MORFE_sym reduced dynamics and nonlinear mappings - Expressions from latex automatic output

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This file is intended to document the reduced dynamics and nonlinear mappings expressions given by MORFE_sym via the newly implemented latex output functions. Differences between these expressions and those present on the notes "SymbolicAnalysis.pdf" will be highlighted.
Obs.: I'm following the same notation as in the "Direct parametrisation of invariant manifolds for generic non-autonomous systems including superharmonic resonances" paper, from hereafter referenced as [1]. Specifically, the variables u and v that appear on the notes are denoted here by y_1 and y_2. Since one operates directly from physical space, without diagonalizing the linear part beforehand, the variables y_1 and y_2 that appear on the notes have no counterpart here.

1 Base code

1.1 Unforced cubic Duffing oscillator

1.1.1 Undamped CNF

Reduced dynamics (up to order 11):

$$\dot{z}_1 = i\omega_1 z_1 + i\frac{3h}{2\omega_1} z_1^2 z_2 - i\frac{51h^2}{16\omega_1^3} z_1^3 z_2^2 + i\frac{1419h^3}{128\omega_1^5} z_1^4 z_2^3 - i\frac{47505h^4}{1024\omega_1^7} z_1^5 z_2^4 + i\frac{438825h^5}{2048\omega_1^9} z_1^6 z_2^5 \tag{1}$$

$$\dot{z}_2 = -i\omega_1 z_2 - i\frac{3h}{2\omega_1} z_1 z_2^2 + i\frac{51h^2}{16\omega_1^3} z_1^2 z_2^3 - i\frac{1419h^3}{128\omega_1^5} z_1^3 z_2^4 + i\frac{47505h^4}{1024\omega_1^7} z_1^4 z_2^5 - i\frac{438825h^5}{2048\omega_1^9} z_1^5 z_2^6 \tag{2}$$

Nonlinear mappings (up to order 7):

$$\begin{aligned} y_1 = {} & z_1 + z_2 + \frac{h}{8\omega_1^2} z_1^3 - \frac{3h}{4\omega_1^2} z_1^2 z_2 - \frac{3h}{4\omega_1^2} z_1 z_2^2 + \frac{h}{8\omega_1^2} z_2^3 + \frac{h^2}{64\omega_1^4} z_1^5 - \frac{39h^2}{64\omega_1^4} z_1^4 z_2 + \frac{69h^2}{32\omega_1^4} z_1^3 z_2^2 \\ & + \frac{69h^2}{32\omega_1^4} z_1^2 z_2^3 - \frac{39h^2}{64\omega_1^4} z_1 z_2^4 + \frac{h^2}{64\omega_1^4} z_2^5 + \frac{h^3}{512\omega_1^6} z_1^7 - \frac{73h^3}{512\omega_1^6} z_1^6 z_2 + \frac{1569h^3}{512\omega_1^6} z_1^5 z_2^2 \\ & - \frac{2139h^3}{256\omega_1^6} z_1^4 z_2^3 - \frac{2139h^3}{256\omega_1^6} z_1^3 z_2^4 + \frac{1569h^3}{512\omega_1^6} z_1^2 z_2^5 - \frac{73h^3}{512\omega_1^6} z_1 z_2^6 + \frac{h^3}{512\omega_1^6} z_2^7 \end{aligned} \tag{3}$$

$$\begin{aligned} y_2 = {} & i\omega_1 z_1 - i\omega_1 z_2 + i\frac{3h}{8\omega_1} z_1^3 + i\frac{3h}{4\omega_1} z_1^2 z_2 - i\frac{3h}{4\omega_1} z_1 z_2^2 - i\frac{3h}{8\omega_1} z_2^3 + i\frac{5h^2}{64\omega_1^3} z_1^5 - i\frac{81h^2}{64\omega_1^3} z_1^4 z_2 \\ & - i\frac{69h^2}{32\omega_1^3} z_1^3 z_2^2 + i\frac{69h^2}{32\omega_1^3} z_1^2 z_2^3 + i\frac{81h^2}{64\omega_1^3} z_1 z_2^4 - i\frac{5h^2}{64\omega_1^3} z_2^5 + i\frac{7h^3}{512\omega_1^5} z_1^7 - i\frac{305h^3}{512\omega_1^5} z_1^6 z_2 \\ & + i\frac{2691h^3}{512\omega_1^5} z_1^5 z_2^2 + i\frac{2139h^3}{256\omega_1^5} z_1^4 z_2^3 - i\frac{2139h^3}{256\omega_1^5} z_1^3 z_2^4 - i\frac{2691h^3}{512\omega_1^5} z_1^2 z_2^5 + i\frac{305h^3}{512\omega_1^5} z_1 z_2^6 \end{aligned}$$

$$
- i\frac{7h^3}{512\omega_1^5} z_2^7 \tag{4}
$$

These expressions match the ones on the notes perfectly.

1.1.2 Undamped RNF

Reduced dynamics (up to order 7):

$$
\dot{z}_1 = i\omega_1 z_1 + i\frac{3h}{2\omega_1} z_1^2 z_2 + i\frac{3h}{2\omega_1} z_1 z_2^2 - i\frac{15h^2}{16\omega_1^3} z_1^3 z_2^2 - i\frac{3h^2}{8\omega_1^3} z_1^2 z_2^3 + i\frac{267h^3}{128\omega_1^5} z_1^4 z_2^3 - i\frac{3h^3}{128\omega_1^5} z_1^3 z_2^4 \tag{5}
$$

$$
\dot{z}_2 = -i\omega_1 z_2 - i\frac{3h}{2\omega_1} z_1^2 z_2 - i\frac{3h}{2\omega_1} z_1 z_2^2 + i\frac{3h^2}{8\omega_1^3} z_1^3 z_2^2 + i\frac{15h^2}{16\omega_1^3} z_1^2 z_2^3 + i\frac{3h^3}{128\omega_1^5} z_1^4 z_2^3 - i\frac{267h^3}{128\omega_1^5} z_1^3 z_2^4 \tag{6}
$$

Nonlinear mappings (up to order 7):

$$
\begin{aligned}
y_1 &= z_1 + z_2 + \frac{h}{8\omega_1^2} z_1^3 + \frac{h}{8\omega_1^2} z_2^3 + \frac{h^2}{64\omega_1^4} z_1^5 - \frac{21h^2}{64\omega_1^4} z_1^4 z_2 - \frac{21h^2}{64\omega_1^4} z_1 z_2^4 + \frac{h^2}{64\omega_1^4} z_2^5 \\
&+ \frac{h^3}{512\omega_1^6} z_1^7 - \frac{109h^3}{512\omega_1^6} z_1^6 z_2 + \frac{357h^3}{512\omega_1^6} z_1^5 z_2^2 + \frac{357h^3}{512\omega_1^6} z_1^2 z_2^5 - \frac{109h^3}{512\omega_1^6} z_1 z_2^6 + \frac{h^3}{512\omega_1^6} z_2^7
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
y_2 &= i\omega_1 z_1 - i\omega_1 z_2 + i\frac{3h}{8\omega_1} z_1^3 - i\frac{3h}{8\omega_1} z_2^3 + i\frac{5h^2}{64\omega_1^3} z_1^5 - i\frac{27h^2}{64\omega_1^3} z_1^4 z_2 + i\frac{27h^2}{64\omega_1^3} z_1 z_2^4 - i\frac{5h^2}{64\omega_1^3} z_2^5 \\
&+ i\frac{7h^3}{512\omega_1^5} z_1^7 - i\frac{233h^3}{512\omega_1^5} z_1^6 z_2 + i\frac{195h^3}{512\omega_1^5} z_1^5 z_2^2 - i\frac{195h^3}{512\omega_1^5} z_1^2 z_2^5 + i\frac{233h^3}{512\omega_1^5} z_1 z_2^6 - i\frac{7h^3}{512\omega_1^5} z_2^7
\end{aligned} \tag{8}
$$

The reduced dynamics expressions match the ones on the notes perfectly. For the nonlinear mappings there is a little difference on the coefficient that multiplies $z_1^4 z_2$ and $z_1 z_2^4$ on the y_1 expression. On the notes it is 21 rather than $\frac{21}{64}$. Since all other coefficients match this is probably a typo on the notes.

1.1.3 Damped CNF

Reduced dynamics (up to order 5)

$$
\begin{aligned}
\dot{z}_1 &= (i\delta_1\omega_1 - \xi_1\omega_1)\, z_1 + i\frac{3h}{2\delta_1\omega_1} z_1^2 z_2 + \frac{384h^2\delta_1\xi_1^6 - 744h^2\delta_1\xi_1^4 + 384h^2\delta_1\xi_1^2 - 33h^2\delta_1}{64\delta_1\xi_1^7\omega_1^3 - 144\delta_1\xi_1^5\omega_1^3 + 100\delta_1\xi_1^3\omega_1^3 - 20\delta_1\xi_1\omega_1^3} \\
&\frac{-384ih^2\xi_1^7 + 936ih^2\xi_1^5 - 708ih^2\xi_1^3 + 156ih^2\xi_1}{-64i\xi_1^8\omega_1^3 + 176i\xi_1^6\omega_1^3 - 164i\xi_1^4\omega_1^3 + 56i\xi_1^2\omega_1^3 - 4i\omega_1^3} z_1^3 z_2^2
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\dot{z}_2 &= -\,(i\delta_1\omega_1 + \xi_1\omega_1)\, z_2 - i\frac{3h}{2\delta_1\omega_1} z_1 z_2^2 + \frac{384h^2\delta_1\xi_1^6 - 744h^2\delta_1\xi_1^4 + 384h^2\delta_1\xi_1^2 - 33h^2\delta_1}{64\delta_1\xi_1^7\omega_1^3 - 144\delta_1\xi_1^5\omega_1^3 + 100\delta_1\xi_1^3\omega_1^3 - 20\delta_1\xi_1\omega_1^3} \\
&\frac{+384ih^2\xi_1^7 - 936ih^2\xi_1^5 + 708ih^2\xi_1^3 - 156ih^2\xi_1}{+64i\xi_1^8\omega_1^3 - 176i\xi_1^6\omega_1^3 + 164i\xi_1^4\omega_1^3 - 56i\xi_1^2\omega_1^3 + 4i\omega_1^3} z_1^2 z_2^3
\end{aligned} \tag{10}
$$

Nonlinear mappings (up to order 5):

$$
\begin{aligned}
y_1 &= z_1 + z_2 - \frac{h}{\omega_1^2} z_1^3 + \frac{-3ih\delta_1 - 3h\xi_1}{4\delta_1^2\xi_1\omega_1^2 - 4i\delta_1\xi_1^2\omega_1^2 + 2i\delta_1\omega_1^2} z_1^2 z_2 + \frac{3ih\delta_1 - 3h\xi_1}{4\delta_1^2\xi_1\omega_1^2 + 4i\delta_1\xi_1^2\omega_1^2 - 2i\delta_1\omega_1^2} z_1 z_2^2 - \\
&\frac{h}{\omega_1^2} z_2^3 + \frac{3h^2}{\omega_1^4} z_1^5 + \frac{-12h^2\delta_1\xi_1^2 - 21h^2\delta_1 + 12ih^2\xi_1^3 + 15ih^2\xi_1}{4\delta_1\xi_1^2\omega_1^4 - 2\delta_1\omega_1^4 - 4i\xi_1^3\omega_1^4 + 4i\xi_1\omega_1^4} z_1^4 z_2 + \\
&\frac{-50331648h^2\delta_1\xi_1^{24} + 333447168h^2\delta_1\xi_1^{22} - 965738496h^2\delta_1\xi_1^{20} + 1606287360h^2\delta_1\xi_1^{18} - 1694564352h}{16777216\delta_1\xi_1^{26}\omega_1^4 - 117440512\delta_1\xi_1^{24}\omega_1^4 + 3649044448\delta_1\xi_1^{22}\omega_1^4 - 662700032\delta_1\xi_1^{20}\omega_1^4 + 779485184\delta_1\xi_1^{18}\omega_1^4 - 62167}
\end{aligned} \tag{11}
$$

$$y_2 = \frac{i\omega_1}{\delta_1 - i\xi_1} z_1 - \frac{i\omega_1}{\delta_1 + i\xi_1} z_2 + i\frac{3h}{2\delta_1\omega_1} z_1^2 z_2 - i\frac{3h}{2\delta_1\omega_1} z_1 z_2^2 + \frac{18h^2\delta_1\xi_1 - 18ih^2\xi_1^2 + 9ih^2}{4\delta_1\xi_1^2\omega_1^3 - 2\delta_1\omega_1^3 - 4i\xi_1^3\omega_1^3 + 4i\xi_1\omega_1^3} z_1^4 z_2 +$$
$$\frac{100663296h^2\delta_1\xi_1^{25} - 692060160h^2\delta_1\xi_1^{23} + 2091909120h^2\delta_1\xi_1^{21} - 3656908800h^2\delta_1\xi_1^{19} + 4090429440h^2\delta_1\xi_1^{17} - 30}{16777216\delta_1\xi_1^{26}\omega_1^3 - 117440512\delta_1\xi_1^{24}\omega_1^3 + 364904448\delta_1\xi_1^{22}\omega_1^3 - 662700032\delta_1\xi_1^{20}\omega_1^3 + 779485184\delta_1\xi_1^{18}\omega_1^3 - 6216}$$
(12)

The coefficients on the expressions start to get over-complicated at order 5, and apparently different from the ones in the notes. However, upon simplification with Mathematica, it's possible to obtain for the reduced dynamics:
Reduced dynamics (up to order 5)

$$\dot{z}_1 = (i\delta_1\omega_1 - \xi_1\omega_1) z_1 + i\frac{3h}{2\delta_1\omega_1} z_1^2 z_2 + \frac{3h^2(-11i\delta_1 + 3\xi_1)}{4\ \delta_1^2\omega^3} z_1^3 z_2^2 \quad (13)$$

$$\dot{z}_2 = -(i\delta_1\omega_1 + \xi_1\omega_1) z_2 - i\frac{3h}{2\delta_1\omega_1} z_1 z_2^2 + \frac{3h^2(11i\delta_1 + 3\xi_1)}{4\delta_1^2\omega^3} z_1^2 z_2^3 \quad (14)$$

These expressions are exactly the same from the notes.

1.1.4 Damped RNF

Reduced dynamics (up to order 3):

$$\dot{z}_1 = (i\delta_1\omega_1 - \xi_1\omega_1) z_1 + i\frac{3h}{2\delta_1\omega_1} z_1^2 z_2 + \frac{-6h\delta_1\xi_1 - 6ih\xi_1^2 + 3ih}{2\delta_1\omega_1} z_1 z_2^2 \quad (15)$$

$$\dot{z}_2 = -(i\delta_1\omega_1 + \xi_1\omega_1) z_2 + \frac{-6h\delta_1\xi_1 + 6ih\xi_1^2 - 3ih}{2\delta_1\omega_1} z_1^2 z_2 - i\frac{3h}{2\delta_1\omega_1} z_1 z_2^2 \quad (16)$$

Nonlinear mappings (up to order 3):

$$y_1 = (\delta_1 - i\xi_1) z_1 + (\delta_1 + i\xi_1) z_2 + \frac{4h\delta_1\xi_1^2 - h\delta_1 - 4ih\xi_1^3 + 3ih\xi_1}{\omega_1^2} z_1^3 + \frac{4h\delta_1\xi_1^2 - h\delta_1 + 4ih\xi_1^3 - 3ih\xi_1}{\omega_1^2} z_2^3 \quad (17)$$

$$y_2 = i\omega_1 z_1 - i\omega_1 z_2 \quad (18)$$

The expressions for the reduced dynamics don't match the ones on the notes, so there is no point in continuing further to order 5. The nonlinear mappings are not given on the notes for this case, so no comparison is possible. However, the fact that the y_2 mapping doesn't have any cubic terms doesn't seem to be right.

2 Modified code

From the discrepancy between the reduced dynamics expressions for the damped RNF given on the notes and the ones obtained by the code, a search for a possible bug was conducted. It was found that the quantity $\sigma^{(p,k)}$, defined in [1] and related to resonance conditions, was not being correctly calculated for the autonomous case, and thus affecting on the solution of the homological equations (for example on line 433 of "MORFE_sym.jl").

In order to remedy that, the calculation of $\sigma^{(p,k)}$ was added also for the autonomous case in all of the scripts related to the cubic duffing oscillator (for example on line 442 of "Duffing_cubic_damped_unforced_RNF.jl"). This section contains the updated results.

P.S.: Actually, looking again more closely into the files, the bug was already fixed for the undamped cases, but not for the damped ones! This explains why results for them remain completely unchanged.

2.1 Unforced cubic Duffing oscillator

2.1.1 Undamped CNF

Reduced dynamics (up to order 11):

$$\dot{z}_1 = i\omega_1 z_1 + i\frac{3h}{2\omega_1}z_1^2 z_2 - i\frac{51h^2}{16\omega_1^3}z_1^3 z_2^2 + i\frac{1419h^3}{128\omega_1^5}z_1^4 z_2^3 - i\frac{47505h^4}{1024\omega_1^7}z_1^5 z_2^4 + i\frac{438825h^5}{2048\omega_1^9}z_1^6 z_2^5 \tag{19}$$

$$\dot{z}_2 = -i\omega_1 z_2 - i\frac{3h}{2\omega_1}z_1 z_2^2 + i\frac{51h^2}{16\omega_1^3}z_1^2 z_2^3 - i\frac{1419h^3}{128\omega_1^5}z_1^3 z_2^4 + i\frac{47505h^4}{1024\omega_1^7}z_1^4 z_2^5 - i\frac{438825h^5}{2048\omega_1^9}z_1^5 z_2^6 \tag{20}$$

Nonlinear mappings (up to order 7):

$$\begin{aligned} y_1 &= z_1 + z_2 + \frac{h}{8\omega_1^2}z_1^3 - \frac{3h}{4\omega_1^2}z_1^2 z_2 - \frac{3h}{4\omega_1^2}z_1 z_2^2 + \frac{h}{8\omega_1^2}z_2^3 + \frac{h^2}{64\omega_1^4}z_1^5 - \frac{39h^2}{64\omega_1^4}z_1^4 z_2 + \frac{69h^2}{32\omega_1^4}z_1^3 z_2^2 \\ &+ \frac{69h^2}{32\omega_1^4}z_1^2 z_2^3 - \frac{39h^2}{64\omega_1^4}z_1 z_2^4 + \frac{h^2}{64\omega_1^4}z_2^5 + \frac{h^3}{512\omega_1^6}z_1^7 - \frac{73h^3}{512\omega_1^6}z_1^6 z_2 + \frac{1569h^3}{512\omega_1^6}z_1^5 z_2^2 \\ &- \frac{2139h^3}{256\omega_1^6}z_1^4 z_2^3 - \frac{2139h^3}{256\omega_1^6}z_1^3 z_2^4 + \frac{1569h^3}{512\omega_1^6}z_1^2 z_2^5 - \frac{73h^3}{512\omega_1^6}z_1 z_2^6 + \frac{h^3}{512\omega_1^6}z_2^7 \end{aligned} \tag{21}$$

$$\begin{aligned} y_2 &= i\omega_1 z_1 - i\omega_1 z_2 + i\frac{3h}{8\omega_1}z_1^3 + i\frac{3h}{4\omega_1}z_1^2 z_2 - i\frac{3h}{4\omega_1}z_1 z_2^2 - i\frac{3h}{8\omega_1}z_2^3 + i\frac{5h^2}{64\omega_1^3}z_1^5 - i\frac{81h^2}{64\omega_1^3}z_1^4 z_2 \\ &- i\frac{69h^2}{32\omega_1^3}z_1^3 z_2^2 + i\frac{69h^2}{32\omega_1^3}z_1^2 z_2^3 + i\frac{81h^2}{64\omega_1^3}z_1 z_2^4 - i\frac{5h^2}{64\omega_1^3}z_2^5 + i\frac{7h^3}{512\omega_1^5}z_1^7 - i\frac{305h^3}{512\omega_1^5}z_1^6 z_2 \\ &+ i\frac{2691h^3}{512\omega_1^5}z_1^5 z_2^2 + i\frac{2139h^3}{256\omega_1^5}z_1^4 z_2^3 - i\frac{2139h^3}{256\omega_1^5}z_1^3 z_2^4 - i\frac{2691h^3}{512\omega_1^5}z_1^2 z_2^5 + i\frac{305h^3}{512\omega_1^5}z_1 z_2^6 \\ &- i\frac{7h^3}{512\omega_1^5}z_2^7 \end{aligned} \tag{22}$$

The expressions for both reduced dynamics and nonlinear mappings remain unchanged, and thus the same as in the notes.

2.1.2 Undamped RNF

Reduced dynamics (up to order 7):

$$\dot{z}_1 = i\omega_1 z_1 + i\frac{3h}{2\omega_1}z_1^2 z_2 + i\frac{3h}{2\omega_1}z_1 z_2^2 - i\frac{15h^2}{16\omega_1^3}z_1^3 z_2^2 - i\frac{3h^2}{8\omega_1^3}z_1^2 z_2^3 + i\frac{267h^3}{128\omega_1^5}z_1^4 z_2^3 - i\frac{3h^3}{128\omega_1^5}z_1^3 z_2^4 \tag{23}$$

$$\dot{z}_2 = -i\omega_1 z_2 - i\frac{3h}{2\omega_1}z_1^2 z_2 - i\frac{3h}{2\omega_1}z_1 z_2^2 + i\frac{3h^2}{8\omega_1^3}z_1^3 z_2^2 + i\frac{15h^2}{16\omega_1^3}z_1^2 z_2^3 + i\frac{3h^3}{128\omega_1^5}z_1^4 z_2^3 - i\frac{267h^3}{128\omega_1^5}z_1^3 z_2^4 \tag{24}$$

Nonlinear mappings (up to order 7):

$$\begin{aligned} y_1 &= z_1 + z_2 + \frac{h}{8\omega_1^2}z_1^3 + \frac{h}{8\omega_1^2}z_2^3 + \frac{h^2}{64\omega_1^4}z_1^5 - \frac{21h^2}{64\omega_1^4}z_1^4 z_2 - \frac{21h^2}{64\omega_1^4}z_1 z_2^4 + \frac{h^2}{64\omega_1^4}z_2^5 \\ &+ \frac{h^3}{512\omega_1^6}z_1^7 - \frac{109h^3}{512\omega_1^6}z_1^6 z_2 + \frac{357h^3}{512\omega_1^6}z_1^5 z_2^2 + \frac{357h^3}{512\omega_1^6}z_1^2 z_2^5 - \frac{109h^3}{512\omega_1^6}z_1 z_2^6 + \frac{h^3}{512\omega_1^6}z_2^7 \end{aligned} \tag{25}$$

$$y_2 = i\omega_1 z_1 - i\omega_1 z_2 + i\frac{3h}{8\omega_1}z_1^3 - i\frac{3h}{8\omega_1}z_2^3 + i\frac{5h^2}{64\omega_1^3}z_1^5 - i\frac{27h^2}{64\omega_1^3}z_1^4 z_2 + i\frac{27h^2}{64\omega_1^3}z_1 z_2^4 - i\frac{5h^2}{64\omega_1^3}z_2^5$$
$$+ i\frac{7h^3}{512\omega_1^5}z_1^7 - i\frac{233h^3}{512\omega_1^5}z_1^6 z_2 + i\frac{195h^3}{512\omega_1^5}z_1^5 z_2^2 - i\frac{195h^3}{512\omega_1^5}z_1^2 z_2^5 + i\frac{233h^3}{512\omega_1^5}z_1 z_2^6 - i\frac{7h^3}{512\omega_1^5}z_2^7 \tag{26}$$

The expressions for both reduced dynamics and nonlinear mappings remain unchanged, and thus the same as in the notes.

2.1.3 Damped CNF

Reduced dynamics (up to order 5):

$$\dot{z}_1 = (i\delta_1\omega_1 - \xi_1\omega_1)\, z_1 + i\frac{3h}{2\delta_1\omega_1}z_1^2 z_2 + \frac{120h^2\delta_1\xi_1^8 - 90h^2\delta_1\xi_1^6 - 129h^2\delta_1\xi_1^4 + 78h^2\delta_1\xi_1^2}{32\delta_1\xi_1^9\omega_1^3 - 40\delta_1\xi_1^7\omega_1^3 - 24\delta_1\xi_1^5\omega_1^3 - 24\delta_1\xi_1^3\omega_1^3}$$
$$\frac{-51h^2\delta_1 - 120ih^2\xi_1^9 + 150ih^2\xi_1^7 + 99ih^2\xi_1^5 + 36ih^2\xi_1^3 - 165ih^2\xi_1}{+56\delta_1\xi_1\omega_1^3 - 32i\xi_1^{10}\omega_1^3 + 56i\xi_1^8\omega_1^3 + 8i\xi_1^6\omega_1^3 - 72i\xi_1^4\omega_1^3 + 56i\xi_1^2\omega_1^3 - 16i\omega_1^3}z_1^3 z_2^2 \tag{27}$$

$$\dot{z}_2 = -\,(i\delta_1\omega_1 + \xi_1\omega_1)\, z_2 - i\frac{3h}{2\delta_1\omega_1}z_1 z_2^2 + \frac{120h^2\delta_1\xi_1^8 - 90h^2\delta_1\xi_1^6 - 129h^2\delta_1\xi_1^4 + 78h^2\delta_1\xi_1^2}{32\delta_1\xi_1^9\omega_1^3 - 40\delta_1\xi_1^7\omega_1^3 - 24\delta_1\xi_1^5\omega_1^3}$$
$$\frac{-51h^2\delta_1 + 120ih^2\xi_1^9 - 150ih^2\xi_1^7 - 99ih^2\xi_1^5 - 36ih^2\xi_1^3 + 165ih^2\xi_1}{-24\delta_1\xi_1^3\omega_1^3 + 56\delta_1\xi_1\omega_1^3 + 32i\xi_1^{10}\omega_1^3 - 56i\xi_1^8\omega_1^3 - 8i\xi_1^6\omega_1^3 + 72i\xi_1^4\omega_1^3 - 56i\xi_1^2\omega_1^3 + 16i\omega_1^3}z_1^2 z_2^3 \tag{28}$$

Nonlinear mappings (up to order 5):

$$y_1 = z_1 + z_2 - \frac{ih}{12\delta_1\xi_1\omega_1^2 + 12i\xi_1^2\omega_1^2 - 8i\omega_1^2}z_1^3 + \frac{-3h\delta_1 + 3ih\xi_1}{4\delta_1\omega_1^2}z_1^2 z_2 + \frac{-3h\delta_1 - 3ih\xi_1}{4\delta_1\omega_1^2}z_1 z_2^2$$
$$+ \frac{ih}{12\delta_1\xi_1\omega_1^2 - 12i\xi_1^2\omega_1^2 + 8i\omega_1^2}z_2^3 + \frac{3ih^2}{960\delta_1\xi_1^3\omega_1^4 - 608\delta_1\xi_1\omega_1^4 + 960i\xi_1^4\omega_1^4 - 1088i\xi_1^2\omega_1^4 + 192i\omega_1^4}z_1^5$$
$$+ \frac{-8100h^2\delta_1\xi_1^7 + 13581h^2\delta_1\xi_1^5 - 6282h^2\delta_1\xi_1^3 + 657h^2\delta_1\xi_1}{124416\delta_1\xi_1^{11}\omega_1^4 - 362880\delta_1\xi_1^9\omega_1^4 + 384480\delta_1\xi_1^7\omega_1^4 - 176832\delta_1\xi_1^5\omega_1^4 + 32352\delta_1\xi_1^3\omega_1^4}$$
$$\frac{-8100ih^2\xi_1^8 + 17631ih^2\xi_1^6 - 12060ih^2\xi_1^4 + 2607ih^2\xi_1^2 - 78ih^2}{-1536\delta_1\xi_1\omega_1^4 + 124416i\xi_1^{12}\omega_1^4 - 425088i\xi_1^{10}\omega_1^4 + 550368i\xi_1^8\omega_1^4 - 331488i\xi_1^6\omega_1^4 + 90528i\xi_1^4\omega_1^4}$$
$$\frac{}{-8864i\xi_1^2\omega_1^4 + 128i\omega_1^4}z_1^4 z_2 + \frac{3145728h^2\delta_1\xi_1^{25} - 8355840h^2\delta_1\xi_1^{23} + 2998272h^2\delta_1\xi_1^{21}}{1048576\delta_1\xi_1^{27}\omega_1^4 - 2883584\delta_1\xi_1^{25}\omega_1^4 + 983040\delta_1\xi_1^{23}\omega_1^4}$$
$$\frac{+6819840h^2\delta_1\xi_1^{19} - 3225600h^2\delta_1\xi_1^{17} - 2859264h^2\delta_1\xi_1^{15} + 900480h^2\delta_1\xi_1^{13}}{+2850816\delta_1\xi_1^{21}\omega_1^4 - 13475840\delta_1\xi_1^{19}\omega_1^4 - 1492992\delta_1\xi_1^{17}\omega_1^4 + 465408\delta_1\xi_1^{15}\omega_1^4 + 489728\delta_1\xi_1^{13}\omega_1^4}$$
$$\frac{+715872h^2\delta_1\xi_1^{11} - 52800h^2\delta_1\xi_1^9 - 82830h^2\delta_1\xi_1^7 - 7662h^2\delta_1\xi_1^5 + 2310h^2\delta_1\xi_1^3}{-24816\delta_1\xi_1^{11}\omega_1^4 - 80496\delta_1\xi_1^9\omega_1^4 - 12640\delta_1\xi_1^7\omega_1^4 + 3360\delta_1\xi_1^5\omega_1^4 + 1104\delta_1\xi_1^3\omega_1^4 + 80\delta_1\xi_1\omega_1^4}$$
$$\frac{+342h^2\delta_1\xi_1 - 3145728ih^2\xi_1^{26} + 9928704ih^2\xi_1^{24} - 6782976ih^2\xi_1^{22} - 6168576ih^2\xi_1^{20}}{-1048576i\xi_1^{28}\omega_1^4 + 3407872i\xi_1^{26}\omega_1^4 - 2293760i\xi_1^{24}\omega_1^4 - 2654208i\xi_1^{22}\omega_1^4 + 2756608i\xi_1^{20}\omega_1^4}$$
$$\frac{+6610944ih^2\xi_1^{18} + 2045952ih^2\xi_1^{16} - 2353920ih^2\xi_1^{14} - 596928ih^2\xi_1^{12} + 373104ih^2\xi_1^{10}}{+1153024i\xi_1^{18}\omega_1^4 - 1221120i\xi_1^{16}\omega_1^4 - 431872i\xi_1^{14}\omega_1^4 + 247280i\xi_1^{12}\omega_1^4 + 111216i\xi_1^{10}\omega_1^4}$$
$$\frac{+115401ih^2\xi_1^8 - 17616ih^2\xi_1^6 - 8118ih^2\xi_1^4 - 312ih^2\xi_1^2 + 69ih^2}{-13312i\xi_1^8\omega_1^4 - 12064i\xi_1^6\omega_1^4 - 1296i\xi_1^4\omega_1^4 + 176i\xi_1^2\omega_1^4 + 32i\omega_1^4}z_1^3 z_2^2$$
$$+ \frac{3145728h^2\delta_1\xi_1^{25} - 8355840h^2\delta_1\xi_1^{23} + 2998272h^2\delta_1\xi_1^{21}}{1048576\delta_1\xi_1^{27}\omega_1^4 - 2883584\delta_1\xi_1^{25}\omega_1^4 + 983040\delta_1\xi_1^{23}\omega_1^4}$$
$$\frac{+6819840h^2\delta_1\xi_1^{19} - 3225600h^2\delta_1\xi_1^{17} - 2859264h^2\delta_1\xi_1^{15} + 900480h^2\delta_1\xi_1^{13}}{+2850816\delta_1\xi_1^{21}\omega_1^4 - 13475840\delta_1\xi_1^{19}\omega_1^4 - 1492992\delta_1\xi_1^{17}\omega_1^4 + 465408\delta_1\xi_1^{15}\omega_1^4 + 489728\delta_1\xi_1^{13}\omega_1^4}$$
$$\frac{+715872h^2\delta_1\xi_1^{11} - 52800h^2\delta_1\xi_1^9 - 82830h^2\delta_1\xi_1^7 - 7662h^2\delta_1\xi_1^5 + 2310h^2\delta_1\xi_1^3}{-24816\delta_1\xi_1^{11}\omega_1^4 - 80496\delta_1\xi_1^9\omega_1^4 - 12640\delta_1\xi_1^7\omega_1^4 + 3360\delta_1\xi_1^5\omega_1^4 + 1104\delta_1\xi_1^3\omega_1^4 + 80\delta_1\xi_1\omega_1^4}$$

$$
\frac{+342h^2\delta_1\xi_1 - 3145728ih^2\xi_1^{26} + 9928704ih^2\xi_1^{24} - 6782976ih^2\xi_1^{22} - 6168576ih^2\xi_1^{20}}{-1048576i\xi_1^{28}\omega_1^4 + 3407872i\xi_1^{26}\omega_1^4 - 2293760i\xi_1^{24}\omega_1^4 - 2654208i\xi_1^{22}\omega_1^4 + 2756608i\xi_1^{20}\omega_1^4}
$$
$$
\frac{+6610944ih^2\xi_1^{18} + 2045952ih^2\xi_1^{16} - 2353920ih^2\xi_1^{14} - 596928ih^2\xi_1^{12} + 373104ih^2\xi_1^{10}}{+1153024i\xi_1^{18}\omega_1^4 - 1221120i\xi_1^{16}\omega_1^4 - 431872i\xi_1^{14}\omega_1^4 + 247280i\xi_1^{12}\omega_1^4 + 111216i\xi_1^{10}\omega_1^4}
$$
$$
\frac{+115401ih^2\xi_1^8 - 17616ih^2\xi_1^6 - 8118ih^2\xi_1^4 - 312ih^2\xi_1^2 + 69ih^2}{-13312i\xi_1^8\omega_1^4 - 12064i\xi_1^6\omega_1^4 - 1296i\xi_1^4\omega_1^4 + 176i\xi_1^2\omega_1^4 + 32i\omega_1^4} z_1^2 z_2^3
$$
$$
+ \frac{-8100h^2\delta_1\xi_1^7 + 13581h^2\delta_1\xi_1^5 - 6282h^2\delta_1\xi_1^3 + 657h^2\delta_1\xi_1}{124416\delta_1\xi_1^{11}\omega_1^4 - 362880\delta_1\xi_1^9\omega_1^4 + 384480\delta_1\xi_1^7\omega_1^4 - 176832\delta_1\xi_1^5\omega_1^4 + 32352\delta_1\xi_1^3\omega_1^4}
$$
$$
\frac{-8100ih^2\xi_1^8 + 17631ih^2\xi_1^6 - 12060ih^2\xi_1^4 + 2607ih^2\xi_1^2 - 78ih^2}{-1536\delta_1\xi_1\omega_1^4 + 124416i\xi_1^{12}\omega_1^4 - 425088i\xi_1^{10}\omega_1^4 + 550368i\xi_1^8\omega_1^4 - 331488i\xi_1^6\omega_1^4 + 90528i\xi_1^4\omega_1^4}
$$
$$
\frac{}{-8864i\xi_1^2\omega_1^4 + 128i\omega_1^4} z_1 z_2^4 - \frac{3ih^2}{960\delta_1\xi_1^3\omega_1^4 - 608\delta_1\xi_1\omega_1^4 - 960i\xi_1^4\omega_1^4 + 1088i\xi_1^2\omega_1^4 - 192i\omega_1^4} z_2^5 \tag{29}
$$

$$
y_2 = \frac{i\omega_1}{\delta_1 - i\xi_1} z_1 - \frac{i\omega_1}{\delta_1 + i\xi_1} z_2 + \frac{3h\delta_1 + 3ih\xi_1}{12\delta_1\xi_1\omega_1 + 12i\xi_1^2\omega_1 - 8i\omega_1} z_1^3 + \frac{6h\delta_1\xi_1 - 6ih\xi_1^2 + 3ih}{4\delta_1\omega_1} z_1^2 z_2
$$
$$
+ \frac{6h\delta_1\xi_1 + 6ih\xi_1^2 - 3ih}{4\delta_1\omega_1} z_1 z_2^2 + \frac{3h\delta_1 - 3ih\xi_1}{12\delta_1\xi_1\omega_1 - 12i\xi_1^2\omega_1 + 8i\omega_1} z_2^3 + \frac{-15h^2\delta_1 - 15ih^2\xi_1}{960\delta_1\xi_1^3\omega_1^3 - 608\delta_1\xi_1\omega_1^3 + 960i\xi_1^4\omega_1^3 - 1088i\xi_1^2 \cdots} \tag{30}
$$

In the above equations, the y_2 expression was cut for practical purposes. Up to order 3, the expression for the reduced dynamics remains the same. However, when the order 5 coefficients are considered, the effect of the modification made to the code shows itself. The nonlinear mapping equations are also, this time starting from order 3, not the same as the ones before. Once again, Mathematica was used in order to simplify the long coefficients, yielding:
Reduced dynamics (up to order 5):

$$
\dot{z}_1 = (i\delta_1\omega_1 - \xi_1\omega_1) z_1 + i\frac{3h}{2\delta_1\omega_1} z_1^2 z_2 - i\frac{3h^2\left(17 - 14\xi_1^2 + 4i\xi_1\delta_1\right)}{8\omega_1^3\delta_1^2\left(-2i\delta_1 + \xi_1\right)} z_1^3 z_2^2 \tag{31}
$$
$$
\dot{z}_2 = -(i\delta_1\omega_1 + \xi_1\omega_1) z_2 - i\frac{3h}{2\delta_1\omega_1} z_1 z_2^2 - i\frac{3h^2\left(17 - 14\xi_1^2 - 4i\xi_1\delta_1\right)}{8\omega_1^3\delta_1^2\left(2i\delta_1 + \xi_1\right)} z_1^2 z_2^3 \tag{32}
$$

Nonlinear mappings (up to order 5):

$$
y_1 = z_1 + z_2 - \frac{ih}{4\omega_1^2\left(3\delta_1\xi_1 + 3i\xi_1^2 - 2i\right)} z_1^3 + \frac{3h\left(-\delta_1 + i\xi_1\right)}{4\delta_1\omega_1^2} z_1^2 z_2 + \frac{3h\left(-\delta_1 - i\xi_1\right)}{4\delta_1\omega_1^2} z_1 z_2^2
$$
$$
+ \frac{ih}{4\omega_1^2\left(3\delta_1\xi_1 - 3i\xi_1^2 + 2i\right)} z_2^3 + \frac{3h^2}{32\omega^4\left(30\xi_1^4 - 34\xi_1^2 + 19i\delta_1\xi_1 - 30i\delta_1\xi_1^3 + 6\right)} z_1^5
$$
$$
+ \frac{3h^2\left(18\xi_1^6 - 144\xi_1^4 + 152\xi_1^2 + 93i\delta_1\xi_1 + 18i\delta_1\xi_1^5 - 135i\delta_1\xi_1^3 - 26\right)}{32\omega^4\left(9\xi_1^6 - 18\xi_1^4 + 5\xi_1^2 + 4\right)} z_1^4 z_2
$$
$$
+ \frac{3h^2\left(63\xi_1^4 - 121\xi_1^2 - 113i\delta_1\xi_1 + 81i\delta_1\xi_1^3 + 46\right)}{16\omega^4\left(9\xi_1^6 - 18\xi_1^4 + 5\xi_1^2 + 4\right)} z_1^3 z_2^2
$$
$$
+ \frac{3h^2\left(63\xi_1^4 - 121\xi_1^2 + 113i\delta_1\xi_1 - 81i\delta_1\xi_1^3 + 46\right)}{16\omega^4\left(9\xi_1^6 - 18\xi_1^4 + 5\xi_1^2 + 4\right)} z_1^2 z_2^3
$$
$$
+ \frac{3h^2\left(18\xi_1^6 - 144\xi_1^4 + 152\xi_1^2 - 93i\delta_1\xi_1 - 18i\delta_1\xi_1^5 + 135i\delta_1\xi_1^3 - 26\right)}{32\omega^4\left(9\xi_1^6 - 18\xi_1^4 + 5\xi_1^2 + 4\right)} z_1 z_2^4
$$
$$
+ \frac{3h^2}{32\omega^4\left(30\xi_1^4 - 34\xi_1^2 - 19i\delta_1\xi_1 + 30i\delta_1\xi_1^3 + 6\right)} z_2^5 \tag{33}
$$
$$
y_2 = \frac{i\omega_1}{\delta_1 - i\xi_1} z_1 - \frac{i\omega_1}{\delta_1 + i\xi_1} z_2 + \frac{3h\left(\delta_1 + i\xi_1\right)}{4\omega_1\left(3\delta_1\xi_1 + 3i\xi_1^2 - 2i\right)} z_1^3 + \frac{3h\left(2\delta_1\xi_1 - 2i\xi_1^2 + i\right)}{4\delta_1\omega_1} z_1^2 z_2
$$

$$
+\frac{3h\left(2\delta_1\xi_1+2i\xi_1^2-i\right)}{4\delta_1\omega_1}z_1z_2^2+\frac{3h\left(\delta_1-i\xi_1\right)}{4\omega_1\left(3\delta_1\xi_1-3i\xi_1^2+2i\right)}z_2^3
$$
$$
-\frac{15ih^2}{32\omega^3\left(15i\xi_1^3+15\delta_1\xi_1^2-6\delta_1-13i\xi_1\right)}z_1^5
$$
$$
+\frac{3h^2\left(-36\xi_1^7+153\xi_1^5-4\xi_1^3+27i\delta_1\xi_1^2-54i\delta_1-36i\delta_1\xi_1^6+135i\delta_1\xi_1^4-113\xi_1\right)}{32\omega^3\left(9\xi_1^6-18\xi_1^4+5\xi_1^2+4\right)}z_1^4z_2
$$
$$
+\frac{3h^2\left(-144\xi_1^5+315\xi_1^3+234i\delta_1\xi_1^2-46i\delta_1-144i\delta_1\xi_1^4-159\xi_1\right)}{16\omega^3\left(9\xi_1^6-18\xi_1^4+5\xi_1^2+4\right)}z_1^3z_2^2
$$
$$
+\frac{3h^2\left(-144\xi_1^5+315\xi_1^3-234i\delta_1\xi_1^2+46i\delta_1+144i\delta_1\xi_1^4-159\xi_1\right)}{16\omega^3\left(9\xi_1^6-18\xi_1^4+5\xi_1^2+4\right)}z_1^2z_2^3
$$
$$
+\frac{3h^2\left(-36\xi_1^7+153\xi_1^5-4\xi_1^3-27i\delta_1\xi_1^2+54i\delta_1+36i\delta_1\xi_1^6-135i\delta_1\xi_1^4-113\xi_1\right)}{32\omega^3\left(9\xi_1^6-18\xi_1^4+5\xi_1^2+4\right)}z_1z_2^4
$$
$$
+\frac{15ih^2}{32\omega^3\left(-15i\xi_1^3+15\delta_1\xi_1^2-6\delta_1+13i\xi_1\right)}z_2^5 \tag{34}
$$

The expressions for the reduced dynamics don't match the ones in the notes. From this observation together with the fact that the expressions in Section 1 for this case are the same as in the notes, it's possible to conclude that probably the results in there were obtained with the code before the bug was corrected.

2.1.4 Damped RNF

Reduced dynamics (up to order 5):

$$
\begin{aligned}
\dot z_1 &= \left(i\delta_1\omega_1-\xi_1\omega_1\right)z_1+i\frac{3h}{2\delta_1\omega_1}z_1^2z_2+\frac{-6h\delta_1\xi_1-6ih\xi_1^2+3ih}{2\delta_1\omega_1}z_1z_2^2\\
&+\frac{432h^2\delta_1\xi_1^5-639h^2\delta_1\xi_1^3+216h^2\delta_1\xi_1+432ih^2\xi_1^6-855ih^2\xi_1^4+483ih^2\xi_1^2-60ih^2}{864\delta_1\xi_1^8\omega_1^3-2376\delta_1\xi_1^6\omega_1^3+2232\delta_1\xi_1^4\omega_1^3-784\delta_1\xi_1^2\omega_1^3+64\delta_1\omega_1^3+864i\xi_1^9\omega_1^3-2808i\xi_1^7\omega_1^3+3312i\xi_1^5\omega_1^3-1656i\xi_1^3\omega_1^3+288i\xi_1\omega_1^3}z_1^3z_2^2\\
&+\frac{69h^2\delta_1\xi_1^3-60h^2\delta_1\xi_1-66ih^2\xi_1^4+90ih^2\xi_1^2-24ih^2}{864\delta_1\xi_1^8\omega_1^3-2376\delta_1\xi_1^6\omega_1^3+2232\delta_1\xi_1^4\omega_1^3-784\delta_1\xi_1^2\omega_1^3+64\delta_1\omega_1^3-864i\xi_1^9\omega_1^3+2808i\xi_1^7\omega_1^3-3312i\xi_1^5\omega_1^3+1656i\xi_1^3\omega_1^3-288i\xi_1\omega_1^3}z_1^2z_2^3
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
\dot z_2 &= -\left(i\delta_1\omega_1+\xi_1\omega_1\right)z_2+\frac{-6h\delta_1\xi_1+6ih\xi_1^2-3ih}{2\delta_1\omega_1}z_1^2z_2-i\frac{3h}{2\delta_1\omega_1}z_1z_2^2\\
&+\frac{69h^2\delta_1\xi_1^3-60h^2\delta_1\xi_1+66ih^2\xi_1^4-90ih^2\xi_1^2+24ih^2}{864\delta_1\xi_1^8\omega_1^3-2376\delta_1\xi_1^6\omega_1^3+2232\delta_1\xi_1^4\omega_1^3-784\delta_1\xi_1^2\omega_1^3+64\delta_1\omega_1^3+864i\xi_1^9\omega_1^3-2808i\xi_1^7\omega_1^3+3312i\xi_1^5\omega_1^3-1656i\xi_1^3\omega_1^3+288i\xi_1\omega_1^3}z_1^3z_2^2\\
&+\frac{432h^2\delta_1\xi_1^5-639h^2\delta_1\xi_1^3+216h^2\delta_1\xi_1-432ih^2\xi_1^6+855ih^2\xi_1^4-483ih^2\xi_1^2+60ih^2}{864\delta_1\xi_1^8\omega_1^3-2376\delta_1\xi_1^6\omega_1^3+2232\delta_1\xi_1^4\omega_1^3-784\delta_1\xi_1^2\omega_1^3+64\delta_1\omega_1^3-864i\xi_1^9\omega_1^3+2808i\xi_1^7\omega_1^3-3312i\xi_1^5\omega_1^3+1656i\xi_1^3\omega_1^3-288i\xi_1\omega_1^3}z_1^2z_2^3
\end{aligned} \tag{36}
$$

Nonlinear mappings (up to order 5):

$$
\begin{aligned}
y_1 &= \left(\delta_1-i\xi_1\right)z_1+\left(\delta_1+i\xi_1\right)z_2+\frac{4ih\delta_1\xi_1^2-ih\delta_1+4h\xi_1^3-3h\xi_1}{12\delta_1\xi_1\omega_1^2+12i\xi_1^2\omega_1^2-8i\omega_1^2}z_1^3+\frac{-4ih\delta_1\xi_1^2+ih\delta_1+4h\xi_1^3-3h\xi_1}{12\delta_1\xi_1\omega_1^2-12i\xi_1^2\omega_1^2+8i\omega_1^2}z_2^3\\
&+\frac{48ih^2\delta_1\xi_1^4-36ih^2\delta_1\xi_1^2+3ih^2\delta_1+48h^2\xi_1^5-60h^2\xi_1^3+15h^2\xi_1}{960\delta_1\xi_1^3\omega_1^4-608\delta_1\xi_1\omega_1^4+960i\xi_1^4\omega_1^4-1088i\xi_1^2\omega_1^4+192i\omega_1^4}z_1^5
\end{aligned}
$$

$$+\frac{-6ih^2\delta_1\xi_1^3-117ih^2\delta_1\xi_1-6h^2\xi_1^4+102h^2\xi_1^2-42h^2}{20736\delta_1\xi_1^8\omega_1^4-41472\delta_1\xi_1^6\omega_1^4+25632\delta_1\xi_1^4\omega_1^4-4896\delta_1\xi_1^2\omega_1^4+128\delta_1\omega_1^4+20736i\xi_1^9\omega_1^4 -51840i\xi_1^7\omega_1^4+43776i\xi_1^5\omega_1^4-13824i\xi_1^3\omega_1^4+1152i\xi_1\omega_1^4}z_1^4z_2$$

$$+\frac{6ih^2\delta_1\xi_1^3+117ih^2\delta_1\xi_1-6h^2\xi_1^4+102h^2\xi_1^2-42h^2}{20736\delta_1\xi_1^8\omega_1^4-41472\delta_1\xi_1^6\omega_1^4+25632\delta_1\xi_1^4\omega_1^4-4896\delta_1\xi_1^2\omega_1^4+128\delta_1\omega_1^4-20736i\xi_1^9\omega_1^4 +51840i\xi_1^7\omega_1^4-43776i\xi_1^5\omega_1^4+13824i\xi_1^3\omega_1^4-1152i\xi_1\omega_1^4}z_1z_2^4$$

$$+\frac{-48ih^2\delta_1\xi_1^4+36ih^2\delta_1\xi_1^2-3ih^2\delta_1+48h^2\xi_1^5-60h^2\xi_1^3+15h^2\xi_1}{960\delta_1\xi_1^3\omega_1^4-608\delta_1\xi_1\omega_1^4-960i\xi_1^4\omega_1^4+1088i\xi_1^2\omega_1^4-192i\omega_1^4}z_2^5 \tag{37}$$

$$y_2=i\omega_1z_1-i\omega_1z_2+\frac{-6ih\delta_1\xi_1-6h\xi_1^2+3h}{12\delta_1\xi_1\omega_1+12i\xi_1^2\omega_1-8i\omega_1}z_1^3+\frac{6ih\delta_1\xi_1-6h\xi_1^2+3h}{12\delta_1\xi_1\omega_1-12i\xi_1^2\omega_1+8i\omega_1}z_2^3$$

$$+\frac{-120ih^2\delta_1\xi_1^3+60ih^2\delta_1\xi_1-120h^2\xi_1^4+120h^2\xi_1^2-15h^2}{960\delta_1\xi_1^3\omega_1^3-608\delta_1\xi_1\omega_1^3+960i\xi_1^4\omega_1^3-1088i\xi_1^2\omega_1^3+192i\omega_1^3}z_1^5$$

$$+\frac{12ih^2\delta_1\xi_1^4+387ih^2\delta_1\xi_1^2-54ih^2\delta_1+12h^2\xi_1^5-375h^2\xi_1^3+237h^2\xi_1}{20736\delta_1\xi_1^8\omega_1^3-41472\delta_1\xi_1^6\omega_1^3+25632\delta_1\xi_1^4\omega_1^3-4896\delta_1\xi_1^2\omega_1^3+128\delta_1\omega_1^3+20736i\xi_1^9\omega_1^3 -51840i\xi_1^7\omega_1^3+43776i\xi_1^5\omega_1^3-13824i\xi_1^3\omega_1^3+1152i\xi_1\omega_1^3}z_1^4z_2$$

$$+\frac{-12ih^2\delta_1\xi_1^4-387ih^2\delta_1\xi_1^2+54ih^2\delta_1+12h^2\xi_1^5-375h^2\xi_1^3+237h^2\xi_1}{20736\delta_1\xi_1^8\omega_1^3-41472\delta_1\xi_1^6\omega_1^3+25632\delta_1\xi_1^4\omega_1^3-4896\delta_1\xi_1^2\omega_1^3+128\delta_1\omega_1^3-20736i\xi_1^9\omega_1^3 +51840i\xi_1^7\omega_1^3-43776i\xi_1^5\omega_1^3+13824i\xi_1^3\omega_1^3-1152i\xi_1\omega_1^3}z_1z_2^4+$$

$$\frac{120ih^2\delta_1\xi_1^3-60ih^2\delta_1\xi_1-120h^2\xi_1^4+120h^2\xi_1^2-15h^2}{960\delta_1\xi_1^3\omega_1^3-608\delta_1\xi_1\omega_1^3-960i\xi_1^4\omega_1^3+1088i\xi_1^2\omega_1^3-192i\omega_1^3}z_2^5 \tag{38}$$

Compared to the unmodified code, the expressions for the reduced dynamics remain the same up to order 3, remaining different from the ones in the notes. The ones for the nonlinear mappings have changed, and now include cubic terms. Simplifying the coefficients with Mathematica, one obtains:

Reduced dynamics (up to order 5):

$$\dot{z}_1=(i\delta_1\omega_1-\xi_1\omega_1)z_1+i\frac{3h}{2\delta_1\omega_1}z_1^2z_2+\frac{-6h\delta_1\xi_1-6ih\xi_1^2+3ih}{2\delta_1\omega_1}z_1z_2^2$$
$$+\frac{3h^2\left(-9\xi_1+6\xi_1^3-10i\delta_1+6i\delta_1\xi_1^2\right)}{8\omega_1^3\left(4-7\xi_1^2+3\xi_1^4\right)}z_1^3z_2^2+\frac{3h^2\left(8\xi_1+\xi_1^3-12\xi_1^5-4i\delta_1+5i\delta_1\xi_1^2+12i\delta_1\xi_1^4\right)}{8\omega_1^3\left(4-7\xi_1^2+3\xi_1^4\right)}z_1^2z_2^3 \tag{39}$$

$$\dot{z}_2=-(i\delta_1\omega_1+\xi_1\omega_1)z_2+\frac{-6h\delta_1\xi_1+6ih\xi_1^2-3ih}{2\delta_1\omega_1}z_1^2z_2-i\frac{3h}{2\delta_1\omega_1}z_1z_2^2$$
$$+\frac{3h^2\left(8\xi_1+\xi_1^3-12\xi_1^5+4i\delta_1-5i\delta_1\xi_1^2-12i\delta_1\xi_1^4\right)}{8\omega_1^3\left(4-7\xi_1^2+3\xi_1^4\right)}z_1^3z_2^2+\frac{3h^2\left(-9\xi_1+6\xi_1^3+10i\delta_1-6i\delta_1\xi_1^2\right)}{8\omega_1^3\left(4-7\xi_1^2+3\xi_1^4\right)}z_1^2z_2^3 \tag{40}$$

Nonlinear mappings (up to order 5):

$$y_1=(\delta_1-i\xi_1)z_1+(\delta_1+i\xi_1)z_2+\frac{h\left(4\xi_1^3+4i\delta_1\xi_1^2-i\delta_1-3\xi_1\right)}{4\omega^2\left(3i\xi_1^2+3\delta_1\xi_1-2i\right)}z_1^3+\frac{h\left(4\xi_1^3-4i\delta_1\xi_1^2+i\delta_1-3\xi_1\right)}{4\omega^2\left(-3i\xi_1^2+3\delta_1\xi_1+2i\right)}z_2^3$$
$$+\frac{3h^2\left(16\xi_1^5-20\xi_1^3-12i\delta_1\xi_1^2+i\delta_1+16i\delta_1\xi_1^4+5\xi_1\right)}{32\omega^4\left(30i\xi_1^4-34i\xi_1^2-19\delta_1\xi_1+30\delta_1\xi_1^3+6i\right)}z_1^5$$
$$+\frac{3h^2\left(-2\xi_1^4+34\xi_1^2-39i\delta_1\xi_1-2i\delta_1\xi_1^3-14\right)}{32\omega^4\left(648i\xi_1^9-1620i\xi_1^7+1368i\xi_1^5-432i\xi_1^3-153\delta_1\xi_1^2+4\delta_1+648\delta_1\xi_1^8-1296\delta_1\xi_1^6+801\delta_1\xi_1^4+36i\xi_1\right)}z_1^4z_2$$

$$
+ \frac{3h^2\left(-2\xi_1^4+34\xi_1^2+39i\delta_1\xi_1+2i\delta_1\xi_1^3-14\right)}{32\omega^4\left(-648i\xi_1^9+1620i\xi_1^7-1368i\xi_1^5+432i\xi_1^3-153\delta_1\xi_1^2+4\delta_1+648\delta_1\xi_1^8-1296\delta_1\xi_1^6+801\delta_1\xi_1^4-36i\xi_1\right)} z_1 z_2
$$

$$
+ \frac{3h^2\left(16\xi_1^5-20\xi_1^3+12i\delta_1\xi_1^2-i\delta_1-16i\delta_1\xi_1^4+5\xi_1\right)}{32\omega^4\left(-30i\xi_1^4+34i\xi_1^2-19\delta_1\xi_1+30\delta_1\xi_1^3-6i\right)} z_2^5 \tag{41}
$$

$$
\begin{aligned}
y_2 = {} & i\omega_1 z_1 - i\omega_1 z_2 + \frac{3h\left(-2\xi_1^2-2i\delta_1\xi_1+1\right)}{4\omega\left(3i\xi_1^2+3\delta_1\xi_1-2i\right)} z_1^3 + \frac{3h\left(-2\xi_1^2+2i\delta_1\xi_1+1\right)}{4\omega\left(-3i\xi_1^2+3\delta_1\xi_1+2i\right)} z_2^3 \\
& + \frac{15h^2\left(8i\xi_1^4-8i\xi_1^2+4\delta_1\xi_1-8\delta_1\xi_1^3+i\right)}{32\omega^3\left(30\xi_1^4-34\xi_1^2+19i\delta_1\xi_1-30i\delta_1\xi_1^3+6\right)} z_1^5
\end{aligned}
$$

$$
+ \frac{3h^2\left(4\xi_1^5-125\xi_1^3+129i\delta_1\xi_1^2-18i\delta_1+4i\delta_1\xi_1^4+79\xi_1\right)}{32\omega^3\left(648i\xi_1^9-1620i\xi_1^7+1368i\xi_1^5-432i\xi_1^3-153\delta_1\xi_1^2+4\delta_1+648\delta_1\xi_1^8-1296\delta_1\xi_1^6+801\delta_1\xi_1^4+36i\xi_1\right)} z_1^4 z_2
$$

$$
+ \frac{3h^2\left(4\xi_1^5-125\xi_1^3-129i\delta_1\xi_1^2+18i\delta_1-4i\delta_1\xi_1^4+79\xi_1\right)}{32\omega^3\left(-648i\xi_1^9+1620i\xi_1^7-1368i\xi_1^5+432i\xi_1^3-153\delta_1\xi_1^2+4\delta_1+648\delta_1\xi_1^8-1296\delta_1\xi_1^6+801\delta_1\xi_1^4-36i\xi_1\right)} z_1 z_2
$$

$$
+ \frac{15h^2\left(-8i\xi_1^4+8i\xi_1^2+4\delta_1\xi_1-8\delta_1\xi_1^3-i\right)}{32\omega^3\left(30\xi_1^4-34\xi_1^2-19i\delta_1\xi_1+30i\delta_1\xi_1^3+6\right)} z_2^5 \tag{42}
$$

Since the coefficients from this case are different from the ones in the notes both for the unmodified code and for the modified one, a semi-manual (with the help of Mathematica) check was made in order to verify them (up to order 3 only). The calculations are given in Appendix 1, and all manually calculated coefficients match the ones from Julia. This validates the code with the fixed bug.

2.2 Unforced quadratic-cubic Duffing oscillator

2.2.1 Undamped CNF

Reduced dynamics (up to order 7):

$$
\begin{aligned}
\dot{z}_1 = {} & i\omega_1 z_1 + \frac{-10ig^2+9ih\omega_1^2}{6\omega_1^3} z_1^2 z_2 + \frac{-3140ig^4+8388ig^2h\omega_1^2-1377ih^2\omega_1^4}{432\omega_1^7} z_1^3 z_2^2 \\
& + \frac{-523960ig^6+2186724ig^4h\omega_1^2-1913274ig^2h^2\omega_1^4+114939ih^3\omega_1^6}{10368\omega_1^{11}} z_1^4 z_2^3
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
\dot{z}_2 = {} & -i\omega_1 z_2 + \frac{10ig^2-9ih\omega_1^2}{6\omega_1^3} z_1 z_2^2 + \frac{3140ig^4-8388ig^2h\omega_1^2+1377ih^2\omega_1^4}{432\omega_1^7} z_1^2 z_2^3 \\
& + \frac{523960ig^6-2186724ig^4h\omega_1^2+1913274ig^2h^2\omega_1^4-114939ih^3\omega_1^6}{10368\omega_1^{11}} z_1^3 z_2^4
\end{aligned} \tag{44}
$$

Nonlinear mappings (up to order 5):

$$
\begin{aligned}
y_1 = {} & z_1 + z_2 + \frac{g}{3\omega_1^2} z_1^2 - \frac{2g}{\omega_1^2} z_1 z_2 + \frac{g}{3\omega_1^2} z_2^2 + \frac{2g^2+3h\omega_1^2}{24\omega_1^4} z_1^3 + \frac{10g^2-9h\omega_1^2}{12\omega_1^4} z_1^2 z_2 \\
& + \frac{10g^2-9h\omega_1^2}{12\omega_1^4} z_1 z_2^2 + \frac{2g^2+3h\omega_1^2}{24\omega_1^4} z_2^3 + \frac{2g^3+9gh\omega_1^2}{108\omega_1^6} z_1^4 + \frac{178g^3-333gh\omega_1^2}{108\omega_1^6} z_1^3 z_2 \\
& + \frac{-68g^3+117gh\omega_1^2}{9\omega_1^6} z_1^2 z_2^2 + \frac{178g^3-333gh\omega_1^2}{108\omega_1^6} z_1 z_2^3 + \frac{2g^3+9gh\omega_1^2}{108\omega_1^6} z_2^4 \\
& + \frac{20g^4+180g^2h\omega_1^2+81h^2\omega_1^4}{5184\omega_1^8} z_1^5 + \frac{436g^4-444g^2h\omega_1^2-351h^2\omega_1^4}{576\omega_1^8} z_1^4 z_2 \\
& + \frac{3740g^4-9468g^2h\omega_1^2+1863h^2\omega_1^4}{864\omega_1^8} z_1^3 z_2^2 + \frac{3740g^4-9468g^2h\omega_1^2+1863h^2\omega_1^4}{864\omega_1^8} z_1^2 z_2^3
\end{aligned}
$$

$$+\frac{436g^4-444g^2h\omega_1^2-351h^2\omega_1^4}{576\omega_1^8}z_1z_2^4+\frac{20g^4+180g^2h\omega_1^2+81h^2\omega_1^4}{5184\omega_1^8}z_2^5 \tag{45}$$

$$\begin{aligned}y_2 &= i\omega_1 z_1 - i\omega_1 z_2 + i\frac{2g}{3\omega_1}z_1^2 - i\frac{2g}{3\omega_1}z_2^2 + \frac{2ig^2+3ih\omega_1^2}{8\omega_1^3}z_1^3 + \frac{-10ig^2+9ih\omega_1^2}{12\omega_1^3}z_1^2z_2\\ &+ \frac{10ig^2-9ih\omega_1^2}{12\omega_1^3}z_1z_2^2 + \frac{-2ig^2-3ih\omega_1^2}{8\omega_1^3}z_2^3 + \frac{2ig^3+9igh\omega_1^2}{27\omega_1^5}z_1^4 + \frac{118ig^3-279igh\omega_1^2}{54\omega_1^5}z_1^3z_2\\ &+ \frac{-118ig^3+279igh\omega_1^2}{54\omega_1^5}z_1z_2^3 + \frac{-2ig^3-9igh\omega_1^2}{27\omega_1^5}z_2^4 + \frac{100ig^4+900ig^2h\omega_1^2+405ih^2\omega_1^4}{5184\omega_1^7}z_1^5\\ &+ \frac{356ig^4-492ig^2h\omega_1^2-243ih^2\omega_1^4}{192\omega_1^7}z_1^4z_2 + \frac{-3740ig^4+9468ig^2h\omega_1^2-1863ih^2\omega_1^4}{864\omega_1^7}z_1^3z_2^2\\ &+ \frac{3740ig^4-9468ig^2h\omega_1^2+1863ih^2\omega_1^4}{864\omega_1^7}z_1^2z_2^3 + \frac{-356ig^4+492ig^2h\omega_1^2+243ih^2\omega_1^4}{192\omega_1^7}z_1z_2^4\\ &+ \frac{-100ig^4-900ig^2h\omega_1^2-405ih^2\omega_1^4}{5184\omega_1^7}z_2^5\end{aligned} \tag{46}$$

These expressions are not the same from the notes, since the bug wasn't fixed for the quadratic-cubic case. In particular, the expressions from the notes differ from the ones found for the cubic oscillators when $g = 0$! These expressions, on the other hand, don't, and the cubic oscillator dynamics and mappings are recovered.

2.2.2 Undamped RNF

Reduced dynamics (up to order 7):

$$\begin{aligned}\dot z_1 &= i\omega_1 z_1 + \frac{-10ig^2+9ih\omega_1^2}{6\omega_1^3}z_1^2z_2 + \frac{-10ig^2+9ih\omega_1^2}{6\omega_1^3}z_1z_2^2 + \frac{-1940ig^4+6228ig^2h\omega_1^2-405ih^2\omega_1^4}{432\omega_1^7}z_1^3z_2^2\\ &+ \frac{-1060ig^4+3060ig^2h\omega_1^2-81ih^2\omega_1^4}{216\omega_1^7}z_1^2z_2^3\\ &+ \frac{-242360ig^6+1161444ig^4h\omega_1^2-1114938ig^2h^2\omega_1^4+21627ih^3\omega_1^6}{10368\omega_1^{11}}z_1^4z_2^3\\ &+ \frac{-331400ig^6+1344252ig^4h\omega_1^2-1124982ig^2h^2\omega_1^4-243ih^3\omega_1^6}{10368\omega_1^{11}}z_1^3z_2^4\end{aligned} \tag{47}$$

$$\begin{aligned}\dot z_2 &= -i\omega_1 z_2 + \frac{10ig^2-9ih\omega_1^2}{6\omega_1^3}z_1^2z_2 + \frac{10ig^2-9ih\omega_1^2}{6\omega_1^3}z_1z_2^2 + \frac{1060ig^4-3060ig^2h\omega_1^2+81ih^2\omega_1^4}{216\omega_1^7}z_1^3z_2^2\\ &+ \frac{1940ig^4-6228ig^2h\omega_1^2+405ih^2\omega_1^4}{432\omega_1^7}z_1^2z_2^3\\ &+ \frac{331400ig^6-1344252ig^4h\omega_1^2+1124982ig^2h^2\omega_1^4+243ih^3\omega_1^6}{10368\omega_1^{11}}z_1^4z_2^3\\ &+ \frac{242360ig^6-1161444ig^4h\omega_1^2+1114938ig^2h^2\omega_1^4-21627ih^3\omega_1^6}{10368\omega_1^{11}}z_1^3z_2^4\end{aligned} \tag{48}$$

Nonlinear mappings (up to order 5):

$$\begin{aligned}y_1 &= z_1 + z_2 + \frac{g}{3\omega_1^2}z_1^2 - \frac{2g}{\omega_1^2}z_1z_2 + \frac{g}{3\omega_1^2}z_2^2 + \frac{2g^2+3h\omega_1^2}{24\omega_1^4}z_1^3 + \frac{2g^2+3h\omega_1^2}{24\omega_1^4}z_2^3 + \frac{2g^3+9gh\omega_1^2}{108\omega_1^6}z_1^4\\ &+ \frac{358g^3-495gh\omega_1^2}{108\omega_1^6}z_1^3z_2 + \frac{-26g^3+42gh\omega_1^2}{3\omega_1^6}z_1^2z_2^2 + \frac{358g^3-495gh\omega_1^2}{108\omega_1^6}z_1z_2^3 + \frac{2g^3+9gh\omega_1^2}{108\omega_1^6}z_2^4\\ &+ \frac{20g^4+180g^2h\omega_1^2+81h^2\omega_1^4}{5184\omega_1^8}z_1^5 + \frac{212g^4-268g^2h\omega_1^2-63h^2\omega_1^4}{192\omega_1^8}z_1^4z_2\end{aligned}$$

$$+\frac{212g^4-268g^2h\omega_1^2-63h^2\omega_1^4}{192\omega_1^8}z_1z_2^4+\frac{20g^4+180g^2h\omega_1^2+81h^2\omega_1^4}{5184\omega_1^8}z_2^5 \tag{49}$$

$$\begin{aligned} y_2 = {} & i\omega_1 z_1 - i\omega_1 z_2 + i\frac{2g}{3\omega_1}z_1^2 - i\frac{2g}{3\omega_1}z_2^2 + \frac{2ig^2+3ih\omega_1^2}{8\omega_1^3}z_1^3 + \frac{-2ig^2-3ih\omega_1^2}{8\omega_1^3}z_2^3 + \frac{2ig^3+9igh\omega_1^2}{27\omega_1^5}z_1^4 \\ & + \frac{118ig^3-279igh\omega_1^2}{54\omega_1^5}z_1^3z_2 + \frac{-118ig^3+279igh\omega_1^2}{54\omega_1^5}z_1z_2^3 + \frac{-2ig^3-9igh\omega_1^2}{27\omega_1^5}z_2^4 \\ & + \frac{100ig^4+900ig^2h\omega_1^2+405ih^2\omega_1^4}{5184\omega_1^7}z_1^5 + \frac{556ig^4-852ig^2h\omega_1^2-81ih^2\omega_1^4}{192\omega_1^7}z_1^4z_2 \\ & + \frac{-556ig^4+852ig^2h\omega_1^2+81ih^2\omega_1^4}{192\omega_1^7}z_1z_2^4 + \frac{-100ig^4-900ig^2h\omega_1^2-405ih^2\omega_1^4}{5184\omega_1^7}z_2^5 \end{aligned} \tag{50}$$

The same comments that were made for the CNF style apply here.

Appendix A - Cubic coefficients calculation for the damped cubic duffing oscillator with RNF style

The definition of the first order system is as follows:

$$\mathbf{B}\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{Q}(\mathbf{y}, \mathbf{y}), \tag{51}$$

where quadratic recast is used. The unknown vector, $\mathbf{y}$, matrices $\mathbf{A}$ and $\mathbf{B}$, and the vector $\mathbf{Q}(\mathbf{y}, \mathbf{y})$ for the cubic damped duffing oscillator are given by:

$$\mathbf{y} = [u_1, v_1, r_1]^T, \tag{52}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -\omega^2 & -2\xi\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{53}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{54}$$

$$\mathbf{Q}(\mathbf{y^1}, \mathbf{y^2}) = \left[0, -\frac{h}{2}y_1^1 y_3^2 - \frac{h}{2}y_3^1 y_1^2, -y_1^1 y_1^2\right]^T. \tag{55}$$

For the cubic coefficients, there are four monomials of interest for which to solve the system given by Eq. (62) of [1], associated with the following $\boldsymbol{\alpha}(p, k)$ vectors:

$$\boldsymbol{\alpha}(3,1) = [3,0], \quad \boldsymbol{\alpha}(3,2) = [2,1], \quad \boldsymbol{\alpha}(3,3) = [1,2], \quad \boldsymbol{\alpha}(3,4) = [0,3]. \tag{56}$$

For each one of them, it's necessary to calculate vector $\mathbf{R}^{(p,k)}$, and consequently vectors $\mathbf{Q}^{(p,k)}$ and $\mathbf{N}_3^{(p,k)}$ (in this case, vector $\mathbf{N}_2^{(p,k)}$ is always zero, since the system is autonomous). In order to do so, the following nonlinear mappings and reduced dynamics coefficients from orders 1 and 2 will be used:

$$\mathbf{W}^{(1,1)} = \begin{bmatrix} \delta - i\xi \\ i\omega \\ 0 \end{bmatrix}, \mathbf{W}^{(1,2)} = \begin{bmatrix} \delta + i\xi \\ -i\omega \\ 0 \end{bmatrix}, \tag{57}$$

$$\mathbf{W}^{(2,1)} = \begin{bmatrix} 0 \\ 0 \\ (\delta - i\xi)^2 \end{bmatrix}, \mathbf{W}^{(2,2)} = \begin{bmatrix} 0 \\ 0 \\ 2\left(\delta^2 + \xi^2\right) \end{bmatrix}, \mathbf{W}^{(2,3)} = \begin{bmatrix} 0 \\ 0 \\ (\delta + i\xi)^2 \end{bmatrix}, \tag{58}$$

$$\mathbf{f}^{(2,1)} = \mathbf{f}^{(2,2)} = \mathbf{f}^{(2,3)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{59}$$

The coefficients related to order one were obtained from the right eigenmodes of the system, choosing the same normalization as in Julia, and the ones associated with order two were obtained by solving the homological equations directly with the help of Mathematica (and check with the ones obtained by the Julia code). It's important to noticed that because of Eq. (59), the $\mathbf{N}_3^{(p,k)}$ vectors have only zero components, and thus $\mathbf{R}^{(p,k)} = \mathbf{Q}^{(p,k)}$.

With this, it's possible to proceed with the calculation of the reduced dynamics and nonlinear mapping coefficients for order 3.

For $\boldsymbol{\alpha}(3,1)$:

Calculating the right-hand side:

$$\begin{aligned}\mathbf{R}^{(3,1)} &= \mathbf{Q}\left(\mathbf{W}^{(1,1)},\mathbf{W}^{(2,1)}\right)+\mathbf{Q}\left(\mathbf{W}^{(2,1)},\mathbf{W}^{(1,1)}\right)\\ &= \begin{bmatrix}0\\ -h\left(\delta-i\xi\right)^3\\ 0\end{bmatrix},\end{aligned} \tag{60}$$

the system to be solved becomes:

$$\begin{bmatrix}3\lambda_1 & -1 & 0 & 0 & 0\\ \omega^2 & 3\lambda_1+2\xi\omega & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}W_1^{(3,1)}\\ W_2^{(3,1)}\\ W_3^{(3,1)}\\ f_1^{(3,1)}\\ f_2^{(3,1)}\end{bmatrix}=\begin{bmatrix}0\\ R_2^{(3,1)}\\ 0\\ 0\\ 0\end{bmatrix} \tag{61}$$

since this monomial is non-resonant. Solving this system with the help of Mathematica, one gets the same coefficients as in Eqs. 39, 40, 41 and 42.

For $\boldsymbol{\alpha}(3,2)$:

Calculating the right-hand side:

$$\begin{aligned}\mathbf{R}^{(3,2)} &= \mathbf{Q}\left(\mathbf{W}^{(1,1)},\mathbf{W}^{(2,2)}\right)+\mathbf{Q}\left(\mathbf{W}^{(1,2)},\mathbf{W}^{(2,1)}\right)+\mathbf{Q}\left(\mathbf{W}^{(2,1)},\mathbf{W}^{(1,2)}\right)+\mathbf{Q}\left(\mathbf{W}^{(2,2)},\mathbf{W}^{(1,1)}\right)\\ &= \begin{bmatrix}0\\ -3h\left(\delta^3+\xi^2\delta-i\xi\delta^2-i\xi^3\right)\\ 0\end{bmatrix},\end{aligned} \tag{62}$$

the system to be solved becomes:

$$\begin{bmatrix}2\lambda_1+\lambda_2 & -1 & 0 & \delta-i\xi & \delta+i\xi\\ \omega^2 & 2\lambda_1+\lambda_2+2\xi\omega & 0 & i\omega & -i\omega\\ 0 & 0 & -1 & 0 & 0\\ \omega\left(\xi+i\delta\right) & 1 & 0 & 0 & 0\\ \omega\left(\xi-i\delta\right) & 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}W_1^{(3,1)}\\ W_2^{(3,1)}\\ W_3^{(3,1)}\\ f_1^{(3,1)}\\ f_2^{(3,1)}\end{bmatrix}=\begin{bmatrix}0\\ R_2^{(3,2)}\\ 0\\ 0\\ 0\end{bmatrix} \tag{63}$$

since this monomial is resonant for both reduced dynamics equations. Solving this system with the help of Mathematica, one gets the same coefficients as in Eqs. 39, 40, 41 and 42.

For $\boldsymbol{\alpha}(3,3)$:

Calculating the right-hand side:

$$\begin{aligned}\mathbf{R}^{(3,3)} &= \mathbf{Q}\left(\mathbf{W}^{(1,1)},\mathbf{W}^{(2,3)}\right)+\mathbf{Q}\left(\mathbf{W}^{(1,2)},\mathbf{W}^{(2,2)}\right)+\mathbf{Q}\left(\mathbf{W}^{(2,2)},\mathbf{W}^{(1,2)}\right)+\mathbf{Q}\left(\mathbf{W}^{(2,3)},\mathbf{W}^{(1,1)}\right)\\ &= \begin{bmatrix}0\\ -3h\left(\delta^3+\xi^2\delta+i\xi\delta^2+i\xi^3\right)\\ 0\end{bmatrix},\end{aligned} \tag{64}$$

the system to be solved becomes:

$$\begin{bmatrix}\lambda_1+2\lambda_2 & -1 & 0 & \delta-i\xi & \delta+i\xi\\ \omega^2 & \lambda_1+2\lambda_2+2\xi\omega & 0 & i\omega & -i\omega\\ 0 & 0 & -1 & 0 & 0\\ \omega\left(\xi+i\delta\right) & 1 & 0 & 0 & 0\\ \omega\left(\xi-i\delta\right) & 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}W_1^{(3,1)}\\ W_2^{(3,1)}\\ W_3^{(3,1)}\\ f_1^{(3,1)}\\ f_2^{(3,1)}\end{bmatrix}=\begin{bmatrix}0\\ R_2^{(3,3)}\\ 0\\ 0\\ 0\end{bmatrix} \tag{65}$$

since this monomial is resonant for both reduced dynamics equations. Solving this system with the help of Mathematica, one gets the same coefficients as in Eqs. 39, 40, 41 and 42.

For $\boldsymbol{\alpha}(3,3)$:

Calculating the right-hand side:

$$\begin{aligned}\mathbf{R}^{(3,3)} &= \mathbf{Q}\left(\mathbf{W}^{(1,2)},\mathbf{W}^{(2,3)}\right)+\mathbf{Q}\left(\mathbf{W}^{(2,3)},\mathbf{W}^{(1,2)}\right)\\ &= \begin{bmatrix} 0 \\ -h\left(\delta+i\xi\right)^3 \\ 0 \end{bmatrix},\end{aligned} \tag{66}$$

the system to be solved becomes:

$$\begin{bmatrix} 3\lambda_2 & -1 & 0 & 0 & 0 \\ \omega^2 & 3\lambda_2+2\xi\omega & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} W_1^{(3,1)} \\ W_2^{(3,1)} \\ W_3^{(3,1)} \\ f_1^{(3,1)} \\ f_2^{(3,1)} \end{bmatrix} = \begin{bmatrix} 0 \\ R_2^{(3,4)} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{67}$$

since this monomial is non-resonant. Solving this system with the help of Mathematica, one gets the same coefficients as in Eqs. 39, 40, 41 and 42.