Midterm homework problems

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1 Divergence and Laplacian

1. We have the definition of Christoffel symbols:

$$\Gamma^k_{ij} = \frac{\partial \mathbf{e}_i}{\partial x^j} \cdot \mathbf{e}^k$$

Then we have that:

$$\boldsymbol{\nabla} \cdot \mathbf{V} = \partial_i (V^j \mathbf{e_j})^i = \frac{\partial V^i}{\partial x^i} + \Gamma^i_{ij} V^j = V^i_{,i} + \frac{1}{2} g^{im}(g_{mi,j} + g_{mj,i} - g_{ij,m}) V^j$$

Then:

...

2. Since the determinant is an invariant scalar of the matrix then from the relation: $g^{\mu\nu} = g^{-1} c^{\mu\nu}$ we know that c transforms in the exact same way as g does. Since g is a tensor then so is c.

3. We have that:

$$g = \sum_\nu g_{\mu\nu} c^{\mu\nu} \text{ hence } \frac{\partial g}{\partial g_{\mu\nu}} = \frac{\partial}{\partial g_{\mu\nu}} \sum_{\nu'} g_{\mu\nu'} c^{\mu\nu'} = c^{\mu\nu}$$

4. We have that:

$$g^{\mu\nu} g_{\mu\nu,\gamma} = \partial_\gamma \log g$$

We have that:

$$\partial_\gamma g(g_{\mu\nu}) = (\partial_\gamma g) g_{\mu\nu} + g \partial_\gamma g_{\mu\nu}$$

We have that:

$$\partial_\gamma g = \frac{\partial}{\partial g_{\mu\nu}} \frac{\partial g_{\mu\nu}}{\partial \gamma} g = \frac{\partial}{\partial g_{\mu\nu}} g_{\mu\nu,\gamma} g = g_{\mu\nu,\gamma} c^{\mu\nu}$$

Hence:

...

5. We start from the end and we differentiate to obtain:

$$\frac{1}{\sqrt{|g|}} \partial_\gamma (\sqrt{|g|} V^\gamma) = V^\gamma{}_{,\gamma} + \frac{1}{\sqrt{|g|}} V^\gamma \frac{1}{2\sqrt{|g|}} \partial_\gamma |g| = V^\mu{}_{,\mu} + \frac{1}{2} V^\gamma \frac{\partial_\gamma |g|}{|g|} = V^\mu{}_{,\mu} + \frac{1}{2} V^\gamma \log |g|$$

Now using question 4 we re-obtain the formula of question 1 and this concludes the proof.

6. Using the above formula by replacing: $V_\gamma = f_{,\gamma}$ (hence $V^\gamma = g^{\gamma\mu} f_{,\mu}$) we obtain:

$$\nabla^2 f = \frac{1}{\sqrt{|g|}} \partial_\gamma (\sqrt{|g|} f^{,\gamma}) = \frac{1}{\sqrt{|g|}} \partial_\gamma (\sqrt{|g|} g^{\gamma\mu} f_{,\mu})$$

7. In spherical coordinates we have that:

$$[g_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

Then in order to apply the previous formula we need to compute g and $[g^{\mu\nu}]$. We have quite simply:

$$g = r^4 \sin^2\theta \text{ and } g^{\mu\mu} = \frac{1}{g_{\mu\mu}} \text{ and } g^{\mu\nu} = 0 \text{ otherwise.}$$

Plugging this in the previous formula we obtain:

$$\nabla^2 f = \frac{1}{r^2 \sin\theta}\partial_\gamma(r^2 \sin\theta g^{\gamma\mu} f_{,\mu}) = \frac{1}{r^2 \sin\theta}\left(\partial_r(r^2 \sin\theta f_{,r}) + \partial_\theta(\sin\theta f_{,\theta}) + \partial_\varphi(\frac{1}{\sin\theta} f_{,\varphi})\right)$$

Now simplifying the derivatives gives:

$$\begin{aligned}\nabla^2 f &= \frac{1}{r^2 \sin\theta}\left(\sin\theta \partial_r(r^2 f_{,r}) + \partial_\theta(\sin\theta f_{,\theta}) + \frac{1}{\sin\theta}\partial_\varphi f_{,\varphi}\right)\\ &= \frac{1}{r^2}\partial_r(r^2 f_{,r}) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta f_{,\theta}) + \frac{1}{r^2\sin^2\theta}\partial_\varphi f_{,\varphi}\end{aligned}$$

8. Repeating an identical argument but using:

$$[g_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{pmatrix}$$

Gives us immediately that:

$$\nabla^2 f = \frac{1}{r}\partial_\gamma(r g^{\gamma\mu} f_{,\mu}) = r^{-1}(\partial_z(r f_{,z} + \partial_r(r f_{,r}) + \partial_\phi(r^{-1} f_{,\phi})) = f_{,zz} + r^{-1} f_{,r} + f_{,rr} + r^{-2} f_{,\phi\phi}$$

2 Rotating coordinate frame.

1. We have that:

$$t = t \quad \text{and} \quad z = z' \quad \text{and} \quad r = r' \quad \text{and} \quad \phi = \phi' - \Omega t$$

Hence we immediately get that:

$$\mathrm{d}t = \mathrm{d}t \quad \text{and} \quad \mathrm{d}z = \mathrm{d}z' \quad \text{and} \quad \mathrm{d}r = \mathrm{d}r' \quad \text{and} \quad \mathrm{d}\phi = \mathrm{d}\phi' - t\mathrm{d}\Omega - \Omega\mathrm{d}t = \mathrm{d}\phi' - \Omega\mathrm{d}t$$

Where in the last equality we add the assumption that we place ourselves in a rotating frame at constant angular velocity. Now plugging this in the expression for a line element we obtain:

$$\begin{aligned}\mathrm{d}s^2 &= -c^2\mathrm{d}t^2 + (\mathrm{d}z')^2 + (\mathrm{d}r')^2 + (r')^2(\mathrm{d}\phi')^2 = -c^2\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\phi + \Omega\mathrm{d}t)^2\\ &= -c^2\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + r^2\Omega^2\mathrm{d}t^2 + 2r^2\Omega\mathrm{d}\phi\mathrm{d}t\\ &= (r^2\Omega^2 - c^2)\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + 2r^2\Omega\mathrm{d}t\mathrm{d}\phi\end{aligned}$$

Hence we also get:

$$[g_{\mu\nu}] = \begin{pmatrix} (r^2\Omega^2 - c^2) & 0 & 0 & r^2\Omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ r^2\Omega & 0 & 0 & r^2 \end{pmatrix}$$

2. The inverse can be immediately obtained through it's cofactor formulation and gives:

$$[g^{\mu\nu}] = \begin{pmatrix} -\frac{1}{c^2} & 0 & 0 & \frac{\Omega}{c^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\Omega}{c^2} & 0 & 0 & \frac{c^2 - r^2\Omega^2}{c^2 r^2} \end{pmatrix} \quad \text{and} \quad g = -c^2 r^2$$

3. We have that:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Omega t & \sin\Omega t & 0 \\ 0 & -\sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Now notice that the transition matrix is orthogonal hence we immediately have that:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Omega t & -\sin\Omega t & 0 \\ 0 & \sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Hence we obtain immediately that:

$$\begin{aligned}\mathrm{d}s^2 &= -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = -c^2\mathrm{d}t^2 + (\mathrm{d}(x'\cos\Omega t - y'\sin\Omega t))^2 + (\mathrm{d}(x'\sin\Omega t + y'\cos\Omega t))^2 + (\mathrm{d}z')^2\\ &= -c^2\mathrm{d}t^2 + (\cos\Omega t\mathrm{d}x' - x'\Omega\mathrm{d}t\sin\Omega t - \sin\Omega t\mathrm{d}y' - y'\Omega\mathrm{d}t\cos\Omega t)^2 = ...\end{aligned}$$

4. We have that:

$$\begin{pmatrix} -1+h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & 1+h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & 1+h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & 1+h_{33} \end{pmatrix} = \begin{pmatrix} -(1-(x^2+y^2)\Omega^2) & \Omega y & -\Omega x & 0 \\ \Omega y & 1 & 0 & 0 \\ -\Omega x & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence we get:

$$[h_{\mu\nu}] = \begin{pmatrix} (x^2+y^2)\Omega^2 & \Omega y & -\Omega x & 0 \\ \Omega y & 0 & 0 & 0 \\ -\Omega x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$