4.2

Let $\lambda \in \mathbb{R}^+$ and $N \in \mathbb{N}^+_*$.

Let $\forall i \in \{1, \dots, N\} \quad X_i$ be independent random variables st $X_i \sim Poisson(\lambda)$ i.e,

$$\forall i \in \{1, \dots, N\} \quad \forall k \in \mathbb{N}^+ \quad P[X_i = k \mid \lambda] = \frac{\lambda^k e^{-\lambda}}{k!}$$

Let us define $\underline{X} = (X_1, \dots, X_N)$. Because the $(X_i)_{\{1,\dots,N\}}$ are independent, we have that

$$P[\underline{X} \mid \lambda] = P[X_1, \dots, X_N \mid \lambda] = \prod_{i=1}^{N} P[X_i \mid \lambda]$$

4.2.a

For a given sample $\underline{x} = (x_1, \dots, x_N)$, we note the likelihood $P[X_1 = x_1, \dots, X_N = x_N \mid \lambda] = P[x_1, \dots, x_N \mid \lambda]$ for simplicity.
The maximum likelihood estimate $\hat{\lambda}_{ML}$ is the same as maximum the log-likehood estimate and the log-likehood function is concave in λ, therefore, we can obtain $\hat{\lambda}_{ML}$ by solving:

$$\frac{d \ln P[x_1, \dots, x_N \mid \lambda]}{d\lambda}(\hat{\lambda}_{ML}) = 0$$

We have

$$\ln P[x_1, \dots, x_N \mid \lambda] = \ln \prod_{i=1}^{N} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \sum_{i=1}^{N} \ln \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

$$= \sum_{i=1}^{N} x_i \ln \lambda - \lambda - \sum_{k=1}^{N} \ln x_i$$

We calculate the derivative and set it to 0 for $\hat{\lambda}_{ML}$:

$$\frac{d \ln P[x_1, \dots, x_N \mid \lambda]}{d\lambda} = \sum_{i=1}^{N} \left(\frac{x_i}{\lambda} - 1 \right) = \frac{1}{\lambda} \sum_{i=1}^{N} x_i - N$$

$$\frac{d \ln P[x_1, \dots, x_N \mid \lambda]}{d\lambda}(\hat{\lambda}_{ML}) = 0 \Leftrightarrow \frac{1}{\lambda} \sum_{i=1}^{N} x_i - N = 0$$

$$\Leftrightarrow \hat{\lambda}_{ML} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

$\hat{\lambda}_{ML}$ is the sample mean.

4.2.b

In this case $\hat{\lambda}_{ML} = \frac{3+6+3+4}{4} = 4$.

4.2.c

If we suppose the rate perfectly determined, then

$$P[X < 2 \mid \hat{\lambda}_{ML}] = P[0 \mid \hat{\lambda}_{ML}] + P[1 \mid \hat{\lambda}_{ML}] \approx 9.2\%$$