Symmetries in Physics

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Chapter 1

TD1

1.1 Problem 1 Cayley tables

1.1.1

Suppose that an element appears more than once in a given row or column. Then we have that:

$$\exists g, g_i, g_j, g_k \in \mathcal{G}, \quad g = g_i \cdot g_j \wedge g = g_i \cdot g_k \Rightarrow g_j = g_k$$

Since no two elements in a row can be mapped to the same element of the group then a row is a map $\mathcal{G} \to \mathcal{G}$ which from the point above is injective then since it is an endomorphism it necessarily must be a bijection and hence a permutation of $\mathcal{G}$. Therefore each element appears once and exactly once.

1.1.2

Refer above.

1.2 Problem 2 The group D_3

1.2.1

The elements of D_3 are $e = Id$, $r = (B, C, A)$, $r^2 = (C, A, B)$, $s_1 = (A, C, B)$, $s_2 = (B, A, C)$, $s_3 = (C, B, A)$. Then the table is given by:

	e	r	r^2	s_1	s_2	s_3
e	e	r	r^2	s_1	s_2	s_3
r	r	r^2	e	s_2	s_3	s_1
r^2	r^2	e	r	s_3	s_1	s_2
s_1	s_1	s_2	s_3	e	r	r^2
s_2	s_3	s_1	s_2	r^2	e	r
s_3	s_2	s_3	s_1	r	e	r^2

1.2.2

The subgroups of D_3 are $\{e, r, r^2\} = \langle r \rangle$, $\langle s_1 \rangle$, $\langle s_2 \rangle$, $\langle s_3 \rangle$, $\{e\}$.

1.3 Problem 3 Lagrange's theorem.

Let $\mathcal{H}$ be a subgroup of $\mathcal{G}$. Then notice that $\mathcal{G}/\mathcal{H}$ is the set of the cosets of $\mathcal{G}$ by the congruence modulo $\mathcal{H}$. However from Exercise 1 and 2 we know that every coset is in bijection with $\mathcal{H}$. Furthermore since the congruence is an equivalence relation it must be that $\mathcal{G}$ is equal to the reunion of the cosets. Hence we have that:

$$|\mathcal{G}/\mathcal{H}| \cdot |\mathcal{H}| = |\mathcal{G}|$$

The result follows.

1.4 Problem 4 Modular arithmetics

1.4.1

Notice that for any $k \in \mathbb{Z}$ we have that $k\mathbb{Z}$ is a subgroup of $\mathbb{Z}$. The quotient groups are $\mathbb{Z}_{k\mathbb{Z}}$ which are the well-known integers modulo k with the addition modulo k.

1.4.2

Let $|g|$ be the smallest integer such that $g^{|g|} = e$. Such an integer must exist so long as the group to which g pertains is finite. Then notice that for any $k \in \mathbb{Z}$ we have that: $g^{|g| \cdot k} = (g^{|g|})^k = e^k = e$. Hence $|g|\mathbb{Z} \subseteq P_g$. Now let $k \in \mathbb{Z}$ such that $g^k = e$. From construction it must be that $k > |g|$ hence by doing the euclidean division we get that : $k = |g| \cdot \ell + r$. Hence: $g^{|g| \cdot \ell + r} = e \Rightarrow e^{\ell} \cdot g^r = e \Rightarrow g^r = e$. However unless $r = 0$ this is impossible since $r < |g|$ would be a contradiction.

1.4.3

Notice that necessarily $\langle g \rangle$ is a subgroup of cardinality $|g|$ of $\mathcal{G}$ hence from the Lagrange theorem we know that $|g|$ divides $|\mathcal{G}|$.

1.4.4

Let a group $\mathcal{G}$ of order p where p is prime. Then from the previous question we know that all elements of $\mathcal{G}$ must be of order p. However if one element is of order p and $\mathcal{G}$ is of order p it must be that $\mathcal{G}$ is generated by a single element, call it g. Then the obvious homomorphism concludes the proof:

$$\begin{aligned} h : \mathcal{G} &\to \mathbb{Z}/p\mathbb{Z} \\ g^k &\mapsto k \mod p \end{aligned}$$

Chapter 2

TD2

2.1 All finite groups up to 5 elements.

The only group of size 1 is the trivial group which is isomorphic to $\mathbb{Z}_{/1\mathbb{Z}}$. The group of size 2 is isomorphic to $\mathbb{Z}_{/2\mathbb{Z}}$. The group of size 3 is:

	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Which is clearly isomorphic to $\mathbb{Z}_{/3\mathbb{Z}}$. The groups of size 4 are:

	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

Which are respectively isomorphic to $\mathbb{Z}_{/4\mathbb{Z}}$ and $\mathbb{Z}_{/2\mathbb{Z}} \times \mathbb{Z}_{/2\mathbb{Z}}$. Finally the only possible group of size 5 is given by $\mathbb{Z}_{/5\mathbb{Z}}$.

2.2 Union of groups.

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two groups. Then let $g \in \mathcal{G}_1$ and $h \in \mathcal{G}_2$ (w.l.o.g.) then $gh \in \mathcal{G}_1 \cup \mathcal{G}_2 \leftrightarrow gh \in \mathcal{G}_1$ (w.l.o.g.). However since $g \in \mathcal{G}_1$ then $g^{-1} \in \mathcal{G}_1$ and hence we would have that $h \in \mathcal{G}_1$. Hence we have that $\mathcal{G}_1 \cup \mathcal{G}_2$ is a group if and only if $\mathcal{G}_1 \leqslant \mathcal{G}_2$ or vice-versa.

2.3 Quotient groups.

1. Let $\pi : g \mapsto g\mathcal{H}$ which is a natural isomorphism. Then let $\mathcal{G}' = \pi^{-1}(A)$. Since $\mathcal{H} = \mathrm{Ker}\pi$ we have that $\mathcal{H} \lhd \mathcal{G}'$. Then it is immediate from definition that: $\mathcal{G}'/A = \pi(\mathcal{G}') = \pi \circ \pi^{-1}(A) = A$.

2. ...

3. Notice that the isomorphism $\psi : g \mapsto \varphi_g$ has that $\mathrm{Ker}\psi = Z(\mathcal{G})$ furthermore we know that $\mathcal{G}/\mathrm{Ker}\psi \cong \mathrm{Im}\psi$ and hence we immediately get that $\mathcal{G}/Z(\mathcal{G}) \cong \mathrm{Inn}(\mathcal{G})$.

Chapter 3

TD3

Chapter 4

TD 4

4.1 Symmetric Group

1. ...
2. ...
3. ...
4. ...
5. Notice that:

$$\tau\sigma\tau^{-1}(\tau(j_i)) = \tau\sigma(j_i) = \begin{cases} \tau(j_i) & \text{if} \quad i > k \\ \tau(j_{i+1}) & \text{if} \quad i \leq k \end{cases}$$

   Q.E.D.

6. We prove this in two steps. We start by proving that two elements with the same cycle type are conjugated then we prove that two elements which are conjugate have the same cycle type. Let σ_1 and σ_2 who have cycle type $\{\ell_1, \ell_2, \cdots, \ell_k\}$ not that the order in the ℓ_i are arbitrary since disjoint cycles can be permutated. Then from question 5 let $j_i^{(c_n)}$ be the elements of the cycle c of length ℓ_c of σ_n. Then we define:

$$\tau(j_i^{(c_1)}) = j_i^{(c_2)}$$

   Note that this is from construction necessarily a surjective and injective mapping and therefore is a permutation of the elements. Then from construction we get the desired result.
   Then from question 5 still we know that the cycle type of a permutation is invariant under conjugation hence necessarily two elements with different cycle types cannot be conjugate of one another.

7. ...
8. ...

4.2 Representations

1. Notice that for any irreducible representations $(r', Z(G))$ we have that:

$$r'[g]r[g] = r[g]r'[g], \quad \forall g \in Z(G)$$

   Since for any element in $Z(G)$ we must have that:

$$r[g_1 \cdot g_2] = r[g_2 \cdot g_1] = r[g_1]r[g_2] = r[g_2]r[g_1]$$

   From the definition of $Z(G)$. Then from Schur's Lemma we can conclude that $r[g] = \lambda \mathrm{Id}_{\mathcal{E}}$.

2.