ICFP M1 - Relativistic quantum mechanics and introduction to quantum field theory – TD n° 2
Relativistic kinematics

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Reminder and notations

We denote the contravariant coordinates of 4-vectors as $\underline{A} = (a^0, \vec{a}) = (a^0, a^1, a^2, a^3)$. The scalar product in Minkowski space is $\underline{A} \cdot \underline{B} = \vec{a} \cdot \vec{b} - a^0 b^0$, with $\vec{a} \cdot \vec{b}$ the usual scalar product in $\mathbb{R}^3$. We shall write $\underline{A}^2 = \underline{A} \cdot \underline{A} = \vec{a}^2 - (a^0)^2$. A 4-vector $\underline{A}$ is said to be time-like if $\underline{A}^2 < 0$, space-like if $\underline{A}^2 > 0$, light-like if $\underline{A}^2 = 0$. Under a special Lorentz transformation (boost) representing a change of reference frame a 4-vector $\underline{A}$ is transformed to $\underline{A}'$, with

$$\begin{pmatrix} a'^0 \\ a'^1 \\ a'^2 \\ a'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} \tag{1}$$

if the new reference frame is in translation at velocity v along the first spatial axis, with $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$. Such a transformation is an isometry of this Minkowski space, $\underline{A}' \cdot \underline{B}' = \underline{A} \cdot \underline{B}$.

1 Properties of time-like vectors

We call $\mathcal{C}_+ = \{\underline{A} \ : \ \underline{A}^2 < 0 \text{ and } a^0 > 0\}$ the set of time-like 4-vectors with positive time component.

1. What is the sign of $\underline{A} \cdot \underline{B}$ if $\underline{A}, \underline{B} \in \mathcal{C}_+$?
2. Deduce that $\mathcal{C}_+$ is stable under the addition, i.e. if $\underline{A}, \underline{B} \in \mathcal{C}_+$ then $\underline{A} + \underline{B} \in \mathcal{C}_+$, and more generically under linear combinations with positive coefficients.
3. Show that $\mathcal{C}_+$ is stable under special Lorentz transformations.
4. If $\underline{A} \in \mathcal{C}_+$, show that there exists a Lorentz special transformation such that $\underline{A}' = (a'^0, \vec{0})$. Give the parameters of the boost.
5. Show that if $\underline{A}_1$, $\underline{A}_2$, ..., $\underline{A}_n$ are all in $\mathcal{C}_+$, then
$$\sqrt{-(\underline{A}_1 + \cdots + \underline{A}_n)^2} \geq \sqrt{-\underline{A}_1^2} + \cdots + \sqrt{-\underline{A}_n^2} \ . \tag{2}$$
Hint : exploit the invariance of $\underline{A}^2$ under Lorentz transformations. Compare (2) with the triangular inequality for usual 3-vectors.

2 Applications to 4-momenta

The 4-momentum of a particle is denoted $\underline{P}$ (in a given reference frame). It is a conserved quantity in all the reactions considered below, and transforms as a 4-vector under the Lorentz group.

1. Recall the components of $\underline{P}$, and the value of $\underline{P}^2$.
2. Deduce that for a massive particle, $\underline{P} \in \mathcal{C}_+$.
3. Consider a system of n particles of masses m_1, ..., m_n and 4-momenta $\underline{P}_1$, ..., $\underline{P}_n$. The total momentum is denoted $\underline{P} = \underline{P}_1 + \cdots + \underline{P}_n$. Justify the existence of a reference frame in which $\vec{p} = 0$. The energy of the system in this « center of momentum frame » is denoted E^*. Give a lowerbound on E^*.

3 Decays of particles

1. A particle of mass M, at rest in the reference frame of study, decays into n particles of mass m_1, …, m_n. Give a condition on the masses for this desintegration to be possible.
2. In this question we consider the case $n = 2$, and try to determine the 4-momenta $\underline{P}_1$ and $\underline{P}_2$ of the particles created.
   (a) Count the number of unknowns and of equations of the problem.
   (b) Show that the energies of the two particles are
   $$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} c^2 , \qquad E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M} c^2 . \tag{3}$$
   (c) Show that their kinetic energies are
   $$E_{\text{kin},1} = c^2 \Delta M \left(1 - \frac{m_1}{M} - \frac{\Delta M}{2M}\right) , \qquad E_{\text{kin},2} = c^2 \Delta M \left(1 - \frac{m_2}{M} - \frac{\Delta M}{2M}\right) , \tag{4}$$
   with $\Delta M = M - m_1 - m_2$. Compute $E_{\text{kin},1} + E_{\text{kin},2}$, and give an interpretation of your result.
3. When $n \geq 3$ there are not enough conservation laws to determine the energies of all the particles. Show however that the kinetic energy of the i-th particle is upperbounded as
   $$E_{\text{kin},i} \leq c^2 \Delta M \left(1 - \frac{m_i}{M} - \frac{\Delta M}{2M}\right) , \tag{5}$$
   where $\Delta M = M - m_1 - \cdots - m_n$. *Hint :* denote $\underline{Q} = \sum_{j \neq i} \underline{P}_j$, compute the 4-norm of $\underline{P}_i = \underline{P} - \underline{Q}$, and use the triangular inequality (2) to bound $\underline{Q}^2$.

4 Creations of particles

The goal of particle accelerators is to create new particles through the collisions of lighter ones (for instance in the LHC proton-proton collisions are studied). The heavier the particles to be created, the higher the energy to be given to the colliding ones : there is indeed a « reaction threshold » for these processes to be kinematically possible. We consider in this exercise only processes with two incoming particles, $1 + 2 \to 3 + 4 + \ldots$, and denote M_{tot} the sum of the masses of the particles produced.

1. Explain how, from the point of view of kinematics, one can exploit the results established previously on the decay of particles to study these reaction thresholds, and deduce a necessary and sufficient condition on E^*, the energy of the incoming particles in their center of momentum frame, for the reaction to be kinematically possible. Compare qualitatively E^* to the energy of the incoming particles in any other frame.
2. The two colliding particles of mass m_1 and m_2 move in opposite directions along the same axis, their energies in the laboratory frame where the experiment is performed are denoted E_1 and E_2, and their kinetic energies $E_{\text{kin},1}$ and $E_{\text{kin},2}$. Express E^* in terms of m_1, m_2, $E_{\text{kin},1}$ and $E_{\text{kin},2}$.
3. Simplify your result in the following two cases :
   (a) when the particle 2 is at rest in the laboratory frame ; show that the kinematic condition becomes
   $$E_{\text{kin},1} \geq \frac{M_{\text{tot}}^2 - (m_1 + m_2)^2}{2m_2} c^2 . \tag{6}$$
   (b) when the two colliding particles have the same mass and move at the same velocity (in opposite directions) in the laboratory frame.

4. There exists two main classes of particle accelerators : those in which two beams of particles are accelerated in opposite directions and made to collide, and those in which one beam of particle is accelerated towards a fixed target (in the laboratory frame). Discuss the advantages and defects of these two configurations, studying in particular the scaling of E^* in the limit where the kinetic energy of the accelerated particles is much larger than their mass energy.
5. We consider now the particular case where the two incoming particles give birth to two outcoming particles, i.e. the reaction $1 + 2 \to 3 + 4$. We denote m_i and $\underline{P}_i$ the corresponding masses and 4-momenta.
   (a) All the Lorentz invariants quantities that can be relevant for the process are the 16 scalar products $\underline{P}_i \cdot \underline{P}_j$. Show that all of them can be computed from the masses of the particles and from the three variables
   $$s = -(\underline{P}_1 + \underline{P}_2)^2 \ , \qquad t = -(\underline{P}_1 - \underline{P}_3)^2 \ , \qquad u = -(\underline{P}_1 - \underline{P}_4)^2 \ , \tag{7}$$
   known as the Mandelstam variables. In particular the energy in the center of momentum frame is $E^* = c\sqrt{s}$.
   (b) Show that $s + t + u = (m_1^2 + m_2^2 + m_3^2 + m_4^2)\, c^2$.
   (c) When all the masses of the particles are equal to m, show that $s \geq 4m^2c^2$ and $t, u \leq 0$.

5 The case of massless particles

1. A massless particle (a photon for instance) has no center of momentum frame. Justify briefly why.
2. Can one envision a process in which a photon decays into a pair of massive particles ?
3. Can one define a center of momentum frame for a system of one massive and one massless particles ?