ICFP M1 - Relativistic quantum mechanics and introduction to quantum field theory – TD n° 1
Some basics of representation theory

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Invariance requirements provide very strong guiding principles in the construction of physical theories. Indeed the relativity principle expressed as Lorentz invariance, and more abstract symmetries in gauge theories, put very strong constraints on the possible physical interactions. These symmetry transformations have mathematically the structure of a group G, which we recall is a set endowed with a composition law (denoted as addition or multiplication depending on the case) that is associative, possesses a neutral element e such that $eg = ge = g$ for all elements g of the group, and in which every element g has an inverse g^{-1} such that $g^{-1}g = gg^{-1} = e$.

Under the action of a symmetry group various physical quantities can transform in different ways. Take for instance a rotation R of a system, which transforms the spatial coordinates according to $\vec{r} \to \vec{r}\,' = R\vec{r}$. A scalar field $\varphi(\vec{r})$ will be transformed into $\varphi'(\vec{r}\,') = \varphi(R^{-1}\vec{r}\,')$, while a vector field $\vec{A}(\vec{r})$ is not just an arbitrary collection of d functions of the coordinates : its components are linearly combined under such a transformation according to $\vec{A}(\vec{r}) \to \vec{A}'(\vec{r}\,') = R\vec{A}(R^{-1}\vec{r}\,')$. One has thus to distinguish between the symmetry group G and its action onto the various types of physical quantities. The latter has mathematically the structure of a representation of a group G, which is a vector space V, with for each element g of the group an invertible linear application $\rho(g) \in \mathrm{GL}(V)$, that respects the group structure in the sense that $\rho(gh) = \rho(g)\rho(h)$, $\rho(g^{-1}) = \rho(g)^{-1}$ and $\rho(e) = \mathrm{Id}_V$. It is thus crucial to classify the possible representations of a symmetry group, in particular the irreducible ones (a representation is said to be irreducible if it does not contain a proper subspace invariant under the action of ρ), as they will be the basic building blocks of any theory possessing this symetry group.

In this problem we first study some properties of matrix groups, before considering the representations of the simplest non-trivial continuous group, related to the rotations of $\mathbb{R}^3$.

1 Matrix groups

1. We recall that $\mathrm{U}(n)$ denotes the set of unitary $n \times n$ complex matrices M, i.e. such that $MM^\dagger = M^\dagger M = \mathrm{Id}_{\mathbb{C}^n}$; these are the isometries of $\mathbb{C}^n$ for the canonical inner product.
   (a) Check that $\mathrm{U}(n)$ forms a group with the usual matrix multiplication law.
   (b) How many reals are necessary to parametrize $\mathrm{U}(n)$?
   (c) What are the possible values of the determinant of an unitary matrix ?
   (d) $\mathrm{SU}(n)$ is the subset of $\mathrm{U}(n)$ restricted to the matrices of determinant 1. Check that it forms a subgroup of $\mathrm{U}(n)$ (called the special unitary group). How many parameters has it ?
   (e) An unitary matrix can always be diagonalized in an orthonormal basis. What are the possible values of its eigenvalues ? Deduce that any unitary matrix M can be written as $M = \exp[iH]$, with H an Hermitian matrix. What is the additional constraint on H to ensure that $M \in \mathrm{SU}(n)$? Check the consistency with your answers to the previous questions.

2. Similarly $\mathrm{O}(n)$ denotes the group of real orthogonal matrices, with $MM^T = M^T M = \mathrm{Id}_{\mathbb{R}^n}$, that hence preserves the canonical scalar product on $\mathbb{R}^n$, and $\mathrm{SO}(n)$ its subgroup in which the determinant is equal to 1.
   (a) How many real parameters encode a matrix of $\mathrm{O}(n)$?
   (b) What are the possible values of the determinant of such a matrix ?
   (c) How many real parameters encode a matrix of $\mathrm{SO}(n)$?
   (d) Give a simple example of a matrix which is in $\mathrm{O}(n)$ but not in $\mathrm{SO}(n)$.
   (e) What are the conditions on the real matrix A for $M = \exp[A]$ to be orthogonal ? What is the determinant of such an M ?

2 The relationship between SO(3) and SU(2)

1. From the analysis above deduce that a matrix M of $\mathrm{SU}(2)$ can be written $M = \exp[-i\vec{\theta}\cdot\vec{S}]$ where $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$ is a vector of three real parameters and $\vec{S} = \frac{\vec{\sigma}}{2}$ with $\vec{\sigma}$ the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{1}$$

Check the following commutation rules (known as the Lie algebra of the group) for these matrices,

$$\left[\frac{\sigma_1}{2}, \frac{\sigma_2}{2}\right] = i\,\frac{\sigma_3}{2} , \qquad \left[\frac{\sigma_2}{2}, \frac{\sigma_3}{2}\right] = i\,\frac{\sigma_1}{2} , \qquad \left[\frac{\sigma_3}{2}, \frac{\sigma_1}{2}\right] = i\,\frac{\sigma_2}{2} , \tag{2}$$

which can be written in a more compact form as

$$\left[\frac{\sigma_a}{2}, \frac{\sigma_b}{2}\right] = i\,\epsilon_{abc}\,\frac{\sigma_c}{2} , \tag{3}$$

with ϵ_{abc} the completely antisymmetric tensor and with an implicit summation over repeated indices. A more complete analysis shows that every matrix M of $\mathrm{SU}(2)$ can be written in an unique way as $M = \exp[-i\vec{\theta}\cdot\vec{S}]$ with $||\vec{\theta}|| \le 2\pi$ (with the identification of all vectors of norm 2π to have the uniqueness). One of the consequences of this bijection is that $\mathrm{SU}(2)$ is a so-called compact group.

2. Show similarly that a matrix of $\mathrm{SO}(3)$ can be parametrized as $M = \exp[-i\vec{\theta}\cdot\vec{K}]$, with

$$K_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \qquad K_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} , \qquad K_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4}$$

Check that these matrices verify the same commutation rules $[K_a, K_b] = i\,\epsilon_{abc} K_c$. Interpret the matrix $e^{-i\theta K_j}$ as a transformation of $\mathbb{R}^3$.

3. One can thus construct a mapping between the two groups $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ by identifying the parameters $\vec{\theta}$ introduced in the two previous questions ; this mapping preserves the group structure because the generators of infinitesimal transformations $\vec{\sigma}/2$ and $\vec{K}$ obey the same commutation rules. However, the two groups are not globally the same : compute for instance $e^{-i\theta S_3}$ and show that both $\mathrm{Id}_{\mathbb{C}^2}$ and $-\mathrm{Id}_{\mathbb{C}^2}$ are sent to $\mathrm{Id}_{\mathbb{R}^3}$ by this mapping. Indeed $\mathrm{SO}(3)$ is connected but not simply connected (there exists closed paths inside $\mathrm{SO}(3)$ which cannot be continuously deformed into a point), while $\mathrm{SU}(2)$ is simply connected, forming thus its universal covering group.

3 Representations of SU(2)

We want now to describe the representations of SU(2), we shall thus introduce a finite-dimensional complex vector space V, on which any element g of SU(2) is represented by a linear operator $\rho(g)$. As discussed previously SU(2) is generated by S_1, S_2, S_3 ; we thus consider the corresponding representants $\widehat{J}_1$, $\widehat{J}_2$ and $\widehat{J}_3$, that are linear operators on V obeying the commutation rules

$$[\widehat{J}_a, \widehat{J}_b] = i\,\epsilon_{abc}\,\widehat{J}_c \ , \tag{5}$$

and define the mapping ρ by $\rho(\exp[-i\vec{\theta}\cdot\vec{S}]) = \exp[-i\vec{\theta}\cdot\vec{\widehat{J}}]$. We restrict ourselves to unitary representations, i.e. we introduce an Hilbert structure on V and assume that the $\widehat{J}_a$ are Hermitian, in such a way that $\rho(g)$ is unitary for all g.

From the three operators $\widehat{J}_1$, $\widehat{J}_2$ and $\widehat{J}_3$ we further define

$$\widehat{J}^2 = (\widehat{J}_1)^2 + (\widehat{J}_2)^2 + (\widehat{J}_3)^2 \ , \qquad \widehat{J}_+ = \widehat{J}_1 + i\widehat{J}_2 \ , \qquad \widehat{J}_- = \widehat{J}_1 - i\widehat{J}_2 \ . \tag{6}$$

1. Show that $\widehat{J}^2$ is hermitian, and that it commutes with $\widehat{J}_1$, $\widehat{J}_2$ and $\widehat{J}_3$. Hence V admits an orthonormal basis of vectors which are simultaneously eigenvectors of $\widehat{J}^2$ and $\widehat{J}_3$.
2. Show that the eigenvalues of $\widehat{J}^2$ are positive, and can thus be written $j(j+1)$ with $j \geq 0$.
3. Suppose that $|v\rangle$ is a common eigenvector of $\widehat{J}^2$ and $\widehat{J}_3$, with eigenvalues denoted $j(j+1)$ and m, respectively. Show that $\widehat{J}_+|v\rangle$, if it is non-zero, is also a common eigenvector of $\widehat{J}^2$ and $\widehat{J}_3$; what are the associated eigenvalues ? Same question for $\widehat{J}_-|v\rangle$.
4. Argue that for V to be finite dimensional there must be at least one common eigenvector $|v_0\rangle$ of $\widehat{J}^2$ and $\widehat{J}_3$ such that $\widehat{J}_+|v_0\rangle = 0$. We assume for the moment that there exists only one such $|v_0\rangle$ (modulo multiplication by a constant of course). We shall denote $j(j+1)$ and m_0 its eigenvalues for $\widehat{J}^2$ and $\widehat{J}_3$ respectively.
5. Give an expression of $\widehat{J}_-\widehat{J}_+$ in which $\widehat{J}^2$ appears. Conclude that $j(j+1) = m_0(m_0+1)$.
6. Argue similarly that the finite-dimensionality of V implies the existence of an integer $k \geq 0$ such that $(\widehat{J}_-)^{k+1}|v_0\rangle = 0$. We take the smallest k with this property and denote $|w_0\rangle = (\widehat{J}_-)^k|v_0\rangle$, and m_0' its eigenvalue under $\widehat{J}_3$. Relate m_0' and m_0.
7. Compute $\widehat{J}_+\widehat{J}_-$, and show that $j(j+1) = m_0'(m_0'-1)$.
8. Conclude that $m_0 = j = -m_0' = \frac{k}{2}$, with k the integer introduced above.
9. We denote $|j,j\rangle$ a normalized vector proportional to $|v_0\rangle$, and $|j,m\rangle$ the orthonormal basis of common eigenvectors of $\widehat{J}^2$ and $\widehat{J}_3$ obtained by successive applications of $\widehat{J}_-$ to $|j,j\rangle$, then properly normalized. Write down the action of $\widehat{J}_+$ and $\widehat{J}_-$ on this basis (choosing the phases of the vectors in order to have real coefficients), and deduce the one of $\widehat{J}_1$ and $\widehat{J}_2$. Simplify your results in the special cases $j=0$ and $j=1/2$.
10. We now come back on the assumption made in question 4 : show that any finite-dimensional unitary representation of SU(2) can be decomposed as a direct sum of irreducible sub-representations (we recall that a representation is said irreducible if it does not contain a proper subspace invariant under the action of ρ), each of them characterized by an integer or half-integer j, called its spin, and of dimension $2j+1$. To do this use the fact that each eigenspace of $\widehat{J}^2$ is a sub-representation (i.e. is stable under the action of the group), and construct recursively an orthonormal basis of these eigenspaces by exploiting the eigenvectors of $\widehat{J}_3$ that are cancelled by $\widehat{J}_+$.
11. Actually even infinite-dimensional unitary representations can be decomposed as an infinite direct sum of such representations ; this is a property called semi-simplicity, or complete irreducibility, a consequence of the compacity of SU(2). To convince you that this statement is reasonable, consider the Hilbert space of functions from $\mathbb{R}^3$ to $\mathbb{C}$ you studied in quantum mechanics to describe the wavefunction of a particle, and the angular momentum operator $\vec{L} = \vec{R} \wedge \vec{P}$. Show, from properties of the position and momentum operators, that the commutation rules

of the components of $\vec{L}$ corresponds precisely to the Lie algebra of SU(2). Conclude that the spherical harmonics decomposition of an arbitrary function on the unit sphere corresponds to the decomposition in irreducible representations described above.

12. Spherical harmonics only exist for integer spins. Explain this by computing $e^{2i\pi \widehat{J}_3}$ in the basis $\{|j,m\rangle\}_{m=-j,\dots,j}$ for j integer and half-integer. This phenomenon can be related to the global difference between SO(3) and SU(2) discussed previously.
13. As we have seen above one can construct new representations of a group by forming the direct sums of the representations. Another way is to take the tensor product of two representations. Give some examples in quantum mechanics of such tensor product constructions. A part of representation theory, that is called the Clebsch-Gordan problem, is devoted to the decomposition of tensor product representations as sums of irreducible representations; it is encountered in quantum mechanics for the addition of angular momenta. Recall without proof the form of this decomposition for the product of two irreducible representations of SU(2) of spin j_1 and j_2; check its consistency in terms of the dimensions of the vector spaces; specialize your result to $j_1 = j_2 = 1/2$.