HW2 - Probability

Marco Biroli

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1 Change of variables

1. From the change of variable theorem we know that:

$$f_{U,V}(u,v) = f_{X,Y}(uv, v(1-u))|J|^{-1}$$

Where:

$$J = \begin{vmatrix} \frac{y}{(x+y)^2} & -\frac{x}{(x+y)^2} \\ 1 & 1 \end{vmatrix} = \frac{1}{x+y} = v^{-1}$$

Then replacing in the definition and using the fact that X and Y are independent and hence we can split the joint law we get that:

$$\begin{aligned} f_{U,V}(u,v) &= \frac{uv^{k-1}}{(k-1)!}e^{-uv}1_{\mathbb{R}^+}(uv)\frac{v^{k-1}(1-u)^{k-1}}{(k-1)!}e^{-v(1-u)}1_{\mathbb{R}^+}(v(1-u))v \\ &= \frac{e^{-v}\sqrt{v^2}\left((1-u)uv^2\right)^{k-1}}{((k-1)!)^2}1_{\mathbb{R}^+}(uv)1_{\mathbb{R}^+}(v-uv) \end{aligned}$$

Then integrating for u on $\mathbb{R}$ gives:

$$f(v) = ...$$

2. ...

2 Order statistics

1. Let $(\Omega_i, \mathcal{F}_i, P_i)$ be the probability space of X_i then define the product probability space as $(\Omega, \mathcal{F}, P)$. Let $(\Omega, \mathcal{F}, P)$ also be the probability space of T. Then we define:

$$\begin{aligned} X_T : \Omega &\longrightarrow \mathbb{R} \\ \mathbf{x} &\longmapsto \mathbf{x}_{T(\mathbf{x})} \end{aligned}$$

Then let $B \in \mathcal{B}(\mathbb{R})$ then we have that:

$$\{\mathbf{x} \in \Omega : X_T(\mathbf{x}) \in B\} \subset \bigotimes_{i \in [\![1,n]\!]} \{x_i \in \Omega_i : X_i(x_i) \in B\} \in \mathcal{F}$$

Where the belonging to $\mathcal{F}$ follows from the definition of the product σ-algebra.

2. I think that $(X_{(1)}, \cdots, X_{(n)})$ is ill-defined since there exists no clear order relation on functions which might not even come from the same space. I assume that what was meant was that:

$$\forall \omega \in \Omega, \exists \sigma \in \mathfrak{S}_n, \sigma(X(\omega)) = \sigma\left((X_1(\omega_1), \cdots, X_n(\omega_n))\right) = (X_{\sigma(1)}(\omega_{\sigma(1)}), \cdots, X_{\sigma(n)}(\omega_{\sigma(n)})) \quad \text{is in increasing order.}$$

Since we have a finite list of real numbers we know from the constructions of the real numbers we can order it. Then we define the permutation σ_ω as the one which sets them in the right order and in case of parity the smaller index goes first. Then we have that σ is a random variable defined as:

$$\begin{aligned} \sigma : \Omega &\longrightarrow \mathfrak{S}_n \\ \omega &\longmapsto \sigma_\omega \end{aligned}$$

We furthermore have that σ is injective and therefore measurable. Hence σ is a well-defined random variable.

3. From the previous question we write $(X_{(1)}, \cdots, X_{(n)}) = \sigma(X)$. Then notice that:

$$f_{\sigma(X)}(\mathbf{x})\mathrm{d}\mathbf{x} = \sum_{\mu\in\mathfrak{S}_n} f_X(\mu^{-1}(\mathbf{x}))\mathrm{d}\mathbf{x} = \sum_{\mu\in\mathfrak{S}_n} f_X(\mu(\mathbf{x}))\mathrm{d}\mathbf{x} = \sum_{\mu\in\mathfrak{S}_n}\prod_{i=1}^n f_{X_i}(\mu(\mathbf{x})_i)\mathrm{d}\mathbf{x}$$

Where on the last equality we used that the X_i are independent. Then since the X_i are identically distributed we have that $\forall i,\ f_{X_i} = f_{X_1}$. Now since the product commutes we have that the terms inside the sum are all equal up to a permutation of the terms, hence:

$$\sum_{\mu\in\mathfrak{S}_n}\prod_{i=1}^n f_{X_i}(\mu(\mathbf{x})_i)\mathrm{d}\mathbf{x} = \sum_{\mu\in\mathfrak{S}_n}\left(\prod_{i=1}^n f_{X_1}(x_i)\mathrm{d}x_i\right) = n!\left(\prod_{i=1}^n f_{X_1}(x_i)\mathrm{d}x_i\right) = n! f_X(\mathbf{x}')1_{\mathbf{x}'=\mu(\mathbf{x})}\mathrm{d}\mathbf{x}'$$

Where we are free to chose any $\mu \in \mathfrak{S}_n$ since the terms in the product commute. If we fix ourselves with the choice $\mu = \sigma$ we get:

$$\sum_{\mu\in\mathfrak{S}_n}\prod_{i=1}^n f_{X_i}(\mu(\mathbf{x})_i)\mathrm{d}\mathbf{x} = n! f_X(\sigma(\mathbf{x})) = n! f_X(\mathbf{x}')1_{\mathbf{x}'=\sigma(\mathbf{x})}\mathrm{d}\mathbf{x}$$

Call μ the function that maps $X_1, \cdots, X_n$ to $X_1, \cdots, X_n - X_{n-1}$. Then plugging this in the definition of the expectancy we get:

$$\begin{aligned}E[\varphi(\mu(\sigma(X)))] &= \int_{\mathbf{x}\in\Omega}\varphi(\mu(\sigma(X(\mathbf{x}))))f_{\mu(\sigma(X))}(\mu(\mathbf{x}))\mathrm{d}\mathbf{x} = n!\int_{\mathbf{x}\in\Omega}\varphi(\mu(\sigma(X(\mathbf{x}))))f_X(\mathbf{x}')1_{\mathbf{x}'=\sigma(\mathbf{x})}\mathrm{d}\mathbf{x}\\ &= n!\int_{\mathbf{x}\in\sigma(\Omega)}\varphi(\mu(X(\mathbf{x})))f_X(\mathbf{x})\mathrm{d}\mathbf{x} = n!\mathbb{E}[\mu(\varphi(X))1_\sigma] \text{ where } 1_\sigma(\mathbf{x}) = \begin{cases}1 & \text{if } \mathbf{x} = \sigma(\mathbf{x})\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

4. From the Block grouping theorem we know that if $X_1, \cdots, X_n$ are independent then $X_1, X_2 - X_1, \cdots, X_n - X_{n-1}$ are independent. Hence taking $\varphi(\mu(\mathbf{x})) = \prod_{i=1}^n g_i(\mu(\mathbf{x_i}))$ where all the f_i are measurable we get that:

$$\mathbb{E}\left[\prod_{i=1}^n g_i(\mu(\sigma(X))_i)\right] = \mathbb{E}\left[n!\,1_\sigma\prod_{i=1}^n g_i(\mu(X)_i)\right] = n!\prod_{i=1}^n\int_{\mathbf{x}\in\Omega} f_{X_1}(g_i(\mu(X(\mathbf{x}))_i))P(\mathbf{x} = \sigma(\mathbf{x}))\mathrm{d}\mathbf{x} = \prod_{i=1}^n \mathbb{E}[g_i(\mu(\sigma(X))_i]$$

Hence the $\mu(\sigma(X))$ are independent. Notice that in the first equality we used question 3, in the second equality we used the independence of $\mu(X)$ and in the third we simply used that $P(\mathbf{x} = \sigma(\mathbf{x})) = \frac{1}{n!}$ and then recontract the integral into an expectancy. Then we have that $X_{(1)} = \min_i X_i$ hence:

$$F_{X_{(1)}}(x) = 1 - \prod_{i=1}^n P(X_i > x) = 1 - \prod_{i=1}^n e^{-\alpha x} = 1 - e^{-\alpha n x}$$

So $X_{(1)}$ follows an exponential law of parameter $n\alpha$. Now consider $X_{(i+1)} - X_{(i)}$. We have that this can be re-written as:

$$X_{(i+1)} - X_{(i)} = \min_{i\in[\![1,n]\!], X_i > X_{(i)}} X_i - X_{(i)}$$

However notice that:

$$P(X_i = x + y | X_i > x) = \frac{P(X_i = x + y \cap X_i > x)}{P(X_i > x)} = \frac{\alpha e^{-\alpha(x+y)}}{e^{-\alpha x}} = \alpha e^{-\alpha y} = P(X_i = y)$$

Hence we get that:

$$X_{(i+1)} - X_{(i)} = \min_{i\in[\![1,n-i]\!]} X_i \sim \mathrm{Exp}(\alpha(n-i))$$

5. It is well know that the expectancy of an exponential random variable of parameter α is given by $\frac{1}{\alpha}$. Hence from the previous question we have that:

$$\mathbb{E}[X_{(i+1)} - X_{(i)}] = \frac{1}{\alpha(n-i)} \quad\text{and}\quad \mathbb{E}[X_{(1)}] = \frac{1}{\alpha n}$$

Denote by $u_i = \mathbb{E}[X_{(i)}]$ then we have that:

$$u_1 = \frac{1}{\alpha n} \quad\text{and}\quad u_{i+1} = u_i + \frac{1}{\alpha(n-i)} = \sum_{\ell=0}^{i}\frac{1}{\alpha(n-\ell)}$$

Now the sum can be written as:

$$\sum_{\ell=0}^{i} \frac{1}{\alpha(n-\ell)} = \frac{1}{\alpha}\left(\sum_{\ell=0}^{n-1} \frac{1}{n-\ell} - \sum_{\ell=i+1}^{n-1} \frac{1}{n-\ell}\right) = \frac{1}{\alpha}\left(\sum_{\ell=1}^{n} \frac{1}{\ell} - \sum_{\ell=1}^{n-i-1} \frac{1}{\ell}\right) = \frac{1}{\alpha}\left(\sum_{\ell=1}^{n} \frac{1}{\ell} - \gamma - (\sum_{\ell=1}^{n-i-1} \frac{1}{\ell} - \gamma)\right)$$

Now using the definition of the digamma function we have that:

$$\sum_{\ell=0}^{i} = \frac{1}{\alpha}\left(\frac{\Gamma'(n+1)}{\Gamma(n+1)} + \frac{\Gamma'(n-i)}{\Gamma(n-i)}\right)$$

6. Notice that:

$$f_{X_{(k)}} = f_{X_{(1)}+(X_{(2)}-X_{(1)})+\cdots+(X_{(k)}-X_{(k-1)})} = f_{X_{(1)}} \star f_{X_{(2)}-X_{(1)}} \star \cdots \star f_{X_{(k)}-X_{(k-1)}}$$

Or in other words if we denote by Y_j exponential random variables of parameter α we have that:

$$X_{(k)} = \sum_{i=1}^{k-1} X_{(i)} - X_{(i-1)} 1_{i>1} = \sum_{i=1}^{k} \frac{Y_i}{n-i+1}$$

7. In general we have that:

$$F_{X_{(k)}}(x) = P\left(\max_{i\in\mathcal{I}} X_i < x \wedge \min_{i\in[\![1,n]\!]\setminus\mathcal{I}} X_i > x \big| |\mathcal{I}| = k\right) = \binom{n}{k} F_{X_1}(x)^k (1-F_{X_1}(x))^{n-k}$$