Pattern Recognition and Machine Learning
Assignment 1 - HMM Signal Source

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1 Verify the MarkovChain and HMM Sources

1.1 Verify that $P(S_t = j)$, $j \in 1, 2$ for $t = 1, 2, 3...$

$$q = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}; A = \begin{bmatrix} 0.99 & 0.01 \\ 0.03 & 0.97 \end{bmatrix}; B = \begin{bmatrix} b_1(x) \\ b_2(x) \end{bmatrix};$$

We find the stationary probability distribution by solving the following system of equations:

$$\mathbf{p}^T = \mathbf{p}^T A \Leftrightarrow \mathbf{p} = A^T \mathbf{p} \tag{1}$$

$$\begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} 0.99 & 0.01 \\ 0.03 & 0.97 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \tag{2}$$

$$\left\{ \begin{array}{c} \pi_1 = 0.99\pi_1 + 0.01\pi_2 \\ \pi_2 = 0.03\pi_1 + 0.97\pi_2 \\ \pi_1 + \pi_2 = 1 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} 0.01\pi_1 = 0.03\pi_2 \\ \pi_1 + \pi_2 = 1 \end{array} \right. \Leftrightarrow \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} \tag{3}$$

1.2 Generate a sequence of T = 10000 state integer number from the test Markov chain

We thereafter calculate the relative frequency of occurrences of the two states, the result obtained was:

$P(S_t = 1) = 0.7448$
$P(S_t = 2) = 0.2552$
We see that the **rand**-function of the the Markov chain seem to work.

1.3 Verifying the HMM rand method

Here we use $b_1(x) \sim N(0, 1)$ and $b_2(x) \sim N(3, 2)$. We first calculate $E[X_t]$ and $var[X_t]$ theoretically:

$$\mu_X = E[X] = E_S[E_X[X|S]] = \sum_{s=1}^{2} P(S=s)E_X[X|S=s] =$$

$$= \sum_{s=1}^{2} P(S=s)E_X[b_s(X)] = 0.75*0+0.25*3 = 0.75$$

and

$$var[X] = E_S[var_X[X|S]] + var_S[E_X[X|S]] =$$

$$= \sum_{s=2}^{2} P(S=s)var_X[b_s(X)] + \sum_{s=2}^{2} P(S=s)(E_X[X|S=s]-\mu_X)^2 =$$

$$= 0.75*1+0.25*2^2+0.75*(0-0.75)^2+0.25*(3-0.75)^2 = 3.4375$$

1.4 Generating observation-sequences

Running the **rand** function of the HMM class generates values quite close to the theoretical ones, but varies a lot between runs. Because of this we couldn't really tell if it worked or not. Thus we increased the length of the sample sequence. We got the following measured results:

Mean	Variance	Samples
0.8432	3.6633	10000
0.7730	3.3913	10000
0.7748	3.5184	100000
0.7218	3.3652	1000000
0.7469	3.4305	10000000

We see that with 10000000 samples we get really close to the theoretical values.

1.5 HMM Behaviour

The HMM behaviour is displayed in Figure 2. We see that there clearly are two distributions, one that oscillates around the mean 0 and one that oscillates around the mean 3. As described by the stationary distribution we're mostly in the state with mean 0.

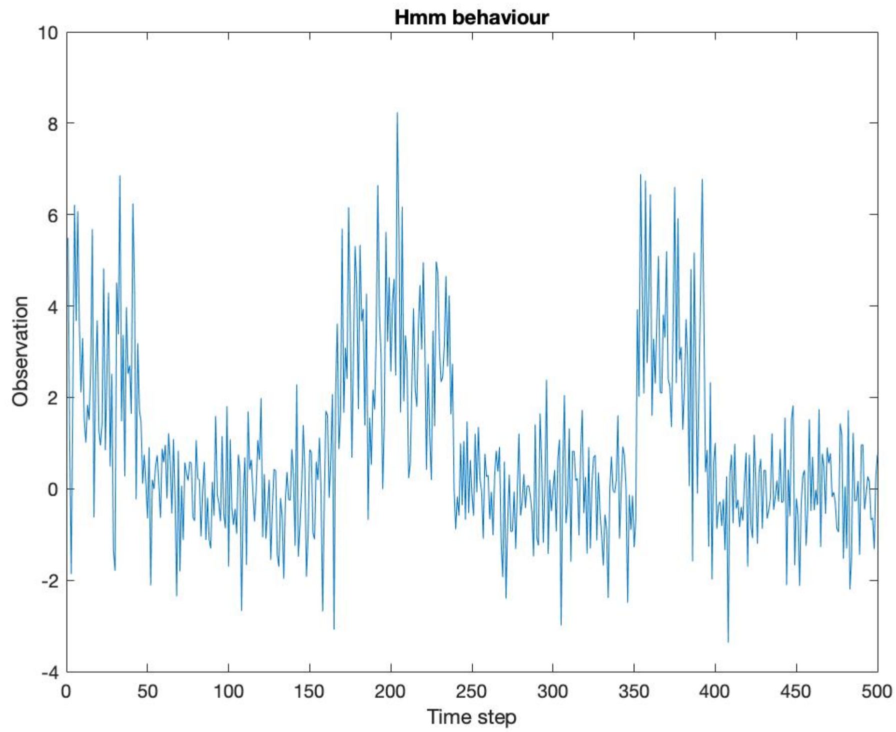

Figure 1: A series of 500 contiguous samples X_t from the HMM specified in section 1.1.

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1.6 HMM Behaviour 2

The HMM behaves similarly in such a way that they both transition between a less varying and a more varying state. The difference is that in this HMM the mean of the states are the same. This time it's harder to distinguish between the different states. The only difference is the larger variance in the second state, which means that outputs with high absolute value are more likely from the second state while the values closer to zero are more likely from the first state. Because of this it's possible to estimate the state sequence but the certainty will be low.

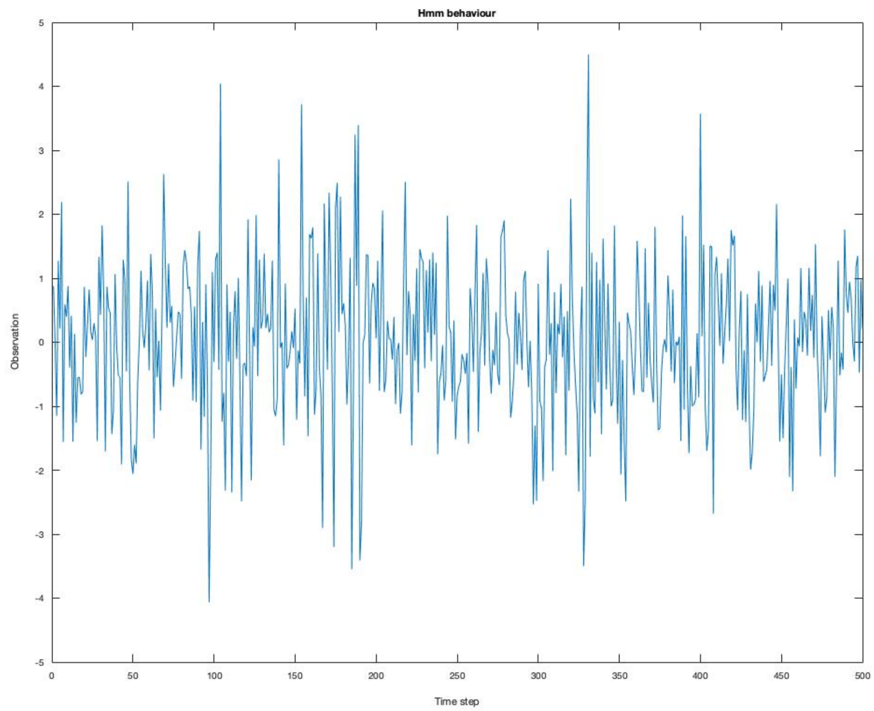

Figure 2: A series of 500 contiguous samples X_t from the HMM specified in section 1.1 but this time with $\mu_1 = \mu_2 = 0$.

1.7 Testing finite-duration HMM

We create finite duration HMM seen below. Note that there are 6 rows and 7 columns in the transition matrix, the 7th state is the exit state.

$$q = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; A = \begin{bmatrix} 0.7 & 0.3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.7 & 0.3 \end{bmatrix}$$

$$b_i(X_t) \sim N(\begin{bmatrix} i & i & i \end{bmatrix}, \begin{bmatrix} i & 0.5 & 0.5 \\ 0.5 & i & 0.5 \\ 0.5 & 0.5 & i \end{bmatrix})\ where\ i = 1, ..., 6.$$

Now generating T = 100 states and observations using the **rand**-function we get the states and observations displayed in Figure 3 and 4. We see that we never go right in the state-chain and it ends (by entering the exit state) before the specified number of samples are generated. For the observations the mean is increasing and also the variance. As described by the Gaussian distributions. Running it multiple times we get the following lengths of the test-sequences: 32, 15, 17, 27. The lengths seem reasonable since the expected length of the sequence should be $0.7/0.3 + 0.8/0.2 + 0.9/0.1 + 0.7/0.3 + 0.7/0.3 = 20$.

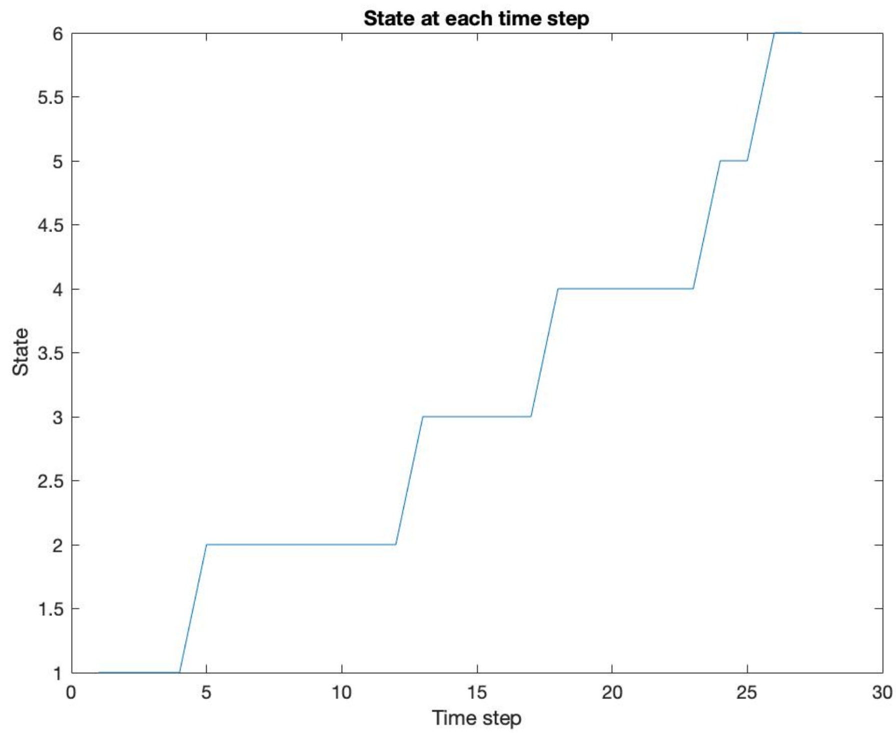

Figure 3: States generated by the HMM using the settings described above.

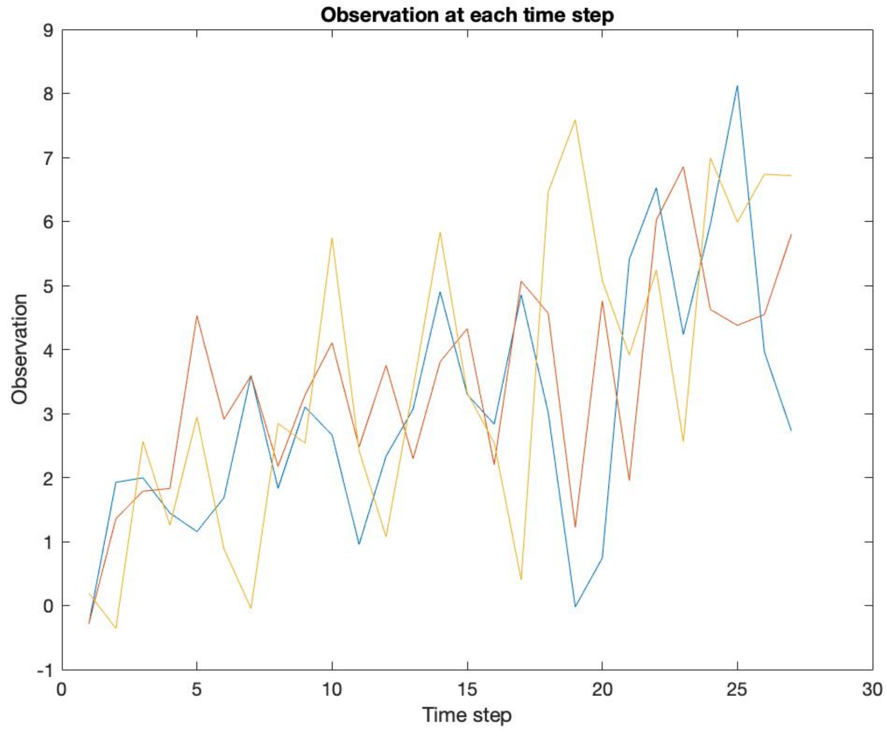

Figure 4: Observations generated by the HMM using the settings described above. Here each color describes the output of one specific element in the vector of generated observations.

1.8 Gaussian vector distributions

This was done in the finite duration HMM test above.