Advanced Quantum Physics
Week 4

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The Stern-Gerlach experiment

The Stern-Gerlach experiment has shown that particles possess an intrinsic angular momentum, *the spin*, that is closely analogous to the angular momentum of a classically spinning object, but that takes only certain quantized values. It has also shown that only one component of a particle's spin can be measured at the same time: a measurement along the z-axis will change the state of the spin along the x direction. The experiment was conducted by the German physicists Otto Stern and Walter Gerlach in 1922.

Magnetic moments in classical physics

In classical physics, a charged particle with charge q moving in a magnetic field $\vec{B}$ with a velocity $\vec{v}$ experiences a Lorentz force

$$\vec{F} = q\vec{v} \times \vec{B}$$

If the particle rotates in a current loop, it generates a magnetic moment $\vec{\mu}$ orthogonal to the plane of the current loop

$$\vec{\mu} = iS\vec{u},$$

where i is the current and S is the surface of the current loop. When this magnetic moment is placed in a magnetic field $\vec{B}$ it is subject to a torque

$$\vec{\Gamma} = \vec{\mu} \times \vec{B}$$

The potential energy of a magnetic moment in a magnetic field is

$$W = -\vec{\mu} \cdot \vec{B}$$

If the field is homogeneous, the torque will try to align the magnetic moment with the magnetic field. If the field is inhomogeneous, there is an additional force acting on the magnetic moment, pushing it to the areas where the field is largest (resp. smallest) if $\vec{\mu} \cdot \vec{B} > 0$ (resp. $\vec{\mu} \cdot \vec{B} < 0$)

$$\vec{F} = -\vec{\nabla} W = |\vec{\mu}| \cos\theta \, \vec{\nabla} B,$$

where θ is the angle between the magnetic moment and $\vec{B}$.

In a classical picture of an atom, the motion of the electron about the nucleus will generate a magnetic moment $\vec{\mu}$. But this motion also defines the angular momentum. These two quantities are therefore proportional

$$\vec{L} = \vec{r} \times \vec{p} = Rmv\vec{u}$$
$$\vec{\mu} = iS\vec{u} = \frac{qv}{2\pi R}\pi R^2 \vec{u} = \frac{qRv}{2}\vec{u}$$

where $\vec{u}$ is a unit vector orthogonal to the plane of motion of the electron and m the mass of the electron. We see that

$$\vec{\mu} = \gamma_0 \vec{L} \qquad \gamma_0 = \frac{q}{2m}$$

where γ_0 is the *gyromagnetic ratio*. The constraint that the magnetic moment is proportional to the angular momentum leads to a particular behavior when it is placed in a uniform magnetic field. Unlike a compass that would align itself with the field, the magnetic moment will start precessing about the magnetic field. This effect is known as the *Larmor precession*. Indeed,

$$\frac{d\vec{L}}{dt} = \vec{\Gamma} = \vec{\mu} \times \vec{B} \qquad \frac{d\vec{\mu}}{dt} = \gamma_0 \vec{\mu} \times \vec{B} = \vec{\omega}_0 \times \vec{\mu},$$

where $\omega_0 = -\gamma_0 B$ is the Larmor frequency. If the field is along the z direction

$$\begin{cases} \dot{\mu}_x = -\omega_0 \mu_y \\ \dot{\mu}_y = -\omega_0 \mu_x \\ \dot{\mu}_z = 0 \end{cases} \qquad \begin{cases} \mu_x(t) = \mu_\perp \cos(\omega_0 t + \varphi) \\ \mu_y(t) = \mu_\perp \sin(\omega_0 t + \varphi) \\ \mu_z(t) = \text{const} \end{cases}$$

The magnetic moment is precessing about the magnetic field at a frequency ω_0.

The Stern-Gerlach experiment

In the Stern-Gerlach experiment, silver atoms are heated in a furnace and the resulting beam is sent through an inhomogeneous magnetic field (in the z direction), as shown in the figure below

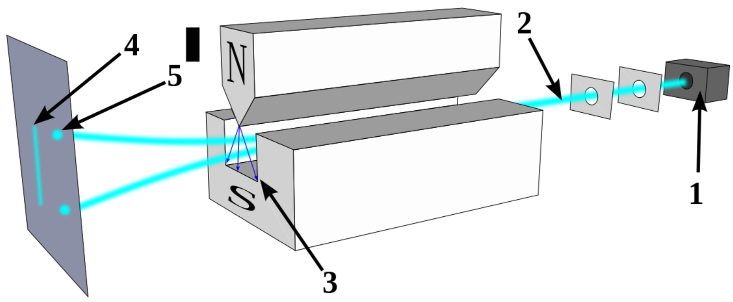


Figure 1: Stern–Gerlach experiment: Silver atoms traveling through an inhomogeneous magnetic field, and being deflected up or down depending on their spin; (1) furnace, (2) beam of silver atoms, (3) inhomogeneous magnetic field, (4) classically expected result, (5) observed result [from wikipedia]

In a classical physics description, the magnetic moment of the atoms would start precessing quickly about the magnetic field (typically $B = 0.1T, \omega_0/2\pi = 1GHz$). On average $\langle\mu_x\rangle = \langle\mu_y\rangle = 0$ and the magnetic moments experience a force along the z axis that is proportional to the constant projection of the magnetic moment along z

$$\vec{F} = \mu_z \vec{\nabla} B$$

Classically, the silver atoms are expected to have randomly distributed orientations of the magnetic moment. As a consequence they should be deflected randomly in the vertical direction and create some density distribution on the detector screen. Instead, the particles passing through the Stern–Gerlach apparatus only leave two spots. They are deflected either up or down by a specific amount as if their magnetic moment was taking only two possible values along the z axis:

$$\mu_z \simeq \pm\mu_B \qquad \mu_B = \frac{\hbar q}{2m},$$

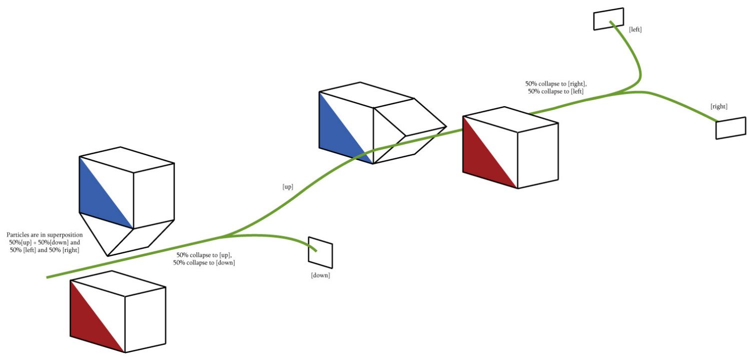
where μ_B is the Bohr magneton. One may think that this result could maybe be explained by the quantization of the orbital angular momentum. However, there are several proofs that this cannot be the case. For example, if the magnetic moment came from an orbital angular momentum, one would expect to see an odd number of spots. Indeed, we have seen that ℓ must take integer values for an orbital angular momentum. Also the Stern-Gerlach experiment was repeated using hydrogen atoms in their ground state. These atoms have a fully spherical wave function for which $\hat{L}_z|\Psi\rangle = 0$ in contradiction with the experiment. It quickly became clear that one must assume that the electron has an intrinsic angular momentum, *the spin*, that has a purely quantum origin.

Quantum mechanical description

Let us try to describe this experiment with quantum mechanics. The Stern-Gerlach apparatus can be seen as a device that performs a measurement of the intrinsic angular moment along the z axis. We call the associated observable $\hat{S}_z$. The experiment has two possible outcomes that we associate to two possible eigenvalues $\pm\hbar/2$ of the operator $\hat{S}_z$. The reason for this convention will become obvious later. The minimal Hilbert space to describe the experiment is therefore of dimension two and is spanned by the eigenstates $|+\rangle_z$ and $|-\rangle_z$ associated to the eigenvalues $\pm\hbar/2$. In this basis, the observable $\hat{S}_z$ is simply

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The choice to perform the measurement along the z axis is arbitrary. We thus expect that measurements along any direction $\vec{u}$ will lead to the same two possible outcomes $\pm\hbar/2$ for the observable $\hat{S}_{\vec{u}}$ associated to the intrinsic angular moment along the axis $\vec{u}$. Now let us imagine we put two Stern-Gerlach devices in series as in the figure below.



The first device is oriented along the z axis. The outcoming beam corresponding to particles that have an intrinsic angular momentum $-\hbar/2$ is stopped while the other beam, corresponding to $+\hbar/2$, is directed into the second Stern-Gerlach device oriented this time along the x axis. The possible outcomes of the second measurements are $\pm\hbar/2$ and there is no reason for one outcome to be more favorable than the other. The beam therefore splits equally in two and the average value of $\hat{S}_x$ is 0. Let us try to find the expression of $\hat{S}_x$ in the basis $\{|+\rangle_z, |-\rangle_z\}$

$$\hat{S}_x = \begin{pmatrix} {}_z\langle+|\hat{S}_x|+\rangle_z & {}_z\langle+|\hat{S}_x|-\rangle_z \\ {}_z\langle-|\hat{S}_x|+\rangle_z & {}_z\langle-|\hat{S}_x|-\rangle_z \end{pmatrix}$$

Clearly, the trace of this matrix is $-\hbar/2 + \hbar/2 = 0$ and its determinant is $(-\hbar/2)(\hbar/2) = -\hbar^2/4$. Moreover the average value of a measurement of $\hat{S}_x$ in the state $|+\rangle_z$ vanishes for the reasons discussed above and thus ${}_z\langle+|\hat{S}_x|+\rangle_z = 0$. The matrix must be of the form

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & e^{-i\phi_x} \\ e^{i\phi_x} & 0 \end{pmatrix}$$

It is easy to see that one can redefine the $|+\rangle_z$, $|-\rangle_z$ states by multiplying them by a phase. This leads to a new ϕ_x'. By convention, one chooses the $\{|+\rangle_z, |-\rangle_z\}$ basis such that $\phi_x = 0$. The matrix for $\hat{S}_x$ is then

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The same reasoning considering the y direction shows that ${}_z\langle +|\hat{S}_y|+\rangle_z = 0$. The operator $\hat{S}_y$ associated to the intrinsic angular momentum along y will thus have the general form

$$\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & e^{-i\phi_y} \\ e^{i\phi_y} & 0 \end{pmatrix}$$

But this time, the choice for the general phase of the $\{|+\rangle_z, |-\rangle_z\}$ basis has already been made. Instead, we can use the fact that we would obtain a zero average value for $\hat{S}_y$ on a beam of particles with $+\hbar/2$ intrinsic angular momentum along x, i.e. ${}_x\langle +|\hat{S}_y|+\rangle_x = 0$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & e^{-i\phi_y} \\ e^{i\phi_y} & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\hbar}{4}2\cos\phi_y = 0$$

We see that we must have $\phi_y = \pm\pi/2$. For direct Cartesian axes, we use $\phi_y = \pi/2$ and

$$\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Measurements along an arbitrary direction $\vec{u}$

We have established the expression for the observables $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$ of the intrinsic angular momentum along x, y, z. It seems natural to suppose that the observable for a measurement along a general direction $\vec{u}$ will have the following expression

$$\hat{S}_{\vec{u}} = u_x\hat{S}_x + u_y\hat{S}_y + u_z\hat{S}_z = \hat{\vec{S}}\cdot\vec{u}$$

One can check that the eigenvalues of this observable are $\pm\hbar/2$. A calculation of the probabilities of the outcomes, for example on an eigenstate of $\hat{S}_z$ with eigenvalue $\hbar/2$, are compatible with the results seen in the experiment.

Summary

The Stern-Gerlach experiment forces us to conclude that the silver atoms have an intrinsic angular momentum, *the spin*, described by a state in a two-dimensional Hilbert space. The observables associated to a measurement of the spin in the x, y, z directions are $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ and always yield $\hbar/2$ or $-\hbar/2$. In the basis $\{|+\rangle_z, |-\rangle_z\}$ of the eigenstates of $\hat{S}_z$ the observables read

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These matrices, without the $\hbar/2$ prefactor, are known as the *Pauli matrices*. They obey the following commutation relations

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z \qquad \left[\hat{S}_y, \hat{S}_z\right] = i\hbar\hat{S}_x \qquad \left[\hat{S}_z, \hat{S}_x\right] = i\hbar\hat{S}_y$$

They are therefore a representation of the angular momentum Lie algebra in the subspace associated to $j = 1/2$. In other words, $|+\rangle_z = |j = 1/2, m = 1/2\rangle$ and $|-\rangle_z = |j =$

$1/2, m = -1/2\rangle$ in our previous notation. The magnetic moment is related to the spin through the gyromagnetic ratio γ

$$\hat{\vec{\mu}} = \gamma \hat{\vec{S}}$$

In the Stern-Gerlach experiment, it is the isolated $5s$ electron that carries the magnetic moment and $\gamma \simeq 2\gamma_0 = q/m_e$. We see that within the quantum mechanical treatment the gyromagnetic ratio is twice as big as the value expected from classical physics. Let us mention here that also protons and neutrons carry a magnetic moment. However, their corresponding gyromagnetic ratios are $\gamma_p \simeq 5.59 q_p/2m_p$ and $\gamma_n \simeq -3.83 q_p/2m_n$ which yield a negligible magnetic moment with respect to the electron.

Finally, let us finish by emphasizing that the three observables of the spin along x, y, z do not commute. This means that a measurement along $\hat{S}_x$ will influence the properties of the state along the z axis. For example, if a particle is in the state $|+\rangle_z$ and is later measured along x with a value $+\hbar/2$, this new state will have equal probability to be measured with $+\hbar/2$ or $-\hbar/2$ if it is measured along z.

Properties of the spin $\frac{1}{2}$

The state of a spin-1/2 is entirely determined by a vector in a two-dimensional Hilbert space. It is very common to use the two eigenstates $|+\rangle_z$ and $|-\rangle_z$ of the $\hat{S}_z$ operator as a basis for this Hilbert space. Any state is then a linear combination

$$|\Sigma\rangle = \alpha|+\rangle_z + \beta|-\rangle_z,$$

with $|\alpha|^2 + |\beta|^2 = 1$. The ket $|\Sigma\rangle$ is often called a *spinor*. The operators, in turn, can always be written as 2×2 matrices. We have

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Bloch sphere

Let us consider the observable $\hat{\vec{S}} \cdot \vec{u}$ that measures the spin along the direction $\vec{u} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. It is written

$$\hat{S}_{\vec{u}} = \hat{\vec{S}} \cdot \vec{u} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}$$

This observable has eigenvalues $\pm\hbar/2$

$$\left(\hat{\vec{S}} \cdot \vec{u}\right)|\pm\rangle_{\vec{u}} = \pm\frac{\hbar}{2}|\pm\rangle_{\vec{u}}$$

and the eigenstates have the following expression

$$|+\rangle_{\vec{u}} = e^{-i\varphi/2}\cos\frac{\theta}{2}|+\rangle_z + e^{i\varphi/2}\sin\frac{\theta}{2}|-\rangle_z$$
$$|-\rangle_{\vec{u}} = -e^{-i\varphi/2}\sin\frac{\theta}{2}|+\rangle_z + e^{i\varphi/2}\cos\frac{\theta}{2}|-\rangle_z$$

When the spin observable is measured in the $|+\rangle_{\vec{u}}$ state is yields the average value

$$_{\vec{u}}\langle+|\hat{\vec{S}}|+\rangle_{\vec{u}} = \frac{\hbar}{2}\vec{u}$$

This result can be obtained by doing an explicit calculation or by observing that the direction $\vec{u}$ has nothing special. Therefore it must be equivalent to a measurement along z that clearly yields $_z\langle+|\hat{\vec{S}}|+\rangle_z = \vec{z}\,\hbar/2$.

It is easy to convince yourself that, up to a global phase, any state of a two-dimensional Hilbert space can be written as

$$|\Sigma\rangle = e^{-i\varphi/2}\cos\frac{\theta}{2}|+\rangle_z + e^{i\varphi/2}\sin\frac{\theta}{2}|-\rangle_z$$

This expression has the same form as the one for the $|+\rangle_{\vec{u}}$ eigenstate. This means that any state can be written as $|+\rangle_{\vec{u}}$ for a well chosen unit vector $\vec{u}$. This property allows to represent a spin state $|\Sigma\rangle$ with a point on the unit sphere, called the *Bloch sphere*. Because $_{\vec{u}}\langle+|\hat{\vec{S}}|+\rangle_{\vec{u}} = \frac{\hbar}{2}\vec{u}$ we see that the knowledge of the average value of $\hat{\vec{S}}$ allows to immediately identify $\vec{u}$ (and the opposite is also true).

Rotations of a spin state

The spin observables obey the commutation relations of the angular momentum Lie algebra. They can therefore be thought of as infinitesimal generators of rotations. While the interpretation of a rotation in $\mathbb{R}^3$ was very clear, the action of a rotation in the spin space is not necessarily obvious. So let us act with the rotation operator of an angle α about z on a general spinor

$$\begin{aligned}\hat{R}_{z,\alpha}|\Sigma\rangle &= \exp\left(-\frac{i}{\hbar}\alpha\hat{S}_z\right)|\Sigma\rangle = \exp\left(-\frac{i}{\hbar}\alpha\hat{S}_z\right)\left[e^{-i\varphi/2}\cos\frac{\theta}{2}|+\rangle_z + e^{i\varphi/2}\sin\frac{\theta}{2}|-\rangle_z\right]\\ &= e^{-i\alpha/2}e^{-i\varphi/2}\cos\frac{\theta}{2}|+\rangle_z + e^{i\alpha/2}e^{i\varphi/2}\sin\frac{\theta}{2}|-\rangle_z\\ &= e^{-i(\varphi+\alpha)/2}\cos\frac{\theta}{2}|+\rangle_z + e^{i(\varphi+\alpha)/2}\sin\frac{\theta}{2}|-\rangle_z = |\Sigma'\rangle\end{aligned}$$

Let us assume that $|\Sigma\rangle$ is described by a unit vector $\vec{u}$, and that the state $|\Sigma'\rangle$ after the action of the rotation is described by the unit vector $\vec{u'}$. We see from the calculation above, that $\vec{u'}$ is a rotation of $\vec{u}$ by an angle α about the z axis. In other words, the rotation operator transforms the spin state in such a way that the average value of $\hat{\vec{S}}$ rotates by the corresponding angle. Note, however, that unlike rotations in $\mathbb{R}^3$, a state $|\Sigma\rangle$ picks up a minus sign when rotated by 2π

$$\hat{R}_{z,2\pi}|\Sigma\rangle = -|\Sigma\rangle$$

This has no consequences for physical observables that will be invariant under a 2π rotations. But in order to recover the exact same quantum state, one has to perform a 4π rotation. In the mathematical jargon, the spin observables $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$ are the generators of the Lie algebra of the $SU(2)$ group. This group is almost the same as $SO(3)$, except for the fact that you need to rotate by 4π to obtain the identity. One says that $SU(2)$ is a double cover of $SO(3)$.

Spin $\frac{1}{2}$ in a uniform magnetic field

Let us consider a spin in a uniform magnetic field. The magnetic moment will try to align with the magnetic field. Just as in the classical case, the Hamiltonian describing this coupling is

$$\hat{\mathcal{H}}_{\text{Zeeman}} = -\hat{\mu}\cdot\vec{B},$$

where $\vec{B}$ is the magnetic field and $\hat{\mu}$ is the magnetic moment observable. This coupling is sometimes referred to as *the Zeeman interaction*. For a spin-$\frac{1}{2}$ and a magnetic field $\vec{B} = (0, 0, B_0)$ the Hamiltonian can be written as

$$\hat{\mathcal{H}}_{\text{Zeeman}} = \frac{\hbar\omega_0}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

where $\omega_0 = -\gamma B_0$ is the Larmor frequency. Let us suppose that at $t = 0$ the state is given by

$$|\Sigma(t=0)\rangle = e^{-i\varphi/2}\cos\frac{\theta}{2}|+\rangle_z + e^{i\varphi/2}\sin\frac{\theta}{2}|-\rangle_z$$

For $t > 0$, the evolution dictated by $\hat{\mathcal{H}}_{\text{Zeeman}}$ is simply

$$\begin{aligned}
|\Sigma(t)\rangle &= e^{-i\varphi/2} e^{-i\omega_0 t/2} \cos\frac{\theta}{2} |+\rangle_z + e^{i\varphi/2} e^{i\omega_0 t/2} \sin\frac{\theta}{2} |-\rangle_z \\
&= e^{-i(\varphi+\omega_0 t)/2} \cos\frac{\theta}{2} |+\rangle_z + e^{i(\varphi+\omega_0 t)/2} \sin\frac{\theta}{2} |-\rangle_z \\
&= \hat{R}_{z,\omega_0 t} |\Sigma(t=0)\rangle
\end{aligned}$$

We can see that the average value of $\hat{\vec{S}}$ precesses about the z axis at the Larmor frequency. This is exactly the same as in the classical picture.

Nuclear magnetic resonance

Nuclear magnetic resonance (NMR) is an experimental technique that takes advantage of the selective absorption of high-frequency electromagnetic radiation by certain atomic nuclei subjected to an appropriately strong uniform magnetic field. This phenomenon was discovered by Isidor Rabi and the technique was further developed by Felix Bloch and Edward Purcell.

Experimental setup

In an NMR experiment, a sample is placed in two magnetic field: a uniform static field $\vec{B}_0 = (0, 0, B_0)$ in the z direction and a rotating field $\vec{B}_1 = B_1(\cos\omega t, \sin\omega t, 0)$ in the xy plane. These two magnetic fields define the Larmor frequency $\omega_0 = -\gamma B_0$ and $\omega_1 = -\gamma B_1$. Here γ is the gyromagnetic ratio of the nuclei that are investigated. In general $|\omega_1| \ll |\omega_0|$ and $\omega \sim \omega_0$. For concreteness, we will suppose that $\omega_0 < 0$ so that in the absence of a rotating field, the lowest energy state is $|+\rangle_z$.

Evolution equations

The Hamiltonian for this system is

$$\hat{\mathcal{H}}(t) = -\gamma \vec{B} \cdot \hat{\vec{S}} = \omega_0 \hat{S}_z + \omega_1 \left(\cos(\omega t)\hat{S}_x + \sin(\omega t)\hat{S}_y\right)$$

Using the Ehrenfest theorem, we see that the evolution equations for the average value of the spin operator $\hat{\vec{S}}$ are

$$\begin{aligned}
\frac{d}{dt}\langle S_x\rangle(t) &= \frac{1}{i\hbar}\langle[\hat{S}_x, \hat{\mathcal{H}}(t)]\rangle = -\gamma\left(-B_0\langle S_y\rangle + B_y(t)\langle S_z\rangle\right) \\
\frac{d}{dt}\langle S_y\rangle(t) &= \frac{1}{i\hbar}\langle[\hat{S}_y, \hat{\mathcal{H}}(t)]\rangle = -\gamma\left(B_0\langle S_x\rangle - B_x(t)\langle S_z\rangle\right) \\
\frac{d}{dt}\langle S_z\rangle(t) &= \frac{1}{i\hbar}\langle[\hat{S}_z, \hat{\mathcal{H}}(t)]\rangle = -\gamma\left(B_x(t)\langle S_y\rangle - B_y(t)\langle S_x\rangle\right)
\end{aligned}$$

which can be rewritten as

$$\frac{d}{dt}\langle\vec{S}\rangle(t) = \gamma\,\langle\vec{S}\rangle(t) \times \vec{B}(t)$$

This equation for the average value of the spin operator is exactly the same as in classical physics.

Rabi oscillations

Solving the evolution equations shows that the spin motion is the combination of a fast precession about the z axis and a slow precession about the rotating field. One can then ask what is the probability $\mathcal{P}_-$ to measure the spin in the $|-\rangle_z$ state. The average value of $\hat{S}_z$ is related to the probability $\mathcal{P}_-$ through

$$\langle S_z \rangle = (1 - \mathcal{P}_-)\,\frac{\hbar}{2} - \mathcal{P}_-\frac{\hbar}{2}$$

from which we find

$$\mathcal{P}_-(t) = \frac{1}{2} - \frac{\langle S_z \rangle}{\hbar} = \frac{\omega_1^2}{\Omega^2}\sin^2\frac{\Omega t}{2},$$

We see that $\mathcal{P}_-(t)$ is oscillating at the *Rabi frequency* Ω defined by $\Omega^2 = (\omega_0 - \omega)^2 + \omega_1^2$. These oscillations are called *Rabi oscillations*. On average, the probability to be in the $|-\rangle_z$ state is

$$\mathcal{P}_- = \overline{\mathcal{P}_-(t)} = \frac{1}{2}\frac{\omega_1^2}{(\omega_0 - \omega)^2 + \omega_1^2}$$

As a function of ω/ω_0, the probability to see a transition to the $|-\rangle_z$ state is a Lorentzian centered around 1 with a width $2\omega_1$. This is an example of a resonant process: When the rotating field rotates at the Larmor frequency, the system has a big probability to have a transition. This is quite natural if one thinks about the energy levels of the system in the absence of the rotating field. In this case the eigenenergies are $\pm\omega_0\hbar/2$. These eigenstates are separated by an energy $\hbar|\omega_0|$. When the rotating field precesses at the frequency $\omega = \omega_0$ it can transfer exactly the right amount $\hbar|\omega_0|$ of energy to create a transition from the occupied lower energy state to the empty $|-\rangle_z$ state. On can say that the field is absorbed by the two-level system. This phenomenon appears in many other quantum mechanical systems that are described by a two-level Hamiltonian.

NMR typical protocol

How can the results found above be useful to learn something about, e.g. the structure of molecules? The NMR protocol typically work in three stages:

- The magnetic nuclear spins are aligned (polarized) by applying the strong constant magnetic field B_0.
- This alignment is perturbed in some way. This can be done by applying a weak oscillating magnetic field, usually referred to as a radio-frequency (RF) pulse.
- The precession of the nuclear spins around B_0 can be measured by the voltage it induces in a detection coil. A Fourier transform of this NMR signal will allow to observe precisely resonances at the nuclei's intrinsic Larmor frequency. Because this frequency is influenced by the chemical environment of the nuclei, the resulting frequency spectrum can be used to characterize the object being sampled.

There are many variants of how NMR is performed, but the general idea is always the same. Nuclear magnetic resonance spectroscopy is widely used to determine the structure of organic molecules in solution, to study molecular physics, crystals as well as non-crystalline materials. NMR is also routinely used in advanced medical imaging techniques, such as in magnetic resonance imaging (MRI).