usual rules apply: Always let the variable represent a number, and use the math words for clues about adding, subtracting, and so on.

Warming up to age

Before getting into some serious algebra word problems, here are a few warm-ups to get you in the mood.

The Problem: A 40-year-old man married a 25-year-old woman. The woman died at the age of 40, and her husband was so saddened that he wept for years after that. He died 5 years after he stopped weeping, on his 80th birthday. How many years was he a widower?



First, just do some mental calculations. The couple was married when he was 40 and she was 25; he was 15 years older than she was. If she died when she was 40, then he was still 15 years older than she was, so he was 55. It doesn't matter how many years he wept and that for 5 years he had stopped weeping. The only figuring you need to do is to take the age when he died, 80, and subtract the age he was when his wife died, 55. Subtracting $80 - 55 = 25$, so he was a widower for 25 years.

The next problem involves more counting and logic than algebra to figure out the ages.

The Problem: A man has three children. The oldest child is three times as old as the youngest. The second child is six years older than the youngest and six years younger than the oldest. How old are the children?



The second or middle child's age is halfway between two numbers that are six years away in either direction. So the ages could be 1, 7, and 13; 2, 8, and 14; 3, 9, and 15; and so on. Go back to the beginning of the problem. The oldest child is three times as old as the youngest, so the age of the oldest has to be a multiple of 3. The multiple has to be larger than 13, so the oldest is 15, 18, 21, or some other multiple of three. Also, the oldest is 12 years older than the youngest. So what number can you multiply by 3 and have it turn out to be 12 larger than the original number? It's the number 18, that you want. If the children are 6, 12, and 18, then the oldest is three times as old as the youngest.

Okay, so you can't stand not using algebra and an equation to do this problem. I give. Here goes: Let the middle child's age be x . Then the oldest child's age is $x + 6$ and the youngest child's age is $x - 6$. If the oldest child's age is equal to three times the youngest child's age, you write that as $x + 6 = 3(x - 6)$. Distributing the 3 on the right, the equation becomes $x + 6 = 3x - 18$. Subtract x from each side and add 18 to each side, and you get $24 = 2x$. Dividing by 2, $12 = x$. The middle child is x years old, or 12, the oldest is $x + 6$ years older, or 18, and the youngest is $x - 6$ years old, or 6.



Brothers and sisters, I have some

If each child in a certain family has at least four brothers and three sisters, then what is the smallest number of children that the family can have?

Answer: There have to be at least nine children in the family — five boys and four girls.

Making age an issue

Traditional age word problems use algebraic expressions to write comparisons such as *twice the age of* or *four years older than* and then solve for one or more person's age. You'll see lots of parentheses in the equations that are first written so that the meaning is clearly defined and the people's ages are clearly identified. Age problems can get pretty wordy and confusing, so you want to be sure that you've written something to parallel the wording.

The Problem: To be a member of a certain club, you have to meet its stringent age requirement. Three more than twice your age must be equal to 21 more than your age. What is the age requirement (in simpler terms)?



First, decide if the club is worth it. That decided, now tackle the problem. Let x represent your age. Write *three more than twice your age* as $3 + 2x$. Then *21 more than your age* becomes $21 + x$. Set the two expressions equal to one another, and the equation to solve is $3 + 2x = 21 + x$. Subtracting x and 3 from each side, you get $x = 18$.

You double your pleasure and double your fun when you introduce twins into the mix. The ages of twins is the same (I won't get into the special case of twins born on two different days even though minutes apart), so you have to be sure to include the age of the twins twice into the problem.

The Problem: The mother of a pair of twins is 30 years older than they are. If the sum of the ages of all three of them (the mother and the twins) is 12 greater than the mother's age, then how old is the mother?



Let the age of the twins be represented with x . Then their mother's age is represented with $x + 30$. Adding the ages of the three people together, you get the expression $x + x + (x + 30)$. Because this sum is 12 greater than the age of the mother, you set this sum equal to $(x + 30) + 12$. The final equation and its solution is as follows.

$$x + x + (x + 30) = (x + 30) + 12$$

$$3x + 30 = x + 42$$

$$2x = 12$$

$$x = 6$$

The age of the twins is 6, so their mother is $6 + 30 = 36$ years. The sum of all three people is $6 + 6 + 36 = 48$, and 48 is 12 greater than the mother's age of 36.

The Problem: Libin is three times as old as Larry. If the sum of their ages is 16 more than twice Larry's age plus 4, then how old is Libin?



Even though the problem asks for Libin's age, it'll be easier to let the variable, x , represent Larry's age and then answer the question after solving the equation. Letting x represent Larry's age, then $3x$ represents Libin's age. The sum of their ages is $x + 3x$. You write *16 more than twice Larry's age plus 4* with numbers and a variable in exactly that same order: $16 + 2x + 4$. Now write the entire equation with the two expressions separated by an equal sign and solve.

$$x + 3x = 16 + 2x + 4$$

$$4x = 20 + 2x$$

$$2x = 20$$

$$x = 10$$

So Larry is 10 years old, making Libin 30 years old.

Going Back and Forth into the Future and the Past

Age problems get even more exciting when you get out your crystal ball and make predictions about the future or look back in a photo album to reminisce about the past. The equations used to solve age problems about the future and past get more complicated because the same amount of time has to be added to or subtracted from each variable expression.

Looking to the future

How old will you be in seven years? Some of us grimace at the thought of another significant birthday, but it's better than the alternative. Add seven to your age, and you know how many candles go on your birthday cake.

The Problem: Jake is six years older than Jack. In 7 years, the sum of their ages will be 52. How old are Jake and Jack now?



Let Jack's age be represented with x and Jake's age represented with $x + 6$. (An alternative would be to let Jake's age be x and Jack's be $x - 6$; most people prefer to work with addition instead of subtraction.) Now deal with *in 7 years*. Add 7 to *both* ages, so Jack's age will be $x + 7$ and Jake's age will be $x + 6 + 7$ or $x + 13$ in seven years. The sum of their ages in seven years will be 52, so the equation and its solution are as follows:

$$\begin{aligned}(x + 7) + (x + 13) &= 52 \\ 2x + 20 &= 52 \\ 2x &= 32 \\ x &= 16\end{aligned}$$

So Jack is 16, and Jake, who is 6 years older, is 22. In seven years, Jack will be $16 + 7 = 23$ and Jake will be $22 + 7 = 29$. The sum $23 + 29 = 52$.

Some age problems also use comparisons where one person is twice as old or four times as old in a number of years.

The Problem: Molly is 30 years older than Margaret. In 18 years, Molly will be twice as old as Margaret. How old will Molly be when she's twice Margaret's age?



Let Margaret's age be represented by x and Molly's age be represented by $x + 30$. In 18 years, Margaret will be $x + 18$ and Molly will be $(x + 30) + 18 = x + 48$. At that time, 18 years from now, Molly will be twice as old as Margaret, so you write that Molly's age, $x + 48 = 2(x + 18)$, twice Margaret's age. Solving the equation, you multiply through by 2 on the right to get $x + 48 = 2x + 36$. Subtract x from each side and subtract 36 from each side to get $12 = x$. So Margaret is 12 years old now, and Molly, who is 30 years older than Margaret is $12 + 30 = 42$ years old. In 18 years, when Molly is twice Margaret's age, Molly will be $42 + 18 = 60$ and Margaret will be $12 + 18 = 30$.



Catching up in age

In an old Abbott and Costello movie, there's a discussion about the ages of a husband and wife. When they marry, the husband is four times as old as his bride. Eight years later he's three times as old as she is. And 24 years after that he's only

twice her age. Abbott asks Costello how long it'll take for her to catch up with him in age. Of course, that's impossible, but what were the ages of the couple when they married? (Try to keep it legal in most states.)

Answer: If he was 64 and she was 16 when they were married, then he was 4 times her age. Eight years later, he was 72 and she was 24 — his age was three times hers. Twenty-four years after that, he was 96 and she was 48, twice her age.

The Problem: Stan is four times as old as Steve. In ten years, Stan will be twice as old as Steve. How old are Stan and Steve?



Let Steve's age be represented by x and Stan's age by $4x$. In ten years, Steve will be $x + 10$ and Stan will be $4x + 10$. Because Stan's age in ten years will be twice Steve's, you write that $4x + 10 = 2(x + 10)$. Multiplying through by 2 on the right, $4x + 10 = 2x + 20$. Subtract $2x$ from each side and 10 from each side to get $2x = 10$, or $x = 5$. So Steve is 5 and Stan is four times that, or 20. In ten years, Steve will be 15 and Stan will be 30.

Going back in time

Looking back on things, you recall those good times with close friends and what you did when you were all younger — by two years or ten years or more. Just as with adding years in word problems, when you have so many years ago in a problem, you subtract the same number of years from each age involved in the situation.

The Problem: Today, Ernest's age is four years more than twice Christine's age. Three years ago, the sum of their ages was 31. How old is Christine right now?



Let x represent Christine's age right now. That means that Ernest's age is represented by $4 + 2x$. To adjust for *three years ago*, you subtract 3 from each age, so Christine was $x - 3$ and Ernest was $4 + 2x - 3 = 2x + 1$ years old. The sum of those two ages, three years ago, was 31. The equation and solution are:

$$\begin{aligned}(x - 3) + (2x + 1) &= 31 \\ 3x - 2 &= 31 \\ 3x &= 33 \\ x &= 11\end{aligned}$$

Christine is 11 years old right now, so Ernest is $4 + 2(11) = 26$ years old. Three years ago, Christine was 8 and Ernest was 23. The sum $8 + 23 = 31$.

With multiple births comes multiples of the same age. Each person involved has his age adjusted in the same way.

The Problem: Three years ago, the sum of the ages of the quintuplets and their older sister was 155. If Sis is five years older than her siblings, then how old are the quints today?



Name that child

Clara's mother had four children. The first was named April, the second was named May, and the third was named June. What was the name of her fourth child?

Answer: Did you fall for it? Her fourth child had to be Clara.



First, *quintuplets* refers to five siblings born at the same time. Designate the variable x as the age of the quints today. Their sister's age is then $x + 5$ years. Three years ago, the quints were $x - 3$ years old, and their sister was $x + 5 - 3$ or $x + 2$ years old. The sum of the ages of these six children is written by multiplying the quints' age by five and adding the sister's age: $5(x - 3) + (x + 2)$, and that sum is set equal to 155 for the equation.

$$\begin{aligned} 5(x - 3) + (x + 2) &= 155 \\ 5x - 15 + x + 2 &= 155 \\ 6x - 13 &= 155 \\ 6x &= 168 \\ x &= \frac{168}{6} = 28 \end{aligned}$$

The quints are each 28 today. Their sister is five years older, so she's 33 years old. Three years ago, the quints were 25 and Sis was 30. Multiplying 25 \times five you get 125. Add 30, and the sum of the six ages is 155.

The Problem: Amanda's age is two years more than twice Bill's age. Nine years ago, her age was three years more than three times Bill's age. How old will they be in five years?



Let Bill's age be represented with x so that Amanda's age can be represented with $2 + 2x$. Nine years ago, Bill was $x - 9$ years old and Amanda was $2 + 2x - 9$ or $2x - 7$ years old. Amanda's age nine years ago was equal to *three years more than Bill's age* at that time. The equation to use is $2x - 7 = 3 + 3(x - 9)$. Notice that both ages in the equation are those representing nine years ago.

$$\begin{aligned} 2x - 7 &= 3 + 3(x - 9) \\ 2x - 7 &= 3 + 3x - 27 \\ 2x - 7 &= 3x - 24 \\ 17 &= x \end{aligned}$$

Bill is 17 years old right now, and Amanda is $2 + 2(17)$ or 36 years old. In five years, Bill will be 22 and Amanda will be 41.

Facing Some Challenges of Age

My Funky Winkerbean story, at the beginning of this chapter, is an example of how some age problems can seem awfully complicated and difficult. The problems in this section may seem impossible at first, but, by taking them apart and putting them back together again as logical equations, you see how those challenges of age aren't all that bad after all.

The Problem: Mike and Ike are good friends and like to challenge new acquaintances with the following riddle, saying that they'll pick up the tab at the restaurant if the riddle can be solved. Here's what they pose to the unsuspecting: *Mike is twice as old as Ike was when Mike was as old as Ike is now. Right now, Mike is 36 years old. How old is Ike right now?*



The most common plan of action is to let Ike's age be represented with x and then write some expressions and equations involving that variable. In this case, because you know Mike's age already, it makes more sense to figure out how many years ago the riddle is referring to — solve for x . Here's what you know:

Mike is twice as old as Ike was . . . so, since Mike is 36, Ike was 18.

Mike was as old . . . letting x be the number of years ago, Mike was $36 - x$.

Ike is now . . . the number of years since Ike was 18 is $18 + x$.

Does that look like sleight-of-hand? Look at Table 15-1, showing possible ages of Mike and Ike if x is the number of years since Ike was 18.

Table 15-1

Mike and Ike's Ages

| x Years | Mike's Age ($36 - x$) | Ike's Age ($18 + x$) |
|-----------------------------|---|--|
| 1 | $36 - 1 = 35$ | $18 + 1 = 19$ |
| 2 | $36 - 2 = 34$ | $18 + 2 = 20$ |
| 3 | $36 - 3 = 33$ | $18 + 3 = 21$ |
| : | : | : |
| 8 | $36 - 8 = 28$ | $18 + 8 = 26$ |
| 9 | $36 - 9 = 27$ | $18 + 9 = 27$ |

Nine years ago, Mike was 27, which is the same age that Ike is now. When Mike was 27, Ike was 18 (or half Mike's age now). If I had just written the equation $36 - x = 18 + x$ and solved it to get $18 = 2x$ or $x = 9$, you might not have believed me. For the naturally skeptical, Table 15-1 helps.

The next problem uses more number theory than algebra. Haul out your prime factorizations to do this problem.



The *prime factorization* of a number is the unique set of all the prime numbers whose product is that number. For example, the prime factorization of the number $105 = 3 \times 5 \times 7$. The prime factorization of $100 = 2^2 \times 5^2$.

The Problem: Mike and Ike are back at it again. They point to a family with three children enjoying their dinner at a nearby table. They offer to buy dinner for that entire family and the person they're making the bet with if this poor, unsuspecting person can figure out the riddle (otherwise, this person pays). The riddle: *At the nearby table, the product of the ages of the three children is 72, and the sum of their ages is today's date. How old are the children?* The bet-taker thinks for a while and then says, "But that isn't enough information!" Mike takes pity and offers him a clue, "Okay, I'll tell you that the oldest child doesn't like pizza." Can you determine the ages of the children?



Haul out another list or chart. The prime factorization of the number 72 is $2^3 \times 3^2$. Use the factors to figure out all the different numbers that divide 72 evenly — and include the number 1 (which isn't a prime number, but is a factor of 72). All the possible ages — numbers that divide 72 evenly are: 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, and 72. Using these numbers, you create a list of the possible ages of the three children from combinations that multiply to give you 72. For example, 72 and 1 and 1 multiply to give you 72. The numbers aren't very probable for the ages of three children, but I'm sure it's happened. Another combination is 36 and 2 and 1. Making this list is fine, but a table is more helpful, because you want the *sum* of the ages, too. Table 15-2 lists all of the possible ages and their sums. Notice that the ages multiplied together are always 72.

Table 15-2 Ages of Three Children and the Sum of Their Ages

| Child 1 | Child 2 | Child 3 | Sum of Ages |
|----------------|----------------|----------------|--------------------|
| 72 | 1 | 1 | 74 |
| 36 | 2 | 1 | 39 |
| 24 | 3 | 1 | 28 |
| 18 | 4 | 1 | 23 |

(continued)

Table 15-2 (continued)

| <i>Child 1</i> | <i>Child 2</i> | <i>Child 3</i> | <i>Sum of Ages</i> |
|----------------|----------------|----------------|--------------------|
| 18 | 2 | 2 | 22 |
| 12 | 6 | 1 | 19 |
| 12 | 3 | 2 | 17 |
| 9 | 8 | 1 | 18 |
| 9 | 4 | 2 | 15 |
| 8 | 3 | 3 | 14 |
| 6 | 6 | 2 | 14 |
| 6 | 4 | 3 | 13 |

You see all the possibilities for the ages. So, if the bet-taker had worked out all the possible ages and sums and compared the sum with today's date, he could have answered the question — unless today is the 14th. That's why he needed more information. By giving the hint that the oldest child doesn't like pizza, the answer to the riddle is clear. The children must be 8, 3, and 3 years old, because the other combination with a sum of 14 is 6, 6, and 2 where there are *two* older children.

One more age problem to consider mixes in a little number sense with ages now and in the past.

The Problem: Ben is a teenager. Bob is one-third as old as Ben was when Bob was half as old as Ben is now. How old are Ben and Bob?



Identical applications

Two men apply for the same job. They are identical in appearance. They have the same mother and father and birth date. The person doing the

hiring asks, "Are you two twins?" They respond, quite honestly, "No, we aren't." How is this possible.

Answer: The men are two children from a set of triplets or quadruplets or quintuplets, and so on.



First, you can narrow down Ben's age to being either 14, 16, or 18. This conclusion comes from the fact that he's a teenager and Bob is half his age; Ben's age has to be evenly divisible by two. Of course, this assumes that the problem only deals with whole numbers, and that's a good assumption. You can just play around with the three guesses to see whatever numbers work, or you can be a bit more systematic. Let x represent the number of years ago in the problem:

$$\text{Bob's age} = \frac{1}{3}(\text{Ben's age} - x)$$

Bob is one-third as old as Ben was x years ago.

$$\frac{1}{3} \text{Ben's age} - \text{Bob's age} = \frac{1}{3}x$$

$\text{Ben's age} - 3\text{Bob's age} = x$, simplifying the equation by distributing the fraction and then multiplying each term by 3

$\text{Ben's age} - x = \frac{1}{2}\text{Ben's age}$, when Bob was half as old as Ben is now.

$\frac{1}{2}\text{Ben's age} = x$, simplifying the previous equation in terms of Ben's age.

$\text{Ben's age} - 3\text{Bob's age} = x$

$\frac{1}{2}\text{Ben's age} = x$, so

$\text{Ben's age} - 3\text{Bob's age} = \frac{1}{2}\text{Ben's age}$, letting the two equations ending in x equal one another and simplifying

$\frac{1}{2}\text{Ben's age} = 3\text{Bob's age}$

$\text{Ben's age} = 6\text{Bob's age}$

So, if Ben is six times as old as Bob and Ben is a teenager, then Ben must be 18 years old and Bob must be 3.

Chapter 16

Taking the Time to Work on Distance

In This Chapter

- ▶ Figuring the distance traveled as the sum of two different distances
- ▶ Equating two distances that have different rates of speed
- ▶ Emptying or filling a tank with open intake and outflow valves
- ▶ Divvying up the work with work problems

This chapter deals with a lot of moving around. People and cars and trains are moving around all the time. Liquids in containers are moving from one place to another. The first problems involve moving around and the distance traveled. The typical distance problems use the relationship that distance is equal to rate times time, $d = rt$. By using this equation, you deal with trains meeting somewhere in the middle of nowhere and wives catching up to forgetful husbands.

Another type of moving around occurs with the classic work problems — how many hands it takes for a lighter workload — and how long it takes when everyone works together.

And, lastly, you'll find a lot of moving around when fluids move from one container to another. If you're trying to fill a tub while the tub is leaking, you have to work quickly to find out how long it takes to actually make the tub overflow.

Summing Up the Distances

These first distance problems have a common theme: that the sum of the distances that two people travel, along the same straight line, equals the total distance. Sometimes, people are traveling toward one another, and the sum

of the distances is how far each has traveled until they meet. Other times, people are traveling in opposite directions, and you sum up the distances traveled to compute how far apart they are.

Meeting somewhere in the middle

Two lovers spy each other on opposite sides of the room and run toward one another. They meet somewhere between where they started running — and where they meet depends on how fast each can run. You can assume that there are no chairs to dodge in their mad dashes. The setup used to solve this problem is pretty much the same as a problem involving two bulls rushing at one another from either side of the arena or two trains approaching one another from opposite directions along the same track.



Keep in mind the formula $d = rt$ or $distance = rate \times time$. When two different distances are added together to equal a total distance, the individual rates and times are multiplied together first and then the products are added together:

$$\begin{aligned} d &= d_1 + d_2 \\ d &= r_1 t_1 + r_2 t_2 \end{aligned}$$

Making good time

With distance problems, you can solve for the distance traveled or the speed at which objects are traveling or the amount of time spent. The two problems in this section involve solving for how much time it takes to reach a goal.

The Problem: Betsy and Bart see each other from opposite sides of a gymnasium that measures 440 feet across. They both start running toward one another at the same time. If Betsy can run at the rate of 4 feet per second and Bart runs at 7 feet per second, then how long does it take for them to meet?



The total distance to be covered is 440 feet. Set that amount equal to the sum of the distances that each runner covers. Betsy can run at the rate of 4 feet per second, and she runs for t seconds. Bart can run at 7 feet per second and also runs for t seconds. So your equation is $440 = 4t + 7t$. Simplify on the right to get $440 = 11t$. Divide each side of the equation by 11, and you get that $t = 40$ seconds. It'll take less than a minute for the two to reach one another, with Betsy running $4(40) = 160$ feet and Bart running $7(40) = 280$ feet.

In the preceding problem, the two people see each other and start running at the same time. What if one starts before the other, and they run for different amounts of time?

The Problem: Jon leaves Chicago at noon and heads south toward Bloomington traveling at 45 mph. Jane leaves Bloomington heading north for Chicago at 1 p.m., traveling at 55 mph. If Chicago and Bloomington are 145 miles apart, then what time will they meet?



The total distance to be traveled is 145 miles. If Jon drives for t hours, then Jane, who left an hour later, will drive for $t - 1$ hours. Take each rate in terms of miles per hour and multiply it by the respective amount of time traveled. Jon will drive for $45t$ miles and Jane will travel for $55(t - 1)$ miles. Add the two distances together and set the sum equal to 145. Solve the equation for t .

$$\begin{aligned} 45t + 55(t - 1) &= 145 \\ 45t + 55t - 55 &= 145 \\ 100t - 55 &= 145 \\ 100t &= 200 \\ t &= 2 \end{aligned}$$

Because t is the number of hours that Jon drives, he drives for two hours and Jane drives for one hour less or one hour. In any case, they meet up at 2 p.m.

Speeding things up

Distance is equal to rate times time. Participants in these distance problems may travel for the same amount of time or different amounts of time. They can travel at the same speed or different speeds. The next two problems let you determine how fast things are moving.

The Problem: Two trains leave the same station traveling in opposite directions. The first train leaves at 2 p.m. The second train leaves a half-hour later and travels at a speed averaging 15 miles per hour faster than the first train. By 8 o'clock that evening, they're 600 miles apart. How fast are the two trains traveling?



Use the same $d = rt$ format, setting the sum of the distances traveled equal to 600. The first train travels for 6 hours at rate r , and the second train, leaving a half-hour later, only travels for 5.5 hours but at a greater speed, $r + 15$. The equation and solution:

$$\begin{aligned} 6r + 5.5(r + 15) &= 600 \\ 6r + 5.5r + 82.5 &= 600 \\ 11.5r + 82.5 &= 600 \\ 11.5r &= 517.5 \\ r &= \frac{517.5}{11.5} = 45 \end{aligned}$$

The first train is traveling at 45 mph. The second train is traveling 15 mph faster, or 60 mph.



Taxi driver

Imagine that you're a taxi driver in Chicago and that you're driving a 1994 yellow cab. Your passengers are college students, and they want to travel 4 miles from home to school. You're driving

30 mph, and the tank is half full of fuel. One mile into the trip, the tank is down to two-fifths full of fuel, and the trip is over 15 minutes later. What is the name and age of the cab driver?

You are the taxi driver. What are your name and age? There's your answer!

All that was unnecessary; it's beginning of the story. It's hard to imagine that

Answers Did you divide by one-half or multiply by two? How big was the tank?

The Problem: Alberto can bicycle 2 miles per hour less than twice as fast as Ollie, so Alberto didn't leave for the rally until two hours after Ollie left. If the total distance they traveled was 504 miles and if Ollie traveled for 10 hours, then how fast can Alberto bicycle?



Let the rate at which Ollie can bicycle be r . Then Alberto's rate is $2r - 2$. If Ollie traveled for 10 hours, then Alberto traveled for 8 hours. The total distance is equal to the sum of the distances the two traveled. Ollie traveled $10r$ miles, and Alberto traveled $8(2r - 2)$ miles.

$$\begin{aligned}10r + 8(2r - 2) &= 504 \\10r + 16r - 16 &= 504 \\26r - 16 &= 504 \\26r &= 520 \\r &= \frac{520}{26} = 20\end{aligned}$$

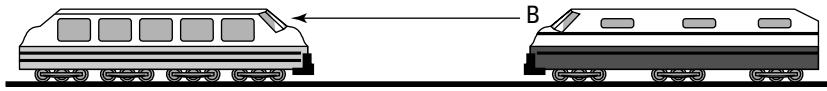
Ollie bicycles at 20 mph, so Alberto bicycles at $2(20) - 2 = 40 - 2 = 38$ mph.

Making a beeline

You can solve for the time it takes to travel, and you can solve for how fast cars or trains travel. Determining the total distance that someone or something travels can also be very interesting. Consider the story of *Super Bee* who flies at the speed of 90 mph. He leaves the engine of a westbound train that's traveling at 60 mph and flies until he reaches an eastbound train that's traveling at 75 mph along the same track. After barely touching the eastbound train, he flies back to the westbound train, touches it, and flies back

and forth and back and forth between the trains until the trains finally meet. Figure 16-1 shows you the two trains and *Super Bee*, flying between them.

Figure 16-1:
Super Bee
flies back
and forth
between the
speeding
trains



The Problem: How far does a bee travel if it flies back and forth between trains that are approaching each other on the same track — one train traveling at 60 mph and the other traveling 75 mph — if the bee flies at 90 mph and, when it started this journey, the trains were 648 miles apart?



You could determine how far the bee had to fly until it flew from the first train to the second train — while the trains are getting closer to one another — and then how far it had to fly back to the first train, and so on, until the trains meet. Or, a much simpler way to do this is to figure out how long it will take the trains to meet if they start 648 miles apart and are approaching each other at the rate of 135 mph (the sum of 60 mph and 75 mph). When you've determined how much time it takes, then you know how long the bee has been flying and can multiply the time and the rate of 90 mph to get the distance. Because $d = rt$, in this case $648 = 135t$. Dividing each side of the equation by 135, you get that $t = 4.8$ hours — the time it takes the trains to meet. Multiplying 90×4.8 , you get a distance of 432 miles. That's one tired bee.

Equating the Distances Traveled

Many distance problems have the scenario that one person catches up with another or one train or plane leaves later and finally passes the first one. The common thread or theme for these problems is that the distance traveled by the two participants is the same. You equate the distances. Some problems have you solve for time — how long it took to catch up. Or you may solve for how fast one or the other is traveling. And the question may even be about the distance — how far they traveled before arriving at the same place. In each case, though, the equation setup is the same.



The formula is $d = rt$ or *distance = rate × time*. And when two distances are equated, the individual rates and times are multiplied together first and then set equal to one another:

$$d_1 = d_2$$

$$r_1 t_1 = r_2 t_2$$

Making it a matter of time

When two people leave the same place at different times, one has to travel more quickly than the other to catch up — assuming that they're both using the same route.

The Problem: Henry left for work at 7 a.m. and drove at an average speed of 45 mph. Unfortunately, he forgot to put some important papers in his briefcase. At 7:30, Betty found the papers, jumped in her car, and chased after Henry. She averaged 55 mph. How long did it take for Betty to catch up to Henry?



The distance that Henry drives and the distance that Betty drives is the same — at the moment Betty catches up to Henry. So multiply the rate that Henry drives times how long he drives and equate that to the rate that Betty drives times her amount of time. Let the amount of time that Henry drives be represented by t hours. Then Betty drives for $t - 0.5$ hours. Multiplying rate times time, the equation and solution are:

$$45t = 55(t - 0.5)$$

$$45t = 55t - 27.5$$

$$27.5 = 10t$$

$$t = \frac{27.5}{10} = 2.75$$

Henry drove for 2.75 hours, or 2 hours and 45 minutes. If Betty drove for half an hour less than that, she drove for 2 hours and 15 minutes.

You can solve directly for the amount of time that Betty traveled by letting Betty's time be t and Henry's time be $t + 0.5$. The equation is just a bit different, but the answer will still come out that Betty drove for 2.25 or 2 hours and 15 minutes.

In the classic story of the tortoise and the hare, the tortoise is slow and steady, and the hare is just too sure of himself. Consider the next problem as something that may have been the specific details of that story.



Trains in a tunnel

At 7 o'clock, a train enters a tunnel on the only track that runs through that tunnel. Another train enters the exact same tunnel at 7 o'clock on the same day. There is no way for the trains

to pass, go around, over or under one another in the tunnel. However, it was possible for the trains to make it to the other end of the tunnel untouched. How can that be?

Answer: One train entered the tunnel at 7 a.m. and the other at 7 p.m.



The Problem: The tortoise and the hare are both poised at the starting line of a 1-mile race. The tortoise can muster up a full 1-mile-per-hour speed, and the hare boasts a hopping speed of 8-miles-per-hour. The starter's gun sounds, and both take off at their top speed. After one minute, the hare decides he has time to take a nap and snoozes for the next 54 minutes. He wakes up and makes a dash for the finish line. How long does it take him to catch up to the tortoise?

The distance that both the tortoise and the hare travel are the precise moment that the hare catches up to the tortoise. The rates are 1-mile-per-hour for the tortoise and 8-miles-per-hour for the hare. The tortoise moves along for t hours, and the hare has t minus 54 minutes (which has to be changed to miles per hour) so it's $t - \frac{54}{60} = t - \frac{9}{10} = t - 0.9$ hours.

Setting the two distances equal to one another and solving,

$$1t = 8(t - 0.9)$$

$$t = 8t - 7.2$$

$$7.2 = 7t$$

$$t = \frac{7.2}{7} \approx 1.03$$

So the distances are equal and the hare catches up in a little more than one hour. But that's too late. The tortoise crossed the finish line after one hour, and the hare hadn't caught up to him yet, so the tortoise wins, yet again.

Speeding things up a bit

When you want to catch up with your older brother, you need to move faster than he's moving, if he left before you.

The Problem: Don left home with his fishing gear and headed for his favorite spot at 6 a.m. Don bicycles at 6 mph. Don's younger brother, Doug, left home at 6:30 and caught up with Don at 6:45 a.m.. How fast did Doug have to travel to catch up to Don at that time?



The distance traveled by the brothers is the same, so set the products of the rates and respective times equal to one another. Don bicycled at 6 mph for 45 minutes. Doug traveled for just 15 minutes. Changing 45 minutes to 0.75 hour and 15 minutes to 0.25 hour, then the only unknown is Doug's rate of speed, which is represented by r . The equation $6(0.75) = r(0.25)$ sets the two distances equal to one another. Simplifying on the left, you get $4.5 = 0.25r$. Dividing each side by 0.25, you get that $r = 18$ mph. Doug must be riding a bike or driving.

The Problem: Carole left Tampa one hour after Warren. Carole caught up to Warren 200 miles from Tampa, because she drove at a rate of speed that's 125 percent Warren's speed. How fast did Carole and Warren drive?



Both Carole and Warren drove 200 miles. Let r represent Warren's speed and t represent how long Warren drove to make that 200 miles. For Warren, you write the equation $200 = rt$. Carole drove the same 200 miles, but she drove faster and for one hour less than Warren. For Carole, you write the equation $200 = (1.25r)(t - 1)$. Going back to Warren's equation, solve for t in terms of the 200 and r , and replace the t in Carole's equation with that expression.

$$\begin{aligned} 200 &= rt, t = \frac{200}{r} \\ 200 &= (1.25r)(t - 1) \\ 200 &= (1.25r)\left(\frac{200}{r} - 1\right) \end{aligned}$$

Next, distribute the $1.25r$ over the terms in the parentheses and solve for r .

$$\begin{aligned} 200 &= 1.25r \times \frac{200}{r} - 1.25r \times 1 \\ 200 &= 250 - 1.25r \\ 1.25r &= 50 \\ r &= \frac{50}{1.25} = 40 \end{aligned}$$

Warren drove at 40 mph, so it took him 5 hours ($200 \div 40$) to drive that 200 miles. Carole drove at 125 percent of Warren's speed, so she drove at 50 mph. It took her four hours ($200 \div 50$) to drive the same distance — one hour less than the time it took Warren.

Solving for the distance

The main element of the $distance = rate \times time$ problems is the distance. You solve for rate or time when given the distance or given the fact that the distances traveled by the participants are equal. Just as important is finding out just how far people have traveled.

The Problem: Paula caught a bus to travel from school to home. Three hours later, her brother, Ken left school to go home driving his own car. If Ken drove 20 miles per hour faster than the bus, and arrived home at the same time as Paula, how far is it from school to home?



The distances are the same. You solve for the rate and time and then compute the distance. Letting r represent the rate at which Paula is traveling on the bus, Ken is driving at the rate of $r + 20$ mph. The time that Paula traveled is t hours, and Ken traveled $t - 3$ hours. Letting the two distances be the same, you get the equation $rt = (r + 20)(t - 3)$. Multiplying on the right, you get $rt = rt - 3r + 20t - 60$. Subtracting rt from each side and adding 60 to each side, the equation becomes $60 = 20t - 3r$. It appears that there are many different times and rates that fit the solution as it's given. Using the equation $60 = 20t - 3r$ you can try out some values for t and r . Some possibilities are shown in Table 16-1.

Table 16-1 Trying Out Some Values for $60 = 20t - 3r$

| <i>t Hours</i> | <i>r mph</i> | $20t - 3r = 60$ | <i>What This Means</i> |
|----------------|--------------|------------------|---|
| 3 | 0 | $60 - 0 = 60$ | Paula traveled 3 hours at 0 mph. Doesn't make sense. |
| 4.5 | 10 | $90 - 30 = 60$ | Paula traveled 4.5 hours at 10 mph. Ken drove at 30 mph. |
| 6 | 20 | $120 - 60 = 60$ | Paula traveled 6 hours at 20 mph. Ken drove at 40 mph. |
| 7.5 | 30 | $150 - 90 = 60$ | Paula traveled 7.5 hours at 30 mph. Ken drove at 50 mph. |
| 9 | 40 | $180 - 120 = 60$ | Paula traveled 9 hours at 40 mph. Ken drove at 60 mph. |
| 10.5 | 50 | $210 - 150 = 60$ | Paula traveled 10.5 hours at 50 mph. Ken drove 70 mph. |
| 12 | 60 | $240 - 180 = 60$ | Paula traveled 12 hours at 60 mph. Ken drove 80 mph. |

Several of the entries in the table are reasonable scenarios for Paula and Ken. To actually finish the problem, you need a little more information. Assume, for instance, that Ken drove 60 mph. That means that the bus Paula is riding on is driving at 40 mph. The equation now becomes $40t = 60(t - 3)$, which multiplies out to be $40t = 60t - 180$. Simplifying, you get $180 = 20t$ or $t = 9$. If Paula rode 9 hours at 40 mph, then the distance from school to home is $40 \times 9 = 360$ miles.

When you're racing against someone bigger or older or faster, and she wants to make a contest of it, you can ask for a *head start*. Applying this situation to a distance problem, you equate the distances by adding or subtracting the head start to make the finish come out the same. Getting a head start helps to even the playing field. It makes a race more of a contest and more interesting if the runners appear to have an equal chance of winning.

The Problem: The contestant from Kenya can run the 10,000-meter race at an average of 6 meters per second. The contestant from Ethiopia has a best time so far of 5.5 meters per second. How far back should the runner from Kenya start to have the expectation that they would cross the finish line at the same time if they both start at the same time?



In this problem, the distances aren't really equal. You have to add on x number of meters to represent the additional distance that the runner from Kenya must cover. The rates are also different, but the times will be the same, if they start at the same time and finish at the same time. Take the distance formula, $d = rt$ and solve for t , giving you $t = \frac{d}{r}$. Now set the time it takes the runner from Kenya equal to the time for the runner from Ethiopia. Then change the times to distances and rates.

$$t_k = t_e$$

$$\frac{d_k}{r_k} = \frac{d_e}{r_e}$$

Replace the denominators with the respective rates, and let the distances be 10,000 for the Ethiopian runner and $10,000 + x$ for the Kenyan runner. Then solve for x by cross-multiplying.

$$\begin{aligned} \frac{10,000 + x}{6} &= \frac{10,000}{5.5} \\ 5.5(10,000 + x) &= 6(10,000) \\ 55,000 + 5.5x &= 60,000 \\ 5.5x &= 5,000 \\ x &= \frac{5,000}{5.5} \approx 909.09 \end{aligned}$$

The runner from Kenya will have to start over 900 meters behind the other runner. And that's assuming that she can run her usual pace for an extra 900 meters.

Working It Out with Work Problems

Work problems usually involve two or more people pitching in together to make a lighter load for everyone involved. Different people work at different rates of speed, so some people accomplish more of the total job than others.

Here are the overriding principals or procedures to use in doing these problems:

- ✓ Let x represent the amount of time needed to do the whole job.
- ✓ Let $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ represent how much can be accomplished in *one day* by the person who can do the whole thing in 2, 3, 4, . . . days, respectively (other time units also apply).
- ✓ Let $\frac{x}{2}, \frac{x}{3}, \frac{x}{4}, \dots$ represent how much of the *whole job* is accomplished by the person who can do the job in 2, 3, 4, . . . days, respectively.
- ✓ Add up all the fractions with x in the numerator and set them equal to the number 1 for the whole job (or a fraction for part of the job).

A traditional problem is that two or more people get together to paint a house. You assume that everyone has a paint brush, paint, ladder, and any other supplies needed to do the job. Working together, they reduce the amount of time necessary to complete the whole job.

The Problem: Tom, Dick, and Harry arrive early one morning at the job site and get ready to paint a huge, old, Victorian mansion. Tom, working by himself, could paint the whole house in 14 days. It would take Dick 10 days to do the job by himself. And Harry could do the job in 8 days. How long does it take for the three men to do the job working together?



Let x represent the amount of time needed to do the whole job. Then Tom will do $\frac{x}{14}$ of the job, Dick will do $\frac{x}{10}$ of the job, and Harry will do $\frac{x}{8}$ of the job. Add the fractions together and set them equal to 1.

$$\frac{x}{14} + \frac{x}{10} + \frac{x}{8} = 1$$

Solve for x by first multiplying both sides of the equation by 280 and then simplifying.

$$\begin{aligned}\frac{x}{14} \times 280^{20} + \frac{x}{10} \times 280^{28} + \frac{x}{8} \times 280^{35} &= 1 \times 280 \\ 20x + 28x + 35x &= 280 \\ 83x &= 280 \\ x &= \frac{280}{83} \approx 3.37\end{aligned}$$

It will take the men not quite three and a half days to do the job by working together.

In some instances, though, the whole crew doesn't show up at the same time.

The Problem: A three-man crew can harvest the field in six hours, while a four-man crew can harvest the field in four hours. If the three-man crew worked for one hour and then were joined by the fourth man, how long will it take the four-man crew to finish the job? How long does it take from start to finish to do the whole job?



First, determine how much is accomplished by the three-man crew before the fourth man shows up. Then determine how long it'll take to finish up the job. If the three-man crew can do the harvesting in six hours, then in one hour, they've done $\frac{1}{6}$ of the job and have $\frac{5}{6}$ left to finish. Let x represent how long it'll take to finish the job, divide by 4 (the amount of time it takes the four-man crew to do the whole job) and set $\frac{x}{4}$ equal to the fraction of the job that's left. Solve for x by cross-multiplying.

$$\begin{aligned}\frac{x}{4} &= \frac{5}{6} \\ 6x &= 20 \\ x &= \frac{20}{6} = 3 \frac{1}{3}\end{aligned}$$

It'll take another 3 hours and 20 minutes to finish the job. Add that to the hour spent by the three-man crew, and the whole harvesting process took 4 hours and 20 minutes.

What if someone leaves before finishing the job? After all, people get tired or have other commitments. Have you ever been left to finish up a project that many other people had started? I'll at least make this next job something fun to do.

The Problem: Sarah, Sue, and Sybil are making chocolate-chip cookies for the annual club bake sale. Working alone, it would take Sarah 8 hours to do all the baking. Sue could do the whole job in 10 hours, and it would take Sybil 12 hours by herself. They all started working early in the morning. But, after 2 hours, Sybil said that she had to leave for an appointment and wouldn't be back. One hour after that, Sue got tired of listening to Sarah's griping and left. How long did it take Sarah to finish up the job by herself, after the other two left?



First, figure out how much of the job got accomplished during the first two hours, before Sybil left. If Sarah can do the job alone in 8 hours, then she did $2\left(\frac{1}{8}\right) = \frac{2}{8} = \frac{1}{4}$ in that two hours. Sue can do the job alone in 10 hours, so, in that first two hours, she did $2\left(\frac{1}{10}\right) = \frac{2}{10} = \frac{1}{5}$ of the job. And Sybil did $2\left(\frac{1}{12}\right) = \frac{2}{12} = \frac{1}{6}$ in that first two hours. Add the three fractions together — $\frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{15 + 12 + 10}{60} = \frac{37}{60}$ — and you see that they're over half finished with the job in the first two hours.

Next, determine how much more is accomplished in the next hour, with just Sarah and Sue working. Add $\frac{1}{8} + \frac{1}{10} = \frac{5 + 4}{40} = \frac{9}{40}$ to get how much of the job is done by the two working together for an hour. Add the amount from the first two hours to this amount, and you get the total amount of the job that's been accomplished so far: $\frac{37}{60} + \frac{9}{40} = \frac{74 + 27}{120} = \frac{101}{120}$. To determine how much time it'll take Sarah to finish the job, first subtract what's been done from 1, and then set that equal to $x \div 8$.

$$\begin{aligned} 1 - \frac{101}{120} &= \frac{19}{120} \\ \frac{x}{8} &= \frac{19}{120} \\ \frac{x}{8} &= \frac{19}{120} \\ 15x &= 19 \\ x &= \frac{19}{15} = 1\frac{4}{15} \end{aligned}$$

It'll take Sarah another 1 hour and 16 minutes (changing the fractional remainder to minutes) to finish up the job by herself.

Incoming and Outgoing

Filling and draining pools and tanks are just other variations on work problems. When you have more than one water pipe pumping water into a tank, it takes less time than one pipe doing the filling by itself. An added twist to these problems is that the tanks can be emptying, too. If your pool has a leak in it, then water is going out one hole at the same time it's coming in a pipe.

The Problem: A backyard swimming pool is being filled by two different hoses. One hose can fill the pool in 10 hours, and it takes the other hose 14 hours to fill the pool. How long will it take for the two hoses to fill the pool if they're turned on at the same time?



Let x represent the amount of time it'll take the two hoses, working together, to fill the pool. Then write two fractions, one with x divided by 10 and the other with x divided by 14 to represent how much of the whole job each hose can accomplish. Add the two fractions together, set the sum equal to 1, and solve for x .

$$\begin{aligned} \frac{x}{10} + \frac{x}{14} &= 1 \\ \frac{x}{10} \times 70^7 + \frac{x}{14} \times 70^5 &= 1 \times 70 \\ 7x + 5x &= 70 \\ 12x &= 70 \\ x &= \frac{70}{12} = \frac{35}{6} = 5\frac{5}{6} \end{aligned}$$

It'll take 5 hours and 50 minutes to fill the pool with the two hoses.

The Problem: A pool takes six hours to fill and eight hours to drain. The drain was accidentally left open for the first three hours while the pool was filling and then closed. How long did it take for the pool to be filled?



In the first three hours, the intake was $\frac{3}{6} = \frac{1}{2}$, which is obtained by dividing 3 by the 6 hours it takes to fill the pool. At the same time, the outgo was $\frac{3}{8}$, dividing 3 by the 8 hours it takes to drain. Subtract the two fractions to get a net filling of $\frac{1}{2} - \frac{3}{8} = \frac{4}{8} - \frac{3}{8} = \frac{1}{8}$, which is the fractional amount that the pool is full right now. Subtract that fraction from 1 to get the amount still needed to be filled: $1 - \frac{7}{8} = \frac{1}{8}$.

Let x represent the number of hours it now takes to complete the filling of the pool. Write the fraction dividing x by 6, set it equal to the fraction of the pool needed to be filled, and solve for x .

$$\begin{aligned}\frac{x}{6} &= \frac{7}{8} \\ 8x &= 42 \\ x &= \frac{42}{8} = 5\frac{2}{8} = 5\frac{1}{4}\end{aligned}$$

It took another 5 hours and 15 minutes to fill the pool after the drain was closed, so the total time was 8 hours and 15 minutes.

The Problem: In the tank of a water tower, the intake valve closes automatically when the tank is full and opens up again when $\frac{3}{5}$ of the water is drained off. The intake fills the tank in 4 hours, and the outlet drains the tank in 12 hours. If the outlet is open continuously, then how long a time is it between two different instances when the tank is completely full?



Start with the first instance that the tank is full. The intake valve closes, and only the outlet valve is working, emptying the tank. You need to determine how long it takes for the outlet valve to empty the tank by $\frac{3}{5}$, leaving it $\frac{2}{5}$ full. Divide x , the amount of time it'll take to lower the amount in the tank, by 12.

$$\begin{aligned}\frac{x}{12} &= \frac{3}{5} \\ 5x &= 36 \\ x &= \frac{36}{5} = 7\frac{1}{5}\end{aligned}$$

It takes 7 hours and 12 minutes for the amount of water in the tank to reach the level at which the intake valve switches on. Now you determine how long it takes to fill the tank again. Let x represent the total amount of time needed. Divide x by 4 for the intake value and subtract the fraction formed by dividing x by 12 for the outlet. Set the difference equal to $\frac{3}{5}$, the amount of the tank that needs to be filled. Solve for x by first multiplying each term by 60, the common denominator.

$$\begin{aligned}\frac{x}{4} - \frac{x}{12} &= \frac{3}{5} \\ \frac{x}{4} \times 60^{15} - \frac{x}{12} \times 60^5 &= \frac{3}{5} \times 60^{12} \\ 15x - 5x &= 36 \\ 10x &= 36 \\ x &= \frac{36}{10} = 3\frac{3}{5}\end{aligned}$$

It'll take another 3 hours and 36 minutes to fill the tank. Add the 7 hours and 12 minutes to get the tank to the level where the intake kicks in, and add the 3 hours and 36 minutes. It's a total of 10 hours and 48 minutes between times that the tank is full.

Chapter 17

Being Systematic with Systems of Equations

In This Chapter

- ▶ Writing more than one equation for the problem
 - ▶ Solving systems of equations using substitution
 - ▶ Finding the break-even point in profits
 - ▶ Solving systems with multiple equations
-

Writing an equation to use for solving a story problem is more than half the battle. Once you have a decent equation involving a variable that represents some number or amount, then the actual algebra needed to solve the equation is typically pretty easy.

Many word problems lend themselves to more than one equation with more than one variable. It's easier to write two separate equations, but it takes more work to solve them for the unknowns. And, in order for there to be a solution at all, you have to have at least as many equations as variables.

Most of the problems in this chapter deal with the more typical two-equation solutions, but I include a section on dealing with three or more equations, too.

Writing Two Equations and Substituting

Word problems often deal with how many of two or more coins, how many ducks and elephants, how much to invest in this or that, how many red and green jelly beans, and so on. You let variables represent the numbers of coins or ducks or dollars or jelly beans. When working with two different equations written about the same situation, then you have two different variables and need to do some algebra to knock that down to one equation. That's where substitution comes in.

Solving systems by substitution

A system of two linear equations, such as $2x + 3y = 31$ and $5x - y = 1$ is usually solved by *elimination* or *substitution*. (Refer to *Algebra For Dummies* if you want a full explanation of each type of solution method.) For the problems in this chapter, I use the *substitution* method, to solve for a variable. This means that you change the format of one of the equations so that it expresses what one of the variables is equal to in terms of the other, and then you *substitute* into the other equation. For example, you solve for y in terms of x in the equation $3x + y = 11$ if you subtract $3x$ from each side and write the equation as $y = 11 - 3x$.

Consider solving the system $2x + 3y = 31$ and $5x - y = 1$. First go to the second equation and rewrite it with y on one side and everything else on the other side. You choose the second equation, because the y variable has a coefficient of -1 . Having a coefficient of 1 or -1 is desirable, because you can avoid working with fractions.



A *coefficient* is a factor or multiplier of a variable. The term $3x$ has a coefficient of 3 , and the term kx has a coefficient of k .

To solve for y in terms of x in $5x - y = 1$, you first add y to each side of the equation and then subtract 1 from each side.

$$\begin{aligned} 5x - y &= 1 \\ 5x &= y + 1 \\ 5x - 1 &= y \end{aligned}$$

Now substitute into the first equation. Because $y = 5x - 1$, replace the y in the first equation with $5x - 1$ and solve for x .

$$\begin{aligned} 2x + 3(5x - 1) &= 31 \\ 2x + 15x - 3 &= 31 \\ 17x - 3 &= 31 \\ 17x &= 34 \\ x &= \frac{34}{17} = 2 \end{aligned}$$

You determine that $x = 2$. Now determine what y is equal to by putting the 2 in for x in the equation $y = 5x - 1$. You get that $y = 5(2) - 1 = 10 - 1 = 9$. So the solution of the system of equations is that $x = 2$ and $y = 9$.

The rest of this chapter deals with how to use substitution in systems of equations to solve word problems.

Working with numbers and amounts of coins

Some problems involving coins are more easily solved using two equations rather than just one, as you find in Chapter 8. If you're nickeling and diming, then you let n represent the number of nickels and d represent the number of dimes. And if you have a total of ten coins, then $n + d = 10$. The rest of the information in the problem helps you write the other equation.

The Problem: You have ten coins in nickels and dimes. You have two less than five times as many dimes as nickels. How much money do you have?



First, you solve for the number of each type of coin. Then you determine what the total worth is, multiplying the number of nickels by 5¢ and dimes by 10¢. Let the number of nickels be represented by n and the number of dimes be represented by d . You write that the total number of coins is ten with the equation $n + d = 10$. You write that the number of dimes is two less than five times the number of nickels with the equation $d = 5n - 2$. As you see, the second equation already has d in terms of n and -2 , so substitute $5n - 2$ in for d in the first equation and solve for n .

$$\begin{aligned} n + (5n - 2) &= 10 \\ 6n - 2 &= 10 \\ 6n &= 12 \\ n &= 2 \end{aligned}$$

You have two nickels, so there are $5(2) - 2 = 10 - 2 = 8$ dimes. Two nickels are 10¢, and eight dimes are 80¢, so you have a total of 90¢.

The Problem: You have a total of 40 coins in nickels and quarters. If you double the number of quarters and add 12, you'll get the same number of coins as you'd have if you multiply the number of nickels by 4 and subtract 4. How much money do you have in quarters?



First, determine how many nickels and quarters by writing one equation where the sum of the number of each type of coin is 40. Then write another equation involving the relationship between the number of each coin. You can determine how much money you have in quarters when you know how many quarters you have.

Let n represent the number of nickels and q represent the number of quarters. Your first equation is $n + q = 40$. The second equation sets the two relationships between the number of coins equal to one another. Let *double the number of quarters and add 12* be represented by $2q + 12$, and

let *multiply the number of nickels by 4 and subtract 4* be represented by $4n - 4$. The second equation reads: $2q + 12 = 4n - 4$. You can do some simplifying and rearranging with the second equation, but I'm going to just solve for q in the first equation and substitute into the second equation. If $n + q = 40$, then $q = 40 - n$. Substituting and solving for q ,

$$\begin{aligned} 2q + 12 &= 4n - 4 \\ 2q + 12 &= 4(40 - q) - 4 \\ 2q + 12 &= 160 - 4q - 4 \\ 2q + 12 &= 156 - 4q \\ 6q + 12 &= 156 \\ 6q &= 144 \\ q &= \frac{144}{6} = 24 \end{aligned}$$

You have 24 quarters, giving you $24(0.25) = \$6$. Does the number of coins check out? If you have 40 coins in all, and 24 are quarters, then you have 16 nickels. Doubling the number of quarters and adding 12 gives you $2(24) + 12 = 48 + 12 = 60$. Multiplying the number of nickels by 4 and subtracting 4, you get $4(16) - 4 = 64 - 4 = 60$. It checks.

Figuring out the purchases of fast food

You're sent out to pick up some refreshments for the guys working on a project. You've gotten their orders and collected the money, but you've lost the piece of paper with the exact listing of what everyone wants. Good thing you know how to solve word problems with numbers of items and cost per item. Math saves the day, yet again.

The Problem: You collected a total of \$25 for hamburgers and soft drinks. The hamburgers cost \$2.50 each, and the soft drinks cost \$1.50 each. Also, the number of hamburgers ordered is three less than twice as many soft drinks. How many hamburgers and how many soft drinks were ordered?



Write two equations — one involving the number of each item times its respective cost, and the other involving the total number of items ordered. Let the number of hamburgers be represented by h and the number of soft drinks be represented by d . Multiplying h by 2.50 and d by 1.50, the total for the whole order is $2.50h + 1.50d = 25$. The number of hamburgers, h , is three less than twice the number of soft drinks, $2d - 3$. Set $h = 2d - 3$. Now replace the h in the first equation with $2d - 3$ and solve for h .

$$\begin{aligned}
 2.50(2d - 3) + 1.50d &= 25.00 \\
 5d - 7.50 + 1.50d &= 25.00 \\
 6.50d - 7.50 &= 25.00 \\
 6.50d &= 32.50 \\
 d &= \frac{32.50}{6.50} = 5
 \end{aligned}$$

So 5 soft drinks were ordered. The number of hamburgers is $2(5) - 3 = 10 - 3 = 7$ hamburgers. Five soft drinks at \$1.50 each is \$7.50. Seven hamburgers at \$2.50 each is \$17.50, for an order that totals \$25.

The Problem: Last night, the Fish House Diner sold some crab-cake dinners and shrimp baskets for a total of \$705. The crab-cake dinners cost \$9, and the shrimp baskets cost \$11.50. If you multiply the number of crab-cake dinners by three and add it to twice the number of shrimp baskets, you get 180 dinners. How many of each were sold?



Let the number of crab-cake dinners be represented with c and the number of shrimp baskets be represented with b . Multiply the number of each type item by its price and set the sum equal to \$705. The equation is $9c + 11.50b = 705$. Now write an equation involving the relationship between the numbers of orders. Three times the number of crab-cake dinners is $3c$, and twice the number of shrimp baskets is $2b$. So $3c + 2b = 180$. This system of equations doesn't have a variable with a coefficient of 1 or -1.

$$9.00c + 11.50b = 705$$

$$3c + 2b = 180$$

You could solve for c or b in one of the equations by dividing each term in the equation by the coefficient of the respective variables. For instance, solving for b in the second equation, you'd get $b = 90 - \frac{3}{2}c$. A better move would be to take advantage of the fact that the c variable has coefficients in one equation that is a multiple of the coefficient in the other equation. Just multiply each of the terms in the second equation by 3 and solve for $9c$.

$$9c + 6b = 540$$

$$9c = 540 - 6b$$

Now replace the $9c$ in the first equation with $540 - 6b$, and solve for b .

$$\begin{aligned}
 (540 - 6b) + 11.50b &= 705.00 \\
 540 + 5.50b &= 705.00 \\
 5.50b &= 165.00 \\
 b &= \frac{165.00}{5.50} = 30
 \end{aligned}$$



Weighing in on a million dollars

You're on one of those game shows where you get to pick a box or a curtain or some container so you can win lots of money. You make your choice and see the words *One Million* flashing in bright neon. You're now told that, if you can

lie on the floor with \$1 million in \$1 bills on your chest for one minute, then you get to take all that money home. Your other choice is to take \$50,000 in cash. Which do you choose?

survive to spend the money. Take the \$50,000 and run — fast!

Weighs 1 pound. Consequently, \$1 million weighs over 1 ton. You'll never

Answer: A \$1 bill weighs 1 gram. There are 454 grams in a pound, so \$454

They sold 30 shrimp baskets. The number of crab-cake dinners is found by solving $3c + 2b = 180$ for c after replacing the b with 30.

$$\begin{aligned}3c + 2b &= 180 \\3c + 2(30) &= 180 \\3c + 60 &= 180 \\3c &= 120 \\c &= 40\end{aligned}$$

So 40 crab-cake dinners and 30 shrimp baskets were sold.

Breaking Even and Making a Profit

Anyone in business will tell you that she's interested in making a profit. Oh, well, the movie and musical *The Producers* violates that premise, but you don't hold with make-believe. In general, the profit from a venture is computed by taking the revenue earned and subtracting the cost that it takes to earn the revenue. Different factors keep pulling the costs up and the prices down, so it's a real balancing act to be in business nowadays and make that profit.

Finding the break-even point

The *break-even point* in business is when the *revenue* (the money brought in) is equal to the *cost* (the money spent to earn the revenue). When a business gets past the break-even point, it shows a profit. To find the break-even point,

you determine when the revenue and the costs are equal. You use a system of equations to solve the problem.

The Problem: You decide to go into the sandal-making business. Your startup costs are \$1,600, and it costs you \$40 per pair of sandals to produce them. You write your total cost function as $C = 1,600 + 40x$, where x is the number of pairs of sandals that you produce. The price at which you sell the sandals is dependent on the number of pairs you sell, so there isn't a fixed price (you lower the price to be able to sell more). In this case, the amount of revenue you get from selling x pairs of sandals is found with $R = 100x - 0.5x^2$. What is the break-even point? How many pairs of sandals do you have to produce and sell to start making a profit?



Consider some cost and revenue values. Table 17-1 shows the revenue, total cost, and net result for $x = 10, 20, 30$, and 40 pairs of sandals.

Table 17-1 Revenue, Cost, and Net Result

| x | $R = 100x - 0.5x^2$ | $C = 1,600 + 40x$ | $Difference: R - C$ |
|-----|---------------------|-------------------|---------------------|
| 10 | 950 | 2,000 | -1,050 |
| 20 | 1,800 | 2,400 | -600 |
| 30 | 2,550 | 2,800 | -250 |
| 40 | 3,200 | 3,200 | 0 |

When 40 pairs of sandals are produced, the revenue and the cost are the same, so $x = 40$ is the break-even point.

Making this table is helpful, but it isn't very practical. A better method is to solve the system of equations. The two equations involved are $C = 1,600 + 40x$ and $R = 100x - 0.5x^2$. Replace the C and the R with y , letting the y represent the amount of money spent in the cost function and the amount of money earned in the revenue function. You want to find out when those two amounts are the same to find the break-even point. Then replace the y in $y = 1,600 + 40x$ with $100x - 0.5x^2$ from the equation $y = 100x - 0.5x^2$. Solve for the value of x .

$$1,600 + 40x = 100x - 0.5x^2$$

$$0.5x^2 - 60x + 1,600 = 0$$

You end up with a quadratic equation. You can factor the equation or you can use the quadratic formula. (See the Cheat Sheet for the formula.) I show you factoring, in the following equation, after multiplying every term by 2 so that the coefficient of the x^2 term becomes a 1.

$$\begin{aligned}1x^2 - 120x + 3,200 &= 0 \\(x - 40)(x - 80) &= 0 \\x = 40 \text{ or } x &= 80\end{aligned}$$

You end up with two different solutions. The first solution is the one shown in Table 17-1. After you hit sales of 40 pairs of sandals, you've broken even and can start to make money. Unfortunately, if you make and sell more than 80 pairs, you start to lose money. Your expenses get too high, and it becomes too costly to produce and sell at that level.

Determining the profit

The break-even point can be where you start seeing some profit to your venture. After you've reached the point where increased sales will start making you some money, you earn profit. The amount of the profit is determined by subtracting the cost from the revenue. $P = R - C$. What level of sales will earn a particular amount of profit?

The Problem: A businessman wants to know when the sale of a particular item reaches a profit level of \$2,000. The revenue equation is $R = 200x - 0.4x^2$, and the cost to produce x items is determined with $C = 4,000 + 100x$. How many items have to be produced and sold to net a profit of \$2,000?



Use the equation $P = R - C$ and replace the P with 2,000. Put the revenue equation and cost equation in their respective positions and solve the equation for x .

$$\begin{aligned}2,000 &= (200x - 0.4x^2) - (4,000 + 100x) \\2,000 &= 200x - 0.4x^2 - 4,000 - 100x \\2,000 &= 100x - 0.4x^2 - 4,000 \\0.4x^2 - 100x + 6,000 &= 0\end{aligned}$$

The quadratic equation is solved by either factoring or using the quadratic formula. I choose to factor, but I'll multiply each term by 2.5, first, to make the coefficient of the x^2 term equal to 1.

$$\begin{aligned}(2.5)0.4x^2 - (2.5)100x + (2.5)6,000 &= 0 \\1x^2 - 250x + 15,000 &= 0 \\(x - 100)(x - 150) &= 0\end{aligned}$$

When $x = 100$ or $x = 150$, the profit is \$2,000. At either level, the difference between the revenue and cost is \$2,000.

Mixing It Up with Mixture Problems

Mixture problems are covered in great detail in Chapter 14. You mix so many quarts of one substance with quarts of another substance. Some rather interesting solutions or mixture problems are made possible by introducing the second equation and solving the system.

Gassing up at the station

It's no easy choice when you pull up at the gas station to fill up your vehicle. First, you have to take a deep breath about the price, and then you have to choose between regular gas, premium gas, or even gas that contains ethanol.

The Problem: Stefanie finds that she gets 19 miles per gallon with regular gas that costs \$2.70 per gallon and 23 miles per gallon with the premium gas that costs \$3.15 per gallon. She paid a total of \$104.40 for gas on a trip of 748 miles. How many gallons of each type gas did she buy?



One of the equations you need deals with cost, and the other deals with the number of miles. The common element in both equations is the number of gallons of each type of gas — and the number of gallons answers the question, too. Let r represent the number of gallons of regular gas and p represent the number of gallons of premium gas. The total cost, $\$104.40 = \$2.70r + \$3.15p$. The total number of miles, $748 = 19r + 23p$. None of the coefficients of the variables is equal to 1, so you have to make a choice as to which variable to solve for. Because the coefficient 19 is the smallest number, I opt to solve for r in the second equation and replace the r in the first equation with that equivalence in terms of p .

$$19r + 23p = 748$$

$$19r = 748 - 23p$$

$$r = \frac{748}{19} - \frac{23}{19} p$$

$$2.70r + 3.15p = 104.40$$

$$2.70\left(\frac{748}{19} - \frac{23}{19} p\right) + 3.15p = 104.40$$

Distributing the 2.70 and simplifying don't make for very pretty computations, but a calculator makes short work of all the operations. I choose to find a common denominator to combine the fractions and decimals, because the decimal you get with a denominator of 19 just keeps repeating. Then you solve for p by multiplying each side of the equation by the reciprocal of its coefficient.

$$\begin{aligned}
 \frac{2,019.6}{19} - \frac{62.1}{19} p + 3.15p &= 104.40 \\
 -\frac{62.1}{19} p + 3.15p &= 104.40 - \frac{2,019.6}{19} \\
 -\frac{62.1}{19} p + \frac{315}{100} p &= \frac{10,440}{100} - \frac{2,019.6}{19} \\
 \frac{-225}{1,900} p &= \frac{-3,600}{1,900} \\
 p &= \frac{-3,600}{1,900} \times \frac{1,900}{-225} = 16
 \end{aligned}$$

Stefanie used 16 gallons of premium gas. Substitute the 16 for p in the equation $19r + 23p = 748$, and you get $19r + 23(16) = 748$. This simplifies to $19r + 368 = 748$. Subtracting 368 from each side, the equation becomes $19r = 380$. Dividing each side by 19, you get that $r = 20$. She bought 20 gallons of regular gas.

Backtracking for all the answers

Problems involving mixtures have the upfront answers of how many gallons of this or how many items of that. Applications of these problems often involve more than just the numbers. In a gallons-of-gasoline problem, you may want to know the average cost per gallon for the trip. In a fund-raising project, you probably want to know something about the total profit.

The Problem: A service group is selling candy bars and bags of almonds for a fund-raiser. They're selling the candy bars (which cost them 40 cents each) for \$1 and the bags of almonds (which cost them 50¢ each) for \$1.25. Their total receipts (revenue) for the sale of 1,350 items was \$1,500. How many of each item did they sell, and what was their profit?



Write an equation about the total receipts by multiplying the amount charged by the number of each type of item. Write an equation about the total number of items by adding the number of each type together and setting it equal to 1,350. Letting the number of candy bars be represented by c and the number of bags of almonds be represented by a , the two equations are: $\$1c + \$1.25a = \$1,500$ and $c + a = 1,350$. Solve for c in the second equation and substitute the equivalent into the first equation.

$$\begin{aligned}
 c + a &= 1,350 \\
 c &= 1,350 - a \\
 1.00(1350 - a) + 1.25a &= 1,500
 \end{aligned}$$

Now simplify the equation, subtract 1,350 from each side, and solve for a by dividing each side by the coefficient.



Cows and hens

In a recent *Ask Marilyn* column in the Sunday Parade magazine, the following question was posed: If a barnyard of 30 animals — cows and

hens — had a total of 74 feet, then how many cows and how many hens were in the barnyard?

Answer Most people reading the column didn't have the benefit of being able to write two equations and solving them. The answer is 7 cows and 23 hens. Did you do it without a system of equations? Do you want to know the equations needed? Okay, they're $c + h = 30$ and $4c + 2h = 74$.

$$\begin{aligned}1,350 - a + 1.25a &= 1,500 \\1,350 + 0.25a &= 1,500 \\0.25a &= 150 \\a &= 600\end{aligned}$$

The group sold 600 bags of almonds. That means that $1,350 - 600 = 750$ candy bars were sold. To figure out the cost for these items, multiply 750 times 40¢ and 600 times 50¢. Their cost was $750(\$0.40) + 600(\$0.50) = \$300 + \$300 = \$600$. If the total receipts were \$1,500, then subtract the \$600 from \$1,500 to get \$900 profit.

Making Several Comparisons with More Than Two Equations

Some situations have more than two or three things acting on an outcome. The type of gasoline used may involve not only the cost of the fuel and the mileage the car gets on the fuel, but it also may involve the availability of the fuel or the speed at which the pump flows. When a problem involves three or more different influences, a system of equations involving three or more equations is used.

Picking flowers for a bouquet

Florists make up bouquets by mixing complementary colors and types of flowers and greenery. The cost of the bouquet depends on the different types of flowers and materials used.

The Problem: A bouquet is to contain 24 flowers, made up of roses, carnations, and daisies. The person buying the bouquet wants twice as many carnations as roses and doesn't want to spend more than \$8.20. If roses cost 50¢ each, carnations 25¢ each, and daisies 40¢ each, then what composition of flowers should be used to match that maximum price?



Write three different equations: one involving the total number of flowers, one involving the total cost for the bouquet, and one involving the relationship between the number of carnations and roses. Letting the number of roses be represented by r , the number of carnations represented by c , and the number of daisies by d , the three equations are: $r + c + d = 24$, $0.50r + 0.25c + 0.40d = 8.20$, and $c = 2r$. You can hone this system of three equations down to two equations by substituting $2r$ for c in each of the first two equations. Then you have the two equations:

$$\begin{aligned} r + 2r + d &= 24 \\ 0.50r + 0.25(2r) + 0.40d &= 8.20 \end{aligned}$$

Simplifying, the two equations become:

$$\begin{aligned} 3r + d &= 24 \\ 1r + 0.40d &= 8.20 \end{aligned}$$

Solve for d in the first equation and substitute the equivalence into the second equation.

$$\begin{aligned} d &= 24 - 3r \\ 1r + 0.40(24 - 3r) &= 8.20 \\ r + 9.6 - 1.20r &= 8.20 \\ 9.6 - 0.20r &= 8.20 \\ -0.20r &= -1.40 \\ r &= \frac{-1.40}{-0.20} = 7 \end{aligned}$$

The number of roses is 7, so twice as many carnations is 14. That leaves room for 3 daisies.

Coming up with a game plan for solving systems of equations

In the example in the preceding section, one of the equations in the system of equations had a convenient relationship. The equation $c = 2r$ allowed for a quick substitution back into the other two equations, creating a nice system

of equations that solved by substitution. When you aren't quite so lucky to have a problem with such a nice feature to it, you need a more general game plan for solving.

When solving a system of three or more linear equations, the following guidelines will help with the solution:

- ✓ Select variables to represent the different amounts or units.
- ✓ Create a table or chart with the variables and their descriptions down one side and the different relationships (prices, numbers, and so on) across the top.
- ✓ Write equations using the variables and relationships from the table.
- ✓ Solve for one of the variables in one of the equations and substitute back into the other equations. Repeat the process until you have one equation with just one variable.

Graphing calculators and spreadsheets are very useful when doing problems involving a large number of variables. The following problem shows you something quite manageable using the substitution method.

The Problem: Sherrill bought 40 dozen cookies for \$350. The chocolate-chip cookies cost \$10 per dozen; the oatmeal-raisin cookies cost \$9 per dozen; and the peanut-butter cookies and snickerdoodles each cost \$8 per dozen. Sherrill got two more dozen chocolate-chip cookies than peanut-butter cookies, and she bought twice as many snickerdoodles as oatmeal-raisin cookies. How many dozen of each type did she buy?



This problem is solved by writing four different equations. Let c represent how many dozen chocolate-chip cookies, r represent the number of dozen of oatmeal-raisin cookies, p the number of dozen of peanut-butter cookies, and d the number of dozen snickerdoodles. Make a table representing the different relationships (see Table 17-2).

Table 17-2 Arranging 40 Dozen Cookies

| <i>Cookie Type</i> | <i>Dozen Of</i> | <i>Cost Per Dozen</i> | <i>Two More</i> | <i>Twice</i> |
|---------------------------|------------------------|------------------------------|------------------------|---------------------|
| Chocolate chip | c | \$10 | | $p + 2$ |
| Oatmeal raisin | r | \$9 | | |
| Peanut butter | p | \$8 | | |
| Snickerdoodle | d | \$8 | | $2r$ |
| Total | 40 | \$350 | | |

Now write equations using the columns of the table. In the “Dozen Of” column, you add up the dozens and let the sum be 40, giving you:

$$c + r + p + d = 40$$

Next, write an equation multiplying the prices times their respective types and having the total be \$350:

$$10c + 9r + 8p + 8d = 350$$

Next, write the relationship between the dozens of chocolate-chip and peanut-butter cookies: $c = p + 2$. And finally, write the relationship between the dozens of oatmeal-raisin and snickerdoodles: $d = 2r$. The following shows the system of equations written in *rectangular* form, with all the same variables lined up under one another. This is the form you’d use in a calculator, replacing all the blank spots with zeros.

$$\begin{array}{rcl} c + r + p + d & = & 40 \\ 10c + 9r + 8p + 8d & = & 350 \\ c - p & = & 2 \\ 2r - d & = & 0 \end{array}$$

Using substitution, you replace the ds in the first two equations with $2r$, and you replace c in the first two equations with $p + 2$.

$$\begin{aligned} (p+2) + r + p + 2r &= 40 \\ 10(p+2) + 9r + 8p + 8(2r) &= 350 \end{aligned}$$

Then simplify the two equations that now have only the variables p and r in them.

$$\begin{aligned} 2p + 3r &= 38 \\ 18p + 25r &= 330 \end{aligned}$$

Solve for p in the first equation and substitute the equivalence into the second equation. Then solve the equation for the value of p .

$$\begin{aligned} 2p &= 38 - 3r \\ p &= 19 - \frac{3}{2}r \\ 18\left(19 - \frac{3}{2}r\right) + 25r &= 330 \\ 342 - 27r + 25r &= 330 \\ -2r &= -12 \\ r &= 6 \end{aligned}$$



Drawing money out of a bag

A bag contains ten \$5 bills, ten \$10 bills and ten \$20 bills. Jack is blindfolded and told that he may draw bills out of the bag, one at a time, until he has three of the same denomination of bill.

He can keep all the money drawn from the bag until he has to stop. What is the greatest amount of money Jack may draw from the bag?

Answer: Jack can get, at most \$90, if he draws two \$5 bills, two \$10 bills,

Sherrill bought 6 dozen oatmeal-raisin cookies. She bought twice as many (or 12 dozen) snickerdoodles. Replacing r with 6 in the equation $2p + 3r = 38$, you get $2p + 18 = 38$, which becomes $2p = 20$, or $p = 10$. She bought 10 dozen peanut-butter cookies. And, if she bought two more dozen chocolate chip than peanut butter, she bought 12 dozen chocolate-chip cookies.

Solving Systems of Quadratic Equations

A quadratic equation occurs when you have one or more terms containing a variable raised to the second power. For example, the equation $y = 3x^2 + 2x$ is a quadratic equation. Solving systems of equations involving quadratic equations leads to some interesting results. You usually end up with more than one solution to the system. Sometimes more than one solution means that there's more than one answer. Often, though, it means that one of the answers fits the situation and the other is just *extraneous* — it satisfies the system of equations, but it doesn't really have any meaning in the practical problem.

Counting on number problems

A common type of problem involving systems of equations is a *number problem*, where you solve for a certain number that has a particular relationship to one or more other numbers. You incorporate operations such as addition and multiplication, and you often use squares of numbers or their reciprocals.

The Problem: The sum of two numbers is 40, and the product of those same two numbers is 204. What are the numbers?



Let the two numbers be represented by the variables x and y . Writing that their sum is 40, the equation is $x + y = 40$. You write that their product is 204 with $xy = 204$. Solve for y in the first equation to get $y = 40 - x$. Now replace the y in the second equation with $40 - x$, simplify the equation by setting it equal to 0, and solve the quadratic equation by factoring.

$$\begin{aligned}x(40-x) &= 204 \\40x - x^2 &= 204 \\0 &= x^2 - 40x + 204 \\0 &= (x-34)(x-6)\end{aligned}$$

You get that x is either 34 or 6. These are the two numbers whose sum is 40 and whose product is 204.

The Problem: The difference between two positive numbers is 20. If you square the larger number and subtract ten times the smaller number from the square, you get 575. What are the two numbers?



Let the two numbers be represented by x and y . Write their difference as $x - y = 20$. Squaring the larger number and subtracting ten times the smaller, you get $x^2 - 10y = 575$. Why does x have to be the larger number? It's because the two numbers are positive, and the result of subtracting y from x has to be a positive 20.

Now, solve for x in the first equation to get $x = 20 + y$. Substitute this into the second equation and solve for y .

$$\begin{aligned}(20+y)^2 - 10y &= 575 \\400 + 40y + y^2 - 10y &= 575 \\y^2 + 30y - 175 &= 0 \\(y-5)(y+35) &= 0\end{aligned}$$

The two solutions of the equation are $y = 5$ and $y = -35$. You discard the $y = -35$, because the problem asks for positive numbers. If $y = 5$, then $x = 20 + 5 = 25$. The two numbers are 25 and 5.

The Problem: One integer is four smaller than another. The sum of their reciprocals is $\frac{5}{24}$. What are the numbers?



Let the two numbers be represented by x and $x - 4$. Writing that the sum of the reciprocals of these two numbers is equal to 24, you have: $\frac{1}{x} + \frac{1}{x-4} = \frac{5}{24}$. Technically, this is a system-of-equations problem, because you could write that the second number, y , has the relationship $y = x - 4$. The substitution is done here almost automatically.

Now multiply each term by the common denominator, $24x(x - 4)$, simplify, and solve the quadratic equation.

$$\begin{aligned}\frac{1}{x} \times 24x(x - 4) + \frac{1}{x-4} \times 24x(x-4) &= \frac{5}{24} \times 24x(x - 4) \\ 24x - 96 + 24x &= 5x^2 - 20x \\ 0 &= 5x^2 - 68x + 96 \\ 0 &= (5x - 8)(x - 12)\end{aligned}$$

The solutions of the quadratic equation are $\frac{8}{5}$ and 12. The first solution isn't an integer, so discard it as being extraneous. However, when $x = 12$, you have an integer and another that's four smaller, which is 8. And, when you add their reciprocals, you get

$$\frac{1}{12} + \frac{1}{8} = \frac{2}{24} + \frac{3}{24} = \frac{5}{24}$$

Picking points on circles

The equation of a circle with its center at the origin is $x^2 + y^2 = r^2$, where r is the radius of the circle. It isn't always easy to pick nice numbers — integers or fractions — that are coordinates of the points on a circle. For example, if a circle has a radius of 5, you can use 3s and 4s for the xs and ys. The points $(3, 4)$, $(-3, 4)$, $(-4, -3)$ and so on work. So do combinations of 0s and 5s. The number of *nice* coordinates is limited.

You can find an infinite number of nice *fractions* that work for the coordinates of the points on a *unit circle* (a circle with a radius of 1).

Let m be any rational number, and the ordered pair $\left(\frac{1-m^2}{1+m^2}, \pm \frac{2m}{1+m^2}\right)$ is a point on the unit circle.

Using the formula to find a few points, let $m = 2$, for example. Then $\left(\frac{1-2^2}{1+2^2}, \pm \frac{2 \cdot 2}{1+2^2}\right) = \left(\frac{-3}{5}, \pm \frac{4}{5}\right)$ are on the unit circle and satisfy the equation $x^2 + y^2 = 1$. The \pm symbol just means that you can let the y -coordinate be either positive or negative — you get two points for the input value.

The formula for these points on the unit circle comes from solving a system of equations.

The Problem: Find the coordinates of the points on the unit circle with its center at the origin that are shared with any line going through the point $(-1,0)$ on that circle. Distinguish between the different lines that go through $(-1,0)$ by letting m represent the slopes of the different lines.

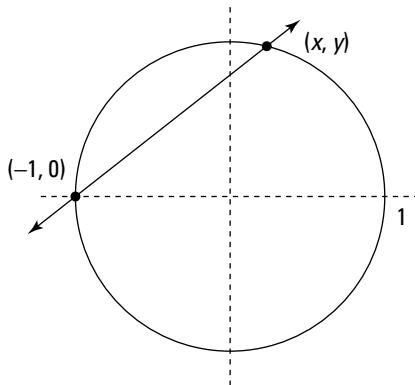


Figure 17-1:
A line
intersects
the unit
circle.



Find the common solution of the circle, $x^2 + y^2 = 1$ and the line $y = mx + b$. The particular line that goes through the point $(-1,0)$, has an equation that can be written a little more specially. Finding the slope of this line, m , using the slope formula and the points (x, y) and $(-1,0)$, you get $m = \frac{y - 0}{x - (-1)} = \frac{y}{x + 1}$. Solving this equation for y , you get that the equation of the line is $y = m(x + 1)$.

$$\begin{aligned}m &= \frac{y}{x + 1} \\m(x + 1) &= y\end{aligned}$$

Using this new form for the equation, substitute the $m(x + 1)$ for y in the equation of the circle.

$$\begin{aligned}x^2 + (m(x + 1))^2 &= 1 \\x^2 + m^2(x^2 + 2x + 1) &= 1 \\x^2 + m^2x^2 + 2m^2x + m^2 - 1 &= 0 \\(1 + m^2)x^2 + 2m^2x + (m^2 - 1) &= 0\end{aligned}$$

The equation you end up with after substitution is a quadratic equation of the form $ax^2 + bx + c = 0$ where the $a = 1 + m^2$, $b = 2m^2$, and $c = m^2 - 1$. Use the quadratic formula to solve for x .

$$\begin{aligned}x &= \frac{-2m^2 \pm \sqrt{(2m^2)^2 - 4(1+m^2)(m^2-1)}}{2(1+m^2)} \\&= \frac{-2m^2 \pm \sqrt{4m^4 - 4m^4 + 4}}{2(1+m^2)} \\&= \frac{-2m^2 \pm \sqrt{4}}{2(1+m^2)} = \frac{-2m^2 \pm 2}{2(1+m^2)} = \frac{-m^2 \pm 1}{1+m^2}\end{aligned}$$

You only need the $+$ part of the \pm portion of the formula, because the $-$ part gives you the point $(-1,0)$, which you already have. So the x -coordinate of a point on the graph of the unit circle is $x = \frac{-m^2+1}{m^2+1}$. You get the y -coordinate by putting this x -coordinate into the equation for the unit circle and solving for y . I'll leave that little goodie for you to do.

Part IV

Taking the Shape of Geometric Word Problems

The 5th Wave

By Rich Tennant



In this part . . .

Geometry: You either love it or hate it. Geometry word problems really are your friends. They almost always involve a formula that you insert and then solve. The main challenge to doing word problems involving geometry is in finding the right formula for the right problem. Draw a picture, insert the given numbers, and you're on your way!

Chapter 18

Plying Pythagoras

In This Chapter

- ▶ Applying the Pythagorean theorem to everyday situations
- ▶ Mixing it up with rates and times in a right triangle
- ▶ Using more than one right triangle to solve a problem

If you've taken a math class, you've probably heard of the Pythagorean theorem. You may not remember what it does, but you probably remember it has something to do with triangles. In this chapter, I reintroduce you to Pythagoras and his theorem. I cover applications where you assume that buildings are at right angles with the ground and that streets are perpendicular to one another at the corner. These assumptions aren't always completely accurate, but the answers you get using these assumptions are close enough that you can forgive a little variance.



The Pythagorean theorem says that, in a right triangle whose two shorter sides (or *legs*) measure a and b , and whose longest side (or *hypotenuse*) measures c , it's always true that $a^2 + b^2 = c^2$. Figure 18-1 shows you how these parts go together in a right triangle.

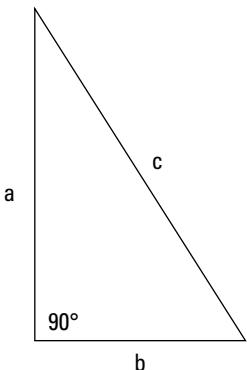


Figure 18-1:
The longest
side is
opposite the
right angle.

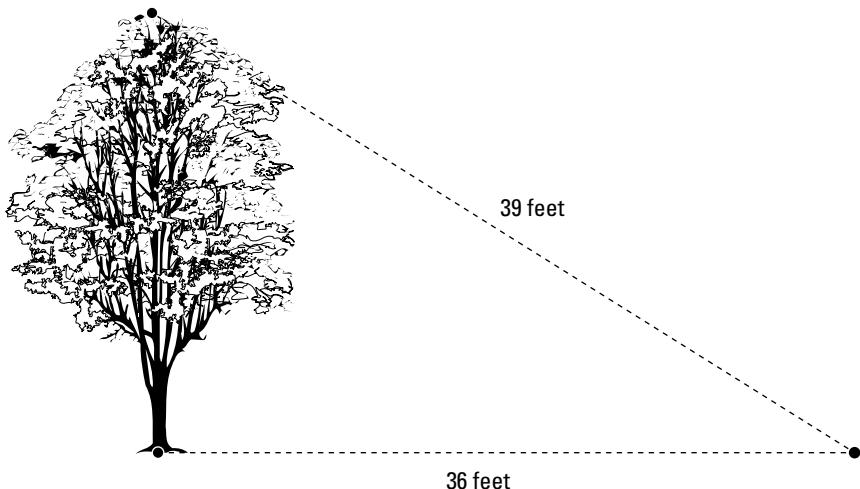
Finding the Height of an Object

You measure your own height with a tape measure or yardstick, and you measure the walls in a room by standing on a chair and using a tape measure or other instrument. But what about finding the height of a tree or building that's too tall to reach the top of or inaccessible to you? Enter, our hero, Mr. Pythagoras and the Pythagorean theorem.

Determining the height of a tree

You want to decorate the tree in your front yard and put a light on the top. You can rent a cherry picker that reaches 20 feet into the air (and you're brave enough to get into the basket). Right now, you have a kite string stretching from the top of the tree to a point on the ground. Figure 18-2 shows you the string and the tree and the point on the ground.

Figure 18-2:
Figuring out
how tall the
tree is.



The Problem: How tall is a tree if a kite string stretched from the top of the tree to a point on the ground measures 39 feet, and the distance from the base of the tree to that same point on the ground measures 36 feet? (Will you be able to put your star on the top of the tree if you can reach 20 feet high?)



You want to use the Pythagorean theorem, but must be careful to assign the right numbers to the different variables. Assume that the tree is perpendicular to the ground, making the right angle opposite the kite string. The hypotenuse is 39 feet, and one of the shorter sides is 36 feet. Using $a^2 + b^2 = c^2$,

and letting $a = 36$ and $c = 39$, the equation reads $36^2 + b^2 = 39^2$. Simplifying, you get $1,296 + b^2 = 1,521$. Subtract 1,296 from each side, and you find that $b^2 = 225$. Taking the square root of each side, the solution of the equation is that $b = \pm 15$. You ignore the negative answer — you want a positive height. So, if the tree is only 15 feet high, you'll be able to reach the top to attach the star.

Sighting a tower atop a mountain

You're waiting patiently in line to catch a ride to the top of a mountain so you can ski down. You know the height of the mountain and the length of the cable carrying the skiers on their T-bars. But just how tall is that tower that's sitting on top of the mountain?

The Problem: A 1,230-foot cable stretches from the starting point of a ski lift to the top of a tower that sits on the highest point of a ski slope (see Figure 18-3). The mountain is 1,150 feet tall, and the starting point of the ski lift is 270 feet from a point directly under the tower. How tall is the tower?

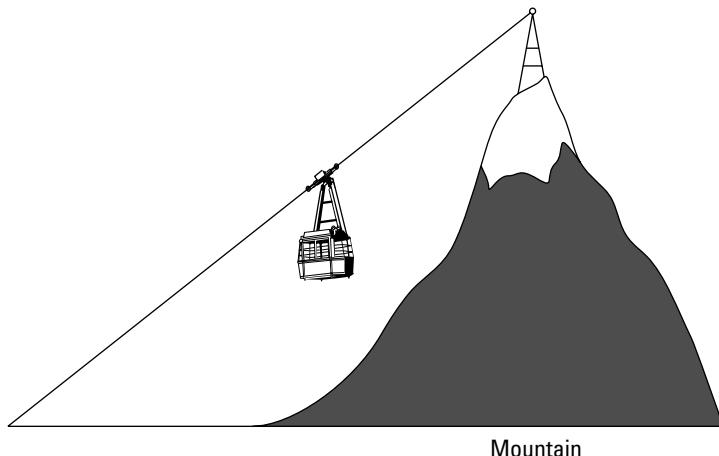


Figure 18-3:
The cable is
attached to
the top of
the tower.



When you determine the combined height of the tower and the top of the ski slope, you can subtract the height of the ski slope from that combined height to get the height of the tower. The hypotenuse of this right triangle is the cable carrying the skiers. Using $a^2 + b^2 = c^2$, let a represent the total height, b represent the horizontal distance, and c represent the length of the cable. Your equation becomes $a^2 + 270^2 = 1,230^2$. Simplifying, you get $a^2 + 72,900 = 1,512,900$. Subtracting 72,900 from each side, $a^2 = 1,440,000$. The positive square root of 1,440,000 is 1,200. Because the mountain is 1,150 feet tall, the tower must be $1,200 - 1,150 = 50$ feet tall.

Finding the height of a window

How tall is the window in a building? You can't go inside the building to take the measurements, but you have information about some distances outside. The distance from the top of the window and the bottom of the window to a single point on the ground are known numbers.

The Problem: From a point on the ground 33 feet away from the base of a building, the distance from the point to the top of a window is 65 feet, and the distance from the same point to the bottom of the window is 55 feet (see Figure 18-4). How tall is the window?

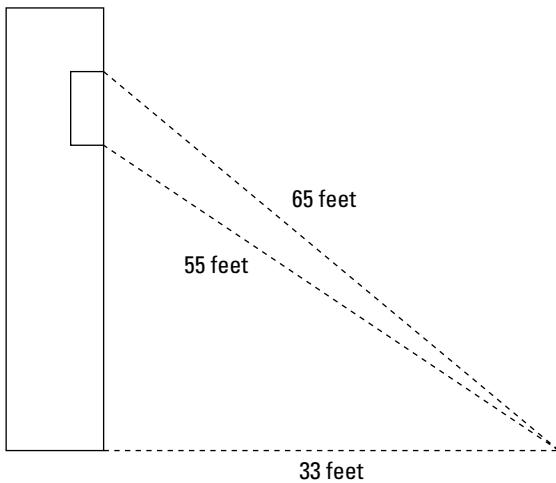


Figure 18-4:
The window
is on the
side of the
building.



Two different right triangles are in play with this problem. You have one right triangle with a hypotenuse of 55 feet and side of 33 feet, and another right triangle with a hypotenuse of 65 feet and same side of 33 feet. Find the missing side of each right triangle and, then, find the difference between the two sides to determine the height of the window. The smaller right triangle is solved using $a^2 + 33^2 = 55^2$, and the larger right triangle is solved with $a^2 + 33^2 = 65^2$. Solving the triangles,

$$\begin{array}{ll} a^2 + 33^2 = 55^2 & a^2 + 33^2 = 65^2 \\ a^2 + 1,089 = 3,025 & a^2 + 1,089 = 4,225 \\ a^2 = 1,936 & a^2 = 3,136 \\ a = 44 & a = 56 \end{array}$$

The difference between the two lengths for the vertical side is $56 - 44 = 12$ feet high.

Determining Distances between Planes

Planes travel at different speeds and different heights. A traffic controller has the responsibility of keeping the planes a safe distance from one another. Air traffic control is just another opportunity for Pythagoras to be of assistance.

Working with the distance apart

Two fighter jets leave the same airbase at the same time and travel at right angles from one another. How far apart are they after a certain amount of time?

The Problem: Jet A leaves the airport at 4 p.m. traveling due east at 550 mph. Jet B leaves the same airport at 4 p.m. traveling due south at 480 mph. How far apart are the jets at 7 p.m.?



Let the distances traveled by the jets be the two legs of a right triangle. Use $d = rt$ to determine the distance traveled by each jet in the three hours that they flew. The eastbound jet traveled $550 \times 3 = 1,650$ miles, and the southbound jet traveled $480 \times 3 = 1,440$ miles. Using the Pythagorean theorem, $1,650^2 + 1,440^2 = c^2$, where c is the distance between the two jets. Simplifying, you get that $2,722,500 + 2,073,600 = c^2$ or $4,796,100 = c^2$. Taking the square root of each side and considering only the positive root, you get that $c = 2,190$. The two jets are 2,190 miles apart after three hours.

What if two jets leave at *different* times, and you want to determine the speed at which a particular jet is traveling? Use the distance formula and the Pythagorean theorem.

The Problem: One jet leaves the airport at noon traveling north at 800 mph. Another jet leaves the same airport at 2 p.m. traveling west. At 3 p.m., the two jets are 2,500 miles apart (see Figure 18-5). What is the speed of the westbound jet?

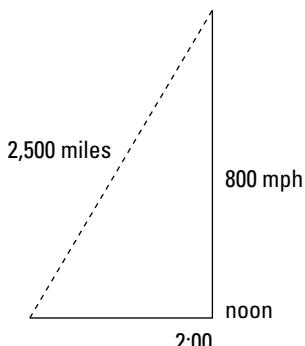


Figure 18-5:
The planes
are 2,500
miles apart.



The hypotenuse of the right triangle formed is 2,500 miles. The jet traveling north flew for three hours. Using $d = rt$, $d = 800 \times 3 = 2,400$ miles. The jet traveling west flew for one hour, so $d = r \times 1$ or $d = r$. After you find that distance, d , you have the rate of the jet. Using the Pythagorean theorem and letting the horizontal (westward) distance be d , you get $2,400^2 + d^2 = 2,500^2$. Simplifying, $5,760,000 + d^2 = 6,250,000$. Subtracting 5,760,000 from each side, $d^2 = 490,000$ or $d = 700$. The westbound jet is traveling at 700 mph.

Taking into account the wind blowing

When the plane you're traveling in has a good tailwind, you make even better time than anticipated. A headwind slows things down a bit. The blowing wind is taken into consideration when figuring how fast a plane is moving and in what direction it's actually heading.

The Problem: Two planes leave the same airport at the same time. The plane traveling due east is moving at 190 mph but has a headwind blowing against it at 40 mph. The plane traveling due north has a tail wind of 70 mph to help it along. At the end of one hour, the planes are 250 miles apart. How fast is the northbound plane traveling (what's its speed)?



Use the Pythagorean theorem with the eastbound and northbound distances as the legs of the triangle and the hypotenuse as the 250-mile distance apart that the planes are after an hour. The rate of the eastbound plane is $190 - 40$ or 150 mph. Let the speed of the northbound plane be represented with r , and add 70 to that amount. Writing the values in their appropriate places in the Pythagorean theorem, you have $150^2 + (r + 70)^2 = 250^2$. Squaring the two numbers and the binomial and solving for r ,

$$\begin{aligned} 150^2 + (r + 70)^2 &= 250^2 \\ 22,500 + r^2 + 140r + 4,900 &= 62,500 \\ r^2 + 140r + 27,400 &= 62,500 \\ r^2 + 140r - 35,100 &= 0 \\ (r + 270)(r - 130) &= 0 \\ r &= -270 \text{ or } 130 \end{aligned}$$

The answer that $r = -270$ is discarded as being extraneous. The northbound plane is traveling at 130 miles per hour.

Figuring Out Where to Land the Boat

A fisherman is out in his boat, a certain distance from the shoreline. He needs to come ashore, walk down the beach a bit, and then walk inland to his home.

The rate at which he can row is different from the rate at which he can walk. Where he comes ashore affects how far he rows and how far he walks. In Figure 18-6, you see a generalized picture for the problems presented in this section.

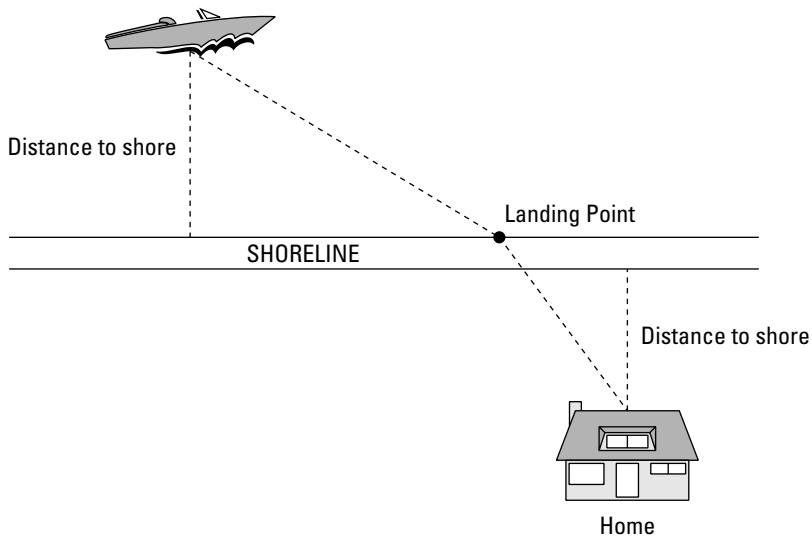


Figure 18-6:
Row to
shore; then
walk home.

Conserving distance

If a fisherman rows straight to shore, then walks along the beach until his home is the closest distance to the shoreline and walks inland, then he actually will have traveled farther than necessary. He probably will have taken more time than necessary, too. Aiming for a point down the shore to land his boat and then walking diagonally to his home will save time.

Computing the distance saved

Comparing the distance traveled along the sides of a right triangle and comparing the sum to the distance along the hypotenuse requires an application of the Pythagorean theorem. Applying this to a fisherman and his trip home, you find the difference in the distance traveled to get home the *long way* and the *shortcut*. The long way would be to go straight to shore, walk along the shore until he's opposite his home, and then walk to his home. The shortcut involves heading for a point down the beach and then traveling diagonally to and from that point.

The Problem: Fisherman Fred is 3 miles off shore in his boat. His home is 19 miles down the shore and 8 miles inland. How much shorter is it to cut diagonally toward a point 4 miles down the beach and then travel diagonally from that point to his home?



The long way and the shortcut involve the sides and hypotenuses of two right triangles. In Figure 18-7, you see the sides of the right triangle as being the long way and the diagonals as the shortcut.

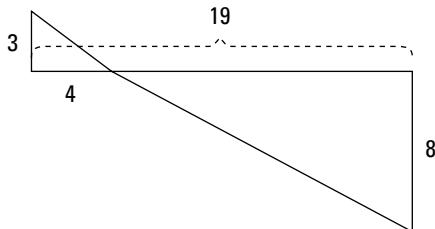


Figure 18-7:
It's shorter
to use the
diagonals.

If Fred goes directly to shore (3 miles) and then travels along the shore to the point where he can head straight inland toward home, he travels a total of $3 + 19 + 8 = 30$ miles. Heading diagonally for a point 4 miles down the shore and then diagonally across the land toward home, the distances are the hypotenuses of two right triangles. The first right triangle has legs measuring 3 and 4 miles. Using the Pythagorean theorem, $3^2 + 4^2 = c^2$. $25 = c^2$, so c , or the hypotenuse, is 5 miles. The other triangle has one side that's 8 miles long. The other side is $19 - 4$ or 15 miles long. Using the Pythagorean Theorem, $8^2 + 15^2 = c^2$. Squaring the numbers and adding, $64 + 225 = 289 = 17^2$. So that hypotenuse is 17. Add the 5 and 17 to get 22 miles, which is 8 miles shorter than going along the sides.

Considering rate and time

Two other considerations of taking a shorter route are the time that can be saved and the rates at which each leg of the journey are traveled. Some people can walk faster than they can row a boat, but others can really move through the water. Also, if the boat has a motor, then water travel may be the more efficient, but a bicycle or car can change the whole scenario.

The Problem: Kayaker Katie is 7 miles from shore, paddling along on a bright, sunny day. She needs to get to the store to pick up some sunscreen. The store is 32 miles down the coast and 6 miles inland. Katie can paddle two-and-a-half times as fast as she can walk, so she lands x miles down the shore and walks directly to the store. If Katie spends as much time paddling as walking, then how far down the shore does she land? Figure 18-8 shows you the relative positions and helps you see the triangles formed.

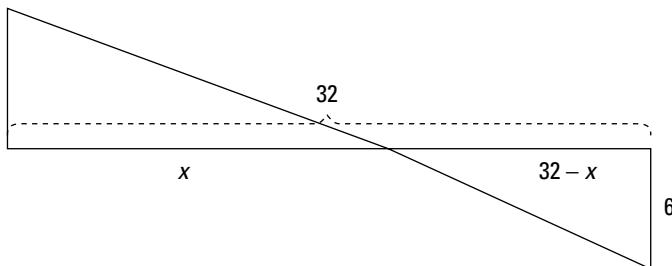


Figure 18-8:
Katie lands
 x miles
down the
coast.



Find the two distances along the hypotenuses (in terms of x) using the Pythagorean theorem. After finding the distances, use the distance formula $d = rt$, and solve for t , $t = \frac{d}{r}$. The time it took for each part of the trip along a hypotenuse was the same, so the two distances divided by their respective rates are equal to one another. Because Katie paddles more quickly than she walks, let the rate at which she walks be r and the rate at which she paddles be $2.5r$.

First, writing the distance paddled, $7^2 + x^2 = c_p^2$, so $c_p = \sqrt{7^2 + x^2}$. Then, to compute the distance walked, subtract x from 32 for the length of one side of the second triangle. The hypotenuse of this second triangle is $6^2 + (32 - x)^2 = c_w^2$, or

$$\begin{aligned}c_w &= \sqrt{6^2 + (32 - x)^2} \\&= \sqrt{36 + 1,024 - 64x + x^2} \\&= \sqrt{1,060 - 64x + x^2}\end{aligned}$$

Now, setting the two times equal to one another by dividing the distances by their respective rates, $\frac{\sqrt{49 + x^2}}{2.5r} = \frac{\sqrt{1,060 - 64x + x^2}}{r}$. Solve this equation for x by multiplying both sides by r , squaring both sides, cross-multiplying, and solving the resulting equation.

$$\begin{aligned}r \times \frac{\sqrt{49 + x^2}}{2.5r} &= \frac{\sqrt{1,060 - 64x + x^2}}{r} \times r \\ \left(\frac{\sqrt{49 + x^2}}{2.5} \right)^2 &= \left(\frac{\sqrt{1,060 - 64x + x^2}}{1} \right)^2 \\ \frac{49 + x^2}{6.25} &= \frac{1,060 - 64x + x^2}{1} \\ 49 + x^2 &= 6,625 - 400x + 6.25x^2 \\ 0 &= 6,576 - 400x + 5.25x^2\end{aligned}$$

The quadratic equation can be factored, but the factors aren't easy to come by. Using the quadratic formula, while messy, is quicker in this case. You get that $x = 24$ or x is a number greater than 52. The second number doesn't make any sense and is extraneous, because the total distance down the coast is only 32 miles. Katie landed 24 miles down the coast.

The Problem: Shrimper Stanley is 10 miles from shore in his boat. The store where he sells his daily catch of shrimp is 60 miles down the coast and x miles inland. Stanley lands his boat 24 miles down the coast and bicycles directly to the store. His boat moves at 13 miles per hour, and he bicycles at 15 miles per hour. If the trip from his boat to the store takes a total of 5 hours, then how far is it from the shore to the store? Figure 18-9 shows you the layout of the distances and speeds.

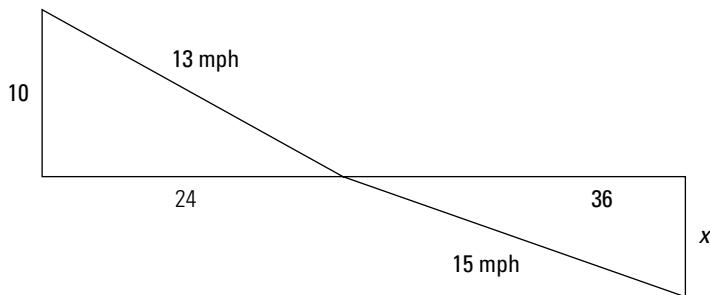


Figure 18-9:
The shrimp
are on ice
for the five-
hour trip.



First determine how far Stanley traveled in his boat. After you have the distance traveled in the boat, you determine how much time he spent in the boat by dividing by 13 mph. Whatever time is left is what he spent bicycling, so you can compute the distance bicycled and then the distance from the shore to the store.

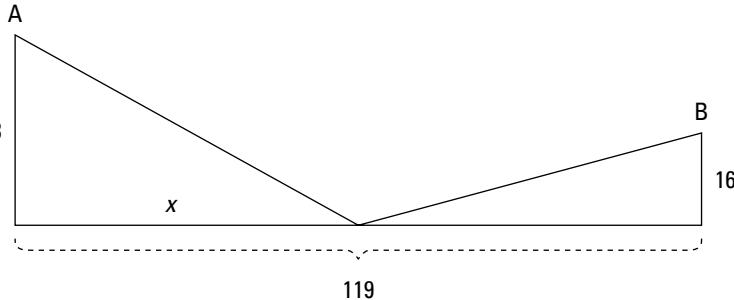
Completing the right triangle involving the boat, $10^2 + 24^2 = 100 + 576 = 676 = 26^2$. The distance traveled in the boat is 26 miles. Dividing 26 by 13, you have that Stanley traveled for 2 hours in his boat. The total trip was 5 hours, so he spent 3 hours bicycling. The distance bicycling is equal to the rate of 15 mph times the time, so Stanley bicycled 45 miles. The distance of 45 miles is the hypotenuse of a right triangle. Solving for the distance from the shore to the store, subtract $60 - 24 = 36$ miles along the shore. Then $36^2 + x^2 = 45^2$. $1,296 + x^2 = 2,025$, $x^2 = 729$. The square root of 729 is 27, so x (the distance to the store) is 27 miles.

Placing Things Fairly and Economically

When towns are separated by a number of miles but want to share things like electrical transformers or new shopping centers, the placement of the equipment or buildings has to take into account distances, convenience, and economics.

The Problem: Town A is 33 miles from the main highway, and Town B is 16 miles from that same highway. The two towns are 119 miles apart. A developer wants to build a mall along that highway and connecting roads that go directly from the mall to the two towns. Where should he place the mall so that it's the same distance from each town? Figure 18-10 shows you a layout for the towns and mall, so you can see how the triangles work in the situation.

Figure 18-10:
The two towns are the same distance from the mall.



The two towns will be the same distance from the mall if the hypotenuse of each right triangle is the same. Let the distance along the highway from the point closest to Town A and the new mall be represented by x . Then the distance along the highway from the mall to the place closest to Town B is $119 - x$. Use the x and $119 - x$ to represent the distances along the highway. Then solve for the diagonal distance in each triangle using the Pythagorean theorem. Set the two hypotenuses equal to one another. Square both sides of the equation to get rid of the radicals. Simplify and subtract the squared term from each side. Solve the resulting linear equation for x .

$$\begin{aligned}
 \sqrt{33^2 + x^2} &= \sqrt{16^2 + (119 - x)^2} \\
 (\sqrt{33^2 + x^2})^2 &= (\sqrt{16^2 + (119 - x)^2})^2 \\
 33^2 + x^2 &= 16^2 + (119 - x)^2 \\
 1,089 + x^2 &= 256 + 14,161 - 238x + x^2 \\
 238x &= 256 + 14,161 - 1,089 \\
 238x &= 13,828 \\
 x &= \frac{13,828}{238} = 56
 \end{aligned}$$

The mall should be placed 56 miles down the highway from Town A and $119 - 56 = 63$ miles down the highway from Town B. The distance directly from each town to the mall is $\sqrt{33^2 + 56^2} = \sqrt{1,089 + 3,136} = \sqrt{4,225} = 65$ miles.

Sometimes it's more economical to have the direct or diagonal distances from different towns to a centralized point be different from one another — if it saves money by reducing the total distance of the two.

The Problem: A utility needs to place a transformer along an existing power line that will serve two different customers. The two customers are 12 and 24 miles from the power line, and the customers are 42 miles apart. The utility has three possible locations for the transformer: 9 miles from the closest customer, 16 miles from the closest customer, or 35 miles from that customer. Which location will provide a transformer that's the shortest *total* distance from the two customers? It's hard to picture all this without a sketch; Figure 18-11 shows you how this all sorts out.

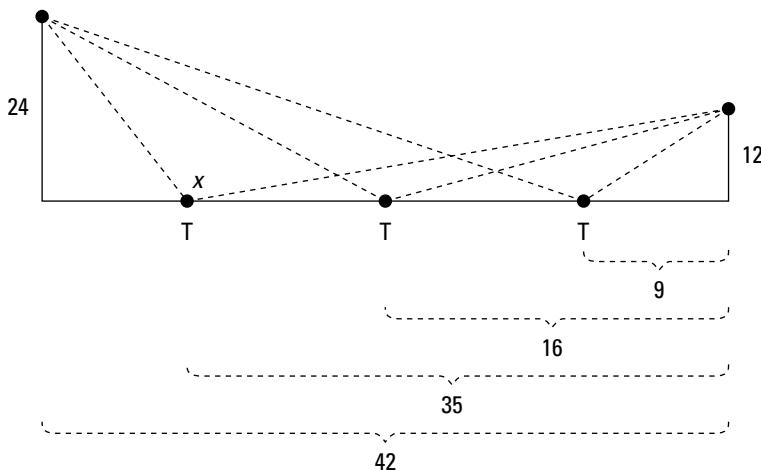


Figure 18-11:
Placing the
transformer
along the
power line.



Determine the sum of the lengths of the hypotenuses of the right triangles for each placement of the transformer.

When it's 9 miles from the closest customer, the first triangle has sides 9 and 12, and the second triangle has sides 33 and 24.

When it's 16 miles from the closest customer, the first triangle has sides 16 and 12, and the second triangle has sides 26 and 24.

When it's 35 miles from the closest customer, the first triangle has sides 35 and 12, and the second triangle has sides 37 and 24.

Now, determining the sums of the diagonal distances (hypotenuses):

$$\sqrt{9^2 + 12^2} + \sqrt{33^2 + 24^2} = \sqrt{225} + \sqrt{1,665} \approx 15 + 40.80 = 55.80$$

$$\sqrt{16^2 + 12^2} + \sqrt{26^2 + 24^2} = \sqrt{400} + \sqrt{1,252} \approx 20 + 35.38 = 55.38$$

$$\sqrt{35^2 + 12^2} + \sqrt{7^2 + 24^2} = \sqrt{1,369} + \sqrt{625} \approx 37 + 25 = 62$$

Placing the transformer on the highway 16 miles from the closest customer results in a slightly shorter total distance from the two customers.

Watching the Tide Drift Away

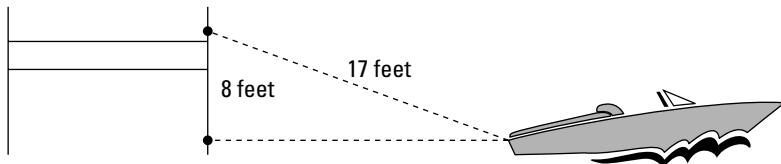
When a boat is tied to a dock, enough line (rope) has to be attached to account for changes in the distance from the boat to the dock occurring due to the tides. When the tide comes in, the boat sits higher and is closer to the dock. When it goes out, the distance is greater. And the movement of the tide will make the boat change positions.

The Problem: A boat is tied to a dock with a rope that's 17 feet long. If the height difference from where the rope is tied to the dock and where it's tied to the boat is 8 feet, then how far from the dock is the boat?



Let the height difference be vertical and the distance from the boat to the dock be horizontal — giving you a right triangle whose hypotenuse is the rope stretched from the dock to the boat. Figure 18-12 shows those vertical and horizontal distances — and the rope as hypotenuse of the triangle.

Figure 18-12:
The tide
is out.



Using the Pythagorean theorem, and letting the distance from the dock to the boat be represented by a , you have $8^2 + a^2 = 17^2$. Simplifying, you get $a^2 = 289 - 64 = 225$. So $a = 15$ feet.

When a boat moves away from a dock because of the movement of the water, the rope that's on a slant and attached to the boat pulls away at a different rate of speed than the speed at which the boat and tide are moving.

The Problem: The tide is moving out at a rate of 2 feet per second. The height difference between where a rope is connected to the dock and the boat is 8 feet. How far out did the boat drift after 5 seconds, how long is the rope extended, and how fast is the rope playing out? Do the same computations for 10 seconds and 20 seconds.



Find the distance the boat has moved and use it to complete the right triangle. The hypotenuse is the length of the rope. Divide the length of the rope by the time elapsed to determine the average rate at which the rope is playing out.

After 5 seconds, at a rate of 2 feet per second, the boat has moved 10 feet. Using the height of the dock as 8 feet and the distance from the dock as 10 feet, then $8^2 + 10^2 = 164 = c^2$. The value of c (length of the rope) is about 12.81 feet. Divide that by 5 seconds, and the rope is playing out (being pulled out) at an average rate of about 2.56 feet per second.

Now do the same thing for 10 seconds. At 2 feet per second, the distance is 20 feet. Completing the right triangle, $8^2 + 20^2 = 464 = c^2$. This time c is about 21.54 feet. Divide that by 10 seconds, and the rope is playing out at about 2.15 feet per second.

Finally, computing for 20 seconds, $8^2 + 40^2 = 1664 = c^2$. This time c is about 40.79 feet. Dividing by 20, the rope is playing out at about 2.04 feet per second.

Did you notice that, as the length of the hypotenuse gets closer to the horizontal length, the rates get closer, too?

Chapter 19

Going around in Circles with Perimeter and Area

In This Chapter

- ▶ Computing the perimeter and area of polygons
 - ▶ Coming full circle with the circumference and area of a circle
 - ▶ Putting shapes together to create interesting structures
 - ▶ Using the distance formula in Cartesian coordinates to measure sides
-

Areas and perimeters of *polygons* (geometric figures with segments for sides) and circles have many practical applications. You need the perimeter of your yard before you order new fencing. You want the total area of your living room before purchasing carpeting. You need to know the area of a walkway around your circular pool when you're choosing between cement or gravel. The challenges to doing problems involving perimeter and area are in determining what type of figure you have and then in finding and applying the correct formula.

Keeping the Cows in the Pasture

Cows and sheep are standard fare when it comes to problems involving fencing in a pasture. You may have a set amount of fencing available, or you may need a particular minimum area for your herd. The pasture doesn't have to be rectangular, but rectangles are nice figures to work with in these problems.

Working with a set amount of fencing

You have several rolls of fencing and you need to create a rectangular area for your herd of cattle. How can you best make use of the fencing? In this case, the better or best use is when you can create the largest possible area.

In Figure 19-1, you see how 60 feet of fencing is used to create several different rectangular regions, each with a different area.

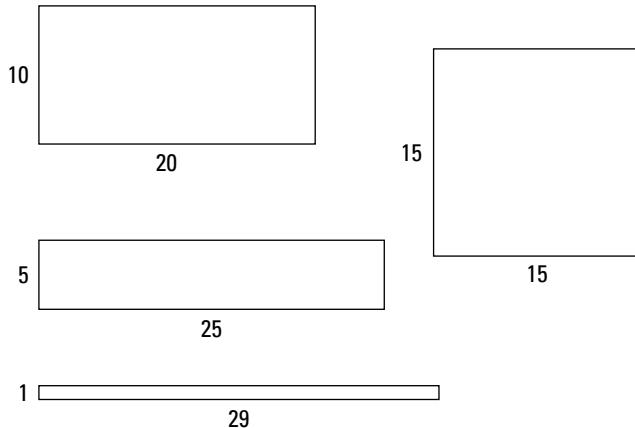


Figure 19-1:
Sixty feet
of fencing
creates
several
areas.

The areas of the figures created by the same 60 feet of fencing are 200 square units, 125 square units, 29 square units, and 225 square units. The way a certain amount of fencing is used makes a huge difference in the area.

The Problem: You have 680 feet of fencing and want to create a rectangular pasture whose length is 10 feet more than twice its width. What dimensions do you make your pasture?



The perimeter of a rectangular figure is determined with the formula $P = 2l + 2w$ where l and w represent the length and width of the rectangle. If your length is to be *10 more than twice the width*, then let w represent the width and $10 + 2w$ represent the length. Replace the perimeter, P , with 680 and solve the equation for w .

$$\begin{aligned} 680 &= 2(10 + 2w) + 2w \\ 680 &= 20 + 4w + 2w \\ 680 &= 20 + 6w \\ 660 &= 6w \\ w &= \frac{660}{6} = 110 \end{aligned}$$

The width of the pasture is 110 feet. Replacing the w with 110 in $10 + 2w$, you get that the length is $10 + 2(110) = 10 + 220 = 230$ feet. Checking the perimeter, you get that $P = 2(230) + 2(110) = 460 + 220 = 680$ feet.

Having a nice, rectangular pasture or yard is fine and dandy, but what if you need to keep the little boy sheep separated from the little girl sheep — or some other such arrangement?

The Problem: You have 1,200 yards of fencing and want to create a rectangular pasture that has two dividing fences running down the middle. If the width is to be 30 feet less than the length, then what is the area of the pasture?



Figure 19-2 shows you the layout of the fencing. Let the length be represented by x and the width be represented by $x - 30$. Solve for the length; then determine the width. And, finally, compute the area by multiplying the length times the width.

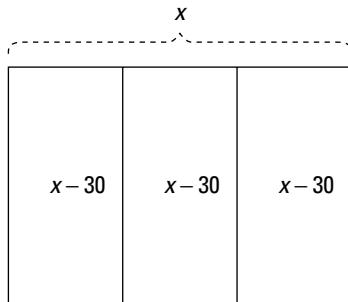


Figure 19-2:
The dividers
keep the
sheep
separated.

The total perimeter plus the dividers is equal to 1,200 yards. So you add up all the fencing to get $1,200 = x + x + x - 30 + x - 30 + x - 30 + x - 30 = 6x - 120$. The equation is now $1,200 = 6x - 120$. Solving for x by adding 120 to each side, $6x = 1,320$. Now, dividing each side by 6, you get that $x = 220$. The length is 220 yards. The width is 30 yards less than that or 190 yards. The area is $220 \times 190 = 41,800$ square yards.

Aiming for a needed area

You're told that llamas need a certain amount of grazing area in order to thrive. Armed with the area constraints, you can go about figuring out how to create the needed room for a hungry llama.

The Problem: You need to create a rectangular area encompassing 6,400 square feet, in which the length is 40 feet less than three times the width. One side of the rectangular area will be along a river, so you don't need to put a fence there. What is the least amount of fencing needed to partition off that rectangular area?



The two different scenarios are:

- ✓ Have the length of the area be along the river.
- ✓ Let the width be along the river.

Look at Figure 19-3 for the two different layouts. The w represents the width, and the $3w - 40$ represents the length.

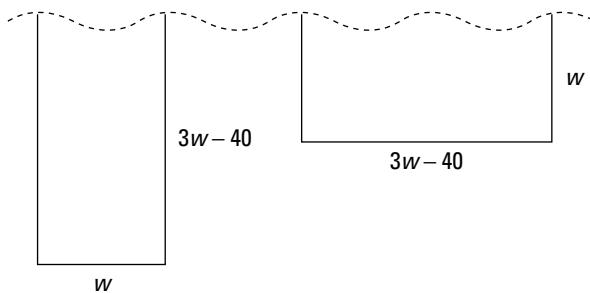


Figure 19-3:
Use the
river instead
of fencing.

In both cases, the area is length times width or $(3w - 40)w$. So solve the equation $A = lw$, replacing the A with 6,400.

$$\begin{aligned} 6400 &= (3w - 40)w \\ 6400 &= 3w^2 - 40w \\ 0 &= 3w^2 - 40w - 6400 \\ 0 &= (3w - 160)(w + 40) \\ w &= \frac{160}{3}, -40 \end{aligned}$$

The two solutions of the quadratic equation are $53\frac{1}{3}$ and -40 . The negative solution makes no sense, of course. Letting the width be $53\frac{1}{3}$, then the length, which is 40 less than three times that is 120 feet. The dimensions of the area needed are $53\frac{1}{3}$ by 120.

Now determine which layout will use the least amount of fencing. If you let the width be along the river, then you need two lengths and one width of the fencing or $293\frac{1}{3}$ feet. If you let the length run along the river, then you need one length and two widths or $226\frac{2}{3}$ feet — the better choice for economy's sake.

Getting the Most Out of Your Resources

You can't always create a rectangular yard or area, but the rectangular shape does seem to be the most popular with most builders and their projects. If you could use any shape you wanted, you could do a lot better with your resources because you could economize on fencing and maximize the area enclosed.

Triangulating the area

A triangular area requires just three sides of fencing. The area of a triangle is found using one of two formulas, depending on whether you have a perpendicular measure from one of the sides to the opposite vertex or just the measures of the sides.



The area of the triangle shown in Figure 19-4 is computed:

$$A = \frac{1}{2}bh, \text{ or}$$

$A = \sqrt{s(s-a)(s-b)(s-c)}$, where s is half the total perimeter.
(This is called *Heron's formula*.)

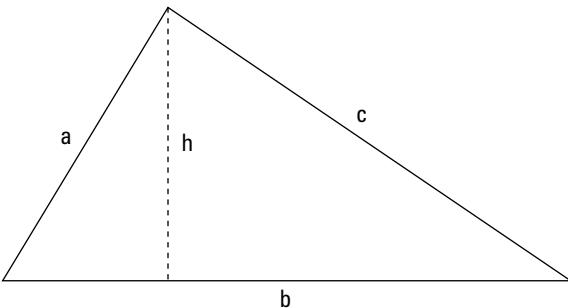


Figure 19-4:
A triangle's
height is
perpendicu-
lar to the
base.

The Problem: You have 72 feet of fencing and need to enclose an area that's triangular. Which triangle will give you more area: a right triangle that has sides measuring 18, 24, and 30 feet, or an equilateral triangle that's 24 feet on each side?



Find the area of the right triangle using the two perpendicular sides as the base and height. The area is $A = \frac{1}{2}(18)(24) = 216$ square feet. To find the area of the equilateral triangle, use Heron's formula. The perimeter of that triangle is 72 feet (the amount of fencing you have), so half the perimeter, s , is 36 feet. Computing the area of the equilateral triangle, $A = \sqrt{36(36 - 24)(36 - 24)(36 - 24)} = \sqrt{36(12)^3} \approx 249.42$ square feet. So the equilateral triangle has the greater area, even though the perimeter of each triangle is the same.

Squaring off with area

In the “Working with a set amount of fencing” section, earlier in this chapter, you see how the same amount of fencing creates several different areas. What size rectangle creates the greatest area? Does the shape of the rectangle or proportion of the lengths of the sides of a rectangle depend on the perimeter? Or is there some optimum shape?

The Problem: A rectangle is to have a perimeter of 36 feet and the greatest area possible. What are the dimensions of the rectangle that has the greatest area?



Let the width of the rectangle be 1 foot, 2 feet, 3 feet, and so on. For each width, determine the length of the rectangle by subtracting the width from 18. You subtract from 18 because the perimeter of a rectangle is $P = 2(l + w)$. When the perimeter is 36 feet, you get $36 = 2(l + w)$. Dividing each side of the equation by 2, $18 = l + w$, and $l = 18 - w$. When you get the width and length, compute the area of the resulting rectangle. Table 19-1 lays it all out.

Table 19-1 Rectangles with a Perimeter of 36 Feet

| Width | Length ($18 - w$) | Area ($l \times w$) |
|--------|---------------------|-----------------------|
| 1 foot | 17 feet | 17 square feet |
| 2 feet | 16 feet | 32 square feet |
| 3 feet | 15 feet | 45 square feet |
| 4 feet | 14 feet | 56 square feet |
| 5 feet | 13 feet | 65 square feet |
| 6 feet | 12 feet | 72 square feet |
| 7 feet | 11 feet | 77 square feet |
| 8 feet | 10 feet | 80 square feet |

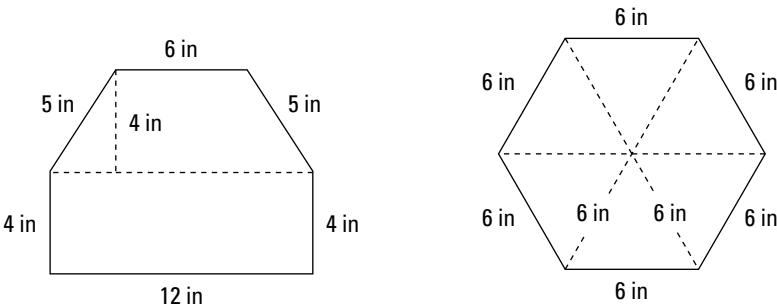
| Width | Length ($18 - w$) | Area ($l \times w$) |
|---------|---------------------|-----------------------|
| 9 feet | 9 feet | 81 square feet |
| 10 feet | 8 feet | 80 square feet |

You see that the area values start decreasing when you pass the point where the rectangle is a square. A rectangle that's actually a square has the greatest possible area for a given perimeter. This statement is most easily proved using calculus. For now, the demonstration with the table should suffice.

Taking the hex out with a hexagon

A hexagon is a six-sided polygon. A *regular* hexagon is special, because all the sides are the same measure, and all the interior angles are 120 degrees. An even more special feature of the regular hexagon is the fact that it's made up of six equilateral triangles, all nestled together. In Figure 19-5, you find two hexagons, one made up of a rectangle topped by a trapezoid, and the other a regular hexagon. You get to find out which has the greater area for a set amount of perimeter.

Figure 19-5:
Hexagons
have six
sides.



The Problem: Each of the hexagons shown in Figure 19-5 has a perimeter of 36 inches. Which has the greater area?



The house-shaped hexagon on the left is a rectangle topped by a trapezoid. Add the two areas together to get the total area. The area of the rectangle is $12 \times 4 = 48$ square inches. The area of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$, which is half the height of the trapezoid times the sum of the two parallel bases.

In the case of the trapezoid in the figure, the area is $A = \frac{1}{2}(4)(6 + 12) = 36$ square inches. Add the area of the rectangle and the area of the trapezoid together to get $48 + 36 = 84$ square inches.

The regular hexagon is made up of six equilateral triangles; the sides of each of the triangles is 6 inches. Use Heron's formula to find the area of one of the triangles, and then just multiply by 6. The area of one of the triangles is $A = \sqrt{9(9-6)(9-6)(9-6)} = \sqrt{9(3)^3} = \sqrt{243} \approx 15.59$ square inches. Multiply that area by 6 to get 93.54 square inches. The regular hexagon has the greater area.

Coming full circle with area

Each of the problems in this section deals with making the most of perimeter to get the biggest possible area. A common theme that you find is that a *regular* polygon has the greatest area of any other polygon of its type. Also, the more sides you add to a regular polygon, the more the polygon seems to resemble a circle. In this section, you compare the area of a hexagon that has a perimeter of 36 inches with the area of a circle that has a circumference of 36 inches.

The Problem: Which has the greater area: a regular hexagon with a perimeter of 36 inches or a circle with a circumference of 36 inches?



First, refer to the problem in the preceding section, “Taking the hex out with a hexagon,” and you find that a regular hexagon with a perimeter of 36 inches has an area of about 93.54 square inches. To find the area of a circle that has a *circumference* of 36 inches, you need to find the radius of the circle, first.

The circumference of a circle is found with: $C = 2\pi r$, where r is the radius. The area of a circle is found with: $A = \pi r^2$, where r is the radius.

If the circumference of a particular circle is 36 inches, then $2\pi r = 36$. Dividing each side of the equation by 2π , you get that r is about 5.73 inches. Use 5.73 as the radius in the formula for the area of the circle, and you get that the area is about 103.15 square inches. The area of the circle is almost 10 square inches larger than that of the hexagon. The area of a circle will always be greater than a polygon with the same perimeter.

Putting in a Walk-Around

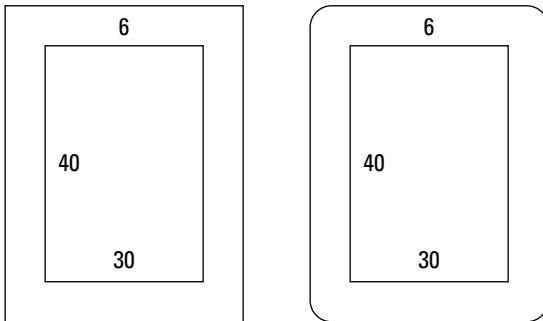
You've finally hit the big time and decide you can afford to put in a pool and party area in the backyard. A pool has water in it — well, let's hope so. With water comes mud and a mess, so you need to put a nice cement walk around the perimeter of the pool.

Determining the area around the outside

When you have an existing pool or other area that needs to be surrounded, then you take measurements of the structure in the middle and determine what you want around the outside — how wide and how deep.

The Problem: You have a rectangular pool that's 40 feet long and 30 feet wide. You want to put in a cement walkway that's 6 feet wide on all four sides, and the corners will be 6-foot squares. How many square feet of cement will you need? (Refer to Figure 19-6 which shows both the square corners for this problem and rounded corners for the next problem.)

Figure 19-6:
The pools
have
cement
walkways.



To figure the total area of the cement walkway, divide the walkway into four rectangles. Include the square corners in rectangles that go across the top and bottom. The rectangles across the top and bottom now have dimensions 42 feet by 6 feet (the 42 comes from $30 + 6 + 6$ for the two ends). Multiplying 42×6 , you get 252 square feet for each of the two sections. The side sections are 40 feet by 6 feet, so each of their areas are $40 \times 6 = 240$ square feet. Double each section type and add the areas together: $2(252) + 2(240) = 504 + 480 = 984$ square feet.

But what if you want curved corners? It may be that you want to make mowing easier by making the corners curves. Or, perhaps, you think that the curved corners are more aesthetically pleasing. Also, there's always the chance that you're just very frugal and want to save some money.

The Problem: You want a 6-foot cement walkway around the outside of your 40 foot by 30 foot rectangular pool. The walkway is to be 6-feet wide at the corners, too, so they'll be pieces of circles with a radius of 6 feet. How many square feet, total, will your walkway contain? (Refer to Figure 19-6 to see what this type walkway would look like.)



The four corners of the walkway are each one-fourth of the same 6-foot-radius circle. Just find the area of a circle with a radius of 6 feet, and add it onto the four rectangular sections. The four rectangular sections have dimensions matching the sides of the pool. Two sections are 40×6 , and the other two are 30×6 .

The total area of the walkway is the area of the circle plus twice the area of each rectangular section.

$$\begin{aligned} A &= \pi(6)^2 + 2(40 \times 6) + 2(30 \times 6) \\ &\approx 113.10 + 480 + 360 \\ &= 953.10 \end{aligned}$$

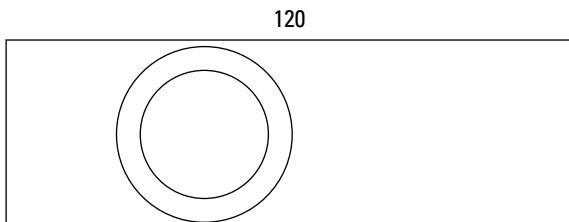
The total area is about 953 square feet. That's about 31 square feet less than the walkway with square corners.

Adding up for the entire area

You're contemplating putting in a circular above-ground swimming pool, but you're not sure whether you have enough room. There's not only the pool itself, but also the 3-foot-wide deck to consider.

The Problem: Your backyard is 120 feet long and 40 feet wide. The neighborhood ordnance dictates that a pool and its surround can't take up more than 25 percent of your yard. How big a pool can you get if you're going to put a 3-foot-wide deck around a circular pool? (Figure 19-7 shows you a possible scenario for the pool and the yard.)

Figure 19-7: 40
The pool fits
in the yard.



First, determine the total square footage allowed by the ordnance. Then try to maximize the size of the pool. Your main constraint will be the width of the yard — you may have to settle for a smaller pool than allowed or go for another shape. Your total square footage is $120 \times 40 = 4,800$ square feet.



Avoiding wet feet

You have a fish pond on your property that's a perfect circle with a diameter of 100 feet. In the exact center of your fish pond, there's a flagpole. You want to tie a rope to that flagpole, but your rope is only 101 feet long, and you don't

have a boat, a raft, or even waders. You're a clever sort, though, and you figure out how to tie the rope to the flagpole without getting your feet wet. How do you do it?

loop. Pull on the end of the rope until it tightens around the flag pole. loop. Pull on the end of your rope, and push the end that was tied to the peg through the edge of the pond. Now, holding the other end of the rope, walk all the way around the pond and back to the peg. Make a loop in the rope. Full on the end of your rope to the peg. Now, holding the other end of the rope, full on the end of your rope until it tightens around the flag pole.

Answer: You first find a peg or small stick and push it into the ground just at the

Twenty-five percent of 4,800 square feet is 1,200 square feet. To get the biggest pool possible, you want the area of the pool plus the deck to be 1,200 square feet (25 percent). Let the radius of your pool be represented by r . Add a 3-foot deck, and the total radius is $r + 3$. Now use the area formula to solve for r .

$$\begin{aligned} A &= \pi(\text{radius})^2 \\ 1,200 &= \pi(r+3)^2 \\ \frac{1,200}{\pi} &= (r+3)^2 \\ \sqrt{\frac{1,200}{\pi}} &= \sqrt{(r+3)^2} \\ \sqrt{\frac{1,200}{\pi}} &= r+3 \\ \sqrt{\frac{1,200}{\pi}} - 3 &= r \end{aligned}$$

The radius of the pool should be about 16.54 feet. Add the 3-foot deck, and it makes the total 19.54 feet. The pool plus the deck just fit the width of the yard.

Creating a Poster

You see posters all over the place announcing events. Garage sale signs crop up every spring, and fundraising dinners are plastered all over the place before the big night. Creating an eye-catching yet economical poster is an art — which is why some people take courses to learn how to do it effectively. Color is important, but so is the use of white space. The problems in this section assume that you have a certain portion of the poster dedicated to the print and pictures and the rest of it is a plain, white border.

Starting with a certain amount of print

You're in charge of printing up campaign posters for your favorite candidate. The person running for office has a lot to say and wants to devote 120 square inches of the poster to information about her position on the issues. You get to create the most economical size poster (use the least amount of print).

The Problem: What are the dimensions of the poster that uses the least amount of material if you have to include 120 square inches of print, 2-inch borders on the sides, and 3-inch borders on the top and bottom? (To make it more reasonable to solve, the measures all have to be whole numbers.)



Determine the different dimensions that rectangular areas of printed material can be in to get 120 square inches. Then determine the amount of border material that has to be added to each rectangle.

Figure 19-8 shows two possibilities for shapes of the resulting poster. Each poster has printed material in the middle taking up a total of 120 square inches. After you determine a height and width that gives you an area of 120 square inches, add 6 inches to the height (3 on the top and 3 on the bottom) and 4 inches to the width (2 and 2). Compute the new total area by multiplying the two new dimensions together. Look at Table 19-2 for the possibilities. (I haven't included choices like 120 by 1 or 60 by 2 because they're impractical.)

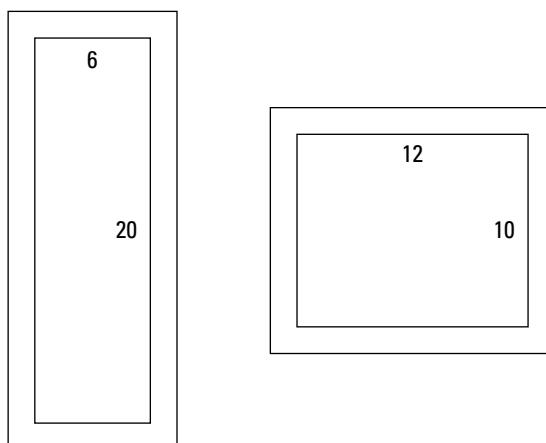


Figure 19-8:
Posters with
a white
border.

Table 19-2**Comparing Sizes of Posters**

| <i>Printed Height</i> | <i>Printed Width</i> | <i>Height + 6</i> | <i>Width + 4</i> | <i>Total Area</i> |
|-----------------------|----------------------|-------------------|------------------|-------------------|
| 40 | 3 | 46 | 7 | 322 square inches |
| 30 | 4 | 36 | 8 | 288 square inches |
| 24 | 5 | 30 | 9 | 270 square inches |
| 20 | 6 | 26 | 10 | 260 square inches |
| 15 | 8 | 21 | 12 | 252 square inches |
| 12 | 10 | 18 | 14 | 252 square inches |
| 10 | 12 | 16 | 16 | 256 square inches |
| 8 | 15 | 14 | 19 | 266 square inches |
| 6 | 20 | 12 | 24 | 288 square inches |
| 5 | 24 | 11 | 28 | 308 square inches |
| 4 | 30 | 10 | 34 | 340 square inches |
| 3 | 40 | 9 | 44 | 396 square inches |

Two different sizes yield the least amount of material: either a poster that's 21 by 12 or a poster that's 18 by 14. Probably a rectangle that's 18 by 14 is more aesthetically pleasing to the eye and would be the choice.



Climbing the ladder

Pete the painter is on a ladder leaning up against the wall that he's working on at the time. He starts on the middle rung of the ladder, goes up

four rungs, down nine rungs, up three rungs and then up ten more rungs to reach the top bar of the ladder. How many rungs are on this ladder?

ANSWER: There are 17 rungs on the ladder. Think of the middle rung as being you to +8. That's the top of the ladder. Take 2×8 and add 1 for the middle rung you to +8. That's the top of the ladder. Down nine rungs you to -2, and up ten more rungs from there takes you to -5. Up three rungs you to -2, and up one more rungs giving you to 16 + 1 = 17 rungs.

Working with a particular poster size

You're to create a poster with a certain number of square inches. The white border around the printed material is to be 2 inches on each side and 3 inches across the top and bottom. Now you have to determine the amount of material left for pictures and words.

The Problem: You are to create a poster with a total area of 300 square inches. In the center of the poster you'll put the printed material in a rectangular box with 2-inch borders on either side. You'll also put 3-inch borders on the top and bottom. The material in the borders is included in the total area of 300 square inches. If the width of the poster is to be $\frac{3}{4}$ the length, then what are the dimensions of the rectangular area containing the printed material? (Refer to Figure 19-9 to help you picture the situation.)

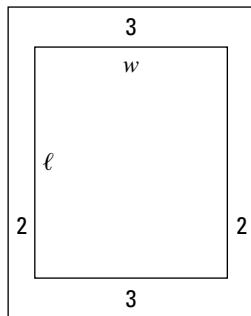


Figure 19-9:
Figuring
out the
dimensions.



Let the dimensions of the print in the poster be $l \times w$. Adding in the white borders, the dimensions of the overall poster are $(l + 6)$ by $(w + 4)$. The width of the poster is to be $\frac{3}{4}$ the length, so replace the $(w + 4)$ with $\frac{3}{4}(l + 6)$. If the poster is to have an area of 300 square inches, then multiply the length times the width and set it equal to 300.

$$\begin{aligned}
 (l + 6) \left(\frac{3}{4}(l + 6) \right) &= 300 \\
 \frac{3}{4}(l + 6)^2 &= 300 \\
 \frac{4}{3} \times \frac{3}{4}(l + 6)^2 &= 300 \times \frac{4}{3} \\
 (l + 6)^2 &= 400 \\
 \sqrt{(l + 6)^2} &= \sqrt{400} \\
 l + 6 &= 20 \\
 l &= 14
 \end{aligned}$$

So the inside length of the poster (for the written part) is to be 14 inches. Add 6 to get the outside length of 20 inches. The width is to be $\frac{1}{4}$ of the length, which makes it 15 inches. The inside width (for the printed material) is 15 inches less the border of 4 inches or 11 inches. To recap: The whole poster is to be 20 by 15 inches, giving you 300 square inches. The inside rectangle is 14 by 11 inches, giving you 154 square inches to put all the print and pictures.

Shedding the Light on a Norman Window

Houses have windows to let in the light and fresh air. The most common shape for a window is a rectangle, but you also see round windows and hexagonal windows and other creative shapes. A *Norman window* is made up of two geometric shapes: a semicircle on top of a rectangle. One side of the rectangle is the same measure as the semicircle's diameter. Figure 19-10 shows two Norman windows.

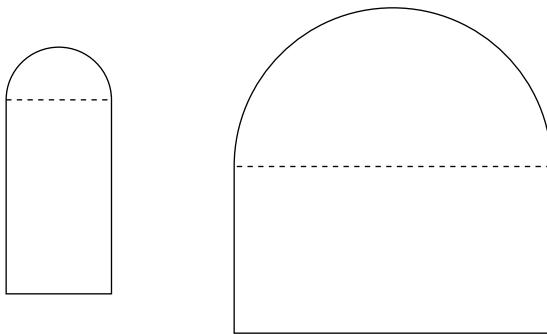


Figure 19-10:
The curve is
on the top.

Maximizing the amount of light

A window lets in light and fresh air. Adding a semicircle to the top of a rectangular window not only lets in more light, but adds a decorative touch.

The Problem: You're planning on putting in a Norman window in a room on the north side of your home. You want as much light as possible to be let in through the window. The rectangular part of the window is to be 6 feet by 4 feet, and the semicircle will sit on top. Which will let in more light: if the rectangular base is 4 feet and the sides 6 feet, or if the rectangular base is 6 feet with 4 foot sides? (Refer to Figure 19-10 for approximate figures.)



Find the total area of each window. The rectangular part of each window is 24 square feet, so the main interest is in the respective areas of the semicircles on top. The window with a 4-foot base has a semicircle on top with a diameter of 4 feet — or a radius of 2 feet. The area of a circle is found with $A = \pi r^2$. The semicircle has half the area of the full circle, so the area of this semicircle is $\frac{1}{2} \pi (2)^2 = 2\pi \approx 6.28$ square feet. The window with a 6-foot base has a semicircle on top whose radius is 3 feet. So the area of that semicircle is $\frac{1}{2} \pi (3)^2 = \frac{9}{2}\pi \approx 14.13$ square feet. Clearly, the wider window will let in more light.

The areas of the two Norman windows in the preceding problem are different by almost 8 square feet — a fairly large difference. You may be surprised to know that the perimeters of these two windows aren't nearly so different.

The Problem: Which has the greater perimeter: a Norman window with a 4-by-6-foot base or a Norman window with a 6-by-4-foot base?



The perimeter of a Norman window consists of the three sides of the rectangle and the circumference of the semicircle. The circumference of a full circle is found by multiplying the diameter by π . Because you only want half that area, you multiply half the diameter (the radius) by π . The Norman window with the base of 4 feet and sides of 6 feet has a semicircle with radius 2 feet, so the total perimeter is $4 + 6 + 6 + 2\pi = 16 + 2\pi \approx 22.28$ feet. The Norman window with the base of 6 feet and sides of 4 feet has a semicircle with radius 3 feet, so the total perimeter is $6 + 4 + 4 + 3\pi = 14 + 3\pi \approx 23.42$ feet. The perimeters of these two windows differ by just a little over 1 foot.

Making the window proportional

What if you want the area of the rectangular part of a Norman window to be equal to the area of the semicircular part? This arrangement just may be more esthetically pleasing to you.

The Problem: How long should the base of a Norman window be if the two sides of the rectangle are 2 feet high and the area of the rectangle is to be equal to the area of the semicircle?



First, let the length of the base be represented by x . Because the area of a rectangle is length times width, the area of the rectangle is $2x$. You want the area of the semicircle also to be equal to $2x$. The area of the semicircle is found by taking half the area of a circle whose radius is half the base of the rectangle, so the radius is $\frac{x}{2}$ and the area of the semicircle is $\frac{1}{2}\pi\left(\frac{x}{2}\right)^2$. Set the area of the rectangle equal to the area of the semicircle and solve for x .

$$\begin{aligned}
 2x &= \frac{1}{2} \pi \left(\frac{x}{2} \right)^2 \\
 2x &= \frac{1}{2} \pi \left(\frac{x^2}{4} \right) \\
 2x &= \frac{\pi x^2}{8} \\
 16x &= \pi x^2 \\
 0 &= \pi x^2 - 16x \\
 0 &= x(\pi x - 16)
 \end{aligned}$$

Two different values of x satisfy the equation. Obviously, $x = 0$ isn't going to work — it's extraneous. But setting the factor in the parentheses equal to 0, you get that $x = \frac{16}{\pi}$, which is about 5.1 feet.

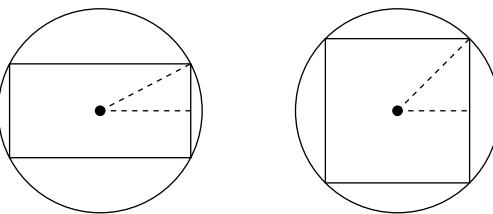
Fitting a Rectangular Peg into a Round Hole

An old adage says that you can't fit a square peg into a round hole. (Or is it a round peg in a square hole?) Mathematicians must have found this statement to be a challenge. No, they still couldn't fit the peg into the hole, but the statement opened up all sorts of questions and answers as to how large a round peg could fit into a square hole or how large a rectangular peg could fit into a semicircular hole. Oh, the possibilities!

Putting rectangles into circles

A rectangle has a set length and width and doesn't fit very neatly into a circle. See Figure 19-11 for some examples of rectangles working their ways into circles.

Figure 19-11:
The circle
has to be
bigger
than the
rectangle.



Just how large a rectangle will fit into a circle, if you have some particular constraints?

The Problem: You want to fit a rectangle with length 8 inches into a circle with radius 5 inches. What width will the rectangle have if each of the vertices of the rectangle are to be on the circle?



Take advantage of the symmetry of a rectangle, the radius of the circle, and good old Pythagoras. First, a rectangle's "center" is where the two diagonals (from opposite corners) intersect. The center of the rectangle is also where the center of the circle is. The radius of this circle is 5 inches, so that's the distance from the center of the rectangle to any of its vertices, because the vertices are on the circle. The length of the rectangle is 8 inches, so a segment drawn from the center of the rectangle horizontally to one of the sides is half that, or 4 inches. Refer to the left circle in Figure 19-11 and you'll see the radius and segment drawn in. The horizontal segment is perpendicular to the side of the rectangle — using the symmetry and angle measures of a rectangle — so the triangle formed is a right triangle. Solving for the length of the third side of the right triangle (I'll call the length b) using the Pythagorean theorem, you get that $4^2 + b^2 = 5^2$ or $b^2 = 25 - 16 = 9$. The length of the third side of the triangle is 3 inches, so the length of the entire side of the rectangle is 6 inches. You can fit an 8-by-6-inch rectangle into a circle whose radius is 5 inches.

A rectangle measuring 8 by 6 inches has an area of 48 square inches. A rectangle with these dimensions is not the largest rectangle that you can fit into a circle with a radius of 5 inches. In calculus, you prove that the largest rectangle that fits into a circle is really a square. So, if that's the case, what size square fits into a particular circle?

The Problem: What is the length of any side of a square that fits into a circle whose radius is 5 inches?



Use the diagonal of the square and the Pythagorean theorem to solve this problem. The diagonal of the square is the same as the diameter of the circle. A circle with a 5-inch radius has a 10-inch diameter. Let the lengths of the sides of the square be x inches long. Because the angles of a square are all right angles, you have a right triangle whose hypotenuse is 10 inches long and whose sides are each x units long. Filling in values in the Pythagorean theorem and solving for x ,

$$\begin{aligned}x^2 + x^2 &= 10^2 \\2x^2 &= 100 \\x^2 &= 50 \\x &= \sqrt{50} = 5\sqrt{2} \approx 7.07\end{aligned}$$

So a square with sides measuring about 7.07 inches is the largest that will fit into the circle. The area of this square is about 49.98 square inches — larger than the rectangle measuring 8 by 6 inches.

Working with coordinate axes



Circles, rectangles, and squares are easily described using the coordinate axes and some points and equations. The distance formula for the coordinate plane allows you to solve for lengths of segments if you have the values of the coordinates at either end.

Using coordinate axes,

- ✓ The equation of a circle with its center at the origin is $x^2 + y^2 = r^2$, where r is the radius of the circle.
- ✓ The distance between the two points (x_1, y_1) and (x_2, y_2) is found with the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

In Figure 19-12, you see a triangle drawn inside a semicircle. Even though a semicircle is only half a full circle, the equation for the circle is used when solving problems involving one of its semicircles.

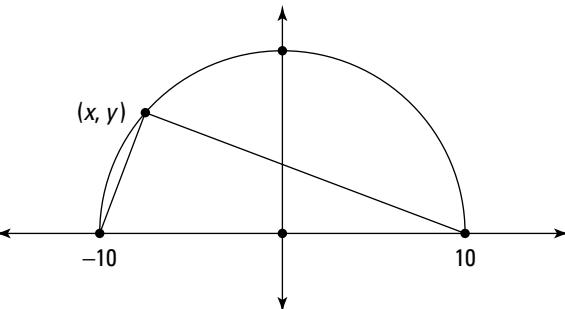


Figure 19-12:
The base of
the triangle
rests on
the axis.

The Problem: A rectangle is inscribed inside a semicircle (its vertices are all on some part of the semicircle). If the semicircle has its center at the origin and a radius of 5 units, then what are the coordinates of the vertices (x,y) in the first quadrant if the length of the rectangle is four times its height?



The y coordinate of the vertex you want also represents the height of the rectangle. The x coordinate is actually half the length, because the x is just the distance from the origin to the right. So the length of the rectangle is represented by $2x$. Writing that *the length is four times the height*, the equation is $2x = 4y$, which simplifies to $x = 2y$.

The equation of a circle with its center at the origin and with a radius of 5 is $x^2 + y^2 = 25$. Replacing the x in this equation with $2y$ and solving for y ,

$$\begin{aligned} (2y)^2 + y^2 &= 25 \\ 4y^2 + y^2 &= 25 \\ 5y^2 &= 25 \\ y^2 &= 5 \\ y &= \pm\sqrt{5} \end{aligned}$$

You use only the positive value for y . Because $x = 2y$, then x must be $2\sqrt{5}$ and the coordinates of the vertex are $(2\sqrt{5}, \sqrt{5})$.

Any triangle that has two of its vertices at the endpoints of a diameter and the third vertex on the circle is a right triangle. You don't prove this fact here, but the next problem allows you to see, from the coordinates of the points, that the rule holds.

The Problem: A triangle has one side along the flat part of a semicircle, with the endpoints of that side of the triangle at the endpoints of that diameter (refer to Figure 19-12). The radius of the semicircle is 10 units. What are the coordinates of the vertex of the triangle if it lies on the circle, and the distance from that vertex to the left endpoint is $2\sqrt{10}$?



You see that the endpoints of the diameter lie along the x -axis. If the radius of the semicircle is 10 units, then the coordinates of the endpoints of the diameter are $(-10, 0)$ and $(10, 0)$. The left endpoint is at $(-10, 0)$. Use the distance formula, and set that distance equal to the value you get by substituting in the points (x, y) and $(-10, 0)$ into the formula.

$$\begin{aligned} 2\sqrt{10} &= \sqrt{(x - (-10))^2 + (y - 0)^2} \\ &= \sqrt{(x + 10)^2 + y^2} \end{aligned}$$

You see both xs and ys in the equation. Use the equation of the circle with its center at the origin and a radius of 10 by solving for y^2 . Then you can substitute the equivalence for y^2 into the equation and solve for x .

$$\begin{aligned} x^2 + y^2 &= 100 \\ y^2 &= 100 - x^2 \end{aligned}$$

Substituting, squaring both sides, and solving for x ,

$$\begin{aligned}2\sqrt{10} &= \sqrt{(x+10)^2 + 100 - x^2} \\(2\sqrt{10})^2 &= (\sqrt{(x+10)^2 + 100 - x^2})^2 \\4 \times 10 &= (x+10)^2 + 100 - x^2 \\40 &= x^2 + 20x + 100 + 100 - x^2 \\40 &= 20x + 200 \\-160 &= 20x \\-8 &= x\end{aligned}$$

The x -coordinate is -8 . Substituting back into the equation for the circle, you get that $y^2 = 100 - (-8)^2 = 100 - 64 = 36$. Taking the square root of each side, you get that y is either $+6$ or -6 . The point you're looking for is above the x -axis, so you want the $+6$. The coordinates of the vertex are $(-8, 6)$.

Back to the fact that you have a right triangle here, you can show that it's true by using the Pythagorean theorem on the lengths of the sides of the triangle. The side along the x -axis is 20 units long. The side that goes from the point $(-8, 6)$ to $(-10, 0)$ is $\sqrt{40}$ units long. (You can check me by using the distance formula.) And the side that goes from the point $(-8, 6)$ to $(10, 0)$ is $\sqrt{360}$ units long. Plugging these values into the Pythagorean theorem,

$$\begin{aligned}(\sqrt{40})^2 + (\sqrt{360})^2 &= 20^2 \\40 + 360 &= 400\end{aligned}$$

Chapter 20

Volumizing and Improving Your Surface

In This Chapter

- ▶ Working with surface areas and volumes of solid figures
 - ▶ Considering practical applications with prisms
 - ▶ Doing what you *can* with a cylinder
-

Three-dimensional figures are a part of our life, so they are measured, weighed, and stacked. A prism is a solid figure with a top and bottom that are exactly the same. Pyramids and cones come to a single point or vertex.

In this chapter, you see how problems are posed and answered by considering the various solid figures and their properties. The surface area of a solid figure is a measure of the number of squares covering the outside, and the volume of a solid figure is a measure of the number of cubes you can fit inside. The respective formulas for surface area and volume are a big help, but just as important is some common sense and being able to picture the situation and the solid figure being used.

The Pictures Speak Volumes

The volume of a prism is found by multiplying the area of the base of the prism by its height. The top and bottom or two bases of a prism are always congruent (exactly the same). The sides are always rectangles. The prisms being considered in this chapter are technically *right rectangular prisms*, meaning that the sides are perpendicular to the bases — the sides don't slant like the Leaning Tower of Pisa — and the bases are parallel to one another.

Boxing up rectangular prisms

A rectangular prism has rectangles for its bases as well as its sides. Because all the surfaces of a rectangular prism are rectangles, you can actually turn it so that any pair of parallel sides are the bases. Pick your favorites.



The volume of a right rectangular prism is equal to the area of the base times the height. Because the base is also a rectangle, the volume formula is more conveniently written $V = lwh$, where l is the length of the base rectangle, w is the width of the base rectangle, and h is the height or the distance between the two bases.



The Problem: What is the height of a rectangular prism that has a square base 6 inches on a side and a volume of 288 cubic inches?

Use the formula for the volume of a prism replacing the V with the 288 cubic inches, the l with 6, and the w with 6. Then solve for h .

$$\begin{aligned} V &= lwh \\ 288 &= 6 \times 6 \times h \\ 288 &= 36h \\ \frac{288}{36} &= h, h = 8 \end{aligned}$$

The height is 8 inches.

The Problem: A rectangular prism has a length that's twice the width and a height that's twice the length. If its volume is eight times the height, then what is the volume?



You use the formula for the volume, even though you don't really have a numerical value to put in for V . The length, height, and volume can all be expressed in terms of the width. First, if *the length is twice the width* you write that $l = 2w$. Because the height is *twice the length*, you write $h = 2l = 2(2w) = 4w$. And the volume is *eight times the height*, so $V = 8h = 8(4w) = 32w$.

Now, using the formula for the volume, $V = lwh$, filling in the equivalences and solving for w ,

$$\begin{aligned} V &= lwh \\ 32w &= (2w)(w)(4w) \\ 32w &= 8w^3 \\ 4w &= w^3 \\ 0 &= w^3 - 4w \\ 0 &= w(w^2 - 4) = w(w - 2)(w + 2) \end{aligned}$$

The solutions for w in the equation are 0, 2, and -2. You discard the 0 and -2, because, even though they're solutions of the equation, they don't answer the question. Letting $w = 2$, you get that the volume is $32(2) = 64$ cubic units. The width of the prism is 2 units, the length is twice that or 4 units, and the height is 4 times the width, or 8 units.

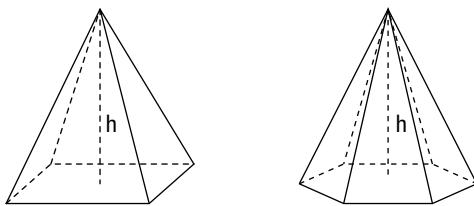
Venturing out with pyramids

A pyramid is a solid figure that has a *polygon* (a figure with segments for its sides) for a base and sides that are triangles — all meeting in one point called the *vertex*.



The volume of a pyramid is found by multiplying the area of the base of the pyramid by its height and taking one-third of that product. The height of a pyramid is the perpendicular distance from the center of the base up to the vertex. Figure 20-1 shows a square pyramid and a hexagonal pyramid. Traditionally, the name of the pyramid is determined by the shape of the base.

Figure 20-1:
Pyramids
are found in
Egypt and
jewelry
settings.



The Problem: The volume of a pyramid with a square base is 216 cubic inches. If the sides of the square base are 9 inches, then what is the pyramid's height?



Use the formula for the volume of a pyramid, $V = \frac{1}{3}Bh$ and replace the V with 216. The area of the base, B , is the area of a square. The area of a square is found by squaring the length of a side. If the sides of the square base are 9 inches, then the area of the base is 81 square inches. Replace the B in the formula with 81 and solve for h .

$$\begin{aligned} 216 &= \frac{1}{3}(81)h \\ 216 &= 27h \\ \frac{216}{27} &= h \\ h &= 8 \end{aligned}$$

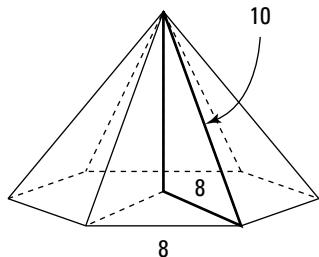
The pyramid is 8 inches high.



The Problem: A hexagonal-based pyramid has sides (of the base) measuring 8 inches. The length along any edge of the pyramid from a base corner to the vertex is 10 inches. What is the pyramid's volume?

You need to find the pyramid's height. Think of the height of the pyramid as being one side of a right triangle whose hypotenuse is the 10-inch edge. (See Figure 20-2 for a sketch of the pyramid and embedded right triangle.)

Figure 20-2:
The right
triangle's
hypotenuse
is an edge
of the
pyramid.



The hexagonal base is made up of equilateral triangles whose sides are all 8 inches, so the distance from the outer corner to the center of the hexagon is 8 inches. You have a right triangle whose hypotenuse is 10 inches with one leg of 8 inches. Using the Pythagorean theorem, you get that the measure of the other leg is 6 inches. That other leg is the height of the pyramid.

Now, armed with the height of the pyramid, 6 inches, you need to find the area of the base. Each of the six equilateral triangles making up the hexagonal base has sides of 8 inches. Use Heron's formula to find the area of one of the triangles and multiply that area by 6. (If you need a refresher on using Heron's formula to find the area of a triangle, refer to Chapter 10.)

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{12(12-8)(12-8)(12-8)} \\ &= \sqrt{12(4)^3} = \sqrt{12(64)} = \sqrt{3 \cdot 2^2 \cdot 8^2} \\ &= 16\sqrt{3} \end{aligned}$$

Six times this area is $96\sqrt{3}$. Now use the formula for the volume of a pyramid, and insert the area of the base and the height.

$$V = \frac{1}{3}(96\sqrt{3})6 = 192\sqrt{3}$$

The volume is about 333 cubic inches.



Fire-breathing dragon

A man is trapped in a cell that has only two possible exits. Through one of the exits, he'd have to pass through a room constructed from a magnifying glass; the blazing sun would instantly fry

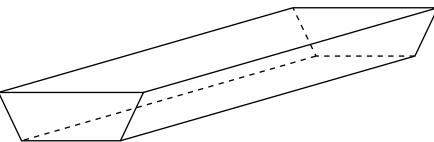
anyone trying to pass through it. The other exit opens onto a fire-breathing dragon. How can the man escape?

Answer: He just has to wait until nightime and go through the first door with the magnifying glass.

Dropping eaves with trapezoidal prisms

A trapezoid is a four-sided polygon with one side parallel to another side. A container with slanted sides so that the top or opening is larger than the bottom might have a trapezoidal cross-section. In Figure 20-3, you see a trapezoidal prism with the two parallel sides on the top and bottom. (Refer to Chapter 10 if you need a refresher on trapezoids and finding their area.)

Figure 20-3:
The
trapezoidal
prism lies on
its side.



The volume of a trapezoidal prism is equal to the area of its base times its height, $V = Bh$. Replacing the B , for the area of the base, with the formula for the area of a trapezoid, the formula for volume becomes

$V = \frac{1}{2} h_1(b_1 + b_2) \cdot h_p$. The subscripts t and p on the height measures are there to differentiate between the height of the trapezoid (the distance between the two parallel bases) and the height of the prism (the distance between the two trapezoidal bases).

The Problem: A watering trough for cattle is a trapezoidal prism. The trapezoid has a bottom base of 2 feet and an opening, at the top, of 3 feet. The trough is 2 feet deep. If the trough contains 60 cubic feet of water, how long is it?



You're looking for the height of the prism — the distance between the two trapezoidal bases on either end of the trough. Find the area of the trapezoidal base (end) and insert it into the formula for the volume. Replace the V in the formula with 60 and solve for the height of the prism.

The area of the trapezoid is $A = \frac{1}{2}(2)(2 + 3) = 1(5) = 5$ square feet. Replacing the V and B in the volume formula with 60 and 5, respectively, you get that $60 = 5h$. Dividing each side of the equation by 5, your answer is that the height of the prism (in this case, the length of the trough) is 12 feet.

Most homes have rain gutters attached at the edge of the roof to catch water coming down the roof surface; the water is then directed to a downspout. The gutters are often trapezoidal prisms where one side of the trapezoid is perpendicular to the two parallel sides so that it forms a flat attachment to the side of the house.

The Problem: An aluminum gutter in the shape of a trapezoidal prism is to go across the 32-foot front portion of a house. If the two parallel sides of the trapezoid are 6 inches and 9 inches, then how deep must the trapezoid be so that the gutter can hold 10 cubic feet of water?



The first chore is to change all the units so they're the same — right now they're in inches and feet. You could change everything to inches, but a cubic foot is $12 \times 12 \times 12$ cubic inches, and that makes the numbers a bit big to work with. Instead, you can change the 6 and 9 inches to feet and work with relatively nice fractions. The area of the trapezoidal end of the gutter is

$A = \frac{1}{2} h \left(\frac{1}{2} + \frac{3}{4} \right) = \frac{1}{2} h \left(\frac{5}{4} \right) = h \left(\frac{5}{8} \right)$ where the inches are changed to fractions of feet and the formula is simplified as much as possible. You're looking for the depth of the gutter, which is the height of the trapezoid in this case. Now use the formula for the volume of the prism, replacing the V with 10, the area of the base, B , with the result of the area of the trapezoid, and the height of the prism, h , with 32 feet. Then solve for h .

$$V = B \times h_p$$

$$10 = h \left(\frac{5}{8} \right) \times 32$$

$$10 = h \left(\frac{5}{8} \right) \times 32^4$$

$$10 = h(20)$$

$$\frac{10}{20} = h$$

The height (depth) of the gutter is half a foot, or 6 inches.

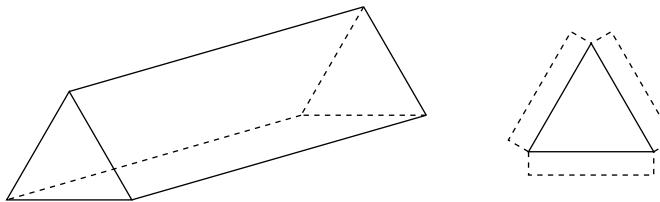
Mailing triangular prisms

You want to mail an umbrella to a friend. An umbrella is long and has a handle. You don't want to use a flat, rectangular prism, because the long package may get bent and damage the umbrella. A long tube or cylinder may work, but an even better idea is to use a triangular prism that has strength from the three sides and flatness to help pack up the umbrella and keep it from flopping about inside.



The surface area of a triangular prism is equal to the sum of the areas of the two bases (the triangles) plus the area of each of the three rectangular sides. Figure 20-4 shows you a triangular prism and, also, what the ends look like if they're extended for tabs so they can be glued to the sides of the prism.

Figure 20-4:
The prism is
formed by
folded
cardboard
and
triangular
ends.



The Problem: You want to mail an umbrella that's 3 feet long with a handle that's 5 inches wide. You choose a mailing container that's a triangular prism made of cardboard. The ends of the prism are equilateral triangles 6 inches on a side, and the length (height) of the prism is 2 inches longer than the umbrella. How much cardboard is needed for the container, if the sides of the triangles are extended by 1 inch for glue tabs?



Break this problem into three different area computations: the areas of the sides of the prisms, the areas of the two triangles that make up the bases, and then the areas of the glue tabs.

Each of the sides of the prism is a rectangle that's 3 feet, 2 inches long and 6 inches wide. The prism has three sides. Changing the length measure to inches, the three rectangles are 38 inches by 6 inches or 228 square inches in area. Three of them makes $3(228) = 684$ square inches.

The triangular bases are equilateral triangles measuring 6 inches on a side. Using Heron's formula (refer back to Chapter 10 for more on this formula), the area of one triangle is

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{9(9-6)(9-6)(9-6)} \\ &= \sqrt{9(3)^3} = \sqrt{9(27)} = \sqrt{3^2 \cdot 3^2 \cdot 3} \\ &= 9\sqrt{3} \end{aligned}$$

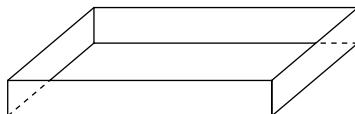
so two of them gives you $2(9\sqrt{3}) = 18\sqrt{3}$ square inches which is about 31 square inches.

The six tabs are each 6 inches by 1 inch or 6 square inches. The six tabs add a total of 36 square inches to the area. So the total area of the entire triangular prism is $684 + 31 + 36 = 751$ square inches.

Folding Up the Sides for an Open Box

A classic calculus problem involves taking a rectangular piece of paper or cardboard or metal, cutting equal squares from the four corners, and folding up the sides to construct an open box. Figure 20-5 shows you how this is done.

Figure 20-5:
Fold up the
sides and
glue the
edges
together.



The Problem: A 9-by-12-inch piece of cardboard is to be made into an open box by cutting slits into square-shaped corners, folding up the sides, and gluing the tabs to the ends. What is the volume of the open box if the corner squares are 2 inches on a side?



The volume of the right rectangular prism that's formed is found with the formula $V = lwh$. The height, h , of the prism is 2 inches — the size of the squares at the corners. The length of the prism is 12 inches minus the two squares in the corners; $12 - 4 = 8$ inches for the length. The width of the prism is

9 inches minus the two corners or $9 - 4 = 5$ inches. The volume of the prism is $V = 8(5)(2) = 80$ cubic inches.

In the calculus problem involving the open box, you get to determine the size of squares in the corners that gives you the largest possible volume. If the squares are small, then the box isn't very deep, but the area of the base is big. If the squares are large, then the box is deep, but the area of the base is small. Calculus allows you to balance the depth and the base area to find the best dimensions. In the next problem, you get to do somewhat the same process with a table of the possible values.

The Problem: A sandbox is to be formed from a 9 foot square piece of metal by cutting equal squares from the corners and folding up the sides. The edges are then welded together to form the box. What size squares should be cut from the corners to form a box that will hold the greatest amount of sand possible (have the greatest volume)?



First, determine the dimensions of the box in terms of the length in feet of the sides of the squares, x . The height is, of course, x . The length and width are both 9 minus the two squares (one on either end) or $9 - 2x$ feet. The volume of the sandbox is $V = lwh = (9 - 2x)(9 - 2x)x$. In Table 20-1, you see measures for x in half-foot increments and the resulting volumes.

Table 20-1

The Volumes of Sandboxes

| x (Height of Sandbox) | $9 - 2x$ (Length and Width of Sandbox) | $(9 - 2x)(9 - 2x)x = V$ |
|-------------------------|--|-----------------------------|
| 0.5 foot | 8 feet | $8(8)(0.5) = 32$ cubic feet |
| 1 foot | 7 feet | $7(7)(1) = 49$ cubic feet |
| 1.5 feet | 6 feet | $6(6)(1.5) = 54$ cubic feet |
| 2 feet | 5 feet | $5(5)(2) = 50$ cubic feet |
| 2.5 feet | 4 feet | $4(4)(2.5) = 40$ cubic feet |

As you see, after 1.5 feet for the height, the volume is decreasing as the squares get bigger. In half-foot increments, the greatest possible volume is obtained when the sandbox is 1.5 feet deep and 6 feet on a side. Using calculus to solve this problem and find the greatest volume you get the same answer. (Of course, I picked a very convenient size box to demonstrate this.)

Following Postal Regulations

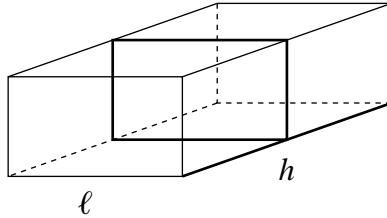
Postal services have regulations regarding the weight and size and contents of packages that you take to be mailed. You may remember taking a package in and watching the clerk wrap a chain around the middle of the box and then holding the remainder of the chain along the height. Regulations dictate that the height plus the *girth* of the box can't exceed 108 inches. The *girth* is that measure around the two shorter dimensions.



The height plus the girth of a rectangular prism is equal to $h + 2l + 2w$.

Refer to Figure 20-6 for a picture of how the height and girth are measured.

Figure 20-6:
The girth is
the distance
around the
middle.



Finding the right size

With the limitations on the total of the height plus the girth, you have to be careful what size package you prepare to send. Everything has to fit into a carton that passes muster. It takes a balance between a package being wide enough and long enough but not too much of either.

The Problem: How tall a box will the postal service accept if it has a base that's 16 inches by 18 inches and if the height plus the girth can't exceed 108 inches?



The measures of 16 and 18 inches represent the width and length of the base. Put these measures into the formula $h + 2l + 2w$ and set the expression equal to 108. Then solve for h .

$$\begin{aligned} h + 2(16) + 2(18) &= 108 \\ h + 32 + 36 &= 108 \\ h + 68 &= 108 \\ h &= 40 \end{aligned}$$

A height of 40 inches is a fairly long box — that's over 3 feet!

If you’re sending something that will fit into any size box and are restricted only by the total volume, then you have more flexibility with the postal rules. For example, if you’re sending Ping-Pong balls or nails, the shape of the box doesn’t matter very much.

The Problem: You have to send wrapped candies that take up 1,000 cubic inches of space. Because of the weight of the candy, the rule is that the height plus the girth can’t exceed 50 inches. If the base of the box is a square, then what are the dimensions of the carton you’ll be using?



Write two equations: one about the volume and the other about the height and girth. Find the common solution or solutions to the system of equations. The volume of this box is found with $V = wwh$, because the base is a square. Replacing the V with 1,000, the equation becomes $1,000 = w^2h$. Write the equation involving the height and girth with $50 = h + 2w + 2w$ or $50 = h + 4w$. Solve the second equation for h , and you get $h = 50 - 4w$. Substitute this equivalence for h into the volume formula and solve.

$$\begin{aligned} 1,000 &= w^2 h \\ 1,000 &= w^2(50 - 4w) \\ 1,000 &= 50w^2 - 4w^3 \\ 4w^3 - 50w^2 + 1,000 &= 0 \\ 2w^3 - 25w^2 + 500 &= 0 \end{aligned}$$

The cubic equation can be factored. You get to use some more powerful algebra in the form of the rational root theorem and synthetic division to find the factors. (These topics are covered thoroughly in *Algebra II For Dummies*.)

$$\begin{aligned} 2w^3 - 25w^2 + 500 &= 0 \\ (w - 10)(2w^2 - 5w - 50) &= 0 \end{aligned}$$

The first factor gives you the answer that $w = 10$. The second factor doesn’t simplify any farther, so you need the quadratic formula to find the solutions.

$$\begin{aligned} w &= \frac{5 \pm \sqrt{25 - 4(2)(-50)}}{2(2)} \\ &= \frac{5 \pm \sqrt{425}}{4} \approx 6.4 \text{ or } -3.9 \end{aligned}$$

Discard the negative number and just consider the solution that $w = 6.4$. It appears that two different sets of dimensions satisfy the requirements that the box contain 1,000 cubic inches of space and that the height plus the girth doesn’t exceed 50 inches.

If $w = 10$, then $h + 4w = 50$ yields $h + 40 = 50$ and $h = 10$. The box is a perfect cube that's 10 by 10 by 10.

If $w = 6.4$, then $h + 4w = 50$ yields $h + 25.6 = 50$ and $h = 24.4$. The box is roughly 6.4 by 6.4 by 24.4. The choice is which works better for the supplier and the customer.

Maximizing the possible volume

If the mailing regulations dictate that you can't send a rectangular box whose height plus girth exceeds 108 inches, then how can you make the most of that 108 inches? What's the greatest volume you can create?

The Problem: What are the dimensions of a rectangular prism (box) with a square base that has the greatest possible volume if the height plus the girth can be at most 108 inches?



Write an expression for the volume using the relationship that $h + 4w = 108$. The base of the box is a square, so w represents both the length and the width. Solving for h , you get $h = 108 - 4w$. Substitute that into the volume formula, and you have $V = w^2(108 - 4w)$. Create a table, trying out different values for w and seeing what volumes you obtain (see Table 20-2).

Table 20-2 **Maximizing the Volume in a Box**

| w (Width) | $108 - 4w$ (Height) | $V = lwh = w^2(108 - 4w)$ |
|-------------|---------------------|--|
| 1 | 104 | $1(104) = 104$ cubic inches (but a silly size for a box) |
| 2 | 100 | $4(100) = 400$ cubic inches |
| 3 | 96 | $9(96) = 864$ cubic inches (still an awfully long box) |
| : | : | : |
| 16 | 44 | $256(44) = 11,264$ cubic inches |
| 17 | 40 | $289(40) = 11,560$ cubic inches |
| 18 | 36 | $324(36) = 11,664$ cubic inches |
| 19 | 32 | $361(32) = 11,552$ cubic inches |

The volumes are starting to decrease, so you settle on the 18-inch square base and height of 36 inches to obtain the greatest possible volume. Are you wondering how I knew to jump to the higher numbers and skip up to where

the width is 18 inches? The answer is simple: calculus! I got impatient with putting in values and used calculus to tell me the exact answer — and then just looked like a hero by going to the right spot.

Making the Most of a 12-Ounce Can

Cylindrical shapes are found in grocery stores, at oil refineries, and attached to space shuttles. A cylinder has circles for its top and bottom and a rectangle wrapped around its sides. The circular and rectangular shapes lend themselves to the formulas needed to find the volume and surface area of a cylinder. The volume of a cylinder is found by multiplying the area of the circular base times the height of the cylinder (the distance between the circles). The surface area is found by adding the areas of the two circles to the area of the rectangle; the width of the rectangle is the same as the circle's circumference.



The formula for the volume of a cylinder is $V = \pi r^2 h$ where r is the radius of the circular base and h is the height between the bases or circles. To find the surface area of a cylinder, use $SA = 2\pi r^2 + 2\pi r h = 2\pi r(r + h)$.

Filling a cylindrical tank

The volume of a cylindrical tank may be measured in cubic feet or cubic yards, or you may get to work with liquid measures such as gallons or quarts. Equivalences are used to change from one unit to another.



Equivalences:

1 cubic foot \approx 7.481 gallons

1 gallon \approx 0.134 cubic foot

1 ounce \approx 1.805 cubic inches

The Problem: A cylindrical tank has a height that's equal to its diameter and it contains $128,000\pi$ cubic feet of liquid. What is the height of the tank?



Use the formula for the volume of a cylinder, replacing the V with the number of cubic feet. The radius of a circle is half the diameter, so, if the height is equal to the diameter, then the height is equal to twice the radius. Replace the h in the volume formula with $2r$ and solve for r .

$$\begin{aligned}
 V &= \pi r^2 h \\
 128,000\pi &= \pi r^2(2r) \\
 128,000\pi &= 2\pi r^3 \\
 \frac{128,000\pi}{2\pi} &= r^3 \\
 64,000 &= r^3 \\
 40 &= r
 \end{aligned}$$

The radius of the tank is 40 feet, so the height is 80 feet.

The Problem: A cylindrical tank is to be filled with vanilla ice cream. The tank is 6 feet tall and has a radius of 4 feet. How many gallons of ice cream will it take to fill the tank?

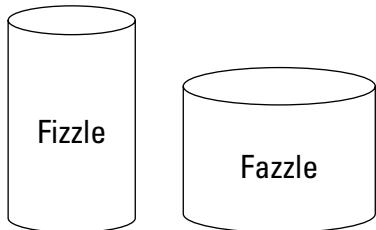


First, find the volume in terms of cubic feet. Then use the equivalence for changing cubic feet to gallons to determine the amount of ice cream needed. The volume of the tank is $V = \pi(4)^2(6) = 96\pi \approx 301.44$ cubic feet. One cubic foot is equal to about 7.481 gallons, so multiply 301.44 by 7.481 to get 2,255.07 gallons of vanilla ice cream. Yum!

Economizing with the surface area

You see 12-ounce containers all over the place in the form of cans for soft drinks, fruit beverages, and more potent brews. The traditional 12-ounce container is about 4.5 inches tall and has a diameter of about 2.5 inches. The shapes and exact sizes vary a bit, but the size is pretty standard. When you go to the grocery store, you need to check the shelves and see how many different heights and diameters of cans all contain 12 ounces. Looks can be deceiving. In Figure 20-7, you see two cylinders drawn to the approximate size of the traditional 12-ounce container and the most *efficient* 12-ounce container. What do you think?

Figure 20-7:
The cans hold the same amount of fluid.



The Problem: A 12-ounce beverage can has a diameter of 2.5 inches and a height of 4.5 inches. The optimum size for a 12-ounce container (one that uses less material to contain the same amount of fluid) is one that has a diameter of 3 inches and a height of 3 inches. How much more material is used to produce the traditional beverage can?



Use the formula for surface area to determine the amount of material needed for each size can.

$$\text{Diameter } 2.5 = \text{radius } 1.25, \text{height } 4.5: SA = 2\pi r(r + h) = 2\pi(1.25)(1.25 + 4.5) \approx 45.14$$

$$\text{Diameter } 3 = \text{radius } 1.5, \text{height } 3: SA = 2\pi r(r + h) = 2\pi(1.5)(1.5 + 3) \approx 42.39$$

The difference in the amount of material is almost 3 square inches. Multiply that by millions of cans, and it's a significant amount of material.

One of the factors that must have determined the size and shape of the traditional 12-ounce container is the cost of materials. Before the days of pop-tops, you had to open the containers with can openers that puncture the top of the can. The material had to be a bit stronger and heavier — and was more expensive.

The Problem: What if the cost of the material on the sides of a 12-ounce container is 1¢ per square inch and the cost of the material on the top and bottom is 2¢ per square inch. How much more does it cost for the material in a cylinder that's 3 inches high with a 1.5-inch radius than it does for a cylinder that's 4.5 inches high with a 1.25-inch radius?



The side of a cylinder is a rectangle that's as long as the circumference of the circle and as high as the height of the cylinder. Multiply the length times the height to get the area. The two bases of the cylinder are circles. Find the area of one and double it. Multiply the areas times their respective costs.

The cylinder with a radius of 1.5 inches has a circumference of $2\pi r = 2\pi(1.5) \approx 9.42$ inches. Multiply that by the height, 3, and the area of the side is about 28.26 square inches. The circular bases have an area of $\pi r^2 = \pi(1.5)^2 \approx 7.07$ square inches. The total cost for the material is $28.26(1) + 2(7.07)(2) = 56.54$ cents.

The cylinder with a radius of 1.25 inches has a circumference of $2\pi r = 2\pi(1.25) \approx 7.85$ inches. Multiply that by the height, 4.5, and the area of the side is about 35.33 square inches. The circular bases have an area of $\pi r^2 = \pi(1.25)^2 \approx 4.91$ square inches. The total cost for the material is $35.33(1) + 2(4.91)(2) = 54.97$ cents.

The difference in the cost is about 1.5¢. Multiply that by millions of cans. . . .

Piling It On with a Conical Sand Pile

A cone has a circle for a base. The sides slant upward to meet at a single point called the vertex. The volume of a cone is one-third the volume of a cylinder that has the same base and same height.



The volume of a circular cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the cone and h is the height — the distance from the center of the circular base to the vertex or top of the cone.



The Problem: What is the height of a cone if its diameter is twice the height and the volume of the cone is 72π cubic inches?

Use the formula for the volume of a cone, replacing the V with 72π and the height, h , with r . Why replace the height measure with r ? The diameter of a circle is twice the radius, so the diameter is equal to $2r$. If the diameter is twice the height, then the diameter, $2r = 2h$. The length of the radius is equal to the height.

$$\begin{aligned} 72\pi &= \frac{1}{3}\pi r^2(r) \\ 72\pi &= \frac{1}{3}\pi r^3 \\ 3 \times 72\pi &= 3 \times \frac{1}{3}\pi r^3 \\ 216\pi &= \pi r^3 \\ \frac{216\pi}{\pi} &= \frac{\pi r^3}{\pi} \\ 216 &= r^3 \\ r &= 6 \end{aligned}$$

The radius is 6 inches. Because the radius and height are the same, the height is also 6 inches.

Have you ever tried to pile sand? It doesn't cooperate all that well. The grains of sand don't stick together unless they're wet. When sand is poured from a container, it tends to form a cone-shaped pile, spreading out farther than it is high.

The Problem: Sand is falling off a conveyer belt and forming a conical shape as the falling sand runs down the sides of the pile. If the height of the pile is always one-third the diameter, then by how much does the volume of the pile change when the pile grows from 10 feet tall to 12 feet tall?



Find the volume of a pile of sand that's 10 feet tall and compare it to the volume of a pile of sand that's 12 feet tall. The pile of sand that's 10 feet tall has a diameter of 30 feet — which means a radius of 15 feet. The pile of sand that's 12 feet tall has a diameter of 36 feet — or a radius of 18 feet.

$$\text{10-foot pile: } V = \frac{1}{3} \pi (15)^2 (10) \approx 2,355 \text{ cubic feet}$$

$$\text{12-foot pile: } V = \frac{1}{3} \pi (18)^2 (12) \approx 4,069 \text{ cubic feet}$$

The pile of sand has grown by over 1,700 cubic feet.

Part V

The Part of Tens

The 5th Wave

By Rich Tennant



"We all know it's a pie, Helen.
There's no need to pipe the
equation 3.141592653 on the top."

In this part . . .

Here you find ten of the most traditional, classic brain-teasers. Tackle and vanquish these problems, and you'll rule the world. (Well, that may be a bit strong, but you can certainly claim dominion over your block.) After that, find out how even *you* could become a mathematician. Sure you could! If the people listed here could conquer mathematics and make their mark, then why not you?

Chapter 21

Ten Classic Brainteasers

In This Chapter

- ▶ Introducing brainteasers involving practical applications
- ▶ Thinking outside the box
- ▶ Presenting ten tests of your true mettle

The math word problems that I present in earlier chapters of this book are categorized, grouped, and discussed for their common elements and special properties. I provide suggestions and procedures for handling the types of problems in each chapter.

The ten brainteasers in this chapter are some that you may have seen elsewhere, some that seem vaguely familiar, and others that are completely new to you. These brainteasers are meant to be fun and mind boggling at the same time. Try to work them out before you peek at the answers. You'll feel so clever.

Three Pirates on an Island



Three pirates arrived on an island after successfully relieving a merchant of his bars of gold. The pirates put their booty in a pile in the center of the island and all fell asleep while guarding the gold. After a while, the first pirate woke up and decided to take his share of the gold and hide it. So he buried his fair share of the gold bars under a palm tree and went back to sleep. The second pirate woke up and took what he thought was *his* fair share and buried those gold bars next to a boulder. He then went back to sleep, too. Then the third pirate awoke, took what he thought was his fair share, and hid it under a boat. He went back to sleep. In the morning, all three pirates woke up and discovered that there were eight bars of gold in the pile. How many bars were in the pile in the beginning?

Answer: Working backward, the 8 bars that remained must have been two-thirds of what was there when the third pirate did the splitting up. So the third pirate saw 12 bars, took his 4, and left 8. If the second pirate left 12 bars, then 12 was two-thirds of what he saw, so he saw 18 bars — he took 6 and left 12. The first pirate left 18 bars, which was two-thirds of what was there in the beginning. One-third of 27 is 9, leaving 18 bars. There were 27 bars of gold when the pirates all went to sleep.

Letter Arithmetic



A father received yet another letter from his son asking for money. The father was tired of doling out the cash. So, instead of money, the father sent the following addition problem for his son to solve.

$$\begin{array}{r}
 S \quad E \quad N \quad D \\
 + \quad M \quad O \quad R \quad E \\
 \hline
 M \quad O \quad N \quad E \quad Y
 \end{array}$$

He said that if his son could figure out which digit each of the letters stood for, he would send his son another installment of cash. What does each letter represent to make this addition problem correct?

Answer: S = 9, E = 5, N = 6, D = 7, M = 1, O = 0, R = 8, Y = 2

$$\begin{array}{r}
 9 \quad 5 \quad 6 \quad 7 \\
 + \quad 1 \quad 0 \quad 8 \quad 5 \\
 \hline
 1 \quad 0 \quad 6 \quad 5 \quad 2
 \end{array}$$

You figure this out with a little trial and error and a smattering of reasoning things out. You're pretty sure that the letter M represents 1, because it's the carryover from adding the number that M represents to the number that S represents. If M represents 1, then S must represent 8 or 9 in order for the sum to be large enough to have a carryover. Work your way backward, trying out different digits for the different letters until you find the unique solution.

Pouring 4 Quarts



A farmer needs to add exactly 4 quarts of weed killer to the fertilizer mix that he's preparing to spread over his field. Unfortunately, he has only a 3-quart container and a 5-quart container, not a 4-quart container. He can't just guess. The accuracy of his measurement is most important. How can he measure exactly 4 quarts with the two containers he has? (There are several possible solutions, but the farmer wants the one that takes the fewest number of steps.)

Answer: The farmer fills the 5-quart container and empties 3 quarts from that container into the 3-quart container, leaving 2 quarts in the 5-quart container. Then he empties the 3-quart container and pours the 2 quarts from the 5-quart container into the 3-quart container, leaving room in the 3-quart container for another quart. He fills the 5-quart container again and pours 1 quart into the 3-quart container that already contains 2 quarts. That leaves exactly 4 quarts in the 5-quart container.

If this is a bit confusing, use Figure 21-1 to visualize it.

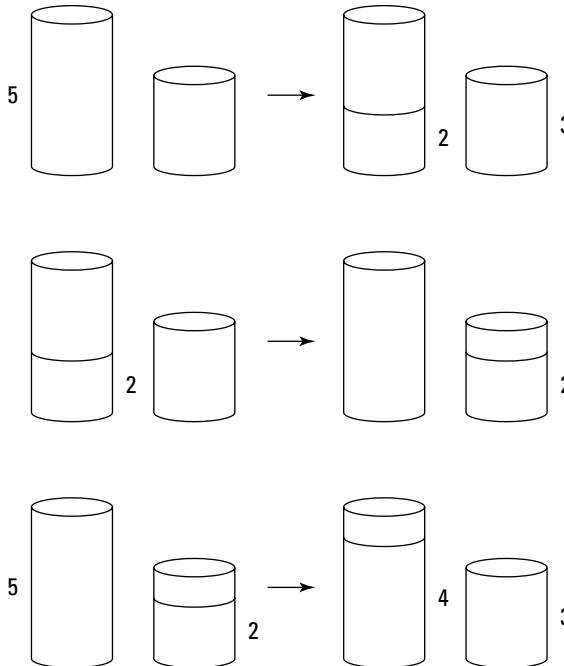


Figure 21-1:
Finding 4
quarts from
5 quarts and
3 quarts.

Magic Square



A magic square is a square array of numbers — three by three, four by four, and so on — such that the sum of the numbers in any column, row, or diagonal always adds up to the same number. Place the digits from 1 through 9 into Figure 21-2 so that each row, column, and diagonal adds up to 15.

| | | |
|--|--|--|
| | | |
| | | |
| | | |

Figure 21-2:
A three-by-
three magic
square.

Answer: Figure 21-3 is a solution for the magic square. You solve this pretty much by trial and error, but you also make some strategic moves, such as putting the number 5 in the middle, because it's the middle value in the list of digits. You need to put the larger digits in positions where they won't be added together. Play around with the numbers a bit, and you'll find the solution. That solution is a pattern for any other list of nine consecutive numbers to be put in a magic square and have a particular sum.

| | | |
|---|---|---|
| 6 | 1 | 8 |
| 7 | 5 | 3 |
| 2 | 9 | 4 |

Figure 21-3:
A solution
for the
magic
square.

Getting Her Exercise



Seven-year-old Caitlin lives on the 50th floor of her apartment building. Every morning she gets on the elevator outside her door and rides down to the ground level where she waits for the school bus to pick her up for school. Every evening Caitlin gets on the elevator on the ground floor, rides to the 40th floor, and walks up the last ten flights of stairs to her apartment. Why does she do this?

Answer: Caitlin can't reach the elevator button for the 50th floor — she can only reach as high as the button for the 40th floor. So she takes the elevator to the 40th floor and walks the rest of the way.

Liar, Liar



You approach two doors. One leads to a fabulous prize, and the other leads to a pile of coal that needs shoveling. If you pick the door to the fabulous prize, you get to take it home. If you pick the other door, you'll be busy for quite a while. Each door has a huge guard, protecting what's on the other side. The guard in front of the prize always tells the truth, and the guard in front of the coal always lies. You're allowed to ask just one question of one of the guards to determine which door leads to the prize. What question do you ask to determine which door goes to the prize?

Answer: You ask a guard, "If I were to ask the other guard which door leads to the prize, what would he say?" And then pick the opposite door of what he tells you.

If you ask that question of the guard who always tells the truth (and who stands in front of the prize), he'll know that the other guard lies and would have answered that you should pick his door (the one with the coal). So the guard telling you the truth will tell you, honestly, that the other guard would get you to pick the door in front of the coal.

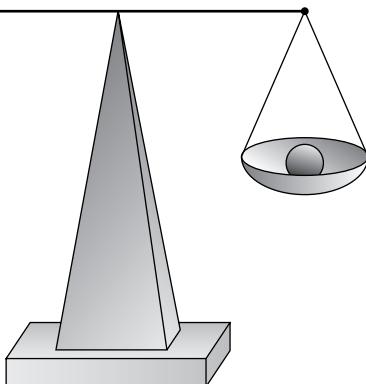
If you ask the question of the guard who always lies (and who stands in front of the coal), he lies and say that the truth-speaking guard would tell you to pick his (the liar's) door. So, in either case, you want to pick the opposite door.

Weighing Nine Nuggets



You're given nine identical-looking nuggets and are told that eight of them are fool's gold and the ninth is real gold. The nuggets are so close in weight that you can't tell, by holding them in your hand, which is the heavier nugget. You have a balance scale (shown in Figure 21-4) that you're allowed to use exactly twice. You can put as many nuggets in the pans of the scale as you want. How can you single out the heavier nugget?

Figure 21-4:
A balance scale is level when the items weigh the same.



Answer: Put three nuggets in one tray and three nuggets in the other tray. If the scale is balanced, then you know that one of the three nuggets not on the scale is the heaviest. If one side of the scale is heavier, then you know that the gold nugget is one of those three. In any case, you've isolated a group of three nuggets that contains the gold nugget. Do another weighing with the three nuggets, putting one nugget on each tray. If the trays are the same height, then the nugget not being weighed is the heaviest and is the gold. If one side is heavier, then you've spotted your gold nugget in that tray.

Where Did the Dollar Go?



Monica, Phoebe, and Rachel went out for lunch together. The waitress brought them a bill for \$30, so each one paid her \$10. When the waitress took the cash to the register, she realized that there'd been an error, and she should have charged them only \$25. So, when she brought the change back, Monica, Phoebe, and Rachel each took \$1 and left the other \$2 as a tip (they hadn't read Chapter 6 of this book and didn't know how to figure out a fair tip). Because each friend paid \$10 and got \$1 back, each person actually paid \$9. Multiply \$9 by 3 and add the \$2 tip, and you get $\$27 + \2 or \$29. What happened to the other dollar?

Answer: This puzzle definitely belongs in the category of *sleight of hand*. Monica, Phoebe, and Rachel actually paid only \$9 each, totaling \$27. The bill was \$25, so subtract \$25 from \$27 and you have the \$2 left to leave for a tip.

How Many Weights?



A merchant has a balance scale (refer to Figure 21-4) and wants to be able to weigh any item between 1 and 50 pounds, to the nearest pound. How many different weights does he need to buy?

Answer: He only needs six different weights to weigh anything from 1 to 50 pounds. The weights are: 1 pound, 2 pounds, 4 pounds, 8 pounds, 16 pounds and 32 pounds. Any number from 1 through 50 (actually, you can go up to 63 pounds with these weights) can be created using combinations of the weights.

Transporting a Fox, a Goose, and Corn



A farmer has a dilemma. She needs to transport her fox, her goose, and her corn across a small river. She only has room for herself and one of the others in her small boat. She'll have to take one across the river, leave it, and go back for the others. But if she leaves the fox alone with the goose, the fox will eat the goose. If she leaves the goose alone with the corn, the goose will eat the corn. How can she get everything across without someone being eaten?

Answer: For simplicity's sake, let the fox be represented by F, the goose by G, and the corn by C. I map it out for you in Table 21-1. The arrows indicate the direction of the boat.

Table 21-1 How the Farmer Gets Her Load across the River

| <i>Step</i> | <i>One Side of the River</i> | <i>Crossing the River</i> | <i>Other Side of the River</i> | <i>Description</i> |
|-------------|------------------------------|---------------------------|--------------------------------|---|
| 1 | FGC | | | All three are on one side of the river with the farmer. |
| 2 | FC | G→ | | She takes the goose across. |
| 3 | FC | ← | G | She leaves the goose and goes back alone. |
| 4 | F | C→ | G | She brings the corn across. |
| 5 | F | ←G | C | She leaves the corn and takes the goose back. |

(continued)

Table 21-1 (continued)

| <i>Step</i> | <i>One Side of the River</i> | <i>Crossing the River</i> | <i>Other Side of the River</i> | <i>Description</i> |
|-------------|----------------------------------|-------------------------------|------------------------------------|---|
| 6 | G | F→ | C | She leaves the goose and takes the fox across. |
| 7 | G | ← | FC | She leaves the fox and goes back alone. |
| 8 | | G→ | FC | She brings the goose back across. |
| | | | FGC | The farmer is on the other side with all three. |

Chapter 22

Ten Unlikely Mathematicians

In This Chapter

- ▶ Identifying a president famous for a mathematical proof
- ▶ Naming a world conqueror who did a proof while in exile
- ▶ Pointing out some contemporary figures

Mathematics has inspired so many people over the centuries. Picture a future U.S. president sitting through a wonderfully exciting meeting of Congress. Was he paying attention? No, he was discovering a proof of the Pythagorean theorem, of course! Other mathematicians came from wealthy families or monasteries. Mathematicians are found in all walks of life. Anyone can be a mathematician with just a bit of curiosity, respect for the basic mathematical rules, and willingness to stick her neck out a bit.

Pythagoras

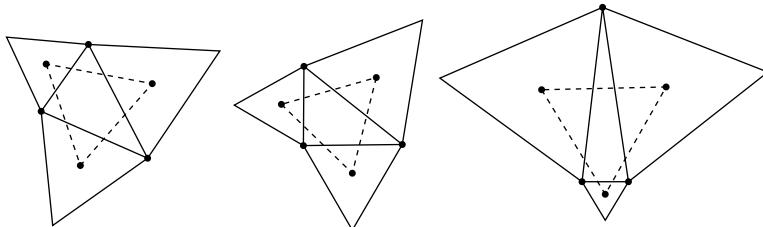
How can Pythagoras be an *unlikely* mathematician when practically everyone knows his name? His contributions to the world of music and mathematics are legendary, although he would have preferred to keep his discoveries a secret. Pythagoras was a prophet and mystic. He established a secret society that had a strict code of conduct. Members of the society were vegetarians who were forbidden from eating beans (lentils). The Pythagoreans, as the society members were called, believed that odd numbers had male attributes and even numbers had female attributes. They based their worship on numerology and let it influence their way of life. The Pythagoreans believed that all numbers were *rational* (could be written as a fraction), even though the Pythagorean theorem belies this with its need for numbers under *radicals* (irrational numbers).

Napoleon Bonaparte

Emperor Napoleon Bonaparte is known for many things, but not many people know how much he contributed to mathematics. Napoleon was always a great supporter of mathematical inquiry and promoted mathematical study whenever he could. One of his better-known contributions is his discovery that, if you construct an equilateral triangle on each side of any other triangle, the centers of those equilateral triangles are the vertices of another equilateral triangle.

Figure 22-1 shows you three different triangles — one scalene, one isosceles, and one right. According to Napoleon, you construct an equilateral triangle (one with all the sides the same measure) on each side of any kind of triangle — the equilateral triangle has its sides measuring the same as that particular side of the base triangle. When the centers of the equilateral triangles are connected by segments, you see that Napoleon's discovered another equilateral triangle formed in the middle.

Figure 22-1:
Napoleon
kept himself
busy while
in exile.



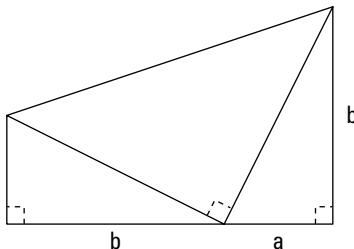
René Descartes

Descartes was born into a wealthy family and received a thorough, general education at a Jesuit college. He studied law for a while but wasn't really all that interested in it. For some years he traveled around with various military campaigns. Descartes wasn't really a professional soldier. He took time off from his accompaniment of military campaigns for some interesting travel and study. He is known as the father of modern philosophy. His most serious interest in mathematics may have coincided with wanting to stay warm. He was traveling with the Bavarian army during a cold, winter campaign, and chose to stay in bed until 10 a.m. thinking about mathematical problems. Doesn't sound like any army I know. In any case, Descartes made huge contributions to mathematics.

President James A. Garfield

President Garfield was the first ambidextrous U.S. president and the second president to be assassinated. He earned money to attend college by driving canal boat teams. He was a classics professor, then college president. He tired of the academic life, so he studied law and became a politician. While a member of the House of Representatives, in 1876, he discovered a novel proof of the Pythagorean theorem. Figure 22-2 shows his construction of a trapezoid, starting with a right triangle. His proof involves the areas of the triangles and trapezoid.

Figure 22-2:
President
Garfield's
proof.



President Garfield was shot in 1881 and died two months later of an infection. Alexander Graham Bell tried to find the bullet with a metal detector that he had invented, but failed — probably due to the metal in the president's bed frame.

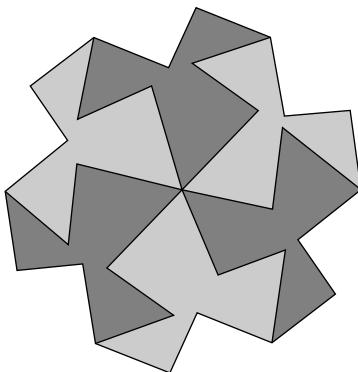
Charles Dodgson (Lewis Carroll)

More famous for his *Alice in Wonderland* and *Alice through the Looking-Glass*, Charles Dodgson is also well-known in mathematics circles for his work in probability, theory of elections, and algebra. Charles Dodgson studied at and later taught at Christ Church College, Oxford. He spent most of his time teaching and tutoring in algebra and developed study materials for students who were struggling with the material. His two *Alice* books have many references to mathematics and logic. Dodgson liked working with children and amused them with his stories and word games — including an early version of Scrabble.

M. C. Escher

M. C. Escher was a Dutch graphic artist. His works are easily recognizable because of the many impossible-looking constructions and titillating tessellations. Figure 22-3 shows an example of the type of tessellation that Escher might have produced. (A *tessellation* is a tiling or filling of the plane with figures that leave no gaps between them.)

Figure 22-3:
An Escher-like
tessellation.



Escher was born in 1903 in the Netherlands and lived in and traveled to many other countries as world and political events affected his life. Although he didn't have any formal mathematical training, his work has strong mathematical components including order and symmetry. His journey to the Alhambra in Spain resulted in his trying to improve upon those artworks using geometric grids as a basis for his own work. His art took many forms from his earliest work in 1937 until his death in 1972.

Sir Isaac Newton

Isaac Newton is probably best known for his discovery of the Law of Gravity, supposedly due to an apple falling on his head. Whether the apple story is true or not, his mathematical discoveries are even more remarkable, because most of his work was done during a two-year period when he had retired to the countryside to think and wait out the bubonic plague that was sweeping Europe. Even more startling is the fact that this two-year period ended with his 25th birthday.

Newton is recognized as the co-inventor of calculus. Both Newton and Gottfried Leibniz discovered calculus at the same time, and independently,

but Newton waited about 20 years to publish his findings, while Leibniz published almost immediately. Newton also discovered three laws of motion, the Corpuscular Theory of Light, and the Law of Cooling. He built telescopes and used them to calculate orbits of planets, but Newton's main interest was really alchemy. After his early discoveries in mathematics and physics, he really did little more to add to the knowledge in these fields.

Marilyn vos Savant

Marilyn vos Savant's column *Ask Marilyn* is found weekly in *Parade* magazine. She started writing this column after being featured in a *Parade* article for her high IQ and then responding to a selection of questions in a follow-up article. Marilyn solves mathematical and logical problems in her column and also answers questions on physics, philosophy, and human nature.

Perhaps you remember her response to what has been dubbed the "Monty Hall problem" (see the nearby sidebar). In a 1990 column, she responded that you'd have a better chance of winning if you switched doors. This lead to all sorts of responses from academics and other readers — much of it criticism of her answer. She was, of course, found to be correct.

Marilyn, in addition to writing her column, is Chief Financial Officer of Jarvik Heart and assists her husband, Robert Jarvik, with cardiovascular disease research.

The Monty Hall problem

Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car, and behind the other two doors are goats. You pick a door, say door #1, and the host, who knows what's behind the doors, opens another door, say door #3, which has a goat. He says to you: "Do you want to pick door #2?" Is it to your advantage to switch your choice of doors?

Marilyn's answer was that you should switch, giving you a two-thirds chance of winning rather than one-third, if you stayed with door #1.

Why was Marilyn correct? Consider the situation where the car is behind door #1 and goats are

behind doors #2 and #3. The game show host is always going to show you one of the goats.

- ✓ If you pick door #1, the host will show one of the goats behind door #2 or door #3, and if you switch, you lose.
- ✓ If you pick door #2, the host will show you door #3, so if you switch to door #1, you win.
- ✓ If you pick door #3, the host will show you door #2, so if you switch to door #1, you win.

You win two-thirds of the time if you switch. This same chance will appear, no matter where you put the car and where you put the goats.

Leonardo da Vinci

When you think of Leonardo da Vinci, you probably think first of the artist. As an artist, Leonardo turned to science as a means of improving his artwork. His study of anatomy and nature led to his remarkably realistic paintings. He was recognized as an inventor, scientist, engineer, musician, mathematician, astronomer, and painter.

His interest in the mathematics of art and nature led him to show how the different parts of the human body are related by the golden rectangle. Leonardo believed that artists should know the laws of nature as well as the rules of perspective.

Martin Gardner

Martin Gardner first came to my attention with his *Mathematical Games* column in *Scientific American*, which he wrote for about 25 years. In addition to this column, he has published over 60 books.

Martin Gardner grew up in Oklahoma, served in the U.S. Navy during World War II, and later earned his bachelor's degree in philosophy from the University of Chicago. He decided to try for a life as a freelance writer after selling a humorous short story to *Esquire*. His second sale was a story based on mathematical topology — a story in science fiction. He is considered to be almost single-handedly responsible for creating the interest in recreational mathematics in the later part of the 20th century. Some subjects that he has popularized are

- ✓ **Flexagon:** A *flexagon* is a hexagon made up from a long strip of equilateral triangles (most easily constructed from adding-machine tape). Folding and refolding reveals the three different faces of a *trihexaflexagon*, the six faces of a *hexahexaflexagon*, and so on.
- ✓ **Polyomino:** A *polyomino* is a grouping of squares — three, four, five, and so on — such that no grouping is the same shape as any other, even when the grouping is flipped or rotated.
- ✓ **Soma cubes:** *Soma cubes* are the three-dimensional versions of polyominoes. Cubes are stacked or otherwise connected, producing different shapes that can't be duplicated by any rotations or flips of the grouping.
- ✓ **Hex:** The game of *Hex* is played on a game board consisting of hexagons. Players take turns choosing hexagons (usually with different-colored game pieces) trying to form a path from one side of the board to the other.

- ✓ **Tangram:** A seven-piece *tangram* starts out as a square. The different pieces — triangles, squares, parallelogram, and so on — are rearranged to form other shapes and pictures.
- ✓ **Penrose tiling:** A *penrose tiling* consists of *rhombi* (a rhombus has all four sides the same measure) that appear to have no pattern or symmetry but that, in fact, have repeated patterns within the tiling.
- ✓ **Fractal:** A *fractal* is a geometric shape that can be continuously broken down into parts that are reduced copies of the original shape. The book *Jurassic Park* introduced the dragon fractal and referred to chaos theory. (I'll leave you with those two topics to search out on your own!)

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