Analysis of nonlinear oscillator equations with normal form, parametrisation of invariant manifold and symbolic computations

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This is a document note that records the work done on explaining how the parametrisation method can be used for analysis of nonlinear oscillator equations, thanks to a symbolic version of an arbitrary order expansion. This gives a powerful tool to analyze numerous problems in vibration theory. The study starts with the normal form style, and a special emphasis is put on the computation of the different variants of normal form: complex, real and oscillator normal form. Analytic expressions up to arbitrary order are given. Analytical backbone curve for CNF. Then the presence of a forcing term is analyzed thanks to a non-autonomous version of the automated solution of the parametrisation methods. Extensions to systems with large number of dofs are highlighted. Parametric resonance is discussed.

1 Normal form for single nonlinear oscillator equations

1.1 Duffing equation with cubic nonlinearity, unforced and undamped

Starting point:

$$\ddot{u} + \omega_1^2 u + h u^3 = 0. \tag{1}$$

The aim of this first section is to show the different styles of normal forms one can use to solve such an osilator like equation. We will distinguish the complex normal form (CNF), the real normal form (RNF) and the oscillator normal form (ONF). Normal form is not unique because of the presence of free functions in the asymptotic developments, letting a freedom for the user to write and analyze the associated normal form. A good way to see this is also related to the interpretation one can do of the resonance relationships.

General case, N oscillators, eigenvalues $\{\pm i\omega_k\}_{k=1,\dots,N}$. Let us denote as:

$$\boldsymbol{\lambda} = [\lambda_1, \ \ \lambda_2, \quad \dots, \lambda_N, \bar{\lambda}_1, \dots, \bar{\lambda}_N], \tag{2}$$

the $2N$ eigenvalues, with $\lambda_k = i\omega_k$, for $k = 1 \dots N$, and $\lambda_{k+N} = \bar{\lambda}_k = -i\omega_k$, for $k = 1 \dots N$.

The general resonance relationship (Poincaré) reads

$$\lambda_k = \sum_{i=1}^{2N} m_i \lambda_i, \tag{3}$$

with $m_i \geq 0$, and $\sum_{i=1}^{2N} m_i = p$, where p is the order considered.

Different interpretations of this relationship are

- CNF: strict interpretation, each eigenvalue is taken separately, so only $\lambda_k = \sum_{i=1}^{2N} m_i \lambda_i$ is a resonance.
- RNF: broader interpretation in the sense

$$|\lambda_k| = \left| \sum_{i=1}^{2N} m_i \lambda_i \right|. \tag{4}$$

This allows to give as resonant a relationhsip between one eigenvalue and its complex conjugate, see later for more details on the Duffing equation.

- ONF : in this case the interpretation is even broader. Since one wants to stick to the oscillator form, the monomials to be considered are the original ones, so (in case of simple Duffing as considered here) only u^2, u^3, etc... Assuming we are close to the linear solution, then $u^3 \sim exp(\pm i\omega t \pm i\omega t \pm i\omega t)$. Looking for resonance relationships leads to search when this term will resonante with the linear one, *e.g.* $exp(\pm i\omega t)$ so roughly speaking the resonance relationship reads

$$\pm i\omega \pm i\omega \pm i\omega = \pm i\omega. \tag{5}$$

And one decalres that the resonance relationship is fulfilled *as soon as one possibility exists in all the combinations given above*, since one cannot split the monomial u^3 into pieces with a linear change of coordinates. One way to rewrite the resonance relationship for the ONF is thus not with absolute value or square, but by writing that *a resoannce relationship is fullfilled as soon as one combination of the next is possible*:

$$\pm i\omega_k = \sum_{i=1}^{2N} m_i(\pm i\omega_i). \tag{6}$$

Further detail will be given in specific case, see below. All these different interpretations lead to classify differently the different monomials, depending on the style used. As a side note, ONF is not implemented in MORFE_Sym... So it is reminded here in order to bridge the gap with the early works on normal form.

Oscillator normal form (ONF) on (45): nothing to do ! Indeed u^3 is the resonant term associated to the trivial resonance. The story is different for CNF and RNF.

Diagonalisation and complexification

In the case of a single Duffing oscillator, the system can be diagonalized thanks to

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i\omega_1 & -i\omega_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{7}$$

resulting in

$$\dot{y}_1 = i\omega_1 y_1 + i\frac{h}{2\omega_1}\left(y_1^3 + 3y_1^2 y_2 + 3y_2 y_1^2 + y_2^3\right), \tag{8}$$

$$\dot{y}_2 = -i\omega_1 y_2 - i\frac{h}{2\omega_1}\left(y_1^3 + 3y_1^2 y_2 + 3y_2 y_1^2 + y_2^3\right). \tag{9}$$

An important property stems from the above equation, is that the second equation is the complex conjugate of the first, setting $\bar{y}_1 = y_2$ and $\bar{y}_2 = y_1$. This means that from these equations, the second one is always the complex conjugate of the first. This property is important and will be kept throughout the process.

To derive the normal form, the nonlinear change of coordinate is introduced as:

$$y_1 = z_1 + a_1 z_1^3 + a_2 z_1^2 z_2 + a_3 z_2 z_1^2 + a_4 z_2^3, \tag{10}$$

$$y_2 = z_2 + b_1 z_1^3 + b_2 z_1^2 z_2 + b_3 z_2 z_1^2 + b_4 z_2^3. \tag{11}$$

Expanding and identifying coefficients of like powers leads to two sets of equations for the 8 unknowns (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4). For the first equation:

$$z_1^3 \textbf{ term:} \quad (i\omega_1 + i\omega_1 + i\omega_1)a_1 = i\omega_1 a_1 + i\frac{h}{2\omega_1} \tag{12}$$

$$z_1^2 z_2 \textbf{ term:} \quad (i\omega_1 + i\omega_1 - i\omega_1)a_2 = i\omega_1 a_2 + i\frac{3h}{2\omega_1} \tag{13}$$

$$z_1 z_2^2 \textbf{ term:} \quad (i\omega_1 - i\omega_1 - i\omega_1)a_3 = i\omega_1 a_3 + i\frac{3h}{2\omega_1} \tag{14}$$

$$z_2^3 \textbf{ term:} \quad (-i\omega_1 - i\omega_1 - i\omega_1)a_4 = i\omega_1 a_4 + i\frac{h}{2\omega_1} \tag{15}$$

The interpretation of the resonance condition takes its full meaning here. The CNF style takes the most strict interpretation, such that only the second line is considered as resonant, meaning that, for the first equation, only $z_1^2 z_2$ will stay as trivially resonant monomial. In CNF, all other three monomials are non resonant and will be elimintaed by the change of coordinate. In the RNF, the different interpretation leads to consider both second and third lines as resonant (interpretation in terms of square). So two terms will stay in the normal form. Finally for the ONF, all terms need to be considered as resonant. As one can see, there is no simple way (using absolute values of squares or whatever) of fulfilling the first and last equation (transform a 3 to a 1 is very difficult...). So the only way to interpret the resonance relationship i the ONF is to say that *as soon as one resonance relationship for the initial monomial is fulfilled, then all terms are considered as resonant*, so even the first and last saying respectively $3i\omega_1 = i\omega_1$ and $-3i\omega_1 = i\omega_1$.

Similar result s and reasoning for the second equation:

$$z_1^3 \textbf{ term:} \quad (i\omega_1 + i\omega_1 + i\omega_1)b_1 = -i\omega_1 b_1 - i\frac{h}{2\omega_1} \tag{16}$$

$$z_1^2 z_2 \textbf{ term:} \quad (i\omega_1 + i\omega_1 - i\omega_1)b_2 = -i\omega_1 b_2 - i\frac{3h}{2\omega_1} \tag{17}$$

$$z_1 z_2^2 \textbf{ term:} \quad (i\omega_1 - i\omega_1 + i\omega_1)b_3 = -i\omega_1 b_3 - i\frac{3h}{2\omega_1} \tag{18}$$

$$z_2^3 \textbf{ term:} \quad (-i\omega_1 + i\omega_1 + i\omega_1)b_4 = -i\omega_1 b_4 - i\frac{h}{2\omega_1} \tag{19}$$

Thanks to MORFE_symbolic, one have direct access to full solutions.

Complex normal form

The mapping

$$u = z_1 + z_2 + \frac{h}{8\omega_1^2}(z_1^3 + z_2^3) - \frac{3h}{4\omega_1^2}(z_1^2 z_2 + z_1 z_2^2), \tag{20}$$

$$v = i\omega_1(z_1 - z_2) + \frac{3ih}{8\omega_1}(z_1^3 + z_1^2 z_2 - z_1 z_2^2 - z_2^3) \tag{21}$$

one remark: good to show the mapping $(y_1, y_2) \longleftrightarrow (z_1, z_2)$ to better show the resonant terms (in the reduced dynamics *vs* in the mapping). Probably some lines to add in julia MORFE_Sym to handle that.

The reduced dynamics

$$\dot{z}_1 = i\omega_1 z_1 + i\frac{3h}{2\omega_1} z_1^2 z_2, \tag{22}$$

$$\dot{z}_2 = -i\omega_1 z_2 - i\frac{3h}{2\omega_1} z_1 z_2^2. \tag{23}$$

Real normal form

The mapping

$$u = z_1 + z_2 + \frac{h}{8\omega_1^2}(z_1^3 + z_2^3) \tag{24}$$

$$v = i\omega_1(z_1 - z_2) + i\frac{3h}{8\omega_1}(z_1^3 + z_2^3) \tag{25}$$

The reduced dynamics

$$\dot{z}_1 = i\omega_1 z_1 + i\frac{3h}{2\omega_1}(z_1^2 z_2 + z_1 z_2^2) \tag{26a}$$

$$\dot{z}_2 = -i\omega_1 z_2 - i\frac{3h}{2\omega_1}(z_1^2 z_2 + z_1 z_2^2) \tag{26b}$$

Important to note the difference between the approaches, CNF keeps only one trivially resonant monomial per line, RNF keeps two and ONF keeps the four, needed to reconstruct the full real displacement. Consequences on the nonlinear mappings.

higher orders

MORFE_symbolic is a powerful tool. Computes to arbitrary order. So the expansions (limited to order three to let the things easy to understand and tractable) can be pushed at the desired order. For illustration order 7.

<u>CNF</u>

Reduced dynamics:

$$\dot{z}_1 = i\omega_1 z_1 + i\frac{3h}{2\omega_1}z_1^2 z_2 - i\frac{51h^2}{16\omega_1^3}z_1^3 z_2^2 + i\frac{1419h^3}{128\omega_1^5}z_1^4 z_2^3 - i\frac{47505h^4}{1024\omega_1^7}z_1^5 z_2^4 + i\frac{438825h^5}{2048\omega_1^7}z_1^6 z_2^5 \tag{27a}$$

$$\dot{z}_2 = -i\omega_1 z_2 - i\frac{3h}{2\omega_1}z_1 z_2^2 + i\frac{51h^2}{16\omega_1^3}z_1^2 z_2^3 - i\frac{1419h^3}{128\omega_1^5}z_1^3 z_2^4 + i\frac{47505h^4}{1024\omega_1^7}z_1^5 z_2^4 - i\frac{438825h^5}{2048\omega_1^7}z_1^6 z_2^5 \tag{27b}$$

One can observe once again that there is only one trivially resonant monomial per order. Beautiful. One could go to very large order so easily...

(corrected, jan. 2023, and up to order 11 to check with Claude's result (which seems to be wrong, missing the nasty term)))

Nonlinear mappings:

$$\begin{aligned} u = {} & z_1 + z_2 + \frac{h}{8\omega_1^2}(z_1^3 + z_2^3) - \frac{3h}{4\omega_1^2}(z_1^2 z_2 + z_1 z_2^2) \\ & + \frac{h^2}{64\omega_1^4}(z_1^5 + z_2^5) - \frac{39h^2}{64\omega_1^4}(z_1^4 z_2 + z_1 z_2^4) + \frac{69h^2}{32\omega_1^4}(z_1^3 z_2^2 + z_1^2 z_2^3) \\ & + \frac{h^3}{512\omega_1^6}(z_1^7 + z_2^7) - \frac{73h^3}{512\omega_1^6}(z_1^6 z_2 + z_1 z_2^6) + \frac{1569h^3}{512\omega_1^6}(z_1^5 z_2^2 + z_1^2 z_2^5) - \frac{2139h^3}{256\omega_1^6}(z_1^4 z_2^3 + z_1^3 z_2^4) \end{aligned} \tag{28a}$$

$$\begin{aligned} v = {} & i\omega_1(z_1 - z_2) + \frac{3ih}{8\omega_1}(z_1^3 + z_1^2 z_2 - z_1 z_2^2 - z_2^3) \\ & + \frac{5ih^2}{64\omega_1^3}(z_1^5 - z_2^5) - i\frac{81h^2}{64\omega_1^3}(z_1^4 z_2 - z_1 z_2^4) - i\frac{69h^2}{32\omega_1^3}(z_1^3 z_2^2 - z_1^2 z_2^3) \\ & + i\frac{7h^3}{512\omega_1^5}(z_1^7 - z_2^7) - i\frac{305h^3}{512\omega_1^5}(z_1^6 z_2 - z_1 z_2^6) + i\frac{2691h^3}{512\omega_1^5}(z_1^5 z_2^2 - z_1^2 z_2^5) + i\frac{2139h^3}{256\omega_1^5}(z_1^4 z_2^3 - z_1^3 z_2^4) \end{aligned} \tag{28b}$$

RNF

The reduced dynamics

$$\dot{z}_1 = i\omega_1 z_1 + i\frac{3h}{2\omega_1}(z_1^2 z_2 + z_1 z_2^2) - i\frac{15h^2}{16\omega_1^3} z_1^3 z_2^2 - i\frac{3h^2}{8\omega_1^3} z_1^2 z_2^3 + i\frac{267h^3}{128\omega_1^5} z_1^4 z_2^3 - i\frac{3h^3}{128\omega_1^5} z_1^3 z_2^4 \tag{29a}$$

$$\dot{z}_2 = -i\omega_1 z_2 - i\frac{3h}{2\omega_1}(z_1^2 z_2 + z_1 z_2^2) + i\frac{15h^2}{16\omega_1^3} z_1^2 z_2^3 + i\frac{3h^2}{8\omega_1^3} z_1^3 z_2^2 - i\frac{267h^3}{128\omega_1^5} z_1^3 z_2^4 + i\frac{3h^3}{128\omega_1^5} z_1^4 z_2^3 \tag{29b}$$

(funny to see that between CNF and RNF the cubic coefficients are the same and deduce easily one from the other, but the story becomes more complicated after !!! sign of the orders percuting one to each other.. What a mechanism !)

The mapping

$$\begin{aligned} u = {} & z_1 + z_2 + \frac{h}{8\omega_1^2}(z_1^3 + z_2^3) + \frac{h^2}{64\omega_1^4}(z_1^5 + z_2^5) - \frac{21h^2}{\omega_1^4}(z_1^4 z_2 + z_1 z_2^4) \\ & + \frac{h^3}{512\omega_1^6}(z_1^7 + z_2^7) - \frac{109h^3}{512\omega_1^6}(z_1^6 z_2 + z_1 z_2^6) + \frac{357h^3}{512\omega_1^6}(z_1^5 z_2^2 + z_1^2 z_2^5) \end{aligned} \tag{30}$$

$$\begin{aligned} v = {} & i\omega_1(z_1 - z_2) + i\frac{3h}{8\omega_1}(z_1^3 - z_2^3) + i\frac{5h^2}{64\omega_1^3}(z_1^5 - z_2^5) - i\frac{27h^2}{64\omega_1^3}(z_1^4 z_2 - z_1 z_2^4) \\ & + i\frac{7h^3}{512\omega_1^5}(z_1^7 - z_2^7) - i\frac{233h^3}{512\omega_1^5}(z_1^6 z_2 - z_1 z_2^6) + i\frac{195h^3}{512\omega_1^5}(z_1^5 z_2^2 - z_1^2 z_2^5) \end{aligned} \tag{31}$$

1.2 Realification

The aim of this section is to discuss the main advantages of the two normal form versions (complex vs real). Link with realification. In order to process the resuts and analyze them, one needs to come back to real coordinates. Different realification schemes are possible. We will underline that

- the main advantage of CNF is to use it in conjunction with polar realification. In this case one obtains analytical formula for the backbone, which is very interesting !
- the main advantage of RNF is to open the door to a more rapid way back to oscillator form equations. This can be easily shown by using realification with cartesian coordinates.

CNF and polar form

Realification with polar form:

$$z_1 = \frac{1}{2}\rho\, \mathrm{e}^{i\alpha} \tag{32a}$$

$$z_2 = \frac{1}{2}\rho\, \mathrm{e}^{-i\alpha} \tag{32b}$$

The main point of the reduced dynamics with CNF is that only one trivially resonant monomial per order is there, and of the form $z_1 z_2^2$, $z_1^2 z_2$ for cubic order, $z_1^2 z_2^3$, $z_1^3 z_2^2$ for quintic, etc... so $z_1^{n+1} z_2^n$ and $z_1^n z_2^{n+1}$ for odd order $2n+1$. So the products are such that exponential terms simplify in the reduced dynamics. Let's show that, first differentiate (32) with respect to time, and substitute with (27) to see what happen up to order 7 (of course one can increase dramatically and easily the order with MORFE_symbolic !!!). One easily arrives at:

$$\dot{\rho} + i\dot{\alpha}\rho = i\omega_1\rho + i\frac{3h}{8\omega_1}\rho^3 - i\frac{51h^2}{256\omega_1^3}\rho^5 + i\frac{1419h^3}{8192\omega_1^5}\rho^7 \tag{33}$$

$$\dot{\rho} - i\dot{\alpha}\rho = -i\omega_1\rho - i\frac{3h}{8\omega_1}\rho^3 + i\frac{51h^2}{256\omega_1^3}\rho^5 - i\frac{1419h^3}{8192\omega_1^5}\rho^7 \tag{34}$$

This is also a direct consequence of the fact that both equations of the reduced dynamics are complex conjugate one to each other for the CNF: stress more on that, very important point! Summing up the two equations one see that all terms vanish such that one obtains, *at arbitrary order*:

$$\dot{\rho} = 0, \tag{35}$$

so ρ the amplitude is simply a constant. Making now the difference one easily writes:

$$\dot{\alpha} = \omega_1 + \frac{3h}{8\omega_1}\rho^2 - \frac{51h^2}{256\omega_1^3}\rho^4 + \frac{1419h^3}{8192\omega_1^5}\rho^6. \tag{36}$$

Simple integration with respect to time gives the backbone curve as

$$\omega_{1,NL} = \omega_1 \left(1 + \frac{3h}{8\omega_1^2}\rho^2 - \frac{512h^2}{256\omega_1^4}\rho^4 + \frac{1419h^3}{8192\omega_1^6}\rho^6\right). \tag{37}$$

Here the process has been pushed to order 7 without any effort but MORFE_sym offers the possibility to go to any order without extra cost. This is a pretty interesting result, which is much more general than that since it can apply to N-dimensional systems reduced along a single NNM, up to arbitrary order: one has all analytic backbones !

NB: make a special part of the code to compute that automatically at the desired order ?

RNF and cartesian realification

The aim of the previous section was to show that the CNF combines perfectly well with a polar representation at the realification stage, since it offers directly a very nice result: the backbone up to any order. In this section we want to underline that the RNF fits well with the cartesian coordinates at the realification stage since it allows going back very close to an oscillaotr equation. but only ONF allows keeping real oscillators throughout the process.

Let's demonstrate that at order three. Very important: it works at order 3 only !!! not at order five... The cartesian coordinates for realification are introduced as:

$$a_1 = z_1 + z_2, \tag{38a}$$

$$a_2 = \frac{z_1 - z_2}{i}. \tag{38b}$$

Differentiating with respect to time, and summing up the two right-hand sides of the reduced dynamics for the RNF given in Eqs. (26), leads to:

$$\dot{a}_1 = -\omega_1 a_2. \tag{39}$$

Interestingly, the nonlinear terms present in (26) are exactly the same on the two lines and just cancel out in this equation. This is the key that allows retrieving something close to an oscillator equation. Unfortunately this is true only at order three. At order 5, some terms are summing up. Which means that this oscillator property is tru for the RNF only at order three but cannot be pushed further ? Or... To dig deeper ?

We continue the calculation only up to order three. making now the difference makes appear:

$$\dot{a}_2 = \omega_1 a_1 + \frac{3h}{\omega_1}(z_1^2 z_2 + z_1 z_2^2). \tag{40}$$

By noting that the inverse linear transform of (41) reads:

$$z_1 = \frac{a_1 + i a_2}{2}, \tag{41a}$$

$$z_2 = \frac{a_1 - i a_2}{2}, \tag{41b}$$

the cubic terms appearing in (40) simplifies to

$$z_1^2 z_2 + z_1 z_2^2 = \frac{1}{4}(a_1^3 + a_1 a_2^2). \tag{42}$$

Such that finally if we stop at order three we get:

$$\ddot{a}_1 + \omega_1^2 a_1 + \frac{3h}{4}(a_1^3 + a_1 a_2^2) = 0 \tag{43a}$$

$$\dot{a}_1 = -\omega_1 a_2 \tag{43b}$$

such that finally a closed equation only for a_1 can be obtained as:

$$\ddot{a}_1 + \omega_1^2 a_1 + \frac{h}{4}(a_1^3 + a_1 \frac{\dot{a}_1^2}{\omega_1^2}) = 0 \tag{44}$$

open questions to go one step further:

- this does not generalize to higher orders... At least not easily ! Maybe we could keep something close to real, but not close to an oscillator...
- possible to add that in MORFE_Sym to compute this automatically (the realification)?
- same RNF as Wagg finally? To double check.

2 Duffing oscillator with quadratic and cubic NL terms

Starting point:

$$\ddot{u} + \omega_1^2 u + g u^2 + h u^3 = 0. \tag{45}$$

Complex normal form

Reduced dynamics, as given by MORFE_Sym, automatically, to any order (here up to order seven):

$$\begin{aligned}\dot{z}_1 = i\omega_1 z_1 &- 3i\frac{2g^2 - h\omega_1^2}{2\omega_1^3} z_1^2 z_2 - i\frac{172g^4 - 312g^2 h\omega_1^2 + 33h^2\omega_1^4}{4\omega_1^7} z_1^3 z_2^2 \\ &- 3i\frac{2856g^6 - 8156g^4 h\omega_1^2 + 4914g^2h^2\omega_1^4 - 215h^3\omega_1^6}{8\omega_1^{11}} z_1^4 z_2^3\end{aligned} \tag{46a}$$

$$\begin{aligned}\dot{z}_2 = -i\omega_1 z_2 &+ 3i\frac{2g^2 - h\omega_1^2}{2\omega_1^3} z_1 z_2^2 + i\frac{172g^4 - 312g^2 h\omega_1^2 + 33h^2\omega_1^4}{4\omega_1^7} z_1^2 z_2^3 \\ &+ 3i\frac{2856g^6 - 8156g^4 h\omega_1^2 + 4914g^2h^2\omega_1^4 - 215h^3\omega_1^6}{8\omega_1^{11}} z_1^3 z_2^4\end{aligned} \tag{46b}$$

Of course letting $g = 0$ it works, one retrieve the same as before.

The nonlinear mapping is more lengthy as compared to Eqs. (28). Indeed, quadratic terms are now present in the nonlinear change of coordinates, and these terms are necessary in order to cancel the non-resonant quadratic term of the reduced dynamics. The mapping is shown up to order 5 only, for the sake of brevity.

$$
\begin{aligned}
u = z_1 + z_2 &- \frac{g}{\omega_1^2}(z_1^2 + z_2^2 + 2z_1z_2) + \frac{2g^2 - h\omega_1^2}{\omega_1^4}\left(z_1^3 + z_2^3 + \frac{3}{2}(z_1^2z_2 + z_1z_2^2)\right) \\
&+ \frac{5g(-g^2 + h\omega_1^2)}{\omega_1^6}(z_1^4 + z_2^4) + \frac{g(-14g^2 + 17h\omega_1^2)}{\omega_1^6}(z_1^3z_2 + z_1z_2^3) + \frac{-18g^3 + 24hg\omega_1^2}{\omega_1^6}z_1^2z_2^2 \\
&+ \frac{14g^4 - 21g^2h\omega_1^2 + 3h^2\omega_1^4}{\omega_1^8}(z_1^5 + z_2^5) \\
&+ \frac{104g^4 - 174g^2h\omega_1^2 + 21h^2\omega_1^4}{2\omega_1^8}(z_1^4z_2 + z_1^3z_2^2 + z_1^2z_2^3 + z_1z_2^4) \qquad (47a)
\end{aligned}
$$

$$
\begin{aligned}
v = i\omega_1(z_1 - z_2) &- \frac{3i(2g^2 - h\omega_1^2)}{2\omega_1^3}(z_1^2z_2 - z_1z_2^2) + \frac{3ig(2g^2 - h\omega_1^2)}{\omega_1^5}(z_1^3z_2 - z_1z_2^3) \\
&- \frac{9i(4g^4 - 4g^2h\omega_1^2 + h^2\omega_1^4)}{2\omega_1^7}(z_1^4z_2 - z_1z_2^4) - \frac{i(104g^4 - 174g^2h\omega_1^2 + 21h^2\omega_1^4)}{2\omega_1^7}(z_1^3z_2^2 - z_1^2z_2^3) \qquad (47b)
\end{aligned}
$$

As a side comment, there is no even orders on the second line. There is surely a good explanation to that...

Real normal form

Reduced dynamics, as given by MORFE_Sym, automatically, to any order (here up to order seven):

$$
\begin{aligned}
\dot{z}_1 = &+i\omega_1 z_1 - \frac{3i(2g^2 - h\omega_1^2)}{2\omega_1^3}(z_1^2z_2 + z_1z_2^2) \\
&- \frac{i(28g^4 - 168g^2h\omega_1^2 - 3h^2\omega_1^4)}{4\omega_1^7}z_1^3z_2^2 - \frac{i(25g^4 - 60g^2h\omega_1^2 + 15h^2\omega_1^4/4)}{\omega_1^7}z_1^2z_2^3 \\
&- \frac{15i(40g^6 - 228g^4h\omega_1^2 + 330g^2h^2\omega_1^4 - h^3\omega_1^6)}{8\omega_1^11}z_1^4z_2^3 \\
&- \frac{3i(1688g^6 - 5052g^4h\omega_1^2 + 3486g^2h^2\omega_1^4 - 131h^3\omega_1^6)}{8\omega_1^11}z_1^3z_2^4, \qquad (48a)
\end{aligned}
$$

$$
\begin{aligned}
\dot{z}_2 = &-i\omega_1 z_2 + \frac{3i(2g^2 - h\omega_1^2)}{2\omega_1^3}(z_1^2z_2 + z_1z_2^2) \\
&+ \frac{i(28g^4 - 168g^2h\omega_1^2 - 3h^2\omega_1^4)}{4\omega_1^7}z_2^3z_1^2 + \frac{i(25g^4 - 60g^2h\omega_1^2 + 15h^2\omega_1^4/4)}{\omega_1^7}z_2^2z_1^3 \\
&+ \frac{15i(40g^6 - 228g^4h\omega_1^2 + 330g^2h^2\omega_1^4 - h^3\omega_1^6)}{8\omega_1^11}z_1^3z_2^4 \\
&+ \frac{3i(1688g^6 - 5052g^4h\omega_1^2 + 3486g^2h^2\omega_1^4 - 131h^3\omega_1^6)}{8\omega_1^11}z_1^4z_2^3. \qquad (48b)
\end{aligned}
$$

And the mapping can also be given, in case...

Oscillator normal form

Not sure so interesting to show that, but, in case. I made the calculation up to order three only.
Starting point at first order:

$$\dot{u} = v, \qquad (49a)$$

$$\dot{v} = -\omega_1^2 u - gu^2 - hu^3. \qquad (49b)$$

The nonlinear mapping at order two reads:

$$u = U - \frac{g}{3\omega_1^1}U^2 - \frac{2g}{3\omega_1^4}V^2, \tag{50a}$$

$$v = V + \frac{2g}{3\omega_1^2}UV. \tag{50b}$$

And the reduced dynamics up to order three

$$\dot{U} = V, \tag{51a}$$

$$\dot{V} = -\omega_1^2 U - \left(h - \frac{2g^2}{3\omega_1^2}\right)U^3 + \frac{4g^2}{3\omega_1^4}UV^2. \tag{51b}$$

Interestingly there are only cubic resonant monomials. So the change of coordinate will not involve any cubic term... The next wtep would be to push up to order 4 and cancel the quartic terms. The dynamics up to order four reads:

$$\dot{U} = V, \tag{52a}$$

$$\dot{V} = -\omega_1^2 U - \left(h - \frac{2g^2}{3\omega_1^2}\right)U^3 + \frac{4g^2}{3\omega_1^4}UV^2 + \left(\frac{hg}{\omega_1^2} - \frac{g^3}{9\omega_1^4}\right)U^4 - \frac{4g^3}{9\omega_1^4}V^2 + \frac{2gh}{\omega_1^4}U^2V^2. \tag{52b}$$

To see if it's worth pushing further, by writing the fourth-order change of coordinate...

3 single oscillator with damping

We start with

$$\ddot{u} + \omega_1^2 u + 2\xi_1\omega_1\dot{u} + hu^3 = 0 \tag{53}$$

The idea is to comment the effect of the damping on the coefficients and see how the previous results are modified with the presence of damping. NB (notation): OK to modify to ω_0, ξ_0, no problem for me ! At worst this can be done at the end...

The eigenvalues in this case are

$$\lambda_{1,2} = -\xi_1\omega_1 \pm i\omega_1\sqrt{1-\xi_1^2} = \omega_1(-\xi_1 \pm i\delta_1), \tag{54}$$

with $\delta_1 = \sqrt{1-\xi_1^2}$, and $|\lambda_{1,2}| = \omega_1$.

3.1 Complex normal form

In the case of the complex normal form, let us start with order 3. The normal form reads in this case:

$$\dot{z}_1 = \lambda_1 z_1 + \frac{3ih}{2\delta_1\omega_1}z_1^2 z_2, \tag{55a}$$

$$\dot{z}_2 = \lambda_2 z_2 - \frac{3ih}{2\delta_1\omega_1}z_1 z_2^2. \tag{55b}$$

Of course, letting $\xi_1 = 0$ leads to $\delta_1 = 1$ and one recover the results of the undamped case. One can also remark that the coefficient of the damped case is very close to the conservative one, it has just been mutliplied by $1/\delta_1 = 1/\sqrt{1-\xi_1^2}$. In case (asymptotic expansion with respect to damping, one could use:

$$\frac{1}{\sqrt{1-\xi_1^2}} \simeq 1 + \frac{\xi_1^2}{2} + \frac{3\xi_1^4}{8} + ... \tag{56}$$

Let us show that using realification with polar form always leads to a nice result, easy to handle and interpret. Using

$$z_1 = \frac{1}{2}\rho_1\,\mathrm{e}^{i\alpha_1}, \tag{57}$$

$$z_2 = \frac{1}{2}\rho_1\,\mathrm{e}^{-i\alpha_1}, \tag{58}$$

one easily arrives at

$$\dot{\rho}_1 = -\omega_1\xi_1\rho_1, \tag{59}$$

which shows (as expected) that the amplitude is exponentially decreasing following the damping rate: $\rho_1(t) = \rho_1^0\,\mathrm{e}^{-\omega_1\xi_1 t}$. On the other hand, the equation for the frequency reads:

$$\dot{\alpha}_1 = \delta_1\omega_1 + \frac{3h}{8\delta_1\omega_1}\rho_1^2. \tag{60}$$

From these two equations, it appears that: (i) the amplitude ρ_1 follows a linear decrease (exponential) decrease driven by the damping ratio, and (ii) the nonlinear (amplitude-depdendent) instantaneous frequency can be defined from the last equation as:

$$\omega_{NL} = \omega_1\left(\delta_1 + \frac{3h}{8\delta_1\omega_1}\rho_1^2\right). \tag{61}$$

Numerical studies to do: show evoluation of conservative vs damped backbone and relationship with FRF!

Next question: does this nice property extend to higher orders straightforwardly? The answer is rather positive, since after symbolic manipulations (at present successful with mathematica, to be implemented in julia MORFE_sym), the CNF reads:

$$\begin{aligned}
\dot{z}_1 &= \lambda_1 z_1 + \frac{3ihz_1^2z_2}{2\delta_1\omega_1} + \frac{3h^2z_1^3z_2^2(-11i\delta_1+3\xi_1)}{4\delta_1^2\omega_1^3}\\
\dot{z}_2 &= \lambda_2 z_2 - \frac{3ihz_1z_2^2}{2\delta_1\omega_1} + \frac{3h^2z_1^2z_2^3(+11i\delta_1+3\xi_1)}{4\delta_1^2\omega_1^3}
\end{aligned} \tag{62}$$

The realification in polar form gives also an interesting result:

$$\begin{aligned}
\dot{\rho} &= -\xi_1\omega_1\rho + \frac{9h^2\xi_1\rho^5}{64\delta_1^2\omega_1^3}\\
\dot{\alpha}_1 &= \delta_1\omega_1 + \frac{3h\rho^2}{8\delta_1\omega_1} - \frac{33h^2\rho^4}{64\delta_1\omega_1^3}
\end{aligned} \tag{63}$$

Interestingly from these two equations, one can observe that:

- the decay rate is now nonlinear, showing that there is a nonlinear damping rate in the normal form variables. One can form the nonlinear damping ξ_{NL} rate as the value given by $\dot{\rho}/\omega_1\rho$, showing linear + nonlinear decay rate as:

  $$\xi_{NL} = \xi_1\left(1 + \frac{9h^2\rho^4}{64\delta_1^2\omega_1^4}\right).$$

  Interestingly, the added term is nonlinear in terms of amplitude, depends on order 4 but has no term on order 3, and depends linearly on the damping ratio ξ_1 that can be factorized.

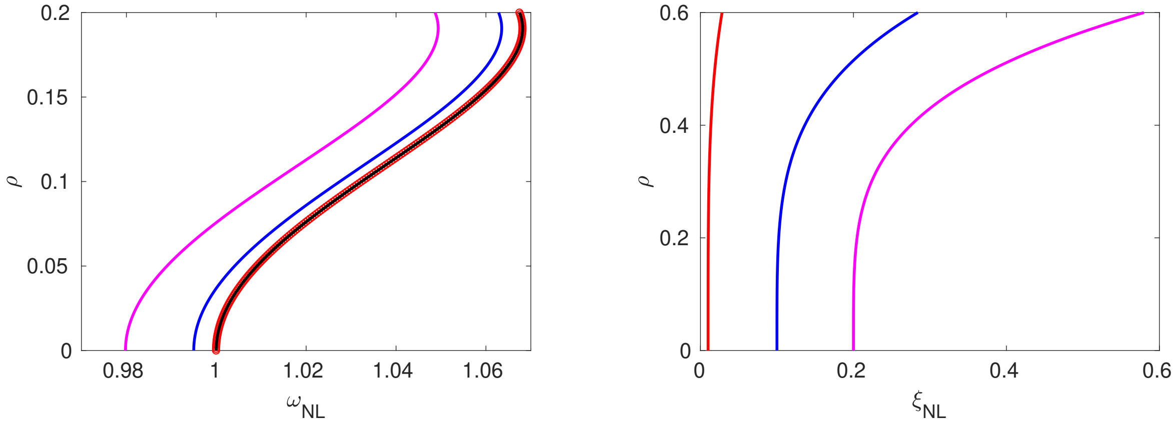

Figure 1: Dependence of the nonlinear instantaneous frequency and nonlinear damping rate as function of amplitude ρ. Comparison between the conservative case (black) and the damped case for increasing value of the damping ratio: $\xi = 0.01$ ($\delta_1 = 0.999$) in red, $\xi = 0.1$ ($\delta_1 = 0.995$) in blue and $\xi = 0.2$ ($\delta_1 = 0.9798$) in magenta. Duffing oscillator with $\omega_1 = 1$ and $h = 10$.

- The instantaneous nonlinear frequency can be defined as well following:
$$\omega_{NL}^d = \omega_1 \left(\delta_1 + \frac{3h\rho^2}{8\delta_1\omega_1^2} - \frac{33h^2\rho^4}{64\delta_1\omega_1^4} \right).$$
This is a very nice expression since it is very close to the undamped case! Recall the conservative case reads:
$$\omega_{NL}^c = \omega_1 \left(1 + \frac{3h\rho^2}{8\omega_1^2} - \frac{33h^2\rho^4}{64\omega_1^4} \right).$$
It is almost equivalent (but not), since replacing ω_1 by the damped oscillation frequency $\delta_1\omega_1$ gives not exactly the same. Indeed for the damped case we can factorize with the damped frequency:
$$\omega_{NL}^d = \delta_1\omega_1 \left(1 + \frac{3h\rho^2}{8(\delta_1\omega_1)^2} - \frac{33h^2\rho^4}{64\delta_1^2\omega_1^4} \right).$$
so a slight difference appear from order 4, which is probably connected to the fact that the nonlienar damping rate appears from this order.

Fig. 1 compares the undamped and damped backbone curves for three increasing values of the damping. For small damping rate, $\xi = 0.01$, the two curves are superimposed. Then the main effect is the shift of the frequency, whereas the change of shape is less pronounced. One can note in particular that due to the fifth-order development, the quintic term is neative inducing a softening effect at higher amplitudes. This should be corrected by considering higher orders. The nonlinear damping ratio are shown also in Fig. 1. One can see that the effect is less important and one needs to go to much marger amplitude to observe the effect.

comment: to be continued with order 7 !!! I think order 7 should be very interesting to show analytically. Then the higher orders could be implemented numerically to see if they do something in the range.

3.2 Real normal form

Using real normal form style one arrives at the following normal form at order 3:
$$\begin{aligned}
\dot{z}_1 &= \lambda_1 z_1 + \frac{3ih}{2\delta_1\omega_1}\left(z_1^2 z_2 + z_1 z_2^2\right) + \frac{3h^2 z_1^3 z_2^2(+i\delta_1 + 3\xi_1)}{4\delta_1^2\omega_1^3} + \frac{3h^2 z_1^2 z_2^3(-5i\delta_1 - 3\xi_1)}{4\delta_1^2\omega_1^3} \\
\dot{z}_2 &= \lambda_2 z_2 - \frac{3ih}{2\delta_1\omega_1}\left(z_1^2 z_2 + z_1 z_2^2\right) + \frac{3h^2 z_1^2 z_2^3(-i\delta_1 + 3\xi_1)}{4\delta_1^2\omega_1^3} + \frac{3h^2 z_1^3 z_2^2(+5i\delta_1 - 3\xi_1)}{4\delta_1^2\omega_1^3}
\end{aligned} \tag{64}$$

Worth to go to order 7, if not too complicated? (with the present expressions it is NOT complicated so rather very nice!!!!). Realification with cartesian coordinates, is there an interest in showing that?

3.3 Oscillator normal form

$$
\begin{aligned}
\dot{z}_1 &= \lambda_1 z_1 + \frac{ihz_1^2 z_2}{2\delta_1\omega_1}(z_1+z_2)^3 \\
\dot{z}_2 &= \lambda_2 z_2 - \frac{ihz_1^2 z_2}{2\delta_1\omega_1}(z_1+z_2)^3
\end{aligned}
\tag{65}
$$

This shows that (as expected) the ONF does not do anytthing to treat the original u^3 which is seen as a single real resonant monomial. This ONF is simply the original equation written with complex coordinates, no terms have been cancelled by the change of coordinates. This follows exactly what was done for the undamped case, following the assumption of light damping we keep throughout the calculation such that resonant monomials of the damped case are the same as the resonant monomials for the conservative case.

4 single oscillator with forcing

In this section I want to cope with the simple problem of a Duffing oscillator + forcing, to see how the non-autonomous terms leads to the appearance of new types of monomials, new forms of resonance etc... On a very simple case we should be able to clearly see how things are appearing.

Here also the main question is that probably we need to select on what we want to focus, because many different points could be looked at. I would suggest to start with:

- primary resonance
- superharmonic resonance
- subhamronic ?
- I would like to see of some terms like parametric excitation can come into play, as the ones we saw with Andrea. But maybe this could happen only with two dofs?

It is clear that the expressions will become tedious but we should focus on the very first term appzaring and their "easy" interpretation in terms of dynamics (eg superhramonic, parametric).

Also comparison of FRF with backbone undamped/damped.

Next steps

Redo close derivations with 2 dofs, allows reducing to single NNM, show results, explain more new things (parametric excitation for example, etc..). general solutions...

Case with internal resonance, do 1:2, 1:3 and 1:5

5 Comparison with multiple scale expansion