$x_n!$ $x_n!$

7.5 observable $\underline{X} = (X_1, \dots, X_t, \dots)$

$\left.\begin{array}{l}\underline{U} = (U_1, \dots, U_t, \dots) \\ \underline{V} = (V_1, \dots, V_t, \dots) \\ \underline{W} = (W_1, \dots, W_t, \dots)\end{array}\right\}$ non observable.

$U_0 = 0$
$U_t = \begin{cases} +1 & U_{t-1} + V_t \geq 0 \\ -1 & U_{t-1} + V_t < 0 \end{cases}$.

recall: $P(a,b,c) = P(a|b,c)P(b|c)P(c)$.

start at T

$$P(\underline{U}, \underline{X} | \lambda, c) = P(U_1, \dots, U_T, X_1, \dots, X_T | \lambda, c) =$$
$$= P(U_T, X_T | U_{T-1}, \dots, U_1, X_{T-1}, \dots, X_1 \; \lambda, c) P(U_{T-1}, X_{T-1} | U_{T-2}, \dots)$$

. . .

$$P(U_i, X_i | U_{i-1}, \dots, U_1, X_{i-1}, \dots, X_1, \lambda, c) =$$
$$= P(X_i | U_i, U_{i-1}, \dots, U_1, X_{i-1}, \dots, X_1, \lambda, c) \, P(U_i | U_{i-1}, \dots, U_1, X_{i-1}, \dots, X_1, \lambda, c) =$$
$$= P(X_i | U_i, c) \, P(U_i | U_{i-1})$$

$$\Rightarrow P(\underline{U}, \underline{X} | \lambda, c) = \left(\prod_{t=2}^{T} \underbrace{P(X_i = x_i | U_i = u_i^c)}_{N(cU_i, \sigma_W^2)} \cdot P(U_i | U_{i-1}) \right) P(X_1 = x_1 | U_1 = u_1^c) \, P(U_1 = u_1 | U_0 = 0)$$

$$Q(c|c^{(t)}) = \sum_{\underline{u} \to \text{all combinations.}} P(\underline{U} = \underline{u} | \underline{X} = \underline{x})^{c^{(t)}} \left(\sum_{t=1}^{T} \log\left(P(X_t = x_t | U_t = u_t, c)\right) + \overbrace{\sum_{t=2}^{T} \log\left(P(U_t = u_t | U_{t-1} = u_{t-1})\right) + \log\left(P(U_1 = u_1 | U_0 = 0)\right)}^{\text{do not depend on } c} \right)$$

all the terms that do not appear in the summand are marginalized:

$$Q'(c|c^{(t)}) = \sum_{t=1}^{T} P(U_t = 1 | \underline{X} = x, c^{(t)}) \log\left(P(X_t = x_t \; | U_t = 1, c)\right) + P(U_t = -1 | \underline{X} = x, c^{(t)}) \log\left(P(X_t = x_t | U_t = -1, c)\right)$$

$$Q'(c|c^{(t)}) = \sum_{t=1}^{T} \gamma_{1,t} \log\left(\frac{1}{\sqrt{2\pi}\sigma_W} e^{-(x_t - c)^2 / 2\sigma_W^2}\right) + \gamma_{2,t} \log\left(\frac{1}{\sqrt{2\pi}\sigma_W} e^{-(x_t + c)^2 / 2\sigma_W^2}\right)$$

Fix log - and ignore parts that do not depend on c:

$$-\sum_{t=1}^{T} \gamma_{1,t} (x_t - c)^2 - \sum_{t=1}^{T} \gamma_{2,t} (x_t + c)^2$$

concave ✓

take derivative and set to $0 \Rightarrow \; -\sum_{t=1}^{T} \gamma_{1,t}(x_t - c) + \sum_{t=1}^{T} \gamma_{2,t}(x_t + c) = 0$.

and therefore:

$$\sum_{t=1}^{T} \underbrace{(\gamma_{1,t} + \gamma_{2,t})}_{\text{each of them add to 1}} c = \sum_{t=1}^{T} (\gamma_{1,t} - \gamma_{2,t}) x_t$$

$$\boxed{c = \frac{1}{T} \sum_{t=1}^{T} (\gamma_{1,t} - \gamma_{2,t}) x_t}$$

7.7 HMM with GMMs

$$\gamma_{im,t} = P[S_t = i \cap U_t = m | x, \lambda] = \gamma_{it} \cdot \frac{w_{im} \, g(x_t, \mu_{im}, C_{im})}{\sum_{k=1}^{M} w_{ik} \, g(x_t, \mu_{ik}, C_{ik})}$$

$\gamma_{i,t} = P(S_t = i | x, \lambda)$