GR - Cosmology TDs

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Chapter 1

TD 1

1.1 Transverse Doppler effect and aberration.

1. ...
2. ...
3. ...

1.2 Relativistic invariants and conserved quantities.

1. (a) The four position is given by $\vec{x} = \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix}$ and we have that the proper time differential is given by: $\mathrm{d}t = \gamma \mathrm{d}\tau$. Hence the four velocity is given by:

$$\vec{u} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \gamma \begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}$$

   (b) We have the simple relation $p^\mu = mu^\mu$. Which gives:

$$\vec{p} = \begin{pmatrix} \gamma mc \\ \gamma m\mathbf{u} \end{pmatrix} = \begin{pmatrix} \frac{E}{c} \\ \gamma m\mathbf{u} \end{pmatrix}$$

   (c) The invariant built from the momentum is given by:

$$p_\mu p^\mu = \eta_{\mu,\nu} p^\mu p^\nu = -m^2c^2$$

   Which gives:

$$m^2c^4 = E^2 - |\vec{p}|^2c^2$$

2. (a) The conserved physical quantities are the energy of the whole system and the momentum of the center of mass. Hence in other words the four momentum is conserved.

   (b) The photon will lose energy and hence it will have a lower frequency $\nu' < \nu$. The four-momenta are given by:

$$p_{e0} = (m, 0, 0, 0), p_{p0} = (E_0, 0, 0, E_0)$$

   And after the collision by:

$$p_{e1} = (E'', 0, p_y, p_z), p_{p1} = (E', 0, -E' \sin\theta, -E' \cos\theta)$$

   (c)

Chapter 2

Chapter 3

3.1 Covariant derivatives and Christoffel symbols

1. (a) We have that:

$$J^{\mu}_{\mu'}J^{\mu'}_{\nu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\mu'}}{\partial x^{\nu}} = \frac{\partial x^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu}$$

   (b) We have that:

$$\left(J^{\mu}_{\sigma'}J^{\mu'}_{\mu}\right)_{,\gamma} = \left(\delta^{\mu}_{\mu}\right)_{,\gamma} = 0 \Rightarrow J^{\mu}_{\sigma',\gamma}J^{\mu'}_{\mu} = -J^{\mu}_{\sigma'}J^{\mu'}_{\mu,\gamma}$$

2. (a) We have that:

$$\partial_{\mu'}\phi'(x') = \frac{\partial}{\partial x^{\mu'}}\phi(x) = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial}{\partial x^{\mu}}\phi(x) = J^{\mu}_{\mu'}\partial_{\mu}\phi(x)$$

   Hence it is a tensor.

   (b) No.

   (c) We have:

$$\begin{aligned}\boldsymbol{\nabla}_{\mu'}A^{\nu'} = \partial_{\mu'}A^{\nu'} = J^{\mu}_{\mu'}J^{\nu'}_{\nu}(\partial_{\mu}A^{\nu} + \Gamma^{\nu}_{\mu\rho}A^{\rho}) &\Leftrightarrow J^{\mu}_{\mu'}J^{\nu'}_{\nu}\partial_{\mu}A^{\mu} + J^{\mu}_{\mu'}J^{\nu'}_{\nu,\mu'}A^{\nu} = J^{\mu}_{\mu'}J^{\nu'}_{\nu}\partial_{\mu}A^{\nu} + J^{\mu}_{\mu'}J^{\nu}_{\nu'}\Gamma^{\nu}_{\mu\rho}A^{\rho} \\ &\Leftrightarrow (J^{\mu}_{\mu'}J^{\nu'}_{\nu,\mu'} - J^{\mu}_{\mu'}J^{\nu'}_{\sigma}\Gamma^{\sigma}_{\mu\nu})A^{\nu} = 0\end{aligned}$$

   Which gives the desired result.

   (d) The exchange of the lower indices changes nothing.

   (e) We consider the invariant quantity $B^{\mu}A_{\mu}$ then:

$$\boldsymbol{\nabla}_{\rho}(A_{\mu}B^{\mu}) = \partial_{\rho}(A_{\mu}B^{\mu}) = (\partial_{\rho}A_{\mu})B^{\mu} + A_{\mu}\partial_{\rho}B^{\mu}$$

   However we also have that:

$$\boldsymbol{\nabla}_{\rho}(A_{\mu}B^{\mu}) = (\boldsymbol{\nabla}_{\rho}A_{\mu})B^{\mu} + A_{\mu}\boldsymbol{\nabla}_{\rho}B^{\mu} = (\boldsymbol{\nabla}_{\rho}A_{\mu})B^{\mu} + A_{\mu}\partial_{\rho}B^{\mu} + A_{\mu}\Gamma^{\mu}_{\rho v}B^{\nu}$$

   Now equating both sides we obtain:

$$(\partial_{\rho}A_{\mu})B^{\mu} = \Gamma^{\sigma}_{\rho M}B^{\mu}A_{\sigma} + \boldsymbol{\nabla}_{\rho}A_{\mu}B^{\mu}$$

   Which gives:

$$\boldsymbol{\nabla}_{\rho}A_{\mu} = \partial_{\rho}A_{\mu} - \Gamma^{\sigma}_{\rho\mu}A_{\sigma}$$

3. (a) We have that:

$$\Gamma^{\rho'}_{\nu'\mu'} = \frac{\partial x^{\rho'}}{\partial X^{\alpha}}\frac{\partial^2 X^{\alpha}}{\partial x^{\mu'}\partial x^{\nu'}} =$$

   (b)

3.2 Geodesics and the shortest path

1. (a) We have that:

$$S = -m\int |\mathrm{d}s| = -m\int \sqrt{\mathrm{d}s_\mu \mathrm{d}s^\mu} = -m\int \sqrt{g_{\mu\nu}\mathrm{d}s^\mu \mathrm{d}s^\nu}$$

Now notice that:

$$\mathrm{d}s = \frac{\mathrm{d}x}{\mathrm{d}\lambda}\mathrm{d}\lambda$$

Hence re-writing on top we obtain:

$$S = -m\int \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\mathrm{d}\lambda$$

(b) The Euler-Lagrange equations are given by:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{-g_{\mu\nu}\dot{x}^\mu}{2\mathcal{L}} =$$