5.5

Let q= $\begin{bmatrix}0.6\\0.1\\0.3\end{bmatrix}$ and A= $\begin{bmatrix}0.9 & 0 & 0.1\\0 & 1 & 0\\0.2 & 0 & 0.8\end{bmatrix}$. We define an HMM model $\lambda = (q, A, B)$ where B defines arbitrary distributions for states 1, 2 and 3.

5.5.a

The state transition matrix is square, therefore the HMM is infinite. When the absorbing state number 2 is reached, the model keeps on generating samples therefore state 2 is not an exit state.

5.5.b

The eigenvalues of A^T are found by solving for $x \in \mathbb{R}$: $\det(A^T - xI_3) = \mid A^T - xI_3 \mid = 0$

$$\begin{aligned}
\mid A^T - xI_3 \mid = 0 &\Leftrightarrow \begin{vmatrix}0.9-x & 0 & 0.2\\ 0 & 1-x & 0\\ 0.1 & 0 & 0.8-x\end{vmatrix} = 0\\
&\Leftrightarrow (0.9-x)(1-x)(0.8-x) - (1-x)0.1 \cdot 0.2 = 0\\
&\Leftrightarrow (1-x)[(0.9-x)(0.8-x) - 0.02] = 0\\
&\Leftrightarrow (1-x)(0.7 - 1.7x + x^2) = 0 \quad \text{(note: 1 is a root !)}\\
&\Leftrightarrow (1-x)(1-x)(0.7-x) = 0
\end{aligned}$$

Therefore the eigenvalues are 1 with algebraic multiplicity 2 and 0.7 with algebraic multiplicity 1.

The L_2-normalized eigenvectors associated with the eigenvalue 1 belong to the normalized eigenspace: $E_1 = \{x \in \mathbb{R} \mid \quad A^T x = x, \|x\|_2^2 = 1\}$. Let us characterize E_1.

$$x = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \in E_1 \Leftrightarrow \begin{cases}A^T x &= x\\ \|x\|_2^2 &= 1\end{cases} \Leftrightarrow \begin{cases}9x_1 + 2x_3 &= 10x_1\\ 10x_2 &= 10x_2\\ x_1 + 8x_3 &= 10x_3\\ x_1^2 + x_2^2 + x_3^2 &= 1\end{cases} \Leftrightarrow \begin{cases}x_1 &= 2x_3\\ x_2 &= x_2\\ x_1^2 + x_2^2 & +x_3^2 = 1\end{cases}$$

Since we have one degree of freedom in the system of equations above we can fix $x_2 = 0$ for a vector $p_1 \in E_1$ and $x_2 = 1$ for a vector $p_2 \in E_1$.

Therefore we have $p_1 = \begin{bmatrix}\frac{2}{\sqrt{5}}\\0\\\frac{1}{\sqrt{5}}\end{bmatrix}$ and $p_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, two L_2-normalized eigenvectors associated with the eigenvalue 1.

Let us denote p_3 a L_2-normalized eigenvector associated with the eigenvalue 0.7. p_3 belongs to the normalized eigenspace: $E_2 = \{x \in \mathbb{R} \mid \quad A^T x = 0.7x, \|x\|_2^2 = 1\}$.
Let us characterize E_2.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in E_2 \Leftrightarrow \begin{cases} A^T x & = 0.7x \\ \|x\|_2^2 & = 1 \end{cases} \Leftrightarrow \begin{cases} 9x_1 + 2x_3 & = 7x_1 \\ 10x_2 & = 7x_2 \\ x_1 + 8x_3 & = 7x_3 \\ x_1^2 + x_2^2 + x_3^2 & = 1 \end{cases} \Leftrightarrow \begin{cases} x_1 + x_3 & = 0 \\ x_2 & = 0 \\ x_1^2 + x_2^2 + x_3^2 & = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 = \frac{1}{\sqrt{2}} \\ x_2 = 0 \\ x_3 = -\frac{1}{\sqrt{2}} \end{cases}$$

Therefore $p_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ is the L_2-normalized eigenvector associated with the eigenvalue 0.7.

5.5.c

The HMM is stationary because A^T has eigenvalues equal to 1.
The stationary state probabilities are the linear combinations of (p_1, p_2) with positive components which sum to 1, *i.e* the vectors $p_s \in \mathbb{R}^3 \quad \exists \alpha \in [0,1] \quad p_s = \alpha p_1 + (1-\alpha) p_2$.

5.5.d

According to the transition matrix:
$\forall t \in \mathbb{N}^+ \quad \forall n \in \mathbb{N}_*^+ \quad p[S_{t+n} = 1 \mid S_t = 2] = 0 \neq p[S_{t+n} = 1 \mid S_t = 1] > 0$.
In words, the probability to reach state 1 at time t+n is different whether we are in state 1 or in state 2 at time t.
Therefore the HMM is not ergodic. Another argument is that the HMM will not converge towards one unique stationary state probability since we showed that there are infinetely many.