Landau-de Gennes Model with Sextic Potentials: Asymptotic Behavior of Minimizers

Wei Wang

School of Mathematical Sciences

Peking University

Joint work with Prof. Zhifei Zhang

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Table of Contents

Liquid Crystals

Liquid crystals (LCs) are anisotropic fluids. The anisotropy arises from the directional nature of the molecular geometry, physical, or chemical properties. They are intermediate between:

- **Crystalline solids**: highly ordered.
- **Isotropic liquids**: fully disordered.

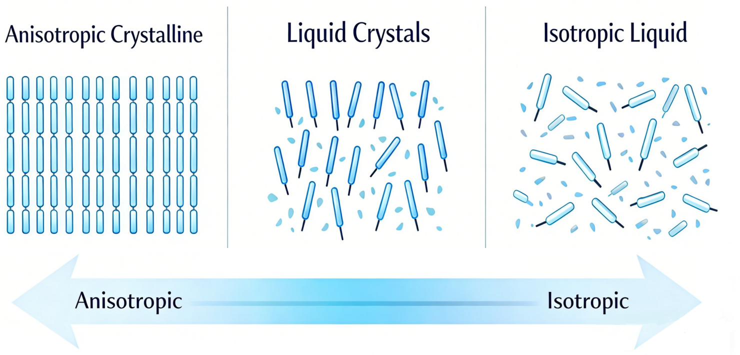

Figure: Order vs. Disorder

Three Phases of Liquid Crystals

- **Nematic**: long-range orientational order (the molecules tend to align parallel to each other); no long-range positional/order correlation of the centers of mass.
- **Cholesteric**: On a larger scale, the director of cholesteric molecules twists in space, forming a helix with a characteristic spatial period.
- **Smectic**: The phase has one-dimensional translational order, resulting in a layered structure.

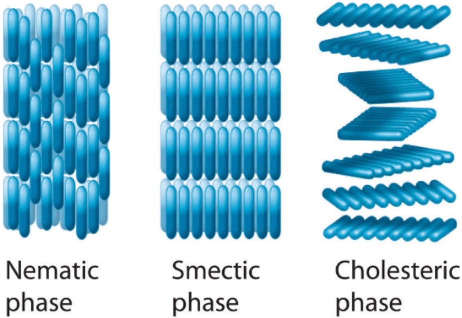

Figure: The Arrangement of Molecules in different Liquid Crystal Phases.

Different Liquid Crystals' Models

- **Vector model**: $\mathbf{n}(x) \in \mathbb{S}^2$ at each material point x.
    - It is simple and useful in many cases.
    - It cannot reflect the head-to-tail symmetry of rod-like molecules with $-\mathbf{n} \sim \mathbf{n}$.
- **Molecular model**: a distribution function $f(x, \mathbf{m})$, which is the number density of molecules with orientation $\mathbf{m} \in \mathbb{S}^2$ at material point x.
    - It provides a more accurate description.
    - The computation is complex.
- **Q-tensor model**: a symmetric, traceless 3×3 matrix $\mathbf{Q}(x)$ for $x \in \mathbb{R}^3$.
    - In this model, one does not assume a preferred direction. It can describe biaxiality.
    - The analysis is also complicated.

Table of Contents

$\mathbf{Q}$-tensor: the Landau-de Gennes Model

The state of a liquid crystal is described by a symmetric, traceless 3×3 matrix:

$$\mathbb{S}_0 := \{\mathbf{Q} \in \mathbb{M}^{3\times 3} : \mathbf{Q} = \mathbf{Q}^{\mathrm{T}}, \operatorname{tr} \mathbf{Q} = 0\}.$$

Eigendecomposition & representation: For orthonormal eigenvectors $\mathbf{n}$ and $\mathbf{m}$, we can express $\mathbf{Q}$ as:

$$\mathbf{Q} = s\left(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I}\right) + r\left(\mathbf{m} \otimes \mathbf{m} - \frac{1}{3}\mathbf{I}\right).$$

Three different cases:

- **Isotropic**: If $s = r = 0$, namely, $\mathbf{Q} = \mathbf{O}$.
- **Biaxial**: If s and r are different and non-zero.
- **Uniaxial**: $\mathbf{Q}$ has a single preferred direction, namely,

$$\mathbf{Q} = t\left(\mathbf{u} \otimes \mathbf{u} - \frac{1}{3}\mathbf{I}\right), \quad \text{for some } t \in \mathbb{R},\ \mathbf{u} \in \mathbb{S}^2.$$

Free Energy Functional

The free energy functional is defined as

$$F(\mathrm{Q},\Omega) := \int_\Omega \left(\mathcal{F}_e(\mathrm{Q}) + \mathcal{F}_b(\mathrm{Q})\right) \mathrm{d}x, \quad \Omega \subset \mathbb{R}^d, \quad d = 2, 3.$$

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- **Elastic energy density**: With elastic constants $L_i \geq 0$ $(i = 1,2,3)$,

$$\mathcal{F}_e(\mathbf{Q}) := \underbrace{\frac{L_1}{2}|\nabla \mathbf{Q}|^2}_{\text{Isotropic term}} + \underbrace{\frac{L_2}{2}\partial_j \mathbf{Q}_{ij}\partial_k \mathbf{Q}_{ik} + \frac{L_3}{2}\partial_k \mathbf{Q}_{ij}\partial_j \mathbf{Q}_{ik}}_{\text{Anisotropic contributions}}.$$

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- **Bulk potential density**: It has the polynomial form given by

$$\begin{aligned}\mathcal{F}_b(\mathbf{Q}) := a_1 - \frac{a_2}{2}\operatorname{tr}\mathbf{Q}^2 + \frac{a_3}{3}\operatorname{tr}\mathbf{Q}^3 + \frac{a_4}{4}(\operatorname{tr}\mathbf{Q}^2)^2 \\ + \frac{a_5}{5}(\operatorname{tr}\mathbf{Q}^2)(\operatorname{tr}\mathbf{Q}^3) + \frac{a_6}{6}(\operatorname{tr}\mathbf{Q}^2)^3 + \frac{a_6'}{6}(\operatorname{tr}\mathbf{Q}^3)^2.\end{aligned}$$

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Stable equilibrium configurations correspond to the minimizers of $F(\cdot,\Omega)$.

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Question (Vanishing elasticity limit)

The asymptotic behavior of minimizers (or critical points) to the quartic Landau-de Gennes model (qLdG) as $\varepsilon \to 0^+$.

The Limit $\varepsilon \to 0^+$: Vacuum Manifold

As $\varepsilon \to 0^+$, the potential term forces $\mathbf{Q}(x)$ to lie in the set of minimizers of $\mathcal{F}_b^{(4)}$, defining the vacuum manifold:

$$\mathcal{N}_u := \left\{ s_* \left(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I} \right) : \mathbf{n} \in \mathbb{S}^2 \right\} \cong \mathbb{R}\mathbf{P}^2.$$

The scalar order parameter s_* is given by

$$s_* := \frac{a_3 + \sqrt{a_3^2 + 24 a_2 a_4}}{4 a_4} \qquad \text{(normalized such that } \mathcal{F}_b^{(4)}(s_*(\mathbf{n} \otimes \mathbf{n} - \mathbf{I}/3)) = 0\text{).}$$

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Limiting energy functional: The limiting functional is the Dirichlet energy for $\mathcal{N}_u$-valued maps, defined by

$$E_0^{(4)}(\mathbf{Q}, \Omega) := \frac{1}{2} \int_\Omega |\nabla \mathbf{Q}|^2 \mathrm{d}x, \tag{qDir}$$

where $\mathbf{Q}(x) \in \mathcal{N}_u$ for a.e. $x \in \Omega$.

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- **Physical link**: This connects the $\mathbf{Q}$-tensor theory to the classical Oseen-Frank model under appropriate boundary conditions.
- **Geometry**: The target manifold $\mathcal{N}_u$ allows for line defects due to its non-trivial fundamental group.

Previous Results: the Energy-Bounded Case

Assumption: Uniform bounds on the energy functional and the L^∞ norm:

$$E_\varepsilon^{(4)}(\mathbf{Q}_\varepsilon, \Omega) + \|\mathbf{Q}_\varepsilon\|_{L^\infty(\Omega)} \leq M, \quad \forall 0 < \varepsilon < 1,$$

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- **Dynamic analysis**: [Wang-Wang-Zhang, *AMRA*, 2017] They analyzed the convergence of the corresponding gradient flow system.

Previous Results: The Logarithmic Energy Case

Assumption: The energy grows logarithmically in terms of ε:

$$E_\varepsilon^{(4)}(\mathbf{Q}_\varepsilon, \Omega) \le M\big(\log(1/\varepsilon) + 1\big), \quad \|\mathbf{Q}_\varepsilon\|_{L^\infty(\Omega)} \le M, \qquad \forall\, 0 < \varepsilon < 1,$$

where $\{\mathbf{Q}_\varepsilon\}$ is a sequence of minimizers. Define $\mu_\varepsilon := \frac{E_\varepsilon^{(4)}(\mathbf{Q}_\varepsilon, \cdot)}{\log(1/\varepsilon)}$ to be the associated Radon measure. Up to a subsequence, $\mu_\varepsilon \rightharpoonup^* \mu_0$ in the sense of Radon measures.

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- **2D analysis**: [Canevari, *ESAIM: COCV*, 2015] The support $\operatorname{supp}\mu_0$ is discrete in Ω. Moreover, there exists $\mathbf{Q}_0 \in C^\infty(\Omega \setminus \operatorname{supp}\mu_0)$ such that up to a subsequence, $\mathbf{Q}_\varepsilon \to \mathbf{Q}_0$ uniformly in $\Omega \setminus \operatorname{supp}\mu_0$.

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Previous Results: The Logarithmic Energy Case

Assumption: The energy grows logarithmically in terms of ε:

$$E_\varepsilon^{(4)}(\mathbf{Q}_\varepsilon,\Omega) \le M\big(\log(1/\varepsilon)+1\big), \quad \|\mathbf{Q}_\varepsilon\|_{L^\infty(\Omega)} \le M, \qquad \forall\, 0<\varepsilon<1,$$

where $\{\mathbf{Q}_\varepsilon\}$ is a sequence of minimizers. Define $\mu_\varepsilon := \frac{E_\varepsilon^{(4)}(\mathbf{Q}_\varepsilon,\cdot)}{\log(1/\varepsilon)}$ to be the associated Radon measure. Up to a subsequence, $\mu_\varepsilon \rightharpoonup^* \mu_0$ in the sense of Radon measures.

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$$E_\varepsilon^{(4)}(\mathbf{Q}_\varepsilon, K) \le C(a_2,a_3,a_4,K,M).$$

This recovers the bounded energy regime away from the singularities.

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  - The analytical techniques for studying these line defects originate from the Ginzburg-Landau model in [Lin-Rivière, *JEMS*, 1999].

Motivation and Problem Setting

Open Problem [Canevari, *ARMA*, 2017]

Can the asymptotic results for the quartic model be generalized to models with a sextic bulk energy density ($a_6, a_6' > 0$)?

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Model Simplification: Following [Huang–Lin, *CVPDE*, 2022], we set $a_3 = a_5 = 0$ for simplicity of exposition. We define the energy functional

$$E_\varepsilon(\mathbf{Q}, \Omega) := \int_\Omega \left(\frac{1}{2} |\nabla \mathbf{Q}|^2 + \frac{1}{\varepsilon^2} f_b(\mathbf{Q}) \right) \mathrm{d}x, \tag{sLDG}$$

where $f_b(\mathbf{Q})$ is the sextic bulk energy density given by

$$f_b(\mathbf{Q}) := a_1 - \frac{a_2}{2} \mathrm{tr}(\mathbf{Q}^2) + \frac{a_4}{4} (\mathrm{tr}(\mathbf{Q}^2))^2 + \frac{a_6}{6} (\mathrm{tr}(\mathbf{Q}^2))^3 + \frac{a_6'}{6} (\mathrm{tr}(\mathbf{Q}^3))^2.$$

- **Coefficients:** $a_2, a_4, a_6, a_6' > 0$.
- **Normalization:** The constant a_1 is chosen such that $\min_{\mathbf{Q} \in \mathbb{S}_0} f_b(\mathbf{Q}) = 0$.

- **Biaxial vacuum manifold**: The bulk potential $f_b(\mathbf{Q})$ vanishes if and only if $\mathbf{Q} \in \mathcal{N}$. The vacuum manifold is given by

$$\mathcal{N} := \{r_*(\mathbf{n} \otimes \mathbf{n} - \mathbf{m} \otimes \mathbf{m}) : (\mathbf{n}, \mathbf{m}) \in \mathcal{M}\},$$

where the radius $r_* > 0$ is determined by

$$4a_6 r_*^4 + 2a_4 r_*^2 - a_2 = 0,$$

and the configuration space is $\mathcal{M} := \{(\mathbf{n}, \mathbf{m}) \in \mathbb{S}^2 \times \mathbb{S}^2 : \mathbf{n} \cdot \mathbf{m} = 0\}$.

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  - **2D analysis**: [Monteil-Rodiac-Van Schaftingen, *AMRA*, 2021; *Math. Ann.*, 2022] investigated the two-dimensional case for generalized Ginzburg–Landau models, including the sextic LdG model.

Main Results: Bounded Energy

Theorem (W.-Zhang, *arXiv:2404.00677*)

Let $\Omega \subset \mathbb{R}^3$ *be a bounded domain, and let* $\{\mathbf{Q}_\varepsilon\}_{0<\varepsilon<1}$ *be local minimizers of* (sLDG) *satisfying*

$$E_\varepsilon(\mathbf{Q}_\varepsilon, \Omega) + \|\mathbf{Q}_\varepsilon\|_{L^\infty(\Omega)} \leq M.$$

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(3) *There exists a locally finite singular set* $\mathcal{S}_{\mathrm{pts}} \subset \Omega$ *such that* $\mathbf{Q}_0 \in C^\infty(\Omega \backslash \mathcal{S}_{\mathrm{pts}}, \mathcal{N})$.

Main Results: Bounded Energy (continued)

(4) For all $j \in \mathbb{Z}_{\geq 0}$, $\mathbf{Q}_{\varepsilon_n} \to \mathbf{Q}_0$ in $C^j_{\mathrm{loc}}(\Omega \backslash \mathcal{S}_{\mathrm{pts}})$. Moreover, for every $\overline{B}_r(x) \subset \Omega \backslash \mathcal{S}_{\mathrm{pts}}$, $\mathbf{Q}_{\varepsilon_n}$ is a smooth solution of

$$\Delta \mathbf{Q}_{\varepsilon_n} = -\frac{1}{2r_*^2}|\nabla \mathbf{Q}_{\varepsilon_n}|^2 \mathbf{Q}_{\varepsilon_n} - \frac{3}{r_*^4}\operatorname{tr}(\nabla \mathbf{Q}_{\varepsilon_n} \nabla \mathbf{Q}_{\varepsilon_n} \mathbf{Q}_{\varepsilon_n})\left(\mathbf{Q}_{\varepsilon_n}^2 - \frac{2r_*^2}{3}\mathbf{I}\right) + \mathbf{R}_n,$$

in $B_{r/2}(x)$, where the remainder $\mathbf{R}_n$ satisfies

$$\|D^j \mathbf{R}_n\|_{L^\infty(B_{r/2}(x))} \leq C\varepsilon_n^2 r^{-j-2},$$

with $C = C(f_b, j, M) > 0$.

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as $n \to +\infty$*, and the following properties hold.*

(1) $\operatorname{supp}(\mu_0) = \mathcal{S}_{\text{line}}$.

(2) $\Omega \cap \mathcal{S}_{\text{line}}$ *is countably* $\mathcal{H}^1$*-rectifiable with* $\mathcal{H}^1(\Omega \cap \mathcal{S}_{\text{line}}) < +\infty$.

Main Results: Logarithmic Energy (continued)

(3) For each subdomain $U \subset\subset \Omega \backslash \mathcal{S}_{\text{line}}$, we have

$$E_{\varepsilon_n}(\mathbf{Q}_{\varepsilon_n}, U) \leq C(f_b, M, U).$$

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   (6b) If $x \in K$ is an endpoint of q segments $\ell_{i_1}, ..., \ell_{i_q}$, let $\mathbf{v}_j$ be the unit direction vector of ℓ_{i_j} pointing outward from x. Then $q \ge 2$, $\sum_{j=1}^q \theta_j \mathbf{v}_j = 0$, and the number of j with $\theta_j = \kappa_*$ is even.

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- **Geometric properties**: Under the specific condition (6b), if $q = 2$, the analysis implies the angle between line segments ℓ_{i_1} and ℓ_{i_2} is exactly π.

Remarks on the Main Results

- **Energy bounds**: By selecting appropriate boundary conditions, both the bounded and logarithmic energy regimes are physically realizable for global minimizers $\{\mathbf{Q}_\varepsilon\}_{0<\varepsilon<1}$.
- **Bounded energy**: The first theorem extends the classic results:
  - H^1 and uniform convergence from [Majumdar-Zarnescu, *ARMA*, 2010].
  - C^j-convergence from [Nguyen-Zarnescu, *CVPDE*, 2013].
- **Logarithmic energy**: The second theorem generalizes the measure-valued convergence and line defect analysis in [Canevari, *ARMA*, 2017] to the sextic model.
- **Geometric properties**: Under the specific condition (6b), if $q = 2$, the analysis implies the angle between line segments ℓ_{i_1} and ℓ_{i_2} is exactly π.
- **Open problem**: While our results focus on minimizers, it remains unknown if these properties hold for general critical points (solutions to Euler-Lagrange equations), similar to the Ginzburg-Landau results in [Bethuel-Brezis-Orlandi, *JFA*, 2001].

Table of Contents

Differences Between Uniaxial and Biaxial Vacuum Manifolds

	Uniaxial	**Biaxial**
Representation	$s_*\left(\mathbf{n}\otimes\mathbf{n}-\frac{\mathbf{I}}{3}\right)$	$r_*(\mathbf{n}\otimes\mathbf{n}-\mathbf{m}\otimes\mathbf{m})$
Eigenvalues	$\frac{2s_*}{3}, -\frac{s_*}{3}, -\frac{s_*}{3}$	$r_*,\ 0,\ -r_*$
Dimension	2	3
Universal covering	$\mathbb{S}^2$	$\mathbb{S}^3$
Fundamental group	$\mathbb{Z}_2$	$Q_8 := \{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}\}$
Topological structure	$\mathbb{S}^2/\mathbb{Z}_2 \cong \mathbb{R}\mathbf{P}^2$	$\mathbb{S}^3/Q_8$
Free homotopy classes	$\mathrm{h}_0, \mathrm{h}_1$	$\mathrm{H}_0 \leftrightarrow \{1\}$, $\mathrm{H}_1 \leftrightarrow \{\pm\mathrm{i}\}$, $\mathrm{H}_2 \leftrightarrow \{\pm\mathrm{j}\}$, $\mathrm{H}_3 \leftrightarrow \{\pm\mathrm{k}\}$, $\mathrm{H}_4 \leftrightarrow \{-1\}$

Table: Comparison between uniaxial and biaxial vacuum manifolds.

Multi-valued Product on Free Homotopy Classes

Definition

Let G be a group. We define a multi-valued product on the set of its conjugacy classes as follows:
Given two conjugacy classes G_1 and G_2, we define $G_1 \cdot G_2$ to be the collection of conjugacy classes containing elements of the form

$$g_1 g_2, \qquad g_1 \in G_1,\ g_2 \in G_2.$$

This operation is commutative, since $g_1 g_2 = (g_1 g_2 g_1^{-1})\, g_1$.

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	H_0	H_1	H_2	H_3	H_4
H_0	H_0	H_1	H_2	H_3	H_4
H_1	H_1	H_0, H_4	H_3	H_2	H_1
H_2	H_2	H_3	H_0, H_4	H_1	H_2
H_3	H_3	H_2	H_1	H_0, H_4	H_3
H_4	H_4	H_1	H_2	H_3	H_0

Table: Multi-valued product on $[\mathbb{S}^1, \mathcal{N}]$.

Energy of Free Homotopy Classes

Let $\mathcal{X} = \mathcal{N}_u$ or $\mathcal{N}$. We define the energy of a free homotopy class $[\alpha]_{\mathcal{X}} \in [\mathbb{S}^1, \mathcal{X}]$ by

$$\mathcal{E}([\alpha]_{\mathcal{X}}) := \inf \left\{ \frac{1}{2} \int_{\mathbb{S}^1} |\beta'(\theta)|^2 \, \mathrm{d}\theta : \ \beta \in H^1(\mathbb{S}^1, \mathcal{X}), \ [\beta]_{\mathcal{X}} = [\alpha]_{\mathcal{X}} \right\}.$$

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We further define the relaxed energy

$$\mathcal{E}^*([\alpha]_{\mathcal{X}}) := \inf \left\{ \sum_{j=1}^{n} \mathcal{E}([\alpha_j]_{\mathcal{X}}) : \ [\alpha]_{\mathcal{X}} \in \prod_{j=1}^{n} [\alpha_j]_{\mathcal{X}} \right\}.$$

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Free homotopy class	h_0	h_1	H_0	H_1	H_2	H_3	H_4
$\mathcal{E}^*$	0	$\frac{\pi s_*^2}{2}$	0	$\frac{\pi r_*^2}{2}$	$\frac{\pi r_*^2}{2}$	πr_*^2	πr_*^2

Sketch of the Proof: Bounded Energy

(1) H^1-**convergence**: A Luckhaus-type lemma adapted to the Landau-de Gennes energy.

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- Obtain C^j-convergence by iterative elliptic estimates.
- Represent the remainder terms via $\mathbf{Y}_\varepsilon$ and h_ε and derive uniform bounds.

Sketch of the Proof: Logarithmic Energy

(1) **Clearing-out property**: Let $\{\mathbf{Q}_\varepsilon\}_{0<\varepsilon<1} \subset H^1(\Omega, \mathbb{S}_0)$ be local minimizers and let $\overline{B}_{2r}(x) \subset \Omega$. Then

$$r^{-1} E_\varepsilon(\mathbf{Q}_\varepsilon, B_{2r}(x)) \ll \log \frac{r}{\varepsilon} \quad \Longrightarrow \quad r^{-1} E_\varepsilon(\mathbf{Q}_\varepsilon, B_r(x)) \lesssim 1.$$

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- **Jerrard-Sandier type lower bound**: Let $A := B_1^2 \setminus B_{1/80}^2$. Assume $\mathbf{Q} \in H^1(B_1^2, \mathbb{S}_0)$ satisfies $\|\mathbf{Q}\|_{L^\infty(B_1^2)} \le M$ and

$$\phi_0(\mathbf{Q}, A) := \operatorname*{ess\,inf}_A \big(\min\{r_*^{-1}(\lambda_1 - \lambda_2),\ r_*^{-1}(\lambda_2 - \lambda_3)\}\big) > 0.$$

  Then

$$E_\varepsilon(\mathbf{Q}, B_1^2) \ge \mathcal{E}^*([\varrho \circ \mathbf{Q}|_{\partial B_1^2}]_{\mathcal{N}})\, \phi_0^2(\mathbf{Q}, A) \log \frac{1}{\varepsilon} - C(f_b, M),$$

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- **Luckhaus-type lemma for logarithmic energy**: interpolation estimates for manifold-valued maps.

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Let $0 < \varepsilon < 1$ and let $\mathbf{Q}_\varepsilon \in H^1(\Omega, \mathbb{S}_0)$ be a local minimizer of (sLDG). Define a non-negative Radon measure μ_ε on $\overline{\Omega}$ by

$$\mu_\varepsilon(U) := \frac{E_\varepsilon(\mathbf{Q}_\varepsilon, U)}{\log \frac{1}{\varepsilon}}, \qquad U \subset \overline{\Omega}.$$

Then there exist $\mu_0 \geq 0$ and $\varepsilon_n \to 0^+$ such that

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- **Rectifiability**: Follows from the results in [Arroyo-Rabasa–De Philippis–Hirsch–Rindler, *GAFA*, 2019] and [Ambrosio-Soner, *Ann. Norm. Pisa*, 1997].

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- $\operatorname{supp} \mu_0$ **is a finite union of line segments**:
  - Densities have finite values.

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- **Discreteness of densities**:
  - Luckhaus-type lemma on cylinders [Lin-Rivière, *JEMS*, 1999].
  - Explicit computation of $\mathcal{E}^*(\mathrm{H}_i)$, $i \in \{1, 2, 3, 4\}$.
- $\operatorname{supp} \mu_0$ **is a finite union of line segments**:
  - Densities have finite values.
  - Classification of 1-dimensional stationary varifolds [Allard-Almgren, *Invent. Math.*, 1976].

(3) **Further properties of the limiting measure**:

- **Discreteness of densities**:
  - Luckhaus-type lemma on cylinders [Lin-Rivière, *JEMS*, 1999].
  - Explicit computation of $\mathcal{E}^*(\mathrm{H}_i)$, $i \in \{1, 2, 3, 4\}$.
- $\operatorname{supp} \mu_0$ **is a finite union of line segments**:
  - Densities have finite values.
  - Classification of 1-dimensional stationary varifolds [Allard-Almgren, *Invent. Math.*, 1976].
- **Endpoint analysis**: Let $x_0 \in \mathcal{S}_{\text{line}}$ be an endpoint. Choose $r > 0$ such that

$$B_{2r}(x_0) \Subset K, \quad \mathcal{S}_{\text{pts}} \cap \partial B_{2r}(x_0) = \emptyset, \quad \mathcal{S}_{\text{line}} \cap B_{2r}(x_0) = \bigcup_{j=1}^{q} \ell_{i_j}.$$

Let $\mathbf{v}_j$ be the unit direction vector of the segment ℓ_{i_j} emanating from x_0, and define $x_j := \ell_{i_j} \cap B_r(x_0)$. Choose $0 < \rho < r/10$ such that the balls $B_\rho(x_j)$ are disjoint, and define $\gamma_j := B_\rho(x_j) \cap \partial B_r(x_0)$. Then

$$\sum_{j=1}^{q} \theta_j \mathbf{v}_j = 0, \qquad \mathrm{H}_0 \in \prod_{j=1}^{q} [\mathbf{Q}_0|_{\gamma_j}]_{\mathcal{N}}. \tag{$\star$}$$

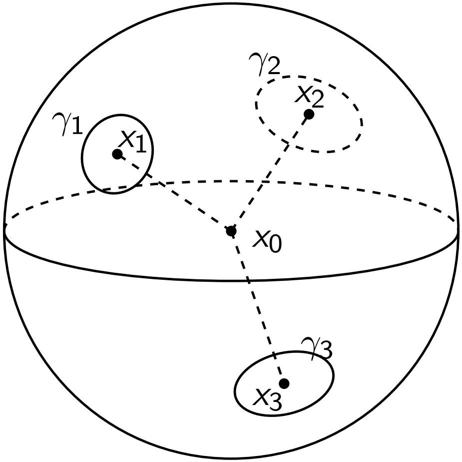

Figure: Geometric configuration near x_0 for $q = 3$.

Proof of (6b)

Let

$$n_0 := \#\{j \in \{1, 2, ..., q\} : \theta_j = \kappa_*\}.$$

Assume that n_0 is odd. Up to a permutation of the indices,

$$\theta_j = \kappa_*, \quad j \in \{1, 2, ..., 2q_0 + 1\},$$

where $q_0 \in \mathbb{Z}_{\geq 0}$ and $2q_0 + 1 \leq q$. We deduce that $[\mathbf{Q}_0|_{\gamma_j}]_{\mathcal{N}} \in \{\mathrm{H}_1, \mathrm{H}_2\}$ for any $j \in \{1, 2, ..., 2q_0 + 1\}$ and $[\mathbf{Q}_0|_{\gamma_j}]_{\mathcal{N}} \in \{\mathrm{H}_3, \mathrm{H}_4\}$ for any $j \in \{2q_0 + 2, ..., q\}$. By the product table, we have

$$\prod_{j=1}^{2q_0+1} [\mathbf{Q}_0|_{\gamma_j}]_{\mathcal{N}} \in \{\mathrm{H}_1, \mathrm{H}_2\},$$

$$\prod_{j=2q_0+2}^{q} [\mathbf{Q}_0|_{\gamma_j}]_{\mathcal{N}} \in \{\mathrm{H}_0, \mathrm{H}_3, \mathrm{H}_4\}.$$

This contradicts $(\star)$, since $\mathrm{H}_0 \notin \mathrm{H}_i \cdot \mathrm{H}_j$ for any $i \in \{1, 2\}$ and $j \in \{0, 3, 4\}$.

Thank you for listening!