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Alessandro Pacco

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Contents

De la forme des arbres

Méthodes d'analyse complexe pour des problèmes d'élasticité bidimensionelle

1.a)

If f is a holomorphic function then it is C^∞ on all of its domain. Then we have that

$$f'(z) = \lim_{|h|\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h_1\to 0, h_2\to 0} \frac{f_1(z_1+h_1, z_2+h_2) - f_1(z_1,z_2) + i[f_2(z_1+h_1,z_2+h_2) - f_2(z_1,z_2)]}{h_1+ih_2}$$

now this limit must be valid both when $h_1 = 0$ and $h_2 \to 0$ and when $h_1 \to 0$ and $h_2 = 0$. From this it follows that

$$f'(z) = \frac{\partial f_1(z_1,z_2)}{\partial z_1} + i\frac{\partial f_2(z_1,z_2}{\partial z_1}$$
$$f'(z) = -i\frac{\partial f_1(z_1,z_2)}{\partial z_2} + \frac{\partial f_2(z_1,z_2)}{\partial z_2}$$

from which the Cauchy conditions follow, i.e.

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}$$
$$\frac{\partial f_1}{\partial z_2} = -\frac{\partial f_2}{\partial z_1}$$

1.b)

Since f_1 and f_2 are C^2, then we can exchange the order of derivation, thus getting

$$\Delta f_1 = \partial^2_{z_1} f_1 + \partial^2_{z_2} f_1 = \partial^2_{z_1 z_2} f_2 + \partial^2_{z_2 z_1}(-f_2) = 0$$

similarly for f_2 we get that

$$\Delta f_2 = \partial^2_{z_1} f_2 + \partial^2 z_2 f_2 = \partial^2_{z_1 z_2}(-f_1) + \partial^2_{z_2 z_1} f_1 = 0$$

2)

At equilibrium we have that

$$\phi + \nabla \cdot (\sigma) = 0$$

where

$$\nabla \cdot (\sigma) = \begin{pmatrix} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} \\ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \end{pmatrix}$$

Componentwise and using Einstein's notation we can therefore write: $\phi_i + \partial_j(\sigma_{ij}) = 0$. Still with Einstein's notation we have that Hooke's law says that

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij}$$

Now it follows that

$$\partial_j \sigma_{ij} = 2\mu \partial_j \epsilon_{ij} + \lambda \partial_i \epsilon_{kk}$$

and using the fact that $\epsilon_{kk} = \boldsymbol{\nabla} \cdot (\mathbf{u})$ and $\partial_j \epsilon_{ij} = \frac{1}{2}\partial_j(\partial_j u_i + \partial_i u_j) = \frac{1}{2}\partial_{jj} u_i + \frac{1}{2}\partial_j \partial_i u_j = \frac{1}{2}\nabla^2 u_i + \frac{1}{2}\partial_j \partial_i u_j$ we get that $\partial_j \sigma_{ij} = 2\mu[\frac{1}{2}\nabla^2 u_i + \frac{1}{2}\partial_j \partial_i u_j] + \lambda \partial_i \boldsymbol{\nabla} \cdot (\mathbf{u})$. Finally it follows that

$$\phi_i + \mu \nabla^2 u_i + \mu \partial_j \partial_i u_j + \lambda \partial_i \boldsymbol{\nabla} \cdot (\mathbf{u}) = 0 \Rightarrow \phi + \mu \nabla^2 \mathbf{u} + (\mu + \lambda) \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot (\mathbf{u})) = 0$$

3)

We have that $\mathbf{u} = (0, 0, \omega)$, from which, using the equation found previously, it follows that

$$\mu \nabla^2(\omega) = 0 \Rightarrow \partial_x^2 \omega + \partial_y^2 \omega = 0$$

where we used the fact that in this case $\boldsymbol{\nabla} \cdot (\mathbf{u}) = 0$. So ω is harmonic

4)

We have that

$$\epsilon = \begin{pmatrix} 0 & 0 & \frac{1}{2}\frac{\partial \omega}{\partial x} \\ 0 & 0 & \frac{1}{2}\frac{\partial \omega}{\partial y} \\ \frac{1}{2}\frac{\partial \omega}{\partial x} & \frac{1}{2}\frac{\partial \omega}{\partial y} & 0 \end{pmatrix}$$

and from $\sigma_{ij} = 2\mu \epsilon_{ij} + \lambda \epsilon_{kk} \delta_{ij}$ we get that

$$\sigma = 2\mu \begin{pmatrix} 0 & 0 & \frac{1}{2}\frac{\partial \omega}{\partial x} \\ 0 & 0 & \frac{1}{2}\frac{\partial \omega}{\partial y} \\ \frac{1}{2}\frac{\partial \omega}{\partial x} & \frac{1}{2}\frac{\partial \omega}{\partial y} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{\partial \mathrm{Im}\Omega}{\partial x} \\ 0 & 0 & \frac{\partial \mathrm{Im}\Omega}{\partial y} \\ \frac{\partial \mathrm{Im}\Omega}{\partial x} & \frac{\partial \mathrm{Im}\Omega}{\partial y} & 0 \end{pmatrix}$$

Finally we have that $\sigma_{yz} + i\sigma_{xz} = \frac{\partial \mathrm{Im}\Omega}{\partial y} + i\frac{\partial \mathrm{Im}\Omega}{\partial x} = \frac{\partial \mathrm{Re}\Omega}{\partial x} + i\frac{\partial \mathrm{Im}\Omega}{\partial x} = \Omega$.

5)

From $\log(z) = \ln(|z|) + i\arg(z)$ we get that $\Omega(z) = -iP\ln(|z|)/2\pi + \arg(z)/2\pi$, which implies that the displacement field has its z component ω given by $\mu\omega(x, y) = -P\ln\left(\sqrt{x^2 + y^2}\right)/2\pi$. Then it follows that $\mu\partial_x \omega = -\frac{P}{2\pi}\frac{x}{x^2+y^2}$ and $\mu\partial_y \omega = -\frac{P}{2\pi}\frac{y}{x^2+y^2}$, and finally

$$\sigma = -\frac{P}{2\pi}\frac{1}{x^2 + y^2}\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ x & y & 0 \end{pmatrix}$$

This corresponds to a situation where

6.a)

With

$$\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & S \\ 0 & S & 0 \end{pmatrix}$$

6.b)

If we call $z = x + iy$, then we have that $\Omega = S(z - R^2/z) = S\left(x - \frac{R^2 x}{x^2+y^2} + i\frac{x^2 y + R^2 y^2 + y^3}{x^2+y^2}\right)$ from which it follows that $\omega(x, y) = \frac{S}{\mu}\frac{x^2 y + R^2 y^2 + y^3}{x^2+y^2}$. The displacement field is given by $\mathbf{u} = (0, 0, \omega)$. Finally we have that

$$\frac{\partial \mathrm{Im}\Omega}{\partial x} = -\frac{2R^2 x y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial \mathrm{Im}\Omega}{\partial y} = \frac{2x^2 y(R^2 + y) + x^4 + y^4}{(x^2 + y^2)^2}$$

so that

$$\sigma = \frac{1}{(x^2 + y^2)^2}\begin{pmatrix} 0 & 0 & -2R^2 x y^2 \\ 0 & 0 & 2x^2 y(R^2 + y) + x^4 + y^4 \\ -2R^2 x y^2 & 2x^2 y(R^2 + y) + x^4 + y^4 & 0 \end{pmatrix}$$

We have the following behaviors:

6.c)

BOOOOOOOHHHHHHH non ne sacciun cazzo

7.a)