Homework 2

Fall 2022: CS 260 Design and Analysis of Algorithms

Instructions: This homework is due on **Monday, November 21, 2022** in class. You can work in groups and should submit the homework handwritten on A4-size papers. Write names of group members clearly on top of the first page as well as the total number of pages. You may have queries related to this homework which should be directed to the TA.

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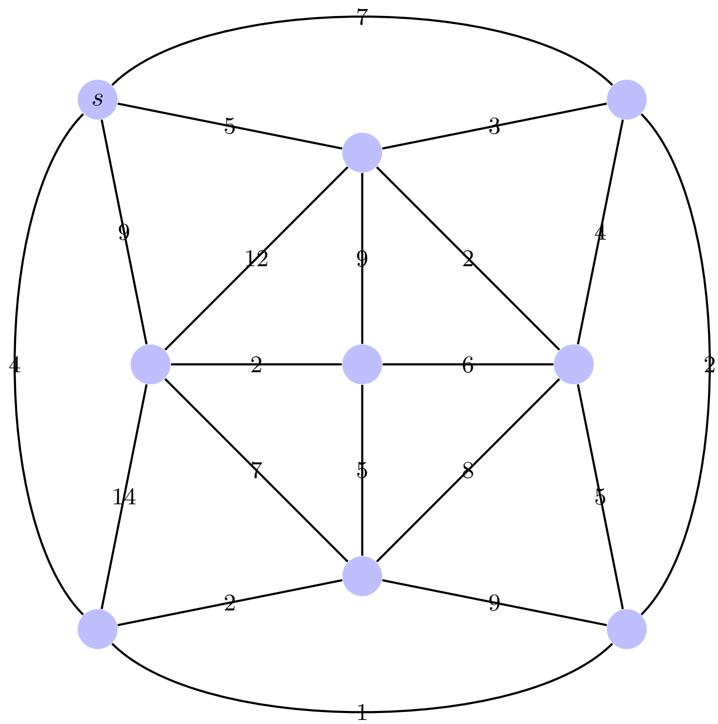
1. **(10 points)** Let G be a connected undirected weighted graph such that no two edges have the same weight. Prove that G has a unique minimum cost spanning tree.

2. **(7 points)** Let $G_1 = (V, E)$ and $G_2 = (V, E)$ be two *directed graphs* with the same structure (i.e., same vertices and same edges). Let the weights of edges in G_1 be *distinct and nonnegative* and the weights of edges in G_2 be *squares* of weights of corresponding edges in G_1. Prove or disprove that *shortest paths* in G_1 and G_2 from some vertex s to any other vertex t constructed by DIJKSTRA's algorithm are the same.

3. **(14 points)** You just moved to KAUST and plan to attend the orientation sessions. The orientation consists of n sessions. Each session i is described by a triplet of integers (S_i, D_i, F_i), where S_i indicates the session start time, D_i represents the duration of the session, and F_i indicates the session finish time. You would like to attend as many sessions as possible. Note that sessions may be overlapping and you can't attend a session after it is already started and you can't leave a session before it finishes. Design an $O(n \log n)$ greedy algorithm that will allow you to do so and prove that your algorithm is correct.

4. **(5 points)** Use PRIM's algorithm to find a minimum spanning tree of the following undirected graph G, starting with the node s.



5. **(5 points)** Apply SELECT algorithm to find the median in the array A of integers. Please use the random numbers at the bottom of the last page in order.

$A =$

12	4	1	17	10	18	5	3	9	19

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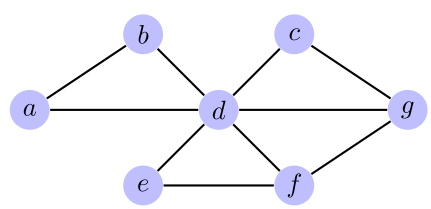
6. **(5 points)** Apply CONTRACTION algorithm to the given graph. Edges should be contracted using the given order of edges such that the edge with the lowest index is contracted first.



Edge numbering: $(a, b) - 1$, $(a, d) - 2$, $(b, d) - 3$, $(c, d) - 4$, $(c, g) - 5$, $(d, g) - 6$, $(d, f) - 7$, $(d, e) - 8$, $(e, f) - 9$, $(f, g) - 10$.

7. **(10 points)** Let $A[1..n]$ be an array of *nonnegative integers* and assume there are exactly k 0's in A, note that $0 < k < n$. Consider the following algorithm and compute the expected number of times Step **(1:)** will be executed.

Input: An array $A[1..n]$ of nonnegative integers.
Output: An array $B[1..k]$ of indices of all 0's in A.

(1:) Generate a random number j in $\{1, \dots, n\}$

(2:) If $A[j] = 0$ then we add j to B if j is not already in B

(3:) Repeat steps **(1:)** and **(2:)** until $|B| = k$

(**4:**) Return B

For example, if $A = [0, 5, 2, 0, 8, 9, 0, 3, 1, 7, 0]$, then one possible solution is $B = [4, 11, 7, 1]$ (the elements in B might not be ordered).

8. **(14 points)** For an undirected graph $G = (V, E)$, a *dominating set* is a subset $S \subseteq V$ such that for each vertex $v \in V$, $v \in S$ or a neighbor u of v is in S. Prove that *the dominating set problem* $\mathrm{DS}(G, k)$ such that the graph G contains a dominating set S of size k is **NP**-complete.

9. **(10 points)** We say that a problem (a language) $L_1 \subseteq A^*$ is a *linear-time reducible* to problem (a language) $L_2 \subseteq B^*$ (written $L_1 < L_2$) if there exists a linear-time computable function $\varphi : A^* \to B^*$ such that for all $\alpha \in A^*$, $\alpha \in L_1$ if and only if $\varphi(\alpha) \in L_2$.

   Suppose we know $L_1 < L_2$ (i.e., L_1 is linear-time reducible to L_2) and $L_2 < L_3$. Assume that the time complexity of L_2 is $\Theta(n^2)$.

   (a) What we can say about the time complexity of L_1? Justify your answer.

   (b) What we can say about the time complexity of L_3? Justify your answer.

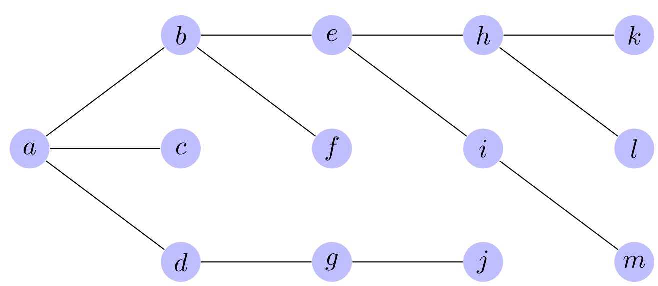
10. **(10 points)** Given an undirected graph $G = (V, E)$, we define a *clique* of size k in G for some positive integer k, as a complete subgraph of G with k vertices. Prove that to find whether a graph G contains a clique of size at least k is an **NP**-complete problem.

11. **(5 points)** For a set cover problem U, F, construct a cover using greedy algorithm:

$$U = \{1, 2, 3, 4, 5, 6, 7\}, \quad F = \{S_1, S_2, S_3, S_4, S_5\},$$

$$S_1 = \{1, 2, 7\},\ S_2 = \{2, 3, 4\},\ S_3 = \{3, 6\},\ S_4 = \{4, 5\},\ S_5 = \{3, 4\}.$$

12. **(5 points)** Find the cardinality of a maximum independent set of the following tree using dynamic programming algorithm.



Note: For Questions 5 and 6. Following is a sequence of random numbers in the interval $[0, 1]$. To get a random number x in the interval $[1, r]$ multiply a number in the following sequence by r and take the ceiling. The random sequence:

0.9474	0.6753	0.2297	0.2868	0.4932	0.6638	0.7089	0.7389	0.7721	0.4464	0.5518	0.8441
0.2151	0.0380	0.4750	0.1149	0.0798	0.6941	0.7539	0.2509	0.3507	0.3278	0.9583	0.4558
0.0953	0.1541	0.2370	0.4146	0.8375	0.9756	0.0187	0.1359	0.2651	0.0282	0.6054	0.9693
0.3329	0.6366	0.4607	0.0881	0.9655	0.0405	0.8081	0.6355	0.6108	0.6023	0.8709	0.0588
0.4438	0.2229	0.4345	0.7614	0.1860	0.8683	0.9245	0.9208	0.9040	0.2328	0.3873	0.9189 .
0.9972	0.4622	0.5659	0.8715	0.2970	0.4622	0.3507	0.1762	0.6662	0.2718	0.0096	0.8994
0.0060	0.2403	0.9021	0.2254	0.6188	0.2925	0.5966	0.2958	0.0103	0.6474	0.7128	0.6137
0.5730	0.2320	0.9132	0.6387	0.9524	0.2398	0.5966	0.7006	0.5001	0.1706	0.1036	0.8058
0.2165	0.4697	0.1688	0.2869	0.1821	0.3031	0.0466	0.3378	0.5366	0.5417	0.2064	0.7757