CS260 PROJECT MIDTERM REPORT: COMPARISON OF COOLEY–TUKEY ALGORITHM AND RADIX-4 FFT ALGORITHM FOR FFT

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1 Introduction

Fourier Transform (FT) is one of the most well-known mathematical transforms that convert a continuous input function from the time domain into the frequency domain. From the perspective of an engineer, with Fourier Transform, a function of analog signal strength against time could be converted into a function against frequency, or vice versa, which means any signal could be seen as a combination of an infinite number of sine waves at different frequencies. For a discrete input function, such as a function describing a digital signal waveform, there is a "discrete version" of the Fourier Transform named Discrete Fourier Transform (DFT), and a discrete function could be regarded as the sum of a finite number of sine waves, depending on the number of the sampling points. In the practical use of Discrete Fourier Transform, we also have its efficient implementation known as Fast Fourier Transform (FFT).

The industry has widely adopted fast Fourier Transform as one of the most commonly-used algorithms. It has demonstrated its massive value in signal processing, especially for wireless communication, as well as digital image and audio processing. For example, Orthogonal frequency division multiplexing (OFDM)[1] and its derived method Orthogonal Frequency Division Multiple Access (OFDMA)[2] is used by modern wireless communication technology such as LTE Mobile Network[3] and Wi-Fi 6 Wireless LAN[4] to improve bandwidth utilization and achieve a higher transmission rate. This method heavily utilizes FFT to modulate and demodulate the signal. Another famous application is image and audio compression. Derived from FFT, Discrete Cosine Transform (DCT) is another transform that can deliver a higher compression ratio than DFT when used by compression algorithms. One Fast DCT implementation is based on FFT with additional pre- and post-processing[5]. So far, many mainstream image and audio compression formats, including MP3 for audio, JPEG for images, and MPEG for video, are DCT-based[6].

Among the FFT implementations, Cooley-Tukey FFT[7] is considered the standard implementation, and there are also several variants, including Radix-4 FFT[8], Rader's FFT[9], Winograd FFT[10], Quantum Fourier transform (QFT)[11], and Split-Radix FFT[12]. This project aims to analyze and re-implement Cooley-Tukey FFT and Radix-4 FFT algorithms. Then we are devoted to making a detailed comparison between them through theoretical and experimental analysis.

2 Discrete Fourier Transform (DFT)

2.1 Mathematical Problem Formulation

In this section, we give a mathematical formulation of discrete Fourier transform (DFT) [13] and explain the naive DFT algorithm in detail.

For brevity, we consider the one-dimensional case of DFT in the following.

Suppose that we have a discrete sequence:

$x_n = x(0), x(1), x(2), ...x(n-1)$ **in length** N**.**

DFT is a function $\mathcal{F}$, which will transform the given sequence x_n to a new sequence,

$X_n = X(0), X(1), X(2), ...X(n-1)$**, also in length** N**.**

Formally, DFT function $\mathcal{F}$ is defined by:

$$X(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n) \cdot e^{-i2\pi \frac{kn}{N}}, \quad n \in \mathbb{Z}, \tag{1}$$

where i is the square root of -1.

The inverse DFT $\mathcal{F}^{-1}$, alias Inverse Discrete Fourier Transform (IDFT), is the function that can reconstruct x_n from X_n with no loss of information. Similarly, we can write this process as:

$$x(k) = \sum_{n=0}^{N-1} X(n) \cdot e^{i2\pi \frac{kn}{N}}, \quad n \in \mathbb{Z}, \tag{2}$$

Moreover, we have Euler's formula:

$$e^{ix} = \cos x + i \sin x \tag{3}$$

Combing Eq. 3 and Eq. 1, we can also write DFT function $\mathcal{F}$ from the complex plane to the trigonometric domain as:

$$X(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n) \cdot \left[\cos(2\pi \frac{kn}{N}) - i \cdot \sin(2\pi \frac{kn}{N})\right] \tag{4}$$

2.2 Time Complexity Analysis of Naive DFT

```
1
2  def dft(x):
3      '''Compute the Discrete Fourier Transform of an input signal x of N samples.'''
4      N = len(x)
5      X = np.zeros(N, dtype=np.complex)
6      # Compute each X[m]
7      for k in range(N):
8          # Compute similarity between x and the m'th basis
9          for n in range(N):
10             X[k] = X[k] + x[n] * np.exp(-2j * np.pi * k * n / N)
11     return X
```

Listing 1: DFT Implementation

Algorithm 1 A naive implementation for DFT

Data: Input discrete sequence $x_n = x(0), x(1), x(2), ...x(n-1)$ in length N
Result: Output sequence $X_n = X(0), X(1), X(2), ...X(n-1)$, also in length N
Init: Initialize $X_n = X(0), X(1), X(2), ...X(n-1)$ as zeros

```
for k ← 0 to N − 1 do
    for n ← 0 to N − 1 do
        X(k) = X(k) + x(n) · e^{−i2π kn/N}
    end
end
return X_n
```

The algorithm procedure and Python version of implementation can be found in Alg. 1 and Lst. 1. Obviously, we have two loops in length N in our implementation. For every output element, we need to traverse every element in the input array. So the time complexity of DFT in 1D case will be:

$$N \times N = N^2 = O(N^2)$$

2.3 DFT Matrix

Considering Eq. 1, if we denote the last term as:

$$W_N = e^{\frac{-i2\pi}{N}} = \cos(\frac{2\pi}{N}) - i\sin(\frac{2\pi}{N}) \tag{5}$$

$$W_N^{kn} = e^{-i2\pi\frac{kn}{N}} = \cos(2\pi\frac{kn}{N}) - i\sin(2\pi\frac{kn}{N}) \tag{6}$$

We can rewrite Eq. 1 using Eq. 6 as:

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x_n \cdot W_N^{kn}, \quad n \in \mathbb{Z}, \tag{7}$$

Using this notation Eq. 6, we can explicitly write DFT as a linear transformation. For instance, if we have x_n with $n = 5$, then $W = W_5$, we can expand DFT as follows:

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 \\ 1 & W^2 & W^4 & W^6 & W^8 \\ 1 & W^3 & W^6 & W^9 & W^{12} \\ 1 & W^4 & W^8 & W^{12} & W^{16} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \end{bmatrix} \tag{8}$$

Another interesting fact is that $g(k) = W_N^{nk}$ is a periodic function with period N. We will find writing DFT in matrix form is more intuitive to lead us to go from DFT to FFT.

2.4 Two-Dimensional DFT

Signal and image processing problems are usually dealing with multi-dimensional data.

Assuming that we are processing an image $f(x, y)$ of size $M \times N$, the two-dimensional DFT is defined as:

$$\mathcal{F}(u, v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x, y)e^{-i2\pi(\frac{ux}{M}+\frac{vy}{N})} \tag{9}$$

where $x, u = 0, 1, 2, ..., M-1$ and $y, v = 0, 1, 2, ..., N-1$.

Similar to the conversion from Eq. 1 to Eq. 2, the inverse of 2D DFT can be written as:

$$f(x, y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} \mathcal{F}(u, v)e^{i2\pi(\frac{ux}{M}+\frac{vy}{N})} \tag{10}$$

where $x, u = 0, 1, 2, ..., M-1$ and $y, v = 0, 1, 2, ..., N-1$.

Comparing Eq. 1 with Eq. 9, we can observe the computing efficiency of transforming high dimensional data from the time domain to the frequency domain becomes tedious with naive DFT.

$$MN \times MN = (MN)^2 = O((MN)^2)$$

That's why developing Fast Fourier transform (FFT) algorithm is essential for practical application. And we'll elaborate on it in the following sections.

3 Description of Algorithms

3.1 Algorithms 1: Cooley–Tukey algorithm

3.1.1 Description

This famous algorithm was invented by Cooley, and Tukey [7], engineers at the IBM research center in the early 1960s. It has had a considerable impact on the development of digital signal processing applications due to its

efficiency [14]. A discrete Fourier transform calculation is a product of a matrix and a vector. It, therefore, requires T^2 multiplications/additions of complex numbers (T is the order of the matrix).

Assuming that a computer performs 10^9 operations per second, a transform calculation on a signal of $T = 10^3$ samples will require 10^{-3}s. A calculation on an image of size $T \times T = 10^6$ will require T^4 or 10^{12} operations which costs fifteen minutes. If we consider processing data in a three-dimensional domain (on vectors of size $T \times T \times T$), we would need T^6 or 10^{18} operations, which require a few decades.

The fast Fourier transform considerably reduces the number of operations: instead of performing T^2 operations, it will suffice to perform $T \log_2 T$. In the three previous examples, we will have to perform 10^4, 2×10^7 and 3×10^{10} operations which will require respectively $10^{-5}s$, $2 \times 10^{-2}s$ and $30s$. To explain this algorithm, we will use recursion by showing that the computation of a Fourier transform of size T comes down to the computation of two Fourier transforms of size $T/2$ followed by $T/2$ multiplications.

We want to calculate for $k = 0, \cdots, T-1$

$$X(k) = \sum_{t=0}^{T-1} x(t) \exp(-2\pi j \frac{k.t}{T}) \tag{11}$$

We pose $t = 2n$ if t is even and $t = 2n+1$ if t is odd. $X(k)$ is then written by posing $N = T/2$

$$X(k) = \sum_{n=0}^{N-1} x(2n) \exp(-2\pi j \frac{k.2n}{T}) + \sum_{n=0}^{N-1} x(2n+1) \exp(-2\pi j \frac{k.(2n+1)}{T}) \tag{12}$$

Let us name the sequences.

$$\begin{gathered} t = 0, \cdots, 2N-1 : x_{2N}(t) = x(t) \\ n = 0, \cdots, N-1 : X_N^{even}(k)(n) = x(2n) \\ n = 0, \cdots, N-1 : X_N^{odd}(k)(n) = x(2n+1) \\ k = 0, \cdots, 2N-1 : X_{2N}(k) = X(k) \end{gathered}$$

With these notations, we get

$$X_{2N}(k) = \sum_{n=0}^{N-1} x_N^{even}(n) \exp\left(-2\pi j \frac{k \cdot n}{N}\right) + \sum_{n=0}^{N-1} x_N^{odd}(n) \exp\left(-2\pi j \frac{k \cdot n}{N}\right) \exp\left(-2\pi j \frac{k}{2N}\right) \tag{13}$$

In the second summation of the right-hand side of the equation, the factor $exp(-2\pi j \frac{k}{2N})$ does not depend on n. We therefore have the following equation for $k = 0, \cdots, 2N-1$.

$$X_{2N}(k) = \left[\sum_{n=0}^{N-1} x_N^{even}(n) \exp\left(-2\pi j \frac{k \cdot n}{N}\right)\right] + \exp\left(-2\pi j \frac{k}{2N}\right) \left[\sum_{n=0}^{N-1} x_N^{odd}(n) \exp\left(-2\pi j \frac{k \cdot n}{N}\right)\right] \tag{14}$$

If $0 \leq k \leq N-1$, we recognize in the two expressions between square brackets the discrete Fourier transforms of the sequences of the even-numbered samples $X_N^{even}(n)$ and the odd-numbered samples $X_N^{odd}(n)$ which we call $X_N^{even}(k)$ and $X_N^{odd}(k)$ respectively.

Therefore, for $k = 0, \cdots, N-1$

$$X_{2N}(k) = X_N^{even}(k) + \exp(-\pi j \frac{k}{N}) X_N^{odd}(k) \tag{15}$$

When $N \leq k \leq 2N-1$, we can write

$$k = \ell + N \tag{16}$$

and notice that

$$\exp(-\pi j \frac{k}{N}) = -\exp(-\pi j \frac{\ell}{N}) \tag{17}$$

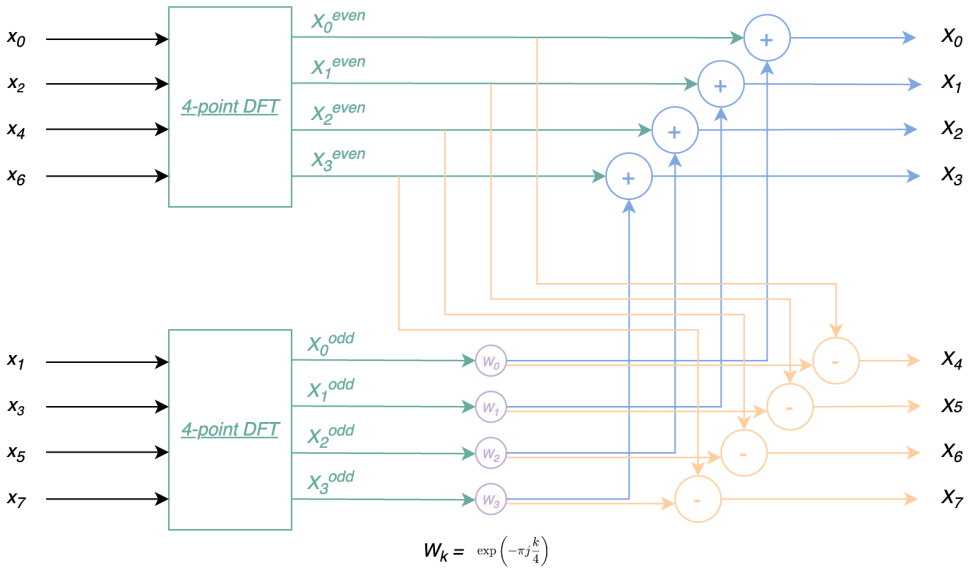

Figure 1: Cooley-Tukey functional diagram

Therefore, for $\ell = 0, \cdots, N-1$, the following equation is used

$$X_{2N}(\ell+N) = \left[\sum_{n=0}^{N-1} X_N^{even}(n) \exp(-2\pi j \frac{(\ell+N).n}{N})\right] + \exp\left(-2\pi j \frac{\ell+N}{2N}\right) \left[\sum_{n=0}^{N-1} X_N^{odd}(n) \exp(-2\pi j \frac{(\ell+N).n}{N})\right] \tag{18}$$

and noting that

$$\exp(-2\pi j \frac{(\ell+N).n}{N}) = \exp(-2\pi j \frac{\ell.n}{N}) \tag{19}$$

we have an analogous writing for $\ell = 0, \cdots, N-1$

$$X_{2N}(\ell+N) = X_N^{even}(\ell) - exp(-\pi j \frac{\ell}{N}) X_N^{odd}(\ell) \tag{20}$$

We can change the name of the variable ℓ to k and combine the two equations $X_{2N}(k)$ and $X_{2N}(k+N)$ for $k = 0, \cdots, N-1$

$$X_{2N}(k) = X_N^{even}(k) + \exp\left(-\pi j \frac{k}{N}\right) X_N^{odd}(k) \tag{21}$$

$$X_{2N}(k+N) = X_N^{even}(k) - \exp\left(-\pi j \frac{k}{N}\right) X_N^{odd}(k) \tag{22}$$

3.1.2 Functionality principal

It is clear that the calculation of the DFT components $X(k)$ in 11 for $k = 0, \cdots, 2N-1$ could be transformed into the calculation of 21 for $k = 0, \cdots, N$ and 22 and for $k = N, \cdots, 2N-1$. The benefit from this method is the reuse of the sub-components $X_N^{even}(k)$ and $X_N^{odd}(k)$ for every $k = 0, \cdots, T/2$ and a corresponding $k+N$ as graphically illustrated in Figure 1.

In addition to reusing the sub-components $X_N^{even}(k)$ and $X_N^{odd}(k)$, the same principle applies to their internal calculation as they are DFTs per se. Accordingly, the problem gets transformed into following the divide and conquer algorithmic paradigm.

3.1.3 Algorithm pseudo-code and complexity analysis

Cooley–Tukey algorithm is an example of divide and conquer algorithms. The time complexity is reduced from $O(N^2)$ of original DFT to $O(NlogN)$.

Cooley–Tukey algorithm divides the N length input into two parts with the length $N/2$. The updating of the result requires $O(N)$ operations. So we have $L(N) = 2L(\frac{N}{2}) + O(N)$. According to Master Theorem, the time complexity is $O(NlogN)$ [15].

The algorithm can be seen as follows[15]:

Algorithm 2 Cooley–Tukey FFT algorithm

Input: An array from $x(0)$ to $x(N-1)$
Output: An array which is the results of DFT of the array from $x(0)$ to $x(N-1)$
$N = length(x)$
if $N = 1$ **then**
  return $X[0]$
else
  $x^{even} = (x_0, x_2, \cdots, x_{N-2})$
  $x^{odd} = (x_1, x_3, \cdots, x_{N-1})$
  $X_{N/2}^{even} = FFT(x_0, x_2, \cdots, x_{N-2})$
  $X_{N/2}^{odd} = FFT(x_1, x_3, \cdots, x_{N-1})$
  for $k = 0$ to $N/2 - 1$ **do**
    $X_N(k) = X_{N/2}^{even}(k) + e^{-j2\pi k/N} X_{N/2}^{odd}(k)$
    $X_N(k + N/2) = X_{N/2}^{even}(k) - e^{-j2\pi k/N} X_{N/2}^{odd}(k)$
  end for
end if
return X_N

3.2 Algorithms 2: Radix-4 FFT Algorithm

3.2.1 Description

The first idea is decimation in time. It means to divide the sequence x(n) in the time domain. Coolkey-Tukey is the Radix-2 time-drawn FFT algorithm, we know that the Radix-2 algorithm divides x(n) into two groups of parity according to the value of n. The purpose is to transform the DFT of computing N points into the DFT of computing 2 points at N/2 point. And then divide into the DFT of computing 4 points at N/4 points. Such operation will be repeated until the sequence is decomposed into the operation of two-point DFT, which is simple addition and subtraction. [8] Therefore, it is easy to extend the conclusion that x(n) can be decomposed into more groups.

Radix-4 FFT algorithm is divide the sequence which length $N = 4^l$ into 4 sequence. As we have explained above, we can transform the DFT of computing N points into the DFT of computing 4 points at N/4 point. Then we transform the DFT of computing N/4 points into the DFT of computing 4 points at N/16 point. In this way, finally, we get the operation of Radix-4 FFT.

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{i2\pi \frac{kn}{N}} \tag{23}$$

By the definition of DFT Eq.23, we could rewrite the formula to the following equation 24.

$$\begin{aligned} X(k) = &\sum_{n=0}^{\frac{N}{4}-1} x(4n) e^{-\left(i\frac{2\pi\cdot(4n)k}{N}\right)} + \sum_{n=0}^{\frac{N}{4}-1} x(4n+1) e^{-\left(i\frac{2\pi(4n+1)k}{N}\right)} \\ &+ \sum_{n=0}^{\frac{N}{4}-1} x(4n+2) e^{-\left(i\frac{2\pi(4n+2)k}{N}\right)} + \sum_{n=0}^{\frac{N}{4}-1} x(4n+3) e^{-\left(i\frac{2\pi(4n+3)k}{N}\right)} \end{aligned} \tag{24}$$

Thus, DFT on a large input can be solved by performing DFT on partitioned inputs four times, and the inputs are split in this way.

$$\mathrm{DFT}_N[x] = \mathrm{DFT}_{\frac{N}{4}}[x_0] + W_N^k \cdot \mathrm{DFT}_{\frac{N}{4}}[x_1] + W_N^{2k} \cdot \mathrm{DFT}_{\frac{N}{4}}[x_2] + W_N^{3k} \cdot \mathrm{DFT}_{\frac{N}{4}}[x_3] \tag{25}$$

$$\begin{aligned} x_0 &= x(0), x(4), x(8), \cdots, x(4N-4) \\ x_1 &= x(1), x(5), x(9), \cdots, x(4N-3) \\ x_2 &= x(3), x(6), x(10), \cdots, x(4N-2) \\ x_2 &= x(4), x(7), x(11), \cdots, x(4N-1) \end{aligned} \tag{26}$$

The Radix-4 butterfly equations 27 are as follows, where $0 <= k <= N/4 - 1$, $r \in \mathbb{Z}$. $X_0, X_1, ..., X_3$ are outputs of DFT.

$$\begin{aligned} X(k + 0N/4) &= X_0(k) + W_N^k X_1(k) + W_N^{2k} X_2(k) + W_N^{3k} X_3(k) \\ X(k + 1N/4) &= X_0(k) + W_N^{k+N/4} X_1(k) + W_N^{2(k+N/4)} X_2(k) + W_N^{3(k+N/4)} X_3(k) \\ X(k + 2N/4) &= X_0(k) + W_N^{k+N/2} X_1(k) + W_N^{2(k+N/2)} X_2(k) + W_N^{3(k+N/2)} X_3(k) \\ X(k + 3N/4) &= X_0(k) + W_N^{k+3N/4} X_1(k) + W_N^{2(k+3N/4)} X_2(k) + W_N^{3(k+3N/4)} X_3(k) \end{aligned} \tag{27}$$

We could simplify Eq. 27 to:

$$\begin{aligned} X(k) &= X_0(k) + W_N^k X_1(k) + W_N^{2k} X_2(k) + W_N^{3k} X_3(k) \\ X(k + N/4) &= X_0(k) - jW_N^k X_1(k) - W_N^{2k} X_2(k) + jW_N^{3k} X_3(k) \\ X(k + N/2) &= X_0(k) - W_N^k X_1(k) + W_N^{2k} X_2(k) - W_N^{3k} X_3(k) \\ X(k + 3N/4) &= X_0(k) + jW_N^k X_1(k) - W_N^{2k} X_2(k) - jW_N^{3k} X_3(k) \end{aligned} \tag{28}$$

3.2.2 Analysis

From above, we can find that there are 12 complex addition operations in Radix-4 FFT.

If the FFT length N = 4^M, DFT can continue recursively decomposed. To determine the total computational cost, considering that the number of stage is $M = log_4(N) = \frac{log_2(N)}{2}$, and each stage need $\frac{N}{4}$ butterflies, the complexity should be:

$$\text{Complex Multiplies} = 3\frac{N}{4}\frac{log_2(N)}{2} = \frac{3}{8}Nlog_2(N) \tag{29}$$

$$\text{Complex Add} = 8\frac{N}{4}\frac{log_2(N)}{2} = Nlog_2(N) \tag{30}$$

The algorithm can be seen as follows 3:

4 Plan of Implementation

We are planning to create three programs in total. The detailed plans are listed as follows:

4.1 Create software for the naive DFT

1. We will use Python to implement the naive DFT according to Alg. 1.
2. We will validate the correctness of our implementation by comparing it to some official softwares like Matlab.
3. The implemented DFT will be our baseline to evaluate our FFT.

Algorithm 3 Radix-4 FFT algorithm

Input: An array from $x(0)$ to $x(N-1)$
Output: An array which is the results of DFT of the array from $x(0)$ to $x(N-1)$
$N = length(x)$
$x_0 = (x(0), x(4), \cdots, x(N-4))$
$x_1 = (x(1), x(5), \cdots, x(N-3))$
$x_2 = (x(2), x(6), \cdots, x(N-2))$
$x_3 = (x(3), x(7), \cdots, x(N-1))$
$X_0 = FFT(x_0)$
$X_1 = FFT(x_1)$
$X_2 = FFT(x_2)$
$X_3 = FFT(x_3)$
$W_N^{kn} = e^{-i2\pi \frac{kn}{N}}$
$W_N^{2kn} = e^{-i2\pi \frac{2kn}{N}}$
$W_N^{3kn} = e^{-i2\pi \frac{3kn}{N}}$
for $k = 0$ to $N/4 - 1$ **do**
    $X(k) = X_0(k) + W_N^k X_1(k) + W_N^{2k} X_2(k) + W_N^{3k} X_3(k)$
    $X(k + N/4) = X_0(k) - jW_N^k X_1(k) - W_N^{2k} X_2(k) + jW_N^{3k} X_3(k)$
    $X(k + N/2) = X_0(k) - W_N^k X_1(k) + W_N^{2k} X_2(k) - W_N^{3k} X_3(k)$
    $X(k + 3N/4) = X_0(k) + jW_N^k X_1(k) - W_N^{2k} X_2(k) - jW_N^{3k} X_3(k)$
end for
return X

4.2 Create software for Cooley-Tukey algorithm

1. We will use Python to implement Cooley-Tukey algorithm.
2. We will validate the correctness of our implementation by comparing it to the naive DFT.
3. We will compare the performance with the naive DFT and Radix-4 FFT.

4.3 Create software for Radix-4 FFT algorithm

1. We will use Python to implement Radix-4 FFT algorithm.
2. We will validate the correctness of our implementation by comparing it to the naive DFT.
3. We will compare the performance with the naive DFT and Cooley-Tukey.

5 Plan of Experiments

1. We will try data samples of different sizes and record the runtime of the algorithms to see if it matches our theoretical analysis.
2. We will record the experiment results to fill Tb. 1.
3. We will visualize the curve of runtime and FLOPS as the data size increase for better comparison of algorithms.
4. We will summarize the results and write our final experiment report.

	Algorithm	DFT		Cooley-Tukey		Radix-4	
		Runtime	FLOPS	Runtime	FLOPS	Runtime	FLOPS
Data Size	1						
	10						
	100						
	1,000						
	10,000						
	100,000						
	1,000,000						

Table 1: Runtime is in seconds, and FLOPS means how many times of addition and multiplication.

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