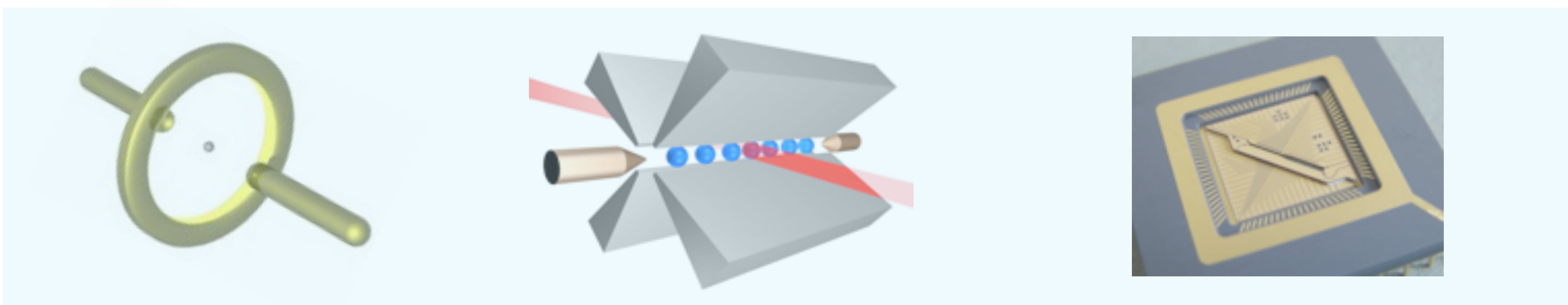
7th International Summer School of the SFB/TRR21 "Control of Quantum Correlations in Tailored Matter"

"Quantum Information Processing with Trapped Ions"

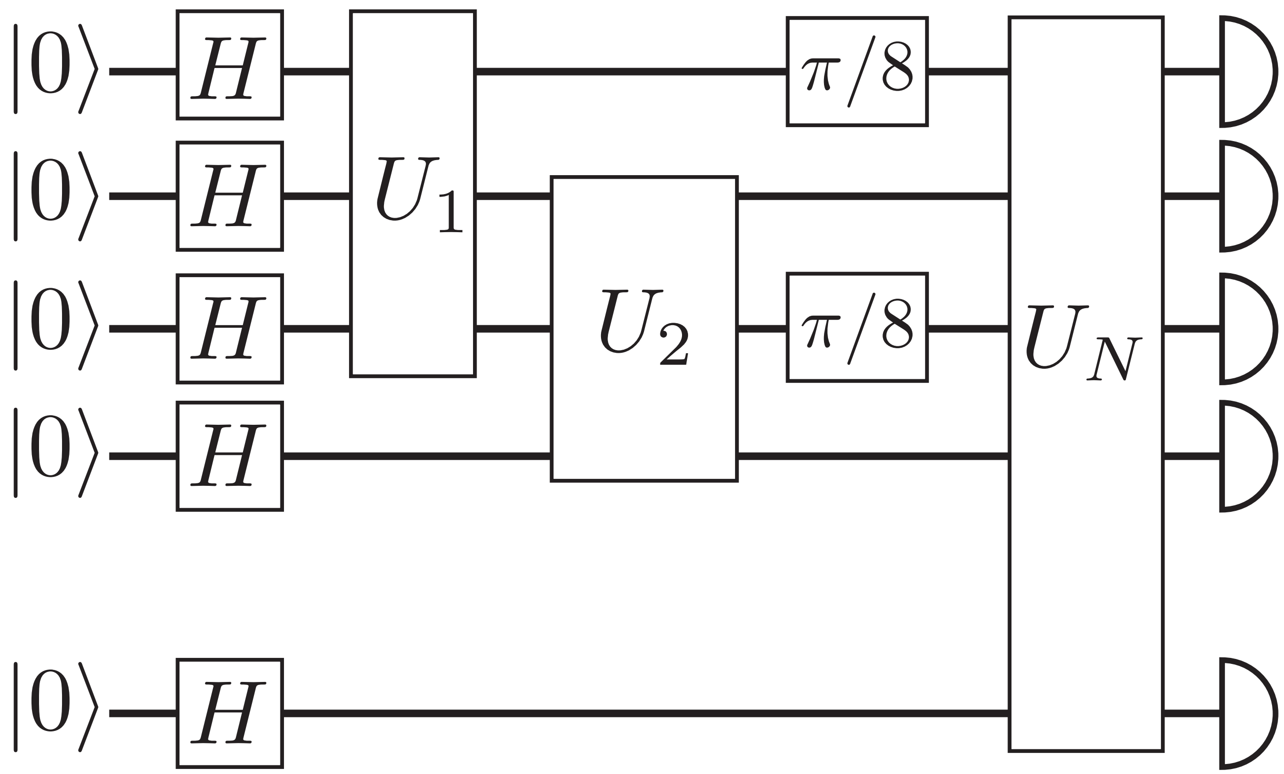


Peter Rabl

Atominstitut
Institute of Atomic and Subatomic Physics

Quantum technologies ...

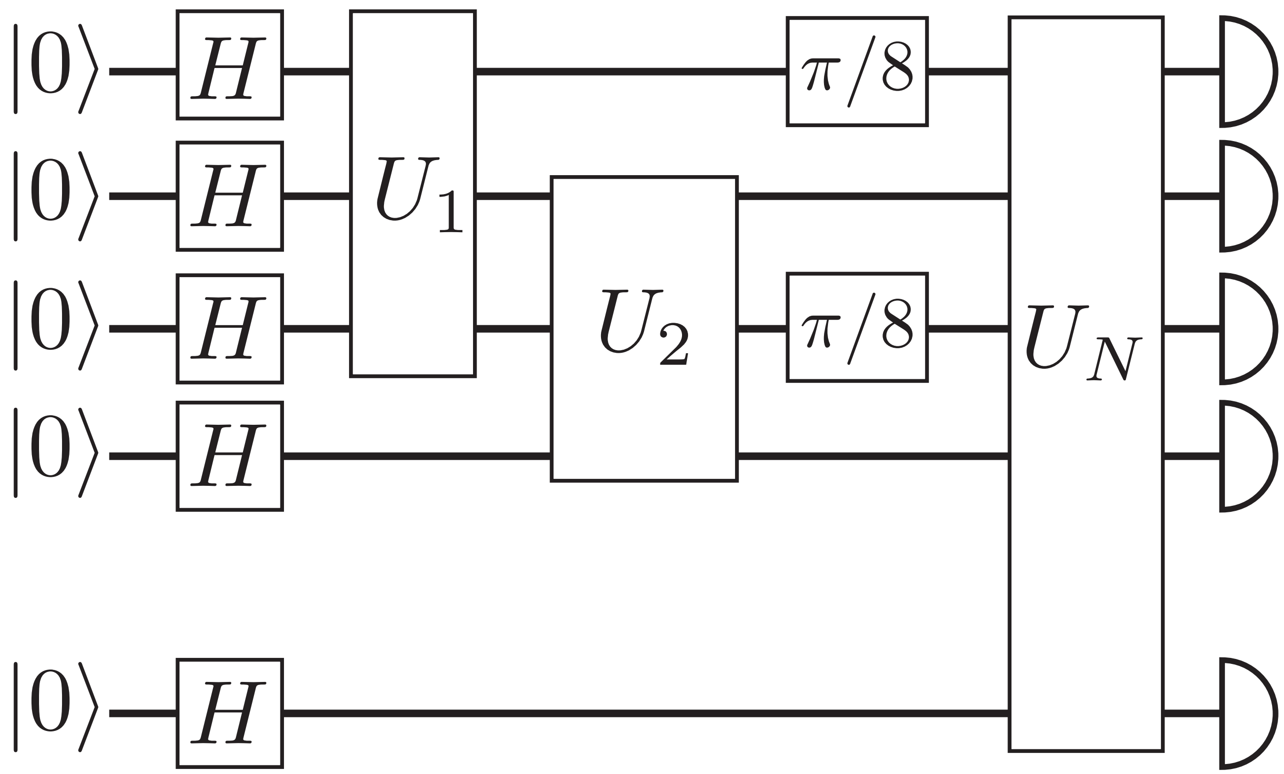
New applications based on quantum superpositions & entanglement !



- *Quantum computing & quantum communication,*
- *Quantum simulators*
- *Quantum enhanced metrology, ...*

Quantum technologies ...

New applications based on quantum superpositions & entanglement !

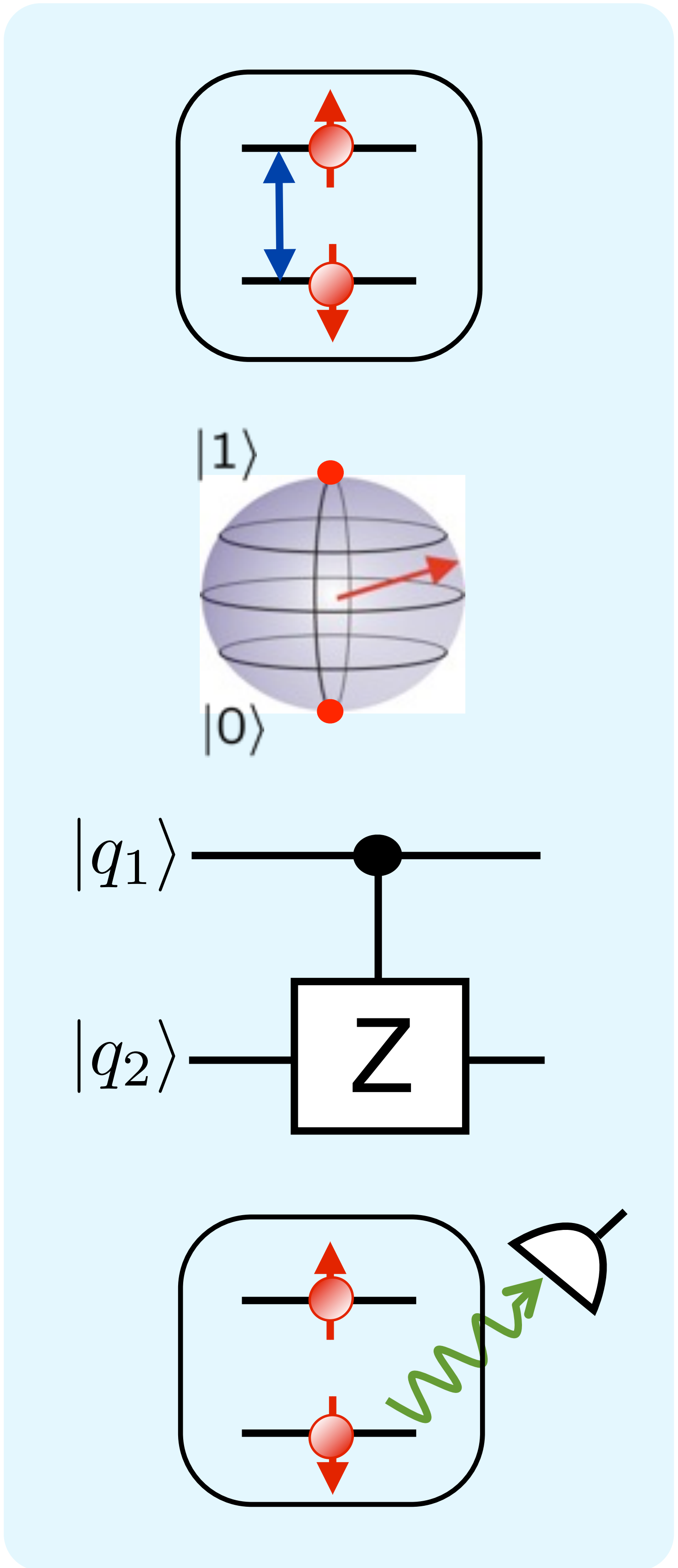


How can we implement these ideas using real physical systems?

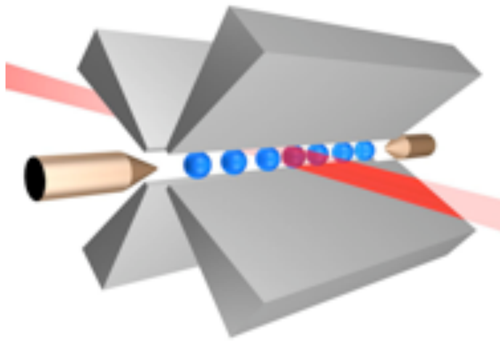
How to build a quantum computer ?

“DiVincenzo criteria”:

1. Well defined qubits
2. Initialization of a pure state
3. Universal set of quantum gates
    - single qubit rotations
    - two qubit gates (CNOT, PHASE, ...)
4. Individual qubit readout
5. Long coherence times

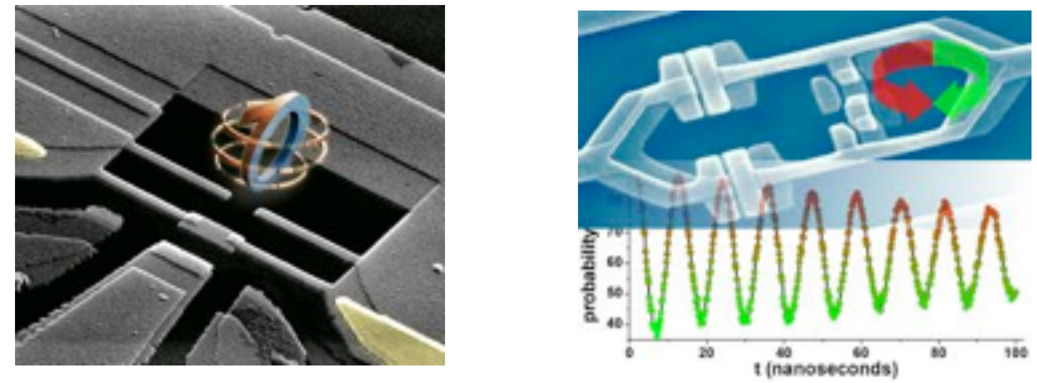


How to build a quantum computer ?

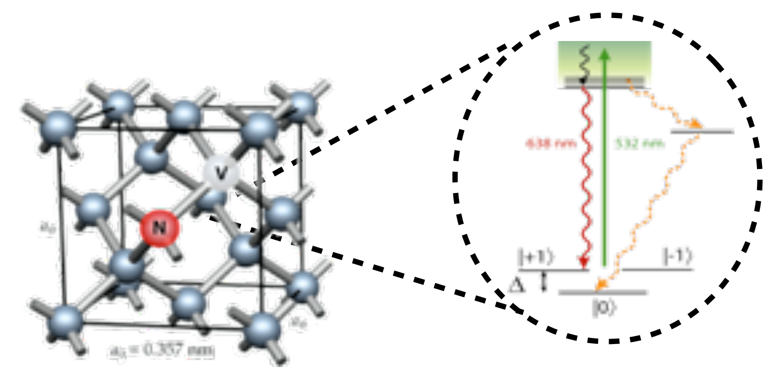


trapped ions

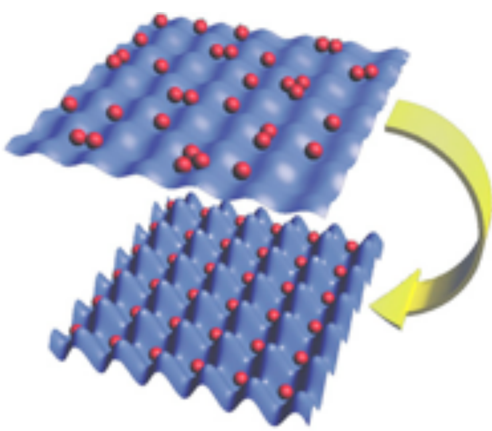
photons



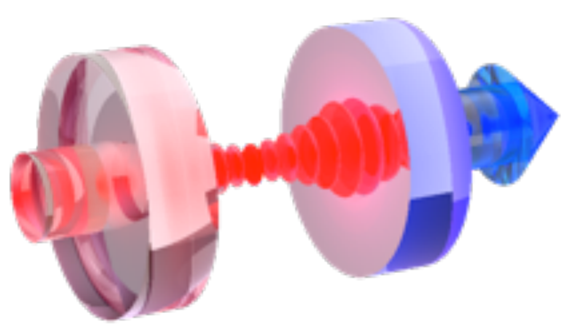
superconducting circuits



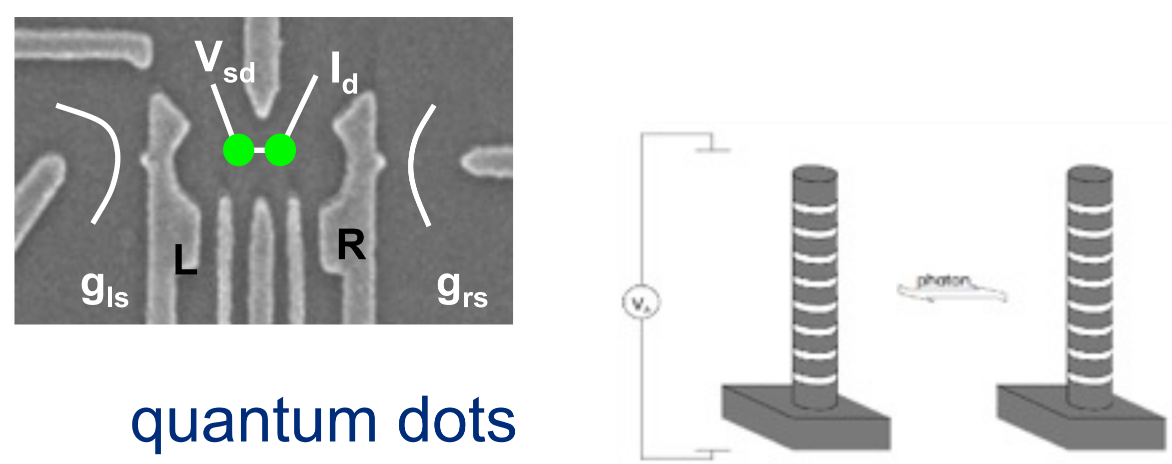
solid state spin qubits



cold atoms

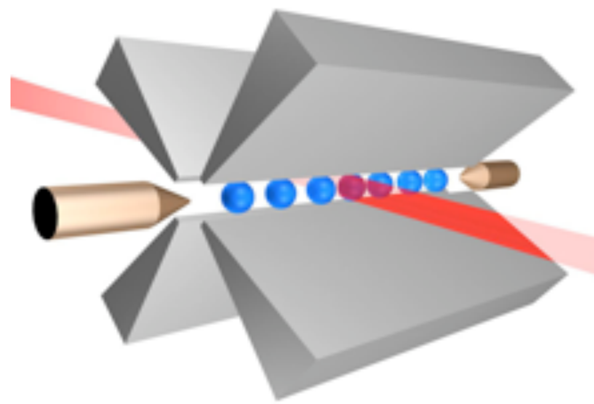


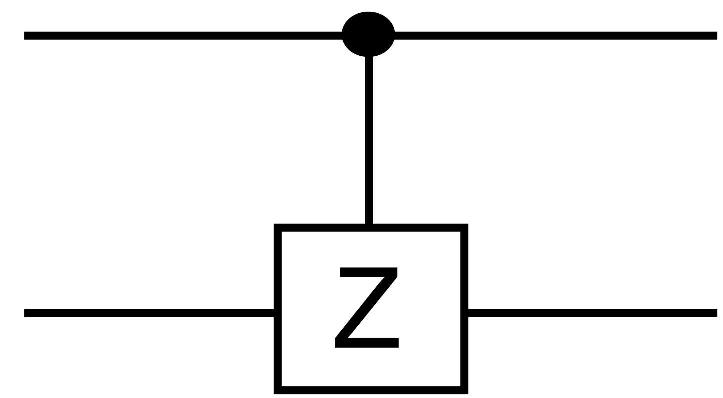
cavity QED



quantum dots

This lecture:
"Quantum information processing with trapped ions"





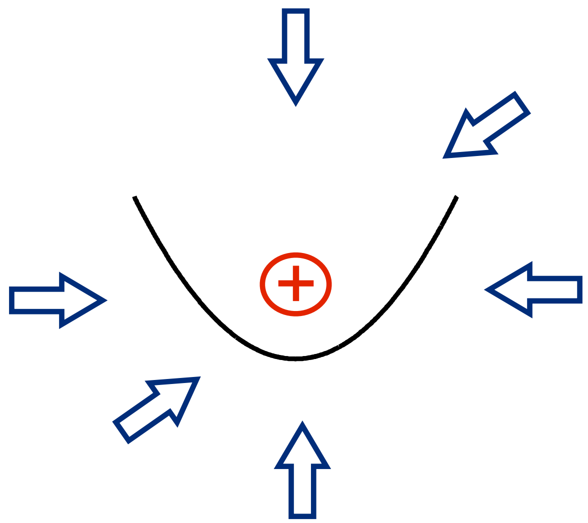
Outline:

- Trapping and manipulating single ions.
- Quantum information processing with trapped ions.
- Geometric gates and spin-models for quantum simulation.
- **Tomorrow:** From trapped ions to macroscopic quantum systems.

Part I:
Trapping and manipulating single ions

How to trap an ion ?

Goal:



3D confinement of a single ion in free space !

How to trap an ion ?

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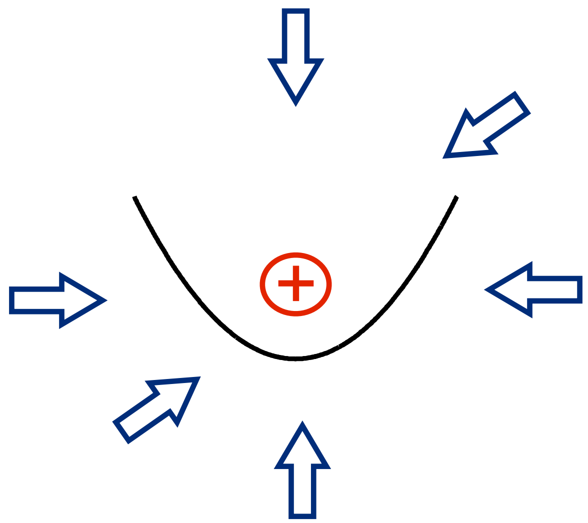


3D confinement of a single ion in free space !

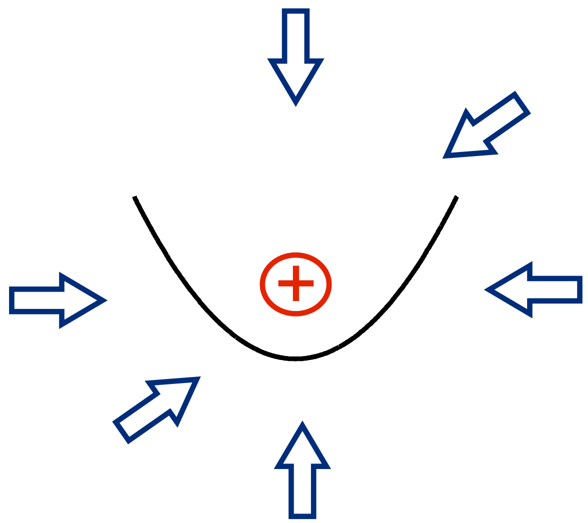
Idea: electrostatic trapping (!?)

$$V_{\mathrm{trap}}(\vec{x}) = e\Phi_{\mathrm{el}}(\vec{x})$$

$$\Phi_{\mathrm{el}}(\vec{x}) \simeq \frac{1}{2}\left(\alpha x^2 + \beta y^2 + \gamma z^2\right)$$

How to trap an ion ?

Goal:



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$$\Phi_{\text{el}}(\vec{x}) \simeq \frac{1}{2}\left(\alpha x^2 + \beta y^2 + \gamma z^2\right)$$

Problem:

Poisson equation in free space:

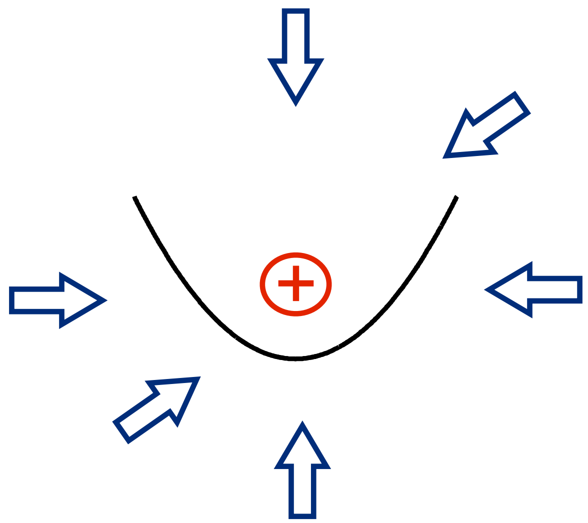
$$\Delta\Phi_{\text{el}} = 0 \qquad \Leftrightarrow \qquad \alpha + \beta + \gamma = 0$$

Saddle point !

No static confinement in all 3 spatial directions !

How to trap an ion ?

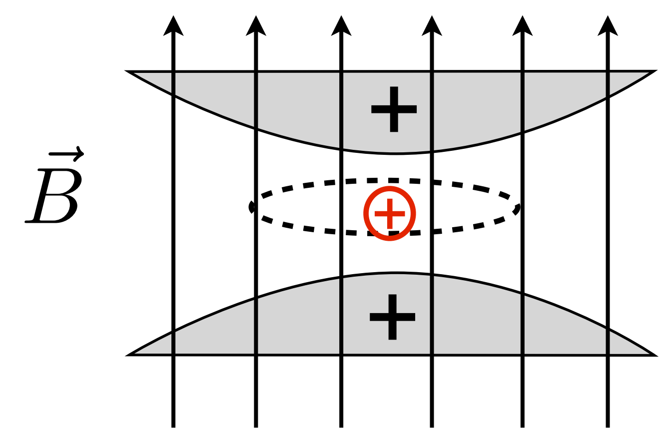
Goal:



3D confinement of a single ion in free space !

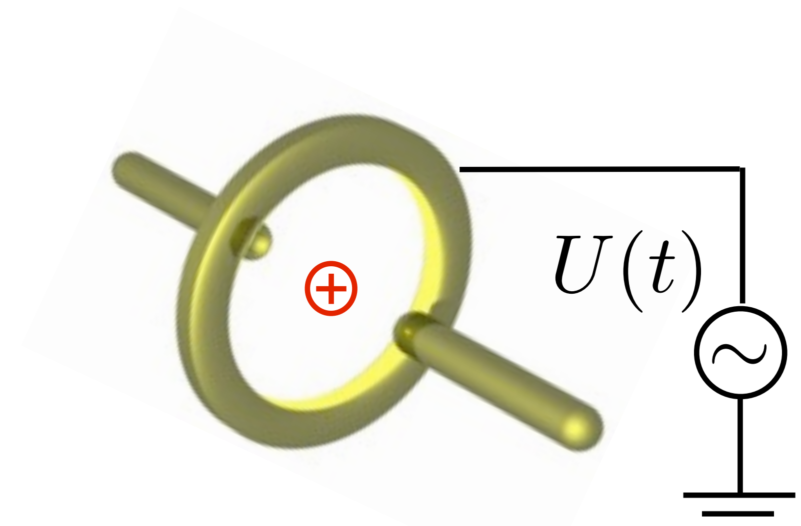
Solution 1: *"Penning trap"* *(Hans Dehmelt)*

$\Rightarrow$ static electric & magnetic fields



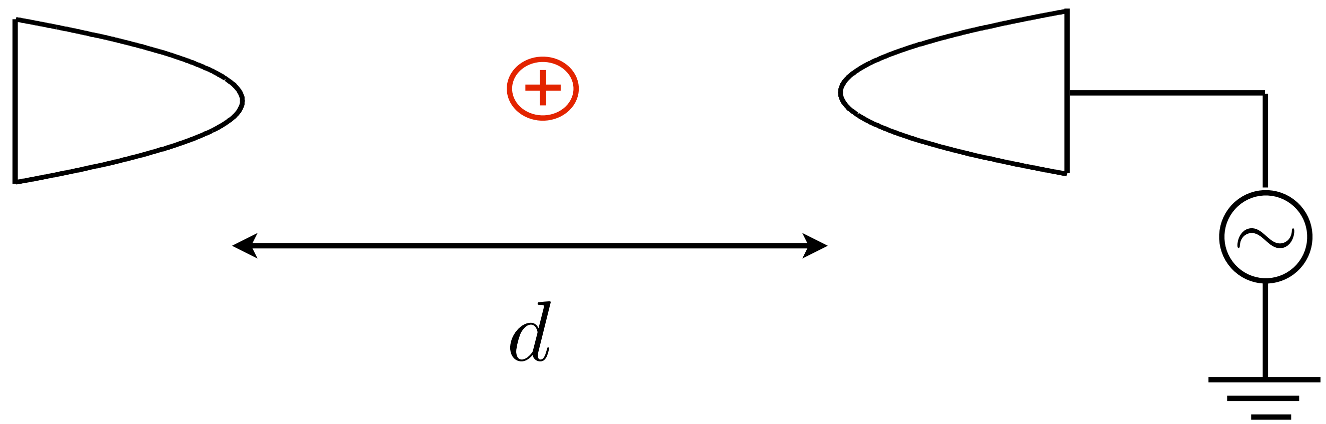
Solution 2: *"Paul trap"* *(Wolfgang Paul)*

$\Rightarrow$ static & oscillating electric fields



Radio-frequency traps

Idea: dynamic confinement of ions ("*Paul trap*"):



radio frequency
(~100 MHz)

$$U(t) = U_0 \cos(\Omega_{\mathrm{rf}} t)$$

equation of motion (1D):

$$\ddot{x}(t) + \left[\kappa^2 \cos(\Omega_{\mathrm{rf}} t)\right] x(t) = 0$$

$$\kappa^2 \approx \frac{eU_0}{md^2}$$

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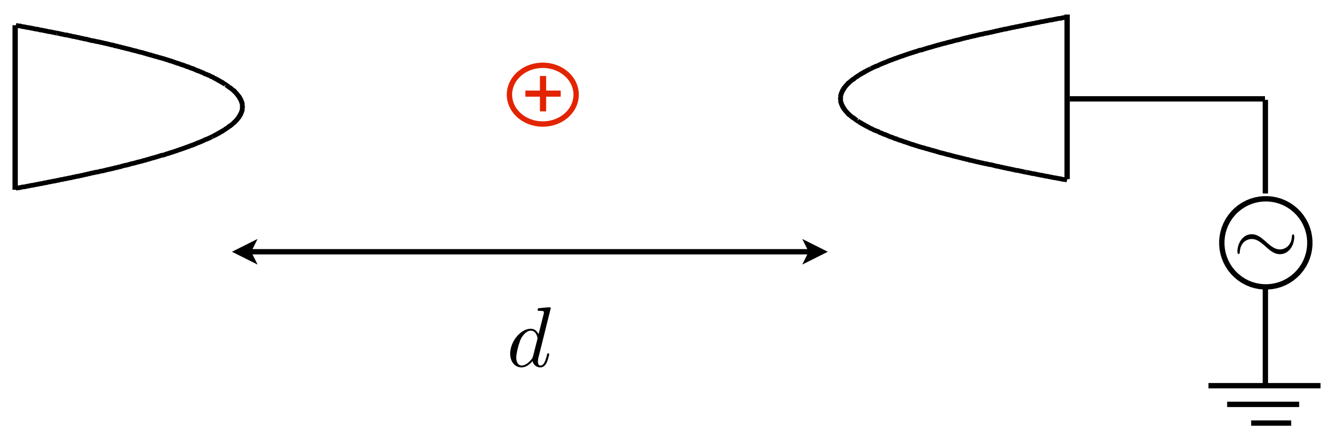

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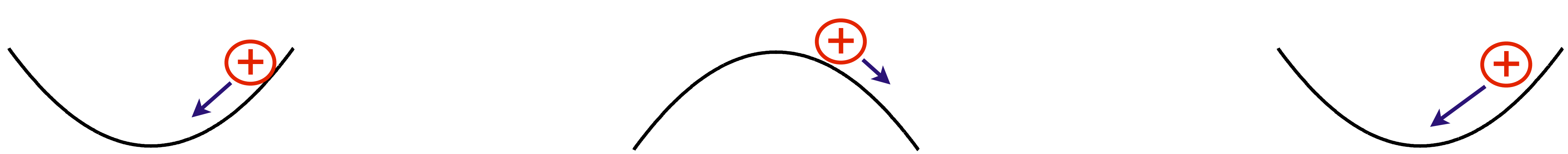
(e ... ion charge, m ... ion mass, d ... trap size, U_0 ... RF voltage)



.... oscillating trapping / anti-trapping potential !

Classical motion of an ion in a RF trap

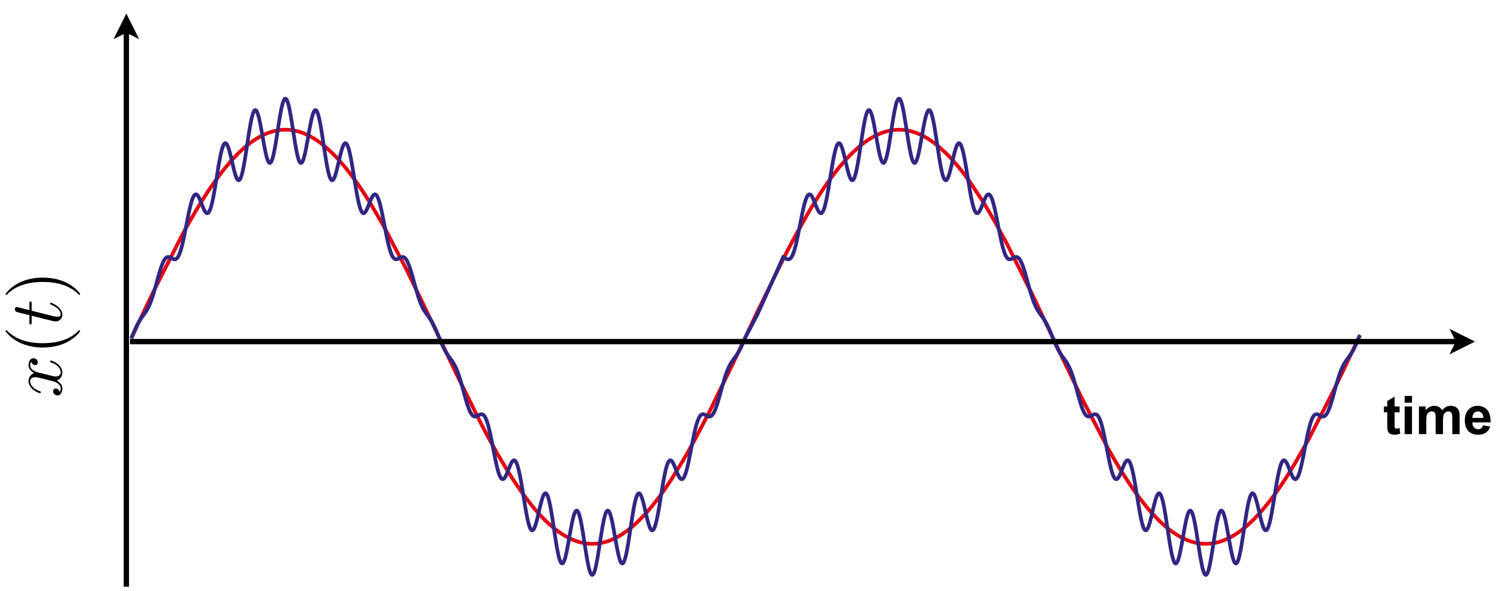
$$\ddot{x}(t) + \left[\kappa^2 \cos(\Omega_{\rm rf} t)\right] x(t) = 0$$

Mathieu equation !

Bound solutions $x(t)$ *for* $\kappa < \Omega_{\rm rf}$:

$$q = 2\kappa^2/\Omega_{\rm rf}^2 \ll 1$$

$$x(t) = A\cos(\omega_t t + \varphi)\left[1 + \frac{q}{2}\cos(\Omega_{\rm rf} t) + \ldots\right]$$



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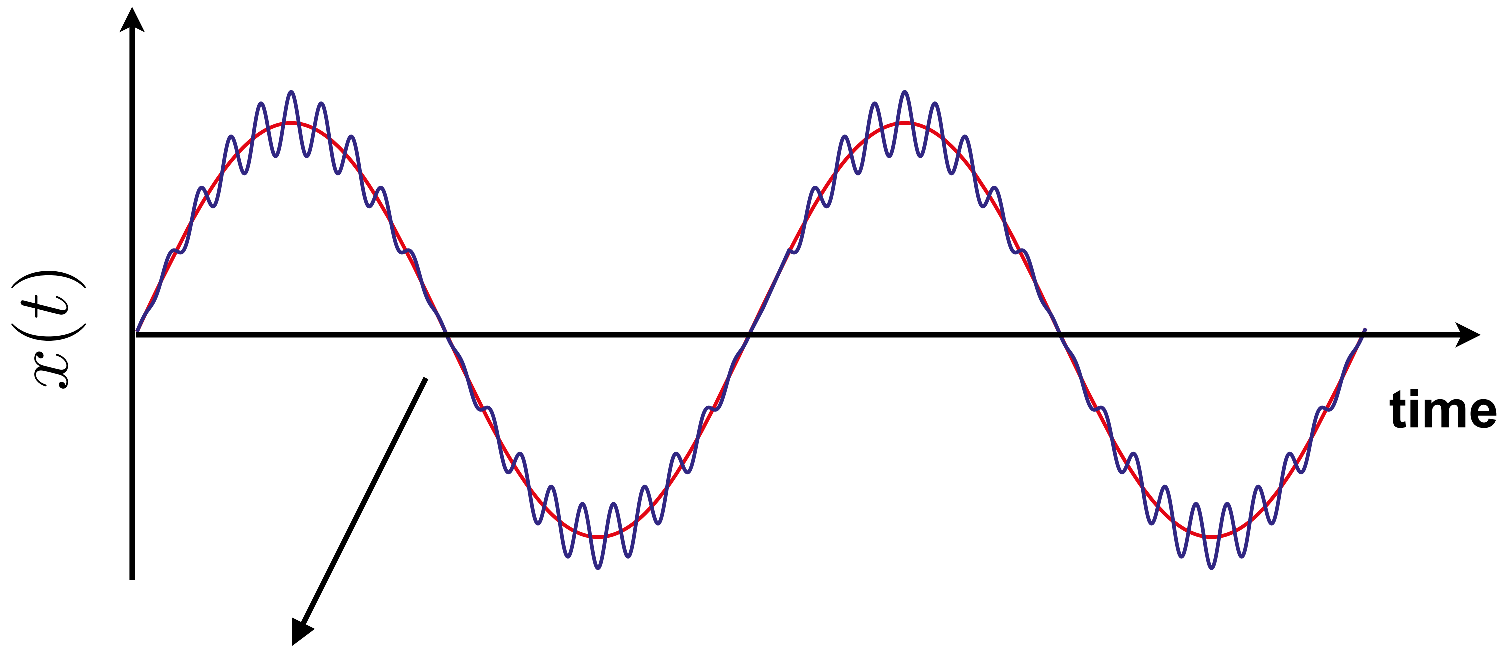

large amplitude ("secular") motion:

$$\sim \omega_t \simeq \frac{1}{\sqrt{2}}\left(\frac{\kappa^2}{\Omega_{\rm rf}}\right) \sim 1 - 10\,\mathrm{MHz}$$

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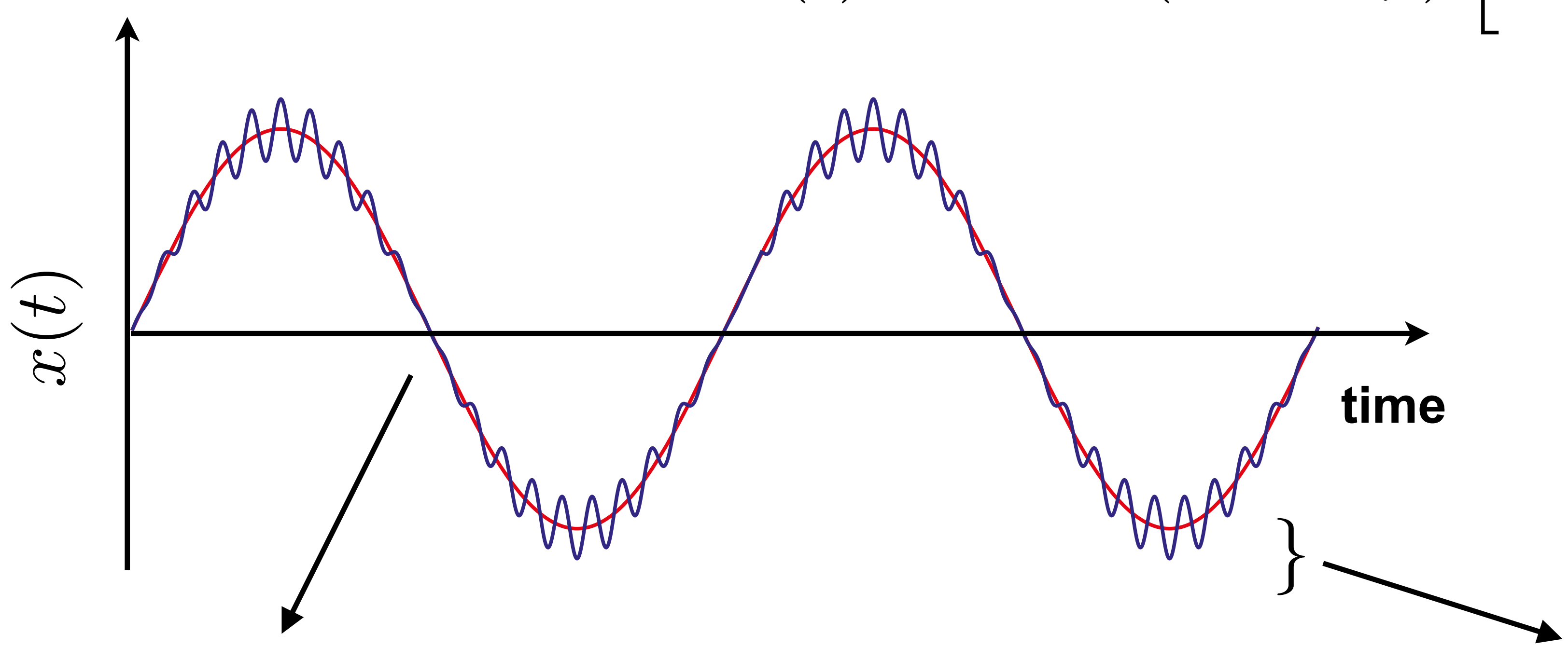


large amplitude ("secular") motion:

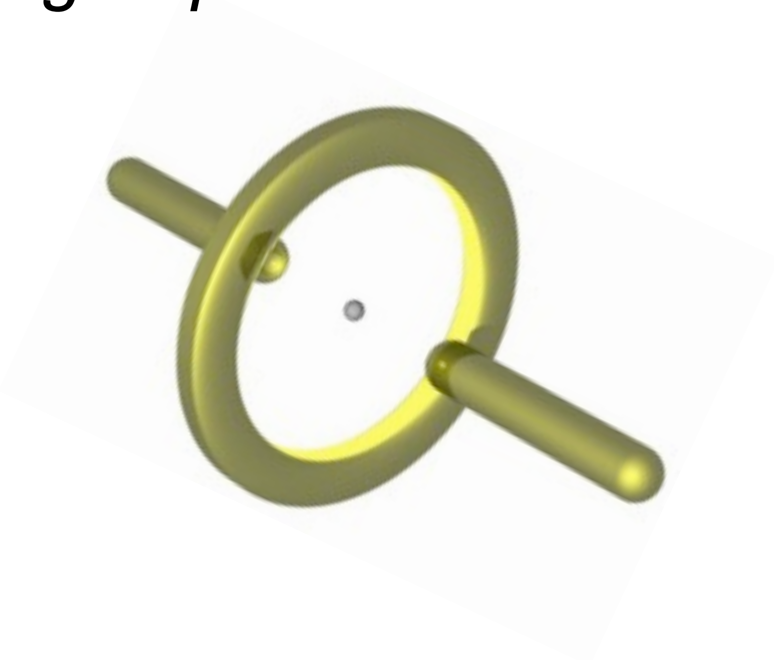
$$\sim \omega_t \simeq \frac{1}{\sqrt{2}}\left(\frac{\kappa^2}{\Omega_{\rm rf}}\right) \sim 1 - 10\,\mathrm{MHz}$$

"micro" motion:

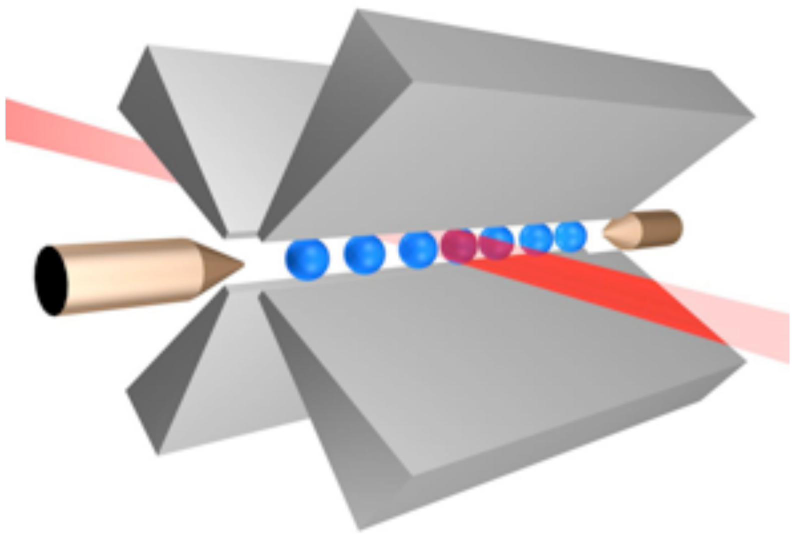
$$\sim \Omega_{\rm rf} \sim 100\,\mathrm{MHz}$$

Ion traps examples ...

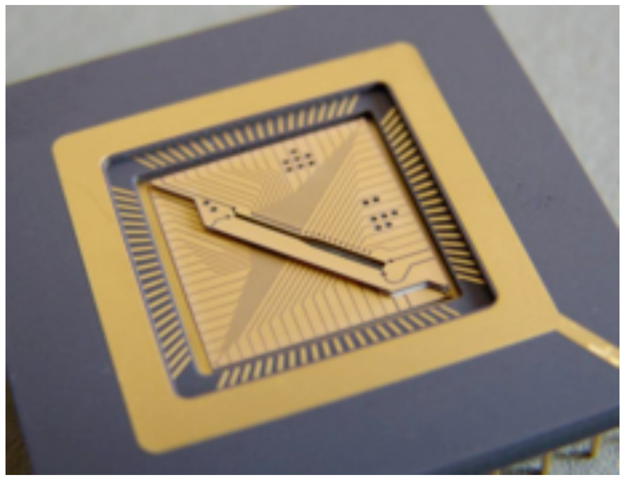
Ring trap:

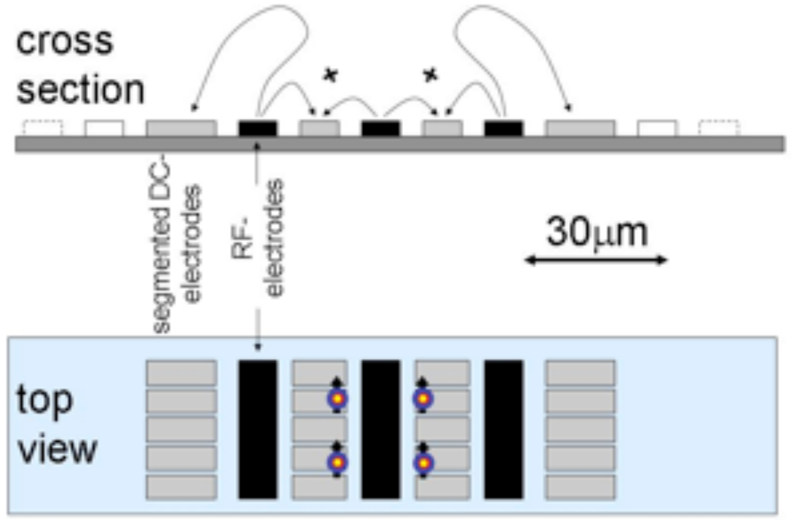


Linear Paul trap:



"Microtraps":





... and many more !

Quantized motion of a trapped ion

Time-dependent Hamiltonian (1D):

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + \frac{m}{2}\kappa^2\cos(\Omega_{\rm rf}t)\hat{x}^2$$

no energy conservation!

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Heisenberg operators:

$$\left.\begin{aligned} \dot{\hat{x}}(t) &= \frac{i}{\hbar}[\hat{H}(t), \hat{x}(t)] = \frac{\hat{p}(t)}{m} \\ \dot{\hat{p}}(t) &= \frac{i}{\hbar}[\hat{H}(t), \hat{p}(t)] = -m\kappa^2 \cos(\Omega_{\mathrm{rf}} t)\hat{x}(t) \end{aligned}\right\} \quad \ddot{\hat{x}}(t) + \kappa^2 \cos(\Omega_{\mathrm{rf}} t)\hat{x}(t) = 0$$

(Mathieu equation)

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(Mathieu equation)

Position operator:

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2m\omega_t}}\left(u(t)\hat{a}^\dagger + u^*(t)\hat{a}\right)$$

$$\ddot{u}(t) + \kappa^2 \cos(\Omega_{\mathrm{rf}} t)u(t) = 0$$

$$u(0) = 1,\ \dot{u}(0) = i\omega_t$$

$$[\hat{a}, \hat{a}^\dagger] = 1$$

(“reference oscillator”, time independent)

Quantized motion of a trapped ion

- Quantum states of a trapped ion:

ground state: $\hat{a}|0\rangle = 0$

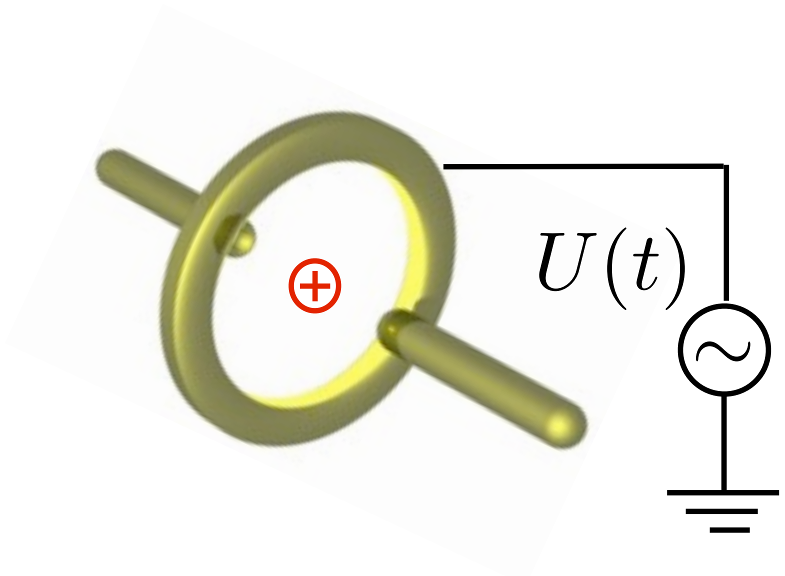
number state: $|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n}}|0\rangle$

$$\left.\begin{array}{l} \hat{a}|0\rangle = 0 \\ |n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n}}|0\rangle \end{array}\right\} \quad |\psi\rangle = \sum_{n=0} c_n |n\rangle$$

- Matrix elements:

$$\langle n+1|\hat{x}(t)|n\rangle \simeq \underbrace{\sqrt{n+1}\sqrt{\frac{\hbar}{2m\omega_t}}e^{i\omega_t t}}_{\text{standard harmonic oscillator}}\left[1+\underbrace{\frac{q}{2}}_{\text{small}}\underbrace{\cos(\Omega_{\rm rf}t)}_{\text{rapidly oscillating}}+\dots\right]$$

Quantized motion of a trapped ion



radio frequency
(~100 MHz)

$$U(t) = U_0 \cos(\Omega_{\rm rf} t)$$

- Effective dynamics of a trapped ion (slow timescales):

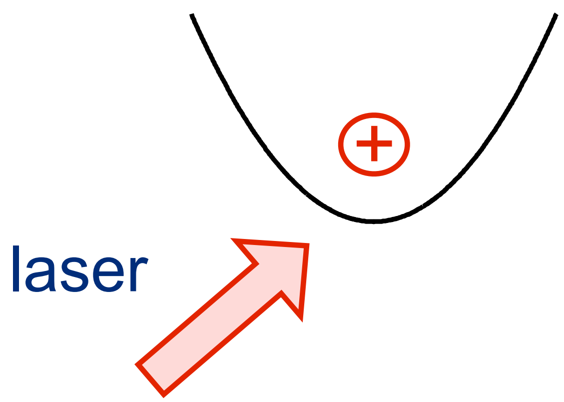
$$H_{\rm trap} \simeq \hbar\omega_x a_x^\dagger a_x + \hbar\omega_y a_y^\dagger a_y + \hbar\omega_z a_z^\dagger a_z$$ ***(3D harmonic oscillator)***

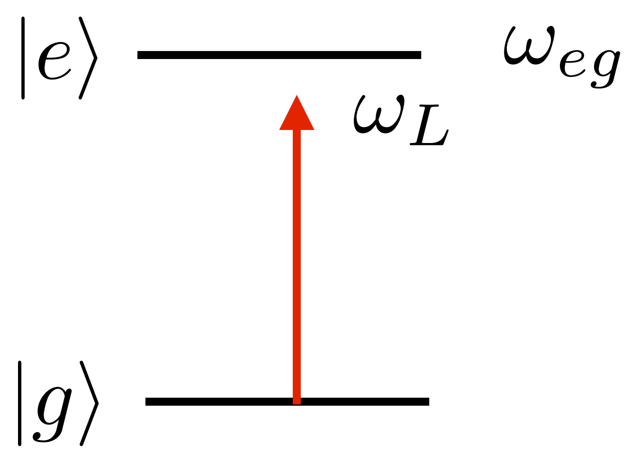
- Corrections:
  high frequencies, electric field offset, large amplitude motion, ...

Details: D. Leibfried et al., Rev. Mod. Phys. ***75****, 281 (2003)*

Manipulating trapped ions with light

Ion-laser interaction





- two level atom driven by a near resonant laser:

$$H = \hbar\omega_{eg}|e\rangle\langle e| + \left(\vec{\mu}_{eg}|e\rangle\langle g| + \vec{\mu}^*_{eg}|g\rangle\langle e|\right)\vec{E}(t,\vec{x})$$

$\vec{\mu}_{eg} = \langle e|\vec{\mu}|g\rangle$ ... *transitions dipole moment*

running wave laser field: $\vec{E}(t,\vec{x}) = 2\vec{E}_0\cos(\omega_L t - \vec{k}_L\vec{x} + \varphi)$

$$H = \hbar\omega_{eg}|e\rangle\langle e| + \hbar\Omega\left(|e\rangle\langle g| + |g\rangle\langle e|\right)\cos(\omega_L t - \vec{k}_L\vec{x} + \varphi)$$

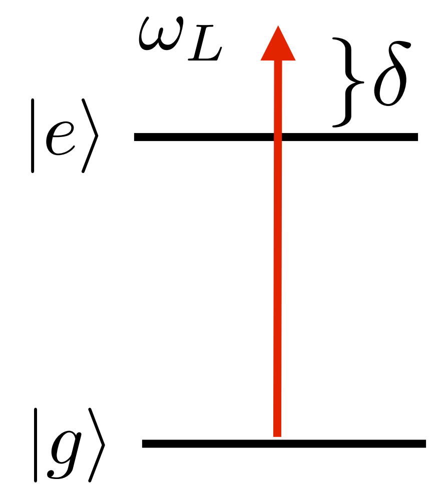
$$\Omega = \frac{2\vec{\mu}_{eg}\vec{E}_0}{\hbar}$$

Rabi frequency

Ion-laser interaction

(1) *"Rotating frame"*: $|\psi\rangle(t) = U_I(t)|\tilde{\psi}\rangle(t)$ $\quad U_I(t) = e^{-i\omega_L t|e\rangle\langle e|}$

$$\tilde{H}(t) = -\hbar\overbrace{(\omega_L - \omega_{eg})}^{\delta}|e\rangle\langle e|$$
$$+\hbar\Omega\left(|e\rangle\langle g|e^{i\omega_L t} + e^{-i\omega_L t}|g\rangle\langle e|\right)\cos(\omega_L t - \vec{k}_L\vec{x} + \varphi)$$



(2) *"Rotating wave approximation"*: $\sim e^{\pm i2\omega_L t} \to 0$

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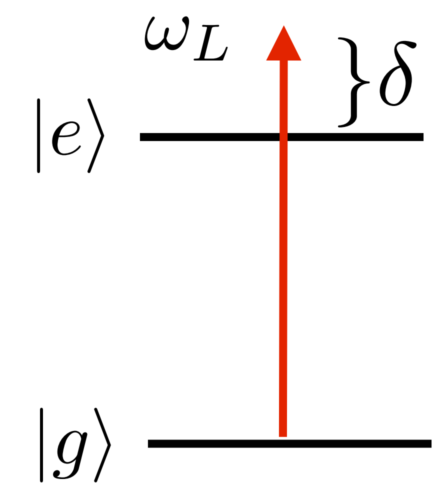

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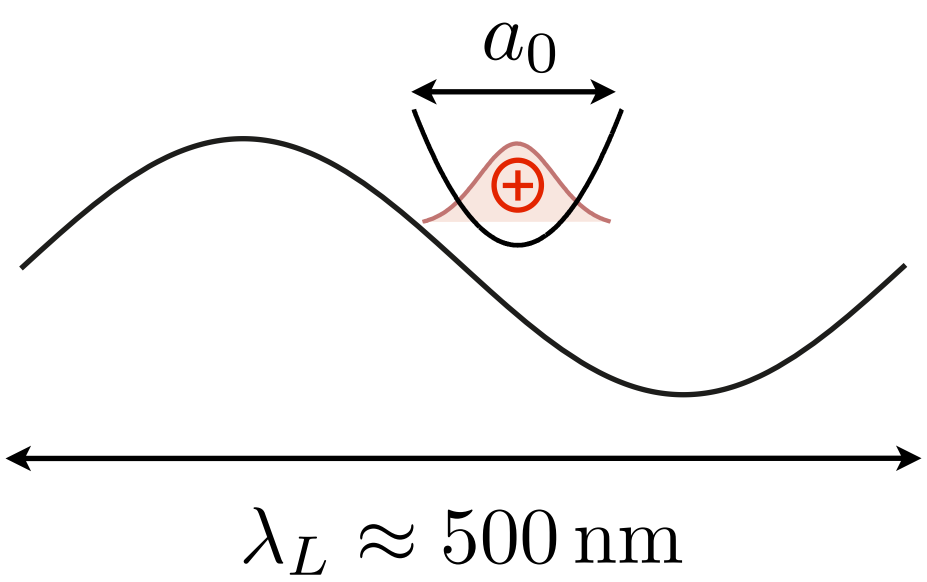
Summary: *Laser driven trapped ion (1D, z-direction)*

$$H = \hbar\omega_t a^\dagger a - \hbar\delta|e\rangle\langle e| + \frac{\hbar\Omega}{2}\left(\sigma_+ e^{ikz}e^{-i\varphi} + e^{-ikz}e^{i\varphi}\sigma_-\right)$$

$$(\,\sigma_+ \equiv |e\rangle\langle g|,\ \sigma_- \equiv |g\rangle\langle e|,\ k \equiv \vec{e}_z \cdot \vec{k}_L\,)$$

Lamb-Dicke limit

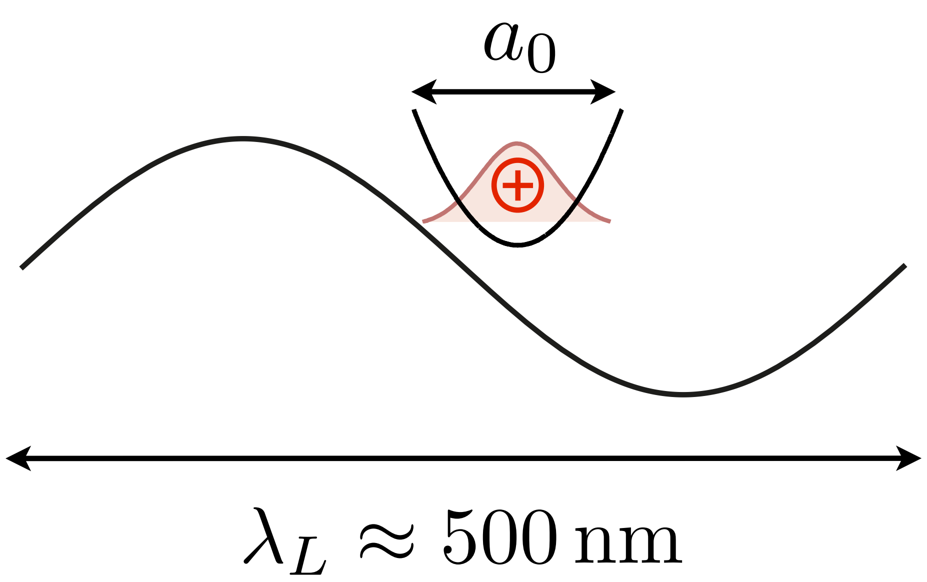


$$a_0 = \sqrt{\frac{\hbar}{2m\omega_t}} \approx 10 - 50\,\mathrm{nm}$$

ground state size

Lamb-Dicke parameter: $\eta = a_0 k_L = 2\pi a_0/\lambda_L \ll 1$

Lamb-Dicke limit



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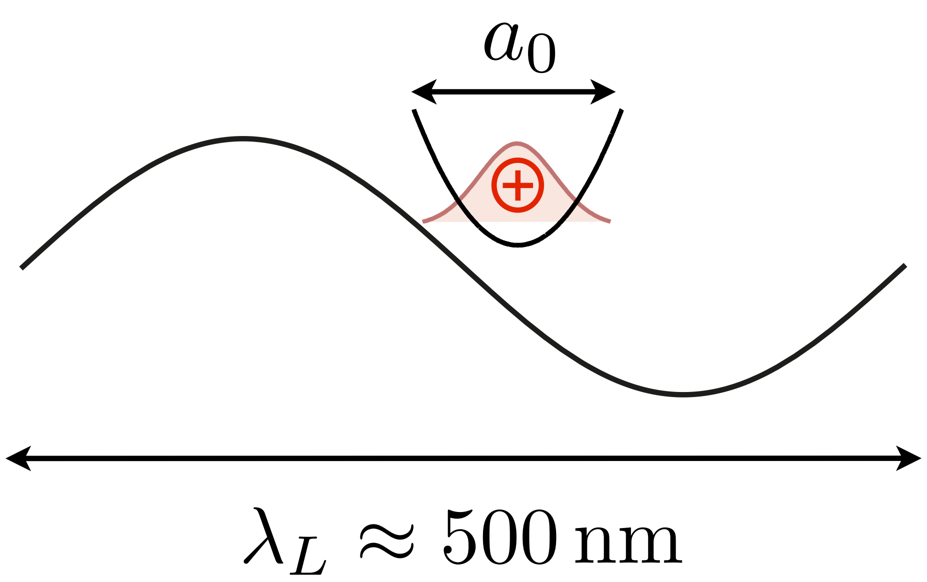
Alternatively: $$\eta^2 = \frac{\hbar k_L^2}{2m\omega_t} = \frac{\hbar^2 k_L^2/(2m)}{\hbar\omega_t} = \frac{E_R}{\hbar\omega_t}$$

recoil energy

trap level spacing

Lamb-Dicke limit



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recoil energy

trap level spacing

Lamb-Dicke limit $\Leftrightarrow$ *strong confinement! ("weak radiation backaction")*

Ion-laser interaction in the LD limit

Laser driven trapped ion (1D, z-direction):

$$H = \hbar\omega_t a^\dagger a - \hbar\delta|e\rangle\langle e| + \frac{\hbar\Omega}{2}\left(\sigma_+ e^{i\eta(a+a^\dagger)} e^{-i\varphi} + e^{-i\overbrace{\eta(a+a^\dagger)}^{k\hat{z}}} e^{i\varphi}\sigma_-\right)$$

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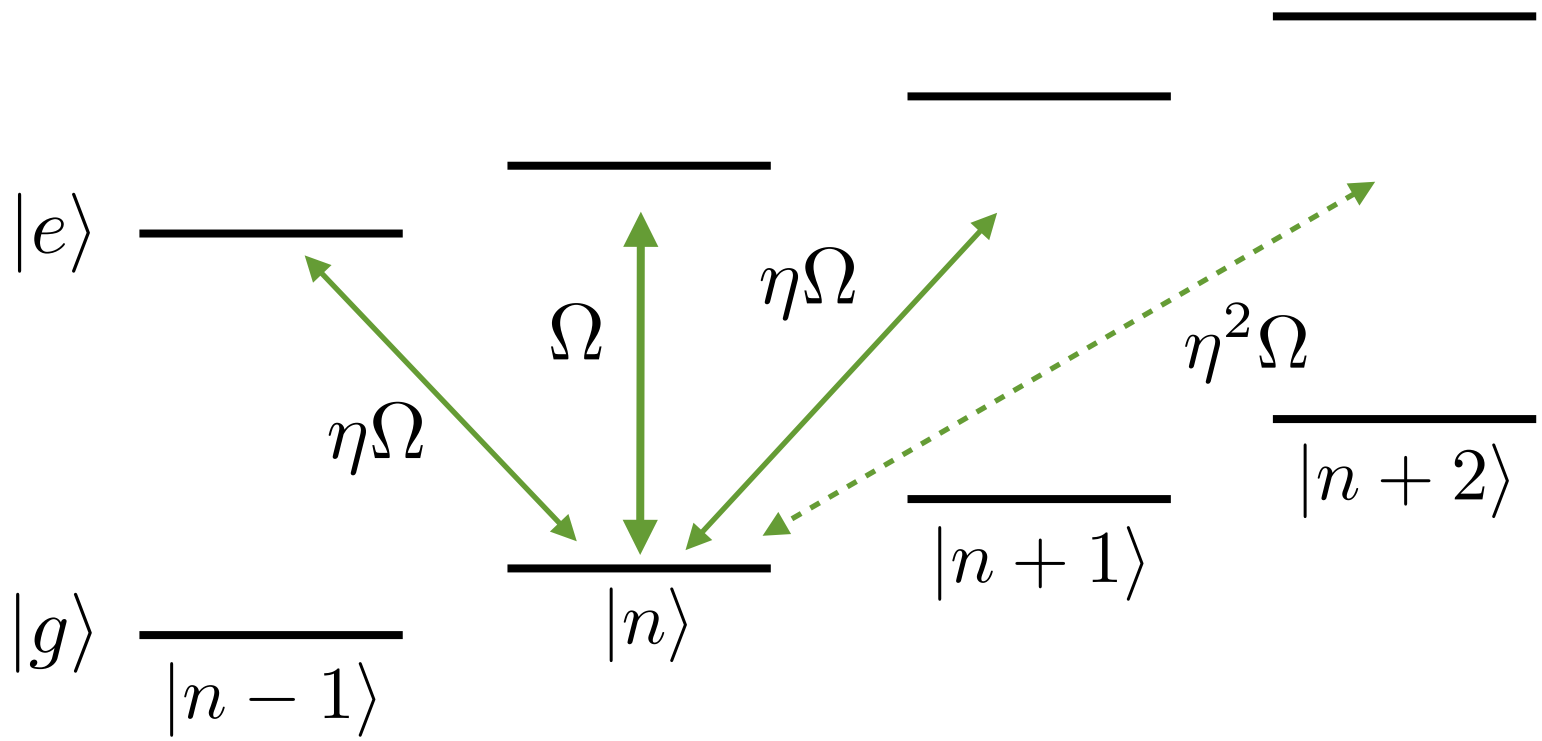
Lamb-Dicke limit: $\eta\sqrt{\bar{n}+1} \ll 1$

$$H = \hbar\omega_t a^\dagger a - \hbar\delta|e\rangle\langle e| + \frac{\hbar\Omega}{2}\left(\sigma_+ e^{-i\varphi} + e^{i\varphi}\sigma_-\right) \qquad \eta^0$$

$$+i\eta\frac{\hbar\Omega}{2}\left(\sigma_+ e^{-i\varphi} - e^{i\varphi}\sigma_-\right)\left(a + a^\dagger\right) \qquad \eta^1$$

$$-\eta^2\frac{\hbar\Omega}{4}\left(\sigma_+ e^{-i\varphi} + e^{i\varphi}\sigma_-\right)\left(a + a^\dagger\right)^2 + \ldots \qquad \eta^2$$

Ion-laser interaction in the LD limit

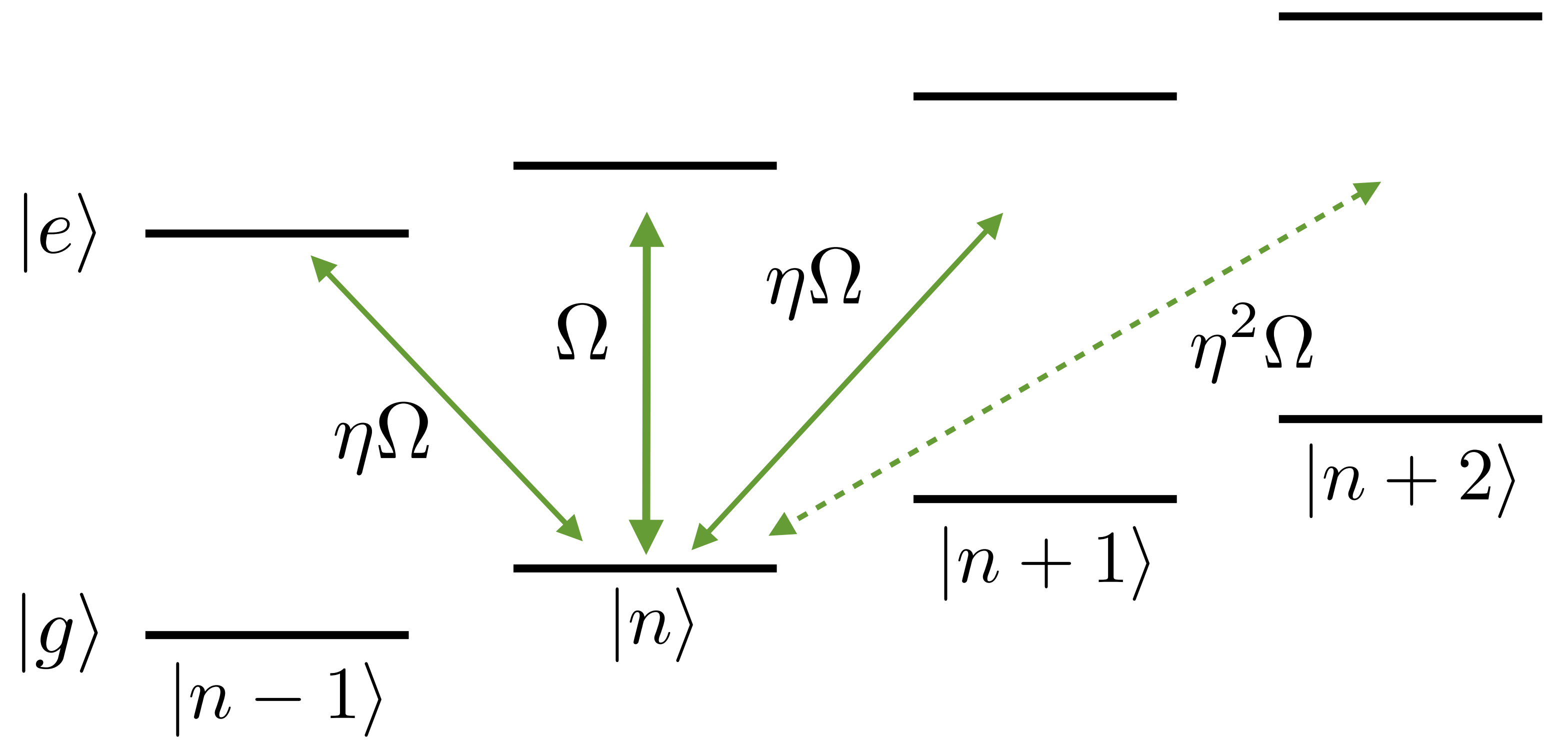


$$H = \hbar\omega_t a^\dagger a - \hbar\delta|e\rangle\langle e| + \frac{\hbar\Omega}{2}\left(\sigma_+ e^{-i\varphi} + e^{i\varphi}\sigma_-\right) \qquad \eta^0$$

$$+i\eta\frac{\hbar\Omega}{2}\left(\sigma_+ e^{-i\varphi} - e^{i\varphi}\sigma_-\right)\left(a + a^\dagger\right) \qquad \eta^1$$

$$-\eta^2\frac{\hbar\Omega}{4}\left(\sigma_+ e^{-i\varphi} + e^{i\varphi}\sigma_-\right)\left(a + a^\dagger\right)^2 + \ldots \qquad \eta^2$$

Ion-laser interaction in the LD limit



rotating frame:

$$H(t) = \frac{\hbar\Omega}{2}\left(\sigma_+ e^{i\eta(ae^{-i\omega_t t}+a^\dagger e^{i\omega_t t})}e^{-i\delta t} + \mathrm{H.c.}\right)$$

$$\simeq \frac{\hbar\Omega}{2}\left(\sigma_+ e^{\underline{-i\delta t}} + i\eta a\sigma_+ e^{-i\underline{(\delta+\omega_t)}t} + i\eta a^\dagger\sigma_+ e^{-i(\delta-\omega_t)t} + \mathrm{H.c.}\right)$$

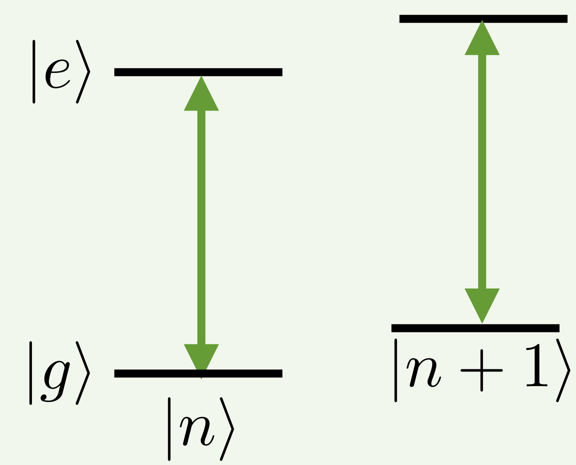
$\Rightarrow$ *Rotating wave approximation*

Carrier, red & blue sideband transitions

Carrier transitions: $\delta = 0$

$$H_{\text{car}} = \frac{\hbar\Omega}{2}\sigma_x$$

(Rabi oscillations)



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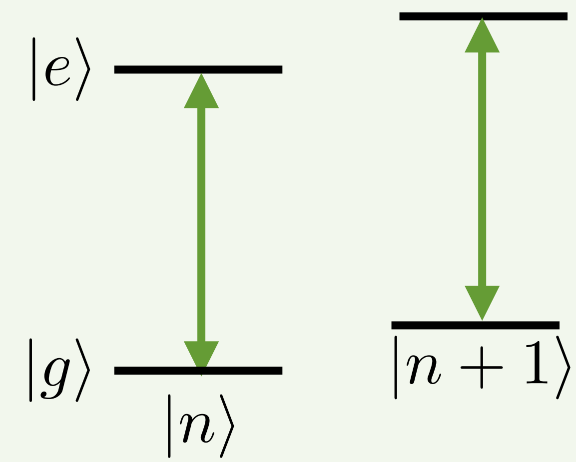
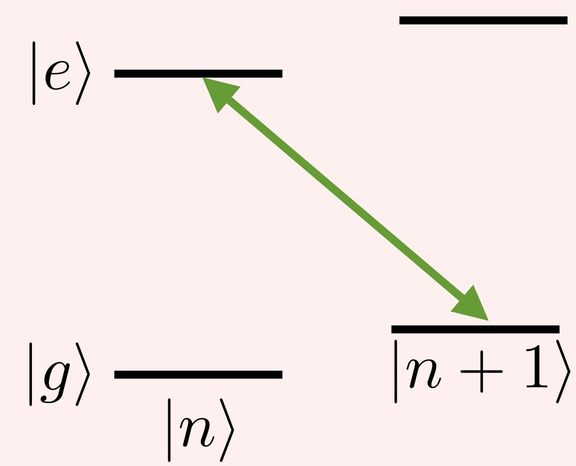


Red sideband transitions: $\delta = -\omega_t$

$$H_{\mathrm{red}} = i\eta\frac{\hbar\Omega}{2}(a\sigma_+ + a^\dagger\sigma_-)$$

(Jaynes-Cummings model)



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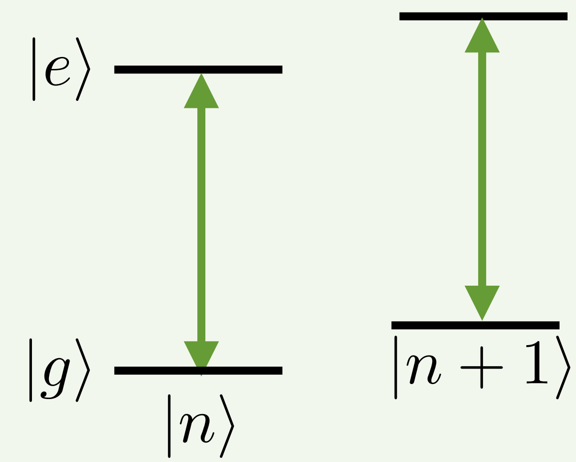

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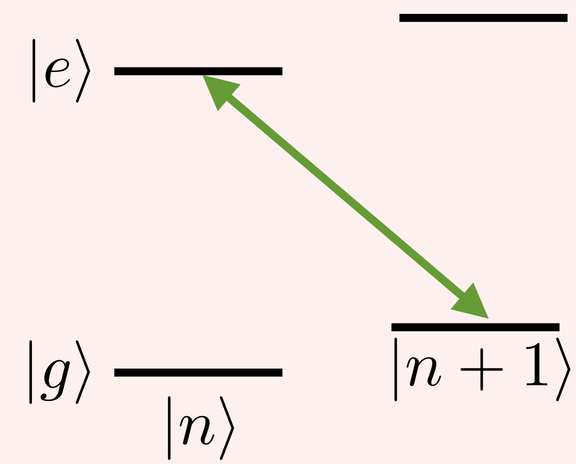
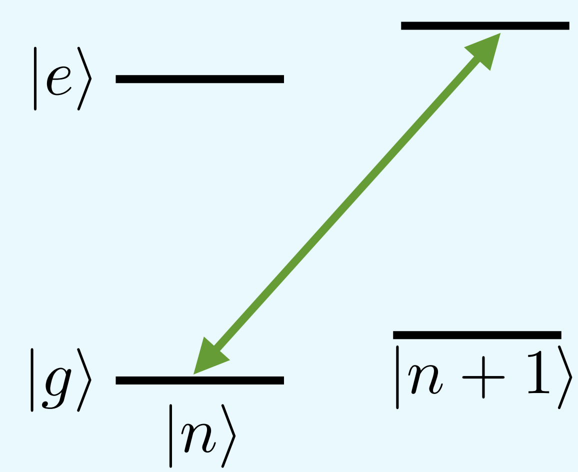


Blue sideband transitions: $\delta = +\omega_t$

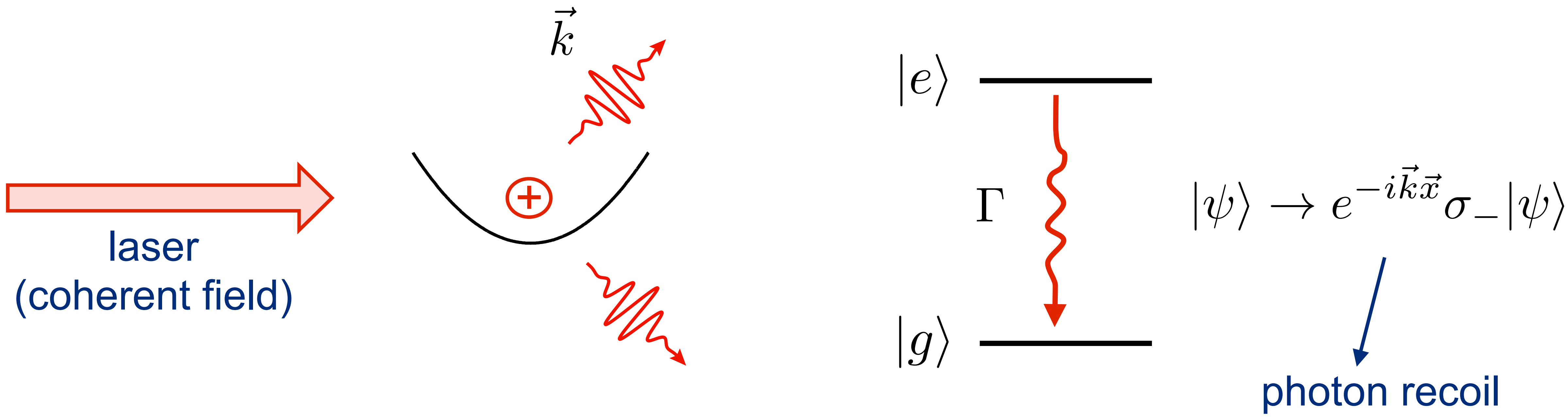
$$H_{\rm blue} = i\eta\frac{\hbar\Omega}{2}(a^\dagger\sigma_+ + a\sigma_-)$$

(Anti-Jaynes-Cummings model)

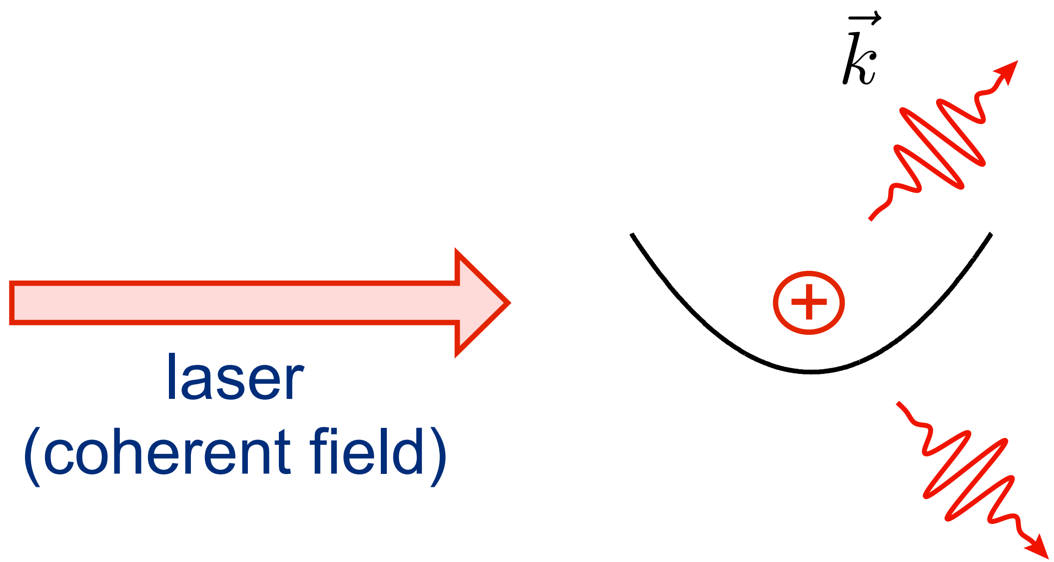


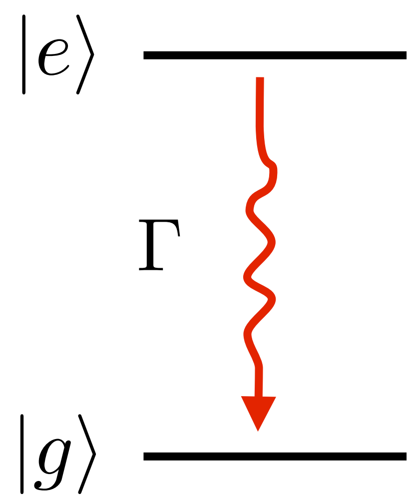
Dissipation & cooling

Spontaneous emission



Spontaneous emission





$$|\psi\rangle \rightarrow e^{-i\vec{k}\vec{x}}\sigma_{-}|\psi\rangle$$

photon recoil

Master equation:

$$\dot{\rho} = -i\left(H_{\mathrm{eff}}\,\rho - \rho H_{\mathrm{eff}}^{\dagger}\right) + \mathcal{J}(\rho)$$

$$H_{\mathrm{eff}} = H - i\frac{\Gamma}{2}|e\rangle\langle e|$$

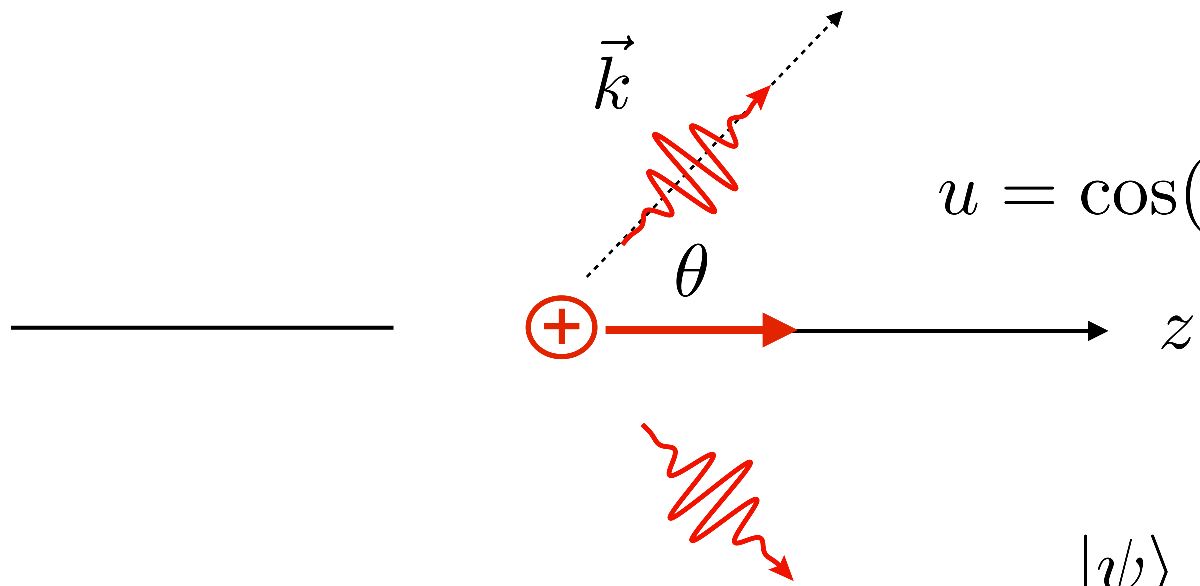
"recycling term":

$$\mathcal{J}(\rho) = \Gamma \int d\Omega_{\vec{n}}\, \Phi(\vec{n}) e^{-ik\vec{n}\cdot\vec{x}} \sigma_{-} \rho \sigma_{+} e^{ik\vec{n}\cdot\vec{x}}$$

dipole emission pattern

Spontaneous emission

1D:



$$u = \cos(\theta)$$

$$|\psi\rangle \rightarrow e^{-i|\vec{k}|z\cos(\theta)}\sigma_-|\psi\rangle$$

Master equation 1D:

$$\dot{\rho} = -i\left(H_{\text{eff}}\rho - \rho H_{\text{eff}}^\dagger\right) + \mathcal{J}(\rho)$$

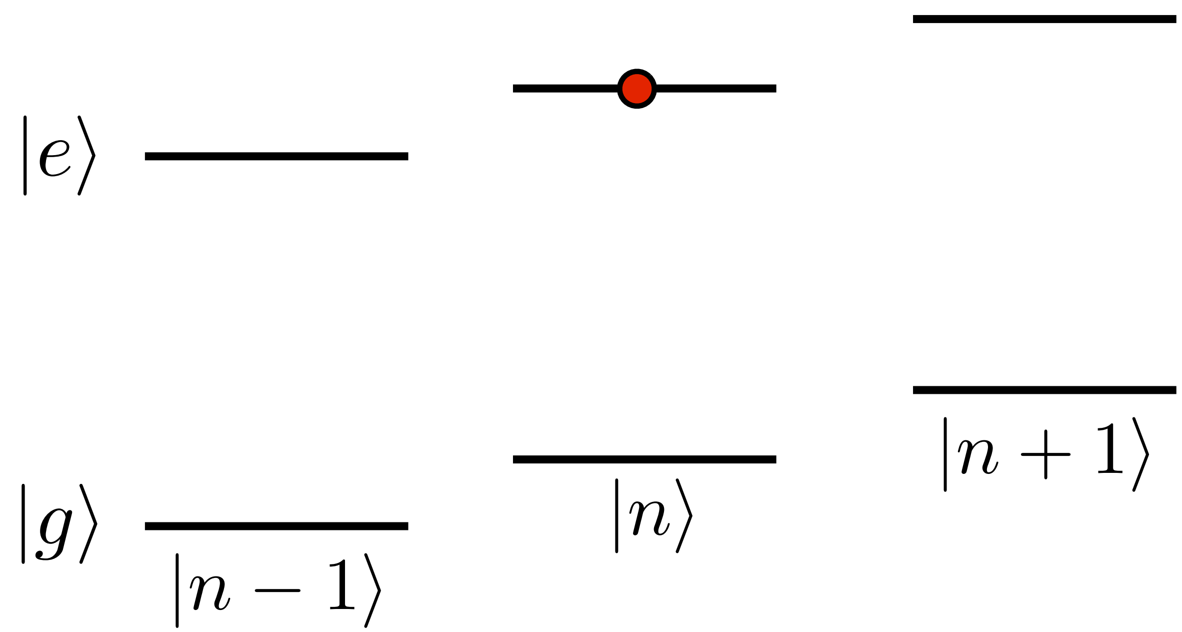
$$H_{\text{eff}} = H - i\frac{\Gamma}{2}|e\rangle\langle e|$$

"recycling term":

$$\mathcal{J}(\rho) = \Gamma\int_{-1}^{1} du\,\Phi(u)e^{-iukz}\sigma_-\rho\sigma_+e^{iukz}$$

dipole emission pattern

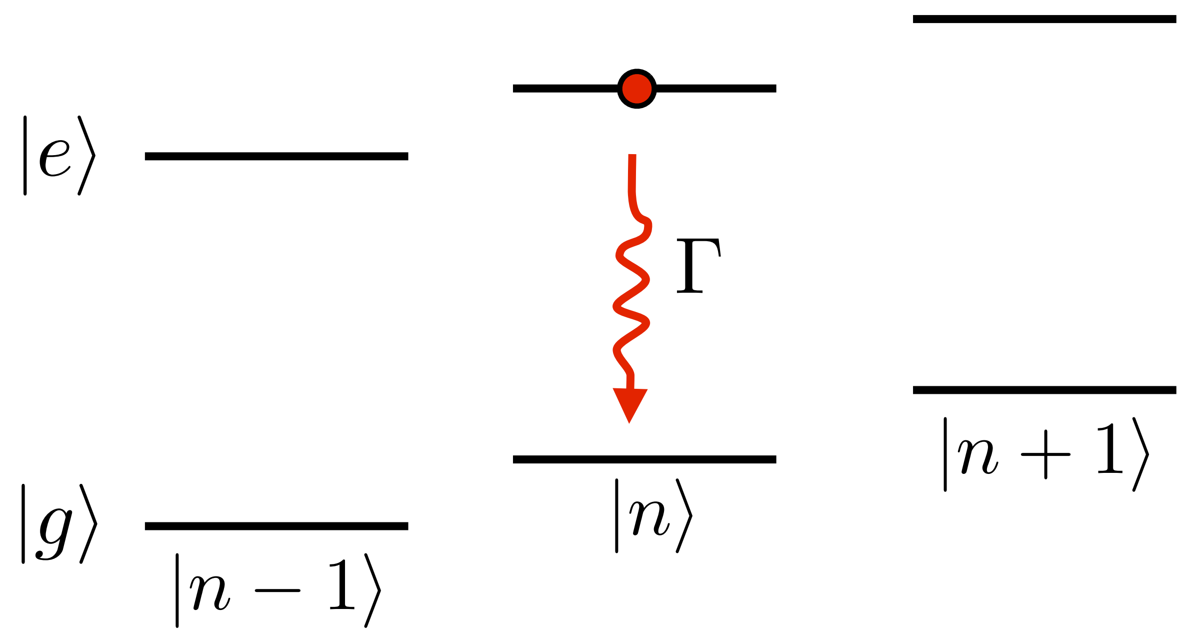
Spontaneous emission: Lamb-Dicke limit



Recycling term (1D, Lamb-Dicke limit):

$$\mathcal{J}(\rho) = \Gamma \int_{-1}^{1} du\, \Phi(u) e^{-iu\eta(a+a^\dagger)} \sigma_- \rho \sigma_+ e^{iu\eta(a+a^\dagger)}$$

Spontaneous emission: Lamb-Dicke limit



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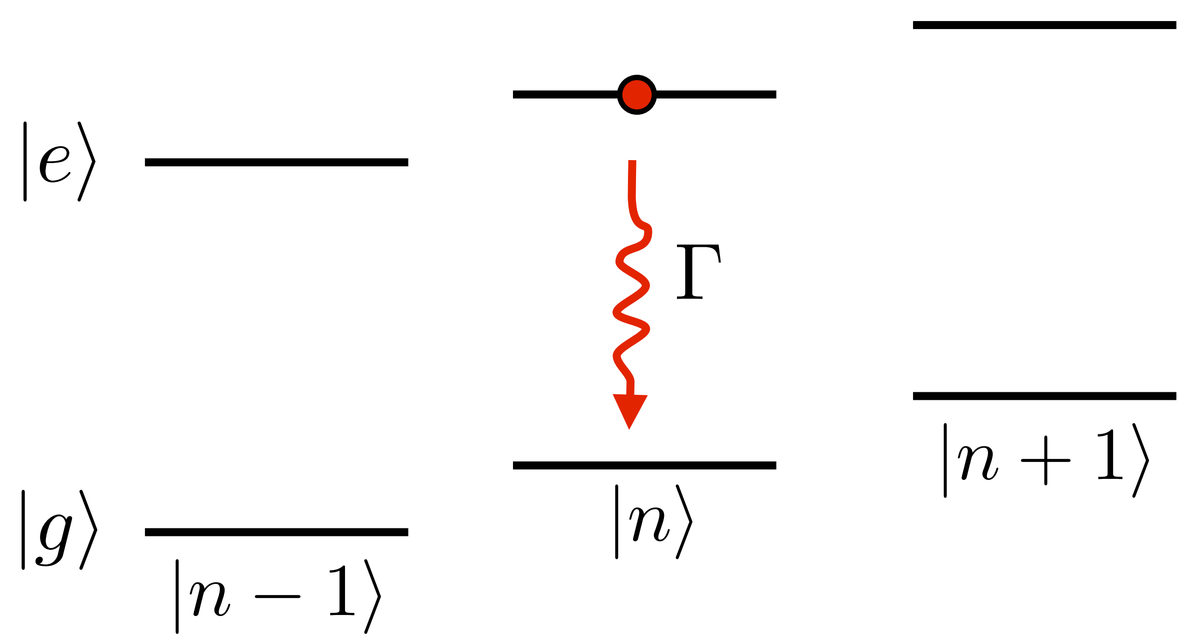
$$\mathcal{J}(\rho) \simeq \Gamma \sigma_- \rho \sigma_+$$

Spontaneous emission: Lamb-Dicke limit



Recycling term (1D, Lamb-Dicke limit):

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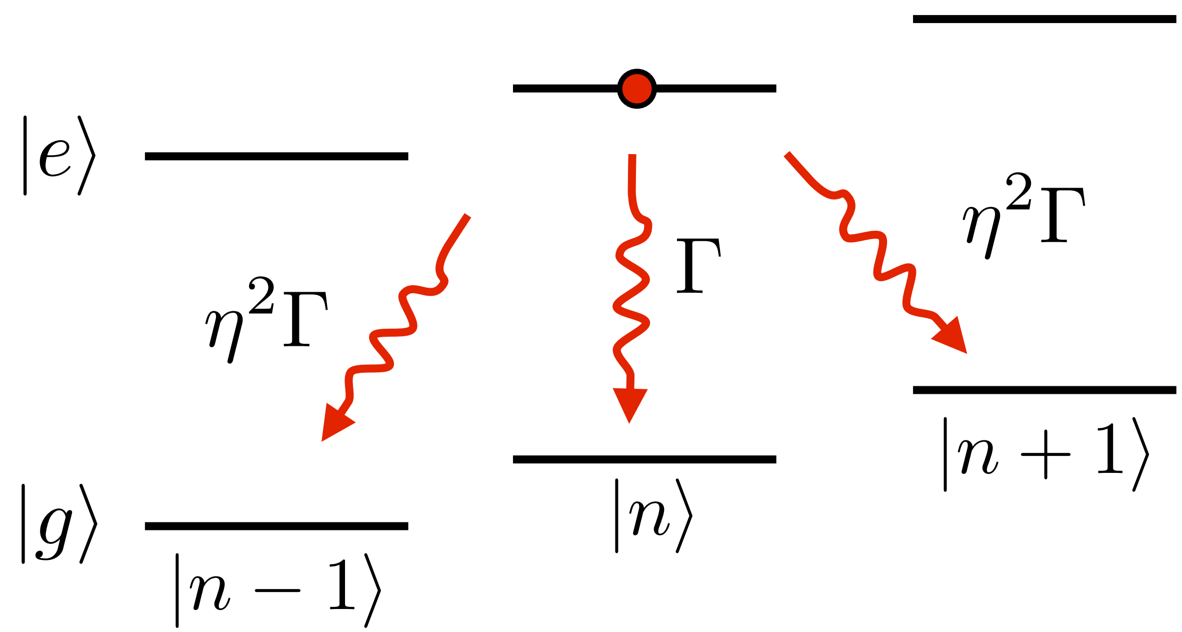
$$\mathcal{J}(\rho) \simeq \Gamma \sigma_- \rho \sigma_+ \; + \; \eta\Gamma \int_{-1}^{1} \underbrace{du\, \Phi(u) u}_{=0} \left(\ldots \right)$$

symmetry !

Spontaneous emission: Lamb-Dicke limit



Recycling term (1D, Lamb-Dicke limit):

$$\mathcal{J}(\rho) = \Gamma \int_{-1}^{1} du\, \Phi(u) e^{-iu\eta(a+a^\dagger)} \sigma_- \rho \sigma_+ e^{iu\eta(a+a^\dagger)}$$

diffusion !

$$\mathcal{J}(\rho) \simeq \Gamma \sigma_- \rho \sigma_+ \; + \; \eta\Gamma \int_{-1}^{1} \underline{du\, \Phi(u) u \,(\; \dots \;)} \; + \eta^2 \alpha \frac{\Gamma}{2} \sigma_- \left(2\bar{z}\rho\bar{z} - \bar{z}^2\rho - \rho\bar{z}^2 \right) \sigma_+$$

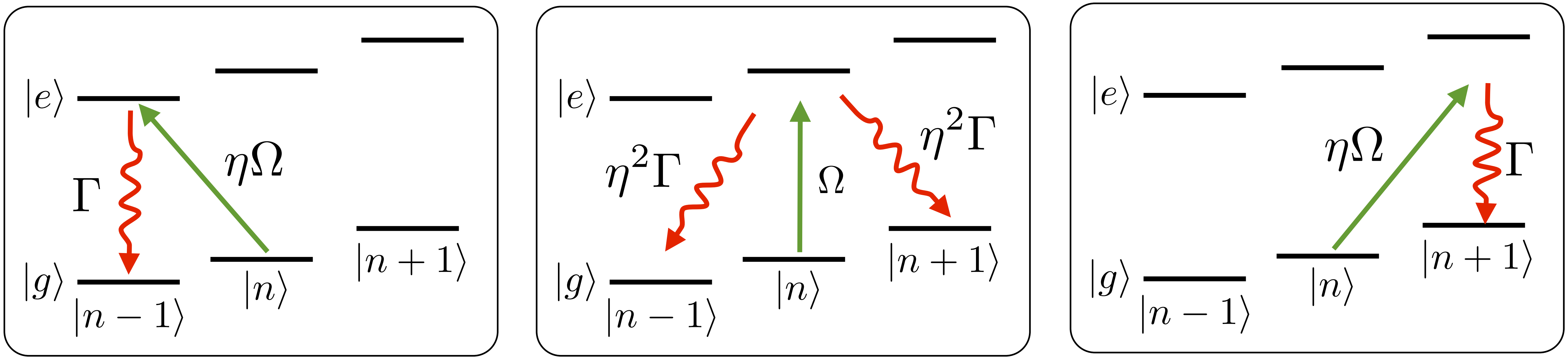
$= 0$

symmetry !

$$\alpha = \int_{-1}^{1} du\, \Phi(u) u^2 = \frac{2}{5}$$

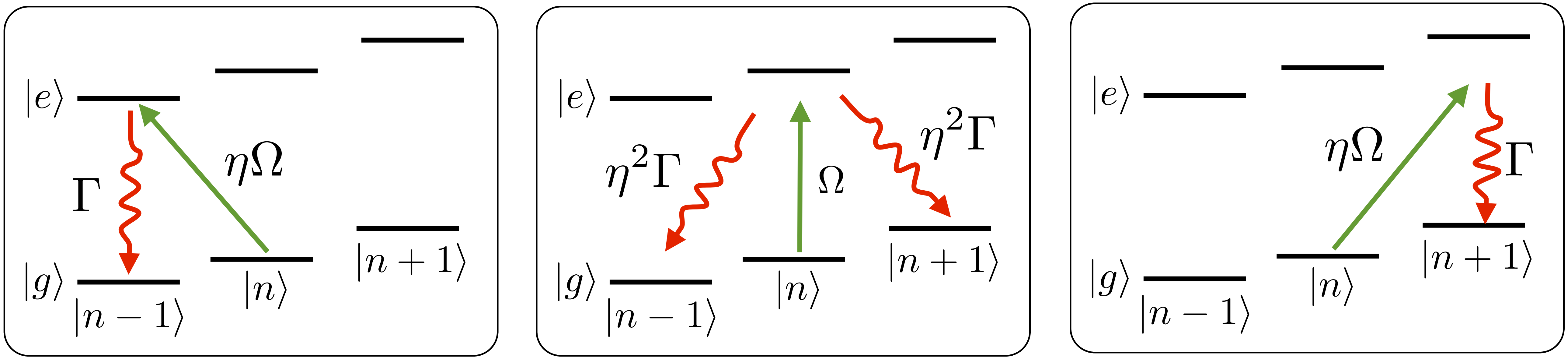
$$\bar{z} = (a + a^\dagger)$$

Laser cooling of trapped ions



cooling **diffusion** **heating**

Laser cooling of trapped ions



cooling **diffusion** **heating**

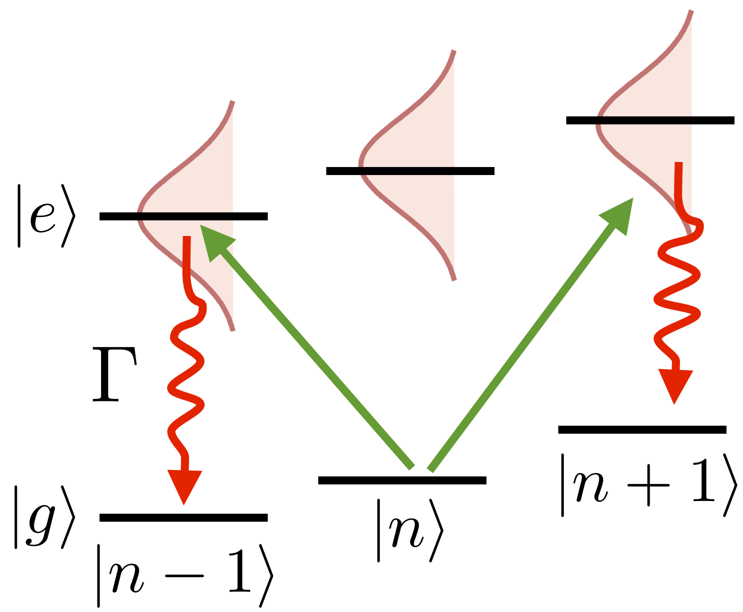
Effective master equation [1]: $\rho_m = \mathrm{Tr}_I\{\rho\}$

$$\dot{\rho}_m = \underbrace{A_-(2a\rho_m a^\dagger - a^\dagger a\rho_m - \rho_m a^\dagger a)}_{\text{(cooling)}} + \underbrace{A_+(2a^\dagger\rho_m a - aa^\dagger\rho_m + \rho_m aa^\dagger)}_{\text{(heating)}}$$

heating / cooling rates: $$A_\pm = \eta^2\Gamma\frac{\Omega^2}{4}\left(\frac{1}{(\delta\mp\omega_m)^2+\Gamma^2/4} + \frac{\alpha}{\delta^2+\Gamma^2/4}\right)$$

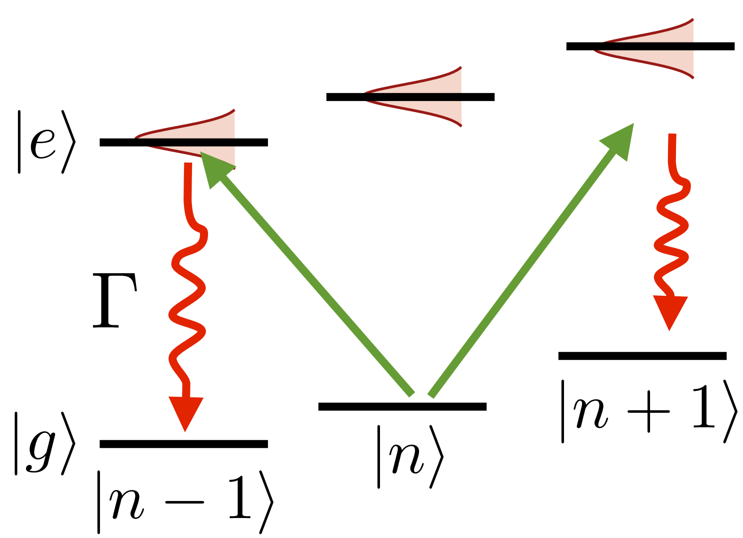
*[1] e.g. J. I. Cirac et al., PRA **46**, 2668 (1992)*

Laser cooling



Doppler cooling: $\Gamma \gg \omega_t$

$$\langle n \rangle = \frac{\Gamma}{4\omega_t}(1+\alpha)$$



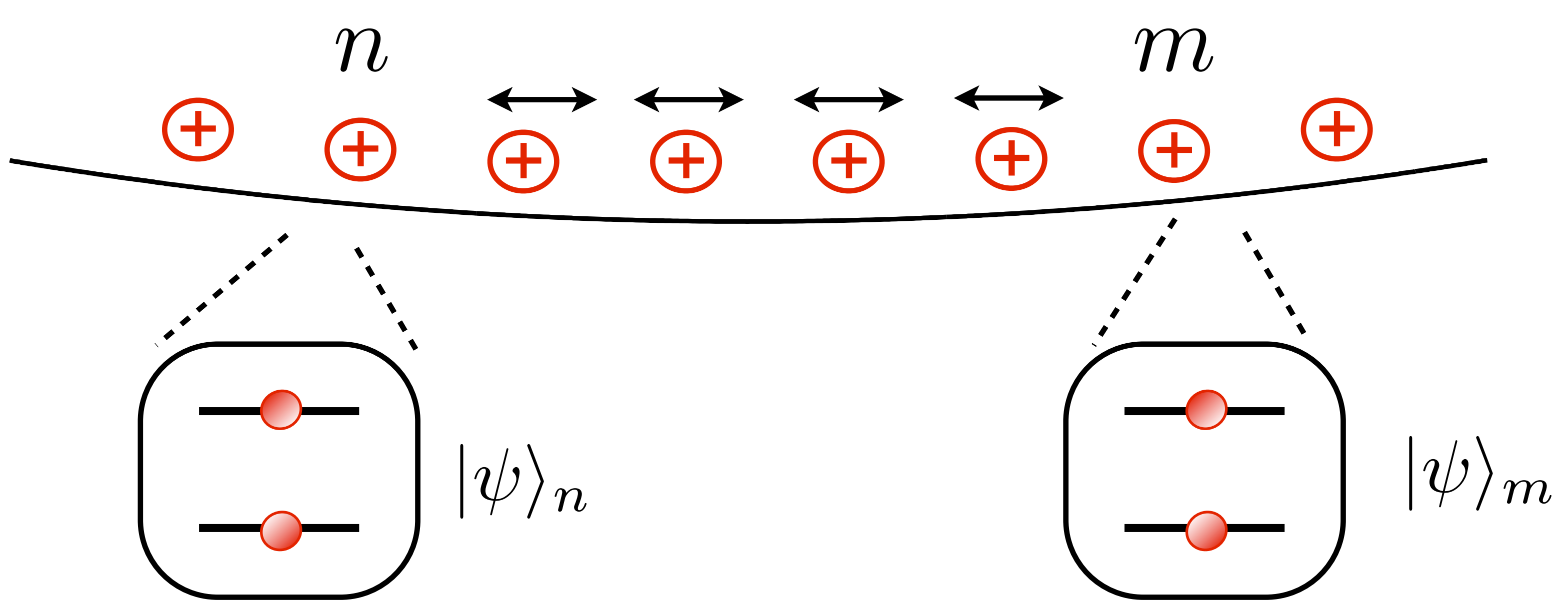
Sideband cooling: $\Gamma \ll \omega_t$

$$\langle n \rangle = \frac{\Gamma^2}{16\omega_t^2}(1+4\alpha) \ll 1^{*)}$$

*) *n < 0.01 in typical trapped ion experiments*

Part II)
Quantum computing with trapped ions

Quantum computing with trapped ions



Basic idea:

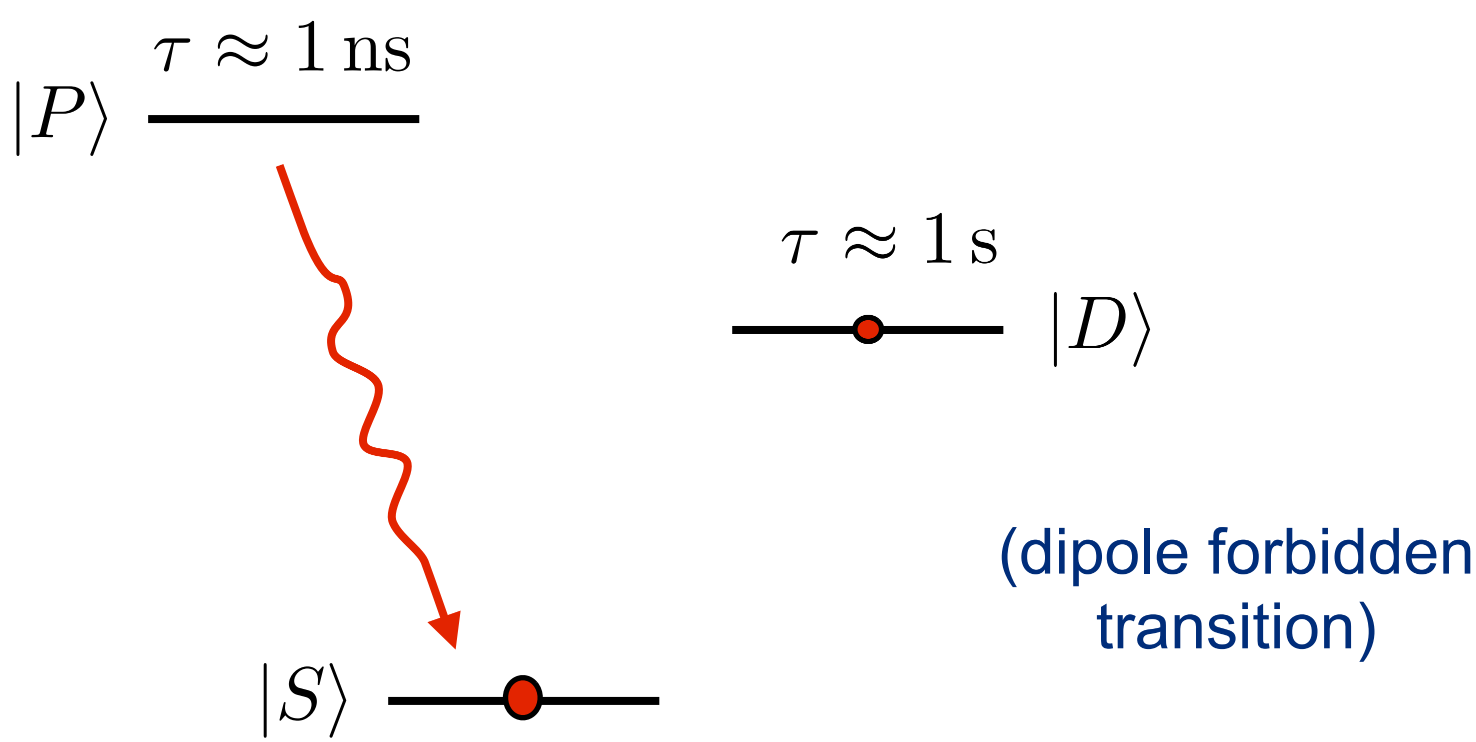
- *Encode qubits in long-lived internal states of trapped ion ions.*
- *Single qubit rotations, initialization and readout using focused lasers.*
- *Two qubit gates using collective phonon modes as a quantum bus.*

Qubits

Qubits

Example: $^{40}\mathrm{Ca}^{+}$

(e.g. R. Blatt, Innsbruck)



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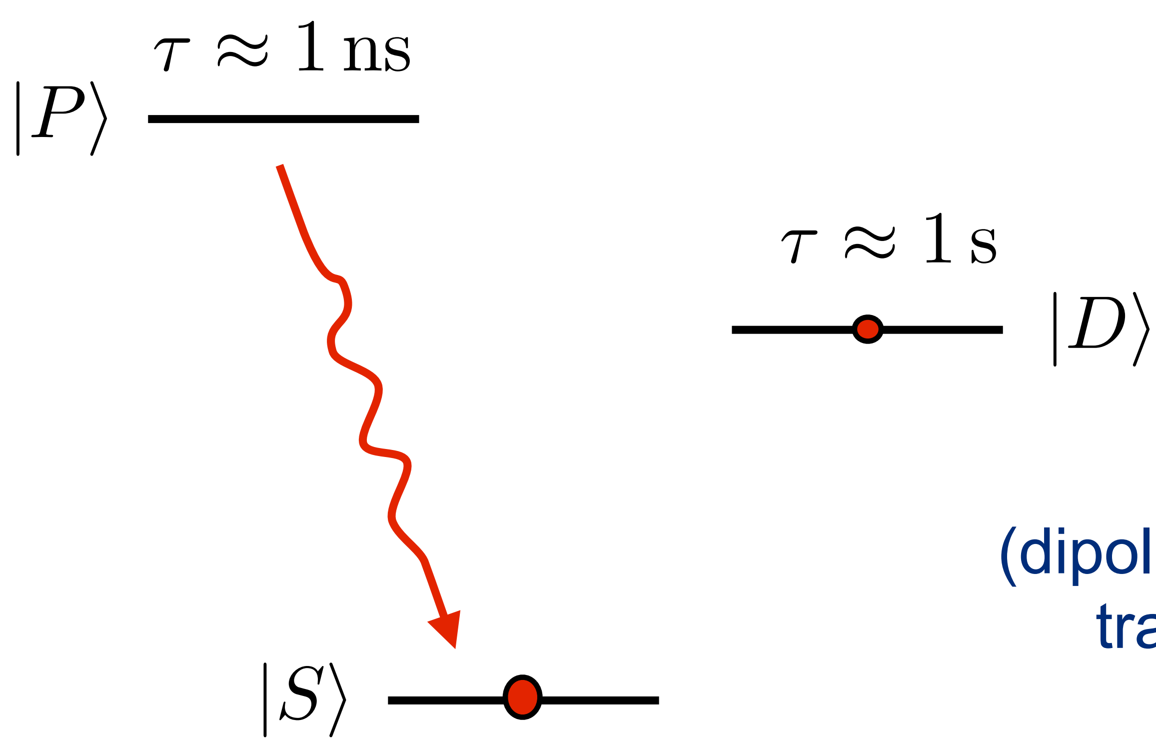

(dipole forbidden transition)

- **Qubit states:** $|0\rangle \equiv |S\rangle$ $|1\rangle \equiv |D\rangle$ (ignoring finestructure)

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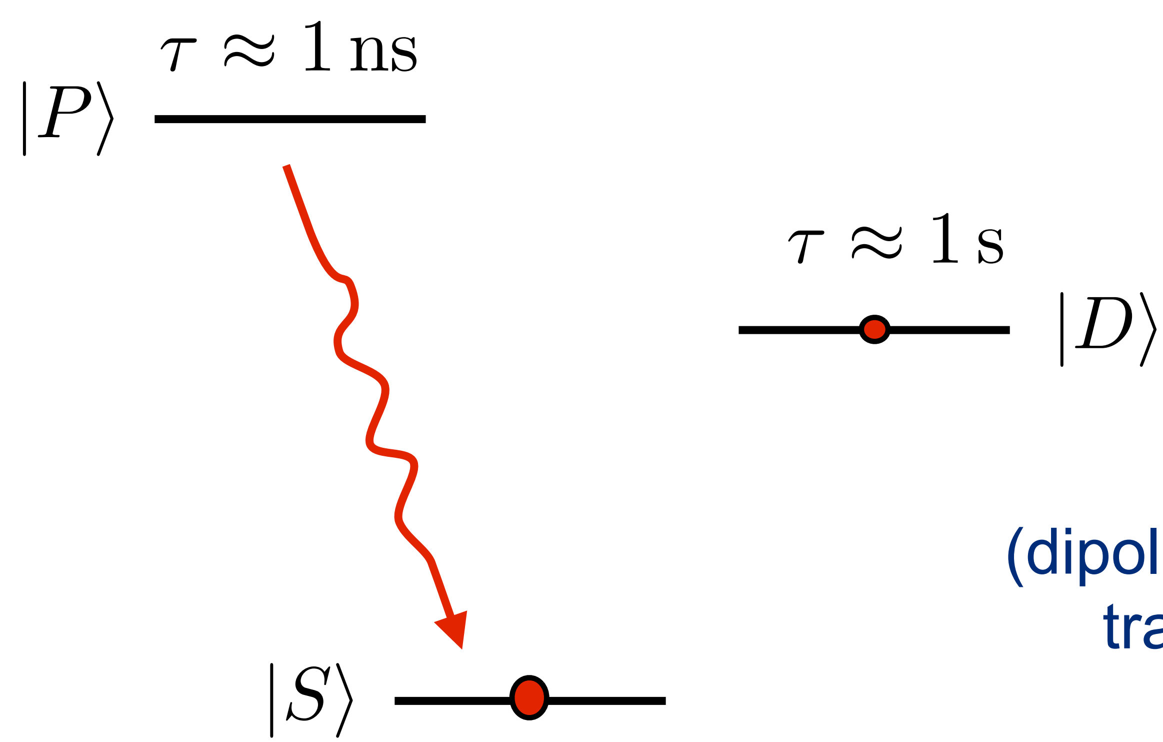

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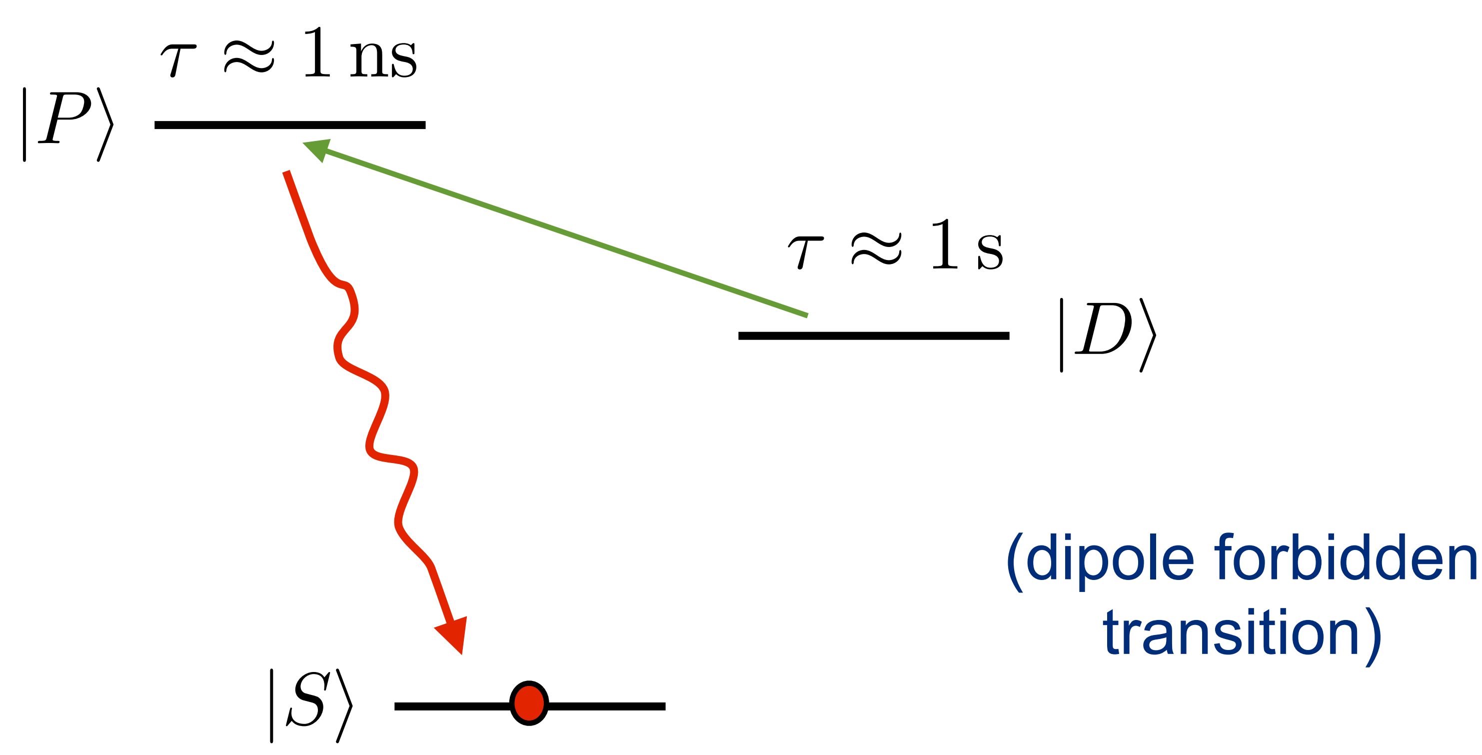

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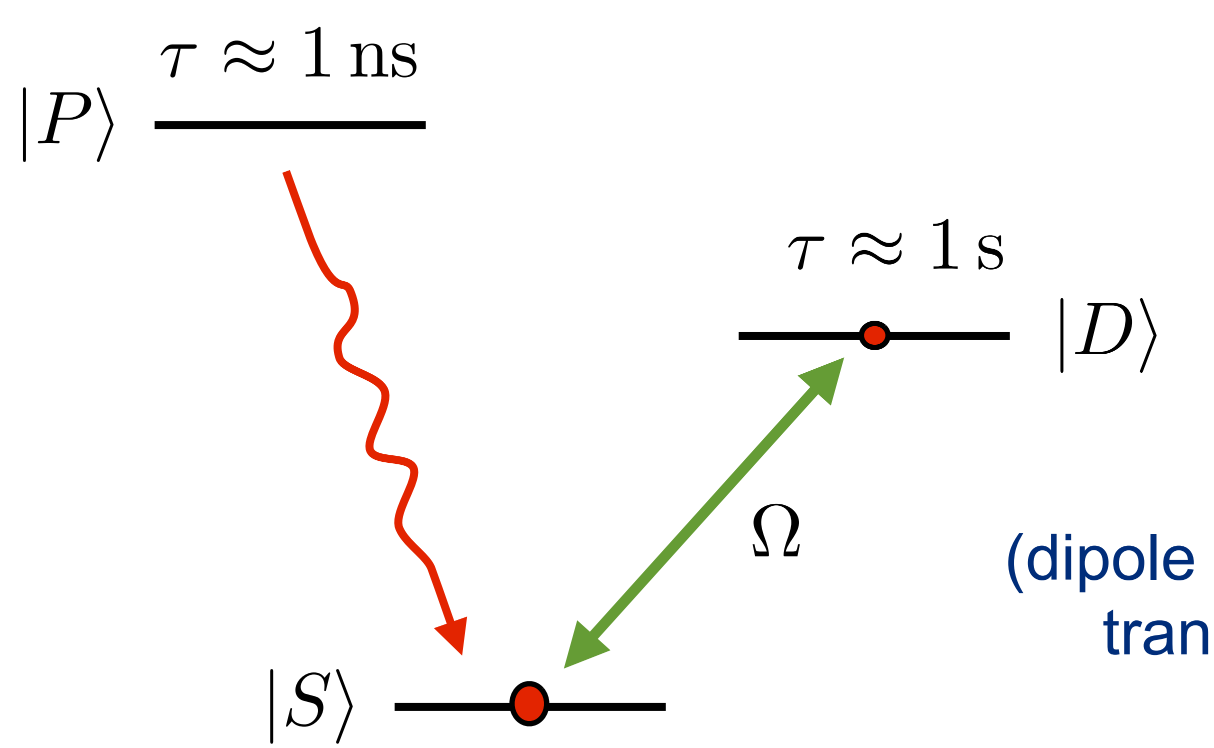

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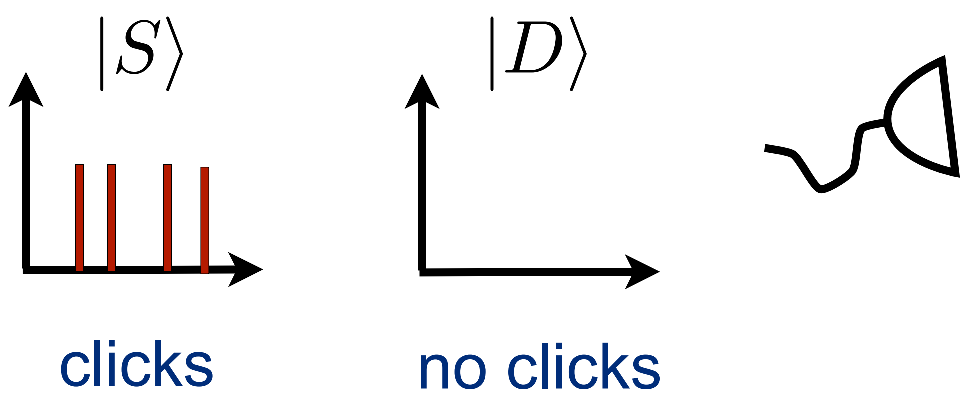

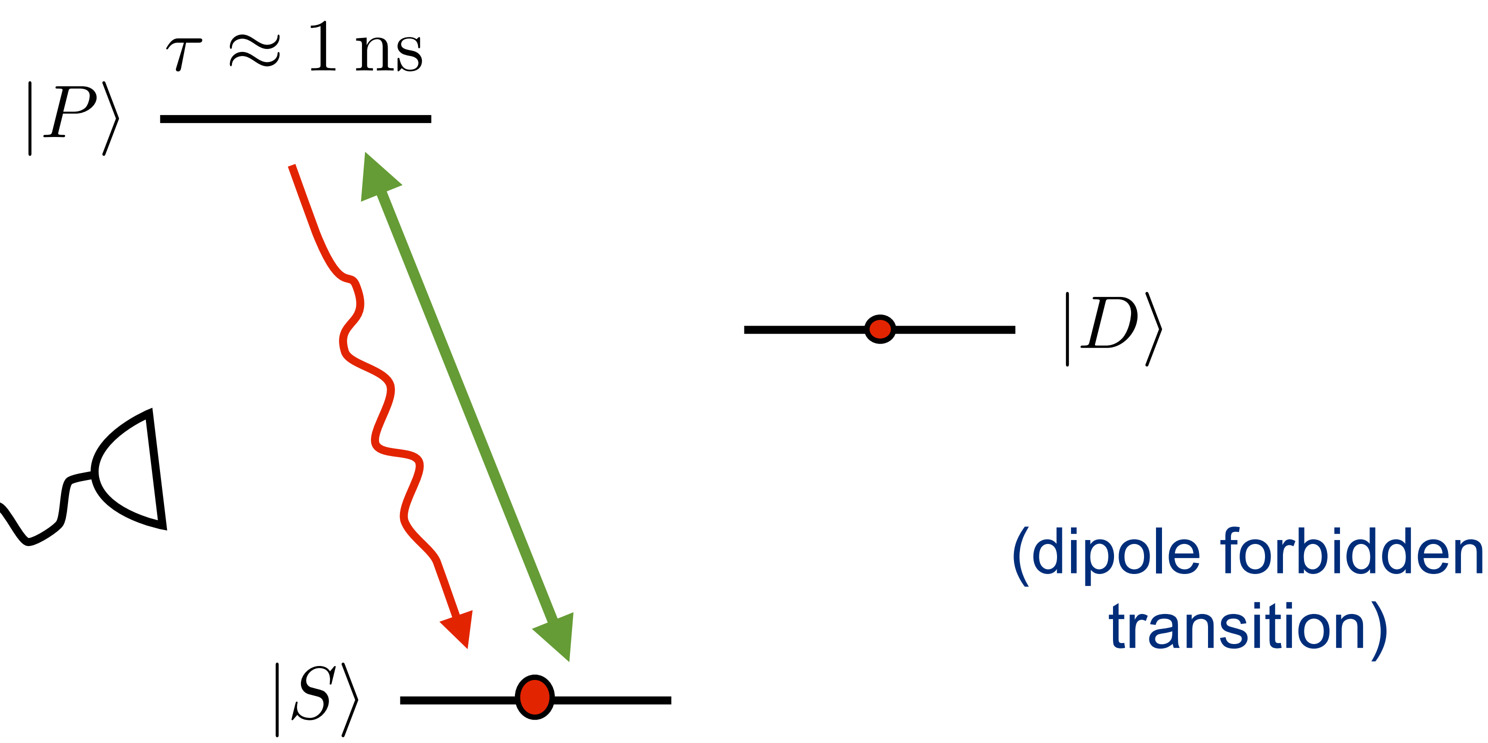

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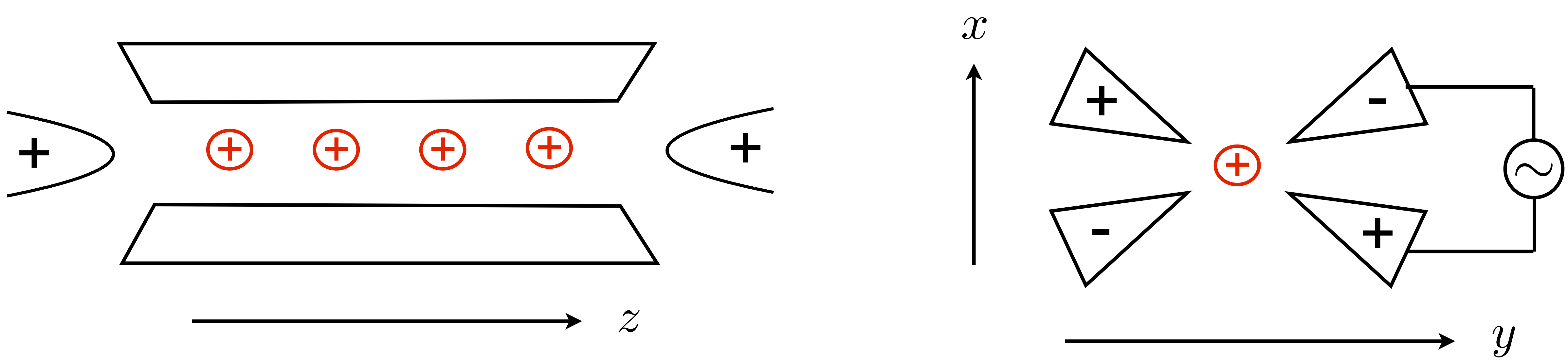
- **Readout:** $\Rightarrow$ *State discrimination from fluorescence signal (>99%).*

Phonon quantum bus

Linear Paul trap



Hamiltonian (N ions):

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + V(\{\vec{r}_i\})$$

Potential energy:

$$V(\{\vec{r}_i\}) = \sum_{i=1}^{N} \frac{1}{2} m \left(\omega_x^2 x_i^2 + \omega_y^2 y_i^2 + \omega_z^2 z_i^2\right) + \frac{e^2}{8\pi\epsilon_0} \sum_{i\neq j} \frac{1}{|\vec{r}_i - \vec{r}_j|}$$

3D trapping potential

Coulomb repulsion

Linear Paul trap

I) Equilibrium configuration: $$\frac{\partial V(\{\vec{r}_i\})}{\partial \vec{r}_{i,\alpha}} = 0 \qquad \Rightarrow \qquad \vec{r}_i^{\,0}$$

Linear Paul trap

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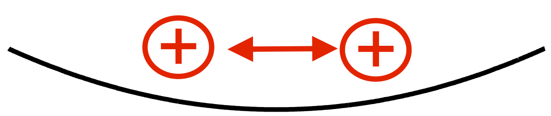
- Example N=2:

$$x^0_{1,2} = y^0_{1,2} = 0$$

$$z^0_1 + z^0_2 = 0$$

$$d = z^0_1 - z^0_2 = \sqrt[3]{\frac{e^2}{4\pi\epsilon_0 m\omega_z^2}}$$



$d \sim 5 - 10\,\mu\mathrm{m}$

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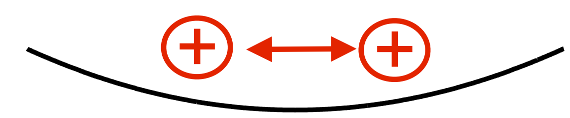
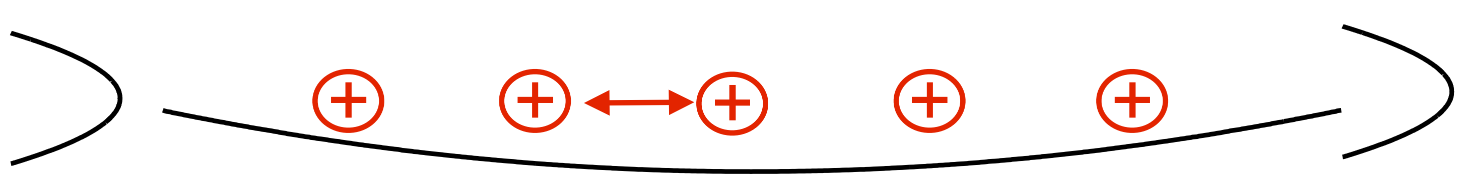


$$d \sim 5 - 10\,\mu\mathrm{m}$$

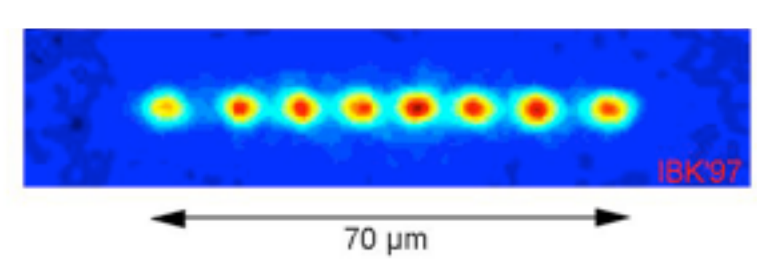
- General N>2 (numerically):



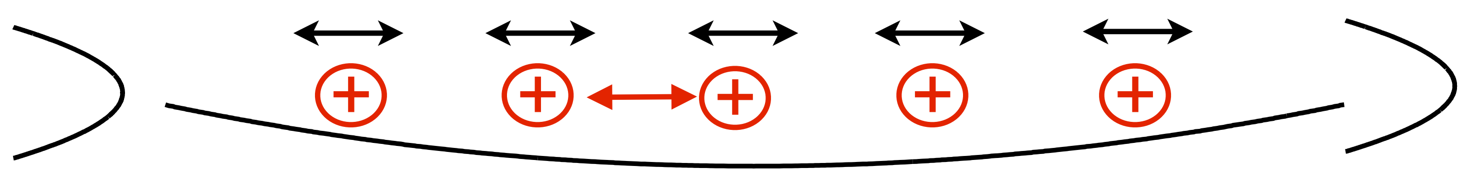
$$d \sim 5 - 10\,\mu\mathrm{m}$$

$$\omega_z \ll \omega_{x,y}$$

- *linear chain of largely spaced ions*
- *individual addressing with lasers !*



Collective phonon modes

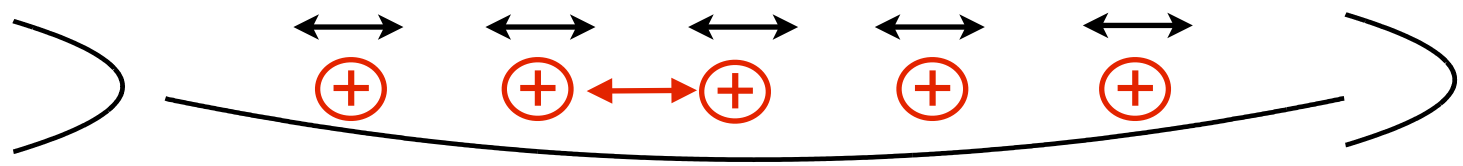


$$d \sim 5 - 10\,\mu\mathrm{m}$$

II) Fluctuations: $\vec{r}_i = \vec{r}_i^{\,0} + \vec{u}_i$

low temperatures: $|\vec{u}_i| \sim 10 - 50\,\mathrm{nm} \ll |\vec{r}_i - \vec{r}_j| \sim 5\,\mu\mathrm{m}$

Collective phonon modes



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low temperatures: $|\vec{u}_i| \sim 10 - 50\,\mathrm{nm} \ll |\vec{r}_i - \vec{r}_j| \sim 5\,\mu\mathrm{m}$

$\Rightarrow$ *harmonic approximation (1D, z-direction):*

$$V(\{\vec{r}_i\}) \simeq V(\{\vec{r}_i^{\,0}\}) + \underbrace{\cancel{\left.\frac{\partial V}{\partial z_i}\right|_{\vec{r}_i^{\,0}}}}_{=0} z_i + \frac{1}{2}\sum_{i,j} \left.\frac{\partial^2 V}{\partial z_i \partial z_j}\right|_{\vec{r}_i^{\,0}} z_i z_j \qquad K_{ij} = \left.\frac{\partial^2 V}{\partial z_i \partial z_j}\right|_{\vec{r}_i^{\,0}}$$

- ***Chain of coupled harmonic oscillators:***

$$\boxed{H = \sum_i \frac{p_i^2}{2m} + \frac{1}{2}\sum_{i,j} K_{ij} z_i z_j}$$

Collective phonon modes

$$H_{\text{phon}} = \sum_i \frac{p_i^2}{2m} + \frac{1}{2}\sum_{i,j} K_{ij} z_i z_j \quad = \quad \sum_n \hbar\omega_n a_n^\dagger a_n$$

$$[a_n, a_m^\dagger] = \delta_{n,m}$$

phonon frequencies

phonon mode operators

Collective phonon modes

$$H_{\text{phon}} = \sum_i \frac{p_i^2}{2m} + \frac{1}{2}\sum_{i,j} K_{ij} z_i z_j \quad = \quad \sum_n \hbar\omega_n a_n^\dagger a_n \qquad [a_n, a_m^\dagger] = \delta_{n,m}$$

phonon frequencies

phonon mode operators

- ***Heisenberg equations:***

$$\left.\begin{aligned} \dot{z}_i(t) &= p_i(t)/m \\ \dot{p}_i(t) &= -\sum_j K_{ij} z_j \end{aligned}\right\} \quad \ddot{z}_i(t) + \sum_j \frac{K_{ij}}{m} z_j = 0$$

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- *Ansatz:*

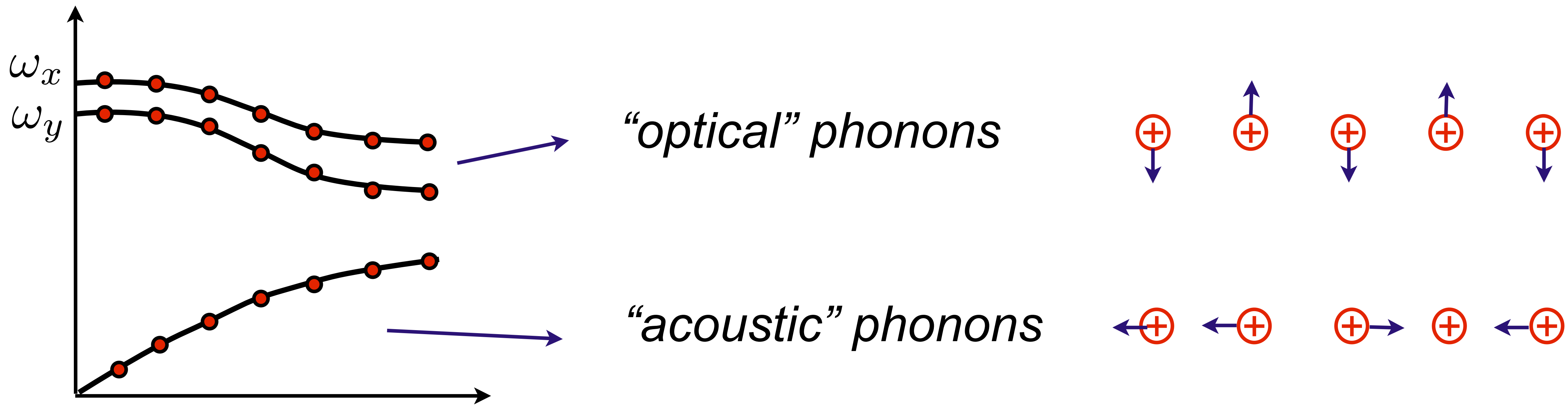
$$z_i(t) = \sum_{n=1}^{N} \sqrt{\frac{\hbar}{2m\omega_n}} c_n(i) \left(a_n e^{-i\omega_n t} + a_n^\dagger e^{i\omega_n t}\right)$$

$\Rightarrow$ *Eigenvalue problem for orthonormal set of modefunctions* $c_n(i)$*:*

$$\omega_n^2 c_n(i) - \sum_j \frac{K_{ij}}{m} c_n(j) = 0 \qquad [z_i, p_j] = i\hbar\delta_{ij} \rightarrow \sum_n c_n(i) c_n(j) = \delta_{ij}$$

Collective phonon modes

Mode spectrum (linear Paul trap):



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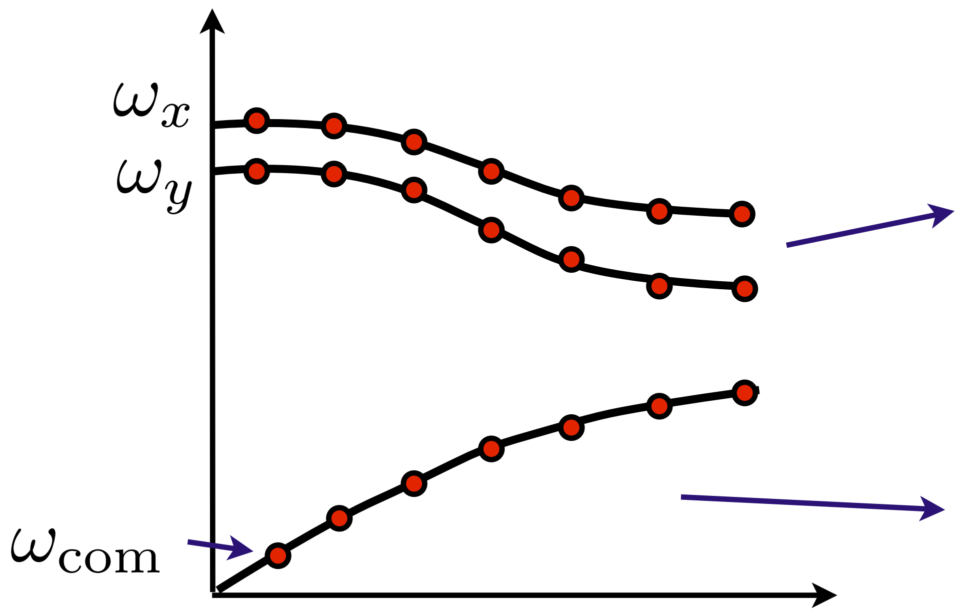
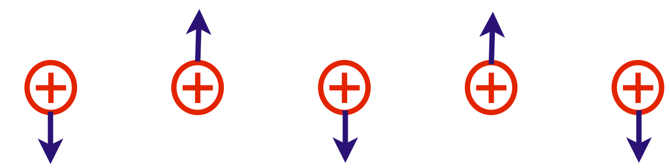
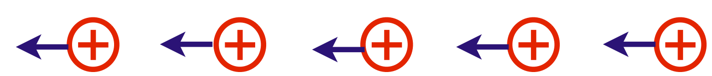


"optical" phonons



"acoustic" phonons

- *center of mass (COM) mode:*



$$\omega_{\mathrm{com}} = \omega_z \qquad c_{\mathrm{com}}(i) = 1/\sqrt{N}$$

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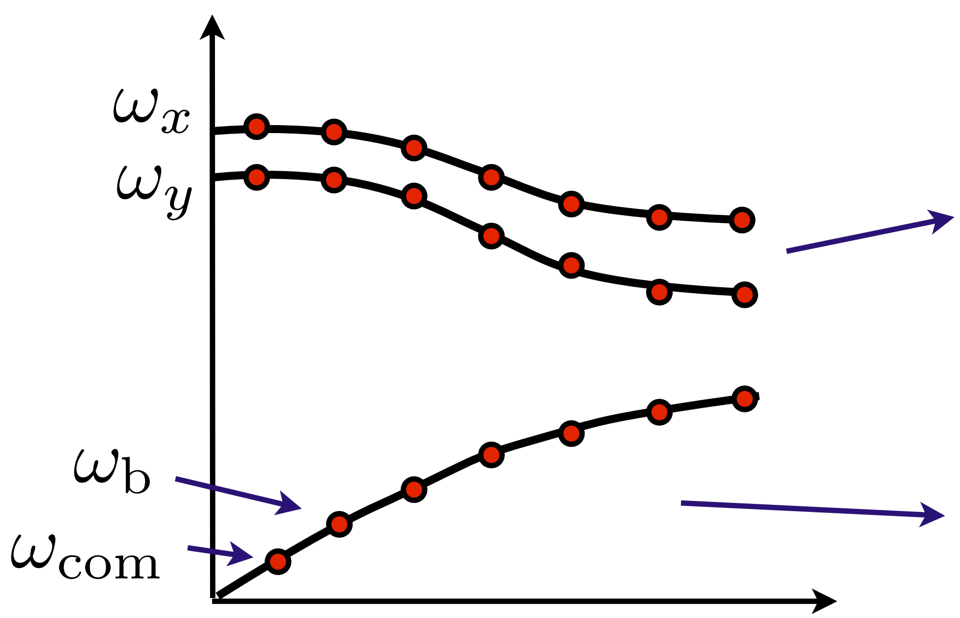

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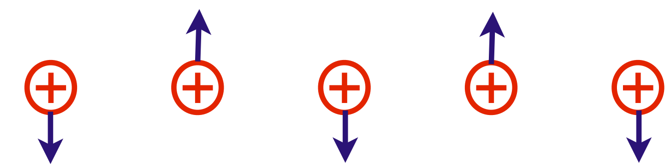

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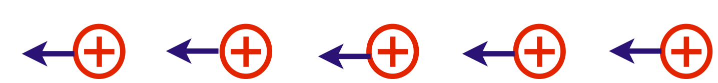

$$\omega_{\rm com} = \omega_z \qquad c_{\rm com}(i) = 1/\sqrt{N}$$

- *breathing mode:*

$$\omega_b = \sqrt{3}\omega_z \qquad c_{\rm b}(i) \sim i$$

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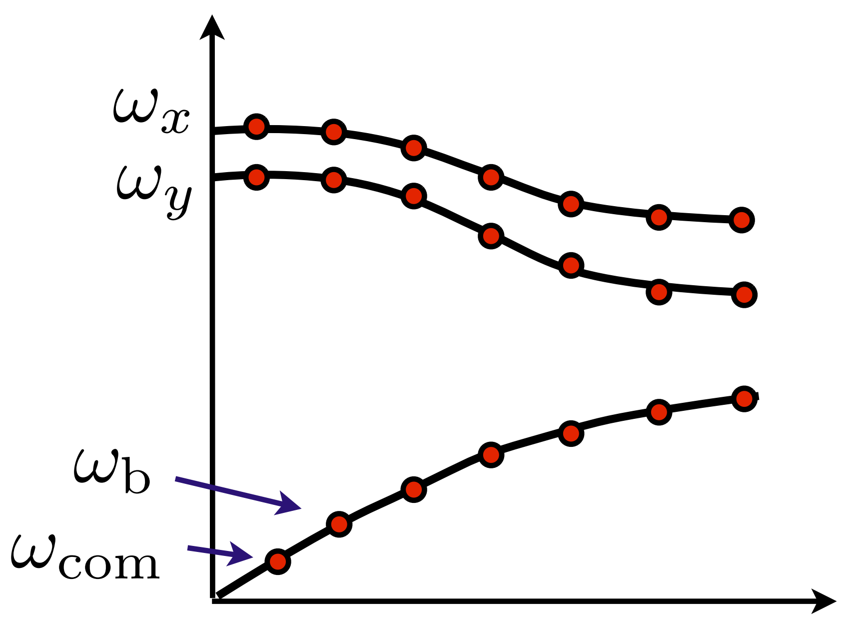
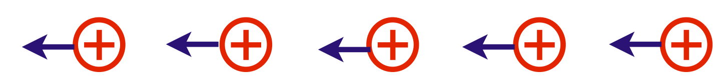


$$\Delta\omega = \omega_b - \omega_{\mathrm{com}} = (\sqrt{3} - 1)\omega_z$$

- *mode spacing (apparently) independent of N*

- *center of mass (COM) mode:*



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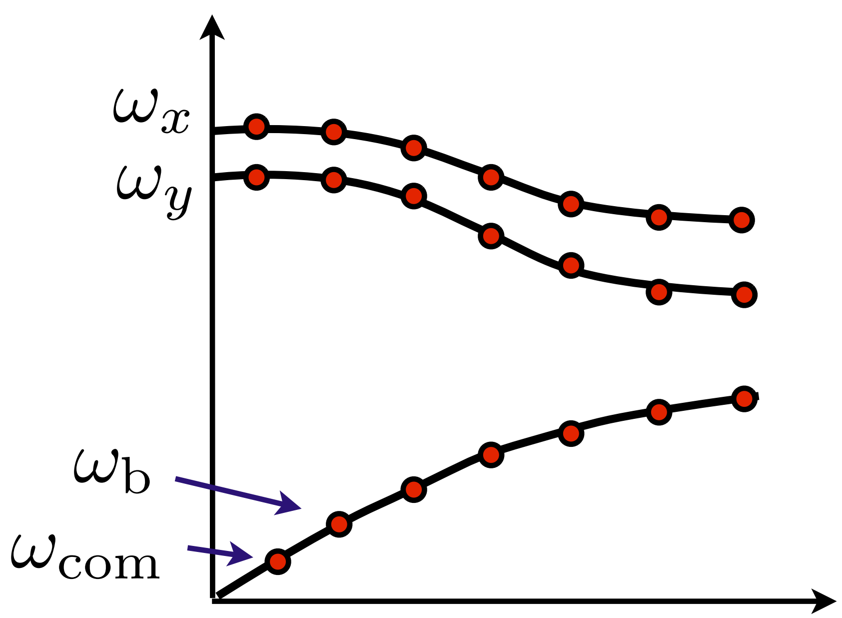

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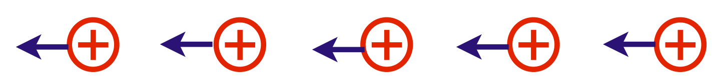

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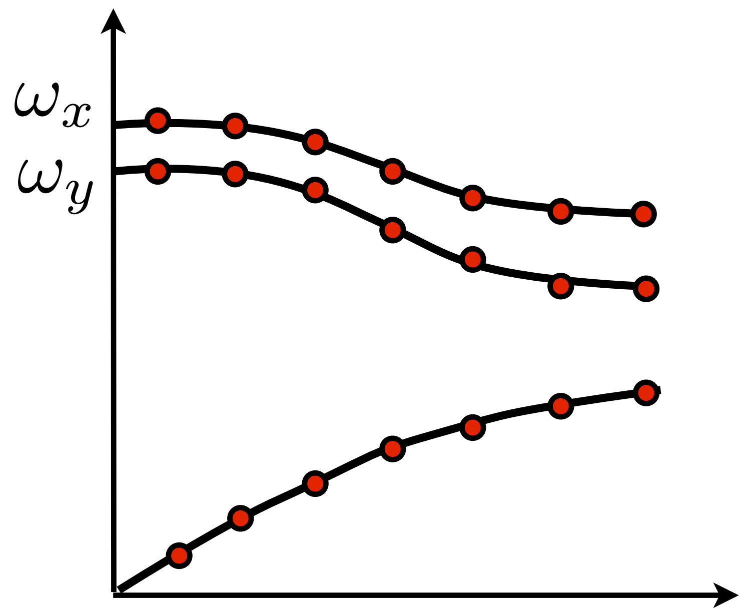
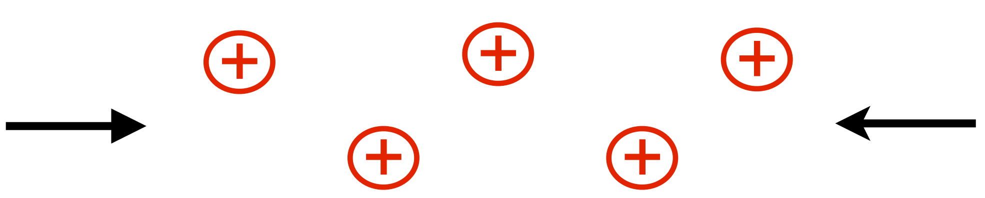


$$\Delta\omega = \omega_b - \omega_{\text{com}} = (\sqrt{3}-1)\omega_z$$

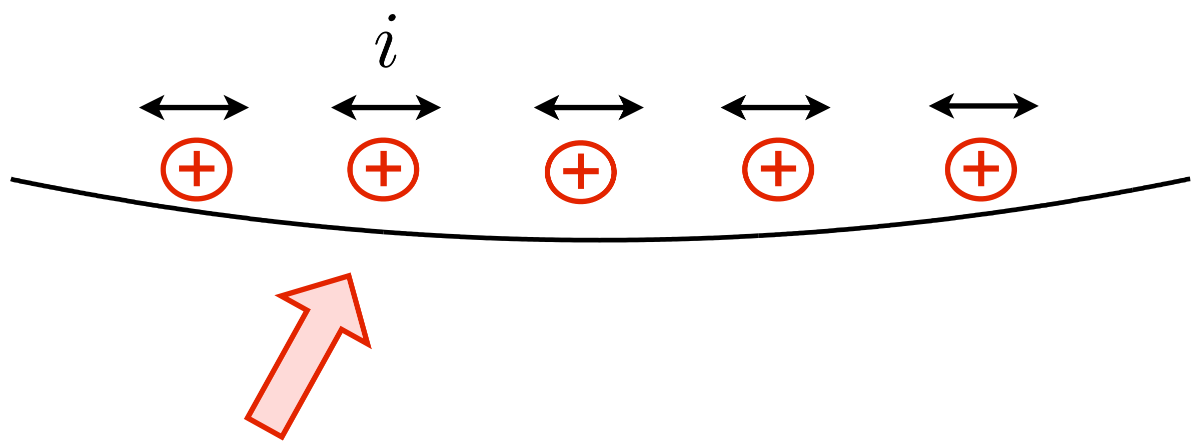
- *mode spacing (apparently) independent of N*

<u>In practice:</u>
With increasing N the trapping frequency ω_z *must be lowered to avoid instabilities.*



zig-zag-instability

Addressing single phonon modes



$$H_i(t) = \frac{\hbar\Omega_i}{2}\left(\sigma_+ e^{ikz_i(t)} e^{-i\delta t} + e^{-ikz_i(t)} e^{i\delta t}\sigma_-\right)$$

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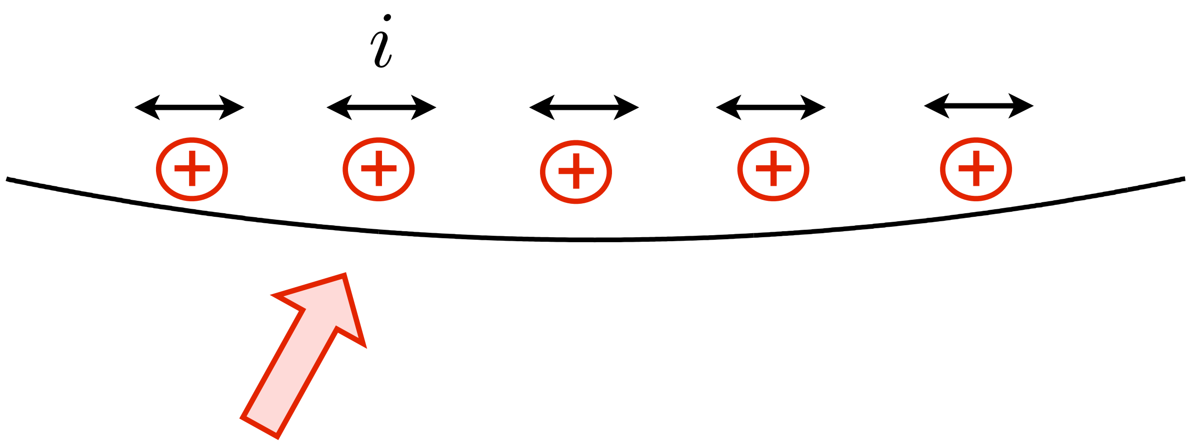

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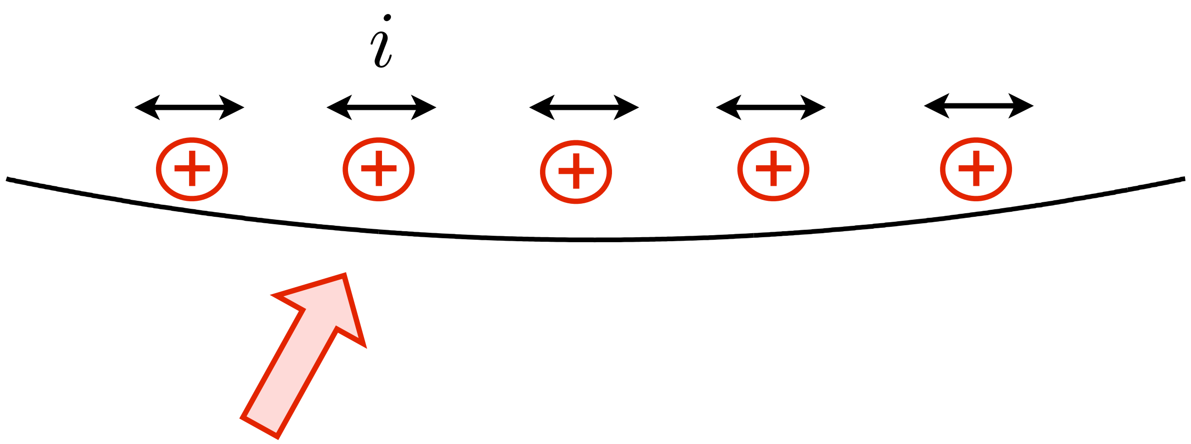
$$\simeq \frac{\hbar\Omega_i}{2}e^{-i\delta t}\sigma_+\left[1 + i\sum_n \eta_n c_n(i)\left(a_n^\dagger e^{-i\omega_n t} + a_n^\dagger e^{i\omega_n t}\right)\right] + \text{H.c.}$$

$\delta = -\omega_n$:

$$\simeq i\eta\frac{\hbar\Omega_i}{2}\left(\sigma_+ a_n + \sigma_- a_n^\dagger\right) + \mathcal{O}\left(e^{i(\omega_n \pm \omega_m)t}\right)$$

Addressing single phonon modes



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$$\simeq \frac{\hbar\Omega_i}{2} e^{-i\delta t}\sigma_+ \left[1 + i\sum_n \eta_n c_n(i)\left(a_n^\dagger e^{-i\omega_n t} + a_n^\dagger e^{i\omega_n t}\right)\right] + \mathrm{H.c.}$$

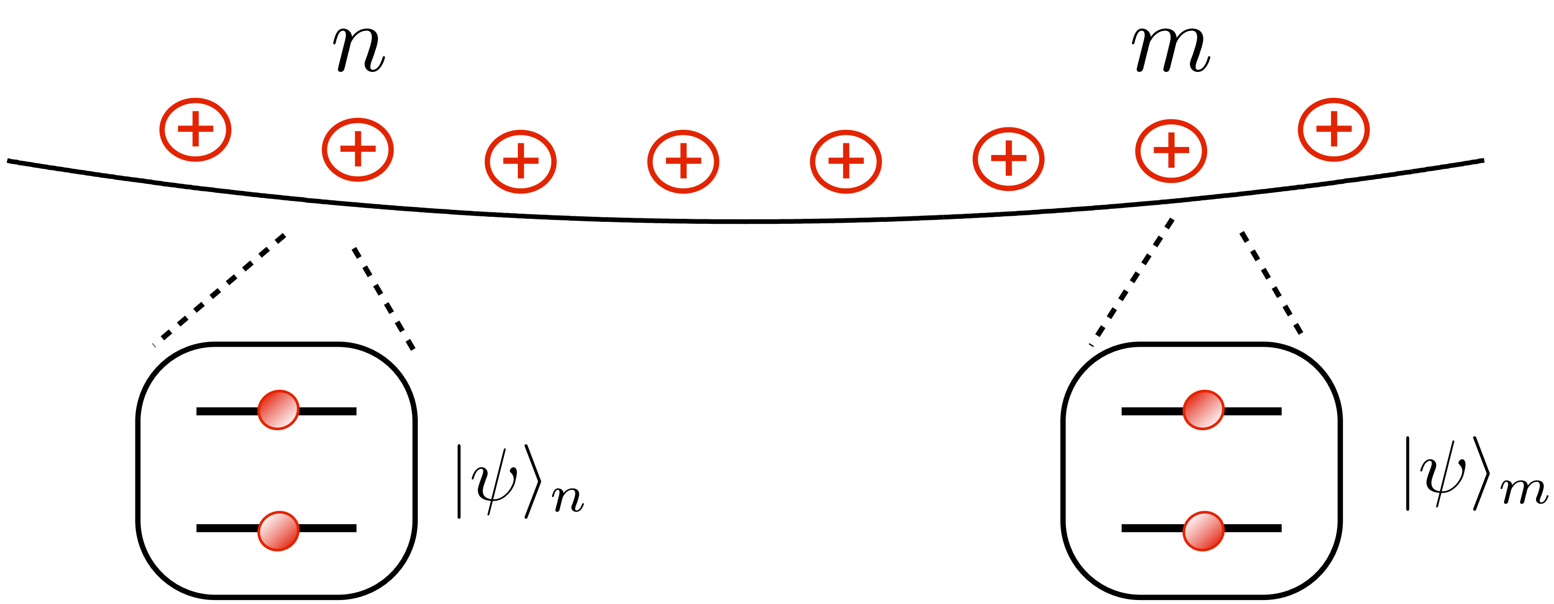
$\delta = -\omega_n :$

$$\simeq i\eta\frac{\hbar\Omega_i}{2}\left(\sigma_+ a_n + \sigma_- a_n^\dagger\right) + \mathcal{O}\left(e^{i(\omega_n \pm \omega_m)t}\right)$$

- *Addressing of individual modes if* $\eta\Omega < |\omega_n - \omega_m|$.
- *Speed of gate operations (below) limited by mode spacing.*

Two qubit gates

Cirac-Zoller gate



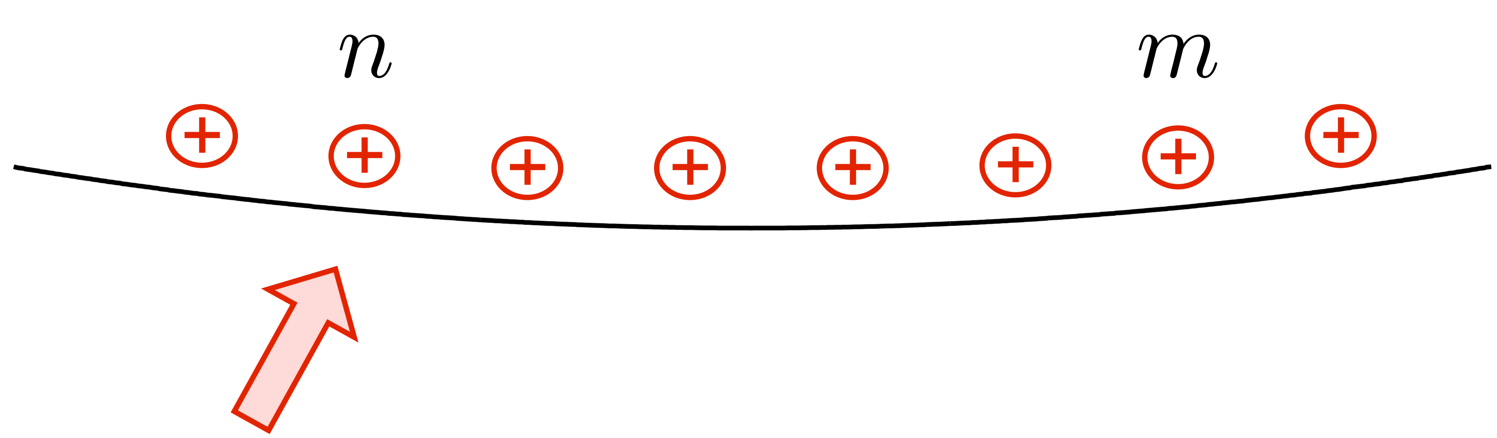
Initial state:

- Ions n and m in arbitrary states $|\psi\rangle_n$ and $|\psi\rangle_m$
- COM mode cooled to the ground state $|0\rangle_p$

Goal: $U = U_{\text{gate}}^{nm} \otimes \mathbb{1}_p$ *e.g.* $U_{\text{PHASE}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

*J. I. Cirac and P. Zoller, PRL **74**, 4091 (1995)*

Cirac-Zoller gate

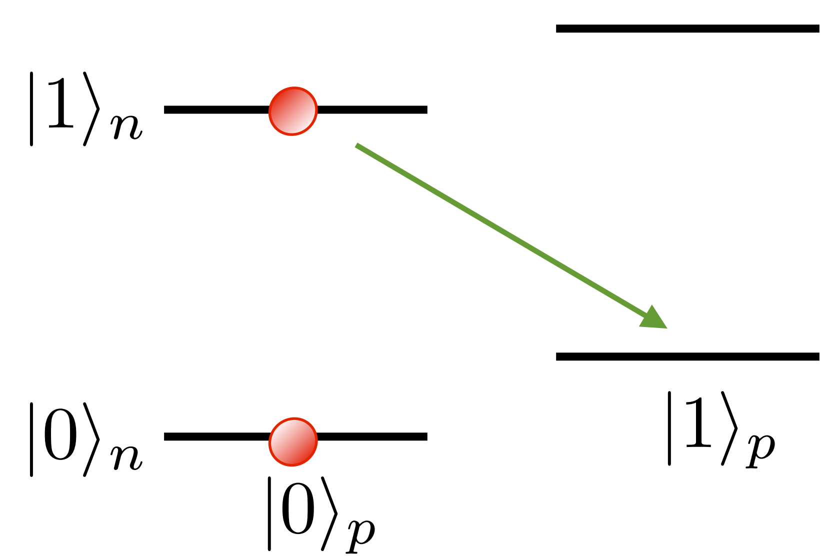


1

$$|\psi\rangle_i = (\alpha_n|0\rangle_n + \beta_n|1\rangle_n)|0\rangle_p|\psi_m\rangle$$

$$\downarrow$$

$$|\psi\rangle_1 = |0\rangle_n(\alpha_n|0\rangle_p + \beta_n|1\rangle_p)|\psi_m\rangle$$



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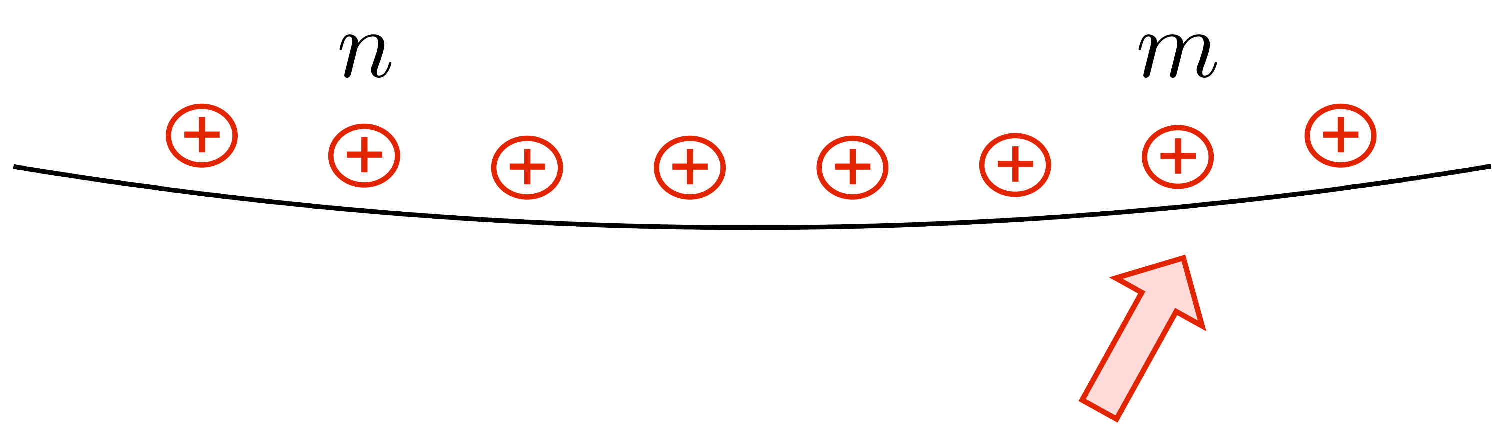

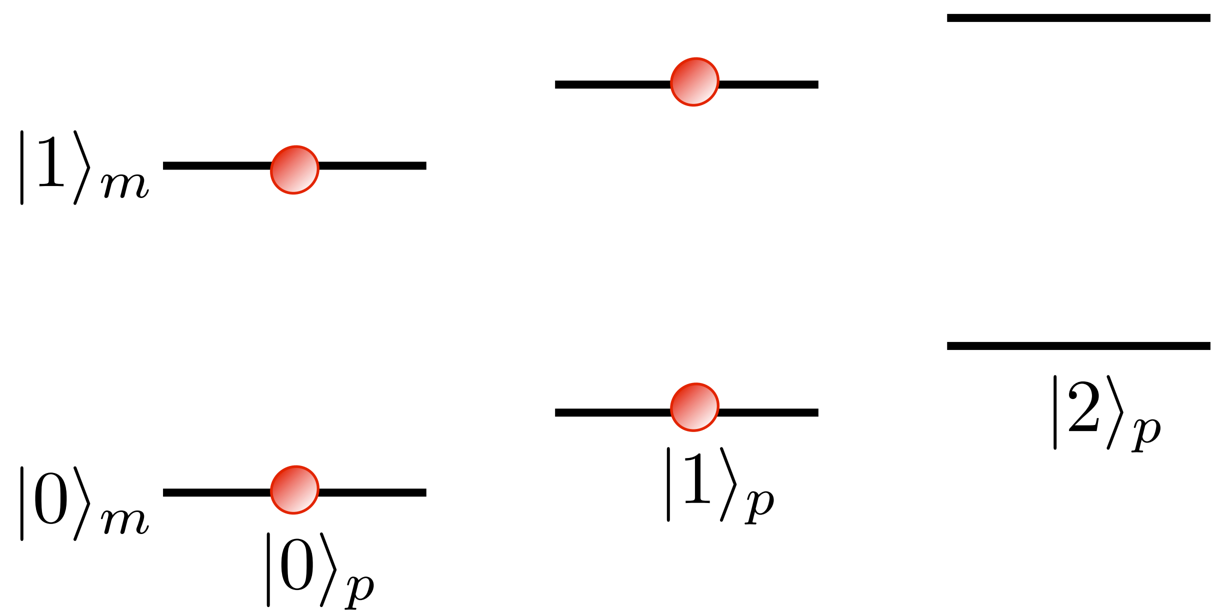

$$|\psi_1\rangle = |0\rangle_n(\alpha_n\alpha_m|0\rangle_p|0\rangle_m + \alpha_n\beta_m|0\rangle_p|1\rangle_m + \beta_n\alpha_m|1\rangle_p|0\rangle_m + \beta_n\beta_m|1\rangle_p|1\rangle_m)$$

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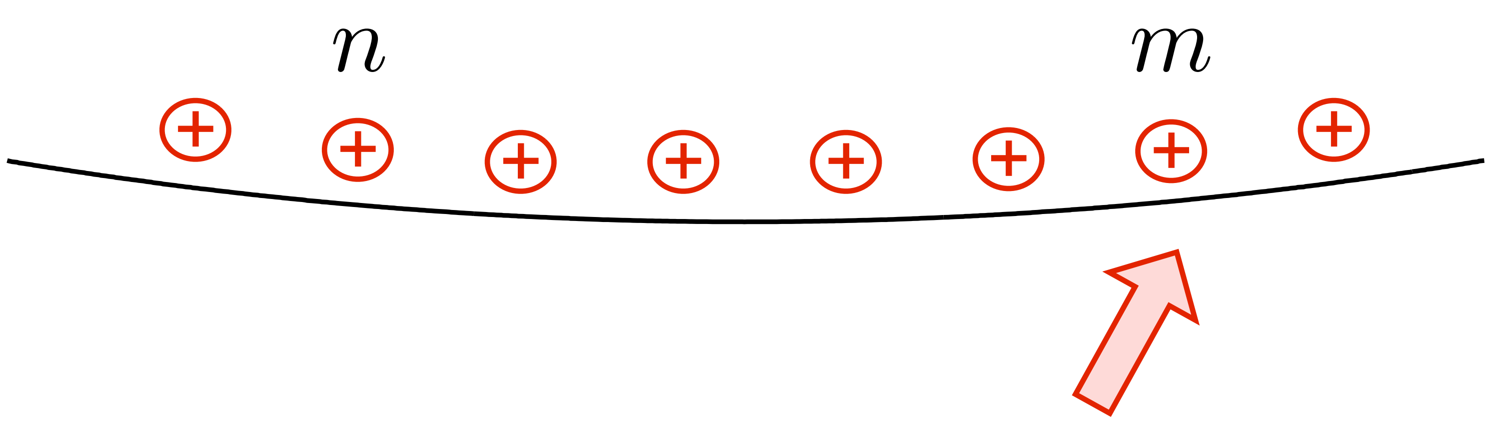
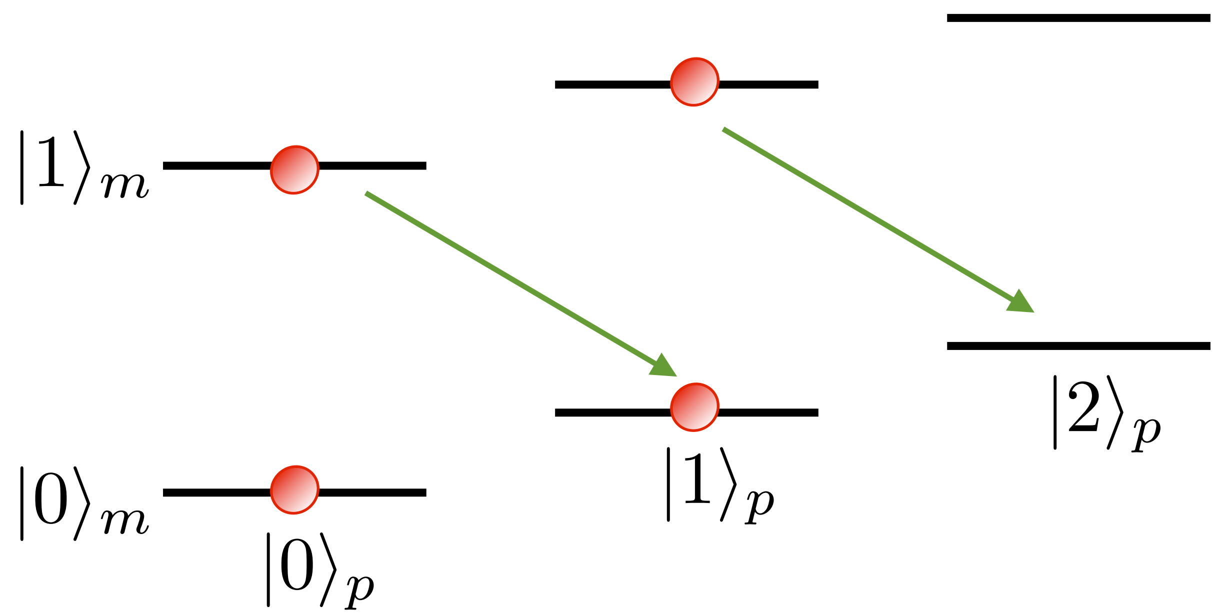
Cirac-Zoller gate





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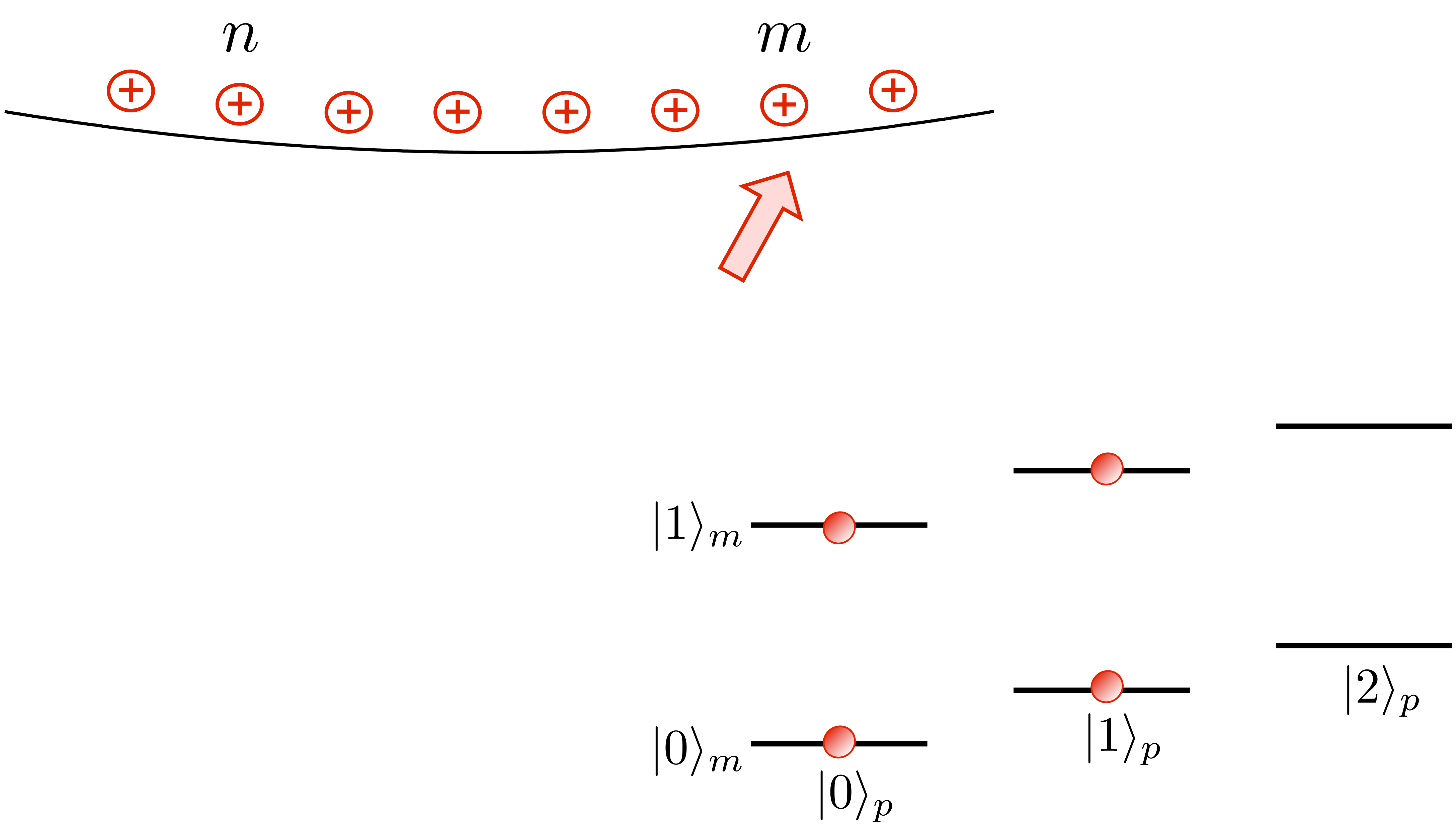
Cirac-Zoller gate



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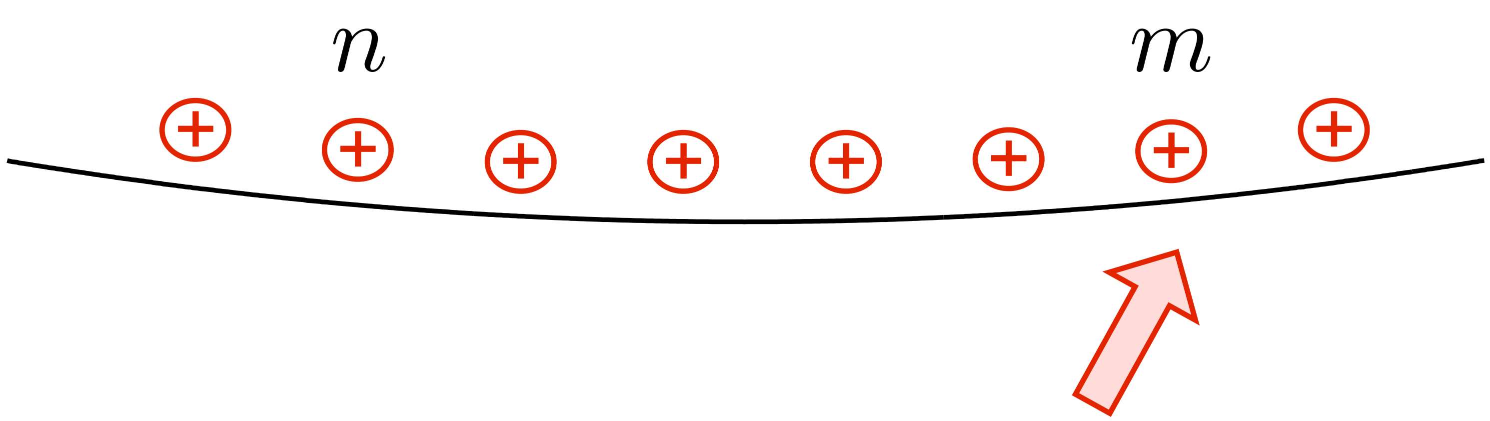
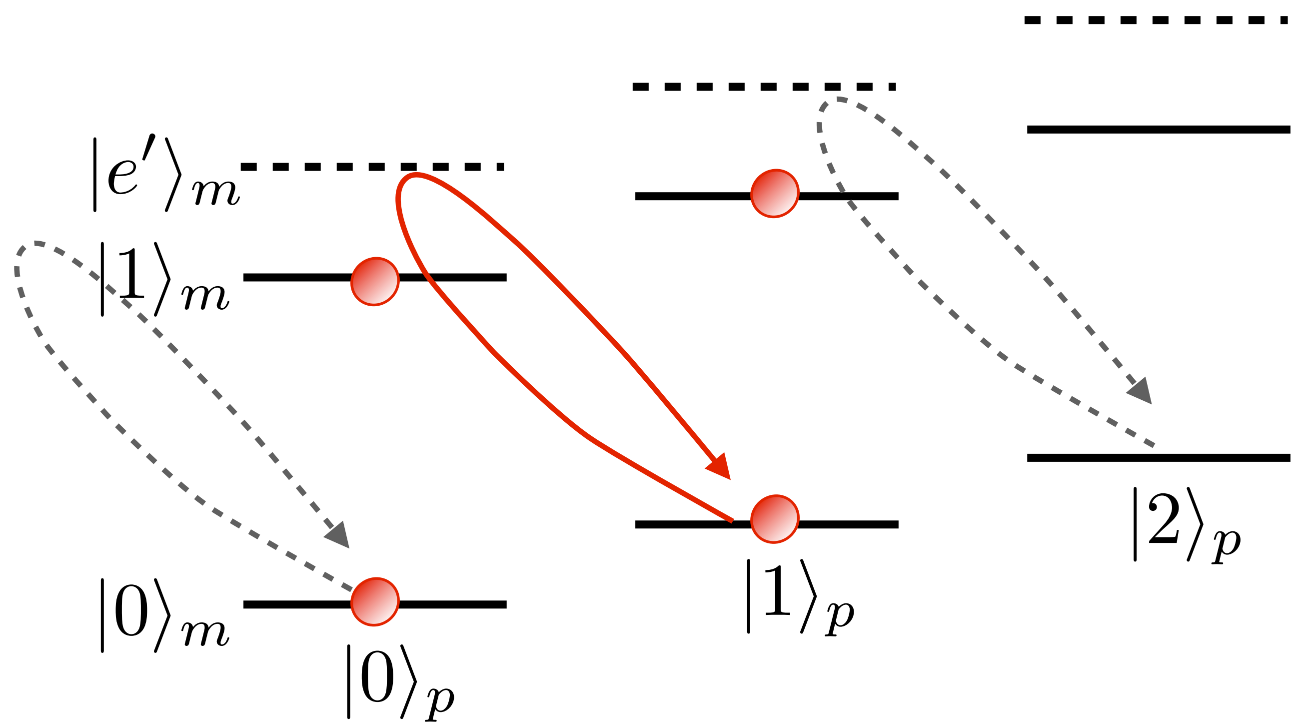
Cirac-Zoller gate



② ***Trick:*** *2π-red sideband rotation using an auxiliary atomic state* $|e'\rangle$.

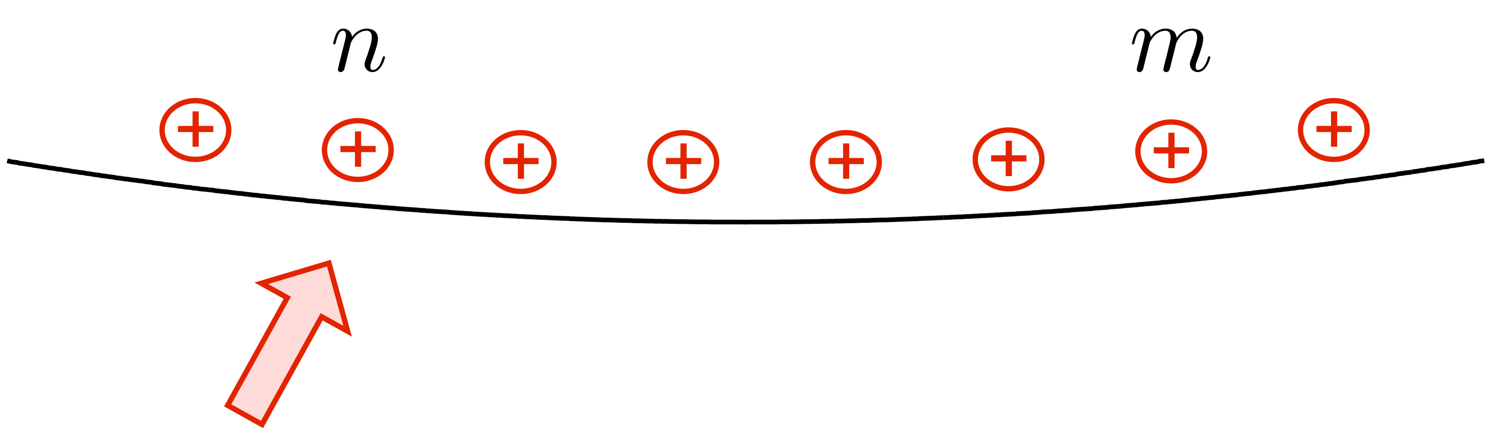
$$R_x(\pi)R_x(\pi) = -1$$



$$|\psi_2\rangle = |0\rangle_n(\underbrace{\alpha_n\alpha_m|0\rangle_p|0\rangle_m + \alpha_n\beta_m|0\rangle_p|1\rangle_m - \beta_n\alpha_m|1\rangle_p|0\rangle_m + \beta_n\beta_m|1\rangle_p|1\rangle_m)}_{\text{entangled !}}$$

*J. I. Cirac and P. Zoller, PRL **74**, 4091 (1995)*

Cirac-Zoller gate

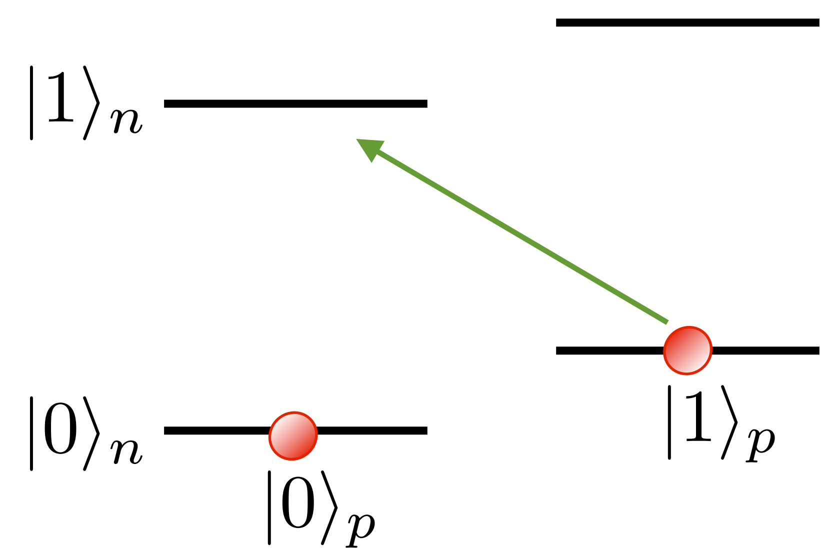


3

$$|\psi_2\rangle = |0\rangle_n \otimes |\Psi\rangle_{p,m}$$

$$\downarrow$$

$$|\psi_3\rangle = |\Psi\rangle_{n,m} \otimes |0\rangle_p$$



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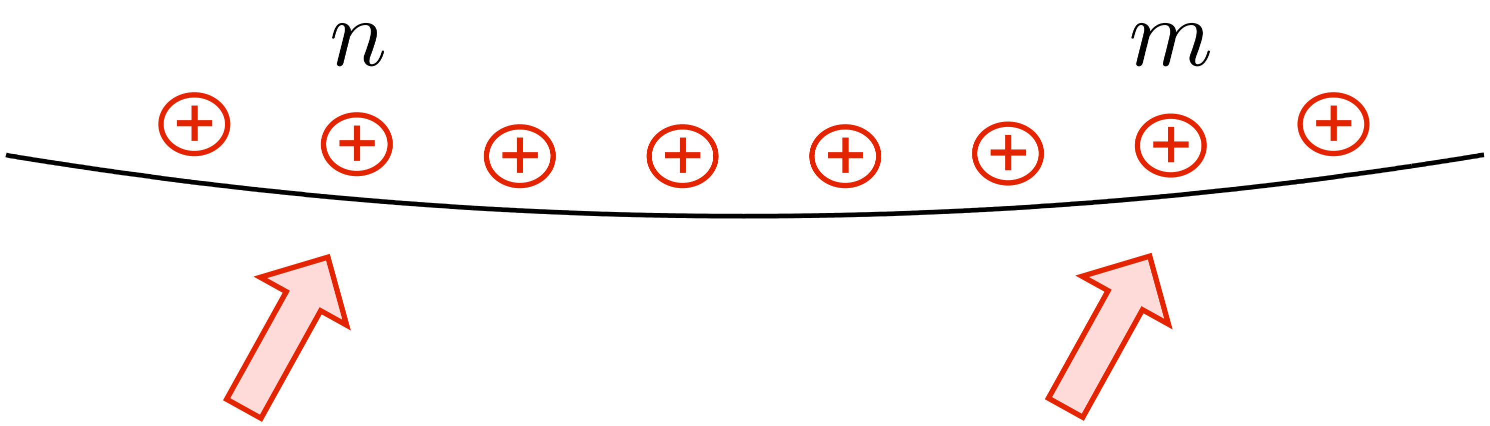
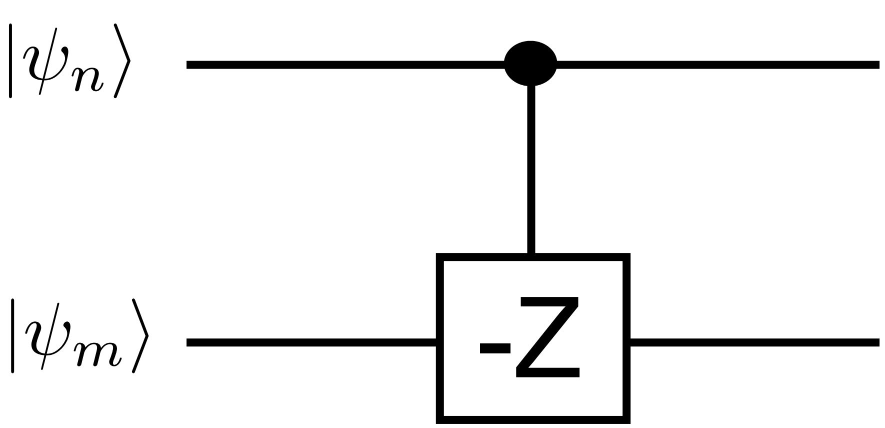
Cirac-Zoller gate: Summary



Final two qubit state:

$$|\psi_f\rangle = (\alpha_n\alpha_m|0\rangle_n|0\rangle_m + \alpha_n\beta_m|0\rangle_n|1\rangle_m - \beta_n\alpha_m|1\rangle_n|0\rangle_m + \beta_n\beta_m|1\rangle_n|1\rangle_m)$$

$$U_{\mathrm{tot}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



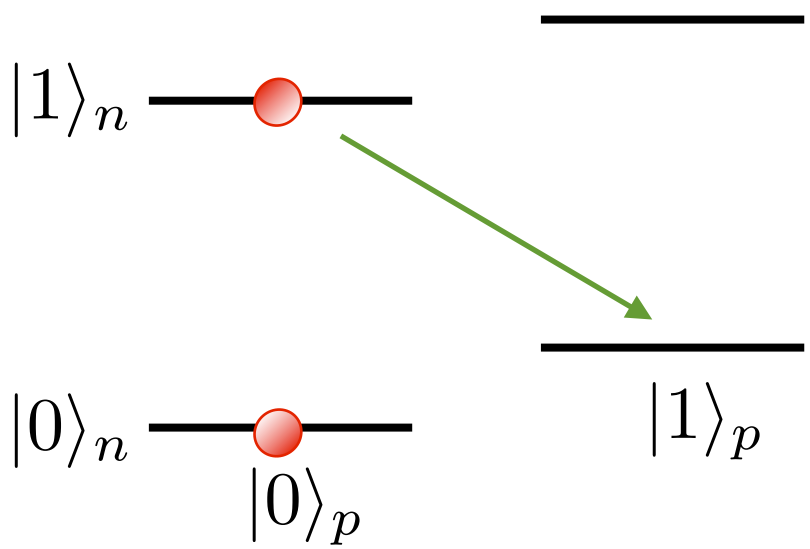
"Controlled Phase Gate"

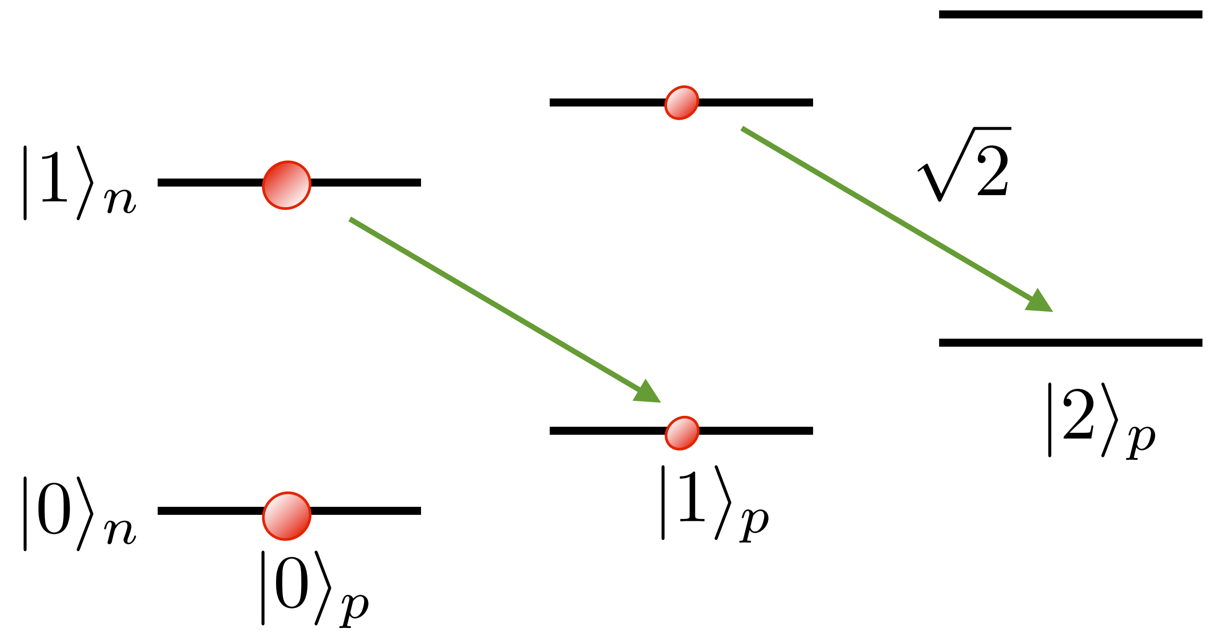
*J. I. Cirac and P. Zoller, PRL **74**, 4091 (1995)*

Geometric gates

“Thermal” gates

- *Resonant qubit phonon coupling:*





$$\langle a_p^\dagger a_p \rangle = 0$$

$$\langle a_p^\dagger a_p \rangle > 0$$

"Thermal" gates

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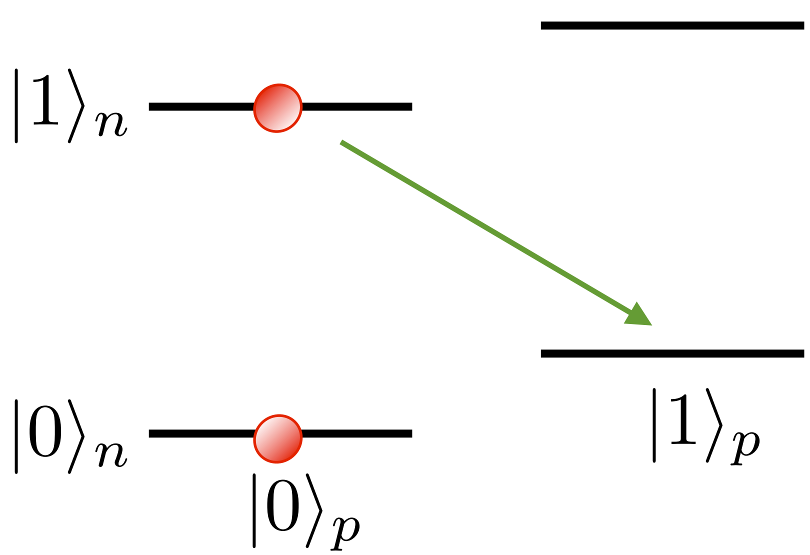

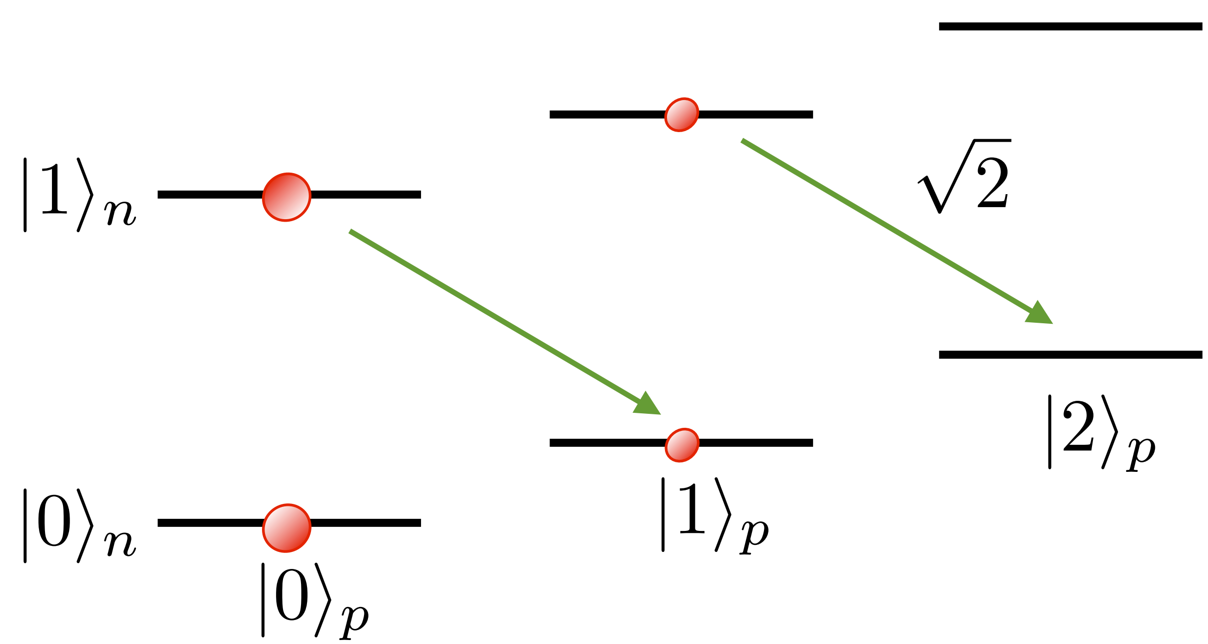

$$\langle a_p^\dagger a_p \rangle = 0 \qquad\qquad \langle a_p^\dagger a_p \rangle > 0$$

- *Off-resonant qubit phonon coupling:*

  - *"Mølmer-Sørensen gate"*
  - *"Geometric gate"*
  - *"Pushing gate / fast gate"*
  - *....*

  } *independent of initial phonon occupation !*

Driven harmonic oscillator (I)

Driven harmonic oscillator:

$$H(t) = \hbar\omega a^\dagger a - \hbar f(t)(a + a^\dagger)$$

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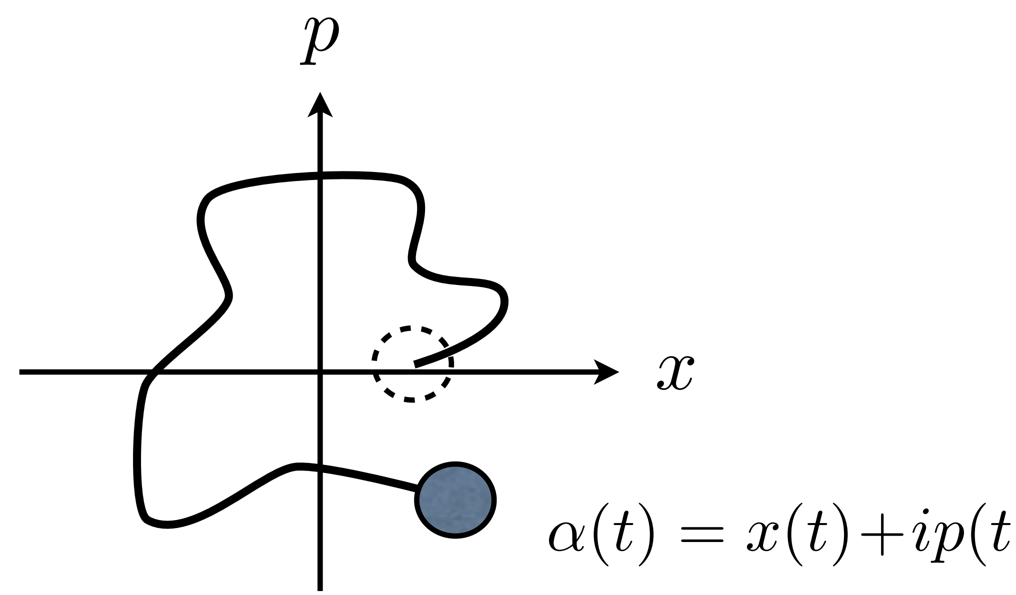
Schrödinger equation:

$$\left.\begin{aligned} |\dot\psi(t)\rangle &= -\frac{i}{\hbar} H(t)|\psi(t)\rangle \\ |\psi(0)\rangle &= |\alpha_0\rangle \end{aligned}\right\}$$



$$|\psi(t)\rangle = e^{i\phi(t)}|\alpha(t)\rangle$$

Driven harmonic oscillator (I)

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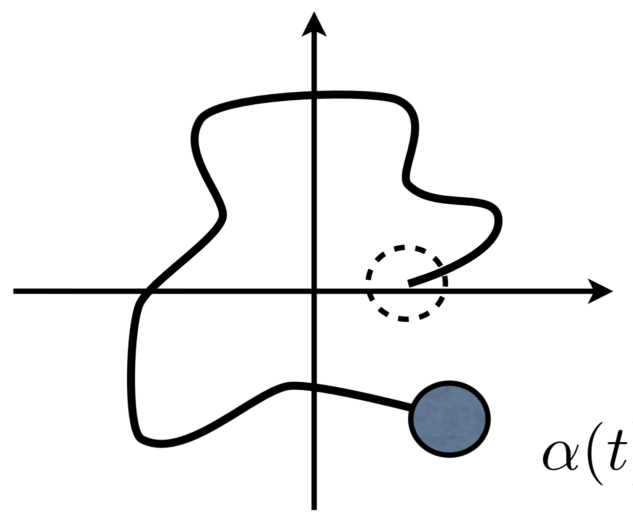

Schrödinger equation:

$$\left.\begin{aligned} |\dot{\psi}(t)\rangle &= -\frac{i}{\hbar} H(t)|\psi(t)\rangle \\ |\psi(0)\rangle &= |\alpha_0\rangle \end{aligned}\right\}$$

$$\boxed{|\psi(t)\rangle = e^{i\phi(t)}|\alpha(t)\rangle}$$

$$\frac{d\alpha}{dt} = -i\omega\alpha + if(t)$$

$$\frac{d\phi}{dt} = \frac{f(t)}{2}(\alpha + \alpha^*)$$

$$\xrightarrow[\text{rotating frame}]{\alpha(t) = \alpha_r(t)e^{-i\omega t}}$$

$$\frac{d\alpha_r}{dt} = ie^{i\omega t} f(t)$$

$$\frac{d\phi}{dt} = \mathrm{Im}\frac{d\alpha_r}{dt}\alpha_r^*$$

Driven harmonic oscillator (I)

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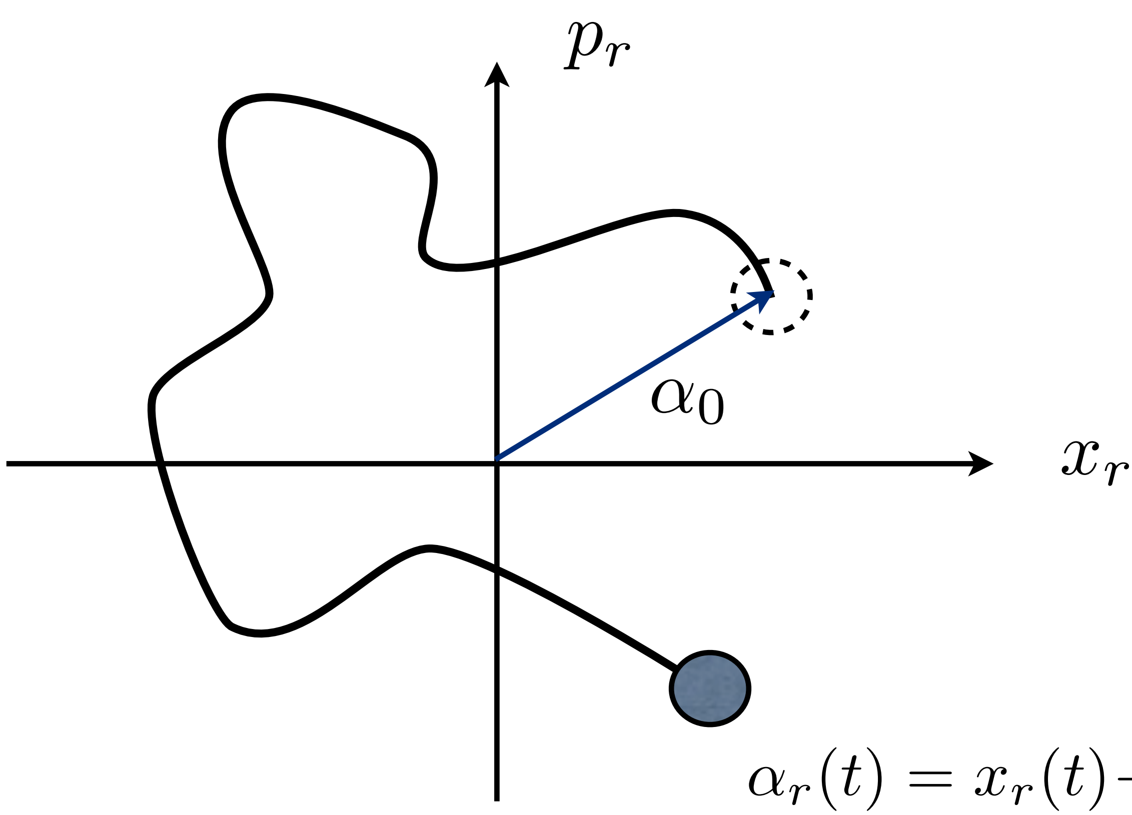
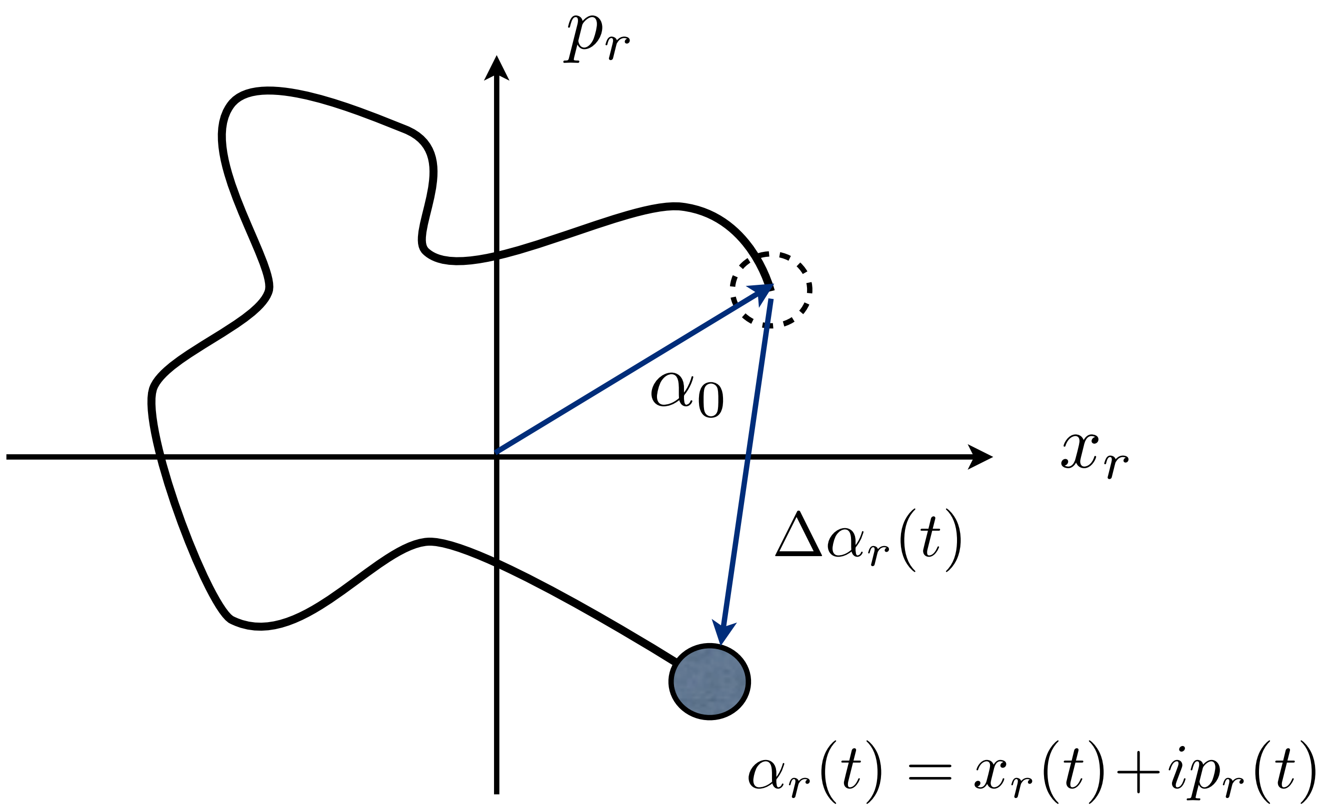


$\alpha_r(t) = x_r(t) + ip_r(t)$

Driven harmonic oscillator (I)

$$\frac{d\alpha_r}{dt} = ie^{i\omega t} f(t)$$

$$\frac{d\phi}{dt} = \mathrm{Im} \frac{d\alpha_r}{dt} \alpha_r^*$$



(I) $$\alpha_r(t) = \alpha_0 + i \int_0^t ds\, e^{i\omega s} f(s) \; = \; \alpha_0 + \Delta\alpha_r(t)$$

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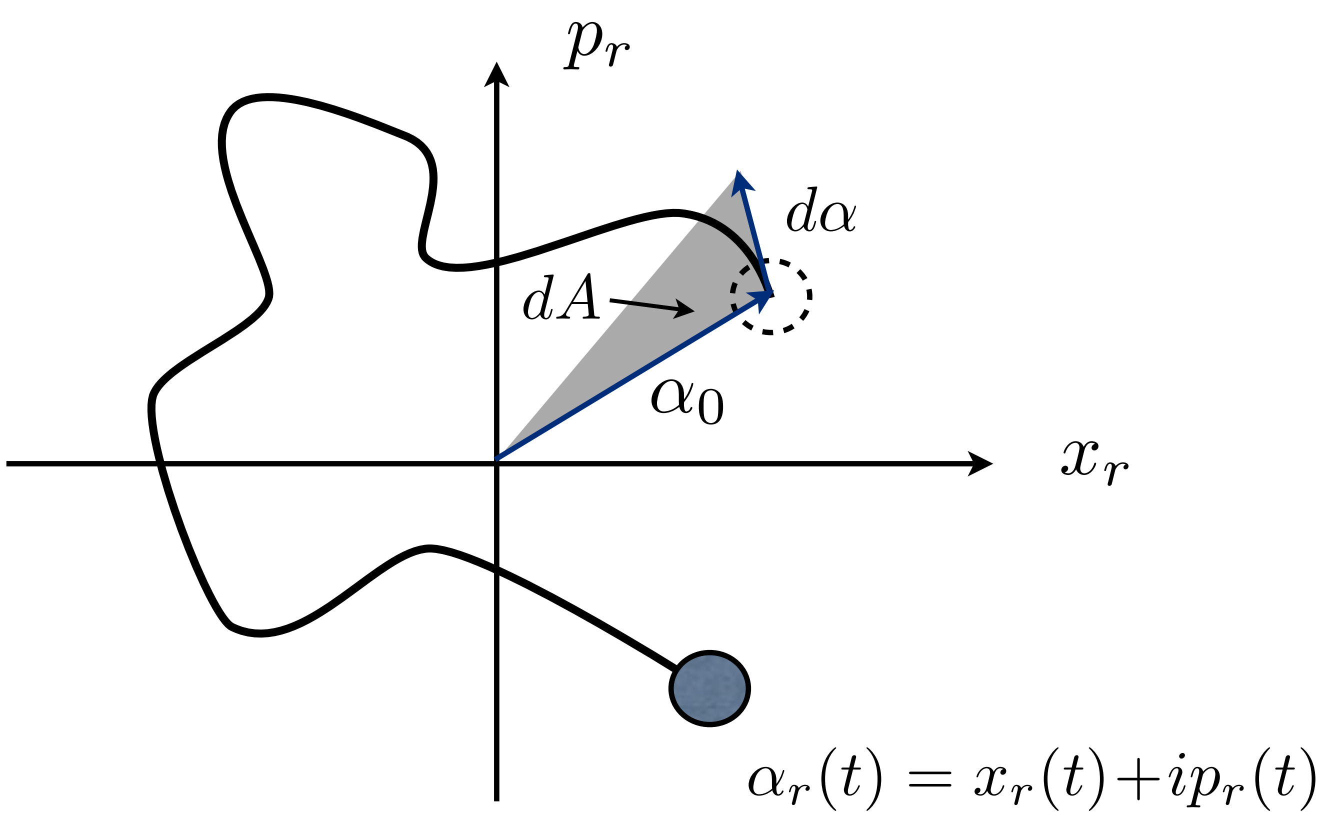

(I) $$\alpha_r(t) = \alpha_0 + i \int_0^t ds\, e^{i\omega s} f(s) \; = \; \alpha_0 + \Delta\alpha_r(t)$$

(II) $$\frac{d\phi}{dt} = \mathrm{Im} \frac{d\alpha_r}{dt} \alpha_r^* = 2\frac{dA}{dt}$$

Driven harmonic oscillator (I)

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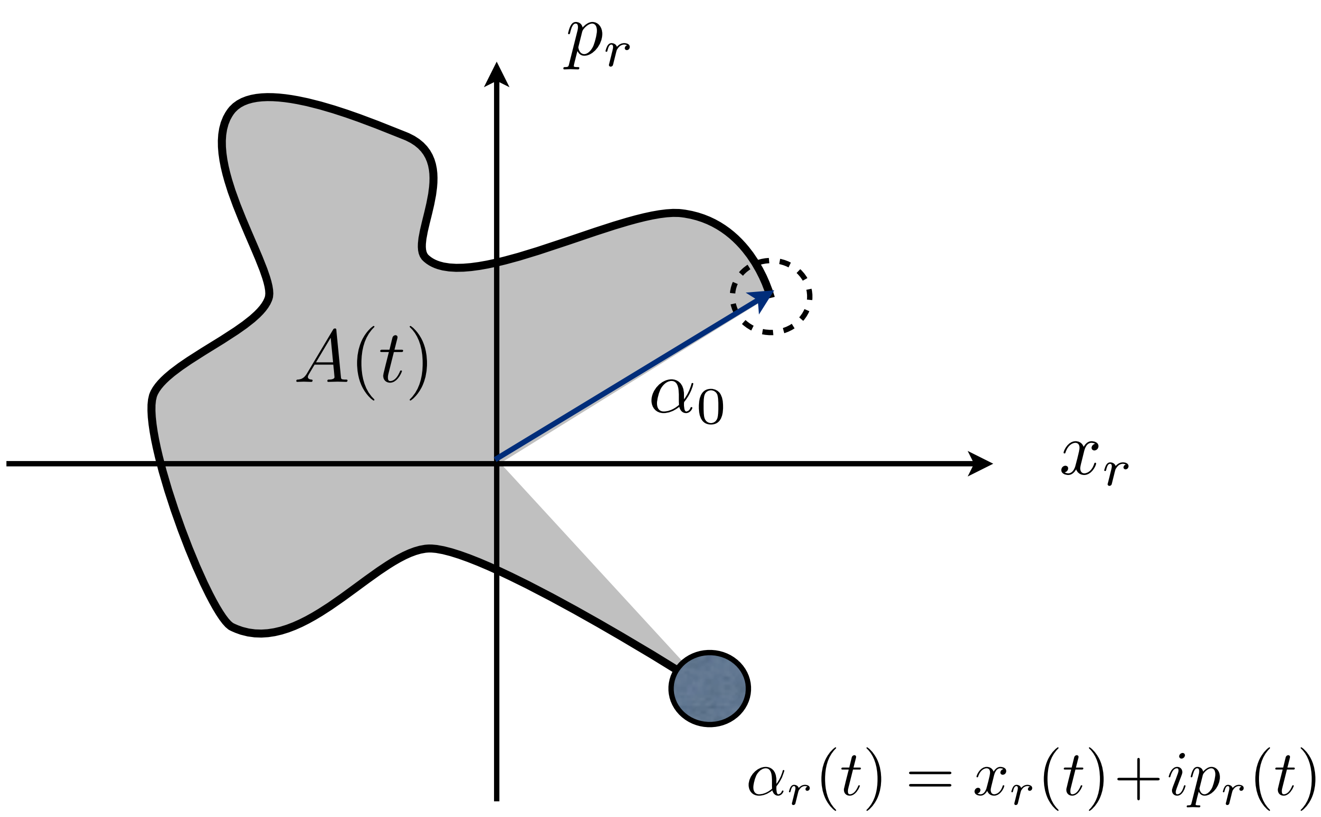
$$\frac{d\phi}{dt} = \mathrm{Im}\,\frac{d\alpha_r}{dt}\alpha_r^*$$



(I) $$\alpha_r(t) = \alpha_0 + i\int_0^t ds\, e^{i\omega s} f(s) \;=\; \alpha_0 + \Delta\alpha_r(t)$$

(II) $$\frac{d\phi}{dt} = \mathrm{Im}\,\frac{d\alpha_r}{dt}\alpha_r^* = 2\frac{dA}{dt} \qquad \Longrightarrow \qquad \phi(t) = 2A(t)$$

geometric phase !

Driven harmonic oscillator (II)

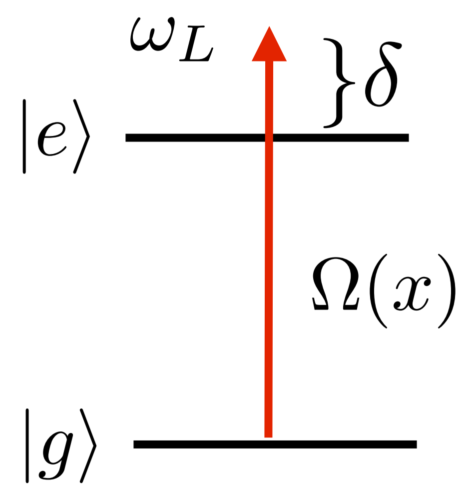
Spin-dependent driving force:

$$H(t) = \hbar\omega a^\dagger a - \hbar f(t)(a + a^\dagger)\sigma_z$$

Driven harmonic oscillator (II)

Spin-dependent driving force:

$$H(t) = \hbar\omega a^\dagger a - \hbar f(t)(a + a^\dagger)\sigma_z$$



AC-Stark effect:

$$H_{\mathrm{Stark}} = \frac{[\Omega(x)]^2}{4\delta}\sigma_z \approx \frac{\Omega^2(0)}{4\delta}\sigma_z + \frac{\nabla\Omega^2}{4\delta}(a + a^\dagger)\sigma_z$$

Driven harmonic oscillator (II)

Spin-dependent driving force:

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$$|\psi_i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|\alpha_0\rangle$$

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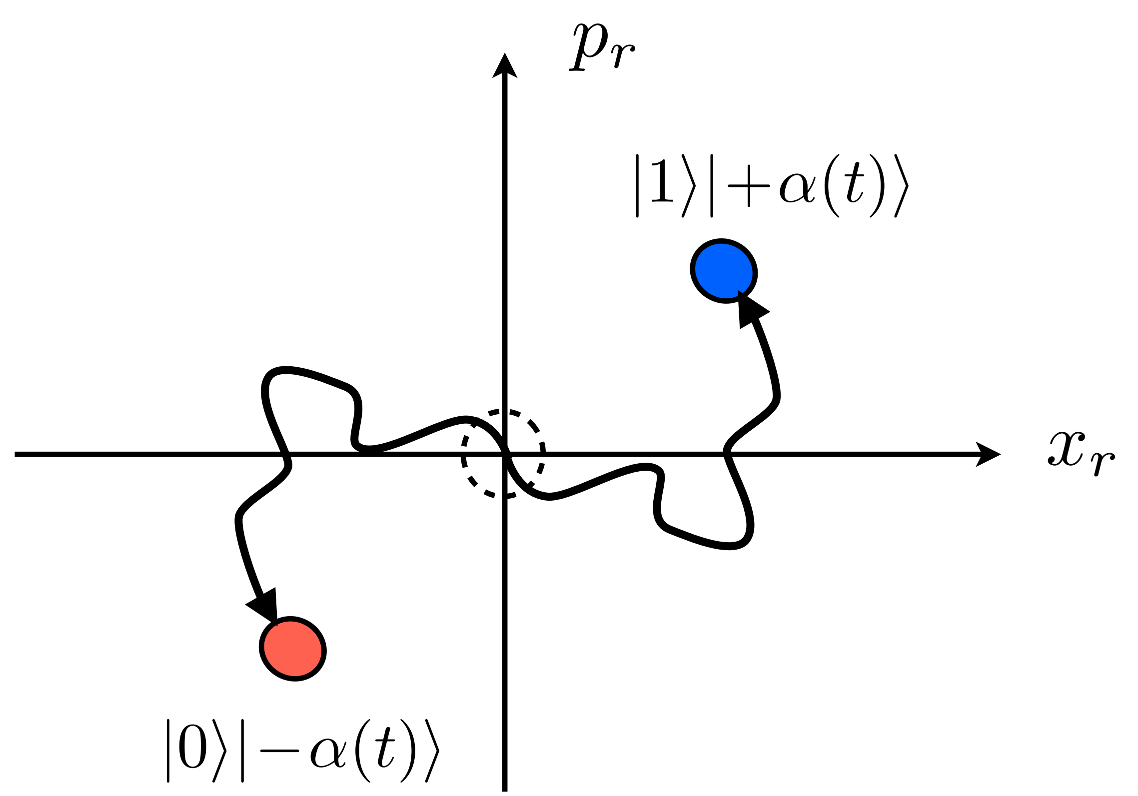

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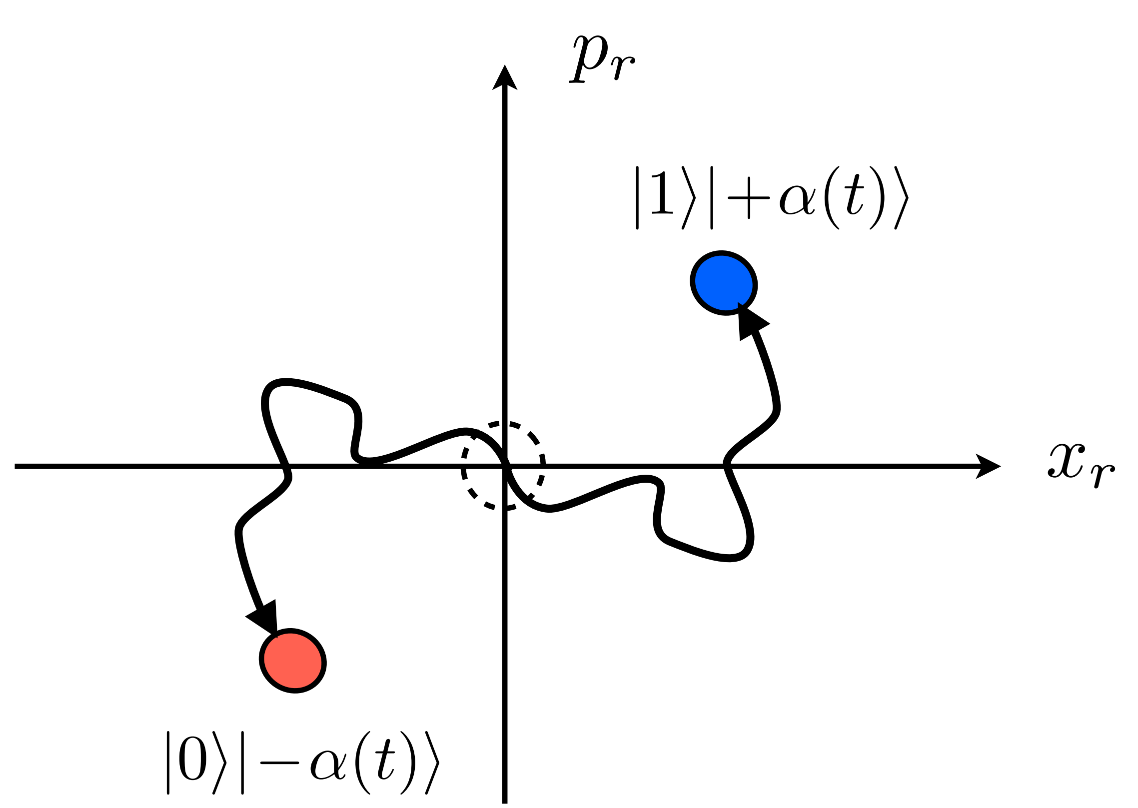

remark: $\phi(t) \sim f^2 \sim \sigma_z^2$ *(global phase)*

Driven harmonic oscillator (III)

Two-spin-dependent driving force:

$$H(t) = \hbar\omega a^\dagger a - \hbar f(t)(a + a^\dagger)(\sigma_z^1 + \sigma_z^2) \qquad (\Rightarrow \ [H(t), \sigma_z^i] = 0\,)$$

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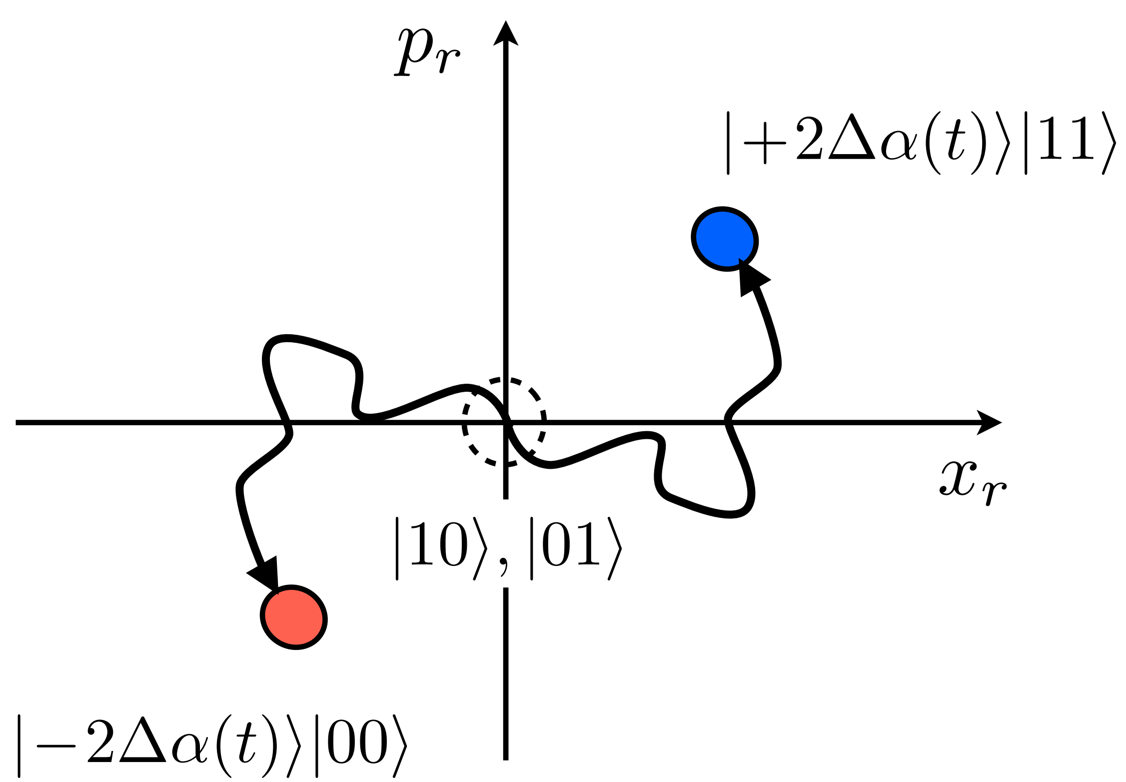

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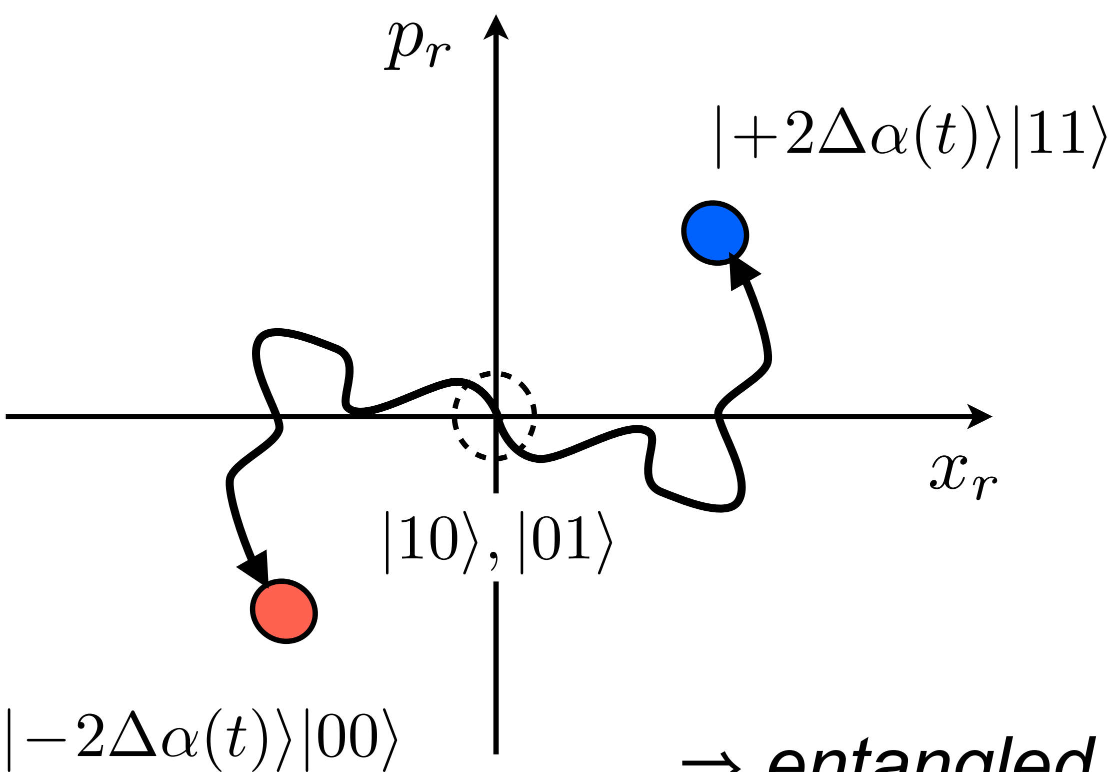

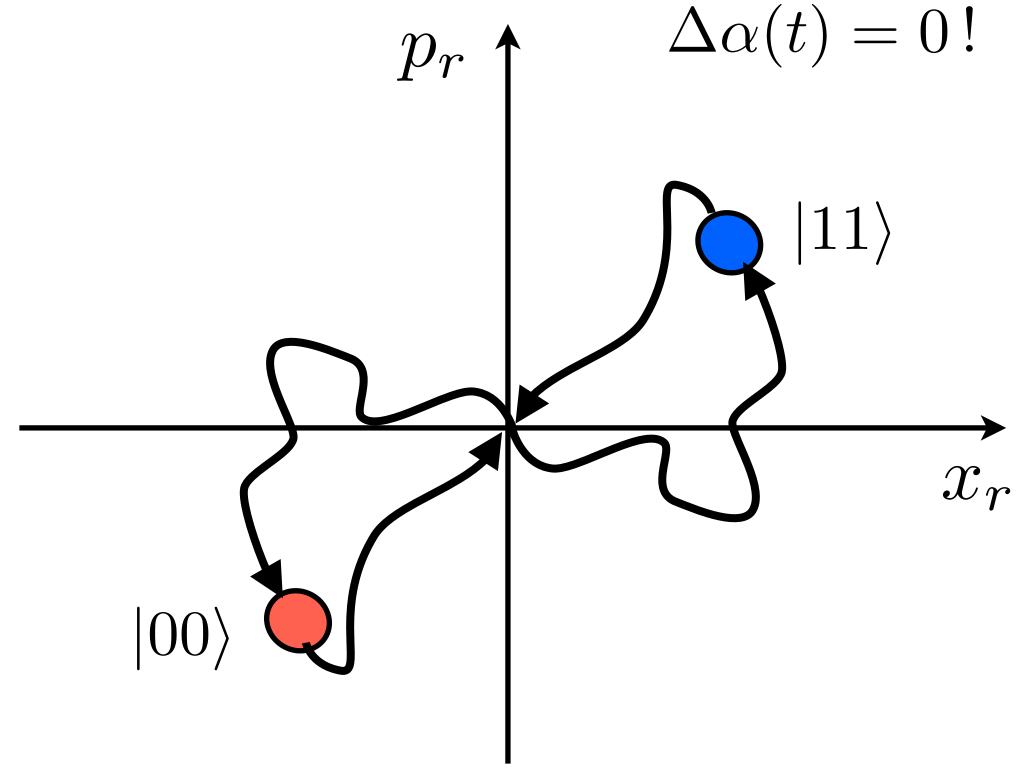

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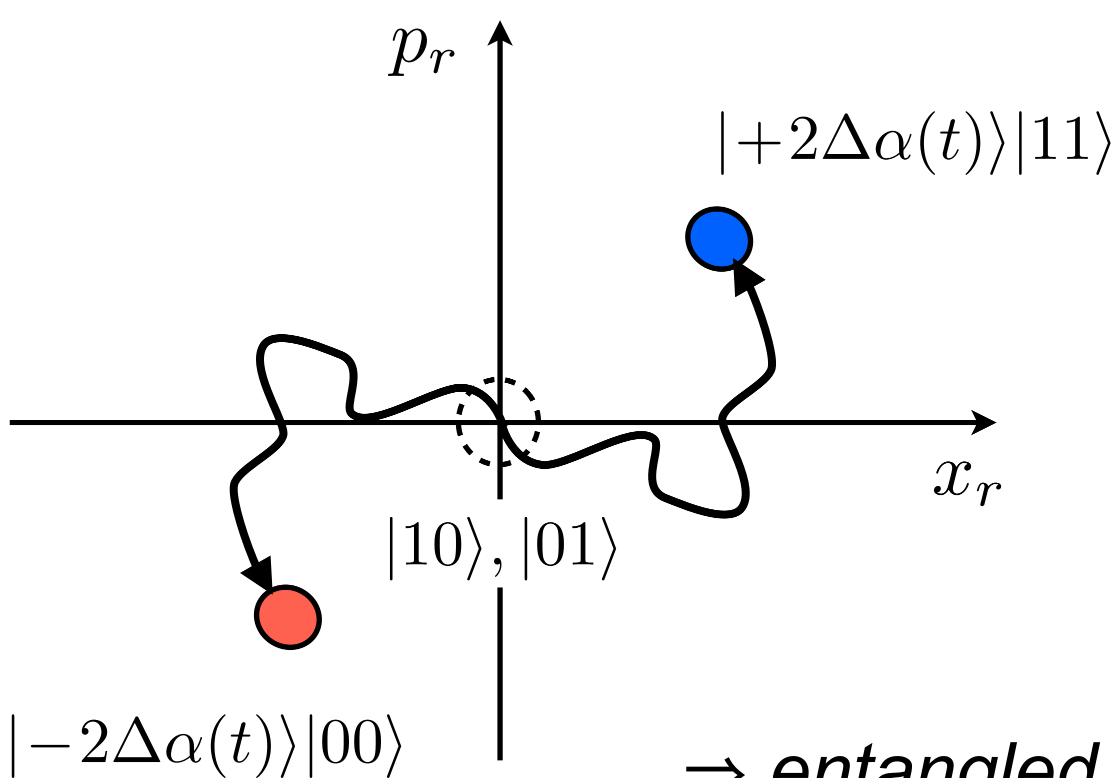

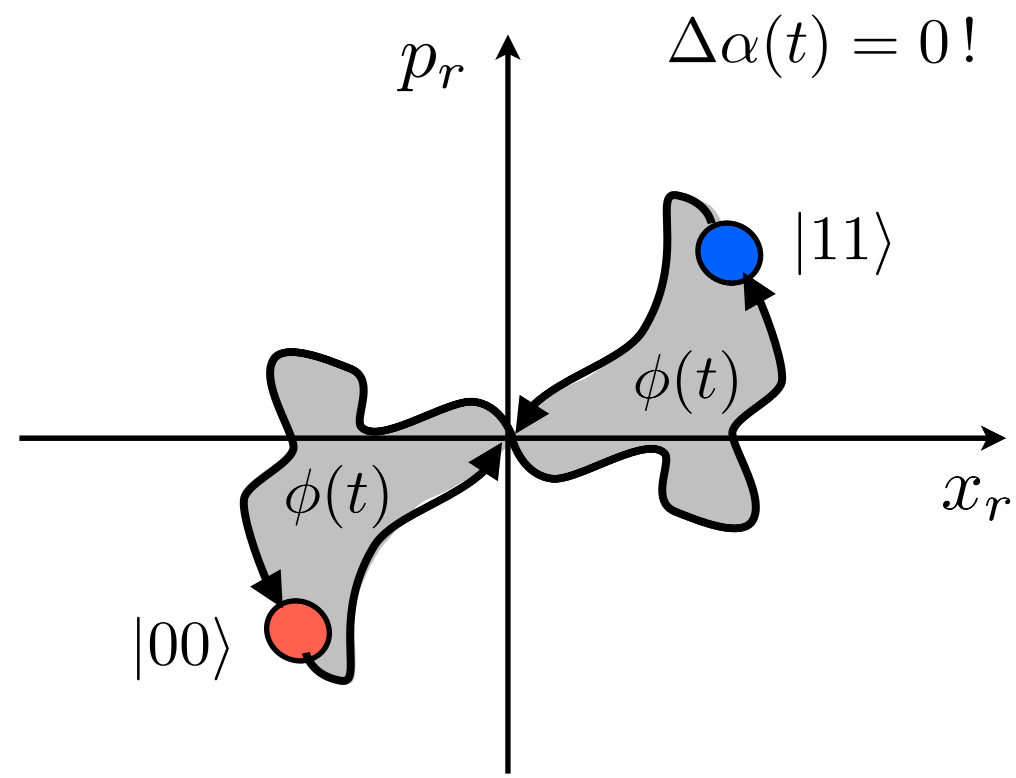

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Geometric gates

Two-spin-dependent driving force:

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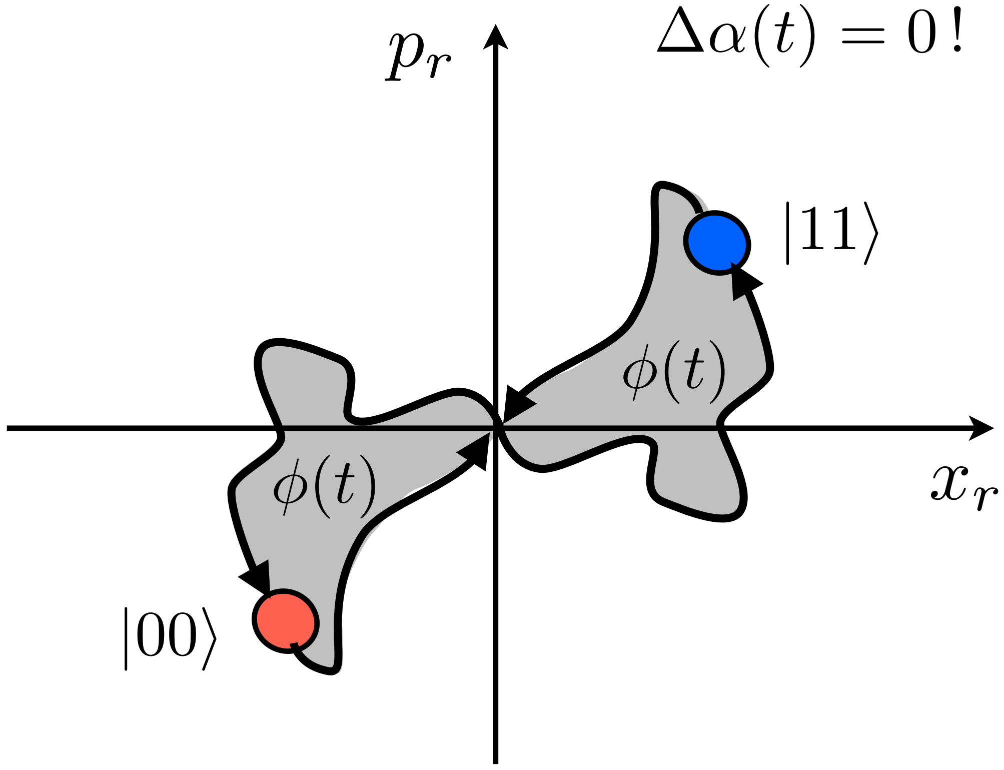
$$\rightarrow |\psi(t)\rangle = \underline{\left[e^{i\phi(t)(\sigma_z^1+\sigma_z^2)^2}|\sigma_z^1, \sigma_z^2\rangle\right]} \otimes |\alpha_0\rangle \longrightarrow$$

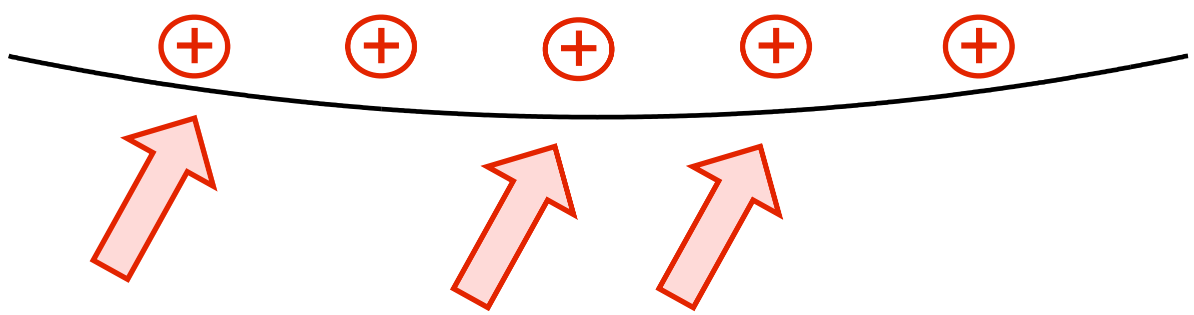
independent of motional state / temperature !!!

$$\boxed{\phi(t) \approx \frac{\pi}{2} \quad \Rightarrow \textit{two qubit gate !}}$$

$$U_{\mathrm{g}} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \equiv U_{\mathrm{PHASE}}$$



Generalized geometric gates



$$H(t) = \hbar \sum_n \omega_n a_n^\dagger a_n - \hbar \sum_i \sum_n f_{i,n}(t)(a_n + a_n^\dagger)\sigma_z^i$$

see e.g.: *J. J. García-Ripoll, P. Zoller, and J. I. Cirac, PRA **71**, 062309 (2005)*

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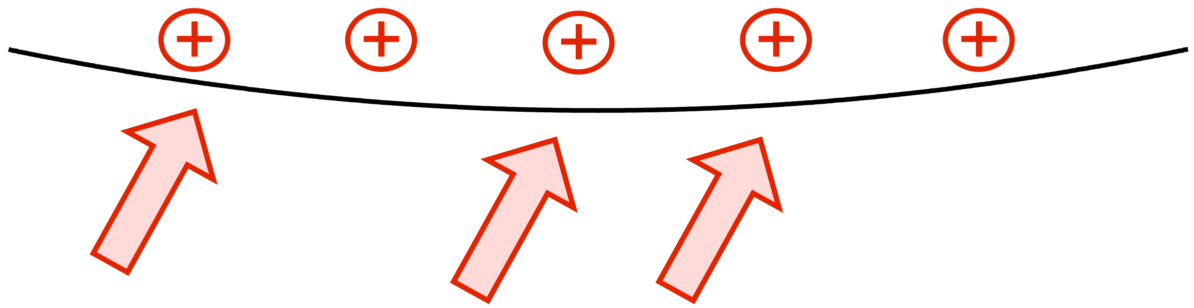


$$H(t) = \hbar \sum_n \omega_n a_n^\dagger a_n - \hbar \sum_i \sum_n f_{i,n}(t)(a_n + a_n^\dagger)\sigma_z^i$$

Disentangling conditions:

$$\Delta\alpha_n(t) = i \sum_i \int_0^t ds\, f_{i,n}(s) e^{i\omega_n s} = 0 \qquad \forall n$$

Effective spin dynamics:

$$U_{\text{spin}} = e^{i \sum_{i,j} \Phi_{ij}(t)\sigma_z^i \sigma_z^j}$$ $\Rightarrow$ *general Ising model*

see e.g.: *J. J. García-Ripoll, P. Zoller, and J. I. Cirac, PRA **71**, 062309 (2005)*

State of the art

- *gate errors:* $\sim 10^{-4} - 10^{-5}$ *(single qubit)*

  $\sim 10^{-2}$ *(two qubit)*

- *single qubit coherence times (clock states):* $\sim 10\,\mathrm{min}$

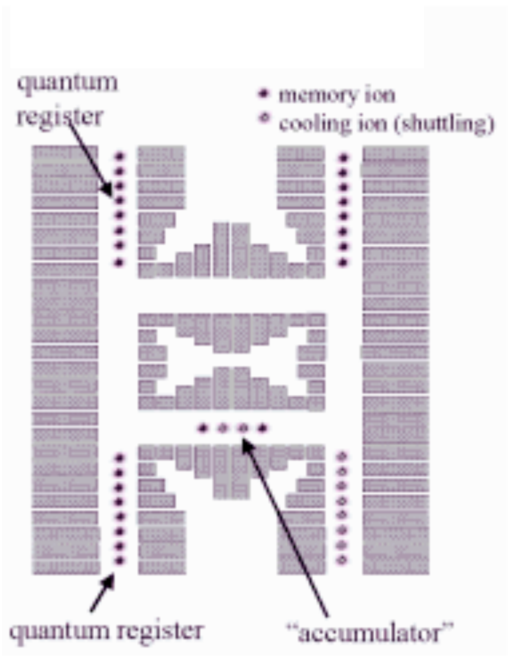
- *entanglement up to 14 ions, ~100 gates operations on 6 qubits*

- *demonstration of basic algorithms, quantum error correction, ..*

- *quantum simulation of spin model ~ 10 ions*

Challenge: scalability !

- *multi-zone traps*
- *microtraps*



Carrier, red & blue sideband transitions

$$H = \underbrace{\hbar\omega_t a^\dagger a - \hbar\delta|e\rangle\langle e|}_{H_0} + \frac{\hbar\Omega}{2}\left(\sigma_+ e^{i\eta(a+a^\dagger)} + \sigma_- e^{-i\eta(a+a^\dagger)}\right)$$

⇒ *Rotating frame:* $H \to U_I^\dagger H U_I + i\hbar\dot{U}_I^\dagger U_I, \qquad U_I = e^{-iH_0 t/\hbar}$

$$(\, U_I^\dagger \sigma_+ U_I = e^{-i\delta t}\sigma_+, \qquad U_I^\dagger a U_I = e^{-i\omega_t t} a \,)$$

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$$(\, U_I^\dagger \sigma_+ U_I = e^{-i\delta t}\sigma_+, \qquad U_I^\dagger a U_I = e^{-i\omega_t t} a \,)$$

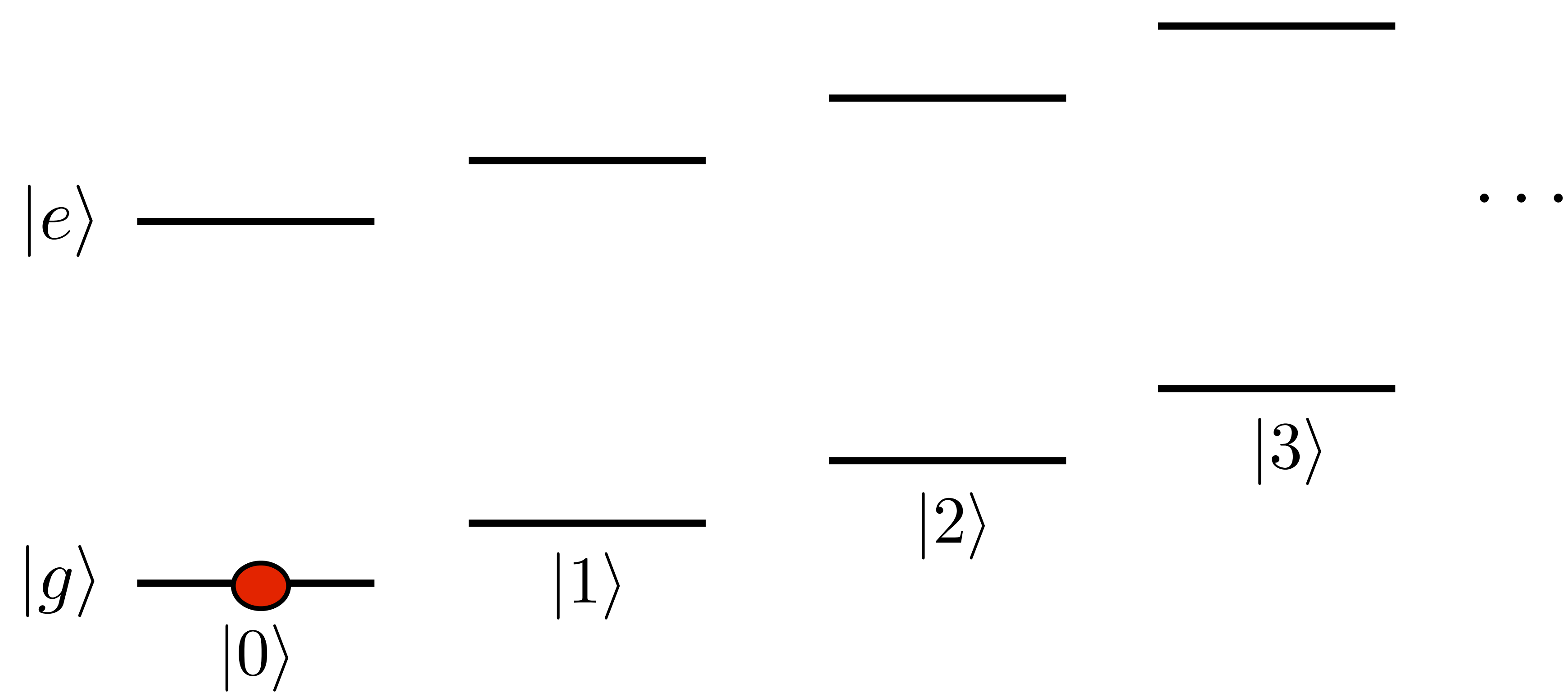
$$H(t) = \frac{\hbar\Omega}{2}\left(\sigma_+ e^{i\eta(a e^{-i\omega_t t} + a^\dagger e^{i\omega_t t})} e^{-i\delta t} + \text{H.c.}\right)$$

$$\simeq \frac{\hbar\Omega}{2}\left(\sigma_+ e^{\underline{-i\delta t}} + i\eta a\sigma_+ e^{-i\underline{(\delta+\omega_t)}t} + i\eta a^\dagger \sigma_+ e^{-i\underline{(\delta-\omega_t)}t} + \text{H.c.}\right)$$

$\Rightarrow$ *Rotating wave approximation*

Quantum state preparation

Quantum state preparation



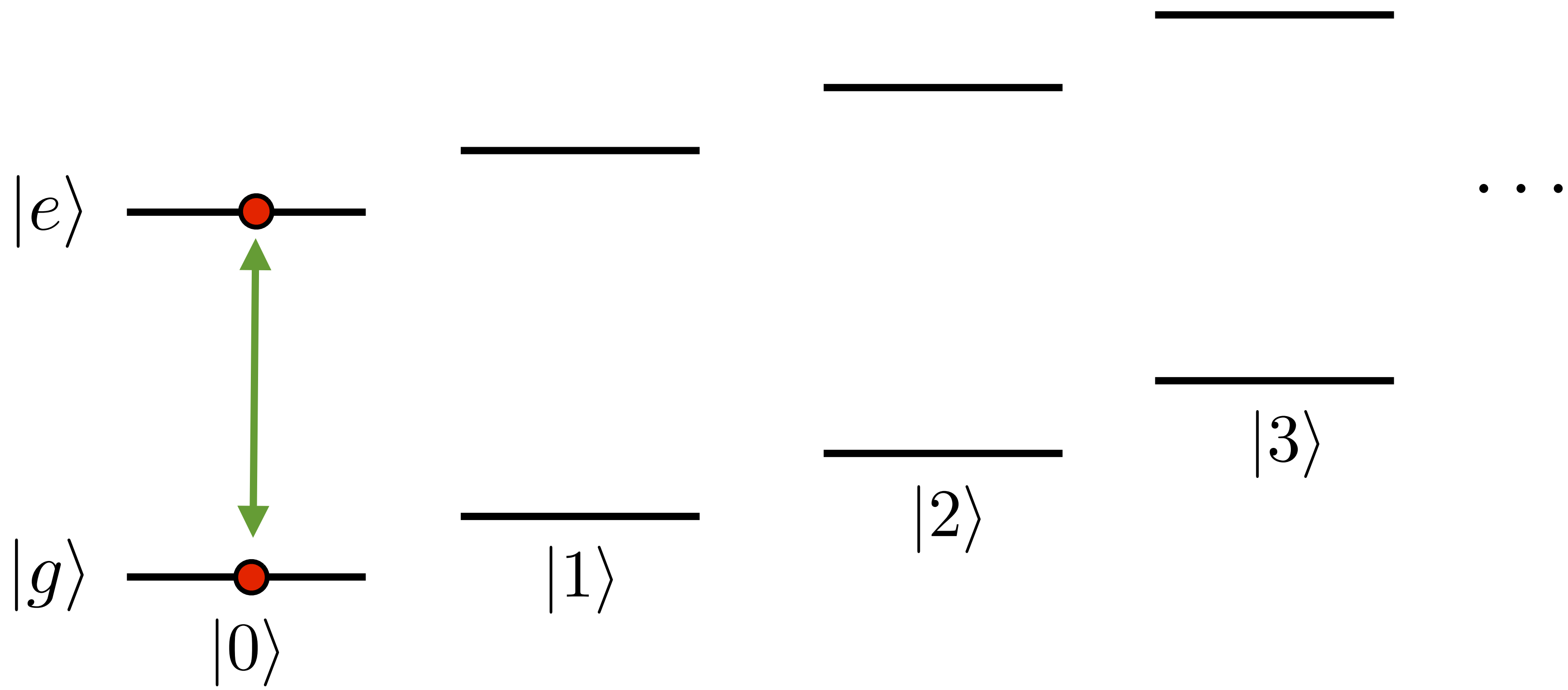
<u>Goal:</u>

$$|\psi\rangle_i = |0\rangle|g\rangle \quad \xrightarrow{\;\;U\;\;} \quad |\psi\rangle_f = \left(\sum_n c_n |n\rangle\right)|g\rangle$$

initial pure state
(e.g. laser cooling)

arbitrary motional
superposition state

Quantum state preparation



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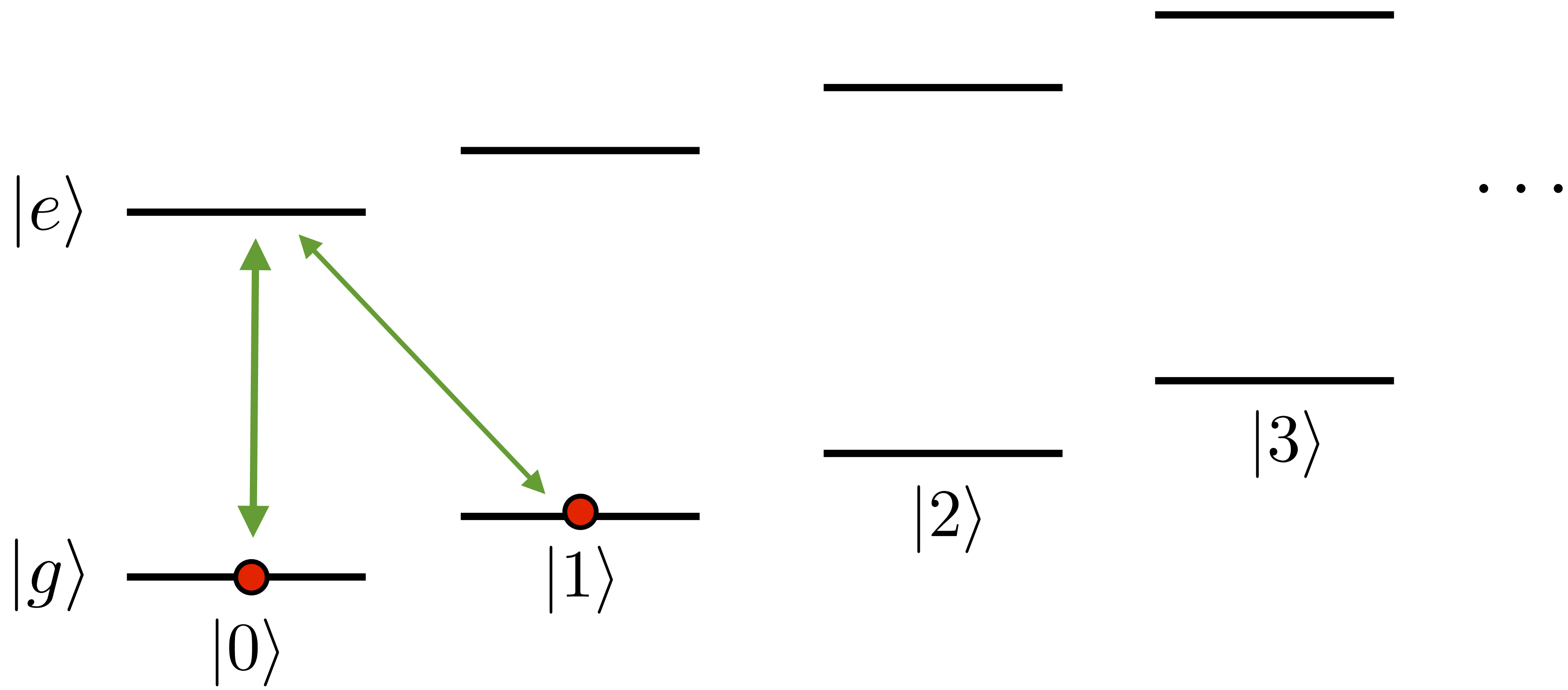

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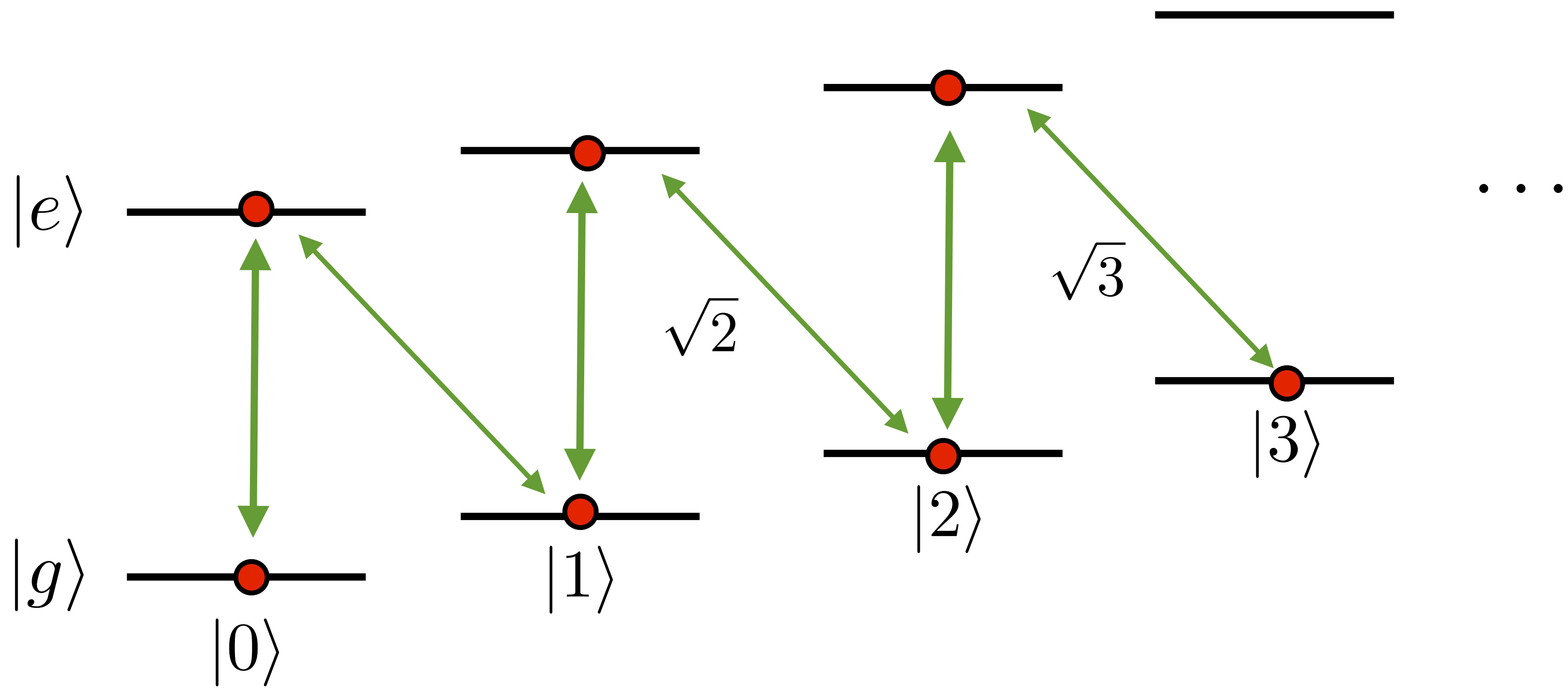

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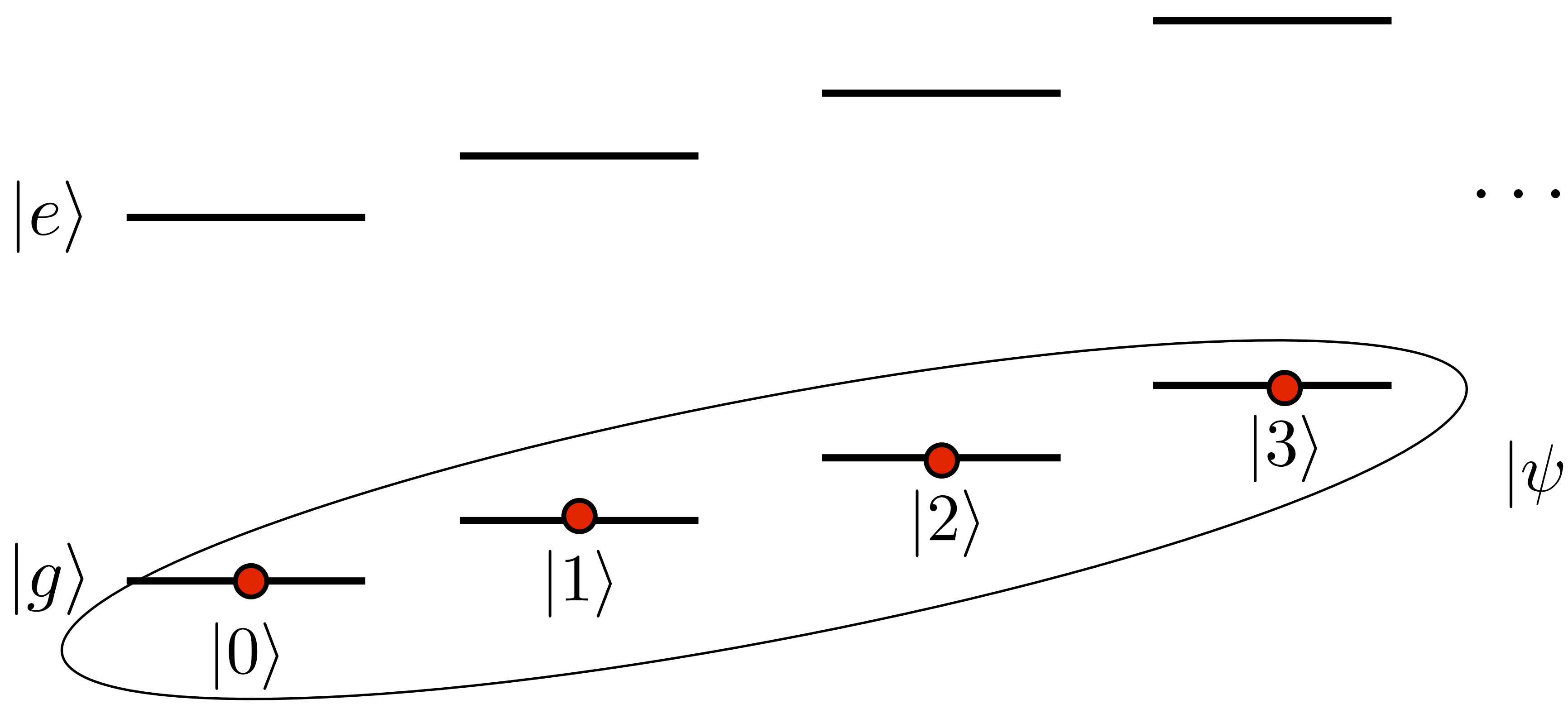


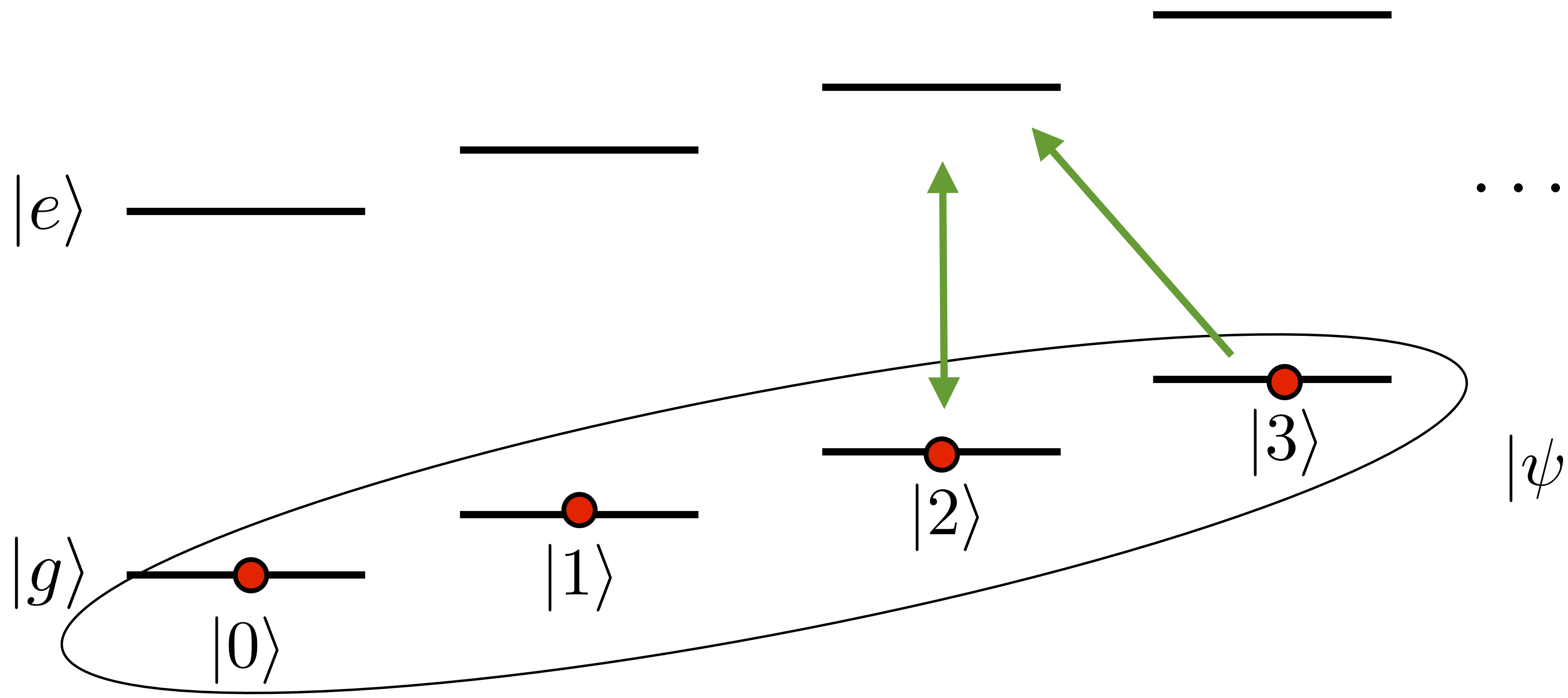
$$|\psi\rangle_m = \left(\sum_{n=0}^{M} c_n |n\rangle\right)$$

Idea (Law & Eberly, PRL (1996)):

$$|\psi\rangle_i = |0\rangle|g\rangle \xleftarrow{U^{-1}} |\psi\rangle_f = \left(\sum_{n=0}^{M} c_n |n\rangle\right)|g\rangle$$

inverse problem !

Quantum state preparation



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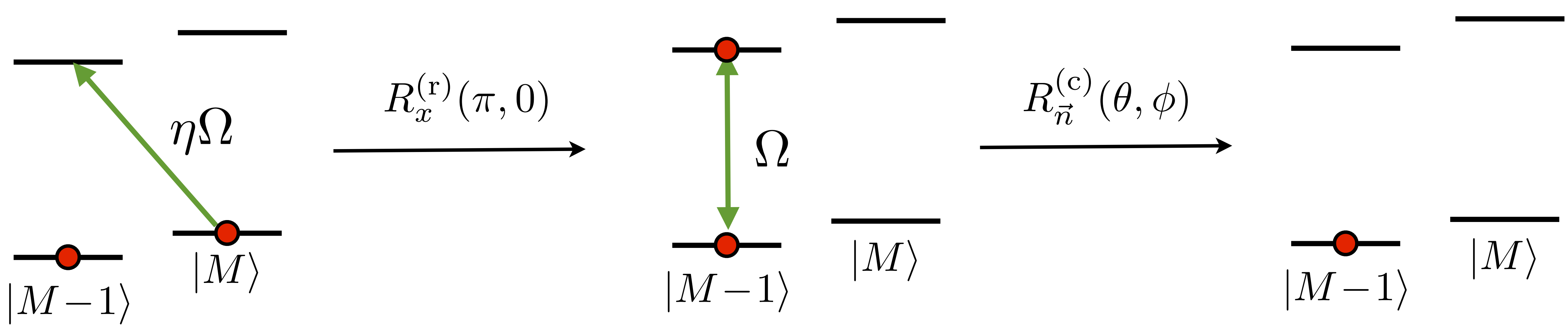
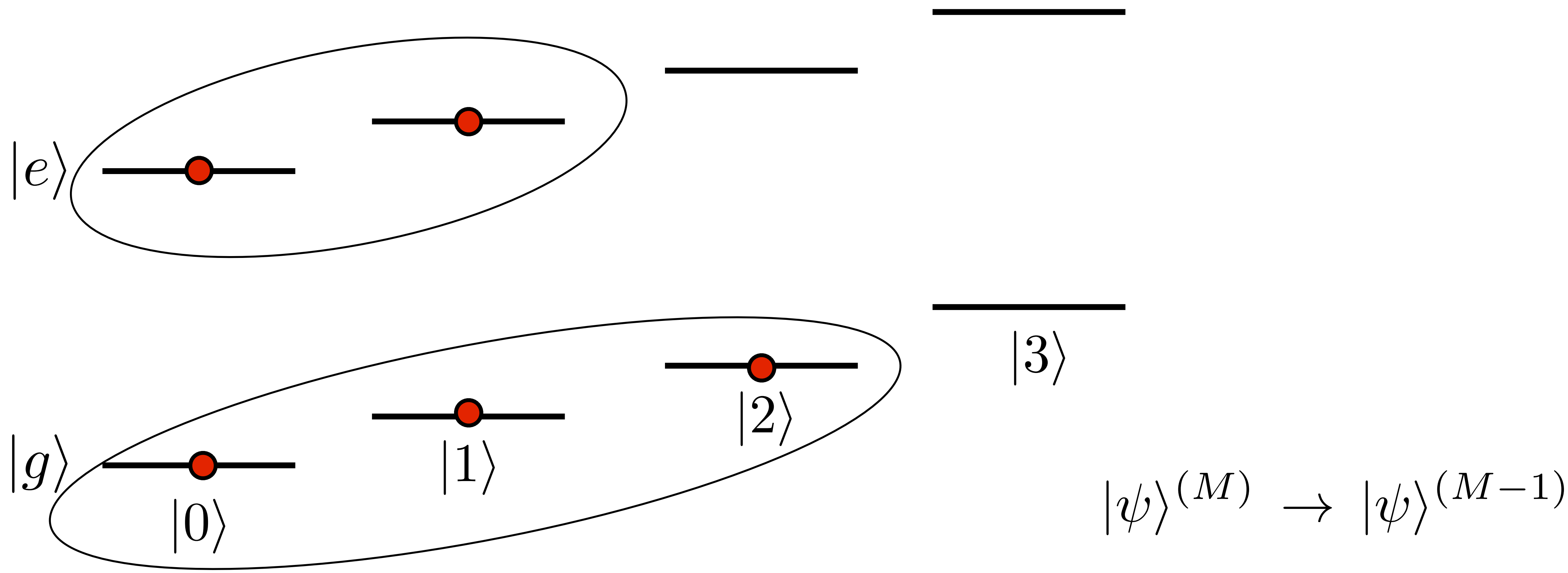
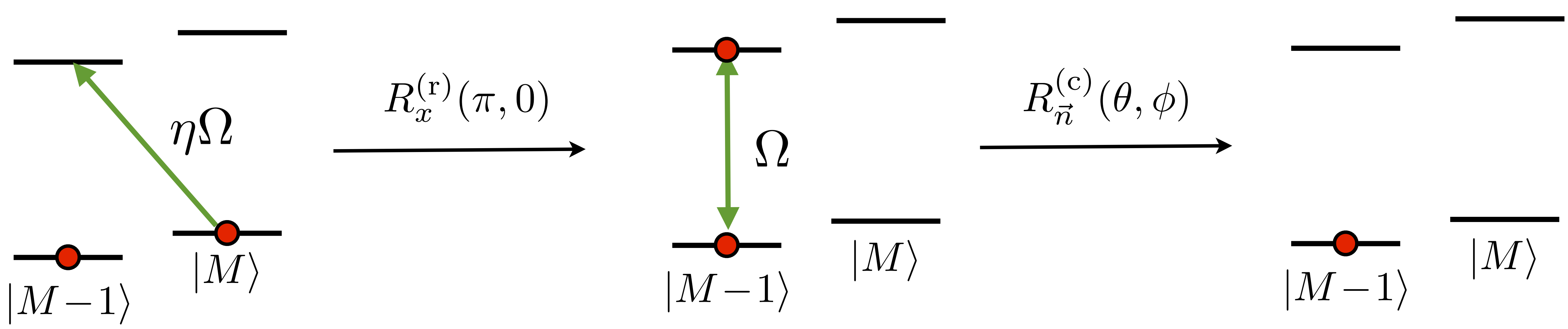


Quantum state preparation





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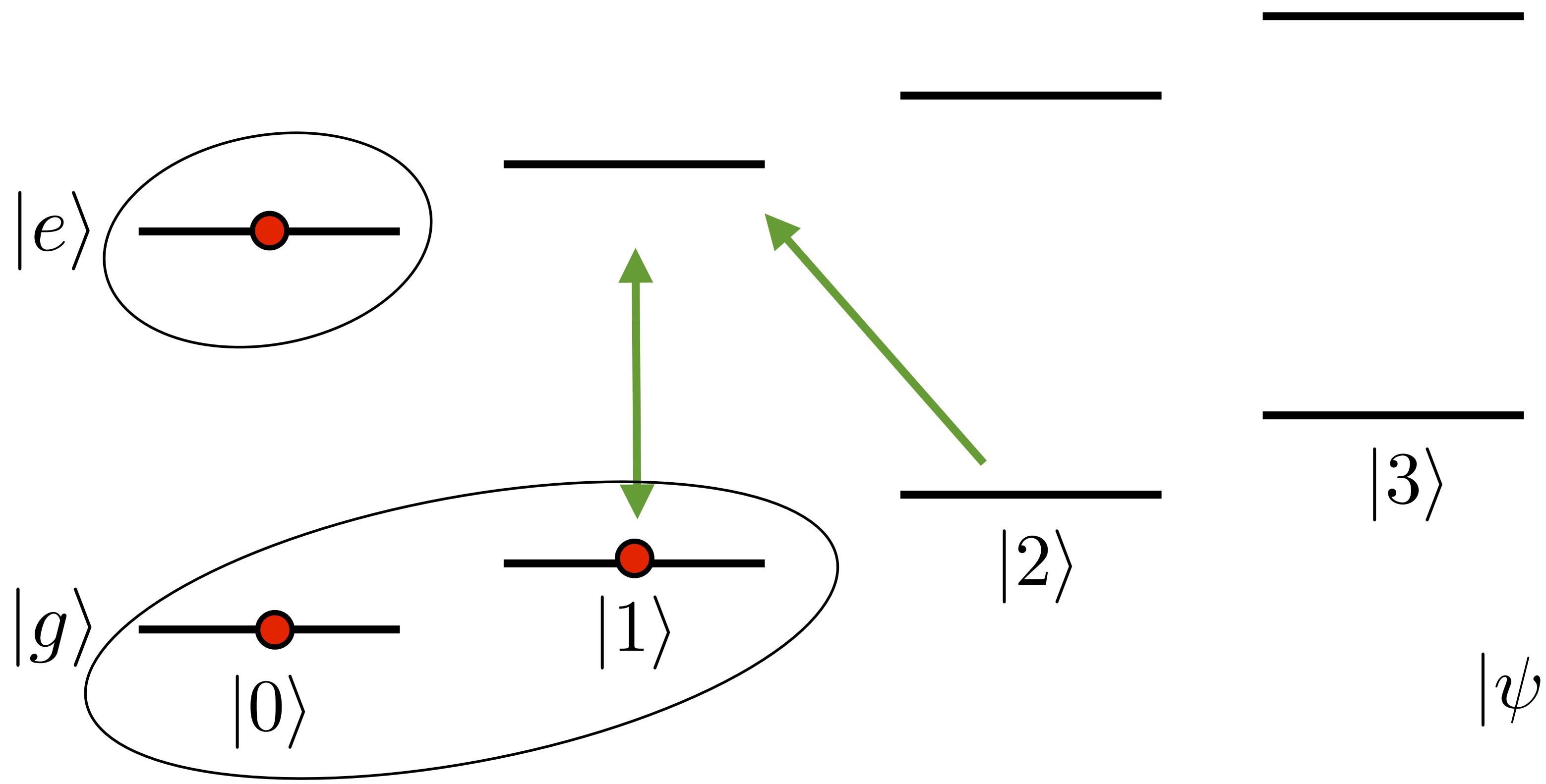
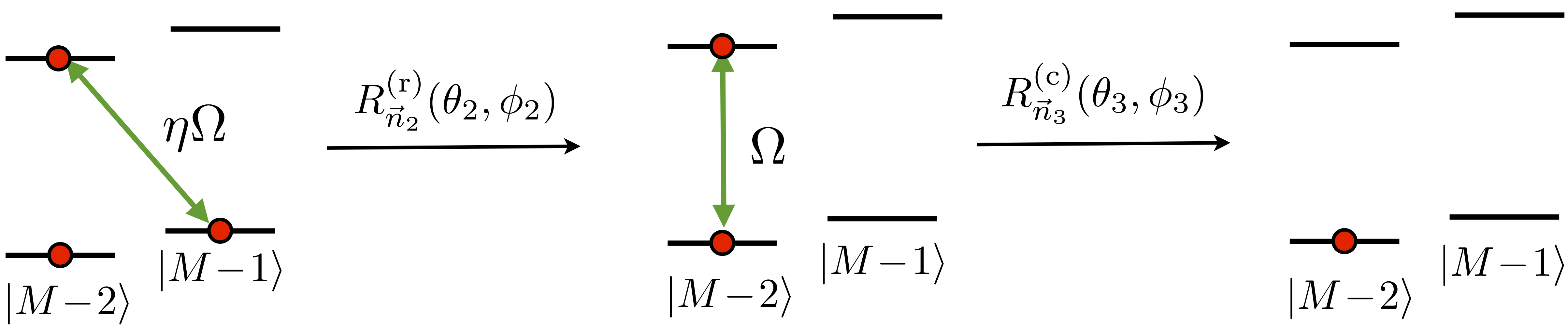


$$|\psi\rangle^{(M-1)} \to |\psi\rangle^{(M-2)}$$



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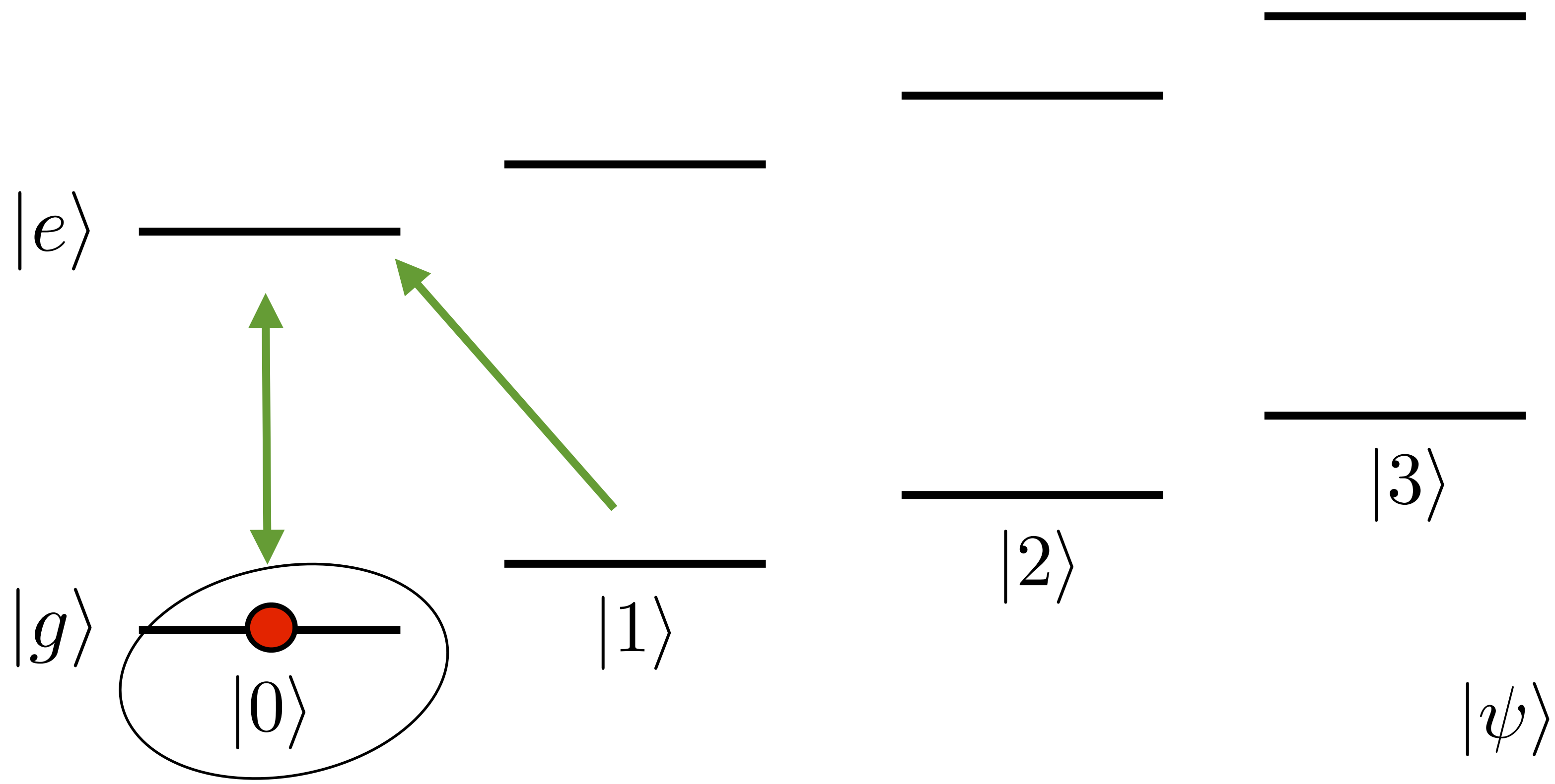


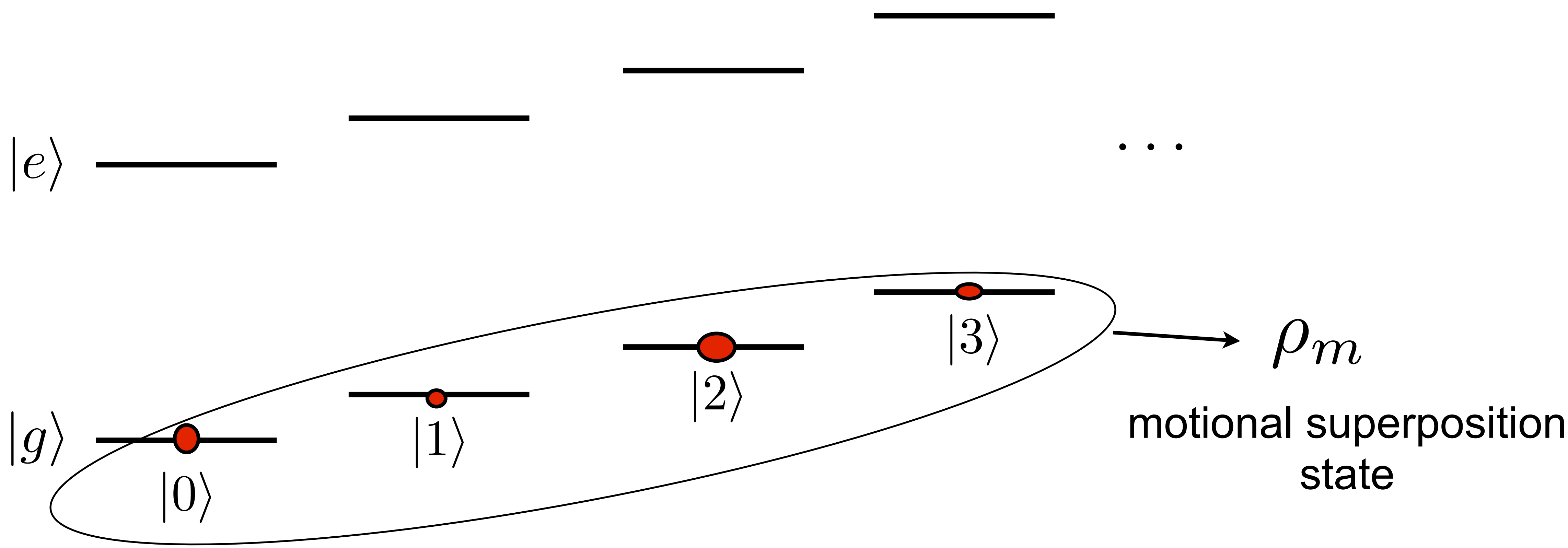
Result: Iterative construction of the inverse state preparation operation !

$$U^{-1} = \ldots R^{(c)}_{\vec{n}_3}(\theta_3, \phi_3) R^{(r)}_{\vec{n}_2}(\theta_2, \phi_2) R^{(c)}_{\vec{n}_1}(\theta_1, \phi_1) R^{(r)}_{x}(\pi, 0)$$

State preparation: $|\psi\rangle_i = |0\rangle|g\rangle \xrightarrow{(U^{-1})^{-1}} |\psi\rangle_f = \left(\sum_{n=0}^{M} c_n |n\rangle\right)|g\rangle$

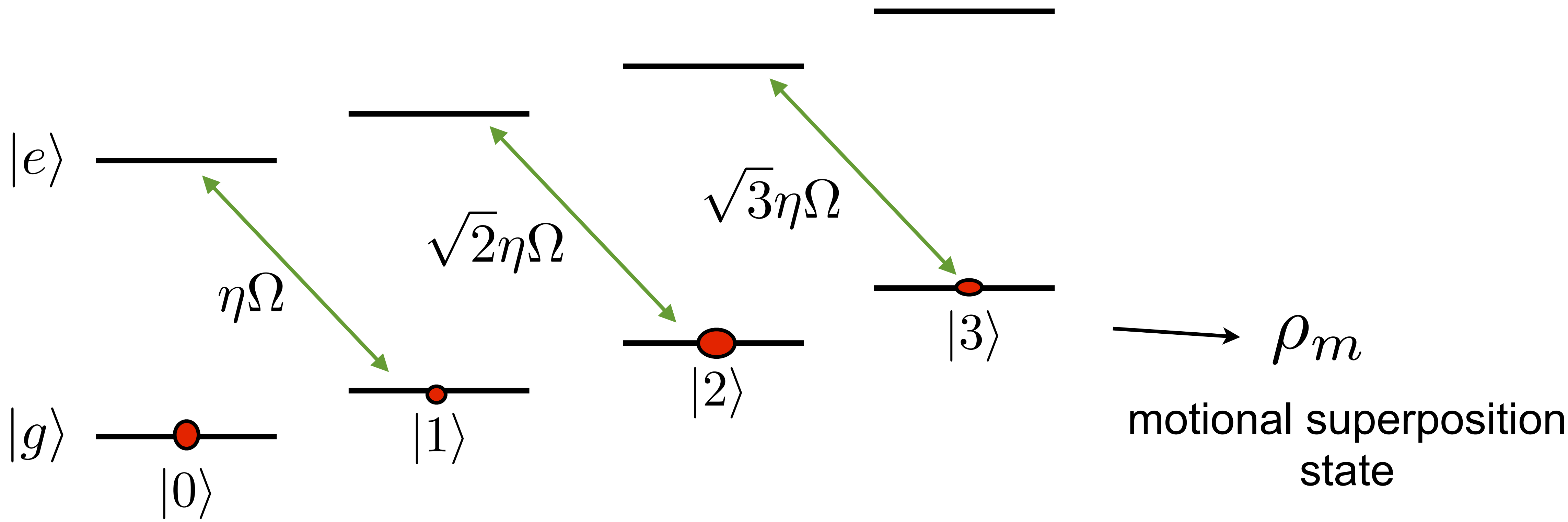
Quantum state tomography

Quantum state tomography



$\Rightarrow$ *How do I "measure"* ρ_m *?*

Step 1: Measuring populations



Apply red-sideband pulse & measure atomic population:

$$P_g(t) = \frac{1}{2}\left(1 + \sum_{n=0}^{\infty} P_n \cos\left(\sqrt{n}gt\right) e^{-\gamma_n t}\right)$$

$$P_n = \langle n|\rho_m|n\rangle$$

$$g = \eta\Omega$$

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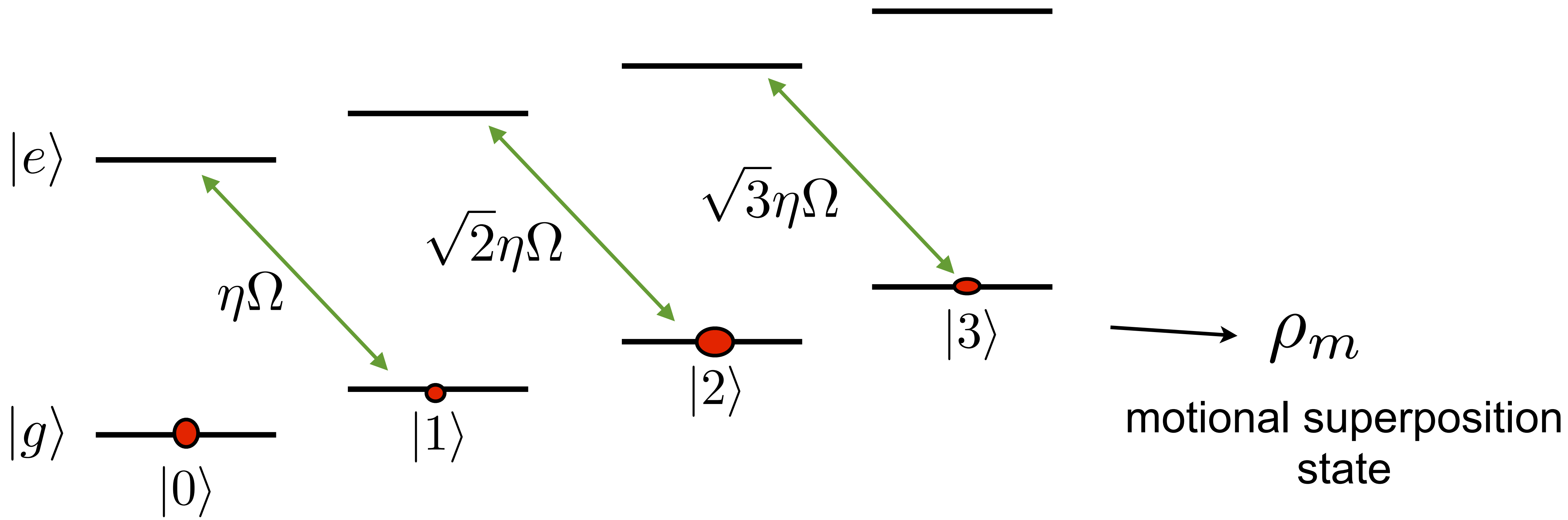

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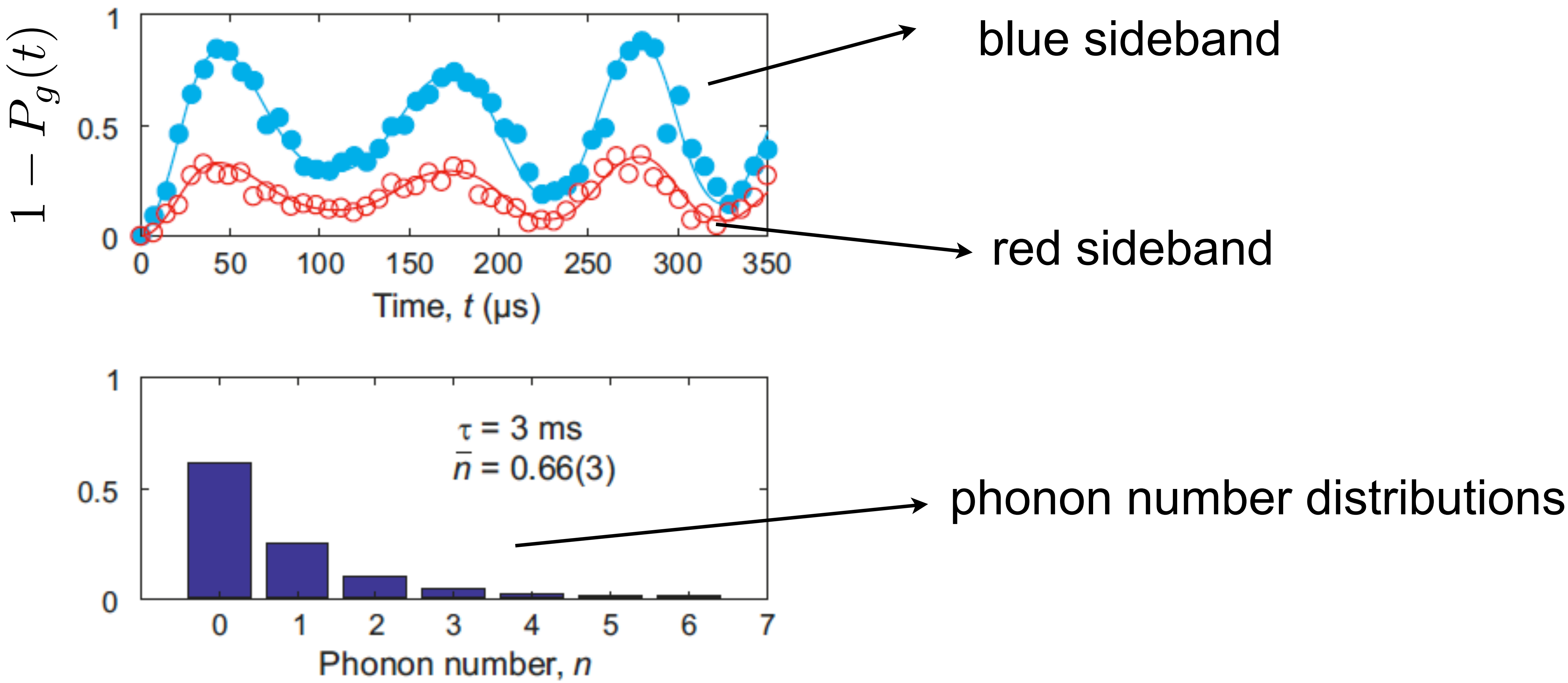

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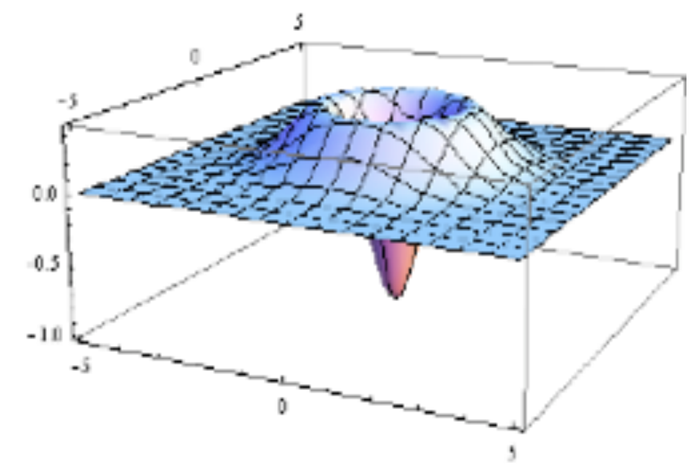
⇒ Extract P_n from Fourier components of $P_g(t)$!

Step 2: Quantum state tomography

Wigner function:

$$W(\alpha) = \frac{1}{\pi^2} \int d^2\beta e^{-i(\alpha\beta^* + \beta\alpha^*)} \mathrm{Tr}\{e^{i(\beta a^\dagger + \beta^* a)} \rho_m\}$$



(Complete information about density operator!)

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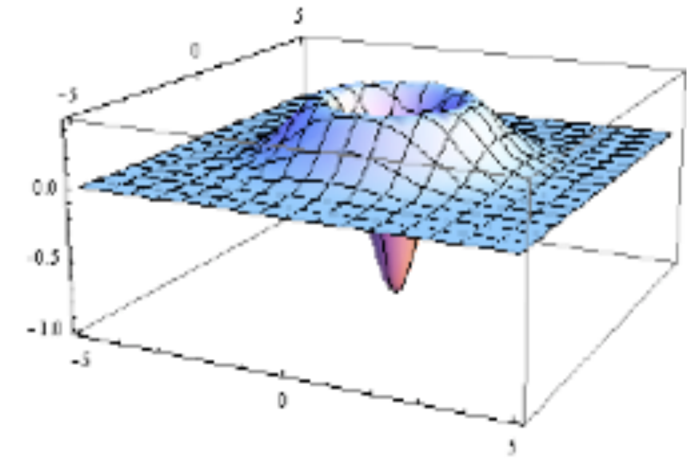

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(Complete information about density operator!)

Identity [1]: $$W(\alpha) = \frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^n P_n(\alpha)$$

$$P_n(\alpha) = \mathrm{Tr}\{\mathcal{D}^\dagger(\alpha)\rho\mathcal{D}(\alpha)\}$$ ... populations of oscillator state displaced by $-\alpha$

[1] e.g.: W. Schleich, "Quantum Optics in Phase Space"

Quantum state preparation & tomography

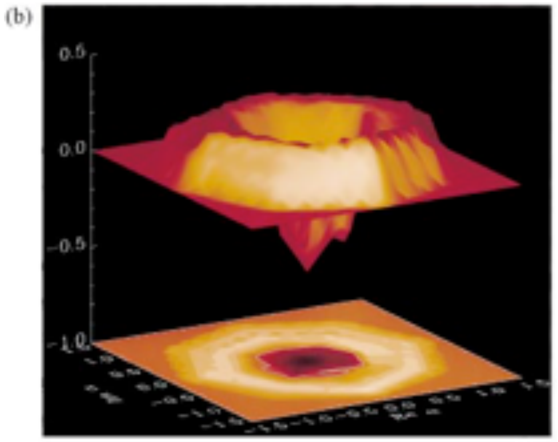
1. Prepare arbitrary motional superposition state ρ_m .

2. Displace the state of the ion by $-\alpha$.

3. Measure populations $P_n(\alpha)$.

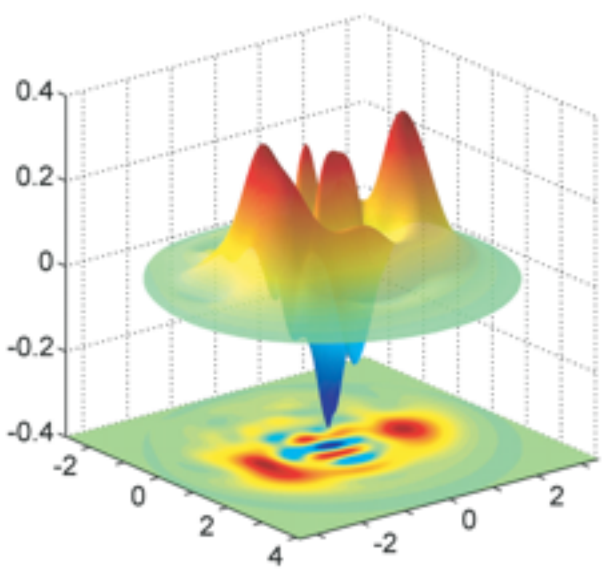
4. Calculate value of $W(\alpha)$ according to $W(\alpha) = \frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^n P_n(\alpha)$.

Quantum state tomography



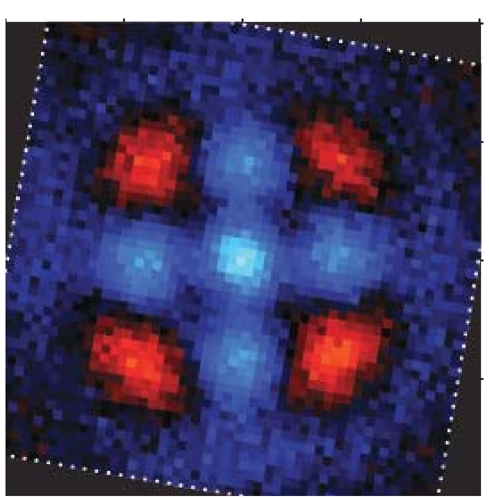
- *Reconstructed Wigner function of a Fock state n=1 of a trapped ion.*

  *D. Leibfried et al., PRL **77,** 4281 (1996)*



- *Reconstructed Wigner function of a Schrödinger cat state of a microwave cavity.*

  *S. Deléglise et al., Nature **455**, 510 (2008)*



- *Reconstructed Wigner function of the state* $|0\rangle + |4\rangle$ *of a superconducting LC circuit.*

  *M. Hofheinz et al., Nature **459**, 546 (2009)*