CS260: ASSIGNMENT 3

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1 Problem 1

1. Yes. We can make limit the computation to stop after 1 hour and output the result that the program arrived at that point.
2. No. whether a program finishes is an undecidable problem (Halt problem).
3. No. This is the undecidable halting problem.
4. Yes if this program cannot alter its instructions by itself. We could do the static analysis.

2 Problem 2

Assume the range of f is set D, i.e. $D = ran(f) = \{y = f(x)\}$.

- Firstly, we can prove that D is an infinite set.
  Since f is unbounded, then
  $$\forall M, \exists x, f(x) > M$$
  Accordingly,
  $$\forall M, \exists f(x), f(x) > M$$
  That means that by construction of D,
  $$\forall M, \exists y \in D, y > M$$
  Similarly, from the other direction (if we suppose f is not only positive, otherwise this line is to be skipped),
  $$\forall M, \exists x, f(x) < M$$
  , Which means
  $$\forall M, \exists y \in D, y < M$$
  Thus, D is infinite.
- Secondly, we can prove that D is a recursive set.
  As we know, a set of natural numbers is called computable, recursive, or decidable if there is an algorithm that takes a number as input, terminates after a finite amount of time, and correctly decides whether the number belongs to the set or not.
  Let z be an element of $D = ran(f)$. We search for the smallest x such that $f(x) > z$.
  We are sure that x exists because f is unbounded (as above it means $\forall M, \exists x, f(x) > M$).

  Since f is non-decreasing and x is the smallest such that $f(x) > z$, then
  $$z \in D = ran(f) \iff z \in \{f(0), \ldots, f(x)\}$$
  .
  We have $\{f(0), \ldots, f(x)\}$ is finite and computable **so every input z we can decide if it is an element of this set or not in a finite amount of time**.
  Accordingly, D is a recursive set.

Therefore, D is an infinite recursive set.

3 Problem 3

Let's say our infinite recursive set is A and $A \subseteq \mathbb{N}$. We can define our function f using the minimization operation μ introduced in our class:

$$f(0) = \mu_y, y \in A;$$
$$f(n+1) = \mu_y, y \in A \; and \; y > f(n);$$

From the definition of μ-operator, we know f is recursive. From how we define f, we know f must be no-decreasing. From how we define f, we know it accepts one variable. From A is infinite, we know f is unbounded.

Therefore, we have shown that every infinite recursive set is the range of a non-decreasing and unbounded function with one variable.

Part of our solution refers to *Mathematical logic: a course with exercises Part II: Recursion theory, Gödel's theorems, set theory, model theory, by René Cori and Daniel Lascar, translated by Donald H. Pelletier. (Oxford University Press)*

4 Problem 4

Based on Post's theorem, L_i is recursive if and only if L_i and $w \backslash L_i$ are recursively enumerable sets $\forall i \in \{1, 2, 3\}$. We have

$$w \backslash L_i = \bigcup_{j \neq i} L_j$$

and L_j are recursively enumerable $\forall j \in \{1, 2, 3\}$. Accordingly, $w \backslash L_i$ are recursively enumerable sets $\forall i \in \{1, 2, 3\}$. In addition, since, again L_i is recursively enumerable $\forall i \in \{1, 2, 3\}$ then, now both $w \backslash L_i$ and L_i are recursively enumerable. Therefore L_i is recursive $\forall i \in \{1, 2, 3\}$.