Math 341 Homework 6

Isaac Boaz

February 13, 2023

Problem 2

Let X be a random variable representing the number of shooting stars per hour. Assume X is Poisson distributed with $E[x] = \lambda$ (i.e: $X \sim Poisson(\lambda)$).

a) Instead of a 1-hour interval, consider an interval of t hours for some $t > 0$. If Y denotes the number of stars in t hours, what is its distribution and parameter value?

$$Y \sim Poisson(t\lambda)$$

b) Calculate the probability that no shooting stars are observed in t hours (i.e $P(Y = 0)$).

$$P(Y = y) = \frac{e^{-t\lambda} \cdot (t\lambda)^y}{y!}$$

$$\begin{aligned} P(Y = 0) &= \frac{e^{-t\lambda} \cdot (t\lambda)^0}{0!} \\ &= \frac{e^{-t\lambda}}{1} \\ &= e^{-t\lambda} \end{aligned}$$

c) Suppose we measure the time until the first shooting star. Let T denote this time. Explain why the event $A = \{T > t\}$ is equivalent to the event $B = \{Y = 0\}$

Event B represents the event that no shooting stars are observed in t hours.

Event A represents the event that the first shooting star is observed after t hours.

Since event A is defined as $T > t$, we know that at least t hours have passed before the first shooting star is observed.

d) Using (c), compute the cdf of T. Then, state the distribution of T and its parameter value.

$$\begin{aligned} A = B &\implies P(A) = P(B) \\ P(T > t) &= P(Y = 0) \\ &= e^{-t\lambda} \end{aligned}$$

$$\begin{aligned} \text{CDF of T: } F(t) &= P(T \leq t) \\ &= 1 - e^{-t\lambda} \end{aligned}$$

Exponential distribution with parameter λ.

Problem 7

Let X be a random variable for the percentage of Associate's degree holders.
Estimated mean = 31.22, std = 5.3.

a) Using the normal distribution, compute the proportion of US countries where at least 40% of residents have an Associate's degree.

$$\begin{aligned} P(X \geq 40) \\ X \sim N(31.22, 5.3^2) \\ \mathcal{Z} = (X-\mu)/\sigma \end{aligned}$$

$$\begin{aligned} P(X \geq 40) = P(\frac{x\mu}{\sigma} &\geq \frac{40-\mu}{\sigma}) \\ P(\mathcal{Z} \geq \frac{40-31.22}{5.3}) \\ &= 1 - P(\mathcal{Z} < 1.657) \\ &= 1 - P(\mathcal{Z} \leq 1.657) \\ &= 1 - \text{pnorm}(1.657) \\ &= 1 - 0.9512403 = 0.0487597 \end{aligned}$$

b) Let Y denote per capita income represented by $Y = 45000 + 1.7X$.

$$Y = aX + b, \mu = au + b, \sigma = a^2\sigma^2$$

The distribution of Y is normal with mean $\mu = 45000 + 1.7 \cdot 31.22 = 45000 + 53.474 = 45053.474$ and standard deviation $\sigma = 1.7 \cdot 5.3 = 8.91$.

c) Calculate c such that $P(|Y - 45053.07| < c) = 0.758$

$$\begin{aligned} P(-c < Y - 45053.07 < c) = 0.758 \\ = P(-c < Y - 45053.07) + P(Y - 45053.07 < c) \\ = P(-c + 45053.07 < Y < c + 45053.07) \\ = P(\frac{-c + 45053.07 - \mu_y}{\sigma_y} < \frac{Y - \mu_y}{\sigma_y} < \frac{c + 45053.07 - \mu_y}{\sigma_y}) \\ = P(\frac{-c}{\sigma_y} < \mathcal{Z} < \frac{c}{\sigma_y}) \\ = 1 - 2P(\mathcal{Z} < \frac{-c}{\sigma_y}) \\ P(\mathcal{Z} < \frac{-c}{\sigma_y}) = \frac{1 - 0.758}{2} \\ = 0.121 \end{aligned}$$

where σ_y is the standard deviation of Y.