# Homework 2

## Fall 2020: CS 260 Design and Analysis of Algorithms

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**Instructions:** This homework is due on **Sunday, November 22, 2020**. You can work in groups and should submit the homework by e-mail to Liuwei Wang. Write names of group members clearly on top of the first page as well as the total number of pages. You may have queries related to this homework which should be directed to Liuwei Wang.

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Office hours: Tuesday, 8:30–10am
`https://kaust.zoom.us/j/95911468021`

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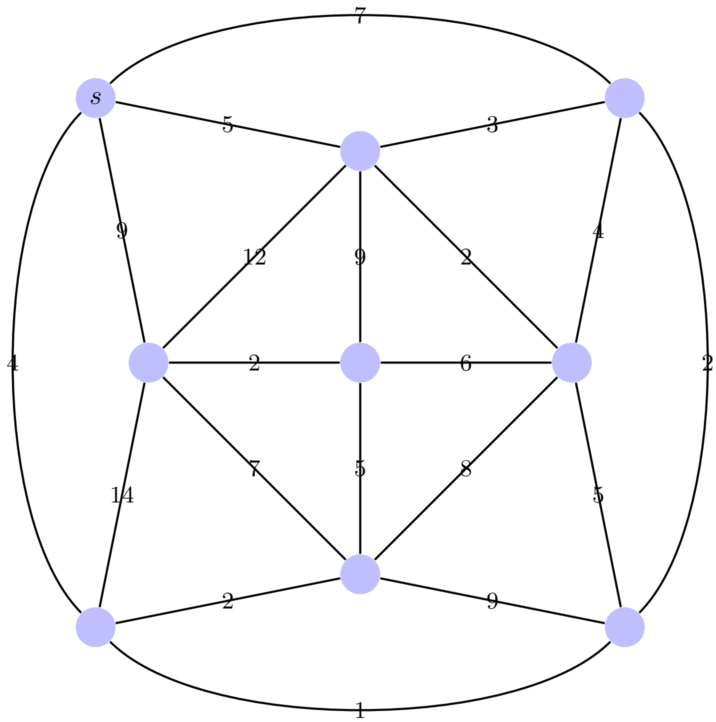
1. **(5 points)** Given an alphabet $\Sigma = \{\mathtt{b}, \mathtt{a}, \mathtt{l}, \mathtt{c}, \mathtt{k}, \mathtt{s}, \mathtt{h}, \mathtt{e}, \mathtt{p}\}$, construct an optimal binary prefix-free code for the sentence `baa baa black sheep` (ignore all whitespace) using Huffman's algorithm.

2. **(12 points)** Let $G$ be a connected undirected weighted graph such that no two edges have the same weight. Prove that $G$ has a unique minimum cost spanning tree.

3. **(7 points)** Let $G_1 = (V, E)$ and $G_2 = (V, E)$ be two *directed graphs* with the same structure (i.e., same vertices and same edges). Let the weights of edges in $G_1$ be *distinct and nonnegative* and the weights of edges in $G_2$ be *squares* of weights of corresponding edges in $G_1$. Prove or disprove that *shortest paths* in $G_1$ and $G_2$ from some vertex $s$ to any other vertex $t$ constructed by DIJKSTRA's algorithm are the same.

4. **(14 points)** You just moved to KAUST and plan to attend the orientation sessions. The orientation consists of $n$ sessions. Each session $i$ is described by a triplet of integers $(S_i, D_i, F_i)$, where $S_i$ indicates the session start time, $D_i$ represents the duration of the session, and $F_i$ indicates the session finish time. You would like to attend as many sessions as possible. Note that sessions may be overlapping and you can't attend a session after it is already started and you can't leave a session before it finishes. Design an $O(n \log n)$ greedy algorithm that will allow you to do so and prove that your algorithm is correct.

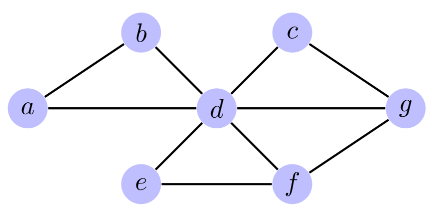
5. **(5 points)** Use PRIM's algorithm to find a minimum spanning tree of the following undirected graph $G$, starting with the node $s$.




6. **(5 points)** Apply SELECT algorithm to find the median in the following array $A$ of integers:

$$A = \boxed{12 \mid 4 \mid 1 \mid 17 \mid 10 \mid 18 \mid 5 \mid 3 \mid 9 \mid 19}\,.$$

7. **(5 points)** Apply CONTRACTION algorithm to the graph




with the following numbering of edges: $(a, b) - 1$, $(a, d) - 2$, $(b, d) - 3$, $(c, d) - 4$, $(c, g) - 5$, $(d, g) - 6$, $(d, f) - 7$, $(d, e) - 8$, $(e, f) - 9$, $(f, g) - 10$. Update the edge indices after each contraction.

8. **(12 points)** Let $A[1..n]$ be an array of *nonnegative integers* and assume there are exactly $k$ 0's in $A$, note that $0 < k < n$. Consider the following algorithm and compute the expected number of times Step **(1:)** will be executed.

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**Input:** An array $A[1..n]$ of nonnegative integers.
**Output:** An array $B[1..k]$ of indices of all 0's in $A$.

(**1:**) Generate a random number $j$ in $\{1, \ldots, n\}$

(**2:**) If $A[j] = 0$ then we add $j$ to $B$ if $j$ is not already in $B$

(**3:**) Repeat steps **(1:)** and **(2:)** until $|B| = k$

(**4:**) Return $B$

For example, if $A = [0, 5, 2, 0, 8, 9, 0, 3, 1, 7, 0]$, then one possible solution is $B = [4, 11, 7, 1]$ (the elements in $B$ might not be ordered).

9. **(15 points)** For an undirected graph $G = (V, E)$, a *dominating set* is a subset $S \subseteq V$ such that, for each vertex $v \in V$, $v \in S$ or a neighbor $u$ of $v$ is in $S$. Prove that *the dominating set problem* $\mathrm{DS}(G, k)$, which requires to check if the graph $G$ contains a dominating set $S$ of size $k$, is NP-complete.

10. **(10 points)** We say that a problem (a language) $L_1 \subseteq A^*$ is a *linear-time reducible* to problem (a language) $L_2 \subseteq B^*$ (written $L_1 < L_2$) if there exists a linear-time computable function $\varphi : A^* \to B^*$ such that for all $\alpha \in A^*$, $\alpha \in L_1$ if and only if $\varphi(\alpha) \in L_2$.

    Suppose we know $L_1 < L_2$ (i.e., $L_1$ is linear-time reducible to $L_2$) and $L_2 < L_3$. Assume that the time complexity of $L_2$ is $\Theta(n^2)$.

    (a) What we can say about the time complexity of $L_1$? Justify your answer.

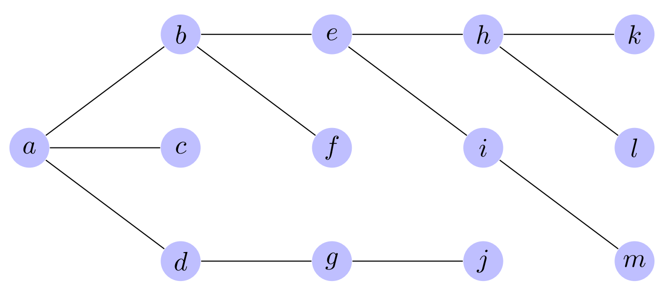
    (b) What we can say about the time complexity of $L_3$? Justify your answer.

11. **(5 points)** For a set cover problem $U, F$, construct a cover using greedy algorithm:

$$U = \{1, 2, 3, 4, 5, 6, 7\}, \quad F = \{S_1, S_2, S_3, S_4, S_5\},$$

$$S_1 = \{1, 2, 7\},\ S_2 = \{2, 3, 4\},\ S_3 = \{3, 6\},\ S_4 = \{4, 5\},\ S_5 = \{3, 4\}.$$

12. **(5 points)** Find the cardinality of a maximum independent set of the following tree using dynamic programming algorithm.




**Note:** For Questions 6 and 7. Following is a sequence of random numbers in the interval $[0, 1]$. To get a random number $x$ in the interval $[1, r]$ multiply a number in the following sequence by $r$ and take the ceiling. The random sequence (use numbers in the order from left to right starting from the first row):

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.9474 | 0.6753 | 0.2297 | 0.2868 | 0.4932 | 0.6638 | 0.7089 | 0.7389 | 0.7721 | 0.4464 | 0.5518 | 0.8441 |
| 0.2151 | 0.0380 | 0.4750 | 0.1149 | 0.0798 | 0.6941 | 0.7539 | 0.2509 | 0.3507 | 0.3278 | 0.9583 | 0.4558 |
| 0.0953 | 0.1541 | 0.2370 | 0.4146 | 0.8375 | 0.9756 | 0.0187 | 0.1359 | 0.2651 | 0.0282 | 0.6054 | 0.9693 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.3329 | 0.6366 | 0.4607 | 0.0881 | 0.9655 | 0.0405 | 0.8081 | 0.6355 | 0.6108 | 0.6023 | 0.8709 | 0.0588 |
| 0.4438 | 0.2229 | 0.4345 | 0.7614 | 0.1860 | 0.8683 | 0.9245 | 0.9208 | 0.9040 | 0.2328 | 0.3873 | 0.9189 . |
| 0.9972 | 0.4622 | 0.5659 | 0.8715 | 0.2970 | 0.4622 | 0.3507 | 0.1762 | 0.6662 | 0.2718 | 0.0096 | 0.8994 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0060 | 0.2403 | 0.9021 | 0.2254 | 0.6188 | 0.2925 | 0.5966 | 0.2958 | 0.0103 | 0.6474 | 0.7128 | 0.6137 |
| 0.5730 | 0.2320 | 0.9132 | 0.6387 | 0.9524 | 0.2398 | 0.5966 | 0.7006 | 0.5001 | 0.1706 | 0.1036 | 0.8058 |
| 0.2165 | 0.4697 | 0.1688 | 0.2869 | 0.1821 | 0.3031 | 0.0466 | 0.3378 | 0.5366 | 0.5417 | 0.2064 | 0.7757 |