HW3 - Probability

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1. From the law of total probability we have that:

$$P(S_n = \mathbf{s}) = \sum_{\mathbf{s}' \in \mathbb{Z}^d} P(S_n = \mathbf{s} | S_{n-1} = \mathbf{s}') P(S_{n-1} = \mathbf{s}')$$

Furthermore we have that (call $B = \{\mathbf{e^{(1)}}, -\mathbf{e^{(1)}}, \cdots, \mathbf{e^{(d)}}, -\mathbf{e^{(d)}}\}$)

$$P(S_n = \mathbf{s} | S_{n-1} = \mathbf{s}') = \frac{1}{2d} 1_{(\mathbf{s}-\mathbf{s}') \in B}$$

Hence the sum reduces to:

$$P(S_n = \mathbf{s}) = \sum_{\mathbf{n} \in B} \frac{1}{2d} P(S_{n-1} = \mathbf{s} - \mathbf{n})$$

We therefore have that:

$$p(\mathbf{s}, n+1) = \frac{1}{2d} \sum_{\mathbf{n} \in B} p(\mathbf{s} - \mathbf{n}, n)$$

Then the Fourier transform is given by:

$$\tilde{p}(\mathbf{k}, n) = \sum_{\mathbf{s} \in \mathbb{Z}^d} p(\mathbf{s}, n) e^{-i\mathbf{k}\cdot\mathbf{s}} = \sum_{\mathbf{s} \in \mathbb{Z}^d} \left(\frac{1}{2d} \sum_{\mathbf{n} \in B} p(\mathbf{s} - \mathbf{n}, n-1) \right) e^{-i\mathbf{k}\cdot\mathbf{s}} = \frac{1}{2d} \sum_{\mathbf{s} \in \mathbb{Z}^d} \sum_{\mathbf{n} \in B} p(\mathbf{s} - \mathbf{n}, n-1) e^{-i\mathbf{k}\cdot\mathbf{s}}$$

Now since all the sums converge we can switch them and we get:

$$\tilde{p}(\mathbf{k}, n) = \frac{1}{2d} \sum_{\mathbf{n} \in B} \sum_{\mathbf{s} \in \mathbb{Z}^d} p(\mathbf{s} - \mathbf{n}, n-1) e^{-i\mathbf{k}\cdot\mathbf{s}} = \frac{1}{2d} \sum_{\mathbf{n} \in B} \sum_{\mathbf{s} \in \mathbb{Z}^d} p(\mathbf{s}, n-1) e^{-i\mathbf{k}\cdot(\mathbf{s}+\mathbf{n})} = \frac{\tilde{p}(\mathbf{k}, n-1)}{2d} \sum_{\mathbf{n} \in B} e^{-i\mathbf{k}\cdot\mathbf{n}} = a(\mathbf{k}) \tilde{p}(\mathbf{k}, n-1)$$

Where we took $a(\mathbf{k}) = \frac{1}{2d} \sum_{\mathbf{n} \in B} e^{-i\mathbf{k}\cdot\mathbf{n}}$. Now we know that:

$$p(\mathbf{s}, 0) = \delta_{\mathbf{s}} \text{ and hence } \tilde{p}(\mathbf{k}, 0) = \sum_{s \in \mathbb{Z}^d} \delta_{\mathbf{s}} e^{-i\mathbf{k}\cdot\mathbf{s}} = 1$$

We therefore have that:

$$\tilde{p}(\mathbf{k}, n) = a(\mathbf{k})^n = \left(\frac{1}{2d} \sum_{\mathbf{n} \in B} e^{-i\mathbf{k}\cdot\mathbf{n}} \right)^n$$

Inversing the Fourier transform gives:

$$p(\mathbf{s}, n) = \int_{BZ} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} \tilde{p}(\mathbf{k}, n) e^{i\mathbf{k}\cdot\mathbf{s}} = \int_{BZ} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} \left(\frac{1}{2d} \sum_{\mathbf{n} \in B} e^{-i\mathbf{k}\cdot\mathbf{n}} \right)^n e^{i\mathbf{k}\cdot\mathbf{s}}$$

2. We have the following (remembering $B = \{\mathbf{e_x}, -\mathbf{e_x}, \mathbf{e_y}, -\mathbf{e_y}\}$ and writing $B' = \{\mathbf{e_x} + \mathbf{e_y}, \mathbf{e_x} - \mathbf{e_y}, \mathbf{e_y} - \mathbf{e_x}, -\mathbf{e_x} - \mathbf{e_y}\}$):

n	$p(\mathbf{s}, n)$	$p_1(\mathbf{s}, n)$
1	$\frac{1}{4}\delta_{\mathbf{s}\in B}$	$\frac{1}{4}\delta_{\mathbf{s}\in B}$
2	$\frac{1}{4}\delta_{\mathbf{s}} + \frac{1}{16}\delta_{\mathbf{s}\in 2B} + \frac{1}{8}\delta_{\mathbf{s}\in B'}$	$\frac{1}{4}\delta_{\mathbf{s}} + \frac{1}{16}\delta_{\mathbf{s}\in 2B} + \frac{1}{8}\delta_{\mathbf{s}\in B'}$
3	$\frac{1}{64}\delta_{\mathbf{s}\in 3B} + \frac{9}{64}\delta_{\mathbf{s}\in B} + \frac{3}{64}\delta_{\lvert\lvert\mathbf{s}\rvert\rvert^2=5}$	$\frac{1}{64}\delta_{\mathbf{s}\in 3B} + \frac{3}{64}\delta_{\lvert\lvert\mathbf{s}\rvert\rvert^2=5} + \frac{5}{64}\delta_{\mathbf{s}\in B}$

3. We have that:

$$\begin{aligned}P(\mathbf{S_{n'}}=\mathbf{s'}|\mathbf{S_n}=\mathbf{s}) &= P(\sum_{i=1}^{n'}\mathbf{X_i}=\mathbf{s'}|\sum_{i=1}^{n}\mathbf{X_i}=\mathbf{s}) = P(\sum_{i=1}^{n}\mathbf{X_i}+\sum_{i=n+1}^{n'}\mathbf{X_i}=\mathbf{s'}|\sum_{i=1}^{n}\mathbf{X_i}=\mathbf{s})\\ &= P(\mathbf{s}+\sum_{i=n+1}^{n'}\mathbf{X_i}=\mathbf{s'}|\sum_{i=1}^{n}\mathbf{X_i}=\mathbf{s}) = P(\mathbf{s}+\sum_{i=n+1}^{n'}\mathbf{X_i}=\mathbf{s'})\\ &= P(\sum_{i=n+1}^{n'}\mathbf{X_i}=\mathbf{s'}-\mathbf{s}) = P(\sum_{i=1}^{n'-n}\mathbf{X_i}=\mathbf{s'}-\mathbf{s})\\ &= P(\mathbf{S_{n'-n}}=\mathbf{s'}-\mathbf{s})\end{aligned}$$

Where in the 4^{th} equality we used the fact that $(X_1,\cdots,X_n)$ is independent from $(X_{n+1},\cdots,X_{n'})$ and in the 6^{th} equality we used the fact that the X_i are identically distributed random variables.

4. We have that (we denote by $A_{\mathbf{s},n}=\{\mathbf{S_1}\neq\mathbf{s},\cdots,\mathbf{S_{n-1}}\neq\mathbf{s},\mathbf{S_n}=\mathbf{s}\}$):

$$P(\mathbf{S_n}=\mathbf{s})=\sum_{k=0}^{n}P(\mathbf{S_n}=\mathbf{s}|A_{\mathbf{s},k})P(A_{\mathbf{s},k})+\delta_{\mathbf{s}}\delta_n=\sum_{k=0}^{n}P(\mathbf{S_{n-k}}=\mathbf{0})p_1(\mathbf{s},k)+\delta_{\mathbf{s}}\delta_n$$

Hence re-writing it we obtain:

$$p(\mathbf{s},n)=\sum_{k=0}^{n}p(\mathbf{0},n-k)p_1(\mathbf{s},k)+\delta_{\mathbf{s}}\delta_n$$

5. From the previous question we have that:

$$\begin{aligned}\hat{p}(\mathbf{s},\lambda)&=\sum_{n\in\mathbb{N}}p(\mathbf{s},n)\lambda^n=\sum_{n\in\mathbb{N}}\left(\sum_{k=0}^{n}p(\mathbf{0},n-k)p_1(\mathbf{s},k)+\delta_{\mathbf{s}}\delta_n\right)\lambda^n=\delta_{\mathbf{s}}\lambda^0+\sum_{k=0}^{+\infty}\sum_{n=k}^{+\infty}p(\mathbf{0},n-k)p_1(\mathbf{s},k)\lambda^{n-k}\lambda^k\\ &=\delta_{\mathbf{s}}+\left(\sum_{k\in\mathbb{N}}\lambda^k p_1(\mathbf{s},k)\right)\left(\sum_{n\in\mathbb{N}}\lambda^n p(\mathbf{0},n)\right)=\delta_{\mathbf{s}}+\hat{p}_1(\mathbf{s},\lambda)\hat{p}(\mathbf{0},\lambda)\end{aligned}$$

Where in the third equality we used Fubbini's theorem in order to exchange the summation.

6. Using Question 1 we get:

$$\hat{p}(\mathbf{s},\lambda)=\sum_{n\in\mathbb{N}}\left(\int_{BZ}\frac{\mathrm{d}\mathbf{k}}{(2\pi)^d}a(\mathbf{k})^n e^{i\mathbf{k}\cdot\mathbf{s}}\right)\lambda^n=\int_{BZ}\frac{\mathrm{d}\mathbf{k}}{(2\pi)^d}e^{i\mathbf{k}\cdot\mathbf{s}}\sum_{n\in\mathbb{N}}(\lambda a(\mathbf{k}))^n=\int_{BZ}\frac{\mathrm{d}\mathbf{k}}{(2\pi)^d}\frac{e^{i\mathbf{k}\cdot\mathbf{s}}}{1-\lambda a(\mathbf{k})}$$

We are allowed to exchange the integral and the sum since from Question 5 we are assured of their convergence.

7. Notice that:

$$p_r=\sum_{n\in\mathbb{N}}p_1(\mathbf{0},n)=\lim_{\lambda\to1^-}\sum_{n\in\mathbb{N}}p_1(\mathbf{0},n)\lambda^n=\lim_{\lambda\to1^-}\hat{p}_1(\mathbf{0},\lambda)=\lim_{\lambda\to1^-}\frac{\hat{p}(\mathbf{0},\lambda)}{\hat{p}(\mathbf{0},\lambda)}-\delta_{\mathbf{0}}\frac{1}{\hat{p}(\mathbf{0},\lambda)}=\lim_{\lambda\to1^-}1-\frac{1}{\hat{p}(\mathbf{0},\lambda)}$$

Now replacing $\hat{p}(\mathbf{0},\lambda)$ with the integral expression found in Question 6 and noticing that up to a re-writing of the exponentials we have that $a(\mathbf{k})=\frac{1}{d}\sum_{i=1}^{d}\cos(k_i)$ we get:

$$p_r=\lim_{\lambda\to1^-}1-\frac{1}{\int_{BZ}\frac{\mathrm{d}\mathbf{k}}{(2\pi)^d}\frac{1}{1-\lambda a(\mathbf{k})}}$$

Which is the desired result.

8. (a) In the case $d=1$ the equation simplifies to:

$$\hat{p}_1(\mathbf{0},\lambda)=1-\frac{1}{\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\frac{1}{1-\lambda\cos(k)}}$$

We now compute the integral:

$$\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{1-\lambda\cos(k)}=\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{1-\lambda\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)}$$

Now by taking $z = e^{i\theta}$ meaning that $\mathrm{d}z = ie^{i\theta}\mathrm{d}\theta$ gives:

$$\int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{1-\lambda\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)} = -i\int_{S^1} \frac{2\mathrm{d}z}{z(2-\lambda(z+\frac{1}{z}))} = -i\int_{S^1} \frac{2\mathrm{d}z}{2z-\lambda z^2-1}$$

Now the integrand admits two poles in:

$$z_0^{\pm} = \frac{1 \pm \sqrt{1-\lambda^2}}{\lambda}$$

However since $0 \leq \lambda < 1$ the only pole inside the unit circle is $z_0 = z_0^-$. Then from the residue theorem we have that:

$$-i\int_{S^1} \frac{2\mathrm{d}z}{2z-\lambda z^2-1} = 2\pi\mathrm{Res}(\frac{2}{2z-\lambda z^2-1}, z_0) = \frac{2\pi}{\sqrt{1-\lambda^2}}$$

Plugging this back on top we get the desired result:

$$\hat{p}_1(\mathbf{0},\lambda) = 1 - \frac{1}{\frac{1}{\sqrt{1-\lambda^2}}} = 1 - \sqrt{1-\lambda^2}$$

(b) Notice that:

$$\hat{p}_1(\mathbf{0},\lambda) = \sum_{n=0}^{+\infty} p_1(\mathbf{0},n)\lambda^n \Rightarrow p_1(\mathbf{0},n) = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\hat{p}_1(\mathbf{0},\lambda)\Big|_{\lambda=0}$$

Notice that this corresponds to the n-th term of the series expansion of $1-\sqrt{1-\lambda^2}$ around 0. Since it is even we already know that all the odd powers must vanish. More generally we have from the generalized binomial formula:

$$1-\sqrt{1-\lambda^2} = 1 - \sum_{k=0}^{+\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!}(-\lambda^2)^k = -\sum_{k=1}^{+\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!}(-1)^k\lambda^{2k}$$

Now we rework a bit the coefficient in front:

$$\frac{1}{2^k}(-1)^{k+1}\frac{(1-2)\cdots(1-2k+2)}{k!} = (-1)^{2k}\frac{1}{2^k}\frac{1\cdot 3\cdots(2k-3)}{k!} = \frac{1}{2^k}\frac{1\cdot 3\cdots(2k-3)k!}{(2k)!}\binom{2k}{k}$$

Which when simplified gives:

$$= \frac{1}{2^k}\frac{1}{2k-1}\frac{1}{2\cdot 4\cdots(2k-2)\cdot 2k}k!\binom{2k}{k} = \frac{1}{2^{2k}}\frac{1}{2k-1}\frac{1}{1\cdot 2\cdots(k-1)\cdot k}k!\binom{2k}{k} = \frac{1}{2^{2k}}\frac{1}{2k-1}\binom{2k}{k}$$

(c) From definition since the q_{2n} are the coefficients in the series expansion of $1-\sqrt{1-\lambda^2}$ around 0 it must be that $q_{2n} \to 0$ as $n \to +\infty$. Now furthermore notice that $E[p_1(\mathbf{0},n)] = \frac{\mathrm{d}}{\mathrm{d}\lambda}\hat{p}_1(0,\lambda)\Big|_{\lambda=1} = +\infty$ so the average time of first return to the origin is undefined.