HW1 - Probability

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1 Generating functions.

1. We have:
$$g_X(s) = E[s^X] = \sum_{n\geq 0} P[X=n]s^n \text{ and hence } g_X(1) = \sum_{n\in\mathbb{N}} P[X=n] = 1$$

2. We know that:
$$|g_X(s)| \leqslant \sum_{n\in\mathbb{N}} s^n$$
Hence g_X is well defined for all $s \in [0,1[$. Furthermore:
$$\left|\frac{\mathrm{d}^k g_X(s)}{\mathrm{d}s^k}\right| \leqslant \sum_{n\in\mathbb{N}} (n+k)^k s^n$$
Hence every derivative also has a radius of convergence of 1.

3. For a Bernouilli variable we get:
$$g_X(s) = (1-p)s^0 + p\cdot s = 1 - p + ps$$

4. Notice that:
$$P[X=i] = \left.\frac{\mathrm{d}^i g_X}{\mathrm{d}s^i}\right|_{s=0}$$
Hence the law of X is fully characterized by g_X.

5. We have:
$$E[X^k] = \sum_{n\in\mathbb{N}} P[X=n]n^k$$
And we also have:
$$\frac{\mathrm{d}^k g_X(s)}{\mathrm{d}s^k} = \sum_{n\geq 0} P[X=n+k](n+k)\cdots(n-1)\cdot s^n$$
Now the radius of convergence of this series is given by:
$$\begin{aligned} R^{-1} &= \limsup_{n\to+\infty} \sqrt[n]{P[X=n+k](n+k)\cdots(n-1)} = \limsup_{n\to+\infty} \sqrt[n]{P[X=n+k](n+k)^k \cdot 1 \cdots \frac{n-1}{n+k}} \\ &= \limsup_{n\to+\infty} \sqrt[n]{P[X=n+k](n+k)^k} = \limsup_{n\to+\infty} \sqrt[n]{P[X=n]n^k} \end{aligned}$$
Hence we see that:
$$f(s) = \sum_{n\in\mathbb{N}} P[X=n]n^k s^n \text{ and } \frac{\mathrm{d}^k g_X(s)}{\mathrm{d}s^k} = \sum_{n\geq 0} P[X=n](n+k)\cdots(n-1)\cdot s^n$$
Have the same radius of convergence. Hence $E[X^k] = \lim_{s\to 1^-} f(x)$ converges if and only if $\lim_{s\to 1^-} \frac{\mathrm{d}^k g_X(s)}{\mathrm{d}s^k}$ converges.

6. We have:
$$\frac{\mathrm{d}g_X(s)}{\mathrm{d}s} = \sum_{n\geqslant 1} P[X=n]ns^{n-1}$$

Hence we get that $E[X] = \frac{\mathrm{d}g_X}{\mathrm{d}s}\Big|_{s=0}$. Now notice that $V[X] = E[X^2 - 2E[X]X + E[X]^2] = E[X^2] - 2E[X]^2 + E[X]^2 = E[X^2] - E[X]^2$. Now notice that:

$$\frac{\mathrm{d}^2 g_X(s)}{\mathrm{d}s^2} = \sum_{n\geq 2} P[X=n]n(n-1)s^{n-2} = \sum_{n\geq 2}(P[X=n]n^2 - P[X=n]n)s^{n-2} = \sum_{n\geq 0} P[X=n]n^2 - \frac{1}{s}\frac{\mathrm{d}g_X(s)}{\mathrm{d}s}$$

Hence we have that $E[X^2] = \left(\frac{\mathrm{d}^2 g_X}{\mathrm{d}s^2} + \frac{\mathrm{d}g_X}{\mathrm{d}s}\right)\Big|_{s=0}$. Putting everything together gives:

$$V[X] = \left(\frac{\mathrm{d}^2 g_X}{\mathrm{d}s^2} + \frac{\mathrm{d}g_X}{\mathrm{d}s}\right)\Big|_{s=0} - \left(\frac{\mathrm{d}g_X}{\mathrm{d}s}\Big|_{s=0}\right)^2$$

7. We have that $g_{S_n}(s) = E[s^{\sum_{i=1}^n X_i}] = E[\prod_{i=1}^n s^{X_i}] = \prod_{i=1}^n E[s^{X_i}] = \prod_{i=1}^n g_{X_i}(s)$.
8. Applying question 8 with the results of question 2 we know that the generating function of a binomial law of parameters (n, p) represented by the r.v. Y is given by:

$$g_Y(s) = \prod_{i=1}^n (1 - p(1+s)) = (1 - p(1+s))^n$$

9. We have that $g_Y(s) = E[s^{\sum_{1\leq i\leq U} X_i}] = E[\prod_{i=1}^U s^{X_i}] = E[\prod_{i=1}^U E[s^{X_i}]] = E[E[s^{X_1}]^U] = g_U \circ g_{X_1}(s)$.
10. See question 2 for the first part of the question. Then we have:

$$\begin{aligned}\int_0^{2\pi} g_X(e^{i\theta})e^{-ik\theta}\mathrm{d}\theta &= \int_0^{2\pi}\sum_{n\in\mathbb{N}} P[X=n]e^{i\theta(n-k)}\mathrm{d}\theta = \sum_{n\in\mathbb{N}} P[X=n]\int_0^{2\pi} e^{i\theta(n-k)}\mathrm{d}\theta \\ &= \sum_{n\in\mathbb{N}} P[X=n]2\pi\delta(n-k) = 2\pi P[X=k]\end{aligned}$$

11. Applying the inversion formula we have that:

$$\begin{aligned}\lim_{n\to+\infty} P[X_n = k] &= \lim_{n\to+\infty}\frac{1}{2\pi}\int_0^{2\pi} g_{X_n}(e^{i\theta})e^{-ik\theta}\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}\lim_{n\to+\infty} g_{X_n}(e^{i\theta})e^{-ik\theta}\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} g_X(e^{i\theta})e^{-ik\theta}\mathrm{d}\theta = P[X=k]\end{aligned}$$

2 Simple random walk.

Fundamental Lemma. We remind here the statement of the Fundamental Lemma seen in the course since it will prove useful multiple times. The Fundamental Lemma states that for a symmetric random walk:

$$P(S_2 \neq 0, \cdots, S_{2n} \neq 0) = P(S_{2n} = 0)$$

Furthermore this follows from the recursive formula given by:

$$P(S_2 \neq 0, \cdots, S_{2n-2} \neq 0, S_{2n} = 0) = P(S_{2n-2} = 0) - P(S_{2n} = 0)$$

Basic probability. We also remind here the basic probability formula for random walks which follows from combinatorics arguments:

$$P(S_n = k) = \binom{n}{\frac{n+k}{2}}\frac{1}{2^n}$$

1. The random walk can reach zero only after an even amount of steps because it must have done k steps up and k steps down for a total of $2k$ steps. Hence T_{2n} must be even.
2. Let $(S_1, \cdots, S_{2n})$ be a path such that $T_{2n} = 2k$ then notice that the unique equivalent path given by $(-S_1, \cdots, -S_{2n})$ has $T_{2n} = 2n - 2k$ hence the conclusion follows.

3. To solve this question we consider an equivalent problem. Notice that number of strictly negative paths from 0 to $2n+1$ is equal to the number of paths equal or below -1 from 1 to $2n+1$ since the first step must be negative and we then must stay at all time below the -1 axis. Then up to a re-scaling of the y-axis we know that this is equal to the negative or zero paths of length $2n$ which is what we are looking for. Then we know from the Fundamental Lemma seen in the course that the number of strictly negative paths from 0 to $2n$ is given by:

$$\#\{S_1 < 0, \cdots, S_{2n} < 0\} = \frac{1}{2}\#\{S_1 \neq 0, \cdots, S_{2n} \neq 0\} = \frac{1}{2}\#\{S_{2n} = 0\}$$

Now to every strictly negative path from 0 to $2n$ we have 2 possible paths from 0 to $2n+1$ since we get an extra 2 choices for the last step. Furthermore one needs not to worry for the last step being zero since it is impossible to return at 0 after an odd number of steps: $2n+1$. Hence we get that:

$$\#\{S_1 \leq 0, \cdots, S_{2n} \leq 0\} = \#\{S_1 < 0, \cdots, S_{2n+1} < 0\} = \frac{1}{2}\#\{S_{2n} = 0\} \cdot 2 = \#\{S_{2n} = 0\}$$

Hence:

$$P(T_{2n} = 0) = P(S_1 \leq 0, \cdots, S_{2n} \leq 0) = P(S_{2n} = 0)$$

Intermediary Result. At the start of question 4 we will first prove the following statement which will turn out to be helpful in question 5. We therefore single it out here for clarity:

$$\frac{P(S_{2n} = 0)}{2r - 1} = P(S_1 \neq 0, \cdots, S_{2n-1} \neq 0, S_{2n} = 0)$$

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4. Let $(S_1, \cdots, S_{2n})$ be a path such that $T_{2n} = 2k$ since $1 \leq k \leq n-1$ we know that there are at least 2 steps in the negative plane and at most $2n-2$ and hence the path must have at least one return to the origin. Call the time of the first such return $2r$. We now want to compute the following:

$$P(S_1 > 0, S_2 > 0, \cdots, S_{2r-1} > 0, S_{2r} = 0) = \frac{1}{4}P(S_1 \geq 0, S_2 \geq 0, \cdots, S_{2r-3} \geq 0, S_{2r-2} = 0)$$

Now using the formula from TD2 we get:

$$P(S_1 > 0, S_2 > 0, \cdots, S_{2r-1} > 0, S_{2r} = 0) = \frac{1}{2}P(S_1 \neq 0, \cdots, S_{2r-1} \neq 0, S_{2r} = 0)$$

Now using the formula from the course from which the fundamental lemma is derived we get that:

$$P(S_1 > 0, S_2 > 0, \cdots, S_{2r-1} > 0, S_{2r} = 0) = 2^{-1}(P(S_{2r-2} = 0) - P(S_{2r} = 0))$$

Now replacing these two probabilities with their expression (also derived in the course) we get:

$$P(S_1 > 0, S_2 > 0, \cdots, S_{2r-1} > 0, S_{2r} = 0) = 2^{-1}\left(\frac{(2r-2)!}{2^{2r-2}(r-1)!^2} - \frac{(2r)!}{2^{2r}r!^2}\right)$$

Now putting everything on the same denominator gives (which also proves the intermediary result):

$$P(S_1 > 0, S_2 > 0, \cdots, S_{2r-1} > 0, S_{2r} = 0) = 2^{-1}\left(\frac{1}{2r-1}\frac{(2r)!}{2^{2r}r!^2}\right) = \frac{2^{-1}}{2r-1}P(S_{2r} = 0)$$

Now there are two possible cases, either the path was positive before the return to 0 and hence we have $2k - 2r$ steps left that need to be positive in the $2n - 2r$ remaining steps. Or the path was negative before the first return to 0 and hence we need $2k$ steps to be positive in the $2n-2r$ steps left. This leads to the following formula:

$$\begin{aligned}
P(T_{2n}=2k) &= \sum_{r=1}^{k} P(S_1>0,\cdots,S_{2r-1}>0,S_{2r}=0)P(T_{2n-2r}=2k-2r) \\
&\quad + \sum_{r=1}^{n-k} P(S_1<0,\cdots,S_{2r-1}<0,S_{2r}=0)P(T_{2n-2r}=2k) \\
&= \sum_{r=1}^{k} \frac{2^{-1}}{2r-1} P(S_{2r}=0)P(T_{2n-2r}=2k-2r) \\
&\quad + \sum_{r=1}^{n-k} P(S_1<0,\cdots,S_{2r-1}<0,S_{2r}=0)P(T_{2n-2r}=2k) \\
&= \sum_{r=1}^{k} \frac{2^{-1}}{2r-1} P(S_{2r}=0)P(T_{2n-2r}=2k-2r) \\
&\quad + \sum_{r=1}^{n-k} \frac{2^{-1}}{2r-1} P(S_{2r}=0)P(T_{2n-2r}=2k)
\end{aligned}$$

Where in the 3^{rd} equality we used the fact that the number of paths strictly above the axis is equal to the number of paths below which follows from a mirror symmetry along the axis. Now multiplying left and right by powers of 2 we get the formula asked for in the homework.

5. Let $\mathcal{H}_n$: "$\forall k \in [\![0,n]\!],\quad P(T_{2n}=2k) = 2^{-2n}\binom{2k}{k}\binom{2n-2k}{n-k}$". We have that $\mathcal{H}_0$ is trivially true. Then take $n \in \mathbb{N}$ such that $\mathcal{H}_m$ is true for $m \in [\![0,n-1]\!]$. As seen in the course we are going to re-write the arcsin law as follows for more compact notation: $P(T_{2n}=2k) = P(S_{2k}=0)P(S_{2n-2k}=0)$. Then we get that:

$$\begin{aligned}
P(T_{2n}=2k) &= \sum_{r=1}^{k} \frac{P(S_{2r}=0)}{2(2r-1)} P(T_{2n-2r}=2k-2r) + \sum_{r=1}^{n-k} \frac{P(S_{2r}=0)}{2(2r-1)} P(T_{2n-2r}=2k) \\
&= \sum_{r=1}^{k} \frac{P(S_{2r}=0)}{2(2r-1)} P(S_{2k-2r}=0)P(S_{2n-2k}=0) + \sum_{r=1}^{n-k} \frac{P(S_{2r}=0)}{2(2r-1)} P(S_{2k}=0)P(S_{2n-2r-2k}=0) \\
&= \frac{P(S_{2n-2k}=0)}{2} \sum_{r=1}^{k} \frac{P(S_{2r}=0)}{2r-1} P(S_{2k-2r}=0) + \frac{P(S_{2k}=0)}{2} \sum_{r=1}^{n-k} \frac{P(S_{2r}=0)}{2r-1} P(S_{2n-2r-2k}=0)
\end{aligned}$$

However now using the intermediary result of question 4 notice that the first sum can be re-written as (and similarly for the second sum):

$$\sum_{r=1}^{k} P(S_1 \neq 0,\cdots,S_{2r-1}\neq 0,S_{2r}=0)P(S_{2k-2r}=0)$$

Notice that every term of the sum corresponds to the probability of the event: A_r : "the walk touches zero for the first time at step $2r$ and finishes at 0 after $2k$ steps in total". Notice furthermore that all the A_i are disjoint since the 'first time' condition necessarily separates them. Hence we can rewrite the sum as:

$$\sum_{r=1}^{k} P(A_r) = P(\bigcup_{i=1}^{k} A_r) = P(S_{2k}=0)$$

An identical reasoning can be applied to the second sum and hence we obtain:

$$\begin{aligned}
P(T_{2n}=2k) &= \frac{P(S_{2n-2k}=0)P(S_{2k=0})}{2} + \frac{P(S_{2k}=0)P(S_{2n-2k}=0)}{2} \\
&= P(S_{2n-2k}=0)P(S_{2k}=0)
\end{aligned}$$

And hence $\{\mathcal{H}_i : i \in [\![0,n-1]\!]\} \Rightarrow \mathcal{H}_n$ and this concludes the proof.

6. (a) The probability that one of the players leads the whole game is given by $P(T_{20}=20)+P(T_{20}=0) = 2P(T_{20}=0) = \frac{46\,189}{131\,072} \approx 0.35$.

(b) The probability of the winner leading at least 16 times during the game is given by $2(P(T_{20} \in \{20, 18, 16\}) = \frac{22\,451}{32\,768} \approx 0.69$.

(c) The probability that both players lead 10 times is given by $P(T_{20} = 10) = \frac{3\,969}{65\,536} \approx 0.06$.

7. Following the same reasoning as in question 4 for $P(S_1 \geq 0, \cdots, S_{2r-1} \geq 0, S_{2r} = 0)$ and replacing r with n we immediately obtain the desired formula which is:

$$P(T_{2n} = 2n, S_{2n} = 0) = \frac{2}{2n+1} P(S_{2n+2} = 0) = \frac{2}{2n+1} \frac{P(S_{2n} = 0)(2n+1)}{2(n+1)} = \frac{P(S_{2n} = 0)}{n+1} = P(T_{2n} = 0, S_{2n} = 0)$$

Where the last equality directly follows from symmetry with the axis.

8. We take the same recursive formula as previously and consider the first term:

$$\begin{aligned}\frac{2^{-1}}{2r-1} P(S_{2r} = 0) \frac{P(S_{2n-2r} = 0)}{n-r+1} &= 2^{-1} \left(\frac{P(S_{2r} = 0)}{2r-1} \right) \left(\frac{P(S_{2n-2r+2} = 0)}{2n-2r+1} \right) 2 \\ &= P(S_2 \neq 0, \cdots, S_{2r-2} \neq 0, S_{2r} = 0) P(S_2 \neq 0, \cdots, S_{2n-2r} \neq 0, S_{2n-2r+2} = 0) \\ &= P(S_2 \neq 0, \cdots, S_{2r-2} \neq 0, S_{2r} = 0, S_{2r+2} \neq 0, \cdots, S_{2n} \neq 0, S_{2n+2} = 0)\end{aligned}$$

Notice that this corresponds to the probability of the event A_r :' a path of length $2n+2$ admits only two zeroes in $2r$ and at the end in $2n+2$.". Notice furthermore that all the A_r are disjoint events and furthermore from the symmetry which comes from flipping the horizontal axis we have that $A_r \cong A_{n-r}$. Hence we get that:

$$\begin{aligned}\sum_{r=1}^{k} P(A_r) + \sum_{r=1}^{n-k} P(A_r) &= P(\bigcup_{r=1}^{k} A_r) + \sum_{r=1}^{n-k} P(A_{n-r}) = P(\bigcup_{r=1}^{k} A_r) + \sum_{r=k}^{n-1} P(A_r) \\ &= P(\bigcup_{r=1}^{k} A_r) + P(\bigcup_{r=k}^{n-1} A_r) = P(\bigcup_{r=1}^{n-1} A_r) = P(S_{2n+2} = 0, \exists r \in [\![1, n-1]\!] S_{2r} = 0)\end{aligned}$$

Notice that we could conclude here since the value is independent from k we already know that normalization will force the correct result. Now notice that this is simply a re-writing of the lamplighter problem seen in the lecture and from the lecture we know that:

$$\begin{aligned}\sum_{r=1}^{k} P(A_r) + \sum_{r=1}^{n-k} P(A_r) &= P(\bigcup_{r=1}^{n-1} A_r) = P(S_{2n+2} = 0, \exists r \in [\![1, n-1]\!] S_{2r} = 0) \\ &= 2\frac{P(S_{2n+2} = 0)}{2n+2-1} = 2\frac{P(S_{2n+2} = 0)}{2n+1} \\ &= 2\frac{P(S_{2n} = 0)(2n+1)}{2(n+1)} \frac{1}{2n+1} \\ &= \frac{P(S_{2n} = 0)}{n+1}\end{aligned}$$

9. The most striking difference in between the results of question 5 and 8 is that the result in question 8 is independent of that value of k whilst the one of question 5 is not.