

HW3 - Probability

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1. From the law of total probability we have that:

$$P(S_n = \mathbf{s}) = \sum_{\mathbf{s}' \in \mathbb{Z}^d} P(S_n = \mathbf{s} | S_{n-1} = \mathbf{s}') P(S_{n-1} = \mathbf{s}')$$

Furthermore we have that (call $B = \{\mathbf{e}^{(1)}, -\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(d)}, -\mathbf{e}^{(d)}\}$)

$$P(S_n = \mathbf{s} | S_{n-1} = \mathbf{s}') = \frac{1}{2d} 1_{(\mathbf{s}-\mathbf{s}') \in B}$$

Hence the sum reduces to:

$$P(S_n = \mathbf{s}) = \sum_{\mathbf{n} \in B} \frac{1}{2d} P(S_{n-1} = \mathbf{s} - \mathbf{n})$$

We therefore have that:

$$p(\mathbf{s}, n+1) = \frac{1}{2d} \sum_{\mathbf{n} \in B} p(\mathbf{s} - \mathbf{n}, n)$$

Then the Fourier transform is given by:

$$\tilde{p}(\mathbf{k}, n) = \sum_{\mathbf{s} \in \mathbb{Z}^d} p(\mathbf{s}, n) e^{-i\mathbf{k} \cdot \mathbf{s}} = \sum_{\mathbf{s} \in \mathbb{Z}^d} \left(\frac{1}{2d} \sum_{\mathbf{n} \in B} p(\mathbf{s} - \mathbf{n}, n-1) \right) e^{-i\mathbf{k} \cdot \mathbf{s}} = \frac{1}{2d} \sum_{\mathbf{s} \in \mathbb{Z}^d} \sum_{\mathbf{n} \in B} p(\mathbf{s} - \mathbf{n}, n-1) e^{-i\mathbf{k} \cdot \mathbf{s}}$$

Now since all the sums converge we can switch them and we get:

$$\tilde{p}(\mathbf{k}, n) = \frac{1}{2d} \sum_{\mathbf{n} \in B} \sum_{\mathbf{s} \in \mathbb{Z}^d} p(\mathbf{s} - \mathbf{n}, n-1) e^{-i\mathbf{k} \cdot \mathbf{s}} = \frac{1}{2d} \sum_{\mathbf{n} \in B} \sum_{\mathbf{s} \in \mathbb{Z}^d} p(\mathbf{s}, n-1) e^{-i\mathbf{k} \cdot (\mathbf{s} + \mathbf{n})} = \frac{\tilde{p}(\mathbf{k}, n-1)}{2d} \sum_{\mathbf{n} \in B} e^{-i\mathbf{k} \cdot \mathbf{n}} = a(\mathbf{k}) \tilde{p}(\mathbf{k}, n-1)$$

Where we took $a(\mathbf{k}) = \frac{1}{2d} \sum_{\mathbf{n} \in B} e^{-i\mathbf{k} \cdot \mathbf{n}}$. Now we know that:

$$p(\mathbf{s}, 0) = \delta_{\mathbf{s}} \text{ and hence } \tilde{p}(\mathbf{k}, 0) = \sum_{\mathbf{s} \in \mathbb{Z}^d} \delta_{\mathbf{s}} e^{-i\mathbf{k} \cdot \mathbf{s}} = 1$$

We therefore have that:

$$\tilde{p}(\mathbf{k}, n) = a(\mathbf{k})^n = \left(\frac{1}{2d} \sum_{\mathbf{n} \in B} e^{-i\mathbf{k} \cdot \mathbf{n}} \right)^n$$

Inversing the Fourier transform gives:

$$p(\mathbf{s}, n) = \int_{BZ} \frac{d\mathbf{k}}{(2\pi)^d} \tilde{p}(\mathbf{k}, n) e^{i\mathbf{k} \cdot \mathbf{s}} = \int_{BZ} \frac{d\mathbf{k}}{(2\pi)^d} \left(\frac{1}{2d} \sum_{\mathbf{n} \in B} e^{-i\mathbf{k} \cdot \mathbf{n}} \right)^n e^{i\mathbf{k} \cdot \mathbf{s}}$$

2. We have the following (remembering $B = \{\mathbf{e}_x, -\mathbf{e}_x, \mathbf{e}_y, -\mathbf{e}_y\}$ and writing $B' = \{\mathbf{e}_x + \mathbf{e}_y, \mathbf{e}_x - \mathbf{e}_y, \mathbf{e}_y - \mathbf{e}_x, -\mathbf{e}_x - \mathbf{e}_y\}$):

n	$p(\mathbf{s}, n)$	$p_1(\mathbf{s}, n)$
1	$\frac{1}{4} \delta_{\mathbf{s} \in B}$	$\frac{1}{4} \delta_{\mathbf{s} \in B}$
2	$\frac{1}{4} \delta_{\mathbf{s}} + \frac{1}{16} \delta_{\mathbf{s} \in 2B} + \frac{1}{8} \delta_{\mathbf{s} \in B'}$	$\frac{1}{4} \delta_{\mathbf{s}} + \frac{1}{16} \delta_{\mathbf{s} \in 2B} + \frac{1}{8} \delta_{\mathbf{s} \in B'}$
3	$\frac{1}{64} \delta_{\mathbf{s} \in 3B} + \frac{9}{64} \delta_{\mathbf{s} \in B} + \frac{3}{64} \delta_{\ \mathbf{s}\ ^2=5}$	$\frac{1}{64} \delta_{\mathbf{s} \in 3B} + \frac{3}{64} \delta_{\ \mathbf{s}\ ^2=5} + \frac{5}{64} \delta_{\mathbf{s} \in B}$

Hence it is generally true that $\sum_{\mathbf{s} \in \mathbb{Z}^d} p(\mathbf{s}, n) = 1$ however $\sum_{\mathbf{s} \in \mathbb{Z}^d} p_1(\mathbf{s}, n) \neq 1$.

3. We have that:

$$\begin{aligned}
P(\mathbf{S}_{\mathbf{n}'} = \mathbf{s}' | \mathbf{S}_{\mathbf{n}} = \mathbf{s}) &= P\left(\sum_{i=1}^{n'} \mathbf{X}_i = \mathbf{s}' \mid \sum_{i=1}^n \mathbf{X}_i = \mathbf{s}\right) = P\left(\sum_{i=1}^n \mathbf{X}_i + \sum_{i=n+1}^{n'} \mathbf{X}_i = \mathbf{s}' \mid \sum_{i=1}^n \mathbf{X}_i = \mathbf{s}\right) \\
&= P\left(\mathbf{s} + \sum_{i=n+1}^{n'} \mathbf{X}_i = \mathbf{s}' \mid \sum_{i=1}^n \mathbf{X}_i = \mathbf{s}\right) = P\left(\mathbf{s} + \sum_{i=n+1}^{n'} \mathbf{X}_i = \mathbf{s}'\right) \\
&= P\left(\sum_{i=n+1}^{n'} \mathbf{X}_i = \mathbf{s}' - \mathbf{s}\right) = P\left(\sum_{i=1}^{n'-n} \mathbf{X}_i = \mathbf{s}' - \mathbf{s}\right) \\
&= P(\mathbf{S}_{\mathbf{n}'-\mathbf{n}} = \mathbf{s}' - \mathbf{s})
\end{aligned}$$

Where in the 4th equality we used the fact that (X_1, \dots, X_n) is independent from $(X_{n+1}, \dots, X_{n'})$ and in the 6th equality we used the fact that the X_i are identically distributed random variables.

4. We have that (we denote by $A_{\mathbf{s},n} = \{\mathbf{S}_1 \neq \mathbf{s}, \dots, \mathbf{S}_{n-1} \neq \mathbf{s}, \mathbf{S}_n = \mathbf{s}\}$):

$$P(\mathbf{S}_{\mathbf{n}} = \mathbf{s}) = \sum_{k=0}^n P(\mathbf{S}_{\mathbf{n}} = \mathbf{s} | A_{\mathbf{s},k}) P(A_{\mathbf{s},k}) + \delta_{\mathbf{s}} \delta_n = \sum_{k=0}^n P(\mathbf{S}_{\mathbf{n}-\mathbf{k}} = \mathbf{0}) p_1(\mathbf{s}, k) + \delta_{\mathbf{s}} \delta_n$$

Hence re-writing it we obtain:

$$p(\mathbf{s}, n) = \sum_{k=0}^n p(\mathbf{0}, n-k) p_1(\mathbf{s}, k) + \delta_{\mathbf{s}} \delta_n$$

5. From the previous question we have that:

$$\begin{aligned}
\hat{p}(\mathbf{s}, \lambda) &= \sum_{n \in \mathbb{N}} p(\mathbf{s}, n) \lambda^n = \sum_{n \in \mathbb{N}} \left(\sum_{k=0}^n p(\mathbf{0}, n-k) p_1(\mathbf{s}, k) + \delta_{\mathbf{s}} \delta_n \right) \lambda^n = \delta_{\mathbf{s}} \lambda^0 + \sum_{k=0}^{+\infty} \sum_{n=k}^{+\infty} p(\mathbf{0}, n-k) p_1(\mathbf{s}, k) \lambda^{n-k} \lambda^k \\
&= \delta_{\mathbf{s}} + \left(\sum_{k \in \mathbb{N}} \lambda^k p_1(\mathbf{s}, k) \right) \left(\sum_{n \in \mathbb{N}} \lambda^n p(\mathbf{0}, n) \right) = \delta_{\mathbf{s}} + \hat{p}_1(\mathbf{s}, \lambda) \hat{p}(\mathbf{0}, \lambda)
\end{aligned}$$

Where in the third equality we used Fubini's theorem in order to exchange the summation.

6. Using Question 1 we get:

$$\hat{p}(\mathbf{s}, \lambda) = \sum_{n \in \mathbb{N}} \left(\int_{BZ} \frac{d\mathbf{k}}{(2\pi)^d} a(\mathbf{k})^n e^{i\mathbf{k} \cdot \mathbf{s}} \right) \lambda^n = \int_{BZ} \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k} \cdot \mathbf{s}} \sum_{n \in \mathbb{N}} (\lambda a(\mathbf{k}))^n = \int_{BZ} \frac{d\mathbf{k}}{(2\pi)^d} \frac{e^{i\mathbf{k} \cdot \mathbf{s}}}{1 - \lambda a(\mathbf{k})}$$

We are allowed to exchange the integral and the sum since from Question 5 we are assured of their convergence.

7. Notice that:

$$p_r = \sum_{n \in \mathbb{N}} p_1(\mathbf{0}, n) = \lim_{\lambda \rightarrow 1^-} \sum_{n \in \mathbb{N}} p_1(\mathbf{0}, n) \lambda^n = \lim_{\lambda \rightarrow 1^-} \hat{p}_1(\mathbf{0}, \lambda) = \lim_{\lambda \rightarrow 1^-} \frac{\hat{p}(\mathbf{0}, \lambda)}{\hat{p}(\mathbf{0}, \lambda)} - \delta_{\mathbf{0}} \frac{1}{\hat{p}(\mathbf{0}, \lambda)} = \lim_{\lambda \rightarrow 1^-} 1 - \frac{1}{\hat{p}(\mathbf{0}, \lambda)}$$

Now replacing $\hat{p}(\mathbf{0}, \lambda)$ with the integral expression found in Question 6 and noticing that up to a re-writing of the exponentials we have that $a(\mathbf{k}) = \frac{1}{d} \sum_{i=1}^d \cos(k_i)$ we get:

$$p_r = \lim_{\lambda \rightarrow 1^-} 1 - \frac{1}{\int_{BZ} \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{1 - \lambda a(\mathbf{k})}}$$

Which is the desired result.

8. (a) In the case $d = 1$ the equation simplifies to:

$$\hat{p}_1(\mathbf{0}, \lambda) = 1 - \frac{1}{\int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{1}{1 - \lambda \cos(k)}}$$

We now compute the integral:

$$\int_{-\pi}^{\pi} \frac{dk}{1 - \lambda \cos(k)} = \int_{-\pi}^{\pi} \frac{d\theta}{1 - \lambda \left(\frac{e^{i\theta} + e^{-i\theta}}{2} \right)}$$

Now by taking $z = e^{i\theta}$ meaning that $dz = ie^{i\theta}d\theta$ gives:

$$\int_{-\pi}^{\pi} \frac{d\theta}{1 - \lambda \left(\frac{e^{i\theta} + e^{-i\theta}}{2} \right)} = -i \int_{S^1} \frac{2dz}{z(2 - \lambda(z + \frac{1}{z}))} = -i \int_{S^1} \frac{2dz}{2z - \lambda z^2 - 1}$$

Now the integrand admits two poles in:

$$z_0^{\pm} = \frac{1 \pm \sqrt{1 - \lambda^2}}{\lambda}$$

However since $0 \leq \lambda < 1$ the only pole inside the unit circle is $z_0 = z_0^-$. Then from the residue theorem we have that:

$$-i \int_{S^1} \frac{2dz}{2z - \lambda z^2 - 1} = 2\pi \text{Res}\left(\frac{2}{2z - \lambda z^2 - 1}, z_0\right) = \frac{2\pi}{\sqrt{1 - \lambda^2}}$$

Plugging this back on top we get the desired result:

$$\hat{p}_1(\mathbf{0}, \lambda) = 1 - \frac{1}{\frac{1}{\sqrt{1 - \lambda^2}}} = 1 - \sqrt{1 - \lambda^2}$$

(b) Notice that:

$$\hat{p}_1(\mathbf{0}, \lambda) = \sum_{n=0}^{+\infty} p_1(\mathbf{0}, n) \lambda^n \Rightarrow p_1(\mathbf{0}, n) = \frac{1}{n!} \frac{d^n}{d\lambda^n} \hat{p}_1(\mathbf{0}, \lambda) \Big|_{\lambda=0}$$

Notice that this corresponds to the n -th term of the series expansion of $1 - \sqrt{1 - \lambda^2}$ around 0. Since it is even we already know that all the odd powers must vanish. More generally we have from the generalized binomial formula:

$$1 - \sqrt{1 - \lambda^2} = 1 - \sum_{k=0}^{+\infty} \frac{\frac{1}{2}(\frac{1}{2} - 1) \cdots (\frac{1}{2} - k + 1)}{k!} (-\lambda^2)^k = - \sum_{k=1}^{+\infty} \frac{\frac{1}{2}(\frac{1}{2} - 1) \cdots (\frac{1}{2} - k + 1)}{k!} (-1)^k \lambda^{2k}$$

Now we rework a bit the coefficient in front:

$$\frac{1}{2^k} (-1)^{k+1} \frac{(1 - 2) \cdots (1 - 2k + 2)}{k!} = (-1)^{2k} \frac{1}{2^k} \frac{1 \cdot 3 \cdots (2k - 3)}{k!} = \frac{1}{2^k} \frac{1 \cdot 3 \cdots (2k - 3) k!}{(2k)!} \binom{2k}{k}$$

Which when simplified gives:

$$= \frac{1}{2^k} \frac{1}{2k - 1} \frac{1}{2 \cdot 4 \cdots (2k - 2) \cdot 2k} k! \binom{2k}{k} = \frac{1}{2^{2k}} \frac{1}{2k - 1} \frac{1}{1 \cdot 2 \cdots (k - 1) \cdot k} k! \binom{2k}{k} = \frac{1}{2^{2k}} \frac{1}{2k - 1} \binom{2k}{k}$$

(c) From definition since the q_{2n} are the coefficients in the series expansion of $1 - \sqrt{1 - \lambda^2}$ around 0 it must be that $q_{2n} \rightarrow 0$ as $n \rightarrow +\infty$. Now furthermore notice that $E[p_1(\mathbf{0}, n)] = \frac{d}{d\lambda} \hat{p}_1(0, \lambda) \Big|_{\lambda=1} = +\infty$ so the average time of first return to the origin is undefined.