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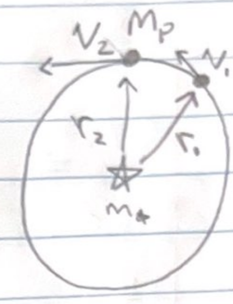
Exoplanets: HW I

$$r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta}$$



1) a. $\frac{dA}{dt} = \frac{L}{2m_p} = \text{constant}$

$E_{TOT} = ?$

$L = ?$

Total energy at each position,

$$E_{TOT} = U_g + K$$

U_g: gravitational potential energy

K: kinetic energy

generally, $U_g = -\frac{GM_\star m_p}{r}$

$$K = \frac{1}{2} m_p V^2$$

So for each position,

$$\boxed{\begin{aligned} E_1 &= -\frac{GM_\star m_p}{r_1} + \frac{m_p V_1^2}{2} \\ E_2 &= -\frac{GM_\star m_p}{r_2} + \frac{m_p V_2^2}{2} \end{aligned}}$$

$E_1 = E_2$ because energy is conserved.

Now total angular momentum at each position,

generally, $L = m r \times v$

$$\Rightarrow \boxed{\begin{aligned} L_1 &= m_p r_1 \times v_1 \\ L_2 &= m_p r_2 \times v_2 \end{aligned}}$$

$L_1 = L_2$ because angular momentum is conserved.

Invoke conservation of energy,

$$E_1 = E_2$$

$$\frac{-GM_\star m_p}{r_1} + \frac{m_p v_1^2}{2} = -\frac{GM_\star m_p}{r_2} + \frac{m_p v_2^2}{2}$$

$$\frac{-GM_\star m_p}{r_1} + \frac{GM_\star m_p}{r_2} = \frac{m_p v_2^2}{2} - \frac{m_p v_1^2}{2}$$

$$GM_\star \cancel{m_p}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = \frac{\cancel{m_p}}{2}(v_2^2 - v_1^2)$$

$$GM_\star\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = \frac{1}{2}(v_2^2 - v_1^2)$$

from angular momentum,

$v_1 = L_1/m_p r_1$, $v_2 = L_2/m_p r_2$

also $L_1 = L_2 = L$

$$GM_\star\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = \frac{1}{2}\left(\left[\frac{L_2}{m_p r_2}\right]^2 - \left[\frac{L_1}{m_p r_1}\right]^2\right)$$

$$GM_\star\left(\frac{1}{r_2}-\frac{1}{r_1}\right)=\frac{L^2}{2}\left(\left[\frac{1}{m_p r_2}\right]^2-\left[\frac{1}{m_p r_1}\right]^2\right)$$

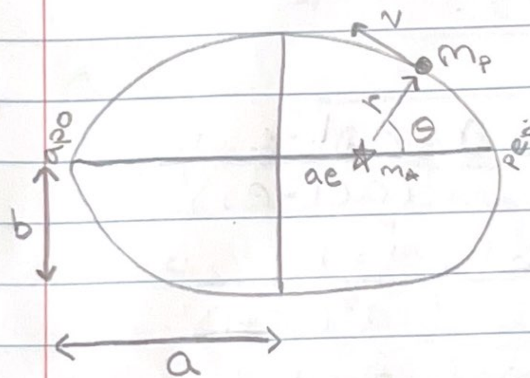
$$GM_\star\left(\frac{1}{r_2}-\frac{1}{r_1}\right)=\frac{L^2}{2m_p^2}\left(\frac{1}{r_2^2}-\frac{1}{r_1^2}\right)$$

solve for L,

$$L^2\left(\frac{1}{r_2^2}-\frac{1}{r_1^2}\right)=GM_\star m_p^2\left(\frac{1}{r_2}-\frac{1}{r_1}\right)$$

Thinking geometrically how r_1 and r_2 relate to a and e,

$$b^2=a^2(1-e^2)$$



we are given,

$$r(\theta)=\frac{a(1-e^2)}{1+e\cos\theta}$$

for simplicity let's choose,

$$r_1=r(\theta=0°)=\frac{a(1-e^2)}{1+e}$$

$$r_2=r(\theta=180°)=\frac{a(1-e^2)}{1-e}$$

$$\frac{1}{r_2}-\frac{1}{r_1}=\frac{1-e}{a(1-e^2)}-\frac{1+e}{a(1-e^2)}=\frac{1-e-1-e}{a(1-e^2)}$$

$$=\frac{-2e}{a(1-e^2)}$$

Try to simplify r_1 and r_2,

$$r_1 = \frac{a(1-e^2)}{1+e} = \frac{a(-e^2+1)}{1+e}$$

$$r_1 = \frac{-ae^2+a}{1+e} = \frac{(-ea+a)\cancel{(e+1)}}{\cancel{(e+1)}}$$

$$r_1 = a(1-e)$$

$$r_2 = \frac{a(1-e^2)}{1-e} = \frac{a(-e^2+1)}{1-e} = \frac{-ae^2+a}{1-e}$$

$$r_2 = \frac{(-ea-a)\cancel{(e-1)}}{-1\cancel{(e-1)}} = \frac{-ea-a}{-1}$$

$$r_2 = a(e+1)$$

$$\frac{1}{r_2^2} - \frac{1}{r_1^2} = \frac{1}{(a(e+1))^2} - \frac{1}{(a(1-e))^2}$$

$$= \frac{1}{a^2(e+1)^2} - \frac{1}{a^2(-e+1)^2}$$

Plug in what I learned about r_1 and r_2 into energy conservation mess,

$$\frac{L^2}{m_p^2}\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right) = GM_\star\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$$\frac{L^2}{m_p^2}\left(\frac{1}{a^2(e+1)^2} - \frac{1}{a^2(1-e)^2}\right) = GM_\star\left(\frac{-2e}{a(1-e^2)}\right)$$

$$\frac{L^2}{m_p^2} = \frac{GM_\star\left(\frac{-2e}{a(1-e^2)}\right)}{\frac{1}{a^2(e+1)^2} - \frac{1}{a^2(1-e)^2}}$$

$(e+1)(e+1) = e^2 + 2e + 1$
$(1-e)(1-e) = 1 - 2e + e^2$

$$\frac{L^2}{m_p^2} = GM_\star \left(\frac{-2ea^2(e+1)^2}{a(1-e)^2} - \frac{-2ea^2(1-e)^2}{a(1-e^2)} \right)$$

$$\frac{L^2}{m_p^2} = GM_\star \left(\frac{-2ea(e^2+2e+1)}{(1-e)^2} + \frac{2ea(e^2-2e+1)}{(1-e)^2} \right)$$

$$\sqrt{\frac{L^2}{m_p^2}} = \sqrt{GM_\star \left(a(1-e^2) \right)}$$

$$\Rightarrow \boxed{\mathcal{L} = \frac{L}{m_p} = \sqrt{GM_\star \cdot a(1-e^2)}}$$

b. $C = ?$

$L_{zero} = \sqrt{GM_\star a} \cdot m_p$

$L_{non\text{-}zero} = \sqrt{GM_\star a(1-e^2)} \cdot m_p$

$C = L_{zero} - L_{non\text{-}zero}$

$C = \left(\sqrt{GM_\star a} - \sqrt{GM_\star a(1-e^2)}\right) m_p$

$C = \left((GM_\star a)^{1/2} - (GM_\star a(1-e^2))^{1/2}\right) m_p$

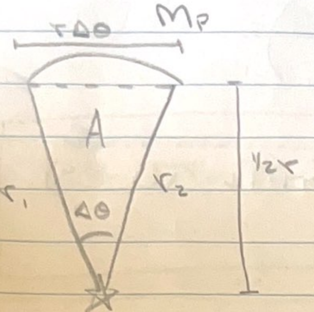
$$\boxed{C = m_p (GM_\star a)^{1/2} \left[1 - (1-e^2)^{1/2} \right]}$$

$\Rightarrow$ a system with many eccentric planets should have a high C.

C. $\Delta\Theta(r, \mathcal{L}) = ?$

$$\frac{dA}{dt} = \frac{L}{2m_p} = \text{constant}$$

$$\mathcal{L} = \frac{L}{m_p} = \left[G m_\star \cdot a(1-e^2)\right]^{1/2}$$



$$\text{area}_{\text{triangle}} = \frac{1}{2} B \cdot H$$

$$dA = \frac{1}{2} r^2 \Delta\Theta$$

$$\frac{dA}{dt} = \frac{L}{2m_p} = \frac{\mathcal{L}}{2}$$

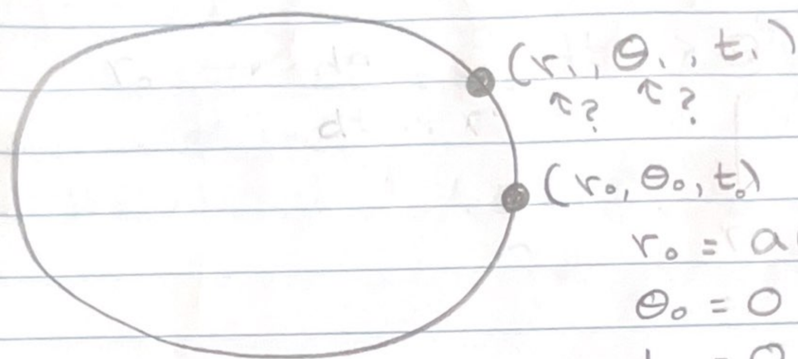
$$dA = \frac{\mathcal{L}}{2} dt \quad \text{and} \quad dA = \frac{r^2}{2} d\Theta$$

$$\frac{\mathcal{L}}{2} dt = \frac{r^2}{2} d\Theta$$

$$\boxed{\frac{d\Theta}{dt} = \frac{\mathcal{L}}{r^2}}$$

d. Starting at, $r_0 = a(1-e)$
$t_0 = 0$
$\theta_0 = 0$

calculate a new r and θ covering the full orbit of the planet



$r_0 = a(1-e)$
$\theta_0 = 0$
$t_0 = 0$

$$\Delta\theta = \theta_1 - \theta_0 = \frac{\mathcal{L}}{r_0^2}\Delta t$$

$$\Rightarrow \theta_1 = \frac{\mathcal{L}}{r_0^2}\Delta t + \theta_0$$

$$r_1 = \frac{a(1-e^2)}{1+e\cos\theta_1}$$

$$\theta_2 = \frac{\mathcal{L}}{r_1^2}\Delta t + \theta_1$$

$$r_2 = \frac{a(1-e^2)}{1+e\cos\theta_2}$$

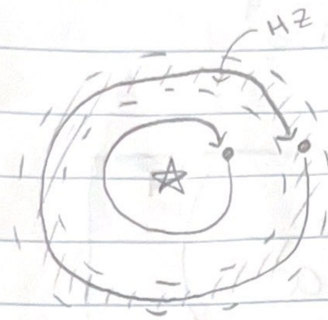
keep going...

How the iterative process should work

☆ See Jupyter notebook on next page ☆

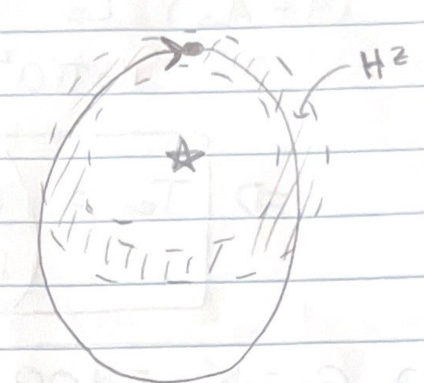
2) Thinking about the "Habitable zone",

<u>circular orbits:</u>



Either in OR out of HZ

<u>elliptical orbits:</u>



can move between in AND out of HZ

a. $T_{eq} = ?$, equillibrium temperature for a planet

$$F_{absorbed} = F_{emitted}$$

Flux absorbed by planet from star

Flux emitted by planet

Start with flux absorbed by planet,

$F_{absorbed} = (1 - A_B) F_\star$, some fraction of light will be reflected.

albedo

and $F_\star = \frac{I}{4}$ and $I = \frac{L_\star}{4\pi a^2}$

$$\Rightarrow F_{absorbed} = (1 - A_B)\frac{L_\star}{16\pi a^2}$$

Now express flux emitted by planet,

$F_{emitted} = \sigma_{SB} T_{eq}^4$

equate the two terms,

$$F_{absorbed} = F_{emitted}$$

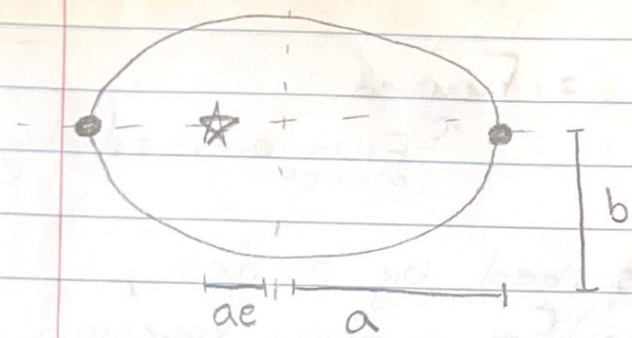
$$(1-A_B)\frac{L_\star}{16\pi a^2} = \sigma_{SB} T_{eq}^4$$

$$\Rightarrow \boxed{T_{eq} = \left(\frac{(1-A_B)L_\star}{16\pi\sigma_{SB}a^2}\right)^{1/4}}$$

b. $e = X = 108/365 = 0.29$ ← April 18

$$\boxed{e = 0.29}$$

c. for large enough e, $T_{peri} \neq T_{api}$



$r_p = ae + a = a(e+1)$

$r_A = a - ae = a(1-e)$

$r_p = 1.3\times10^{13}$ cm, $r_A = 7.4\times10^{12}$ cm

$\left(a = 0.7\,AU,\ A = 0.3 \quad M_\star = M_\odot,\ a = 1.04\times10^{13}\,cm \right)$

at periastron,

$$T_p = \left(\frac{(1-A_B)L_\odot}{16\pi\sigma_{SB}(a(e+1))^2}\right)^{1/4}$$

$$T_p = \left(\frac{(1-0.3)\,3.846\times10^{33}\ erg/s}{16\pi(5.67\times10^{-5}\ g\,s^{-3}K^{-4})(1.3\times10^{13})^2}\right)^{1/4}$$

$$\Rightarrow \boxed{T_{peri} = 273\ K}$$

at apastron,

$$T_A = \left(\frac{(1-0.3)\,3.846 \times 10^{33}\ \mathrm{erg/s}}{16\pi(5.67\times10^{8}\ \mathrm{g\,s^{-1}\,K^{-4}})(7.4\times10^{12}\ \mathrm{cm})^2} \right)^{1/4}$$

$$\Rightarrow \boxed{T_A = 362\ \mathrm{K}}$$

water is a liquid for temperatures,
$273\ \mathrm{K} < T_{eq} < 373\ \mathrm{K}$.

$\Rightarrow$ Yes, my planet exits the habitable zone.

d.

(see Jupyter notebook)

3) a. assume minimum mass solar nebula,

$$\Sigma_0 = 1.7 \times 10^3 \ g \cdot cm^{-2} \text{ at } 1AU$$

$$M_{planetisimals} = \frac{M_{disk}}{100}$$

$$\Sigma_{snowline} = 4\Sigma_0$$

$$\Sigma_p(r) = ? \quad \text{(piecewise fxn)}$$

mmsn power law is,

$$\Sigma(r) = 1700 \left(\frac{r}{1AU}\right)^{-1.5} g \cdot cm^{-2}$$

$$\Sigma_{snow}(r) = 4\Sigma(r) = 6800 \left(\frac{r}{1AU}\right)^{-1.5} g \cdot cm^{-2}$$

$$\Rightarrow \boxed{\Sigma_p(r) = \begin{cases} 1700 \left(\frac{r}{1AU}\right)^{-1.5} g \cdot cm^{-2}, & r < r_{snow} \\ 6800 \left(\frac{r}{1AU}\right)^{-1.5} g \cdot cm^{-2}, & r \geq r_{snow} \end{cases}}$$

(r_{snow} will depend on stellar properties)

b. $M_p(r) = ?$, isolation mass

from lecture,

$$M_p = \frac{8}{\sqrt{3}} \pi^{3/2} M_*^{-1/2} a^3 \Sigma(r)^{3/2}$$

but $\Sigma(r) \propto a^{-3/2} \Rightarrow M_p \propto \frac{a^3}{M_*^{1/2}}$

need to determine r_{snow},

$$T_{eq} = \left(\frac{(1-A_B) L_\odot}{16\pi \sigma_{SB} r^2} \right)^{1/4}$$

solve for r,

$$T_{eq}^4 = \frac{(1-A_B) L_\odot}{16\pi \sigma_{SB} r^2}$$

$$r^2 = \frac{(1-A_B) L_\odot}{16\pi \sigma_{SB} T_{eq}^4}$$

$$r = \sqrt{\frac{(1-A_B) L_\odot}{16\pi \sigma_{SB} T_{eq}^4}}$$

Temperature to freeze water ~ 273 K

$$\Rightarrow r_{snow} = \sqrt{\frac{(1-A_B) L_\odot}{16\pi \sigma_{SB} (273\ \text{K})^4}}$$

In the solar system, $r_{snow} \sim 3$ AU

c. $M_{env} = 1.6 \times 10^{-6} \left[\frac{\rho_m}{(g \cdot cm^{-3})}\right]^{-1/2} \left[\frac{T_{disk}}{K}\right]^{3/2} \cdot \ldots$

$$\ldots \cdot \ln\left(\frac{2\rho_m r}{\Sigma}\right)^{3/2} M_\oplus$$

$$T_{disk}^4 = 6 \times 10^9 \left(\frac{r}{AU}\right)^{-3} K^4$$

$\rho_{m,rocky} = 5.5 \ g/cm^3$ (same as Earth)
$\rho_{m,icy} = 1 \ g/cm^3$

(see Jupyter notebook on next page)

d. Planets closer to the sun require a higher minimum mass needed to accrete a significant atmosphere. Also planets close to the sun are not very massive, hence these planets lack substantial gas envelopes