Landau-de Gennes Model with Sextic Potentials: Asymptotic Behavior of Minimizers

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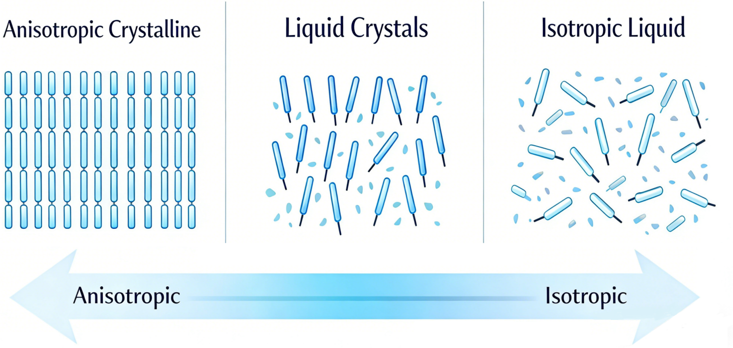
Liquid Crystals

Liquid crystals (LCs) are anisotropic fluids. The anisotropy arises from the directional nature of the molecular geometry, physical, or chemical properties. They are intermediate between:

- **Crystalline solids**: highly ordered
- **Isotropic liquids**: fully disordered



Three Phases of LCs

- **Nematic**: long-range orientational order (the molecules tend to align parallel to each other); no long-range positional/order correlation of the centers of mass.
- **Cholesteric**: On a larger scale, the director of cholesteric molecules twists in space, forming a helix with a characteristic spatial period.
- **Smectic**: The phase has one-dimensional translational order, resulting in a layered structure.

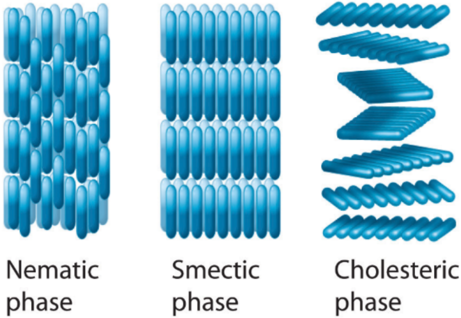


Figure: The Arrangement of Molecules in different Liquid Crystal Phases.

Different Liquid Crystals' Models

- **Vector model**: $\mathbf{n}(x) \in \mathbb{S}^2$ at each material point x.
  - Advantage: Simple and useful in many cases.
  - Drawback: Cannot reflect the head-to-tail symmetry of rod-like molecules with $-\mathbf{n} \sim \mathbf{n}$.
- **Molecular model**: A distribution function $f(x, \mathbf{m})$, which is the number density of molecules with orientation $\mathbf{m} \in \mathbb{S}^2$ at material point x.
  - Advantage: Provides a more accurate description.
  - Drawback: The computation is complex.
- **Q-tensor model**: A 3×3 traceless symmetric matrix $\mathbf{Q}(x)$ for $x \in \mathbb{R}^3$.
  - Advantage: Does not assume a preferred direction. It can describe biaxiality.
  - Drawback: The analysis is also complicated.

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$\mathbf{Q}$-tensor Model (Landau-de Gennes Model)

- Define the set of 3×3 traceless symmetric matrices by
$$\mathbb{S}_0 := \{\mathbf{Q} \in \mathbb{M}^{3\times 3} : \mathbf{Q}^{\mathrm{T}} = \mathbf{Q},\ \operatorname{tr} \mathbf{Q} = 0\}.$$
- Let $\mathbf{Q} \in \mathbb{S}_0$ and $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of $\mathbf{Q}$. Then
$$\mathbf{Q} = \lambda_1 \mathbf{n}_1 \otimes \mathbf{n}_1 + \lambda_2 \mathbf{n}_2 \otimes \mathbf{n}_2 + \lambda_3 \mathbf{n}_3 \otimes \mathbf{n}_3,$$
where $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3 \in \mathbb{S}^2$ satisfy $\mathbf{n}_i \cdot \mathbf{n}_j = \delta_{ij}\,(1 \le i, j \le 3)$. One may rewrite $\mathbf{Q}$ as
$$\mathbf{Q} = s\left(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I}\right) + r\left(\mathbf{m} \otimes \mathbf{m} - \frac{1}{3}\mathbf{I}\right), \quad s, r \in \mathbb{R},$$
where $\mathbf{n}, \mathbf{m}$ are two orthonormal eigenvectors of $\mathbf{Q}$.
- Three different cases:
  - **Isotropic**: If $s = r = 0$, namely, $\mathbf{Q} = \mathbf{O}$.
  - **Biaxial**: If s and r are different and non-zero.
  - **Uniaxial**: $\mathbf{Q}$ has a single preferred direction, namely,
$$\mathbf{Q} = t\left(\mathbf{u} \otimes \mathbf{u} - \frac{1}{3}\mathbf{I}\right), \quad \text{for some } t \in \mathbb{R},\ \mathbf{u} \in \mathbb{S}^2.$$

Free Energy Functional

The free energy functional is defined by

$$F(\mathbf{Q},\Omega) := \int_\Omega (\mathcal{F}_e(\mathbf{Q}) + \mathcal{F}_b(\mathbf{Q}))\,\mathrm{d}x, \quad \Omega \subset \mathbb{R}^d\ (d=2,3).$$

- $\mathcal{F}_e(\mathbf{Q})$: the elastic energy density. For elastic constants $L_i \geq 0$ $(i = 1,2,3)$,

$$\mathcal{F}_e(\mathbf{Q}) = \frac{L_1}{2}|\nabla \mathbf{Q}|^2 + \frac{L_2}{2}\partial_j \mathbf{Q}_{ij}\partial_k \mathbf{Q}_{ik} + \frac{L_3}{2}\partial_k \mathbf{Q}_{ij}\partial_j \mathbf{Q}_{ik}.$$

- $\mathcal{F}_b(\mathbf{Q})$: the bulk potential density. It has the polynomial form as

$$\begin{aligned}\mathcal{F}_b(\mathbf{Q}) := a_1 - \frac{a_2}{2}\operatorname{tr}\mathbf{Q}^2 + \frac{a_3}{3}\operatorname{tr}\mathbf{Q}^3 + \frac{a_4}{4}(\operatorname{tr}\mathbf{Q}^2)^2 \\ + \frac{a_5}{5}(\operatorname{tr}\mathbf{Q}^2)(\operatorname{tr}\mathbf{Q}^3) + \frac{a_6}{6}(\operatorname{tr}\mathbf{Q}^2)^3 + \frac{a_6'}{6}(\operatorname{tr}\mathbf{Q}^3)^2.\end{aligned}$$

  Here $\{a_i\}_{i=1}^6$ and a_6' are non-negative material constants.

Stable equilibrium configurations correspond to minimizers of $F(\cdot,\Omega)$.

Vanishing Elasticity Limit: Background

- A simplified Landau-de Gennes model is obtained by setting

$$L_2 = L_3 = 0, \quad a_5 = a_6 = a_6' = 0.$$

- Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. We recall the quartic Landau-de Gennes energy

$$E_\varepsilon^{(4)}(\mathbf{Q}, \Omega) := \int_\Omega \left(\frac{1}{2} |\nabla \mathbf{Q}|^2 + \frac{1}{\varepsilon^2} \mathcal{F}_b^{(4)}(\mathbf{Q}) \right) \mathrm{d}x, \qquad \text{(qLdG)}$$

where

$$\mathcal{F}_b^{(4)}(\mathbf{Q}) = a_1 - \frac{a_2}{2} \operatorname{tr} \mathbf{Q}^2 - \frac{a_3}{3} \operatorname{tr} \mathbf{Q}^3 + \frac{a_4}{4} (\operatorname{tr} \mathbf{Q}^2)^2, \quad \mathbf{Q} \in \mathbb{S}_0.$$

Here a_1 is chosen such that $\inf_{\mathbf{Q} \in \mathbb{S}_0} \mathcal{F}_b^{(4)}(\mathbf{Q}) = 0$.

- **The observation in physics**: Elastic effects are typically small compared to bulk effects.

Question (Vanishing elasticity limit)

The asymptotic behavior of minimizers (or critical points) to the quartic Landau-de Gennes model (qLdG) as $\varepsilon \to 0^+$.

Limiting Functional

The vacuum manifold is defined by

$$\mathcal{N}_u := \left\{ s_* \left(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I} \right) : \ \mathbf{n} \in \mathbb{S}^2 \right\} = (\mathcal{F}_b^{(4)})^{-1}(0),$$

where

$$s_* := s_*(a_2, a_3, a_4) = \frac{a_3 + \sqrt{a_3^2 + 24 a_2 a_3}}{4 a_4}.$$

Letting $\varepsilon \to 0^+$, the term $\frac{1}{\varepsilon^2}\mathcal{F}_b^{(4)}(\mathbf{Q})$ in (qLdG) forces $\mathbf{Q}$ to take values in $\mathcal{N}_u$.

The limiting energy functional is given by

$$E_0^{(4)}(\mathbf{Q}, \Omega) := \frac{1}{2} \int_\Omega |\nabla \mathbf{Q}|^2 \, \mathrm{d}x, \quad \mathbf{Q} \in H^1(\Omega, \mathcal{N}_u). \tag{qDir}$$

Previous Results: Bounded Energy

Consider the case where $\exists\, M > 0$ such that, for all $\varepsilon \in (0,1)$,

$$E_\varepsilon^{(4)}(\mathbf{Q}_\varepsilon, \Omega) + \|\mathbf{Q}_\varepsilon\|_{L^\infty(\Omega)} \le M.$$

- [Majumdar-Zarnescu, *ARMA*, 2010]: Assume that $\partial\Omega \in C^\infty$ and $\mathbf{Q}_b \in C^\infty(\partial\Omega, \mathcal{N}_u)$. Let $\{\mathbf{Q}_\varepsilon\}_{\varepsilon\in(0,1)}$ be a minimizing sequence of (qLdG) with $\mathbf{Q}_\varepsilon|_{\partial\Omega} = \mathbf{Q}_b$. Then there exists a sequence $\varepsilon_i \to 0^+$ such that
  - $\mathbf{Q}_{\varepsilon_i} \to \mathbf{Q}_0$ in H^1, where $\mathbf{Q}_0$ is a minimizer of (qDir) with $\mathbf{Q}_0|_{\partial\Omega} = \mathbf{Q}_b$.
  - For every $K \subset\subset \Omega \setminus \mathrm{sing}(\mathbf{Q}_0)$, $\mathbf{Q}_{\varepsilon_i} \to \mathbf{Q}_0$ uniformly in K.
- [Nguyen-Zarnescu, *CVPDE*, 2013]: For every $j \in \mathbb{Z}_+$ and every open set $K \subset\subset \Omega \setminus \mathrm{sing}(\mathbf{Q}_0)$, $\mathbf{Q}_{\varepsilon_i} \to \mathbf{Q}_0$ in $C^j(K)$. Moreover, if $\mathbf{Q}_0 \in C^\infty$, then there exists an asymptotic expansion

$$\mathbf{Q}_\varepsilon = \mathbf{Q}_0 + \varepsilon \mathbf{Q}_0^{(1)} + \varepsilon^2 \mathbf{Q}_0^{(2)} + \cdots .$$

- [Contreras-Lamy, *Anal. PDE*, 2022; Feng-Hong, *CVPDE*, 2022]: They extended Nguyen-Zarnescu's results to anisotropic cases with non-zero L_2 and L_3.
- [Wang-Wang-Zhang, *AMRA*, 2017]: analysis of the dynamic case.

Previous Results: Logarithmic Energy

Assume that $\exists M > 0$ such that $\mathbf{Q}_\varepsilon$ minimizes (qLdG), and

$$\frac{E_\varepsilon^{(4)}(\mathbf{Q}_\varepsilon, \Omega)}{\log \frac{1}{\varepsilon}} + \|\mathbf{Q}_\varepsilon\|_{L^\infty(\Omega)} \leq M, \quad \forall\, 0 < \varepsilon < 1.$$

Let $\{\mu_\varepsilon\}_{0<\varepsilon<1} \subset (C(\Omega))'$ be Radon measures defined by

$$\mu_\varepsilon(A) := \frac{E_\varepsilon^{(4)}(\mathbf{Q}_\varepsilon, A)}{\log \frac{1}{\varepsilon}}, \quad \text{for every Borel set } A \subset \overline{\Omega}.$$

Up to a subsequence, $\mu_\varepsilon \rightharpoonup^* \mu_0$ in the sense of measures.

- [Canevari, *ESAIM: COCV*, 2015]: analysis of the two-dimensional case.
- [Canevari, *ARMA*, 2017]: $\mu_0 = \sum_i \frac{\pi s_*^2}{2} \mathcal{H}^1 \llcorner \ell_i$, where $\{\ell_i\}$ are line segments. For every open set $K \subset\subset \Omega \setminus \operatorname{supp}(\mu_0)$, there holds

  $$E_\varepsilon^{(4)}(\mathbf{Q}_\varepsilon, K) \leq C(a, b, c, K, M),$$

  returning to the bounded-energy setting. The idea originates from [Lin-Riviére, *JEMS*, 1999] for the Ginzburg-Landau model.

Motivation of Our Works

Question (Canevari, *ARMA*, 2017)

Can the results for the model (qLdG) be generalized to the model with a sextic bulk energy density and $a_6, a_6' > 0$?

Following [Huang-Lin, *CVPDE*, 2022], for simplicity, we set $a_3 = a_5 = 0$. Accordingly, we consider the energy functional

$$E_\varepsilon(\mathbf{Q}, \Omega) = \int_\Omega \left(\frac{1}{2} |\nabla \mathbf{Q}|^2 + \frac{1}{\varepsilon^2} f_b(\mathbf{Q}) \right) \mathrm{d}x, \qquad \text{(sLDG)}$$

where f_b denotes a sextic bulk energy density given by

$$f_b(\mathbf{Q}) := a_1 - \frac{a_2}{2} \operatorname{tr} \mathbf{Q}^2 + \frac{a_4}{4} (\operatorname{tr} \mathbf{Q}^2)^2 + \frac{a_6}{6} (\operatorname{tr} \mathbf{Q}^2)^3 + \frac{a_6'}{6} (\operatorname{tr} \mathbf{Q}^3)^2,$$

with $a_2, a_4, a_6, a_6' > 0$. Here, a_1 is chosen such that

$$\min_{\mathbf{Q} \in \mathbb{S}_0} f_b(\mathbf{Q}) = 0.$$

- **Biaxial vacuum manifold**: We have $f_b(\mathbf{Q}) = 0$ if and only if

$$\mathbf{Q} \in \mathcal{N} := \{ r_*(\mathbf{n} \otimes \mathbf{n} - \mathbf{m} \otimes \mathbf{m}) : \ (\mathbf{n}, \mathbf{m}) \in \mathcal{M} \},$$

where $r_* > 0$ satisfies

$$4a_6 r_*^4 + 2a_4 r_*^2 - a_2 = 0,$$

and $\mathcal{M}$ is defined by

$$\mathcal{M} := \{(\mathbf{n}, \mathbf{m}) \in \mathbb{S}^2 \times \mathbb{S}^2 : \ \mathbf{n} \cdot \mathbf{m} = 0\}.$$

- **Previous results**:
  - [Allender-Longa, *PRE*, 2008] and [Severing-Saalwächter, *PRL*, 2004]: physical background.
  - [Davis-Gartland Jr., *SIAM J. Numer. Anal.*, 1998]: existence of minimizers.
  - [Huang-Lin, *CVPDE*, 2022]: analysis of the dynamic case with bounded energy, analogous to [Wang-Wang-Zhang, *AMRA*, 2017].
  - [Monteil-Rodiac-Van Schaftingen, *AMRA*, 2021 & *Math. Ann.*, 2022]: the two-dimensional case for generalized Ginzburg-Landau models, including the sextic Landau-de Gennes model.

Main Results: Bounded Energy

Theorem (W.-Zhang, *arXiv:2404.00677*)

Let $\Omega \subset \mathbb{R}^3$ *be a bounded domain, and let* $\{\mathbf{Q}_\varepsilon\}_{0<\varepsilon<1}$ *be local minimizers of* (sLDG) *satisfying*

$$E_\varepsilon(\mathbf{Q}_\varepsilon, \Omega) + \|\mathbf{Q}_\varepsilon\|_{L^\infty(\Omega)} \leq M.$$

Then there exists a sequence $\varepsilon_n \to 0^+$ *and* $\mathbf{Q}_0 \in H^1(\Omega, \mathcal{N})$ *such that:*

(1) $\mathbf{Q}_{\varepsilon_n} \to \mathbf{Q}_0$ *strongly in* $H^1_{\mathrm{loc}}(\Omega, \mathbb{S}_0)$, *and* $\varepsilon_n^{-2} f_b(\mathbf{Q}_{\varepsilon_n}) \to 0$ *in* $L^1_{\mathrm{loc}}(\Omega)$.

(2) $\mathbf{Q}_0$ *is locally energy-minimizing harmonic in* Ω. *Moreover,* $\mathbf{Q}_0$ *is a weak solution of*

$$\Delta \mathbf{Q}_0 = -\frac{1}{2r_*^2}|\nabla \mathbf{Q}_0|^2 \mathbf{Q}_0 - \frac{3}{r_*^4}\operatorname{tr}(\nabla \mathbf{Q}_0 \nabla \mathbf{Q}_0 \mathbf{Q}_0)\left(\mathbf{Q}_0^2 - \frac{2r_*^2}{3}\mathbf{I}\right).$$

(3) *There exists a locally finite singular set* $\mathcal{S}_{\mathrm{pts}} \subset \Omega$ *such that* $\mathbf{Q}_0 \in C^\infty(\Omega \backslash \mathcal{S}_{\mathrm{pts}}, \mathcal{N})$.

(4) For all $j \in \mathbb{Z}_{\geq 0}$, $\mathbf{Q}_{\varepsilon_n} \to \mathbf{Q}_0$ in $C^j_{\mathrm{loc}}(\Omega \backslash \mathcal{S}_{\mathrm{pts}})$. Moreover, for every $\overline{B}_r(x) \subset \Omega \backslash \mathcal{S}_{\mathrm{pts}}$, $\mathbf{Q}_{\varepsilon_n}$ is a smooth solution of

$$\begin{aligned} \Delta \mathbf{Q}_{\varepsilon_n} = & -\frac{1}{2r_*^2} |\nabla \mathbf{Q}_{\varepsilon_n}|^2 \mathbf{Q}_{\varepsilon_n} \\ & - \frac{3}{r_*^4} \operatorname{tr}(\nabla \mathbf{Q}_{\varepsilon_n} \nabla \mathbf{Q}_{\varepsilon_n} \mathbf{Q}_{\varepsilon_n}) \left(\mathbf{Q}_{\varepsilon_n}^2 - \frac{2r_*^2}{3} \mathbf{I} \right) + \mathbf{R}_n, \end{aligned}$$

in $B_{r/2}(x)$, where the remainder $\mathbf{R}_n$ satisfies

$$\|D^j \mathbf{R}_n\|_{L^\infty(B_{r/2}(x))} \leq C \varepsilon_n^2 r^{-j-2},$$

with $C = C(f_b, j, M) > 0$.

Main Results: Logarithmic Energy

Theorem (W.-Zhang, *arXiv:2404.00677*)

Let $\Omega \subset \mathbb{R}^3$ *be a bounded domain, and let* $\{\mathbf{Q}_\varepsilon\}_{0<\varepsilon<1}$ *be local minimizers of* (sLDG) *satisfying*

$$E_\varepsilon(\mathbf{Q}_\varepsilon, \Omega) \leq M(\log \frac{1}{\varepsilon} + 1), \quad \|\mathbf{Q}_\varepsilon\|_{L^\infty(\Omega)} \leq M.$$

Then there is a sequence $\varepsilon_n \to 0^+$ *and a closed set* $\mathcal{S}_{\mathrm{line}} \subset \overline{\Omega}$ *such that*

$$\frac{1}{\log \frac{1}{\varepsilon_n}} \left(\frac{1}{2} |\nabla \mathbf{Q}_{\varepsilon_n}|^2 + \frac{1}{\varepsilon_n^2} f_b(\mathbf{Q}_{\varepsilon_n}) \right) \mathrm{d}x \rightharpoonup^* \mu_0 \text{ in } (C(\overline{\Omega}))'$$

as $n \to +\infty$, *and the following properties hold.*

(1) $\mathrm{supp}(\mu_0) = \mathcal{S}_{\mathrm{line}}$.

(2) $\Omega \cap \mathcal{S}_{\mathrm{line}}$ *is countably* $\mathcal{H}^1$*-rectifiable with* $\mathcal{H}^1(\Omega \cap \mathcal{S}_{\mathrm{line}}) < +\infty$.

Main Results: Logarithmic Energy (continued)

(3) For each subdomain $U \subset\subset \Omega \backslash \mathcal{S}_{\text{line}}$, we have

$$E_{\varepsilon_n}(\mathbf{Q}_{\varepsilon_n}, U) \leq C(f_b, M, U).$$

(4) For $\mathcal{H}^1$-a.e. $x \in \mathcal{S}_{\text{line}} \cap \Omega$, $\lim_{r \to 0^+} \mu_0(\overline{B_r(x)})/(2r) \in \{\kappa_*, 2\kappa_*\}$, where $\kappa_* = \pi r_*^2/2$.

(5) The measure $\mu_0 \llcorner \Omega$ is associated with a 1-dimensional stationary varifold.

(6) For each open set $K \subset\subset \Omega$, one has $\mathcal{S}_{\text{line}} \cap \overline{K} = \{\ell_1, \dots, \ell_p\}$, where $\{\ell_i\}_{i=1}^p$ are closed straight line segments such that for $i \neq j$, ℓ_i and ℓ_j are either disjoint or intersect at a common endpoint. Moreover, $\mu_0 \llcorner \overline{K} = \sum_{j=1}^p \theta_j \mathcal{H}^1 \llcorner \ell_j$ with $\theta_j \in \{\kappa_*, 2\kappa_*\}$.

 (6a) If $\overline{D} \subset K$ is a closed disk with $\mathcal{S}_{\text{line}} \cap D = \{x\}$, $\mathcal{S}_{\text{pts}} \cap \partial D = \emptyset$, and x is not an endpoint of any ℓ_i, then the free homotopy class of $\mathbf{Q}_0|_{\partial B_r^2(x)}$ is non-trivial.

 (6b) If $x \in K$ is an endpoint of q segments $\ell_{i_1}, \dots, \ell_{i_q}$, let $\mathbf{v}_j$ be the unit direction vector of ℓ_{i_j} pointing outward from x. Then $q \geq 2$, $\sum_{j=1}^q \theta_j \mathbf{v}_j = 0$, and the number of j with $\theta_j = \kappa_*$ is even.

Remarks on the Main Results

- By choosing appropriate boundary conditions and considering the corresponding global minimizers, the bounded and logarithmic energy bounds can be satisfied.
- The first theorem extends the H^1 and uniform convergence results in [Majumdar-Zarnescu, *ARMA*, 2010] and the C^j-convergence ($j \in \mathbb{Z}_+$) in [Nguyen-Zarnescu, *CVPDE*, 2013].
- The second theorem extends the convergence properties in [Canevari, *ARMA*, 2017].
- Given (6b), if $q = 2$, then the angle between ℓ_{i_1} and ℓ_{i_2} is π.
- Here the results are established for minimizers; it is still unknown whether similar results hold for solutions of the Euler-Lagrange equations, analogous to the Ginzburg-Landau model in [Bethuel-Brezis-Orlandi, *JFA*, 2001].

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	Uniaxial	**Biaxial**
Representation	$s_*\left(\mathbf{n}\otimes\mathbf{n}-\frac{\mathbf{I}}{3}\right)$	$r_*(\mathbf{n}\otimes\mathbf{n}-\mathbf{m}\otimes\mathbf{m})$
Eigenvalues	$\frac{2s_*}{3},-\frac{s_*}{3},-\frac{s_*}{3}$	$r_*,\ 0,\ -r_*$
Dimension	2	3
Universal covering space	$\mathbb{S}^2$	$\mathbb{S}^3$
Fundamental group	$\mathbb{Z}_2$	$Q_8=\{\pm1,\pm\mathrm{i},\pm\mathrm{j},\pm\mathrm{k}\}$
Conjugacy classes	$\{\bar{0}\},\{\bar{1}\}$	$\{1\},\{\pm\mathrm{i}\}$ $\{\pm\mathrm{j}\},\{\pm\mathrm{k}\},\{-1\},$
Free homotopy classes	$\mathrm{h}_0,\mathrm{h}_1$	$\mathrm{H}_0\leftrightarrow\{1\},$ $\mathrm{H}_1\leftrightarrow\{\pm\mathrm{i}\},\mathrm{H}_2\leftrightarrow\{\pm\mathrm{j}\},$ $\mathrm{H}_3\leftrightarrow\{\pm\mathrm{k}\},\mathrm{H}_4\leftrightarrow\{-1\}$

Table: Uniaxial vs. Biaxial

Multi-valued Product

Definition

Given a group G, we can define a multi-valued product on its conjugacy classes. Given two conjugacy classes of G, denoted by G_1 and G_2, we let $G_1 \cdot G_2$ be the collection of conjugacy classes that contain elements of the form $g_1 g_2$, where $g_1 \in G_1$ and $g_2 \in G_2$. This operation is commutative since $g_1 g_2 = (g_1 g_2 g_1^{-1}) g_1$.

	H_0	H_1	H_2	H_3	H_4
H_0	H_0	H_1	H_2	H_3	H_4
H_1	H_1	H_0, H_4	H_3	H_2	H_1
H_2	H_2	H_3	H_0, H_4	H_1	H_2
H_3	H_3	H_2	H_1	H_0, H_4	H_3
H_4	H_4	H_1	H_2	H_3	H_0

Table: Table of the multi-valued product of $[\mathbb{S}^1, \mathcal{N}]$.

Energy of Free Homotopy Classes

Let $\mathcal{X} = \mathcal{N}_u$ or $\mathcal{N}$. Define the energy of the free homotopy class $[\alpha]_{\mathcal{X}} \in [\mathbb{S}^1, \mathcal{X}]$ by

$$\mathcal{E}([\alpha]_{\mathcal{X}}) := \inf \left\{ \frac{1}{2} \int_{\mathbb{S}^1} |\beta'(\theta)|^2 \mathrm{d}\theta : \beta \in H^1(\mathbb{S}^1, \mathcal{X}),\ [\beta]_{\mathcal{X}} = [\alpha]_{\mathcal{X}} \right\}.$$

We also define

$$\mathcal{E}^*([\alpha]_{\mathcal{X}}) := \inf \left\{ \sum_{j=1}^{n} \mathcal{E}([\alpha_j]_{\mathcal{X}}) : [\alpha]_{\mathcal{X}} \in \prod_{j=1}^{n} [\alpha_j]_{\mathcal{X}} \right\}.$$

Free homotopy class	h_0	h_1	H_0	H_1	H_2	H_3	H_4
$\mathcal{E}^*$	0	$\frac{\pi s_*^2}{2}$	0	$\frac{\pi r_*^2}{2}$	$\frac{\pi r_*^2}{2}$	πr_*^2	πr_*^2

Sketch of the Proof: Bounded Energy

(1) H^1-**convergence**: Luckhaus-type lemma for Landau-de Gennes model.

(2) **Uniform convergence away from** $\mathcal{S}_{\text{pts}}$:
- Uniform boundedness: $\|\mathbf{Q}_\varepsilon\|_{L^\infty} \lesssim 1$.
- Bochner-type inequality: $-\Delta e_\varepsilon \lesssim e_\varepsilon^2$ if $\operatorname{dist}(\mathbf{Q}, \mathcal{N}) \ll 1$.
- Monotonicity formula: $r^{-1} E_\varepsilon(\mathbf{Q}_\varepsilon, B_r(x)) \uparrow$ as $r \uparrow$.

(3) C^j-**convergence away from** $\mathcal{S}_{\text{pts}}$: For each $\mathbf{Q} \in \mathcal{N}$, we have $\mathbf{Q}^3 - r_*^2 \mathbf{Q} = 0$, and $\operatorname{tr} \mathbf{Q}^2 = 2r_*^2$. Define
- $\mathbf{Y}_\varepsilon := \varepsilon^{-2}(\mathbf{Q}_\varepsilon^3 - r_*^2 \mathbf{Q}_\varepsilon)$,
- $h_\varepsilon := \varepsilon^{-2}(\operatorname{tr} \mathbf{Q}_\varepsilon^2 - 2r_*^2)$.

These quantities are specified for biaxial vacuum manifold $\mathcal{N}$, different from the uniaxial case. The method is:
- Calculate $-\Delta \mathbf{Y}_\varepsilon$ and $-\Delta h_\varepsilon$ and establish equations.
- Applying the maximal principle for equations like $-\varepsilon^2 \Delta f + a^2 f = F$, C^j-convergence follows from iterative arguments.
- Represent the remainder with $\mathbf{Y}_\varepsilon$ and h_ε and give corresponding estimates.

Sketch of the Proof: Logarithmic Energy

(1) **Clearing-out property**: For local minimizers $\{\mathbf{Q}_\varepsilon\}_{0<\varepsilon<1} \subset H^1(\Omega, \mathbb{S}_0)$ and $B_{2r}(x) \subset\subset \Omega$, we have

$$r^{-1} E_\varepsilon(\mathbf{Q}, B_{2r}(x)) \ll \log \frac{r}{\varepsilon} \quad \Longrightarrow \quad r^{-1} E_\varepsilon(\mathbf{Q}, B_r(x)) \lesssim 1.$$

- Jerrard-Sandier type estimate: Let $0 < \varepsilon < 1$ and $A := B_1^2 \backslash B_{1/80}^2$. If $\mathbf{Q} \in H^1(B_1^2, \mathbb{S}_0)$ satisfies $\|\mathbf{Q}\|_{L^\infty(B_1^2)} \leq M$ and

  $$\phi_0(\mathbf{Q}, A) = \operatorname*{ess\,inf}_A (\min\{r_*^{-1}(\lambda_1 - \lambda_2), r_*^{-1}(\lambda_2 - \lambda_3)\}) > 0,$$

  then

  $$E_\varepsilon(\mathbf{Q}, B_1^2) \geq \mathcal{E}^*([\varrho \circ \mathbf{Q}|_{\partial B_1^2}]_{\mathcal{N}}) \phi_0^2(\mathbf{Q}, A) \log \frac{1}{\varepsilon} - C(f_b, M),$$

  where $\varrho : \mathbb{S}_0 \to \mathcal{N}$ is the nearest projection.
- Extension property: trivial homotopy class $\Rightarrow$ extension.
- Luckhaus-type lemma for logarithmic energy: interpolations for manifold-valued maps.

Sketch of the Proof: Logarithmic Energy

Let $0 < \varepsilon < 1$ and $\mathbf{Q}_\varepsilon \in H^1(\Omega, \mathbb{S}_0)$ be a local minimizer of (sLDG). Define a non-negative Radon measure μ_ε on $\overline{\Omega}$ by

$$\mu_\varepsilon(U) := \frac{E_\varepsilon(\mathbf{Q}_\varepsilon, U)}{\log \frac{1}{\varepsilon}}$$

for any Borel set $U \subset \overline{\Omega}$. Then there exists $\mu_0 \geq 0$ in $(C(\overline{\Omega}))'$ and $\varepsilon_n \to 0^+$ such that

$$\mu_{\varepsilon_n} \rightharpoonup^* \mu_0 \quad \text{in } (C(\overline{\Omega}))'.$$

Define $\mathcal{S}_{\text{line}} := \operatorname{supp}(\mu_0)$, which is a closed subset of $\overline{\Omega}$.

(2) **Preliminaries of the limiting measure**:

- Dichotomy property: Clearing out property implies that

$$r^{-1}\mu_0(B_{2r}(x)) \ll 1 \quad \Longrightarrow \quad \mu_0(B_r(x)) = 0.$$

- Rectifiability theorem: Comes from results in [Arroyo Rabasa-De Philippis-Hirsch-Rindler, *GAFA*, 2019] and [Ambrosio-Soner, *Ann. Norm. Pisa*, 1997].

Sketch of the Proof: Logarithmic Energy

(3) **More information the limiting measure**:

- Discreteness of densities:
  - Luckhaus-type lemma on the cylinders: [Lin-Rivière, *JEMS*, 1999].
  - Explicit calculations of $\mathcal{E}^*(\mathrm{H}_i)$, $i \in \{1,2,3,4,5\}$.
- Characterization of 1-dimensional stationary varifolds:
  - Divergence-free stress tensor $\Rightarrow$ Stationarity of varifold.
  - Results on 1-dimensional stationary varifolds: [Allard-Almgren, *Invent. Math.*, 1976].
- Key properties at the endpoint: For an endpoint $x_0 \in \mathcal{S}_{\text{line}}$, choose $r > 0$ s.t.

$$B_{2r}(x_0) \subset\subset K, \quad \mathcal{S}_{\text{pts}} \cap \partial B_{2r}(x_0) = \emptyset, \quad \mathcal{S}_{\text{line}} \cap B_{2r}(x_0) = \cup_{j=1}^{q} \ell_{i_j}.$$

Let $\mathbf{v}_j$ be the unit direction vector of the segment ℓ_{i_j} emanating from x_0. Define $x_j := \ell_{i_j} \cap B_r(x_0)$. Choose sufficiently small $0 < \rho < \frac{r}{10}$ s.t. $B_\rho(x_j) \cap B_\rho(x_\ell) = \emptyset$ for any $j \neq \ell$. Define $\gamma_j := B_\rho(x_j) \cap \partial B_r(x_0)$.

$$\sum_{j=1}^{q} \theta_j \mathbf{v}_j = 0, \quad \mathrm{H}_0 \in \prod_{j=1}^{q} [\mathbf{Q}_0|_{\gamma_j}]_{\mathcal{N}}. \qquad (\star)$$

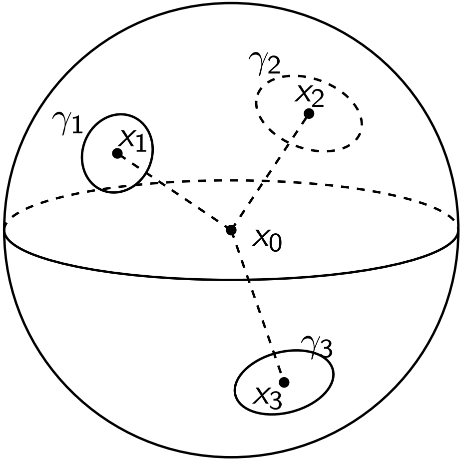

Figure: Geometric configuration near x_0 for $q = 3$.

Proof of (6b)

Let

$$n_0 := \#\{j \in \{1, 2, ..., q\} : \theta_j = \kappa_*\}.$$

Assume that n_0 is odd. Up to a permutation of the indices,

$$\theta_j = \kappa_*, \quad j \in \{1, 2, ..., 2q_0 + 1\},$$

where $q_0 \in \mathbb{Z}_{\geq 0}$ and $2q_0 + 1 \leq q$. We deduce that $[\mathbf{Q}_0|_{\gamma_j}]_{\mathcal{N}} \in \{\mathrm{H}_1, \mathrm{H}_2\}$ for any $j \in \{1, 2, ..., 2q_0 + 1\}$ and $[\mathbf{Q}_0|_{\gamma_j}]_{\mathcal{N}} \in \{\mathrm{H}_3, \mathrm{H}_4\}$ for any $j \in \{2q_0 + 2, ..., q\}$. By the product table, we have

$$\prod_{j=1}^{2q_0+1} [\mathbf{Q}_0|_{\gamma_j}]_{\mathcal{N}} \in \{\mathrm{H}_1, \mathrm{H}_2\},$$

$$\prod_{j=2q_0+2}^{q} [\mathbf{Q}_0|_{\gamma_j}]_{\mathcal{N}} \in \{\mathrm{H}_0, \mathrm{H}_3, \mathrm{H}_4\}.$$

This contradicts $(\star)$, since $\mathrm{H}_0 \notin \mathrm{H}_i \cdot \mathrm{H}_j$ for any $i \in \{1, 2\}$ and $j \in \{0, 3, 4\}$.

Thank you for listening!