TDs - QFT

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Chapter 1

TD1

1.1 Matrix Groups

1.2 The relationship between $SO(3)$ and $SU(2)$.

1.3 Representations of $SU(2)$.

1. An immediate computation yields the desired result
2. Let $|a\rangle$ an eigenvector of $\hat{\mathbf{J}}^2$ then:

$$\hat{\mathbf{J}}^2 |a\rangle = a |a\rangle \Rightarrow \langle a| \hat{\mathbf{J}}^2 |a\rangle = a \langle a|a\rangle \Rightarrow ||\hat{\mathbf{J}} |a\rangle ||^2 = a|| |a\rangle || \Rightarrow a > 0$$

   We propose as a writing for them $j(j+1)$ notice that:

$$j(j+1) = x \Leftrightarrow j^2 + j - x = 0 \Rightarrow j = \frac{-j + \sqrt{j^2 + 4x}}{2}$$

   Hence the writing as $j(j+1)$ is not restrictive and covers all of $\mathbb{R}^+$.
3. Let $|v\rangle$ an eigenvector of $\hat{\mathbf{J}}^2$ and $\hat{\mathbf{J}}_\mathbf{3}$ with eigenvalues $j(j+1)$ and m. Then:

$$\hat{\mathbf{J}}^2 \hat{\mathbf{J}}_+ |v\rangle = \hat{\mathbf{J}}_+ \hat{\mathbf{J}}^2 |v\rangle = j(j+1) \hat{\mathbf{J}}_+ |v\rangle$$

   Since the operator $\hat{\mathbf{J}}^2$ commutes with the $\hat{\mathbf{J}}_\mathbf{i}$. Then:

$$\hat{\mathbf{J}}_\mathbf{3} \hat{\mathbf{J}}_+ |v\rangle = (\hat{\mathbf{J}}_+ \hat{\mathbf{J}}_\mathbf{3} + [\hat{\mathbf{J}}_\mathbf{3}, \hat{\mathbf{J}}_+]) |v\rangle = (m\hat{\mathbf{J}}_+ + i\hat{\mathbf{J}}_\mathbf{2} + 1\hat{\mathbf{J}}_\mathbf{1}) |v\rangle = (m+1)\hat{\mathbf{J}}_+ |v\rangle$$

   Identically for $\hat{\mathbf{J}}_-$ we obtain the same thing but with $m-1$ as the eigenvalue for $\hat{\mathbf{J}}_\mathbf{3}$.
4. Assume that there is no such vector than the ladder operator would span an infinite family of eigenvectors of $\hat{\mathbf{J}}_\mathbf{3}$ and $\hat{\mathbf{J}}_+$ and hence V would be infinite dimensional.
5. We have that:

$$\hat{\mathbf{J}}_- \hat{\mathbf{J}}_+ = \hat{\mathbf{J}}_\mathbf{1}^2 - i[\hat{\mathbf{J}}_\mathbf{1}, \hat{\mathbf{J}}_\mathbf{2}] + \hat{\mathbf{J}}_\mathbf{2}^2 = \hat{\mathbf{J}}^2 - \hat{\mathbf{J}}_\mathbf{3}^2 + \hat{\mathbf{J}}_\mathbf{3}$$

   Then applying this for $|v_0\rangle$ we get:

$$\hat{\mathbf{J}}_- \hat{\mathbf{J}}_+ |v_0\rangle = 0 = (j(j+1) - m_0^2 + m_0) |v_0\rangle \Rightarrow j(j+1) = m_0(m_0+1)$$

6. An identical argument tells us that successive application of the lowering ladder operator must lead to a vanishing state. Then from definition we have that:

$$|w_0\rangle = (\hat{\mathbf{J}}_-)^k |v_0\rangle \Rightarrow m_0' = m_0 - k$$

7. Similarly as before we get the exact same result but with a minus sign.
8. We then have the system:

$$\begin{cases} j(j+1) = m_0(m_0+1) \\ j(j+1) = (m_0-k)(m_0-k-1) \end{cases} \Rightarrow \begin{cases} j(j+1) = m_0(m_0+1) \\ k^2 + k = 2m_0(1+k) \end{cases} \Rightarrow \begin{cases} j = \frac{k}{2} \\ \frac{k}{2} = m_0 \end{cases}$$

9. We have that $\hat{\mathbf{J}_+}$ sends $|j,m\rangle$ to $|j,m+1\rangle$ and similarly $\hat{\mathbf{J}_-}$ sends $|j,m\rangle$ to $|j,m-1\rangle$. Then we get that:

$$\hat{\mathbf{J}_+}|j,m\rangle = x|j,m+1\rangle \Rightarrow \langle j,m|\hat{\mathbf{J}_-}\hat{\mathbf{J}_+}|j,m\rangle = |x|^2 = j(j+1)-m(m+1)$$

Hence we obtain:

$$x = \sqrt{j(j+1)-m(m+1)}$$

Then we have that:

$$\hat{\mathbf{J}}_{\mathbf{1}}|j,m\rangle = \frac{\hat{\mathbf{J}_+}+\hat{\mathbf{J}_-}}{2}|j,m\rangle = \frac{x}{2}\left(|j,m+1\rangle + |j,m-1\rangle\right)$$

Similarly:

$$\hat{\mathbf{J}}_{\mathbf{2}}|j,m\rangle = \frac{\hat{\mathbf{J}_+}-\hat{\mathbf{J}_-}}{2i}|j,m\rangle = \frac{x}{2i}\left(|j,m+1\rangle - |j,m-1\rangle\right)$$

10. Since $\hat{\mathbf{J}}^2$ commutes with the $\hat{\mathbf{J}}_{\mathbf{i}}$ we know that the eigenspaces of $\hat{\mathbf{J}}^2$ are sub-representations of $SU(2)$. We now restrict ourselves to one eigenspace, call it $\tilde{V}_j$ corresponding to the eigenvalue $j(j+1)$. As said previously there must be at least one eigenvector of $\hat{\mathbf{J}}^2$ and $\hat{\mathbf{J}}_{\mathbf{3}}$ which is killed by $\hat{\mathbf{J}_+}$ call it $|j,j,1\rangle$. Then from this eigenvector we can build $|j,m,1\rangle = \hat{\mathbf{J}_-}^{j-m}|j,j,1\rangle$. Which is an irreducible subspace of $\tilde{V}_j$. Then we can write $\tilde{V}_j = V_j^1 \oplus \tilde{V}_j'$. We can then repeat the process on $\tilde{V}_j'$ until we spanned the whole space. Then we have:

$$V = V_0^1 \oplus \cdots \oplus V_0^{n_0} \oplus V_{1/2}^1 \oplus \cdots \oplus V_{1/2}^{n_{1/2}} \oplus \cdots$$

11. We have that $\vec{L} = \vec{R}\wedge\vec{P}$ where $\vec{R}$ and $\vec{P}$ are operators on $L^2(\mathbb{R}^3)$ where $[R_j,P_k] = i\delta_{jk}$. Then we have that $[L_a,L_b] = i\varepsilon_{abc}L_c$. Then the space we describe is $V : \{\psi : S^2 \to \mathbb{C}\}$ and the spherical harmonic decomposition tells us that:

$$\psi(\theta,\varphi) = \sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{\ell} a_{\ell,m}Y_\ell^m(\theta,\varphi)$$

Furthermore we have that:

$$\vec{L}^2 = Y_\ell^m = \ell(\ell+1)Y_\ell^m \quad \text{and} \quad L_3Y_\ell^m = mY_\ell^m$$

Hence the subspace $V_\ell = \mathrm{Span}(Y_\ell^{-\ell},\cdots,Y_\ell^\ell)$ is stable under rotation and $V = V_0\oplus V_1\oplus V_2\oplus\cdots$

12. We have:

$$e^{2i\pi\hat{\mathbf{J}}_{\mathbf{3}}}|j,m\rangle = e^{2i\pi m}|j,m\rangle$$

Now if j is an integer we have that $m\in\mathbb{Z}$ and hence $e^{2i\pi\hat{\mathbf{J}}_{\mathbf{3}}} = \mathrm{Id}$. However if j is a half integer then m is also a half integer and hence $e^{2i\pi\hat{\mathbf{J}}_{\mathbf{3}}} = -\mathrm{Id}$.

13. In QM for example we usually consider the wavefunctions of one particle with no spin we will use the space $L^2(\mathbb{R}^3,\mathbb{C})$ however now if we introduce spin we will consider $L^2(\mathbb{R}^3,\mathbb{C})\otimes\mathbb{C}^2$ or similarly if we consider two particles we need to consider $L^2(\mathbb{R}^3,\mathbb{C})\otimes L^2(\mathbb{R}^3,\mathbb{C})$. Then we know also that:

$$V_{j_1}\otimes V_{j_2} = V_{|j_1-j_2|}\oplus V_{|j_1-j_2|+1}\oplus\cdots\oplus V_{j_1+j_2}$$

Chapter 2

TD2

2.1 Properties of time-like vectors.

1. Let $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{C}_+$. Then $a^0 > ||\vec{a}||$ and similarly for $\mathbf{B}$. Hence $\vec{a}\cdot\vec{b} \leq ||\vec{a}||\cdot||\vec{b}|| \leq a^0 b^0$. Then $\mathbf{A}\cdot\mathbf{B} < 0$.
2. Let $\mathbf{A},\mathbf{B} \in \mathcal{C}_+$ and $\mu,\nu \in \mathbb{R}^+$ then $(\mu\mathbf{A}+\nu\mathbf{B})^2 = \mu^2\mathbf{A}^2 + 2\mu\nu\mathbf{A}\cdot\mathbf{B} + \nu^2\mathbf{B}^2 < 0$. Hence $(\mathbf{A}+\mathbf{B}) \in \mathcal{C}_+$.
3. A special Lorentz transformation is an isometry of the Minkowski space hence $\mathcal{C}_+$ is stable under it.
4. We have that:
$$a^i - \beta^i a^0 = 0 \Rightarrow \beta^i = \frac{a^i}{a^0}$$
5. Suppose by induction that this is true for n the base cases being trivial. Then for $n+1$ note that $\mathcal{C}_+$ is stable under addition so any case can be reduced to the base case $n=2$. We prove this case here:
$$\sqrt{-(\mathbf{A}+\mathbf{B})^2} = \sqrt{-(\mathbf{A'}+\mathbf{B'})^2} = \sqrt{{d^0}^2} = d^0$$
Then ${\mathbf{A_i}}^2 = \vec{a}_i^{\,2} - (a_i^0)^2$ and hence $a_i^0 = \sqrt{-{\mathbf{A_i}}^2 + \vec{a}_i^{\,2}} \geq \sqrt{-{\mathbf{A_i}}^2}$ hence:
$$\sqrt{-(\mathbf{A}+\mathbf{B})^2} \geq \sqrt{-\mathbf{A}^2} + \sqrt{-\mathbf{B}^2}$$

2.2 Applications to 4-momenta

1. $\mathbf{P} = m\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\tau} = (E, m\vec{U})$ and:
$$\mathbf{P}^2 = -E^2 + m^2\vec{U}^2 = -m^2$$
2. We directly have that $P^0 = E > 0$ and $\mathbf{P}^2 = -m^2 < 0$. Hence $\mathbf{P} \in \mathcal{C}_+$.
3. From question 2 of Exercise 1 we know that since $\mathbf{P}_i$ are in $\mathcal{C}_+$ then so is $\mathbf{P}$. Then from question 4 of Exercise 1 we know that there exists a boost transformation such that $\mathbf{P} = (E^*, \vec{0})$. Then using question 5 of Exercise 1 we also know that:
$$E^* \geq \sum_{i=1}^{n} m_i$$

2.3 Decays of particles

1. We must have that $M \geq \sum_{i=1}^{n} m_i$.
2. (a) The number of unknowns are 8 since they are all the components of the two momenta $\mathbf{P}_1$ and $\mathbf{P}_2$. We also have the four equations given by: $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$. Finally we have two more equations ${\mathbf{P_1}}^2 = -m_1^2$ and ${\mathbf{P_2}}^2 = -m_2^2$.

   (b) We have that:
$$ {\mathbf{P_1}}^2 = \mathbf{P}^2 + {\mathbf{P_2}}^2 - 2\mathbf{P}\cdot\mathbf{P_2} \Leftrightarrow -m_1^2 = -M^2 - m_2^2 - 2\left(-ME_2\right) \Leftrightarrow 2ME_2 = M^2 + m_2^2 - m_1^2$$
   Then symmetry gives the desired opposite result.

(c) We have:

$$E_{kin,1} = E_1 - m_1$$

Which immediately gives the desired result after factorization and identically for $E_{kin,2}$. Then:

$$E_{kin,1} + E_{kin,2} = \Delta M$$

In other words all excess mass is converted to kinetic energy.

3. For each new particle we get 4 more unknowns and one more equation so 3 more indeterminates. Now following the hint we write:

$$\mathbf{P} = \sum_j \mathbf{P_j} = \mathbf{P_i} + \mathbf{Q}$$

Then:

$$\mathbf{P_i}^2 = \mathbf{P}^2 + \mathbf{Q}^2 - 2\mathbf{P} \cdot \mathbf{Q} \Leftrightarrow -m_i^2 = -M^2 - 2ME' + \mathbf{Q}^2$$

Then we have:

$$E_i = \frac{M^2 + m_i^2 + \mathbf{Q}^2}{2m} \quad \text{and} \quad E_{kin,i} = \frac{M^2 + m_i^2 - 2Mm_i + \mathbf{Q}^2}{2m}$$

Now using question 5 of Exercise 1 we can bound $\mathbf{Q}^2$ as follows:

$$\sqrt{-\mathbf{Q}^2} \geq \sum_{j \neq i} m_j \Rightarrow \mathbf{Q}^2 \leq -(M - \Delta M - m_i)^2$$

Then re-injecting this above we get the desired inequalities.

2.4 Creations of particles

1.

Chapter 3

TD3

3.1 The Laplace Equation

1. The solution is given by $\frac{q\mathbf{r}}{4\pi}$.

2. Rotationally invariant harmonic functions are given by:

$$\nabla^2 u = 0 \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{n-1}u'(r)\right) = 0 \Leftrightarrow r^{n-1}u'(r) = c \Leftrightarrow u'(r) = c\,r^{1-n} \Leftrightarrow u(r) = \frac{c}{r^{n-2}(n-2)} + c'$$

When $n \neq 2$ in the case where $n = 2$ then we get:

$$u(r) = c\ln r + c'$$

3. We have that:

$$\int_\Omega \mathrm{d}\mathbf{x}[u\nabla^2 v - v\nabla^2 u] = \int_\Omega \mathrm{d}\mathbf{x}\boldsymbol{\nabla}\cdot[u\boldsymbol{\nabla}v - v\boldsymbol{\nabla}u] = \int_{\partial\Omega} \mathrm{d}\mathbf{x}\,\mathbf{n}\cdot[u\boldsymbol{\nabla}v - v\boldsymbol{\nabla}u] = \int_{\partial\Omega}\mathrm{d}\mathbf{x}\left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right]$$

4. We have that:

$$\begin{aligned}\int_{\overline{\mathcal{B}_\varepsilon}} \mathrm{d}\mathbf{x}\,G(\mathbf{x})\nabla^2\varphi(\mathbf{x}) &= \int_{\overline{\mathcal{B}_\varepsilon}} \mathrm{d}\mathbf{x}\left[G(\mathbf{x})\nabla^2\varphi(\mathbf{x}) - \varphi(x)\nabla^2 G(x)\right] \\ &= \int_{\mathcal{C}_\varepsilon} \mathrm{d}\mathbf{x}\,(G(\mathbf{x})(-\mathbf{r})\cdot\boldsymbol{\nabla}\varphi(\mathbf{x}) - \varphi(\mathbf{x})(-\mathbf{r})\boldsymbol{\nabla}G(\mathbf{x})) \\ &= \int_{\partial\Omega} \mathrm{d}\mathbf{x} - \varphi(\mathbf{x})\frac{\partial G}{\partial r} \xrightarrow{\varepsilon\to 0} \varphi(\mathbf{0})\omega_n\varepsilon^{n-1}\frac{\partial G}{\partial r}\Big|_{r=\varepsilon} = \varphi(\mathbf{0})\end{aligned}$$

5. We have:

$$\left\langle G\middle|\nabla^2\varphi\right\rangle = \langle\delta|\varphi\rangle = (-1)^2\left\langle\nabla^2 G\middle|\varphi\right\rangle = \langle\delta|\varphi\rangle$$

3.2 The Helmholtz Equation.

1. We have that:

$$\begin{aligned}(\nabla^2 + k^2)G_\pm(\mathbf{x}) &= -\frac{1}{4\pi}(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + k^2)\frac{e^{\pm ikr}}{r} = \frac{1}{4\pi}\left(\frac{1}{r^2}\frac{\partial i e^{ikr}(i+kr)}{\partial r} + k^2\frac{e^{ikr}}{r}\right) \\ &= -\frac{1}{4\pi}\left(-\frac{e^{ikr}k^2}{r} + k^2\frac{e^{ikr}}{r}\right)\end{aligned}$$

Hence for all $r \neq 0$ where the differential is easily well defined it cancels.

2. Following the same steps as in part 1 questions 3 and 4 we get that $(\nabla^2 + k^2)G_\pm(\mathbf{x}) = \delta(\mathbf{x})$.

3. We can easily deduce that the Green function of $-D$ is given by $-G_\pm$. Then up to taking $k = im$ we get the desired result.

3.3 Fourier transforms

1. We have that:

$$DG = \delta \Leftrightarrow (1 + a_i \boldsymbol{\nabla}^{\mathbf{i}} + \cdots + a_{i_1,\cdots,i_p}\mathbf{grad}^{\mathbf{i_1},\cdots,\mathbf{i_p}})G = \delta \Leftrightarrow (1 + a_j(ip_j) + \cdots + a_{j_1,\cdots,j_p}(ip_{j_1,\cdots,j_p})\tilde{G} = 1$$

2. We have that:

$$\left\langle p(\mathrm{pv}\frac{1}{p} + \alpha\delta(p))\middle|\varphi\right\rangle = \varphi + \alpha \cdot \mathbf{0} \cdot \varphi(\mathbf{0}) = \varphi = \langle 1|\varphi\rangle$$

3. Let $\tilde{G}$ be a solution of (9) then notice that:

$$\left\langle P(\mathbf{p})(\tilde{G}(\mathbf{p}) + \alpha\delta(\mathbf{p} - \mathbf{p_0}))\middle|\varphi\right\rangle = \langle 1|\varphi\rangle + \alpha\,\langle P(\mathbf{p})\delta(\mathbf{p} - \mathbf{p_0})|\varphi\rangle = \langle 1|\varphi\rangle$$

   In terms of $G(\mathbf{x})$ it corresponds to adding a constant.

4. (a) From definition we have that $C_0 = \left\langle \mathrm{pv}\frac{1}{z}\middle| f\right\rangle$.
   (b) From the residue theorem we have that $C_\pm = \left\langle \mathrm{pv}\frac{1}{z} \mp i\pi\delta\middle| f\right\rangle$.

5. True because modifying the integral in a set of measure 0 changes nothing to the value of the integral.

3.4 The wave equation.

1. Let $D = \frac{\partial^2}{\partial t^2} - \nabla^2$ then:

$$\tilde{D} = -\omega^2 - \nabla^2 = -(\nabla^2 + \omega^2)$$

   Then the equation becomes:

$$\tilde{D}\tilde{G}(\omega, \mathbf{x}) = 1 \cdot \delta(\mathbf{x})$$

   Hence $-\tilde{G}(\omega, \mathbf{x})$ is a Green function of the Helmoltz operator.

2. We now know that:

$$\tilde{G}(\omega, \mathbf{x}) = \frac{1}{4\pi}\frac{e^{\pm i\omega|\mathbf{x}|}}{|\mathbf{x}|}$$

   Then doing the inverse Fourier transform we obtain:

$$G(t, x) = \frac{\delta(|\mathbf{x}| \pm t)}{4\pi|\mathbf{x}|}$$

Chapter 4

Representations of the Lorentz group

1. We have that:
$$J_a = \mathcal{J}^L_a + \mathcal{J}^R_a \text{ and } N_a = -i(\mathcal{J}^L_a - \mathcal{J}^R_a)$$
2. We have:
$$[\mathcal{J}^L_a, \mathcal{J}^L_b] = i\varepsilon_{abc}\mathcal{J}^L_c \text{ and } [\mathcal{J}^R_a, \mathcal{J}^R_b] = i\varepsilon_{abc}\mathcal{J}^R_c$$
Then we have that:
$$[\mathcal{J}^L_a, \mathcal{J}^R_b] = 0$$
Hence we have that $L^\uparrow_+$ can be seen as the product of two independent $SU(2)$ groups.
3. The dimension of a j_R representation of $SU(2)$ is $2j_r + 1$ hence the (j_R, j_L) representation of $L^\uparrow_+$ has dimension $(2j_r + 1)(2j_L + 1)$.
4. We have that:
$$i\theta^a J_a + i\nu^a N_a = i\theta_a(J^L_a + J^R_a) + i\nu^a i(J^R_a - J^L_a)$$
Hence we have that:
$$\rho(\Lambda) = e^{i(\theta_a + i\nu_a)\hat{J}^R_a + i(\theta_a - i\nu_a)\hat{J}^L_a} = e^{i(\theta_a + i\nu_a)\hat{J}^R_a} e^{i(\theta_a - i\nu_a)\hat{J}^L_a} \text{ since } [\hat{J}^R_a, \hat{J}^L_a] = 0$$
Hence we have that:
$$\rho(\lambda)\left|j_R, m_R\right\rangle \otimes \left|j_L, m_L\right\rangle = e^{i(\theta_a + i\nu_a)\hat{J}^R_a}\left|j_R, m_R\right\rangle \otimes e^{i(\theta_a - i\nu_a)\hat{J}^L_a}\left|j_L, m_L\right\rangle$$
5. Notice that:
$$\rho(e^{i\nu^a N_a})^\star = \left(e^{\nu^a(\hat{J}^L_a - \hat{J}^R_a)}\right)^\star = e^{\nu^a(\hat{J}^L_a - \hat{J}^R_a)} = \rho(e^{i\nu^a N_a}) \neq \rho(e^{i\nu^a N_a})^{-1}$$
The boost are characterized by a parameter $\phi \in]-\infty, \infty[$ and hence $L^\uparrow_+$ is not compact or similarly $\beta = \frac{v}{c} \in]-1, 1[$ is bounded but not closed and hence not compact.
6. The subgroup of $L^\uparrow_+$ containing only the rotation is the one generated by J_1, J_2, J_3. Or equivalently it is generated by $J^L_a + J^R_a$ and therefore is spanned by the (j_R, j_L) representation. We have:
$$\rho(e^{i\theta_a J_a}) = e^{i\theta_a(\hat{J}^R_a + \hat{J}^L_a)}$$
Hence now defining $\hat{J}_a = \hat{J}^R_a + \hat{J}^L_a$ we have the sum of two angular momenta which we know decomposes $V_{j_R} \otimes V_{j_L}$ to $V_{|j_R - j_L|} \oplus V_{|j_R - j_L| + 1} \otimes \cdots \otimes V_{j_R + j_L}$.
7. The dimension of the $(\frac{1}{2}, \frac{1}{2})$ representation is 4. From the rotation point of view this can be written as $0 \oplus 1$. This looks a lot like the A^μ representation which is given by: $\rho(\Lambda)A = \Lambda A$.
8. We have that:
$$(V_{j^1_R} \otimes V_{j^1_L}) \otimes (V_{j^2_R} \otimes V_{j^2_L}) = \left(\bigoplus_{i=|j^1_R - j^1_L|}^{j^1_R + j^1_L} V_i\right) \otimes \left(\bigoplus_{i=|j^2_R - j^2_L|}^{j^2_R + j^2_L} V_i\right) = \bigoplus_{i,j=|j_R - j_L|}^{j_R + j_L} V_i \otimes V_j$$
In the special case of $j_R = j_L = \frac{1}{2}$ we obtain:
$$\bigoplus_{i,j=0}^{1} V_i \otimes V_j = (0,0) \oplus (0,1) \oplus (1,0) \oplus (1,1)$$
9. We have that:
$$PJ^{\gamma\sigma}$$

Chapter 5

The Klein-Gordon equation

5.1 Basic properties.

1. This reads:
$$(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2)\phi = 0$$
2. Simple chain rule gives the desired result.
3.

5.2 Potential barrier

1. Then we get:
$$((\frac{\partial}{\partial t} + iqV)^2 - \nabla^2 + m^2)\phi = 0$$
2. Plugging in the Ansatz simply transform the ∂^0 into an E.
3. We have that:
$$(-\nabla^2 + m^2)\varphi_-(x) = E^2\varphi_-(x) \quad \text{and} \quad (-\nabla^2 + m^2)\varphi_+(x) = (E - qV_0)^2\varphi_+(x)$$
4. Plugging in the Ansatz we obtain:
$$p'^2 + m^2 = E^2 \quad \text{and} \quad p^2 + m^2 = (E - qV_0)^2$$
Hence we get:
$$p = \pm\sqrt{E^2 - m^2} \quad \text{and} \quad p' = \sqrt{(E - qV_0)^2 - m^2}$$
5. (a) p' is purely imaginary for $E \in [m, qV_0 + m]$. Meaning that if we arrive with less than the energy needed to jump the barrier then the transmitted wave will be evanescent.
   (b) At high enough energies ($E \geq V_0 + m$) there is transmission and so everything is fine. If $E \in [qV_0 - m, qV_0 + m]$ we get reflected. However when $E \in [m, qV_0 - m]$ we still get transmitted ! This is very counter intuitive however the explanation comes from the fact that with a barrier potential this big there is enough energy to create new particles which means our analysis will break down.
6. Since the second derivative must be at least short of a δ then φ must be continuous and the first derivative too. Hence:
$$\begin{cases} 1 + r = t \\ ip(1 - r) = ip't \end{cases} \Leftrightarrow \begin{cases} 1 + r = t \\ p(1 - r) = p'(1 + r) \end{cases} \Leftrightarrow \begin{cases} t = \frac{2p}{p'+p} \\ r = \frac{p-p'}{p'+p} \end{cases}$$
7. The current is given by:
$$j^\mu = -i(\phi^\star\partial^\mu\phi - (\partial^\mu\phi^\star)\phi)$$
Which gives:
$$j^0 = -i(e^{iEt}\varphi^\star(x)(-iE)e^{-iEt}\varphi(x) - (iE)e^{iEt}\varphi^\star(x)\varphi(x)) = -i(-iE|\varphi(x)|^2 + iE|\varphi(x)|^2) = 0$$
And:
$$j^x = -i(te^{ip'x}...$$

Chapter 6

The Dirac equation

6.1 The non-relativistic limit.

1. Splitting the original equation we get:

$$\gamma^0 \frac{\partial \psi}{\partial t} = (\gamma^i \partial_i + m)\psi = (-\gamma^i \cdot \boldsymbol{\nabla} + m)\psi$$

   Now multiplying left and right by γ^0 and using the fact that $(\gamma^0)^2 = -1$ we get:

$$-\frac{\partial \psi}{\partial t} = (-\gamma^0 \gamma^i \boldsymbol{\nabla} + \gamma^0 m)\psi \Rightarrow i\frac{\partial \psi}{\partial t} = (\gamma^0 \gamma^i (-i\boldsymbol{\nabla}) + (-\gamma^0)m)\psi = H\psi$$

   Where $\beta = -\gamma^0$ and $\alpha^i = \gamma^0 \gamma^i$.

2. The Hamiltonian operator will become:

$$H = \alpha \cdot (\mathbf{P} - q\mathbf{A}) + \beta m$$

3. We have that:

$$\{\gamma^i, \gamma^j\} = \gamma^i \gamma^j + \gamma^j \gamma^i = -\begin{pmatrix} \sigma_i \sigma_j & 0 \\ 0 & -\sigma_i \sigma_j \end{pmatrix} - \begin{pmatrix} \sigma_j \sigma_i & 0 \\ 0 & -\sigma_j \sigma_i \end{pmatrix} = 2\eta^{ij}$$

   Trivially $\gamma^0 \gamma^i + \gamma^i \gamma^0 = 0$, and finally:

$$2\gamma^0 \gamma^0 = -2$$

   Then we have:

$$\beta = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix} \quad \text{and} \quad \alpha^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

4. We have:

$$\begin{cases} -m\varphi = -(\sigma \cdot \boldsymbol{\nabla})\chi + im\varphi \\ -m\chi = -(\sigma \cdot \boldsymbol{\nabla})\varphi - im\chi \end{cases}$$