Homework 1
Building Familiarity with Orbits

Kepler figured out orbital motion empirically a long time ago, and Newton revised it within a more physical framework. We derived Kepler's 2nd and 3rd laws in class; now let's have a look at the functional form of Kepler's 1st law.

The proof of the orbital equation of motion is utterly devoid of physical intuition :/ I was pretty saddened by this after digging around for most of an evening, but it's mostly just a bunch of second-semester undergraduate calculus stuff (if you want to see it, pages 15-17 of the Seager textbook show one derivation). What matters is the equation relating the position of an orbiting body as a function of the angle θ, measured from periastron (periapsis), given by:

$$r(\theta) = \frac{a(1- e^2)}{1 + e\cos\theta} \tag{1}$$

1. Next up, we're going to work our way up to an algorithm for computing Keplerian orbits, namely: where is the planet in terms of (r, θ) as a function of time. This will require us to think a bit more about angular momentum and energy. The problem is that so far we only have a geometric relationship between r and θ (Equation 1). In fact, all *anyone* has is this relationship, because the only equation relating the geometrical properties and time is transcendental. You may have seen transcendental equations in your undergrad physics classes in other contexts as well. This is known as the Kepler Problem, and there are papers written all the time introducing new techniques of solving the Kepler Problem. We're going to use one such approach.

   (a) Recall the orbital angular momentum conservation equation that you derived in group work:

   $$\frac{dA}{dt} = \frac{L}{2m_p} = \text{constant} \tag{2}$$

   Draw an elliptical orbit, indicating the position of the star. At each of the two vertices of the planet's orbit, draw the vectors for the planet's velocity ($\vec{v}_1$, $\vec{v}_2$) and position ($\vec{r}_1$, $\vec{r}_1$), respectively. Express the total energy and angular momentum of the planet at each position in terms of the scalar quantities r_1, r_2, v_1, v_2, $M_\circ$, m_p and relevant constants, noting that energy and angular momentum are conserved, and therefore equivalent at each position.
   Start with energy equivalence at these two positions, and math things out to show:

   $$\mathscr{L} = \frac{L}{m_p} = [GM_* \, a(1 - e^2)]^{1/2} \tag{3}$$

   Hint: think geometrically about how r_1 and r_2 relate to a and e, referring back to your labeled ellipse.

   Let's call this $\mathscr{L}$ the specific angular momentum, since it's the angular momentum per unit mass of the planet. Double-check the units on this before proceeding.

(b) Write an expression for a variable C that's defined as difference between these two quantities: L for a planet if its eccentricity were zero, and L for an arbitrary non-zero eccentricity e. This quantity often comes up in exoplanet dynamics papers, like Equation 11 in this one [link]. What is the term for C? Would a system with many eccentric planets have a high or low total C?

(c) The reason we spent all that time generating an equation for $\mathscr{L}$ is that Equation 2 hints at a relationship between θ and time on one side, with angular momentum on the other. Rearrange this equation to give an expression for $\Delta\theta$ as a function of r and $\mathscr{L}$.

(d) Code up an iterative procedure that allows you to calculate a new $r = r_1$ and $\theta = \theta_1$ starting from the periastron point $r_0 = a(1-e)$ at an initial time $t_0 = 0$ and initial angle $\theta_0 = 0$. This procedure should allow you to calculate r and θ at each time-step covering a full orbital period of a planet.

2. One way we could use the position of a planet in its orbit as a function of time would be to determine planetary temperatures. Thinking about the "habitable zone" for planets, we usually imagine a range of semimajor axes where a planet's equilibrium temperature lies between the freezing and boiling temperatures of water. Deriving the equation for the equilibrium temperature of a planet should be familiar from undergrad astrophysics, but we usually start with the assumption of a circular orbit. Let's think about what happens when the orbit *isn't circular*. A planet on an eccentric orbit *could* actually swing in and out of the habitable zone on a single orbit, depending on its heat capacity.

   (a) Dusting the rust off, derive the equation for equilibrium temperature for an exoplanet in a circular orbit with semimajor axis a and stellar luminosity L_*, and albedo A.

   (b) Consider an exoplanet with eccentricity of X, where X=your birthday/number of days in a year. My birthday, for example, is September 17; the 260th day of the year. My exoplanet therefore has an eccentricity of 260/365 = 0.71. Determine the eccentricity of your planet, using your own birthday.

   (c) A planet in a really eccentric orbit will have a very different temperature at periastron than at apastron. Use your equation in part (a) to determine the equilibrium temperature at periastron and apastron for a planet orbiting a Sun-like star, with a semimajor axis a=0.7 AU, and eccentricity e set to the value from part (b). Assume an albedo of 0.3. Does your planet exit the habitable zone at any point?

   (d) The assumption of a planet's temperature *actually* responding instantaneously to varying insolation isn't quite right. There's a delay in the temperature response due to solar forcing; the amount of that delay depends on the heat capacity of the surface material. For a liquid water ocean, for example, this delay is two to three months. Using the code you wrote for 1(d), calculate the instantaneous temperature of the planet at every Δt increment. Plot this temperature versus time, over a full year, then overplot horizontal lines corresponding to the nominal "habitable zone" between the freezing and boiling points of water. Assuming that the planet has a liquid water ocean (that is, you can have a running time-average over 2-3 months), how does this curve change? Look at the table below from Dressing et al. 2010 ("Habitable Climates: The Influence of Eccentricity"). If you had a land planet (assuming that the temperature response time scales linearly with surface heat capacity), how does your temperature curve change? If you wanted to ensure that your planet never departed from the "habitable zone", what is its lowest possible surface heat capacity? What is a material with a similar heat capacity?

Table 1
Model Values for the Surface Heat Capacity C

Surface Type	Effective Heat Capacity (erg cm^{-2} K^{-1})
Land	$C_l = 5.25 \times 10^9$
Ocean	$C_o = 40C_l$
Ice (263 K < T < 273 K)	$C_i = 9.2C_l$
Ice (T < 263 K)	$C_i = 2.0C_l$

3. One simple prediction for why terrestrial planets should form close to the Sun, while giant planets form farther out, can be made by comparing the isolation mass to the minimum mass needed to accrete a significant gas envelope, as a function of distance from the Sun.

   (a) We will start by assuming that the surface density of the disk follows the profile of the Minimum Mass Solar Nebula, with $\Sigma_0 = 1.7 * 10^3$ g cm^{-2} at 1 AU (you'll often see a range of estimates for Σ_0 in the literature). If planetesimals make up 1/100th of the mass of the disk within the snowline and the surface density of planetesimals increases by a factor of 4 at the snowline, write down a piecewise function for the surface density of planetesimals as a function of distance from the Sun, $\Sigma_p(r)$, including units.

   (b) The full equation for the isolation mass (at which planetesimal growth begins to level off) will be derived in lecture before this assignment is due, or you can look it up in de Pater & Lissauer. Using your profile for $\Sigma_p(r)$ from part (a), plot the isolation mass for a growing planet (in units of Earth masses) as a function of distance from the Sun (in units of AU, from 0 to 10 AU).

   (c) The full equation for the minimum mass needed to accrete a significant atmosphere is:

$$M_{\rm env} = 1.6*10^{-6}\ [\rho_{\rm m} / ({\rm g\ cm^{-3}})]^{-1/2}\, [T_{\rm disk} / {\rm K}]^{3/2}\, [\ln (2\, \rho_{\rm m}\, r\ /\ \Sigma\ )]^{3/2}\ M_{\rm Earth}$$

   where ρ_m is the material density of the growing planet (in g cm^{-3}), $T_{\rm disk}$ is the temperature of the disk, Σ is the MMSN disk profile from part (a), and the result is in units of Earth masses.

   Add the curve for $M_{\rm env}$ on the plot from part (b), using a material density appropriate for rocky bodies for radii within the snowline and a material density appropriate for icy bodies for radii beyond the snowline. You'll need an expression for $T_{\rm disk}$! Use this: $T^4_{\rm disk} = 6*10^9(r/{\rm AU})^{-3}$ K^4.

   (d) Per your plot, why should the planets that formed closer to the Sun lack substantial gas envelopes, versus the ones that formed farther away?