Midterm homework problems

Marco Biroli

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1 Divergence and Laplacian

1. We have the definition of Christoffel symbols:
$$\Gamma^k_{ij} = \frac{\partial \mathbf{e}_i}{\partial x^j} \cdot \mathbf{e}^k$$
Then we have that:
$$\boldsymbol{\nabla} \cdot \mathbf{V} = \partial_i (V^j \mathbf{e_j})^i = \frac{\partial V^i}{\partial x^i} + \Gamma^i_{ij} V^j = V^i_{,i} + \frac{1}{2} g^{im}(g_{mi,j} + g_{mj,i} - g_{ij,m})V^j$$
Then:
$$...$$
2. Since the determinant is an invariant scalar of the matrix then from the relation: $g^{\mu\nu} = g^{-1} c^{\mu\nu}$ we know that c transforms in the exact same way as g does. Since g is a tensor then so is c.
3. We have that:
$$g = \sum_\nu g_{\mu\nu} c^{\mu\nu} \text{ hence } \frac{\partial g}{\partial g_{\mu\nu}} = \frac{\partial}{\partial g_{\mu\nu}} \sum_{\nu'} g_{\mu\nu'} c^{\mu\nu'} = c^{\mu\nu}$$
4. We have that:
$$g^{\mu\nu} g_{\mu\nu,\gamma} = \partial_\gamma \log g$$
We have that:
$$\partial_\gamma g(g_{\mu\nu}) = (\partial_\gamma g) g_{\mu\nu} + g \partial_\gamma g_{\mu\nu}$$
We have that:
$$\partial_\gamma g = \frac{\partial}{\partial g_{\mu\nu}} \frac{\partial g_{\mu\nu}}{\partial \gamma} g = \frac{\partial}{\partial g_{\mu\nu}} g_{\mu\nu,\gamma} g = g_{\mu\nu,\gamma} c^{\mu\nu}$$
Hence:
$$...$$
5. We start from the end and we differentiate to obtain:
$$\frac{1}{\sqrt{|g|}} \partial_\gamma (\sqrt{|g|} V^\gamma) = V^\gamma{}_{,\gamma} + \frac{1}{\sqrt{|g|}} V^\gamma \frac{1}{2\sqrt{|g|}} \partial_\gamma |g| = V^\mu{}_{,\mu} + \frac{1}{2} V^\gamma \frac{\partial_\gamma |g|}{|g|} = V^\mu{}_{,\mu} + \frac{1}{2} V^\gamma \log |g|$$
Now using question 4 we re-obtain the formula of question 1 and this concludes the proof.
6. Using the above formula by replacing: $V_\gamma = f_{,\gamma}$ (hence $V^\gamma = g^{\gamma\mu} f_{,\mu}$) we obtain:
$$\nabla^2 f = \frac{1}{\sqrt{|g|}} \partial_\gamma (\sqrt{|g|} f^{,\gamma}) = \frac{1}{\sqrt{|g|}} \partial_\gamma (\sqrt{|g|} g^{\gamma\mu} f_{,\mu})$$
7. In spherical coordinates we have that:
$$[g_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$
Then in order to apply the previous formula we need to compute g and $[g^{\mu\nu}]$. We have quite simply:
$$g = r^4 \sin^2\theta \text{ and } g^{\mu\mu} = \frac{1}{g_{\mu\mu}} \text{ and } g^{\mu\nu} = 0 \text{ otherwise.}$$

Plugging this in the previous formula we obtain:

$$\nabla^2 f = \frac{1}{r^2 \sin\theta}\partial_\gamma(r^2 \sin\theta g^{\gamma\mu} f_{,\mu}) = \frac{1}{r^2 \sin\theta}\left(\partial_r(r^2 \sin\theta f_{,r}) + \partial_\theta(\sin\theta f_{,\theta}) + \partial_\varphi(\frac{1}{\sin\theta} f_{,\varphi})\right)$$

Now simplifying the derivatives gives:

$$\begin{aligned}\nabla^2 f &= \frac{1}{r^2 \sin\theta}\left(\sin\theta\partial_r(r^2 f_{,r}) + \partial_\theta(\sin\theta f_{,\theta}) + \frac{1}{\sin\theta}\partial_\varphi f_{,\varphi}\right) \\ &= \frac{1}{r^2}\partial_r(r^2 f_{,r}) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta f_{,\theta}) + \frac{1}{r^2\sin^2\theta}\partial_\varphi f_{,\varphi}\end{aligned}$$

8. Repeating an identical argument but using:

$$[g_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{pmatrix}$$

Gives us immediately that:

$$\nabla^2 f = \frac{1}{r}\partial_\gamma(r g^{\gamma\mu} f_{,\mu}) = r^{-1}(\partial_z(r f_{,z} + \partial_r(r f_{,r}) + \partial_\phi(r^{-1} f_{,\phi})) = f_{,zz} + r^{-1} f_{,r} + f_{,rr} + r^{-2} f_{,\phi\phi}$$

2 Rotating coordinate frame.

1. We have that:

$$t = t \quad \text{and} \quad z = z' \quad \text{and} \quad r = r' \quad \text{and} \quad \phi = \phi' - \Omega t$$

Hence we immediately get that:

$$\mathrm{d}t = \mathrm{d}t \quad \text{and} \quad \mathrm{d}z = \mathrm{d}z' \quad \text{and} \quad \mathrm{d}r = \mathrm{d}r' \quad \text{and} \quad \mathrm{d}\phi = \mathrm{d}\phi' - t\mathrm{d}\Omega - \Omega\mathrm{d}t = \mathrm{d}\phi' - \Omega\mathrm{d}t$$

Where in the last equality we add the assumption that we place ourselves in a rotating frame at constant angular velocity. Now plugging this in the expression for a line element we obtain:

$$\begin{aligned}\mathrm{d}s^2 &= -c^2\mathrm{d}t^2 + (\mathrm{d}z')^2 + (\mathrm{d}r')^2 + (r')^2(\mathrm{d}\phi')^2 = -c^2\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\phi + \Omega\mathrm{d}t)^2 \\ &= -c^2\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + r^2\Omega^2\mathrm{d}t^2 + 2r^2\Omega\mathrm{d}\phi\mathrm{d}t \\ &= (r^2\Omega^2 - c^2)\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + 2r^2\Omega\mathrm{d}t\mathrm{d}\phi\end{aligned}$$

Hence we also get:

$$[g_{\mu\nu}] = \begin{pmatrix} (r^2\Omega^2 - c^2) & 0 & 0 & r^2\Omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ r^2\Omega & 0 & 0 & r^2 \end{pmatrix}$$

2. The inverse can be immediately obtained through it's cofactor formulation and gives:

$$[g^{\mu\nu}] = \begin{pmatrix} -\frac{1}{c^2} & 0 & 0 & \frac{\Omega}{c^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\Omega}{c^2} & 0 & 0 & \frac{c^2 - r^2\Omega^2}{c^2 r^2} \end{pmatrix} \quad \text{and} \quad g = -c^2 r^2$$

3. We have that:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Omega t & \sin\Omega t & 0 \\ 0 & -\sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Now notice that the transition matrix is orthogonal hence we immediately have that:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Omega t & -\sin\Omega t & 0 \\ 0 & \sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Hence we obtain immediately that:

$$\begin{aligned}\mathrm{d}s^2 &= -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = -c^2\mathrm{d}t^2 + (\mathrm{d}(x'\cos\Omega t - y'\sin\Omega t))^2 + (\mathrm{d}(x'\sin\Omega t + y'\cos\Omega t))^2 + (\mathrm{d}z')^2 \\ &= -c^2\mathrm{d}t^2 + (\cos\Omega t\mathrm{d}x' - x'\Omega\mathrm{d}t\sin\Omega t - \sin\Omega t\mathrm{d}y' - y'\Omega\mathrm{d}t\cos\Omega t)^2 = ...\end{aligned}$$

4. We have that:

$$\begin{pmatrix} -1+h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & 1+h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & 1+h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & 1+h_{33} \end{pmatrix} = \begin{pmatrix} -(1-(x^2+y^2)\Omega^2) & \Omega y & -\Omega x & 0 \\ \Omega y & 1 & 0 & 0 \\ -\Omega x & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence we get:

$$[h_{\mu\nu}] = \begin{pmatrix} (x^2+y^2)\Omega^2 & \Omega y & -\Omega x & 0 \\ \Omega y & 0 & 0 & 0 \\ -\Omega x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3 Frame dragging by a moving rod.

1. We have the equation:

$$\nabla^2\Phi = 4\pi(G_N/c^2)\rho\Theta(R-r)$$

From symmetry arguments we know already that Φ will only be a function of r. Hence we have that by plugging the expression of the laplacian found in part 1 we obtain:

$$\Phi_{,zz} + \Phi_{,rr} + \Phi_{,r}/r + \Phi_{,\phi\phi}/r^2 = \Phi_{,rr} + \Phi_{,r}/r = 4\pi(G_N/c^2)\rho$$

This can be re-written as:

$$\partial_r(r\Phi_{,r}) = 4\pi(G_N/c^2)\rho r \Rightarrow r\Phi_{,r} = 4\pi(G_N/c^2)\rho r^2/2 + c$$

Which gives immediately through integration a solution of the form:

$$\Phi(r) = \pi(G_N/c^2)\rho r^2 + c\log(r) + c'$$

Now the condition $\Phi_{,r}(0) = 0$ ensures that $c = 0$ and the condition $\Phi(R) = 0$ ensures that $c' = -\pi(G_N/c^2)\rho R^2$ hence the final solution is given by:

$$\Phi(r) = \pi(G_N/c^2)\rho(r^2 - R^2)$$

2. We have that:

$$\overline{h_{\mu\nu}} = -4\Phi\delta^0_\mu\delta^0_\nu$$

Hence we get that:

$$h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h = -4\Phi\delta^0_\mu\delta^0_\nu$$

Now we have that:

$$\eta_{\mu\nu}(\overline{h}_{\mu\nu}) = h - \frac{1}{2}Tr(\eta)h = h - 2h = -h$$

Which when plugged in the previous equation gives immediately that:

$$-h = 4\Phi \Leftrightarrow h = -4\Phi$$

Hence we obtain that:

$$h_{\mu\nu}(r) = -2\Phi\delta_{\mu\nu}$$

3. We have that:

$$\mathrm{d}s^2 = (1-2\Phi)(\mathrm{d}r^2 + r^2\mathrm{d}\phi^2)$$

Now we apply the following transformations:

$$r' = (1 + r\Phi_{,r})(1-\Phi)r \quad \text{and} \quad \phi' = (1 - \Phi_{,r}/r)\phi$$

Using our previous results we can re-write this as:

$$r' = (1 + 2\alpha r^2)(1 - \alpha(r^2 - R^2))r \quad \text{and} \quad \phi' = (1 - 2\alpha)\phi$$

Hence we have that:

$$\begin{aligned} \mathrm{d}r' &= \mathrm{d}(r\Phi_{,r})\,(1-\Phi)r + (1+r\Phi_{,r})\,\mathrm{d}(-\Phi)\,r + (1+r\Phi_{,r})(1-\Phi)\mathrm{d}r \\ &= 4\pi(G_N/c^2)\rho r\mathrm{d}r(1-\Phi)r - (1+r\Phi_{,r})\Phi_{,r}\mathrm{d}r + (1+r\Phi_{,r})(1-\Phi)\mathrm{d}r \\ &= \mathrm{d}r\left(\alpha r - \alpha r\Phi - \Phi_{,r} - r\Phi_{,r}^2 + 1 - \Phi + r\Phi_{,r} - r\Phi_{,r}^2\right) \end{aligned}$$

Similarly we get:

$$\mathrm{d}\phi' = (1 - \Phi_{,r}/r)\mathrm{d}\phi - (\mathrm{d}\Phi_{,r})/r\Phi + \Phi_{,r}/r^2\phi\mathrm{d}r = (1 - \Phi_{,r}/r)\mathrm{d}\phi - ()/r\Phi + \Phi_{,r}/r^2\phi\mathrm{d}r$$

4.

5. Hence we obtain:

$$[h_{\alpha'\beta'}] = -2\Phi(r')\begin{pmatrix} 1 & & -2v/c \\ & \mathbf{I} & \\ -2v/c & & 1 \end{pmatrix}$$

6. (a) We use the geodesic equation:

$$-\frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} = \Gamma^z_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$$

Then:

$$\Gamma^\nu_{\mu\lambda} = \frac{1}{2}\eta^{\nu\gamma}(\partial_\mu h_{\gamma\lambda} + \partial_\lambda h_{\gamma\mu} - \partial_\gamma h_{\lambda\mu})$$

We start by computing the Christoffel symbols:

$$\begin{aligned}
\Gamma^z_{00} &= \frac{1}{2}\eta^{z\gamma}(\partial_0 h_{\gamma 0} + \partial_0 h_{\gamma 0} - \partial_\gamma h_{00}) = \partial_0 h_{z0} - \frac{1}{2}\partial_z h_{00} = \partial_0 4\Phi(r)v/c + \partial_z\Phi(r) = 0\\
\Gamma^z_{01} &= \frac{1}{2}\eta^{z\gamma}(\partial_0 h_{\gamma 1} + \partial_1 h_{\gamma 0} - \partial_\gamma h_{10}) = \frac{1}{2}(\partial_0 h_{z1} + \partial_1 h_{z0} - \partial_z h_{10}) = \frac{1}{2}(0 + \partial_x 4\Phi(r)v/c + 0) = 2v/c\,\partial_x\Phi(r)\\
\Gamma^z_{02} &= \frac{1}{2}(\partial_0 h_{z2} + \partial_2 h_{z0} - \partial_z h_{20}) = 2v/c\,\partial_y\Phi(r)\\
\Gamma^z_{03} &= \frac{1}{2}(\partial_0 h_{z3} + \partial_3 h_{z3} - \partial_z h_{30}) = \frac{1}{2}(-2\partial_z\Phi(r) - 4v/c\partial_z\Phi(r)) = -(1-2v/c)\partial_z\Phi(r) = 0\\
\Gamma^z_{11} &= \frac{1}{2}(\partial_1 h_{z1} + \partial_1 h_{z1} - \partial_z h_{11}) = \partial_z\Phi(r) = 0\\
\Gamma^z_{12} &= \frac{1}{2}(\partial_1 h_{z2} + \partial_2 h_{z1} - \partial_z h_{21}) = 0\\
\Gamma^z_{13} &= \frac{1}{2}(\partial_1 h_{z3} + \partial_3 h_{z1} - \partial_z h_{31}) = -\partial_x\Phi(r)\\
\Gamma^z_{22} &= \frac{1}{2}(\partial_2 h_{z2} + \partial_2 h_{z2} - \partial_z h_{22}) = \partial_z\Phi(r) = 0\\
\Gamma^z_{23} &= \frac{1}{2}(\partial_2 h_{z3} + \partial_3 h_{z2} - \partial_z h_{32}) = -\partial_y\Phi(r)\\
\Gamma^z_{33} &= \frac{1}{2}(\partial_3 h_{z3} + \partial_3 h_{z3} - \partial_z h_{33}) = 0
\end{aligned}$$

Hence the geodesic equation becomes:

$$-\ddot{z} = \Gamma^z_{01}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x}{\mathrm{d}\tau} + \Gamma^z_{02}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}y}{\mathrm{d}\tau} + \Gamma^z_{13}\frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}z}{\mathrm{d}\tau} + \Gamma^z_{23}\frac{\mathrm{d}y}{\mathrm{d}\tau}\frac{\mathrm{d}z}{\mathrm{d}\tau}$$

Plugging in the values we get:

$$-\ddot{z} = 2v/c(\partial_x\Phi(r)\gamma\dot{x} + \partial_y\Phi(r)\dot{t}\dot{y}) - \dot{z}(\dot{x}\partial_x\Phi(r) + \dot{y}\partial_y\Phi(r)) = (\dot{\mathbf{x}}\cdot\boldsymbol{\nabla}\Phi)(\frac{2v}{c}\gamma - \dot{z}) = \dot{x}\partial_x\Phi(\frac{2v}{c}\dot{t} - \dot{z})$$

Where in the last equality we used the fact that $\dot{y} = y = 0$ and $\partial_z\Phi = 0$. Now a similar derivation for $\Gamma^t_{\mu\nu}$ and $\Gamma^x_{\mu\nu}$ yields:

$$-\ddot{t} = \frac{2v}{c}\dot{z}\dot{x}\partial_x\Phi \quad\text{and}\quad -\ddot{x} = \partial_x\Phi(\dot{t}^2 - \dot{x}^2 + \dot{z}^2)$$

Hence the final system of equation gives:

$$\begin{cases} -\ddot{z} = \dot{x}\partial_x\Phi\left(\frac{2v}{c}\dot{t} - \dot{z}\right)\\ -\ddot{t} = \frac{2v}{c}\dot{z}\dot{x}\partial_x\Phi\\ -\ddot{x} = \partial_x\Phi(\dot{t}^2 - \dot{x}^2 + \dot{z}^2)\end{cases}$$

Now we can also replace $\partial_x\Phi$ by its value: $\frac{2G_N\pi\rho}{c^2 x}$ (outside of the cylinder) which gives:

$$\begin{cases} -\ddot{z} = \dot{x}\frac{2G_N\pi\rho}{c^2 x}\left(\frac{2v}{c}\dot{t} - \dot{z}\right)\\ -\ddot{t} = \frac{2v}{c}\dot{z}\dot{x}\frac{2G_N\pi\rho}{c^2 x}\\ -\ddot{x} = \frac{2G_N\pi\rho}{c^2 x}(\dot{t}^2 - \dot{x}^2 + \dot{z}^2)\end{cases}$$

Now our physical intuition encourages us to try the Ansatz $t = \tau$ and $x = v_x\tau$. Plugging this in the equation for z gives:

$$-\ddot{z} = \underbrace{\frac{2G_N\pi\rho}{c^2}}_{\alpha}\tau^{-1}\left(\underbrace{\frac{2v}{c}}_{\beta} - \dot{z}\right) \Longleftrightarrow \ddot{z} = \alpha\tau^{-1}(\beta - \dot{z})$$

Now solving the above 2nd order ODE gives directly:

$$z(\tau) = \beta\left(\tau - \frac{\tau^{1+\alpha}}{1+\alpha}\right)$$

Which also gives:

$$\dot{z}(\tau) = \beta(1 - \tau^{\alpha})$$

This then gives the following plot for z:

...

4 Frame-dragging inside a rotating cylinder.

1. From symmetry arguments we know that Φ is only a function of x and y. Hence we have that:

$$\boldsymbol{\nabla}\Phi_{\mu\nu}(\mathbf{x}) = \kappa\int_{-h}^{h}\oint_C \mathrm{d}z'\mathrm{d}x'\mathrm{d}y' T_{\mu\nu}(\mathbf{x}')(\mathbf{x}-\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|^3$$

Now we use the well-know distribution formula:

$$\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{r}}{|\mathbf{r}|^3}\right) = \delta(\mathbf{r})$$

Which immediately gives us that:

$$\nabla^2\Phi_{\mu\nu}(\mathbf{x}) = \int T_{\mu\nu}(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\mathrm{d}\mathbf{x}' = T_{\mu\nu}(\mathbf{x})$$

This is the differential form of Gauss's Law for $\boldsymbol{\nabla}\Phi$. This also tells us that:

$$H2\pi r\boldsymbol{\nabla}\Phi_{\mu\nu}(r) = \oint_{\partial V}\boldsymbol{\nabla}\Phi_{\mu\nu}\cdot\mathrm{d}\mathbf{A} = \iiint_V \nabla^2\Phi_{\mu\nu}\mathrm{d}V = \iiint_V T_{\mu\nu}(\mathbf{x})\mathrm{d}V = T$$

Hence in order to write this as similar to Gauss's Law for electromagnetism we can re-write this as:

$$\boldsymbol{\nabla}\Phi_{\mu\nu}(r) = \frac{T}{2\pi rH}$$

Hence we obtain that:

$$\boldsymbol{\nabla}\Phi = \frac{\rho c^2}{2\pi rH}\begin{pmatrix}\pi r^2H & 0 & \pi r^2H\Omega & 0\\ 0 & 0 & 0 & 0\\ \pi r^2H\Omega & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} = \frac{\rho rc^2}{2}\begin{pmatrix}1 & 0 & \Omega & 0\\ 0 & 0 & 0 & 0\\ \Omega & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

2.