TDs - QFT

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October 21, 2020

Chapter 1

TD1

1.1 Matrix Groups

1.2 The relationship between $SO(3)$ and $SU(2)$.

1.3 Representations of $SU(2)$.

1. An immediate computation yields the desired result
2. Let $|a\rangle$ an eigenvector of $\hat{\mathbf{J}}^2$ then:

   $$\hat{\mathbf{J}}^2 |a\rangle = a |a\rangle \Rightarrow \langle a| \hat{\mathbf{J}}^2 |a\rangle = a \langle a|a\rangle \Rightarrow ||\hat{\mathbf{J}} |a\rangle ||^2 = a|| |a\rangle || \Rightarrow a > 0$$

   We propose as a writing for them $j(j+1)$ notice that:

   $$j(j+1) = x \Leftrightarrow j^2 + j - x = 0 \Rightarrow j = \frac{-j + \sqrt{j^2 + 4x}}{2}$$

   Hence the writing as $j(j+1)$ is not restrictive and covers all of $\mathbb{R}^+$.
3. Let $|v\rangle$ an eigenvector of $\hat{\mathbf{J}}^2$ and $\hat{\mathbf{J}}_\mathbf{3}$ with eigenvalues $j(j+1)$ and m. Then:

   $$\hat{\mathbf{J}}^2 \hat{\mathbf{J}_+} |v\rangle = \hat{\mathbf{J}_+} \hat{\mathbf{J}}^2 |v\rangle = j(j+1) \hat{\mathbf{J}_+} |v\rangle$$

   Since the operator $\hat{\mathbf{J}}^2$ commutes with the $\hat{\mathbf{J}}_\mathbf{i}$. Then:

   $$\hat{\mathbf{J}}_\mathbf{3} \hat{\mathbf{J}_+} |v\rangle = (\hat{\mathbf{J}_+} \hat{\mathbf{J}}_\mathbf{3} + [\hat{\mathbf{J}}_\mathbf{3}, \hat{\mathbf{J}_+}]) |v\rangle = (m\hat{\mathbf{J}_+} + i\hat{\mathbf{J}}_\mathbf{2} + 1\hat{\mathbf{J}}_\mathbf{1}) |v\rangle = (m+1)\hat{\mathbf{J}_+} |v\rangle$$

   Identically for $\hat{\mathbf{J}_-}$ we obtain the same thing but with $m-1$ as the eigenvalue for $\hat{\mathbf{J}}_\mathbf{3}$.
4. Assume that there is no such vector than the ladder operator would span an infinite family of eigenvectors of $\hat{\mathbf{J}}_\mathbf{3}$ and $\hat{\mathbf{J}_+}$ and hence V would be infinite dimensional.
5. We have that:

   $$\hat{\mathbf{J}_-} \hat{\mathbf{J}_+} = \hat{\mathbf{J}_\mathbf{1}}^2 - i[\hat{\mathbf{J}}_\mathbf{1}, \hat{\mathbf{J}}_\mathbf{2}] + \hat{\mathbf{J}_\mathbf{2}}^2 = \hat{\mathbf{J}}^2 - \hat{\mathbf{J}_\mathbf{3}}^2 + \hat{\mathbf{J}}_\mathbf{3}$$

   Then applying this for $|v_0\rangle$ we get:

   $$\hat{\mathbf{J}_-} \hat{\mathbf{J}_+} |v_0\rangle = 0 = (j(j+1) - m_0^2 + m_0) |v_0\rangle \Rightarrow j(j+1) = m_0(m_0+1)$$
6. An identical argument tells us that successive application of the lowering ladder operator must lead to a vanishing state. Then from definition we have that:

   $$|w_0\rangle = (\hat{\mathbf{J}_-})^k |v_0\rangle \Rightarrow m_0' = m_0 - k$$
7. Similarly as before we get the exact same result but with a minus sign.
8. We then have the system:

   $$\begin{cases} j(j+1) = m_0(m_0+1) \\ j(j+1) = (m_0-k)(m_0-k-1) \end{cases} \Rightarrow \begin{cases} j(j+1) = m_0(m_0+1) \\ k^2 + k = 2m_0(1+k) \end{cases} \Rightarrow \begin{cases} j = \frac{k}{2} \\ \frac{k}{2} = m_0 \end{cases}$$

9. We have that $\hat{\mathbf{J}_+}$ sends $|j,m\rangle$ to $|j,m+1\rangle$ and similarly $\hat{\mathbf{J}_-}$ sends $|j,m\rangle$ to $|j,m-1\rangle$. Then we get that:
$$\hat{\mathbf{J}_+}|j,m\rangle = x|j,m+1\rangle \Rightarrow \langle j,m|\hat{\mathbf{J}_-}\hat{\mathbf{J}_+}|j,m\rangle = |x|^2 = j(j+1)-m(m+1)$$
Hence we obtain:
$$x = \sqrt{j(j+1)-m(m+1)}$$
Then we have that:
$$\hat{\mathbf{J}}_{\mathbf{1}}|j,m\rangle = \frac{\hat{\mathbf{J}_+}+\hat{\mathbf{J}_-}}{2}|j,m\rangle = \frac{x}{2}\left(|j,m+1\rangle + |j,m-1\rangle\right)$$
Similarly:
$$\hat{\mathbf{J}}_{\mathbf{2}}|j,m\rangle = \frac{\hat{\mathbf{J}_+}-\hat{\mathbf{J}_-}}{2i}|j,m\rangle = \frac{x}{2i}\left(|j,m+1\rangle - |j,m-1\rangle\right)$$

10. Since $\hat{\mathbf{J}}^2$ commutes with the $\hat{\mathbf{J}}_{\mathbf{i}}$ we know that the eigenspaces of $\hat{\mathbf{J}}^2$ are sub-representations of $SU(2)$. We now restrict ourselves to one eigenspace, call it $\tilde{V}_j$ corresponding to the eigenvalue $j(j+1)$. As said previously there must be at least one eigenvector of $\hat{\mathbf{J}}^2$ and $\hat{\mathbf{J}}_{\mathbf{3}}$ which is killed by $\hat{\mathbf{J}_+}$ call it $|j,j,1\rangle$. Then from this eigenvector we can build $|j,m,1\rangle = \hat{\mathbf{J}_-}^{j-m}|j,j,1\rangle$. Which is an irreducible subspace of $\tilde{V}_j$. Then we can write $\tilde{V}_j = V_j^1 \oplus \tilde{V}_j'$. We can then repeat the process on $\tilde{V}_j'$ until we spanned the whole space. Then we have:
$$V = V_0^1 \oplus \cdots \oplus V_0^{n_0} \oplus V_{1/2}^1 \oplus \cdots \oplus V_{1/2}^{n_{1/2}} \oplus \cdots$$

11. We have that $\vec{L} = \vec{R} \wedge \vec{P}$ where $\vec{R}$ and $\vec{P}$ are operators on $L^2(\mathbb{R}^3)$ where $[R_j, P_k] = i\delta_{jk}$. Then we have that $[L_a, L_b] = i\varepsilon_{abc}L_c$. Then the space we describe is $V : \{\psi : S^2 \to \mathbb{C}\}$ and the spherical harmonic decomposition tells us that:
$$\psi(\theta,\varphi) = \sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{\ell} a_{\ell,m}Y_\ell^m(\theta,\varphi)$$
Furthermore we have that:
$$\vec{L}^2 = Y_\ell^m = \ell(\ell+1)Y_\ell^m \quad \text{and} \quad L_3 Y_\ell^m = mY_\ell^m$$
Hence the subspace $V_\ell = \mathrm{Span}(Y_\ell^{-\ell}, \cdots, Y_\ell^{\ell})$ is stable under rotation and $V = V_0 \oplus V_1 \oplus V_2 \oplus \cdots$

12. We have:
$$e^{2i\pi\hat{\mathbf{J}}_{\mathbf{3}}}|j,m\rangle = e^{2i\pi m}|j,m\rangle$$
Now if j is an integer we have that $m \in \mathbb{Z}$ and hence $e^{2i\pi\hat{\mathbf{J}}_{\mathbf{3}}} = \mathrm{Id}$. However if j is a half integer then m is also a half integer and hence $e^{2i\pi\hat{\mathbf{J}}_{\mathbf{3}}} = -\mathrm{Id}$.

13. In QM for example we usually consider the wavefunctions of one particle with no spin we will use the space $L^2(\mathbb{R}^3,\mathbb{C})$ however now if we introduce spin we will consider $L^2(\mathbb{R}^3,\mathbb{C}) \otimes \mathbb{C}^2$ or similarly if we consider two particles we need to consider $L^2(\mathbb{R}^3,\mathbb{C}) \otimes L^2(\mathbb{R}^3,\mathbb{C})$. Then we know also that:
$$V_{j_1} \otimes V_{j_2} = V_{|j_1-j_2|} \oplus V_{|j_1-j_2|+1} \oplus \cdots \oplus V_{j_1+j_2}$$

Chapter 2

TD2

2.1 Properties of time-like vectors.

1. Let $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{C}_+$. Then $a^0 > ||\vec{a}||$ and similarly for $\mathbf{B}$. Hence $\vec{a} \cdot \vec{b} \leq ||\vec{a}|| \cdot ||\vec{b}|| \leq a^0 b^0$. Then $\mathbf{A} \cdot \mathbf{B} < 0$.
2. Let $\mathbf{A}, \mathbf{B} \in \mathcal{C}_+$ and $\mu, \nu \in \mathbb{R}^+$ then $(\mu\mathbf{A} + \nu\mathbf{B})^2 = \mu^2\mathbf{A}^2 + 2\mu\nu\mathbf{A} \cdot \mathbf{B} + \nu^2\mathbf{B}^2 < 0$. Hence $(\mathbf{A} + \mathbf{B}) \in \mathcal{C}_+$.
3. A special Lorentz transformation is an isometry of the Minkowski space hence $\mathcal{C}_+$ is stable under it.
4. We have that:
$$a^i - \beta^i a^0 = 0 \Rightarrow \beta^i = \frac{a^i}{a^0}$$
5. Suppose by induction that this is true for n the base cases being trivial. Then for $n+1$ note that $\mathcal{C}_+$ is stable under addition so any case can be reduced to the base case $n = 2$. We prove this case here:
$$\sqrt{-(\mathbf{A} + \mathbf{B})^2} = \sqrt{-(\mathbf{A}' + \mathbf{B}')^2} = \sqrt{d^{0\,2}} = d^0$$
Then $\mathbf{A_i}^2 = \vec{a}_i^{\,2} - (a_i^0)^2$ and hence $a_i^0 = \sqrt{-\mathbf{A_i}^2 + \vec{a}_i^{\,2}} \geq \sqrt{-\mathbf{A_i}^2}$ hence:
$$\sqrt{-(\mathbf{A} + \mathbf{B})^2} \geq \sqrt{-\mathbf{A}^2} + \sqrt{-\mathbf{B}^2}$$

2.2 Applications to 4-momenta

1. $\mathbf{P} = m\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\tau} = (E, m\vec{U})$ and:
$$\mathbf{P}^2 = -E^2 + m^2\vec{U}^2 = -m^2$$
2. We directly have that $P^0 = E > 0$ and $\mathbf{P}^2 = -m^2 < 0$. Hence $\mathbf{P} \in \mathcal{C}_+$.
3. From question 2 of Exercise 1 we know that since $\mathbf{P}_i$ are in $\mathcal{C}_+$ then so is $\mathbf{P}$. Then from question 4 of Exercise 1 we know that there exists a boost transformation such that $\mathbf{P} = (E^*, \vec{0})$. Then using question 5 of Exercise 1 we also know that:
$$E^* \geq \sum_{i=1}^{n} m_i$$

2.3 Decays of particles

1. We must have that $M \geq \sum_{i=1}^{n} m_i$.
2. (a) The number of unknowns are 8 since they are all the components of the two momenta $\mathbf{P}_1$ and $\mathbf{P}_2$. We also have the four equations given by: $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$. Finally we have two more equations $\mathbf{P_1}^2 = -m_1^2$ and $\mathbf{P_2}^2 = -m_2^2$.

   (b) We have that:
$$\mathbf{P_1}^2 = \mathbf{P}^2 + \mathbf{P_2}^2 - 2\mathbf{P} \cdot \mathbf{P_2} \Leftrightarrow -m_1^2 = -M^2 - m_2^2 - 2\,(-ME_2) \Leftrightarrow 2ME_2 = M^2 + m_2^2 - m_1^2$$
   Then symmetry gives the desired opposite result.

(c) We have:

$$E_{kin,1} = E_1 - m_1$$

Which immediately gives the desired result after factorization and identically for $E_{kin,2}$. Then:

$$E_{kin,1} + E_{kin,2} = \Delta M$$

In other words all excess mass is converted to kinetic energy.

3. For each new particle we get 4 more unknowns and one more equation so 3 more indeterminates. Now following the hint we write:

$$\mathbf{P} = \sum_j \mathbf{P_j} = \mathbf{P_i} + \mathbf{Q}$$

Then:

$$\mathbf{P_i}^2 = \mathbf{P}^2 + \mathbf{Q}^2 - 2\mathbf{P} \cdot \mathbf{Q} \Leftrightarrow -m_i^2 = -M^2 - 2ME' + \mathbf{Q}^2$$

Then we have:

$$E_i = \frac{M^2 + m_i^2 + \mathbf{Q}^2}{2m} \quad \text{and} \quad E_{kin,i} = \frac{M^2 + m_i^2 - 2Mm_i + \mathbf{Q}^2}{2m}$$

Now using question 5 of Exercise 1 we can bound $\mathbf{Q}^2$ as follows:

$$\sqrt{-\mathbf{Q}^2} \geq \sum_{j \neq i} m_j \Rightarrow \mathbf{Q}^2 \leq -(M - \Delta M - m_i)^2$$

Then re-injecting this above we get the desired inequalities.

2.4 Creations of particles

1.

Chapter 3

TD3

3.1 The Laplace Equation

1. The solution is given by $\frac{q\mathbf{r}}{4\pi}$.

2. Rotationally invariant harmonic functions are given by:

$$\nabla^2 u = 0 \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{n-1}u'(r)\right) = 0 \Leftrightarrow r^{n-1}u'(r) = c \Leftrightarrow u'(r) = c\,r^{1-n} \Leftrightarrow u(r) = \frac{c}{r^{n-2}(n-2)} + c'$$

When $n \neq 2$ in the case where $n = 2$ then we get:

$$u(r) = c \ln r + c'$$

3. We have that:

$$\int_\Omega \mathrm{d}\mathbf{x}[u\nabla^2 v - v\nabla^2 u] = \int_\Omega \mathrm{d}\mathbf{x}\boldsymbol{\nabla}\cdot[u\boldsymbol{\nabla}v - v\boldsymbol{\nabla}u] = \int_{\partial\Omega} \mathrm{d}\mathbf{x}\,\mathbf{n}\cdot[u\boldsymbol{\nabla}v - v\boldsymbol{\nabla}u] = \int_{\partial\Omega} \mathrm{d}\mathbf{x}\left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right]$$

4. We have that:

$$\begin{aligned}
\int_{\overline{\mathcal{B}_\varepsilon}} \mathrm{d}\mathbf{x}\, G(\mathbf{x})\nabla^2\varphi(\mathbf{x}) &= \int_{\overline{\mathcal{B}_\varepsilon}} \mathrm{d}\mathbf{x}\left[G(\mathbf{x})\nabla^2\varphi(\mathbf{x}) - \varphi(x)\nabla^2 G(x)\right] \\
&= \int_{\mathcal{C}_\varepsilon} \mathrm{d}\mathbf{x}\,(G(\mathbf{x})(-\mathbf{r})\cdot\boldsymbol{\nabla}\varphi(\mathbf{x}) - \varphi(\mathbf{x})(-\mathbf{r})\boldsymbol{\nabla}G(\mathbf{x})) \\
&= \int_{\partial\Omega} \mathrm{d}\mathbf{x} - \varphi(\mathbf{x})\frac{\partial G}{\partial r} \xrightarrow{\varepsilon\to 0} \varphi(\mathbf{0})\omega_n\varepsilon^{n-1}\frac{\partial G}{\partial r}\bigg|_{r=\varepsilon} = \varphi(\mathbf{0})
\end{aligned}$$

5. We have:

$$\left\langle G\middle|\nabla^2\varphi\right\rangle = \langle\delta|\varphi\rangle = (-1)^2\left\langle\nabla^2 G\middle|\varphi\right\rangle = \langle\delta|\varphi\rangle$$

3.2 The Helmholtz Equation.

1. We have that:

$$\begin{aligned}
(\nabla^2 + k^2)G_\pm(\mathbf{x}) &= -\frac{1}{4\pi}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + k^2\right)\frac{e^{\pm ikr}}{r} = \frac{1}{4\pi}\left(\frac{1}{r^2}\frac{\partial ie^{ikr}(i+kr)}{\partial r} + k^2\frac{e^{ikr}}{r}\right) \\
&= -\frac{1}{4\pi}\left(-\frac{e^{ikr}k^2}{r} + k^2\frac{e^{ikr}}{r}\right)
\end{aligned}$$

Hence for all $r \neq 0$ where the differential is easily well defined it cancels.

2. Following the same steps as in part 1 questions 3 and 4 we get that $(\nabla^2 + k^2)G_\pm(\mathbf{x}) = \delta(\mathbf{x})$.

3. We can easily deduce that the Green function of $-D$ is given by $-G_\pm$. Then up to taking $k = im$ we get the desired result.

3.3 Fourier transforms

1. We have that:

$$DG = \delta \Leftrightarrow (1 + a_i \boldsymbol{\nabla}^{\mathbf{i}} + \cdots + a_{i_1,\cdots,i_p}\mathbf{grad}^{\mathbf{i_1},\cdots,\mathbf{i_p}})G = \delta \Leftrightarrow (1 + a_j(ip_j) + \cdots + a_{j_1,\cdots,j_p}(ip_{j_1,\cdots,j_p})\tilde{G} = 1$$

2. We have that:

$$\left\langle p(\mathrm{pv}\frac{1}{p} + \alpha\delta(p))\middle|\varphi\right\rangle = \varphi + \alpha \cdot \mathbf{0} \cdot \varphi(\mathbf{0}) = \varphi = \langle 1|\varphi\rangle$$

3. Let $\tilde{G}$ be a solution of (9) then notice that:

$$\left\langle P(\mathbf{p})(\tilde{G}(\mathbf{p}) + \alpha\delta(\mathbf{p} - \mathbf{p_0}))\middle|\varphi\right\rangle = \langle 1|\varphi\rangle + \alpha\,\langle P(\mathbf{p})\delta(\mathbf{p} - \mathbf{p_0})|\varphi\rangle = \langle 1|\varphi\rangle$$

   In terms of $G(\mathbf{x})$ it corresponds to adding a constant.

4. (a) From definition we have that $C_0 = \left\langle \mathrm{pv}\frac{1}{z}\middle|f\right\rangle$.
   (b) From the residue theorem we have that $C_\pm = \left\langle \mathrm{pv}\frac{1}{z} \mp i\pi\delta\middle|f\right\rangle$.

5. True because modifying the integral in a set of measure 0 changes nothing to the value of the integral.

3.4 The wave equation.

1. Let $D = \frac{\partial^2}{\partial t^2} - \nabla^2$ then:

$$\tilde{D} = -\omega^2 - \nabla^2 = -(\nabla^2 + \omega^2)$$

   Then the equation becomes:

$$\tilde{D}\tilde{G}(\omega, \mathbf{x}) = 1 \cdot \delta(\mathbf{x})$$

   Hence $-\tilde{G}(\omega, \mathbf{x})$ is a Green function of the Helmoltz operator.

2. We now know that:

$$\tilde{G}(\omega, \mathbf{x}) = \frac{1}{4\pi}\frac{e^{\pm i\omega|\mathbf{x}|}}{|\mathbf{x}|}$$

   Then doing the inverse Fourier transform we obtain:

$$G(t, x) = \frac{\delta(|\mathbf{x}| \pm t)}{4\pi|\mathbf{x}|}$$

Chapter 4

Representations of the Lorentz group

1. We have that:
$$J_a = \mathcal{J}_a^L + \mathcal{J}_a^R \quad \text{and} \quad N_a = -i(\mathcal{J}_a^L - \mathcal{J}_a^R)$$

2. We have:
$$[\mathcal{J}_a^L, \mathcal{J}_b^L] = i\varepsilon_{abc}\mathcal{J}_c^L \quad \text{and} \quad [\mathcal{J}_a^R, \mathcal{J}_b^R] = i\varepsilon_{abc}\mathcal{J}_c^R$$
Then we have that:
$$[\mathcal{J}_a^L, \mathcal{J}_b^R] = 0$$
Hence we have that $L_+^\uparrow$ can be seen as the product of two independent $SU(2)$ groups.

3. The dimension of a j_R representation of $SU(2)$ is $2j_r + 1$ hence the (j_R, j_L) representation of $L_+^\uparrow$ has dimension $(2j_r + 1)(2j_L + 1)$.

4. We have that:
$$i\theta^a J_a + i\nu^a N_a = i\theta_a(J_a^L + J_a^R) + i\nu^a i(J_a^R - J_a^L)$$
Hence we have that:
$$\rho(\Lambda) = e^{i(\theta_a + i\nu_a)\hat{J}_a^R + i(\theta_a - i\nu_a)\hat{J}_a^L} = e^{i(\theta_a + i\nu_a)\hat{J}_a^R} e^{i(\theta_a - i\nu_a)\hat{J}_a^L} \quad \text{since} \quad [\hat{J}_a^R, \hat{J}_a^L] = 0$$
Hence we have that:
$$\rho(\lambda)\left|j_R, m_R\right\rangle \otimes \left|j_L, m_L\right\rangle = e^{i(\theta_a + i\nu_a)\hat{J}_a^R}\left|j_R, m_R\right\rangle \otimes e^{i(\theta_a - i\nu_a)\hat{J}_a^L}\left|j_L, m_L\right\rangle$$

5. Notice that:
$$\rho(e^{i\nu^a N_a})^\star = \left(e^{\nu^a(\hat{J}_a^L - \hat{J}_a^R)}\right)^\star = e^{\nu^a(\hat{J}_a^L - \hat{J}_a^R)} = \rho(e^{i\nu^a N_a}) \neq \rho(e^{i\nu^a N_a})^{-1}$$
The boost are characterized by a parameter $\phi \in]-\infty, \infty[$ and hence $L_+^\uparrow$ is not compact or similarly $\beta = \frac{v}{c} \in]-1, 1[$ is bounded but not closed and hence not compact.

6. The subgroup of $L_+^\uparrow$ containing only the rotation is the one generated by J_1, J_2, J_3. Or equivalently it is generated by $J_a^L + J_a^R$ and therefore is spanned by the (j_R, j_L) representation. We have:
$$\rho(e^{i\theta_a J_a}) = e^{i\theta_a(\hat{J}_a^R + \hat{J}_a^L)}$$
Hence now defining $\hat{J}_a = \hat{J}_a^R + \hat{J}_a^L$ we have the sum of two angular momenta which we know decomposes $V_{j_R} \otimes V_{j_L}$ to $V_{|j_R - j_L|} \oplus V_{|j_R - j_L|+1} \otimes \cdots \otimes V_{j_R + j_L}$.

7. The dimension of the $(\frac{1}{2}, \frac{1}{2})$ representation is 4. From the rotation point of view this can be written as $0 \oplus 1$. This looks a lot like the A^μ representation which is given by: $\rho(\Lambda)A = \Lambda A$.

8. We have that:
$$(V_{j_R^1} \otimes V_{j_L^1}) \otimes (V_{j_R^2} \otimes V_{j_L^2}) = \left(\bigoplus_{i=|j_R^1 - j_L^1|}^{j_R^1 + j_L^1} V_i\right) \otimes \left(\bigoplus_{i=|j_R^2 - j_L^2|}^{j_R^2 + j_L^2} V_i\right) = \bigoplus_{i,j=|j_R - j_L|}^{j_R + j_L} V_i \otimes V_j$$
In the special case of $j_R = j_L = \frac{1}{2}$ we obtain:
$$\bigoplus_{i,j=0}^{1} V_i \otimes V_j = (0,0) \oplus (0,1) \oplus (1,0) \oplus (1,1)$$

9. We have that:
$$PJ^{\gamma\sigma}$$

Chapter 5

The Klein-Gordon equation

5.1 Basic properties.

1. This reads:
$$(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2)\phi = 0$$
2. Simple chain rule gives the desired result.
3.

5.2 Potential barrier

1. Then we get:
$$((\frac{\partial}{\partial t} + iqV)^2 - \nabla^2 + m^2)\phi = 0$$
2. Plugging in the Ansatz simply transform the ∂^0 into an E.
3. We have that:
$$(-\nabla^2 + m^2)\varphi_-(x) = E^2\varphi_-(x) \text{ and } (-\nabla^2 + m^2)\varphi_+(x) = (E - qV_0)^2\varphi_+(x)$$
4. Plugging in the Ansatz we obtain:
$$p'^2 + m^2 = E^2 \text{ and } p^2 + m^2 = (E - qV_0)^2$$
Hence we get:
$$p = \pm\sqrt{E^2 - m^2} \text{ and } p' = \sqrt{(E - qV_0)^2 - m^2}$$
5. (a) p' is purely imaginary for $E \in [m, qV_0 + m]$. Meaning that if we arrive with less than the energy needed to jump the barrier then the transmitted wave will be evanescent.
   (b) At high enough energies ($E \geq V_0 + m$) there is transmission and so everything is fine. If $E \in [qV_0 - m, qV_0 + m]$ we get reflected. However when $E \in [m, qV_0 - m]$ we still get transmitted ! This is very counter intuitive however the explanation comes from the fact that with a barrier potential this big there is enough energy to create new particles which means our analysis will break down.
6. Since the second derivative must be at least short of a δ then φ must be continuous and the first derivative too. Hence:
$$\begin{cases} 1 + r = t \\ ip(1 - r) = ip't \end{cases} \Leftrightarrow \begin{cases} 1 + r = t \\ p(1 - r) = p'(1 + r) \end{cases} \Leftrightarrow \begin{cases} t = \frac{2p}{p'+p} \\ r = \frac{p-p'}{p'+p} \end{cases}$$
7. The current is given by:
$$j^\mu = -i(\phi^\star\partial^\mu\phi - (\partial^\mu\phi^\star)\phi)$$
Which gives:
$$j^0 = -i(e^{iEt}\varphi^\star(x)(-iE)e^{-iEt}\varphi(x) - (iE)e^{iEt}\varphi^\star(x)\varphi(x)) = -i(-iE|\varphi(x)|^2 + iE|\varphi(x)|^2) = 0$$
And:
$$j^x = -i(te^{ip'x}...$$

Chapter 6

The Dirac equation

6.1 The non-relativistic limit.

1. Splitting the original equation we get:

$$\gamma^0 \frac{\partial \psi}{\partial t} = (\gamma^i \partial_i + m)\psi = (-\gamma^i \cdot \boldsymbol{\nabla} + m)\psi$$

   Now multiplying left and right by γ^0 and using the fact that $(\gamma^0)^2 = -1$ we get:

$$-\frac{\partial \psi}{\partial t} = (-\gamma^0\gamma^i \boldsymbol{\nabla} + \gamma^0 m)\psi \Rightarrow i\frac{\partial \psi}{\partial t} = (\gamma^0\gamma^i(-i\boldsymbol{\nabla}) + (-\gamma^0)m)\psi = H\psi$$

   Where $\beta = -\gamma^0$ and $\alpha^i = \gamma^0\gamma^i$.

2. The Hamiltonian operator will become:

$$H = \alpha \cdot (\mathbf{P} - q\mathbf{A}) + \beta m + qV$$

3. We have that:

$$\{\gamma^i, \gamma^j\} = \gamma^i\gamma^j + \gamma^j\gamma^i = -\begin{pmatrix} \sigma_i\sigma_j & 0 \\ 0 & -\sigma_i\sigma_j \end{pmatrix} - \begin{pmatrix} \sigma_j\sigma_i & 0 \\ 0 & -\sigma_j\sigma_i \end{pmatrix} = 2\eta^{ij}$$

   Trivially $\gamma^0\gamma^i + \gamma^i\gamma^0 = 0$, and finally:

$$2\gamma^0\gamma^0 = -2$$

   Then we have:

$$\beta = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix} \text{ and } \alpha^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

4. We have:

$$\begin{cases} i\dot{\varphi} = (\sigma(\mathbf{P} - q\mathbf{A}))\chi + qV\varphi \\ i\dot{\chi} = (\sigma(\mathbf{P} - q\mathbf{A}))\varphi - 2m\chi + qV\chi \end{cases}$$

5. When $m \to \infty$ the previous equation can be re-written as:

$$\Big\{ (2m + i\tfrac{\partial}{\partial t} - qV)\chi = (\sigma(\mathbf{P} - q\mathbf{A}))\varphi \Rightarrow \chi = \tfrac{1}{2m}(\sigma(\mathbf{P} - q\mathbf{A}))\varphi$$

6. We have that:

$$\begin{aligned}
(\sigma(\mathbf{P} - q\mathbf{A}))^2 &= \sigma_i(P_i - qA_i)\sigma_j(P_j - qA_j) = \sigma_i\sigma_j(P_iP_j - qP_iA_j - qA_iP_j + q^2A_iA_j) \\
&= (\delta_{ij} + i\varepsilon_{ijk}\sigma_k)(P_iP_j - qP_iA_j - qA_iP_j + q^2A_iA_j) \\
&= (\mathbf{P} - q\mathbf{A})^2 + i\varepsilon_{ijk}\sigma_k(P_iP_j - qP_iA_j - qA_iP_j + q^2A_iA_j) \\
&= (\mathbf{P} - q\mathbf{A})^2 - iq\varepsilon_{ijk}\sigma_k(P_iA_j + A_iP_j) \\
&= (\mathbf{P} - q\mathbf{A})^2 - q\varepsilon_{ijk}\sigma_k(\frac{\partial}{\partial x_i}A_j + A_i\frac{\partial}{\partial x_j}) \\
&= (\mathbf{P} - q\mathbf{A})^2 - q\varepsilon_{ijk}\sigma_k(\frac{\partial}{\partial x_i}A_j - A_j\frac{\partial}{\partial x_i}) \\
&= (\mathbf{P} - q\mathbf{A})^2 - q\varepsilon_{ijk}\sigma_k\frac{\partial A_j}{\partial x_i} \\
&= (\mathbf{P} - q\mathbf{A})^2 - q\sigma \cdot \mathbf{B}
\end{aligned}$$

7. In the non relativistic limit we have:

$$i\dot{\varphi} = \frac{1}{2m}(\sigma(\mathbf{P} - q\mathbf{A}))^2\varphi + qV\varphi \Leftrightarrow i\dot{\varphi} = \left(\frac{1}{2m}(\mathbf{P} - q\mathbf{A})^2 - \frac{q\sigma \cdot \mathbf{B}}{2m} + qV\right)\varphi$$

Now writing:

$$\varphi(t, \mathbf{x}) = \begin{pmatrix} \varphi_\uparrow(t, \mathbf{x}) \\ \varphi_\downarrow(t, \mathbf{x}) \end{pmatrix}$$

If $\mathbf{B} = 0$ then $\varphi_\uparrow$ and $\varphi_\downarrow$ are two independent solutions of the Schrodinger equation. However if $\mathbf{B} \neq 0$ and it is not perfectly aligned along $\hat{\mathbf{z}}$ it will introduce some off-diagonal terms which couple $\varphi_\uparrow$ and $\varphi_\downarrow$. Notice that we can rewrite that term as:

$$-\overbrace{2}^{g\text{ Lande}}\ \frac{q}{2m}\underbrace{\frac{\sigma}{2}}_{\delta}\cdot\mathbf{B}$$

6.2 The covariance of the Dirac equation

6.2.1 Transformation of the Dirac equation

1. We have:

$$(\gamma^\mu \partial_\mu + m)\psi(x) = 0$$

Then we have that:

$$(\gamma^\mu \partial'_\mu + m)\psi'(x') = (\gamma^\mu \partial'_\mu + m)D(\Lambda)\psi(x') = (\gamma^\mu D(\Lambda)(\Lambda^{-1})^\nu_\mu \partial_\nu + mD(\Lambda))\psi$$

Which gives:

$$(\gamma^\mu \partial'_\mu + m)\psi' = D(\Lambda)[D(\Lambda)^{-1}\gamma^\mu D(\Lambda)(\Lambda^{-1})^\nu_\mu \partial_\nu + m]\psi$$

Now using (6) we get:

$$(\gamma^\mu \partial'_\mu + m)\psi' = D(\Lambda)[\Lambda^\mu_\nu \gamma^\nu (\Lambda^{-1})^\nu_\mu \partial_\nu + m]\psi = D(\Lambda)(\gamma^\nu \partial_\nu + m)\psi = 0$$

2. We check the condition:

$$D(\Lambda_1\Lambda_2)^{-1}\gamma^\mu D(\Lambda_1\Lambda_2) = D(\Lambda_2)^{-1}D(\Lambda_1)^{-1}\gamma^\mu D(\Lambda_1)D(\Lambda_2) = D(\Lambda_2)^{-1}(\Lambda_1)^\mu_\nu \gamma^\nu D(\Lambda_2) = (\Lambda_1)^\mu_\nu (\Lambda_2)^\nu_\delta \gamma^\delta$$

3. We have:

$$D(\Lambda)^{-1}\gamma^\mu D(\Lambda) = (Id + \frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu)^{-1}\gamma^\mu(Id + \frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu)$$

Now using that:

$$(Id + \varepsilon)^{-1} = Id - \varepsilon + \mathcal{O}(\varepsilon^2)$$

We can simplify the above to:

$$(Id - \frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu)\gamma^\delta(Id + \frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu) = \gamma^\mu - \frac{1}{4}\omega_{\rho\sigma}[\gamma^\rho\gamma^\sigma, \gamma^\mu] = \gamma^\mu - \frac{1}{4}\omega_{\rho\sigma}(\gamma^\rho\{\gamma^\sigma, \gamma^\mu\} - \{\gamma^\rho, \gamma^\mu\}\gamma^\sigma)$$

Which using the anticommutator relation of the γ matrices we can simplify to:

$$\gamma^\mu - \frac{1}{4}\omega_{\rho\sigma}(\gamma^\rho 2\eta^{\sigma\mu} - \gamma^\sigma 2\eta^{\rho\mu})$$

Now using that $\omega_{\rho\sigma}$ is antisymmetric we get:

$$\gamma^\mu + \omega_{\rho\sigma}\eta^{\rho\mu}\gamma^\sigma = \gamma^\mu + \omega^\mu_\sigma\gamma^\sigma = (\Lambda^\mu_\sigma)\gamma^\sigma$$

4. We have that:

$$\gamma^0\gamma^j = \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix} \text{ and } \gamma^i\gamma^j = -\begin{pmatrix} -\sigma_i\sigma_j & 0 \\ 0 & -\sigma_i\sigma_j \end{pmatrix} = \begin{pmatrix} \delta_{ij} + i\varepsilon_{ijk}\sigma_k & 0 \\ 0 & \delta_{ij} + i\varepsilon_{ijk}\sigma_k \end{pmatrix}$$

6.3 The conserved current

1. Since H is Hermitian we have that:
$$H = (H^T)^\star$$
Hence β must be Hermitian and we also have that:
$$\alpha_j^T(-i\partial_j) = (\alpha_j^T(-i\partial_j))^\star = (\alpha_j^T)^\star i\partial_j$$
Hence the α_j matrices must be anti-hermitian.

2. We have:
$$i\frac{\partial \psi^\dagger}{\partial t} = \psi^\dagger H^\dagger$$
Hence:
$$i\frac{\partial \psi^\dagger \psi}{\partial t} = \psi^\dagger H^\dagger \psi + \psi^\dagger H \psi = (\psi^\dagger H \psi)^\dagger + \psi^\dagger H \psi = -i\boldsymbol{\nabla}\cdot\psi^\dagger\alpha\psi$$
Which can be re-written as:
$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j} = 0$$

3. We have that:
$$\partial_0 j^0 = \partial_0\psi^\dagger i\gamma^0 i\gamma^0\psi = -\partial_0\rho$$
Similarly:
$$\partial_\ell j^\ell = \partial_\ell\psi^\dagger i\gamma^0 i\gamma^\ell\psi = -\partial_\ell\psi^\dagger\alpha^\ell\psi$$
Hence we retrieve the expression required.

4. Notice that γ^0 is anti-hermitian whilst the γ^i are hermitian hence:
$$D(\Lambda)^\dagger = Id + \frac{1}{4}\omega_{\mu\nu}(\gamma^\mu\gamma^\nu)^\dagger = 1 + \frac{1}{4}\omega_{\mu\nu}\gamma^0\gamma^\nu\gamma^0\gamma^0\gamma^\mu\gamma^0 = (i\gamma^0)\underbrace{(1+\frac{1}{4}\omega_{\mu\nu}\gamma^\nu\gamma^\mu)}_{D(\Lambda)^{-1}}(i\gamma^0)$$
Hence we get that:
$$\bar{\psi}'(x') = \psi^\dagger(i\gamma^0)D(\Lambda)^{-1}(i\gamma^0)(i\gamma^0) = \bar{\psi}D(\Lambda)^{-1}$$

5. Hence we have that:
$$j'^\mu(x') = \bar{\psi}' i\gamma^\mu\psi' = \bar{\psi}D^{-1}(\Lambda)i\gamma^\mu D(\Lambda)\psi = \bar{\psi}i\Lambda^\mu_\nu\gamma^\nu\psi = \Lambda^\mu_\nu j^\nu$$
Therefore j is a 4-vector.