

Midterm homework problems

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1 Divergence and Laplacian

1. We have the definition of Christoffel symbols:

$$\Gamma_{ij}^k = \frac{\partial \mathbf{e}_i}{\partial x^j} \cdot \mathbf{e}^k$$

Then we have that:

$$\nabla \cdot \mathbf{V} = \partial_i (V^j \mathbf{e}_j)^i = \frac{\partial V^i}{\partial x^i} + \Gamma_{ij}^i V^j = V_{,i}^i + \frac{1}{2} g^{im} (g_{mi,j} + g_{mj,i} - g_{ij,m}) V^j$$

Then:

...

2. Since the determinant is an invariant scalar of the matrix then from the relation: $g^{\mu\nu} = g^{-1} c^{\mu\nu}$ we know that c transforms in the exact same way as g does. Since g is a tensor then so is c .
3. We have that:

$$g = \sum_{\nu} g_{\mu\nu} c^{\mu\nu} \text{ hence } \frac{\partial g}{\partial g_{\mu\nu}} = \frac{\partial}{\partial g_{\mu\nu}} \sum_{\nu'} g_{\mu\nu'} c^{\mu\nu'} = c^{\mu\nu}$$

4. We have that:

$$g^{\mu\nu} g_{\mu\nu,\gamma} = \partial_\gamma \log g$$

We have that:

$$\partial_\gamma g (g_{\mu\nu}) = (\partial_\gamma g) g_{\mu\nu} + g \partial_\gamma g_{\mu\nu}$$

We have that:

$$\partial_\gamma g = \frac{\partial}{\partial g_{\mu\nu}} \frac{\partial g_{\mu\nu}}{\partial \gamma} g = \frac{\partial}{\partial g_{\mu\nu}} g_{\mu\nu,\gamma} g = g_{\mu\nu,\gamma} c^{\mu\nu}$$

Hence:

...

5. We start from the end and we differentiate to obtain:

$$\frac{1}{\sqrt{|g|}} \partial_\gamma (\sqrt{|g|} V^\gamma) = V^\gamma_{,\gamma} + \frac{1}{\sqrt{|g|}} V^\gamma \frac{1}{2\sqrt{|g|}} \partial_\gamma |g| = V^\mu_{,\mu} + \frac{1}{2} V^\gamma \frac{\partial_\gamma |g|}{|g|} = V^\mu_{,\mu} + \frac{1}{2} V^\gamma \log |g|$$

Now using question 4 we re-obtain the formula of question 1 and this concludes the proof.

6. Using the above formula by replacing: $V_\gamma = f_{,\gamma}$ (hence $V^\gamma = g^{\gamma\mu} f_{,\mu}$) we obtain:

$$\nabla^2 f = \frac{1}{\sqrt{|g|}} \partial_\gamma (\sqrt{|g|} f^{,\gamma}) = \frac{1}{\sqrt{|g|}} \partial_\gamma (\sqrt{|g|} g^{\gamma\mu} f_{,\mu})$$

7. In spherical coordinates we have that:

$$[g_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}$$

Then in order to apply the previous formula we need to compute g and $[g^{\mu\nu}]$. We have quite simply:

$$g = r^4 \sin^2 \theta \text{ and } g^{\mu\mu} = \frac{1}{g_{\mu\mu}} \text{ and } g^{\mu\nu} = 0 \text{ otherwise.}$$

Plugging this in the previous formula we obtain:

$$\nabla^2 f = \frac{1}{r^2 \sin \theta} \partial_\gamma (r^2 \sin \theta g^{\gamma\mu} f_{,\mu}) = \frac{1}{r^2 \sin \theta} \left(\partial_r (r^2 \sin \theta f_{,r}) + \partial_\theta (\sin \theta f_{,\theta}) + \partial_\varphi \left(\frac{1}{\sin \theta} f_{,\varphi} \right) \right)$$

Now simplifying the derivatives gives:

$$\begin{aligned} \nabla^2 f &= \frac{1}{r^2 \sin \theta} \left(\sin \theta \partial_r (r^2 f_{,r}) + \partial_\theta (\sin \theta f_{,\theta}) + \frac{1}{\sin \theta} \partial_\varphi f_{,\varphi} \right) \\ &= \frac{1}{r^2} \partial_r (r^2 f_{,r}) + \frac{1}{r^2 \sin \theta} \partial_\theta (\sin \theta f_{,\theta}) + \frac{1}{r^2 \sin^2 \theta} \partial_\varphi f_{,\varphi} \end{aligned}$$

8. Repeating an identical argument but using:

$$[g_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{pmatrix}$$

Gives us immediately that:

$$\nabla^2 f = \frac{1}{r} \partial_\gamma (r g^{\gamma\mu} f_{,\mu}) = r^{-1} (\partial_z (r f_{,z}) + \partial_r (r f_{,r}) + \partial_\phi (r^{-1} f_{,\phi})) = f_{,zz} + r^{-1} f_{,r} + f_{,rr} + r^{-2} f_{,\phi\phi}$$

2 Rotating coordinate frame.

1. We have that:

$$t = t \quad \text{and} \quad z = z' \quad \text{and} \quad r = r' \quad \text{and} \quad \phi = \phi' - \Omega t$$

Hence we immediately get that:

$$dt = dt \quad \text{and} \quad dz = dz' \quad \text{and} \quad dr = dr' \quad \text{and} \quad d\phi = d\phi' - \Omega dt = d\phi' - \Omega dt$$

Where in the last equality we add the assumption that we place ourselves in a rotating frame at constant angular velocity. Now plugging this in the expression for a line element we obtain:

$$\begin{aligned} ds^2 &= -c^2 dt^2 + (dz')^2 + (dr')^2 + (r')^2 (d\phi')^2 = -c^2 dt^2 + dz^2 + dr^2 + r^2 (d\phi + \Omega dt)^2 \\ &= -c^2 dt^2 + dz^2 + dr^2 + r^2 d\phi^2 + r^2 \Omega^2 dt^2 + 2r^2 \Omega d\phi dt \\ &= (r^2 \Omega^2 - c^2) dt^2 + dz^2 + dr^2 + r^2 d\phi^2 + 2r^2 \Omega dt d\phi \end{aligned}$$

Hence we also get:

$$[g_{\mu\nu}] = \begin{pmatrix} (r^2 \Omega^2 - c^2) & 0 & 0 & r^2 \Omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ r^2 \Omega & 0 & 0 & r^2 \end{pmatrix}$$

2. The inverse can be immediately obtained through it's cofactor formulation and gives:

$$[g^{\mu\nu}] = \begin{pmatrix} -\frac{1}{c^2} & 0 & 0 & \frac{\Omega}{c^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\Omega}{c^2} & 0 & 0 & \frac{c^2 - r^2 \Omega^2}{c^2 r^2} \end{pmatrix} \quad \text{and} \quad g = -c^2 r^2$$

3. We have that:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \Omega t & \sin \Omega t & 0 \\ 0 & -\sin \Omega t & \cos \Omega t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Now notice that the transition matrix is orthogonal hence we immediately have that:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \Omega t & -\sin \Omega t & 0 \\ 0 & \sin \Omega t & \cos \Omega t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Hence we obtain immediately that:

$$\begin{aligned} ds^2 &= -c^2 dt^2 + dx^2 + dy^2 + dz^2 = -c^2 dt^2 + (d(x' \cos \Omega t - y' \sin \Omega t))^2 + (d(x' \sin \Omega t + y' \cos \Omega t))^2 + (dz')^2 \\ &= -c^2 dt^2 + (\cos \Omega t dx' - x' \Omega dt \sin \Omega t - \sin \Omega t dy' - y' \Omega dt \cos \Omega t)^2 = \dots \end{aligned}$$

4. We have that:

$$\begin{pmatrix} -1 + h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & 1 + h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & 1 + h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & 1 + h_{33} \end{pmatrix} = \begin{pmatrix} -(1 - (x^2 + y^2)\Omega^2) & \Omega y & -\Omega x & 0 \\ \Omega y & 1 & 0 & 0 \\ -\Omega x & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence we get:

$$[h_{\mu\nu}] = \begin{pmatrix} (x^2 + y^2)\Omega^2 & \Omega y & -\Omega x & 0 \\ \Omega y & 0 & 0 & 0 \\ -\Omega x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3 Frame dragging by a moving rod.

1. We have the equation:

$$\nabla^2 \Phi = 4\pi(G_N/c^2)\rho$$

From symmetry arguments we know already that Φ will only be a function of r . Hence we have that by plugging the expression of the laplacian found in part 1 we obtain:

$$\Phi_{,zz} + \Phi_{,rr} + \Phi_{,r}/r + \Phi_{,\phi\phi}/r^2 = \Phi_{,rr} + \Phi_{,r}/r = 4\pi(G_N/c^2)\rho$$

This can be re-written as:

$$\partial_r(r\Phi_{,r}) = 4\pi(G_N/c^2)\rho r \Rightarrow r\Phi_{,r} = 4\pi(G_N/c^2)\rho r^2/2 + c$$

Which gives immediately through integration a solution of the form:

$$\Phi(r) = \pi(G_N/c^2)\rho r^2 + c \log(r) + c'$$

Now the condition $\Phi_{,r}(0) = 0$ ensures that $c = 0$ and the condition $\Phi(R) = 0$ ensures that $c' = -\pi(G_N/c^2)\rho R^2$ hence the final solution is given by:

$$\Phi(r) = \pi(G_N/c^2)\rho(r^2 - R^2)$$

2. We have that:

$$\overline{h_{\mu\nu}} = -4\Phi\delta_\mu^0\delta_\nu^0$$

Hence we get that:

$$h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h = -4\Phi\delta_\mu^0\delta_\nu^0$$

Now we have that:

$$\eta_{\mu\nu}(\overline{h_{\mu\nu}}) = h - \frac{1}{2}Tr(\eta)h = h - 2h = -h$$

Which when plugged in the previous equation gives immediately that:

$$-h = 4\Phi \Leftrightarrow h = -4\Phi$$

Hence we obtain that:

$$h_{\mu\nu}(r) = -2\Phi\delta_{\mu\nu}$$

3. We have that:

$$ds^2 = (1 - 2\Phi)(dr^2 + r^2 d\phi^2)$$

Now we apply the following transformations:

$$r' = (1 + r\Phi_{,r})(1 - \Phi)r \quad \text{and} \quad \phi' = (1 - \Phi_{,r}/r)\phi$$

Using our previous results we can re-write this as:

$$r' = (1 + 2\alpha r^2)(1 - \alpha(r^2 - R^2))r \quad \text{and} \quad \phi' = (1 - 2\alpha)\phi$$

Hence we have that:

$$\begin{aligned} dr' &= d(r\Phi_{,r})(1 - \Phi)r + (1 + r\Phi_{,r})d(-\Phi)r + (1 + r\Phi_{,r})(1 - \Phi)dr \\ &= 4\pi(G_N/c^2)\rho r dr(1 - \Phi)r - (1 + r\Phi_{,r})\Phi_{,r}dr + (1 + r\Phi_{,r})(1 - \Phi)dr \\ &= dr(\alpha r - \alpha r\Phi - \Phi_{,r} - r\Phi_{,r}^2 + 1 - \Phi + r\Phi_{,r} - r\Phi_{,r}^2) \end{aligned}$$

Similarly we get:

$$d\phi' = (1 - \Phi_{,r}/r)d\phi - (d\Phi_{,r})/r\Phi + \Phi_{,r}/r^2\phi dr = (1 - \Phi_{,r}/r)d\phi - ()/r\Phi + \Phi_{,r}/r^2\phi dr$$

4.

5. Hence we obtain:

$$[h_{\alpha'\beta'}] = -2\Phi(r') \begin{pmatrix} 1 & & -2v/c \\ & \mathbf{I} & \\ -2v/c & & 1 \end{pmatrix}$$

6. (a) We use the geodesic equation:

$$\frac{d^2 z}{d\tau^2} = \Gamma_{\mu\nu}^z \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

Then:

$$\Gamma_{\mu\lambda}^\nu = \frac{1}{2} \eta^{\nu\gamma} (\partial_\mu h_{\gamma\lambda} + \partial_\lambda h_{\gamma\mu} - \partial_\gamma h_{\lambda\mu})$$

We start by computing the Christoffel symbols:

$$\begin{aligned} \Gamma_{00}^z &= \frac{1}{2} \eta^{z\gamma} (\partial_0 h_{\gamma 0} + \partial_0 h_{\gamma 0} - \partial_\gamma h_{00}) = \partial_0 h_{z0} - \frac{1}{2} \partial_z h_{00} = \partial_0 4\Phi(r)v/c + \partial_z \Phi(r) = 0 \\ \Gamma_{01}^z &= \frac{1}{2} \eta^{z\gamma} (\partial_0 h_{\gamma 1} + \partial_1 h_{\gamma 0} - \partial_\gamma h_{10}) = \frac{1}{2} (\partial_0 h_{z1} + \partial_1 h_{z0} - \partial_z h_{10}) = \frac{1}{2} (0 + \partial_x 4\Phi(r)v/c + 0) = 2v/c \partial_x \Phi(r) \\ \Gamma_{02}^z &= \frac{1}{2} (\partial_0 h_{z2} + \partial_2 h_{z0} - \partial_z h_{20}) = 2v/c \partial_y \Phi(r) \\ \Gamma_{03}^z &= \frac{1}{2} (\partial_0 h_{z3} + \partial_3 h_{z0} - \partial_z h_{30}) = \frac{1}{2} (-2\partial_z \Phi(r) - 4v/c \partial_z \Phi(r)) = -(1 - 2v/c) \partial_z \Phi(r) = 0 \\ \Gamma_{11}^z &= \frac{1}{2} (\partial_1 h_{z1} + \partial_1 h_{z1} - \partial_z h_{11}) = \partial_z \Phi(r) = 0 \\ \Gamma_{12}^z &= \frac{1}{2} (\partial_1 h_{z2} + \partial_2 h_{z1} - \partial_z h_{21}) = 0 \\ \Gamma_{13}^z &= \frac{1}{2} (\partial_1 h_{z3} + \partial_3 h_{z1} - \partial_z h_{31}) = -\partial_x \Phi(r) \\ \Gamma_{22}^z &= \frac{1}{2} (\partial_2 h_{z2} + \partial_2 h_{z2} - \partial_z h_{22}) = \partial_z \Phi(r) = 0 \\ \Gamma_{23}^z &= \frac{1}{2} (\partial_2 h_{z3} + \partial_3 h_{z2} - \partial_z h_{32}) = -\partial_y \Phi(r) \\ \Gamma_{33}^z &= \frac{1}{2} (\partial_3 h_{z3} + \partial_3 h_{z3} - \partial_z h_{33}) = 0 \end{aligned}$$

Hence the geodesic equation becomes:

$$\ddot{z} = \Gamma_{01}^z \frac{dt}{d\tau} \frac{dx}{d\tau} + \Gamma_{02}^z \frac{dt}{d\tau} \frac{dy}{d\tau} + \Gamma_{13}^z \frac{dx}{d\tau} \frac{dz}{d\tau} + \Gamma_{23}^z \frac{dy}{d\tau} \frac{dz}{d\tau}$$

Plugging in the values we get:

$$\ddot{z} = 2v/c (\partial_x \Phi(r) \gamma \dot{x} + \partial_y \Phi(r) \gamma \dot{y}) - \dot{z} (\dot{x} \partial_x \Phi(r) + \dot{y} \partial_y \Phi(r)) = (\dot{\mathbf{x}} \cdot \nabla \Phi) \left(\frac{2v}{c} \gamma - \dot{z} \right)$$

4 Frame-dragging inside a rotating cylinder.

1. From symmetry arguments we know that Φ is only a function of x and y . Hence we have that:

$$\nabla \Phi_{\mu\nu}(\mathbf{x}) = \kappa \int_{-h}^h \oint_C dz' dx' dy' T_{\mu\nu}(\mathbf{x}') (\mathbf{x} - \mathbf{x}') / |\mathbf{x} - \mathbf{x}'|^3$$

Now we use the well-know distribution formula:

$$\nabla \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|^3} \right) = \delta(\mathbf{r})$$

Which immediately gives us that:

$$\nabla^2 \Phi_{\mu\nu}(\mathbf{x}) = \int T_{\mu\nu}(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') d\mathbf{x}' = T_{\mu\nu}(\mathbf{x})$$

This is the differential form of Gauss's Law for $\nabla \Phi$. This also tells us that:

$$H2\pi r \nabla \Phi_{\mu\nu}(r) = \oint_{\partial V} \nabla \Phi_{\mu\nu} \cdot d\mathbf{A} = \iiint_V \nabla^2 \Phi_{\mu\nu} dV = \iiint_V T_{\mu\nu}(\mathbf{x}) dV = T$$

Hence in order to write this as similar to Gauss's Law for electromagnetism we can re-write this as:

$$\nabla\Phi_{\mu\nu}(r) = \frac{T}{2\pi rH}$$

Hence we obtain that:

$$\nabla\Phi = \frac{\rho c^2}{2\pi rH} \begin{pmatrix} \pi r^2 H & 0 & \pi r^2 H \Omega & 0 \\ 0 & 0 & 0 & 0 \\ \pi r^2 H \Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{\rho r c^2}{2} \begin{pmatrix} 1 & 0 & \Omega & 0 \\ 0 & 0 & 0 & 0 \\ \Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2.