TDs - QFT

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Chapter 1

TD1

1.1 Matrix Groups

1.2 The relationship between $SO(3)$ and $SU(2)$.

1.3 Representations of $SU(2)$.

1. An immediate computation yields the desired result
2. Let $|a\rangle$ an eigenvector of $\hat{\mathbf{J}}^2$ then:

$$\hat{\mathbf{J}}^2 |a\rangle = a |a\rangle \Rightarrow \langle a| \hat{\mathbf{J}}^2 |a\rangle = a \langle a|a\rangle \Rightarrow ||\hat{\mathbf{J}} |a\rangle ||^2 = a|| |a\rangle || \Rightarrow a > 0$$

We propose as a writing for them $j(j+1)$ notice that:

$$j(j+1) = x \Leftrightarrow j^2 + j - x = 0 \Rightarrow j = \frac{-j + \sqrt{j^2 + 4x}}{2}$$

Hence the writing as $j(j+1)$ is not restrictive and covers all of $\mathbb{R}^+$.

3. Let $|v\rangle$ an eigenvector of $\hat{\mathbf{J}}^2$ and $\hat{\mathbf{J}}_\mathbf{3}$ with eigenvalues $j(j+1)$ and m. Then:

$$\hat{\mathbf{J}}^2 \hat{\mathbf{J}_+} |v\rangle = \hat{\mathbf{J}_+} \hat{\mathbf{J}}^2 |v\rangle = j(j+1) \hat{\mathbf{J}_+} |v\rangle$$

Since the operator $\hat{\mathbf{J}}^2$ commutes with the $\hat{\mathbf{J}}_\mathbf{i}$. Then:

$$\hat{\mathbf{J}}_\mathbf{3} \hat{\mathbf{J}_+} |v\rangle = (\hat{\mathbf{J}_+} \hat{\mathbf{J}}_\mathbf{3} + [\hat{\mathbf{J}}_\mathbf{3}, \hat{\mathbf{J}_+}]) |v\rangle = (m\hat{\mathbf{J}_+} + i\hat{\mathbf{J}}_\mathbf{2} + 1\hat{\mathbf{J}}_\mathbf{1}) |v\rangle = (m+1)\hat{\mathbf{J}_+} |v\rangle$$

Identically for $\hat{\mathbf{J}_-}$ we obtain the same thing but with $m-1$ as the eigenvalue for $\hat{\mathbf{J}}_\mathbf{3}$.

4. Assume that there is no such vector than the ladder operator would span an infinite family of eigenvectors of $\hat{\mathbf{J}}_\mathbf{3}$ and $\hat{\mathbf{J}_+}$ and hence V would be infinite dimensional.
5. We have that:

$$\hat{\mathbf{J}_-} \hat{\mathbf{J}_+} = \hat{\mathbf{J}_\mathbf{1}}^2 - i[\hat{\mathbf{J}}_\mathbf{1}, \hat{\mathbf{J}}_\mathbf{2}] + \hat{\mathbf{J}_\mathbf{2}}^2 = \hat{\mathbf{J}}^2 - \hat{\mathbf{J}_\mathbf{3}}^2 + \hat{\mathbf{J}}_\mathbf{3}$$

Then applying this for $|v_0\rangle$ we get:

$$\hat{\mathbf{J}_-} \hat{\mathbf{J}_+} |v_0\rangle = 0 = (j(j+1) - m_0^2 + m_0) |v_0\rangle \Rightarrow j(j+1) = m_0(m_0+1)$$

6. An identical argument tells us that successive application of the lowering ladder operator must lead to a vanishing state. Then from definition we have that:

$$|w_0\rangle = (\hat{\mathbf{J}_-})^k |v_0\rangle \Rightarrow m_0' = m_0 - k$$

7. Similarly as before we get the exact same result but with a minus sign.
8. We then have the system:

$$\begin{cases} j(j+1) = m_0(m_0+1) \\ j(j+1) = (m_0-k)(m_0-k-1) \end{cases} \Rightarrow \begin{cases} j(j+1) = m_0(m_0+1) \\ k^2 + k = 2m_0(1+k) \end{cases} \Rightarrow \begin{cases} j = \frac{k}{2} \\ \frac{k}{2} = m_0 \end{cases}$$

9. We have that $\hat{\mathbf{J}_+}$ sends $|j,m\rangle$ to $|j,m+1\rangle$ and similarly $\hat{\mathbf{J}_-}$ sends $|j,m\rangle$ to $|j,m-1\rangle$. Then we get that:
$$\hat{\mathbf{J}_+}|j,m\rangle = x|j,m+1\rangle \Rightarrow \langle j,m|\hat{\mathbf{J}_-}\hat{\mathbf{J}_+}|j,m\rangle = |x|^2 = j(j+1) - m(m+1)$$
Hence we obtain:
$$x = \sqrt{j(j+1) - m(m+1)}$$
Then we have that:
$$\hat{\mathbf{J}}_\mathbf{1}|j,m\rangle = \frac{\hat{\mathbf{J}_+} + \hat{\mathbf{J}_-}}{2}|j,m\rangle = \frac{x}{2}\left(|j,m+1\rangle + |j,m-1\rangle\right)$$
Similarly:
$$\hat{\mathbf{J}}_\mathbf{2}|j,m\rangle = \frac{\hat{\mathbf{J}_+} - \hat{\mathbf{J}_-}}{2i}|j,m\rangle = \frac{x}{2i}\left(|j,m+1\rangle - |j,m-1\rangle\right)$$

10. Since $\hat{\mathbf{J}}^2$ commutes with the $\hat{\mathbf{J}}_\mathbf{i}$ we know that the eigenspaces of $\hat{\mathbf{J}}^2$ are sub-representations of $SU(2)$. We now restrict ourselves to one eigenspace, call it $\tilde{V}_j$ corresponding to the eigenvalue $j(j+1)$. As said previously there must be at least one eigenvector of $\hat{\mathbf{J}}^2$ and $\hat{\mathbf{J}}_\mathbf{3}$ which is killed by $\hat{\mathbf{J}_+}$ call it $|j,j,1\rangle$. Then from this eigenvector we can build $|j,m,1\rangle = \hat{\mathbf{J}_-}^{j-m}|j,j,1\rangle$. Which is an irreducible subspace of $\tilde{V}_j$. Then we can write $\tilde{V}_j = V_j^1 \oplus \tilde{V}_j'$. We can then repeat the process on $\tilde{V}_j'$ until we spanned the whole space. Then we have:
$$V = V_0^1 \oplus \cdots \oplus V_0^{n_0} \oplus V_{1/2}^1 \oplus \cdots \oplus V_{1/2}^{n_{1/2}} \oplus \cdots$$

11. We have that $\vec{L} = \vec{R} \wedge \vec{P}$ where $\vec{R}$ and $\vec{P}$ are operators on $L^2(\mathbb{R}^3)$ where $[R_j, P_k] = i\delta_{jk}$. Then we have that $[L_a, L_b] = i\varepsilon_{abc}L_c$. Then the space we describe is $V : \{\psi : S^2 \to \mathbb{C}\}$ and the spherical harmonic decomposition tells us that:
$$\psi(\theta,\varphi) = \sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{\ell} a_{\ell,m}Y_\ell^m(\theta,\varphi)$$
Furthermore we have that:
$$\vec{L}^2 = Y_\ell^m = \ell(\ell+1)Y_\ell^m \quad \text{and} \quad L_3Y_\ell^m = mY_\ell^m$$
Hence the subspace $V_\ell = \mathrm{Span}(Y_\ell^{-\ell}, \cdots, Y_\ell^\ell)$ is stable under rotation and $V = V_0 \oplus V_1 \oplus V_2 \oplus \cdots$

12. We have:
$$e^{2i\pi\hat{\mathbf{J}}_\mathbf{3}}|j,m\rangle = e^{2i\pi m}|j,m\rangle$$
Now if j is an integer we have that $m \in \mathbb{Z}$ and hence $e^{2i\pi\hat{\mathbf{J}}_\mathbf{3}} = \mathrm{Id}$. However if j is a half integer then m is also a half integer and hence $e^{2i\pi\hat{\mathbf{J}}_\mathbf{3}} = -\mathrm{Id}$.

13. In QM for example we usually consider the wavefunctions of one particle with no spin we will use the space $L^2(\mathbb{R}^3,\mathbb{C})$ however now if we introduce spin we will consider $L^2(\mathbb{R}^3,\mathbb{C}) \otimes \mathbb{C}^2$ or similarly if we consider two particles we need to consider $L^2(\mathbb{R}^3,\mathbb{C}) \otimes L^2(\mathbb{R}^3,\mathbb{C})$. Then we know also that:
$$V_{j_1} \otimes V_{j_2} = V_{|j_1-j_2|} \oplus V_{|j_1-j_2|+1} \oplus \cdots \oplus V_{j_1+j_2}$$

Chapter 2

TD2

2.1 Properties of time-like vectors.

1. Let $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{C}_+$. Then $a^0 > ||\vec{a}||$ and similarly for $\mathbf{B}$. Hence $\vec{a} \cdot \vec{b} \leq ||\vec{a}|| \cdot ||\vec{b}|| \leq a^0 b^0$. Then $\mathbf{A} \cdot \mathbf{B} < 0$.
2. Let $\mathbf{A}, \mathbf{B} \in \mathcal{C}_+$ and $\mu, \nu \in \mathbb{R}^+$ then $(\mu\mathbf{A} + \nu\mathbf{B})^2 = \mu^2\mathbf{A}^2 + 2\mu\nu\mathbf{A} \cdot \mathbf{B} + \nu^2\mathbf{B}^2 < 0$. Hence $(\mathbf{A} + \mathbf{B}) \in \mathcal{C}_+$.
3. A special Lorentz transformation is an isometry of the Minkowski space hence $\mathcal{C}_+$ is stable under it.
4. We have that:
$$a^i - \beta^i a^0 = 0 \Rightarrow \beta^i = \frac{a^i}{a^0}$$
5. Suppose by induction that this is true for n the base cases being trivial. Then for $n+1$ note that $\mathcal{C}_+$ is stable under addition so any case can be reduced to the base case $n = 2$. We prove this case here:
$$\sqrt{-(\mathbf{A}+\mathbf{B})^2} = \sqrt{-(\mathbf{A}' + \mathbf{B}')^2} = \sqrt{{d^0}^2} = d^0$$
Then ${\mathbf{A_i}}^2 = \vec{a}_i^{\,2} - (a_i^0)^2$ and hence $a_i^0 = \sqrt{-{\mathbf{A_i}}^2 + \vec{a}_i^{\,2}} \geq \sqrt{-{\mathbf{A_i}}^2}$ hence:
$$\sqrt{-(\mathbf{A}+\mathbf{B})^2} \geq \sqrt{-\mathbf{A}^2} + \sqrt{-\mathbf{B}^2}$$

2.2 Applications to 4-momenta

1. $\mathbf{P} = m\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\tau} = (E, m\vec{U})$ and:
$$\mathbf{P}^2 = -E^2 + m^2\vec{U}^2 = -m^2$$
2. We directly have that $P^0 = E > 0$ and $\mathbf{P}^2 = -m^2 < 0$. Hence $\mathbf{P} \in \mathcal{C}_+$.
3. From question 2 of Exercise 1 we know that since $\mathbf{P}_i$ are in $\mathcal{C}_+$ then so is $\mathbf{P}$. Then from question 4 of Exercise 1 we know that there exists a boost transformation such that $\mathbf{P} = (E^*, \vec{0})$. Then using question 5 of Exercise 1 we also know that:
$$E^* \geq \sum_{i=1}^{n} m_i$$

2.3 Decays of particles

1. We must have that $M \geq \sum_{i=1}^{n} m_i$.
2. (a) The number of unknowns are 8 since they are all the components of the two momenta $\mathbf{P}_1$ and $\mathbf{P}_2$. We also have the four equations given by: $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$. Finally we have two more equations ${\mathbf{P_1}}^2 = -m_1^2$ and ${\mathbf{P_2}}^2 = -m_2^2$.

   (b) We have that:
$$ {\mathbf{P_1}}^2 = \mathbf{P}^2 + {\mathbf{P_2}}^2 - 2\mathbf{P} \cdot \mathbf{P_2} \Leftrightarrow -m_1^2 = -M^2 - m_2^2 - 2\left(-ME_2\right) \Leftrightarrow 2ME_2 = M^2 + m_2^2 - m_1^2$$
   Then symmetry gives the desired opposite result.

(c) We have:

$$E_{kin,1} = E_1 - m_1$$

Which immediately gives the desired result after factorization and identically for $E_{kin,2}$. Then:

$$E_{kin,1} + E_{kin,2} = \Delta M$$

In other words all excess mass is converted to kinetic energy.

3. For each new particle we get 4 more unknowns and one more equation so 3 more indeterminates. Now following the hint we write:

$$\mathbf{P} = \sum_j \mathbf{P_j} = \mathbf{P_i} + \mathbf{Q}$$

Then:

$$\mathbf{P_i}^2 = \mathbf{P}^2 + \mathbf{Q}^2 - 2\mathbf{P} \cdot \mathbf{Q} \Leftrightarrow -m_i^2 = -M^2 - 2ME' + \mathbf{Q}^2$$

Then we have:

$$E_i = \frac{M^2 + m_i^2 + \mathbf{Q}^2}{2m} \quad \text{and} \quad E_{kin,i} = \frac{M^2 + m_i^2 - 2Mm_i + \mathbf{Q}^2}{2m}$$

Now using question 5 of Exercise 1 we can bound $\mathbf{Q}^2$ as follows:

$$\sqrt{-\mathbf{Q}^2} \geq \sum_{j \neq i} m_j \Rightarrow \mathbf{Q}^2 \leq -(M - \Delta M - m_i)^2$$

Then re-injecting this above we get the desired inequalities.

2.4 Creations of particles

1.

Chapter 3

TD3

3.1 The Laplace Equation

1. The solution is given by $\frac{q\mathbf{r}}{4\pi}$.
2. Rotationally invariant harmonic functions are given by:

$$\nabla^2 u = 0 \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{n-1}u'(r)\right) = 0 \Leftrightarrow r^{n-1}u'(r) = c \Leftrightarrow u'(r) = c\,r^{1-n} \Leftrightarrow u(r) = \frac{c}{r^{n-2}(n-2)} + c'$$

   When $n \neq 2$ in the case where $n = 2$ then we get:

$$u(r) = c\ln r + c'$$

3. We have that:

$$\int_{\Omega} \mathrm{d}\mathbf{x}[u\nabla^2 v - v\nabla^2 u] = \int_{\Omega} \mathrm{d}\mathbf{x}\boldsymbol{\nabla}\cdot[u\boldsymbol{\nabla} v - v\boldsymbol{\nabla} u] = \int_{\partial\Omega} \mathrm{d}\mathbf{x}\,\mathbf{n}\cdot[u\boldsymbol{\nabla} v - v\boldsymbol{\nabla} u] = \int_{\partial\Omega} \mathrm{d}\mathbf{x}\left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right]$$

4. We have that:

$$\begin{aligned}
\int_{\overline{\mathcal{B}_\varepsilon}} \mathrm{d}\mathbf{x}\, G(\mathbf{x})\nabla^2\varphi(\mathbf{x}) &= \int_{\overline{\mathcal{B}_\varepsilon}} \mathrm{d}\mathbf{x}\left[G(\mathbf{x})\nabla^2\varphi(\mathbf{x}) - \varphi(x)\nabla^2 G(x)\right] \\
&= \int_{\mathcal{C}_\varepsilon} \mathrm{d}\mathbf{x}\,(G(\mathbf{x})(-\mathbf{r})\cdot\boldsymbol{\nabla}\varphi(\mathbf{x}) - \varphi(\mathbf{x})(-\mathbf{r})\boldsymbol{\nabla}G(\mathbf{x})) \\
&= \int_{\partial\Omega} \mathrm{d}\mathbf{x} - \varphi(\mathbf{x})\frac{\partial G}{\partial r} \xrightarrow{\varepsilon\to 0} \varphi(\mathbf{0})\omega_n\varepsilon^{n-1}\frac{\partial G}{\partial r}\Big|_{r=\varepsilon} = \varphi(\mathbf{0})
\end{aligned}$$

5. We have:

$$\left\langle G\middle|\nabla^2\varphi\right\rangle = \langle\delta|\varphi\rangle = (-1)^2\left\langle\nabla^2 G\middle|\varphi\right\rangle = \langle\delta|\varphi\rangle$$