

Symmetries in Physics

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Chapter 1

TD1

1.1 Problem 1 Cayley tables

1.1.1

Suppose that an element appears more than once in a given row or column. Then we have that:

$$\exists g, g_i, g_j, g_k \in \mathcal{G}, \quad g = g_i \cdot g_j \wedge g = g_i \cdot g_k \Rightarrow g_j = g_k$$

Since no two elements in a row can be mapped to the same element of the group then a row is a map $\mathcal{G} \rightarrow \mathcal{G}$ which from the point above is injective then since it is an endomorphism it necessarily must be a bijection and hence a permutation of \mathcal{G} . Therefore each element appears once and exactly once.

1.1.2

Refer above.

1.2 Problem 2 The group D_3

1.2.1

The elements of D_3 are $e = Id$, $r = (B, C, A)$, $r^2 = (C, A, B)$, $s_1 = (A, C, B)$, $s_2 = (B, A, C)$, $s_3 = (C, B, A)$. Then the table is given by:

| | e | r | r^2 | s_1 | s_2 | s_3 |
|-------|-------|-------|-------|-------|-------|-------|
| e | e | r | r^2 | s_1 | s_2 | s_3 |
| r | r | r^2 | e | s_2 | s_3 | s_1 |
| r^2 | r^2 | e | r | s_3 | s_1 | s_2 |
| s_1 | s_1 | s_2 | s_3 | e | r | r^2 |
| s_2 | s_3 | s_1 | s_2 | r^2 | e | r |
| s_3 | s_2 | s_3 | s_1 | r | e | r^2 |

1.2.2

The subgroups of D_3 are $\{e, r, r^2\} = \langle r \rangle$, $\langle s_1 \rangle$, $\langle s_2 \rangle$, $\langle s_3 \rangle$, $\{e\}$.

1.3 Problem 3 Lagrange's theorem.

Let \mathcal{H} be a subgroup of \mathcal{G} . Then notice that \mathcal{G}/\mathcal{H} is the set of the cosets of \mathcal{G} by the congruence modulo \mathcal{H} . However from Exercise 1 and 2 we know that every coset is in bijection with \mathcal{H} . Furthermore since the congruence is an equivalence relation it must be that \mathcal{G} is equal to the reunion of the cosets. Hence we have that:

$$|\mathcal{G}/\mathcal{H}| \cdot |\mathcal{H}| = |\mathcal{G}|$$

The result follows.

1.4 Problem 4 Modular arithmetics

1.4.1

Notice that for any $k \in \mathbb{Z}$ we have that $k\mathbb{Z}$ is a subgroup of \mathbb{Z} . The quotient groups are $\mathbb{Z}/k\mathbb{Z}$ which are the well-known integers modulo k with the addition modulo k .

1.4.2

Let $|g|$ be the smallest integer such that $g^{|g|} = e$. Such an integer must exist so long as the group to which g pertains is finite. Then notice that for any $k \in \mathbb{Z}$ we have that: $g^{|g| \cdot k} = (g^{|g|})^k = e^k = e$. Hence $|g|\mathbb{Z} \subseteq P_g$. Now let $k \in \mathbb{Z}$ such that $g^k = e$. From construction it must be that $k > |g|$ hence by doing the euclidean division we get that: $k = |g| \cdot \ell + r$. Hence: $g^{|g| \cdot \ell + r} = e \Rightarrow e^\ell \cdot g^r = e \Rightarrow g^r = e$. However unless $r = 0$ this is impossible since $r < |g|$ would be a contradiction.

1.4.3

Notice that necessarily $\langle g \rangle$ is a subgroup of cardinality $|g|$ of \mathcal{G} hence from the Lagrange theorem we know that $|g|$ divides $|\mathcal{G}|$.

1.4.4

Let a group \mathcal{G} of order p where p is prime. Then from the previous question we know that all elements of \mathcal{G} must be of order p . However if one element is of order p and \mathcal{G} is of order p it must be that \mathcal{G} is generated by a single element, call it g . Then the obvious homomorphism concludes the proof:

$$\begin{aligned} h : \mathcal{G} &\rightarrow \mathbb{Z}/p\mathbb{Z} \\ g^k &\mapsto k \pmod{p} \end{aligned}$$

Chapter 2

TD2

2.1 All finite groups up to 5 elements.

The only group of size 1 is the trivial group which is isomorphic to $\mathbb{Z}/1\mathbb{Z}$. The group of size 2 is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The group of size 3 is:

| | | | |
|---|---|---|---|
| | a | b | c |
| a | a | b | c |
| b | b | c | a |
| c | c | a | b |

Which is clearly isomorphic to $\mathbb{Z}/3\mathbb{Z}$. The groups of size 4 are:

| | | | | |
|---|---|---|---|---|
| | a | b | c | d |
| a | a | b | c | d |
| b | b | c | d | a |
| c | c | d | a | b |
| d | d | a | b | c |

| | | | | |
|---|---|---|---|---|
| | a | b | c | d |
| a | a | b | c | d |
| b | b | a | d | c |
| c | c | d | a | b |
| d | d | c | b | a |

Which are respectively isomorphic to $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Finally the only possible group of size 5 is given by $\mathbb{Z}/5\mathbb{Z}$.

2.2 Union of groups.

Let \mathcal{G}_1 and \mathcal{G}_2 be two groups. Then let $g \in \mathcal{G}_1$ and $h \in \mathcal{G}_2$ (w.l.o.g.) then $gh \in \mathcal{G}_1 \cup \mathcal{G}_2 \Leftrightarrow gh \in \mathcal{G}_1$ (w.l.o.g.). However since $g \in \mathcal{G}_1$ then $g^{-1} \in \mathcal{G}_1$ and hence we would have that $h \in \mathcal{G}_1$. Hence we have that $\mathcal{G}_1 \cup \mathcal{G}_2$ is a group if and only if $\mathcal{G}_1 \leq \mathcal{G}_2$ or vice-versa.

2.3 Quotient groups.

1. Let $\pi : g \mapsto g\mathcal{H}$ which is a natural isomorphism. Then let $\mathcal{G}' = \pi^{-1}(A)$. Since $\mathcal{H} = \text{Ker}\pi$ we have that $\mathcal{H} \triangleleft \mathcal{G}'$. Then it is immediate from definition that: $\mathcal{G}'/A = \pi(\mathcal{G}') = \pi \circ \pi^{-1}(A) = A$.
2. ...
3. Notice that the isomorphism $\psi : g \mapsto \varphi_g$ has that $\text{Ker}\psi = Z(\mathcal{G})$ furthermore we know that $\mathcal{G}/\text{Ker}\psi \cong \text{Im}\psi$ and hence we immediately get that $\mathcal{G}/Z(\mathcal{G}) \cong \text{Inn}(\mathcal{G})$.

Chapter 3

TD3