Dynamical Systems TDs.

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Chapter 1

TD1

1.1 Linearization in the vicinity of a fixed point in a planar phase space.

1. We have:
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

   Now the eigenvalues of the matrix are given by:
   $$(a-\lambda)(d-\lambda) - cb = 0 \Rightarrow \lambda = \frac{a+d \pm \sqrt{a^2 + 4bc - 2ad + d^2}}{2} = \frac{p \pm \sqrt{p^2 - 4q}}{2}$$

   Then we have that:
   $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

2. Generally we get:
   $$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = A(0)\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} + B(0)\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

   Hence we have:
   $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = A(t)s_1\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} + B(t)s_2\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \Rightarrow A(t) = e^{s_1 t} \text{ and } B(t) = e^{s_2 t}$$

   In other words introducing $z(t) = x(t) + iy(t)$ we get:
   $$z(t) = e^{i\theta}e^{s_1 t} + e^{i\phi}e^{s_2 t}$$

   Where we have that $\alpha_1 = \cos(\theta)$ and $\beta_1 = \sin(\theta)$. Similarly for α_2, β_2. Then using the definition of s_1 and s_2 we get:
   $$z(t) = (e^{i\theta}e^{\frac{\sqrt{\Delta}}{2}t} + e^{i\phi}e^{-\frac{\sqrt{\Delta}}{2}t})e^{\frac{p}{2}t}$$

3. When $s_1 > 0$ and $s_2 > 0$ then we have that $z(t) \to \texttt{ComplexInfinity}$. When $s_1 < 0$ and $s_2 < 0$ we have that $z(t) \to 0$. When $s_1 > 0$ and $s_2 < 0$ then $z(t)$ will be dominated by the first term and diverge and vice-versa. Finally if s_1 and s_2 are complex with a non zero real part then if any real part is positive we will get a diverging elliptic spiral, if both real parts are negative then we will get an inward elliptic spiral to 0.

4. In the case $\Delta = 0$ we have that $s_1 = s_2$ and we get a perfect circle.

5. By simply plotting $\Delta(p, q)$ one gets the desired regions.

1.2 A predatory-prey model. Volterra's equations

1. Assuming that the probability that a big fish eats a small fish is proportional to the density of small fishes and big fishes we get the equations given in the text. We furthermore assumed that the penalty of dying from an attack is not necessarily equal to the gain in succeeding in an attack.

2. At equilibrium we get:

$$\begin{cases} \mu x = axy \\ \nu y = bxy \end{cases} \Rightarrow \begin{cases} y = \frac{\mu}{a} \\ x = \frac{\nu}{b} \end{cases} \vee \begin{cases} x = 0 \\ y = 0 \end{cases}$$

We start by analyzing the stability of the first equilibrium. Take $x(t) = \frac{\nu}{b} + \varepsilon_x(t)$ and similarly for $y(t)$. Then we get the following equations:

$$\begin{cases} \dot{\varepsilon}_x(t) = \frac{\mu\nu}{b} + \mu\varepsilon_x(t) - a\left(\frac{\nu}{b} + \varepsilon_x(t)\right)\left(\frac{\mu}{a} + \varepsilon_y(t)\right) \\ \dot{\varepsilon}_y(t) = -\frac{\mu\nu}{a} + \nu\varepsilon_y(t) + b\left(\frac{\nu}{b} + \varepsilon_x(t)\right)\left(\frac{\mu}{a} + \varepsilon_y(t)\right) \end{cases}$$

Now restricting ourselves to first order we get:

$$\begin{cases} \dot{\varepsilon}_x(t) = \frac{\mu\nu}{b} + \mu\varepsilon_x(t) - \frac{\mu\nu}{b} - \frac{a\nu}{b}\varepsilon_y(t) - \mu\varepsilon_x(t) \\ \dot{\varepsilon}_y(t) = -\frac{\mu\nu}{a} - \nu\varepsilon_y(t) + \frac{\mu\nu}{a} + \frac{b\mu}{a}\varepsilon_x(t) + \nu\varepsilon_y(t) \end{cases}$$

Which simplifies to:

$$\begin{cases} \dot{\varepsilon}_x(t) = -\frac{a\nu}{b}\varepsilon_y(t) \\ \dot{\varepsilon}_y(t) = \frac{b\mu}{a}\varepsilon_x(t) \end{cases}$$

This is a specific case of exercise 1 where $a = 0, b = -\frac{a\nu}{b}, c = \frac{b\mu}{a}, d = 0$. Then in this case we clearly have $p = 0$ and $q = \mu\nu$ and hence $\Delta = -4\mu\nu$. Hence the solutions are purely imaginary and we know that ε_x and ε_y will stay bounded.

Now we interest ourselves to the second equation. Take $x(t) = \varepsilon_x(t)$ and $y(t) = \varepsilon_y(t)$ then we get:

$$\begin{cases} \dot{\varepsilon}_x(t) = \mu\varepsilon_x(t) \\ \dot{\varepsilon}_y(t) = -\nu\varepsilon_y(t) \end{cases}$$

Now we immediately get that $\varepsilon_y \to 0$ and $\varepsilon_x \to +\infty$ and hence this is an unstable solution.

3. During the stability analysis of the first equilibrium we already exhibited an oscillating solution. Where in the limit of small amplitude oscillations the period was given by $\frac{\pi}{\sqrt{\mu\nu}}$ or equivalently the frequency was given by $2\sqrt{\mu\nu}$.

4. ...

5. Dividing everything together and separating the variables yields:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\nu y + bxy}{\mu x - axy} = -\frac{y}{x}\frac{bx - \nu}{ay - \mu} = -\left(\frac{y}{ay - \mu}\right)\left(\frac{bx - \nu}{x}\right) \Rightarrow \mathrm{d}y\left(\frac{y}{ay - \mu}\right)^{-1} = -\mathrm{d}x\left(\frac{bx - \nu}{x}\right)$$

Which when simplified gives:

$$\mathrm{d}x\left(\frac{bx - \nu}{x}\right) + \mathrm{d}y\left(\frac{ay - \mu}{y}\right) = 0$$

Now integrating gives:

$$V = b\int \mathrm{d}x - \nu\int \frac{\mathrm{d}x}{x} + a\int \mathrm{d}y - \mu\int \frac{\mathrm{d}y}{y} = bx - \nu\ln(x) + ay - \mu\ln(y)$$

6. We simply get that we need to replace μ with $\mu - c\rho$ and ν with $\nu + c\rho$ where ρ is the number of fishermen and c the hunting rate.

1.3   The Van Der Pol oscillator.

1. The one and only equilibrium solution is given by $x = 0$. Then let $x(t) = \varepsilon(t)$. Then simplifying to first order we get:

$$\ddot{\varepsilon}(t) - \lambda\dot{\varepsilon}(t) + \omega_0^2\varepsilon(t) = 0$$

The solution of such an equation is well known and given by:

$$\varepsilon(t) = \alpha e^{\frac{t(\lambda - \sqrt{\lambda^2 - 4\omega_0^2})}{2}} + \beta e^{\frac{t(\lambda + \sqrt{\lambda^2 - 4\omega_0^2})}{2}}$$

Then we see that this is stable if and only if $\lambda \leq 0$.

2. From the ODE we get:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega_0^2 x \frac{\mathrm{d}x}{\mathrm{d}t} = \left(\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega_0^2 x\right)\frac{\mathrm{d}x}{\mathrm{d}t} = (\lambda - x^2)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2$$

3. When $\lambda < 0$ we have $\frac{\mathrm{d}E}{\mathrm{d}t} < 0$ however we know $E(t) \geq 0$ hence we have that $E(t) \to 0$.

4. Taking the Ansatz $x(t) = a\sin(\omega t + \phi)$ and plugging it into the energy equation we get:

$$E = \frac{1}{2}\left[a^2\omega^2\cos^2(\omega t + \phi) + \omega_0^2 a^2\sin^2(\omega t + \phi)\right]$$

Which gives:

$$\frac{a^2}{2}(-2\omega^3\cos(\omega t + \phi)\sin(\omega t + \phi) + 2\omega_0^2 a^2\sin(\omega t + \phi)\cos(\omega t + \phi)) = (\lambda - a^2\sin^2(\omega t + \phi))a^2\omega^2\cos^2(\omega t + \phi)$$

Which simplifies to:

$$a(\omega_0^2 - \omega^2)\sin(\omega t + \phi) + (\frac{a^4}{4} - \lambda)\cos(\omega t + \phi) - \frac{a^3}{2}\omega\cos(3\omega t + \phi) = 0$$

Now neglect the higher harmonics and hence set the last term to 0. Which now gives us that:

$$\omega = \omega_0 \quad \text{and} \quad a^2 = 4\lambda$$

5. We start from (7):

$$\ddot{x} + (x^2 - \lambda)\dot{x} + \omega_0^2 x = 0 \Rightarrow \ddot{x}x + (x^3 - \lambda x)\dot{x} + \omega_0^2 x^2 = 0$$

Now taking the average, integrating by parts and using that the average of the derivatives must be 0 gives:

$$\int \ddot{x}x + \omega_0^2\bar{x}^2 = 0 \Leftrightarrow [x\dot{x}] - \int \dot{x}^2 + \omega_0^2\bar{x}^2 = 0 \Leftrightarrow \bar{\dot{x}}^2 = \omega_0^2\bar{x}^2$$

6. We now take:

$$x(t) = \sum_{n \geq 1} a_n\sin(\omega nt + \phi_n)$$

Then:

$$x^2 = \sum_{n,m} a_n a_m\sin(\omega nt + \phi_n)\sin(\omega mt + \phi_m)$$

Hence:

$$\bar{x}^2 = \sum_{n \geq 1} a_n^2\sin^2(\omega nt + \phi_n)$$

Now repeating something similar for the derivatives we obtain:

$$\omega_0^2\sum_{n \geq 1} a_n^2 = \sum_{n \geq 1} a_n^2 n^2\omega^2 \Rightarrow \omega^2 = \omega_0^2\frac{\sum a_n^2}{\sum a_n^2 n^2}$$

The approximation is valid only if:

$$a_1^2 \gg \sum_{n \geq 2} a_n^2 n^2$$

7. We now take $x = \sqrt{\lambda}y$. Which gives:

$$\sqrt{\lambda}\ddot{y} + (\lambda y^2 - \lambda)\sqrt{\lambda}\dot{y} + \omega_0^2\sqrt{\lambda}y = 0 \Leftrightarrow \ddot{y} + \lambda(y^2 - 1)\dot{y} + \omega_0^2 y = 0$$

Which after integration gives:

$$\ddot{y} + \lambda\left(\frac{\dot{y}^3}{3} - \dot{y}\right) + \omega_0^2 y = 0$$

Where we took the constant equal to zero up to a shift of origin.

8. We define $u = y/\lambda$, $t = \lambda\tau$ and $v = \dot{u}$ we get:

$$\begin{cases} \frac{\mathrm{d}^2 y}{\mathrm{d}\tau^2}\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 + \lambda(\frac{1}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^3 - \frac{\mathrm{d}y}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t}) + \omega_0^2 y = 0 \\ v = \dot{u} \end{cases}$$

Which re-writes as:

$$\begin{cases} \dot{v}\lambda^{-2} + \frac{1}{3}v^3 - v + \omega_0^2 u = 0 \\ v = \dot{u} \end{cases}$$

9. Re-writing the above equation we get:

$$\dot{v} = -\lambda^2 \omega_0^2 (u - (v - \frac{v^3}{3})\frac{1}{\omega_0^2})$$

Hence for both time scales to be on the same level we need:

$$u = \frac{1}{\omega_0^2}\left(v - \frac{v^3}{3}\right) = f(v)$$

10. ...

11.

Chapter 2

TD2

2.1 Instability of linearly coupled oscillators

2.1.1 Stability Analysis

1. We can re-write the system by taking $u = \dot{x}$ and $v = \dot{y}$ then:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega_1^2 & a & 0 & 0 \\ b & -\omega_2^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix}$$

   Diagonalization gives:

$$s^2 = \frac{-\omega_1^2 - \omega_2^2 \pm \sqrt{4ab + (\omega_1^2 - \omega_2^2)^2}}{2}$$

2. If $ab > 0$ and small then the two s_i are negative reals and hence the system is stable. And the difference in between the s_i increases when ab increases.

3. If ab is negative then the system will be unstable if:

$$4ab + (\omega_1^2 - \omega_2^2)^2 \leq 0$$

   Then if $\omega_1 = \omega_2$ the system will be negative for any negative value of ab.

4. ...

5. Because of the symmetry involved in exchanging the two oscillators

2.2 Multiple scales

1.