# Exercise 3.12

Friday, March 29, 2019 3:52 PM

3.12

a,b

$$\mathbb{E}[X_k] = 0$$

$$var(X_k) = \begin{cases} \sigma_h^2 & 1 \le k \le L \\ \sigma_e^2 & L+1 \le k \le 2L \end{cases} \quad \text{if } S=1 \quad \text{viceversa if } S=2.$$

conditioned to the source they are independent

$(x_1,\dots,x_L)$ and $(x_{L+1},\dots,x_{2L})$

$$\max_{i\in\{1,2\}} f_{(x_1,\dots,x_{2L}|i)}((x_1,\dots,x_{2L})|i) =$$

$$= f_{(x_1,\dots,x_L|i)} \, f_{(x_{L+1},\dots,x_{2L}|i)}$$

then
$$f_{x_1,\dots,x_L|i} = \prod_{\ell=1}^{L} \frac{1}{\sqrt{2\pi}(\sigma_i)} e^{-(x_\ell)^2/2\sigma_i^2} \quad \text{with} \quad \sigma_i^2 = \begin{cases} \sigma_h^2 & i=1 \\ \sigma_e^2 & i=2. \end{cases}$$

$$f_{x_{L+1},\dots,x_{2L}|i} = \prod_{\ell=L+1}^{2L} \frac{1}{\sqrt{2\pi}(\bar\sigma_i)} e^{-(x_\ell)^2/2\bar\sigma_i^2} \quad \text{with} \quad \bar\sigma_i^2 = \begin{cases} \sigma_e^2 & i=1 \\ \sigma_h^2 & i=2. \end{cases}$$

by taking logarithm we obtain the same solution:

$$\max_{i\in\{1,2\}} \sum_{\ell=1}^{L} \log\left(\frac{1}{\sqrt{2\pi}\sigma_i}\right) - (x_\ell)^2/2\sigma_i^2 + \sum_{\ell=L+1}^{2L} \log\left(\frac{1}{\sqrt{2\pi}(\bar\sigma_i)}\right) - x_\ell^2/2\bar\sigma_i^2$$

$$\max_{i\in\{1,2\}} L\cdot\log\left(\frac{1}{\sigma_i\bar\sigma_i}\right) - \sum_{\ell=1}^{L} x_\ell^2/2\sigma_i^2 - \sum_{\ell=L+1}^{2L} x_\ell^2/2\bar\sigma_i^2$$

note that for $i=1$ and $i=2$ $\log\left(\frac{1}{\sigma_i\bar\sigma_i}\right)$ is the same.

$$\min_{i\in\{1,2\}} \sum_{\ell=1}^{L} x_\ell^2/2\sigma_i^2 + \sum_{\ell=L+1}^{2L} x_\ell^2/2\bar\sigma_i^2$$

$$\sum_{\ell=1}^{L} \frac{x_\ell^2}{2\sigma_h^2} + \sum_{\ell=L+1}^{2L} \frac{x_\ell^2}{2\sigma_e^2} \overset{2}{\underset{1}{\gtrless}} \sum_{\ell=1}^{L} \frac{x_\ell^2}{2\sigma_e^2} + \sum_{\ell=L+1}^{2L} \frac{x_\ell^2}{2\sigma_h^2}$$

$$\sum_{l=1}^{L}\frac{x_l^2}{2\sigma_h^2} + \sum_{l=L+1}^{2L}\frac{x_l^2}{2\sigma_l^2} \underset{①}{\overset{②}{\gtrless}} \sum_{l=1}^{L}\frac{x_l^2}{2\sigma_l^2} + \sum_{l=L+1}^{2L}\frac{x_l^2}{2\sigma_h^2}$$

$$\sum_{l=1}^{L} x_l^2\left(\frac{1}{\sigma_h^2}-\frac{1}{\sigma_l^2}\right) + \sum_{l=L+1}^{2L} x_l^2\left(\frac{1}{\sigma_l^2}-\frac{1}{\sigma_h^2}\right) \underset{①}{\overset{②}{\gtrless}} 0$$

$$\boxed{\sum_{l=L+1}^{2L} x_l^2 - \sum_{l=1}^{L} x_l^2 \underset{①}{\overset{②}{\gtrless}} 0}$$

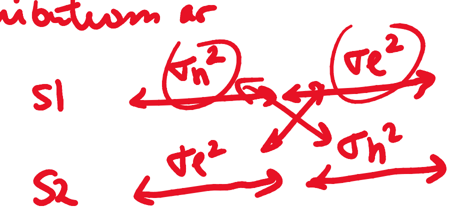
Let $Y \triangleq \sum_{l=L+1}^{2L} x_l^2 - \sum_{l=1}^{L} x_l^2$

**3.12.c** $\mathbb{E}^2[Y|S=1] \overset{?}{=} \mathbb{E}^2[Y|S=2]$ we care about the square because

$$\mathbb{E}[(Y-\mathbb{E}[Y])^2|S=j] = \mathbb{E}[Y^2|S=j] - \mathbb{E}^2[Y|S=j].$$

$$\mathbb{E}^2[Y|S=1] = \left(\mathbb{E}\left[\sum_{l=L+1}^{2L} x_l^2 - \sum_{l=1}^{L} x_l^2 \,|\, S=1\right]\right)^2$$

note that this is because changing the class changes the distribution as




$$\mathbb{E}^2[Y|S=2] = \left(\mathbb{E}\left[\sum_{l=1}^{L} x_l^2 - \sum_{l=L+1}^{2L} x_l^2 \,|\, S=1\right]\right)^2$$

and therefore

$$\mathbb{E}^2[Y|S=1] = \mathbb{E}^2[Y|S=2].$$

An identical argument holds for $\mathbb{E}[Y^2|S=j]$.