Master ENS ICFP - First Year 2020/2021
Relativistic Quantum Mechanics and Introduction to Quantum Field Theory
Mid Term Homework

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1 Some operator identities : 6 points

1. We have that:

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{A^n}{n!} B \sum_{n=0}^{+\infty} \frac{(-A)^n}{n!} = \sum_{n,m=0}^{+\infty} \frac{A^n B (-A)^m}{n!m!} = \sum_{n,m=0}^{+\infty} (-1)^m \frac{A^n B A^m}{n!m!} = \sum_{n=0}^{+\infty} \frac{A^n}{n!} \sum_{m=0}^{+\infty} (-1)^m \frac{B A^m}{m!}$$

...

2. ...

3. We have that:

$$[F, G^\dagger] = [\sum_j f_j a_j, \sum_j g_j^\star a_j^\dagger] = \sum_{j,k=0}^{+\infty} f_j g_k^\star [a_j, a_k^\dagger] = \sum_{j,k=0}^{+\infty} f_j g_k^\star \delta_{jk} = \sum_{j=0}^{+\infty} f_j g_j^\star$$

Furthermore we have that $[F, G^\dagger] \propto \mathrm{Id}$ and therefore we trivially have that $[F,[F,G^\dagger]] = [G^\dagger, [F, G^\dagger]] = 0$. Now applying question 2 we have that:

$$e^{G^\dagger} e^F = e^{-\frac{1}{2}\sum_j f_j g_j^\star} e^{G^\dagger + F} \Rightarrow e^{\frac{1}{2}\sum_j f_j g_j^\star} e^F = \underbrace{e^{\frac{1}{2}\sum_j f_j g_j^\star} e^{-\frac{1}{2}\sum_j f_j g_j^\star}}_{=e^A e^{-A}} e^{G^\dagger + F}$$

Now from Question 1 we have that for any A (trivially $[A, \mathrm{Id}] = 0$) we get:

$$e^A \mathrm{Id} e^{-A} = \mathrm{Id} + \sum_{n=1}^{+\infty} \frac{1}{n!} \cdot 0 = \mathrm{Id}$$

Hence the above formula simplifies to:

$$e^{F + G^\dagger} = e^{\frac{1}{2}\sum_j f_j g_j^\star} e^{G^\dagger} e^F$$

4. Similarly as before let $F = \int \mathrm{d}^3\mathbf{q} f(\mathbf{q}) a(\mathbf{q})$ and $G = \int \mathrm{d}^3\mathbf{q} h(\mathbf{q})^\dagger a(\mathbf{q})$. Then we have that:

$$[F, G^\dagger] = [\int \mathrm{d}^3\mathbf{q} f(\mathbf{q}) a(\mathbf{q}), \int \mathrm{d}^3\mathbf{q} h(\mathbf{q}) a^\dagger(\mathbf{q})] = \int \mathrm{d}^3\mathbf{q} f(\mathbf{q}) h(\mathbf{q}) [a(\mathbf{q}), a^\dagger(\mathbf{q})] = \int \mathrm{d}^3\mathbf{q} f(\mathbf{q}) h(\mathbf{q})$$

A similar direct application of 2 gives the desired result.

2 An example of an asymptotic series

We have that:

$$f(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}x e^{-x^2 - gx^4} \quad \text{hence} \quad |f(g)| < \int_{-\infty}^{+\infty} \mathrm{d}x |e^{-x^2 - gx^4}| = \int_{-\infty}^{+\infty} \mathrm{d}x e^{-x^2 - x^4 \operatorname{Re} g} < \int_{-\infty}^{+\infty} \mathrm{d}x e^{-x^4 \operatorname{Re} g}$$

Hence as long as $\operatorname{Re} g > 0$ this is obviously well defined from the last term and if $\operatorname{Re} g = 0$ this is obviously well defined from the before last term.

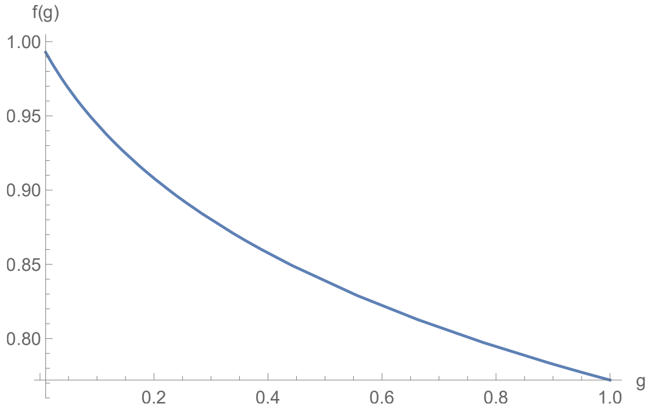

Figure 1: Plot of the numerical values of $f(g)$ for $g \in [0.01, 1]$.

1. This integral admits an exact solution given by:

$$f(g) = \frac{e^{\frac{1}{8g}} K_{\frac{1}{4}}(\frac{1}{8g})}{2\sqrt{\pi g}} \delta_{\mathrm{Re}\, g>0} + \delta_{\mathrm{Re}\, g=0} \quad \text{where} \quad K_n(z) \quad \text{is the modified Bessel function of the second kind.}$$

The plot of the numerical values for $g \in [0.01, 1]$ is given in Figure 1. Then $f(g)$ decreases monotonically when $g > 0$ increases since:

$$\frac{\mathrm{d}}{\mathrm{d}g} f(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}x (-x^4) e^{-x^2 - gx^4} = \frac{-1}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{+\infty} x^4 e^{-x^2 - gx^4}}_{>0 \quad \text{when} \quad g \in \mathbb{R}^+} < 0$$

2. We have that:

$$e^{-gx^4} = \sum_{n=0}^{+\infty} \frac{(-gx^4)^n}{n!}$$

And plugging this in the expression of f and inverting the sum and the integral gives:

$$\frac{1}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \frac{(-g)^n}{n!} \int_{-\infty}^{+\infty} \mathrm{d}x \; x^{4n} e^{-x^2}$$

We notice the integral ressembles strongly the gamma function hence we change variables by taking $u = x^2$ ($\mathrm{d}u = 2\sqrt{u}\mathrm{d}x$) and we get:

$$2^{-1} \int_{-\infty}^{+\infty} \mathrm{d}u u^{2n - \frac{1}{2}} e^{-u} = 2^{-1} \Gamma(2n + \frac{1}{2}) = 2^{-4n} \sqrt{\pi} \frac{\Gamma(4n)}{\Gamma(2n)} \quad \text{from the Legendre duplication formula.}$$

Hence plugging it back up top we obtain:

$$\tilde{f}(g) = \sum_{n=0}^{+\infty} \left(\frac{(-1)^n (4n)!}{n! 2^{4n} (2n)!} \right) g^n$$

Notice that the terms f_n are monotonically increasing in norm and diverge hence the sum does not converge absolutely and $R = 0$ and it also does not converge conditionally. The order of magnitude of the first few terms is ...

3.

3 A relation between Dirac spinors

1. Remember that up to a re-writing we have that:

$$\omega_{ij} = \varepsilon_{ijk} \theta^k \quad \text{and} \quad \omega^{k0} = \nu^k \quad \text{and} \quad \omega_{\mu\nu} = 0 \quad \text{otherwise.}$$

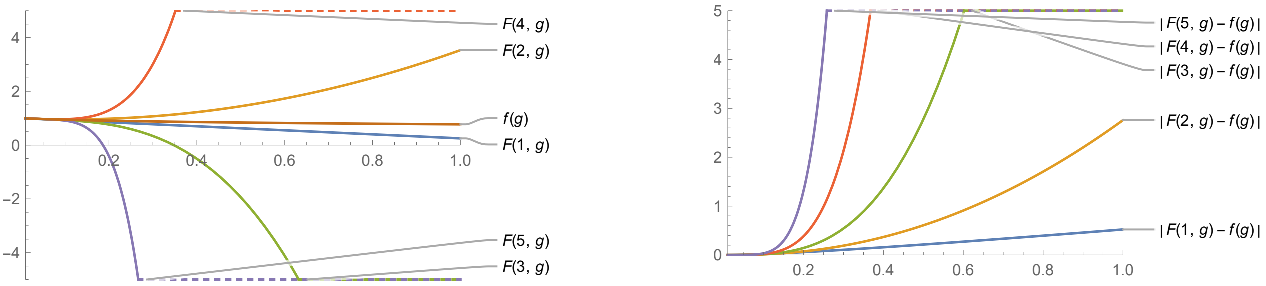

Figure 2: Series approximations of $f(g)$ and their errors for $g \in [0.01, 1]$.

Where the θ^k represent the rotations in the 3 spatial dimensions and the ν^k represent the boosts along the the three spatial directions. Hence the representation of $L(\mathbf{p})$ trivially has that $\theta^k = 0$. Then we have that:

$$i\gamma^0 D(L(\mathbf{p}))i\gamma^0 = i\gamma^0 \exp\left(\frac{1}{4}\omega_{\mu\nu}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\right)i\gamma^0$$

Now in order to simplify we need to bring the product with the gamma matrices inside. To do so we need to remember two properties. Firstly we have that $(i\gamma^0) = (i\gamma^0)^{-1}$ and secondly we have that: $Pe^A P^{-1} = e^{PAP^{-1}}$. Hence applying this formula here we obtain that:

$$i\gamma^0 D(L(\mathbf{p}))i\gamma^0 = \exp\left(\frac{\omega_{\mu\nu}}{4} i\gamma^0(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)i\gamma^0\right) = \exp\left(\frac{\omega_{\mu\nu}}{4}(i\gamma^0\gamma^\mu i\gamma^0 i\gamma^0\gamma^\nu i\gamma^0 - i\gamma^0\gamma^\nu \imath\gamma^0 i\gamma^0\gamma^\mu i\gamma^0)\right)$$

Now we use the formula from the course: $i\gamma^0\gamma^\mu i\gamma^0 = P^\mu_\nu\gamma^\nu$ where P^μ_ν is the parity operator defined in (3.33). Hence this means that the terms in the exponential simplify to:

$$i\gamma^0 D(L(\mathbf{p}))i\gamma^0 = \exp\left(\frac{\omega_{\mu\nu}}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(-1)^{1-\delta_0^\mu}(-1)^{1-\delta_0^\nu}\right) = \exp\left(\frac{\omega_{\mu\nu}(-1)^{\delta_0^\mu+\delta_0^\nu-\delta_0^\mu\delta_0^\nu}}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\right)$$

Hence now we take $\omega'_{\mu\nu} = \omega_{\mu\nu}(-1)^{\delta_0^\mu+\delta_0^\nu-\delta_0^\mu\delta_0^\nu}$, now since $\omega_{00} = 0$ we can discard the term $\delta_0^\mu\delta_0^\nu$ which means that we are left with:

$$\omega'_{\mu\nu} = \omega_{\mu\nu}(-1)^{\delta_0^\mu+\delta_0^\nu} \Leftrightarrow \omega'_{ij} = \omega_{ij} \wedge \omega'_{k0} = -\omega_{k0} \Rightarrow \theta'^k = \theta^k \wedge \nu'^k = -\nu^k$$

Hence we have that:

$$i\gamma^0 D(L(\mathbf{p}))i\gamma^0 = D(L(-\mathbf{p}))$$

2. We have that:

$$(ip_\mu\gamma^\mu + m)u(\mathbf{p},\sigma) = 0 \quad \text{and} \quad (-ip_\mu\gamma^\mu + m)v(\mathbf{p},\sigma) = 0$$

Hence if we take $\mathbf{p} = 0$ and $p_0 = -p^0 = -m$ for a particle at rest the above simplifies to:

$$(-im\gamma^0 + m)u(\mathbf{0},\sigma) = 0 \quad \text{and} \quad (im\gamma^0 + m)v(\mathbf{0},\sigma) = 0$$

Which up to dividing by m simplifies to:

$$(-i\gamma^0 + \mathbf{1})u(\mathbf{0},\sigma) = 0 \quad \text{and} \quad (i\gamma^0 + \mathbf{1})v(\mathbf{0},\sigma) = 0$$

Hence up to a re-writing we have that:

$$i\gamma^0 u(\mathbf{0},\sigma) = u(\mathbf{0},\sigma) \quad \text{and} \quad i\gamma^0 v(\mathbf{0},\sigma) = -v(\mathbf{0},\sigma)$$

Then from the definition of u, v and $D(L(\mathbf{p}))$ we have that $u(\mathbf{p},\sigma) = D(L(\mathbf{p}))u(\mathbf{0},\sigma)$ and identically for v. Hence we have that:

$$i\gamma^0 u(\mathbf{p},\sigma) = i\gamma^0 D(L(\mathbf{p}))u(\mathbf{0},\sigma) = i\gamma^0 D(L(\mathbf{p}))i\gamma^0 i\gamma^0 u(\mathbf{0},\sigma) = D(L(-\mathbf{p}))u(\mathbf{0},\sigma) = u(-\mathbf{p},\sigma)$$

And identically:

$$i\gamma^0 v(\mathbf{p},\sigma) = i\gamma^0 D(L(\mathbf{p}))v(\mathbf{0},\sigma) = i\gamma^0 D(L(\mathbf{p}))i\gamma^0 i\gamma^0 v(\mathbf{0},\sigma) = D(L(-\mathbf{p}))(-v(\mathbf{0},\sigma)) = -v(-\mathbf{p},\sigma)$$

4 Some traces of products of γ-matrices

We have that:

$$\operatorname{tr}\gamma_\mu\gamma_\nu = \operatorname{tr}\{\gamma_\mu,\gamma_\nu\} - \gamma_\nu\gamma_\mu = 2\operatorname{tr}\eta_{\mu\nu}I_4 - \operatorname{tr}\gamma_\nu\gamma_\mu = 2\operatorname{tr}\eta_{\mu\nu}I_4 - \operatorname{tr}\gamma_\mu\gamma_\nu$$

Hence adding on both side we obtain the desired equality:

$$\operatorname{tr}\gamma_\mu\gamma_\nu = \eta_{\mu\nu}\operatorname{tr}I_4 = 4\eta_{\mu\nu}$$

Similarly we have that:

$$\operatorname{tr}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = ...$$

5 Energy levels of a relativistic charged spin-0 particle in a harmonic electrostatic potential

1. From these relation we have that:

$$\begin{aligned} X^2\left|n\right\rangle &= \frac{1}{2m\Omega}\left(a^2 + (a^\dagger)^2 + \{a, a^\dagger\}\right)\left|n\right\rangle \\ &= \frac{1}{2m\Omega}(\sqrt{n}\sqrt{n-1}\left|n-2\right\rangle + \sqrt{n+1}\sqrt{n+2}\left|n+2\right\rangle + (n+1)\left|n\right\rangle + n\left|n\right\rangle) \end{aligned}$$

   Hence we get that:

$$\left\langle n\right| X^4 \left|n\right\rangle = (\left(\left\langle n\right| X^2\right)\left(X^2\left|n\right\rangle\right)) = \frac{1}{(2m\Omega)^2}(n^2 - n + n^2 + 3n + 2 + n + 1 + n) = \frac{2n^2+3n+2}{(2m\Omega)^2}$$

2. We have that:

$$(D_\mu D^\mu + m^2)\Phi = 0 \quad \text{where} \quad D^\mu = \partial^\mu - iqA^\mu$$

   Then taking $\Phi = e^{-iEt}\phi$ and plugging it in we get that:

$$e^{-iEt}(D_iD^i + m^2)\phi + D_0D^0e^{-iEt}\phi = 0$$

   Looking now only at the second term we have that:

$$D_0D^0e^{-iEt}\phi = (\partial_0 + i\frac{m}{2}\omega^2x^2)(\partial^0 - i\frac{m}{2}\omega^2x^2)e^{-iEt}\phi = (\partial_0\partial^0 + \frac{m}{2}\omega^2x^2)e^{-iEt}\phi =$$

   Now looking at the time derivative we obtain:

$$\partial_0\partial^0e^{-iEt}\phi = \partial_0(-iEe^{-iEt}\phi + e^{-iEt}\partial^0\phi) = E^2e^{-iEt}\phi - iEe^{-iEt}\partial^0\phi + e^{-iEt}\partial_0\partial^0\phi$$

3.

6 The axial current

1. We know that $\{\gamma_5, \gamma^\mu\} = 0$ and hence $\gamma_5\gamma^\mu = -\gamma^\mu\gamma_5$. Then we have that:

$$e^{i\varepsilon\gamma_5}\gamma^\mu = \sum_{n\in\mathbb{N}}\frac{1}{n!}(i\varepsilon\gamma_5)^n\gamma^\mu = \sum_{n\in\mathbb{N}}\frac{1}{n!}(-1)^n\gamma^\mu(i\varepsilon\gamma_5)^n = \gamma^\mu\sum_{n\in\mathbb{N}}\frac{(-i\varepsilon\gamma_5)^n}{n!} = \gamma^\mu e^{-i\varepsilon\gamma_5}$$

   Notice also trivially that $(e^{i\varepsilon\gamma_5})^\dagger = e^{-i\varepsilon\gamma_5}$ since $\gamma_5 = \gamma_5^\dagger$. Hence the axial transformation gives:

$$\overline{e^{i\varepsilon\gamma_5}\psi} = (e^{i\varepsilon\gamma_5}\psi)^\dagger i\gamma^0 = \psi^\dagger e^{-i\varepsilon\gamma_5}i\gamma^0 = \psi^\dagger i\gamma^0e^{i\varepsilon\gamma_5} = \overline{\psi}e^{i\varepsilon\gamma_5}$$

2. If we replace ψ by the axial transformed $e^{i\varepsilon\gamma_5}\psi$ we also have to replace $\overline{\psi}$ by the axial transformed $\overline{\psi}e^{i\varepsilon\gamma_5}$. Hence we obtain:

$$S = \int \mathrm{d}^4x\overline{\psi}e^{i\varepsilon\gamma_5}(-\not\partial + iq\not A - m)e^{i\varepsilon\gamma_5}\psi$$

   Now notice that:

$$e^{i\varepsilon\gamma_5}\not a e^{i\varepsilon\gamma_5} = e^{i\varepsilon\gamma_5}a_\mu\gamma^\mu e^{i\varepsilon\gamma_5} = (a_\mu e^{i\varepsilon\gamma_5} + [e^{i\varepsilon\gamma_5}, a_\mu])\gamma^\mu e^{i\varepsilon\gamma_5} = a_\mu + [e^{i\varepsilon\gamma_5}, a_\mu]\gamma^\mu e^{i\varepsilon\gamma_5}$$

Then since $[\partial_\mu, e^{i\varepsilon\gamma_5}] = 0$ and $[A_\mu, e^{i\varepsilon\gamma_5}] = 0$ we have that:

$$S = \int \mathrm{d}^4x \overline{\psi}(-\not{\partial} + iq\not{A} - me^{i2\varepsilon\gamma_5})\psi$$

Hence the action is left unchanged if and only if $m = 0$ or $\varepsilon = 0$. The second case corresponding to the trivial case of reducing the transformation to the identity can be discarded. Now assuming $m = 0$. The infinitesimal transformation corresponding to the axial transformation is given by:

$$\psi \longmapsto e^{i\varepsilon\gamma_5}\psi = (1 + i\varepsilon\gamma_5)\psi = \psi + i\varepsilon\gamma_5\psi$$

Now this transformation conserves both the action and the Lagrangian hence from the formula (4.53) of the notes we have that the corresponding conserved current is given by:

$$j_5^\mu = -\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi^a}\frac{\partial\psi^a}{\partial\varepsilon} = -i\overline{\psi}\gamma^\mu\gamma_5\psi$$

3. The Dirac equations are given by:

$$(\not{\partial} - iq\not{A} + m)\psi = 0 \text{ and } \overline{\psi}(-\overleftarrow{\not{\partial}} + iq\not{A} + m) = 0$$

Then we have that:

$$\partial_\mu j_5^\mu = \partial_\mu(-i\overline{\psi}\gamma^\mu\gamma_5\psi) = -\overline{\psi}(\not{\partial} + \overleftarrow{\not{\partial}})\gamma_5\psi = -\overline{\psi}(iq\not{A} - m - iq\not{A} - m)\gamma_5\psi = 2m\overline{\psi}\gamma_5\psi \propto m$$

7 Supersymmetry

1.