Math methods solutions

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Chapter 4

4.24

$f(z) = \frac{1}{z^4 \sin(\pi z)}$. We have poles in all of $\mathbb{Z}$. In particular we have a simple pole in $z \neq 0$ and a pole of order 5 in $z = 0$, to see this just multiply by $(z-a)($ resp $(z-a)^5)$ $, a \in \mathbb{Z}$ and use that $\sin(x)/x = 1$ for $x \to 0$. Then we want to use the residue theorem. In particular consider the circle γ_N of radius $N + \frac{1}{2}$ (notice that we want to avoid passing through a pole), which will be our loop. Then for $k \neq 0$, performing a simple first order Taylor expansion, we have that $\sin(\pi z) \underset{z \to k}{\sim} (z-k) \frac{\partial}{\partial z} \sin(\pi z)|_{z=k} = (z-k)\pi \cos(\pi k) = (z-k)\pi(-1)^k$. Hence $\text{Res}(f;k) = \lim_{z \to k}(z-k)f(z) = \lim_{z \to k} \frac{(z-k)}{z^4(z-k)\pi(-1)^k} = \frac{(-1)^k}{\pi k^4}$.

For $k = 0$ we have $\text{Res}(f;0) = \frac{1}{4!} \lim_{z \to 0} \frac{\partial^4}{\partial z^4}(z^5 f(z))$. Now we need to express sin with its Taylor series up to the 5-th order: we have that $\sin(\epsilon) = \epsilon - \frac{\epsilon^3}{3!} + \frac{\epsilon^5}{5!} + o(\epsilon^7)$ so that $\frac{1}{\sin(\epsilon)} = \frac{1}{\epsilon}(1 - \frac{\epsilon^2}{3!} + \frac{\epsilon^4}{5!} + o(\epsilon^6))^{-1} = \frac{1}{\epsilon}(1 + \frac{\epsilon^2}{3!} - \frac{\epsilon^4}{5!} + \frac{\epsilon^4}{3!^2} + o(\epsilon^6))$ (here we used that $1/(1-x) \sim 1 + x + x^2$ and pay attention to keep all of the orders up to the 5-th one). Hence

$$f(z) = \frac{1}{z^4 \sin(\pi z)} \underset{z \to 0}{\sim} \frac{1}{\pi z^5} \left(1 + \frac{\pi^2 z^2}{3!} + \pi^4 z^4 (\frac{1}{3!^2} - \frac{1}{5!}) + o(z^6)\right)$$

and so we get $\text{Res}(f;0) = \frac{1}{4!} \lim_{z \to 0} \frac{\partial^4}{\partial z^4}(z^5 f(z)) = \pi^3(\frac{1}{3!^2} - \frac{1}{5!}) = \frac{7\pi^3}{360}$.

Now we use the residue theorem on γ_N . On a

$$\int_{\gamma_N} f(z)dz = \frac{7\pi^3}{360} + \sum_{n=1}^{N} \frac{(-1)^n}{\pi n^4} + \sum_{n=1}^{N} \frac{(-1)^n}{\pi n^4} = \frac{7\pi^3}{360} + \frac{2}{\pi} \sum_{n=1}^{N} \frac{(-1)^n}{n^4}$$

On γ_n we have that $|\sin(\pi z)| \geq \delta > 0$. Hence for any $z \in \gamma_n$ we get that $|f(z)| \leq \frac{1}{\delta(N+1/2)^4}$ so that for $N \to \infty$ we get that

$$\left|\int_{\gamma_N} f(z)dz\right| \leq \frac{2\pi}{\delta(N+\frac{1}{2})^3} \to 0$$

Finally for $N \to \infty$ we get that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{7\pi^4}{720}$.

4.28

1)

We have that $\Gamma(0) = \int_0^\infty du \cdot e^{-u} = 1$. Then we have that $\Gamma(1/2) = \int_0^\infty \frac{du}{\sqrt{u}} e^{-u} = \int_0^\infty \frac{2tdt}{t} e^{-t^2} = 2\int_0^\infty dte^{-t^2} = \sqrt{(\pi)}$, with the substitution $t = \sqrt{u}$.

2)

We have that $\Gamma(x+1) = \int_0^\infty e^{-u}u^x = 0 + x\int_0^\infty duu^{x-1}e^{-u} = x\Gamma(x)$, where we did an integration by parts.

3)

$\Gamma(n) = n!\Gamma(1) = n!$ and

$$\Gamma(n+1/2) = (n-1/2)\Gamma(n-1/2) = \Gamma(1/2)\prod_{k=0}^{n-1}(k+1/2) = \sqrt{\pi}\prod_{k=0}^{n-1}(2k+1)/2 = \frac{\sqrt{\pi}}{2^n} 1 \cdot 3 \cdots (2n-1)$$

$$= \frac{\sqrt{\pi}}{2^n} \cdot \frac{n!}{2 \cdot 4 \cdots (2n)} = \frac{\sqrt{\pi}(2n)!}{2^{2n} n!}$$

4.29

1)

On a $F(x) = \int_a^b e^{xf(t)g(t)}dt$, with f having a maximum in $t = t_0$. THen we have that $F(x) = e^{xf(t_0)} \int_a^b e^{x|f(t)-f(t_0)|}g(t)dt$. The idea is that when $x \to \infty$, $e^{x|f(t)-f(t_0)|}$ is 1 in t_0 and goes exponentially to zero in all of the other values. Hence all of the values of t that contribute to the integral exponentially concentrate near t_0 . Therefore we can write

$$F(x) \sim_{x\to\infty} e^{xf(t_0)}g(t_0) \int_a^b e^{x|f(t)-f(t_0)|}dt$$

If we do an expansion we have that $f(t) = f(t_0) + 1/2f''(t_0)(t - t_0)^2 + O((t - t_0)^3)$, with $f''(t_0) < 0$. Hence

$$F(x) \sim e^{xf(t_0)}g(t_0) \int_a^b e^{1/2(f''(t_0)x(t-t_0)^2}$$

, when $x \to \infty$ the integral in nonzero only near t_0. Hence we can replace $a \to +\infty$ and $b \to \infty$. Then $F(x) \sim_{x\to\infty} e^{xf(t_0)}g(t_0)\sqrt{\frac{2\pi}{-f''(t_0)}}\frac{1}{\sqrt{x}}$

2)

Let's take again the gamma function, we have that

$$\Gamma(x+1) = \int_0^\infty du \cdot e^{-u}u^x = \int due^{-u+x\ln(u)}$$

and $f(u) = -u + x\ln(u)$ is maximal in $u_0 = x$ and $f(u_0) = -x + x\ln(x)$. Then with $t = u/x$ we have that

$$\Gamma(x+1) = \int_0^\infty (dtxx?)e^{-xt}x^x t^x = xx^x \int_0^\infty dte^{-xt+x\ln(t)} = xx^x \int_0^\infty dte^{xg(t)}$$

with $g(t) = -t + \ln(t)$. Notice that g has a maximum which is unique and is located at 1, so that we can apply Laplace method:
$g''(t) = -\frac{1}{t^2} \Rightarrow g''(1) = -1$. Therefore $\Gamma(x+1) \sim_{x\to\infty} xx^x e^{-x}\frac{1}{\sqrt{x}}\sqrt{\frac{2\pi}{-(-1)}}$ so that $\Gamma(x+1) \sim_{x\to\infty} (x/e)^x\sqrt{2\pi x}$.
If $x = n$, $\Gamma(n+1) = n! \sim_{n\to\infty} (n/e)^n\sqrt{2\pi n}$