TD-Probability

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Chapter 1

TD 1

1.1 A strategic choice.

Let $X \in \{0,1\}^3$ (resp. Y) be the random variable corresponding to the results of the matches using the first strategy (resp. the second strategy). Then we have that (let $D = \{(1,1,1),(1,1,0),(0,1,1)\}$):

$$P(X \in D) = a^2b + ab(1-a) + (1-a)ba = ab(2-a)$$

Similarly:

$$P(Y \in D) = b^2a + ba(1-b) + (1-b)ab = ba(2-b)$$

Then since $a > b$ we have that $P(X \in D) < P(Y \in D)$, hence the winning strategy is BAB.

1.2 Derangements

1.2.1

Let E be a finite set and $A, B \subseteq E$. We denote by 1_A the indicator function of A and $\bar{A}$ the complement of A. Then we have that:

$$1_{\bar{A}} = 1 - 1_A \quad \text{and} \quad 1_{A\cap B} = 1_A \cdot 1_B \quad \text{and} \quad 1_{A\cup B} = 1_A + 1_B - 1_{A\cap B}$$

1.2.2

We will prove this by induction on n. The base case $n = 1$ as well as $n = 2$ are trivially satisfied. Now assume that this is satisfied for n then we have that (using the induction hypothesis for $n = 2$):

$$\text{card}\left(\bigcup_{i=1}^{n} A_i \bigcup A_{n+1}\right) = \text{card}\left(\bigcup_{i=1}^{n} A_i\right) + \text{card}(A_{n+1}) - \text{card}\left(\left(\bigcup_{i=1}^{n} A_i\right) \bigcap A_{n+1}\right)$$

Now we develop the last term into:

$$\left(\bigcup_{i=1}^{n} A_i\right) \bigcap A_{n+1} = \bigcup_{i=1}^{n} \left(A_i \bigcap A_{n+1}\right)$$

Now applying the induction hypothesis gives the desired result.

1.2.3

Let A_i be the set of permutations that fixes point i. Then from the inclusion-exclusion principle we have:

$$D_n = n! - \text{card}\left(\bigcup_{i=1}^{n} A_i\right) = n! - \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)! = n! \sum_{k=2}^{n} \frac{(-1)^k}{k!}$$

1.2.4

The probability that no one gets their jacket corresponds to the probability of having a derangement in other words:

$$p_n = \frac{D_n}{n!} = \sum_{k=2}^{n} \frac{(-1)^k}{k!} \overset{n\to\infty}{\longrightarrow} \frac{1}{e}$$

1.2.5

We have that:

$$D_{n,l} = \binom{n}{l} D_{n-l} = \binom{n}{l}(n-l)! \sum_{k=2}^{n-l} \frac{(-1)^k}{k!} = \frac{n!}{l!} \sum_{k=2}^{n-l} \frac{(-1)^k}{k!}$$

Hence the probability that eaxctly l people leave with their jackets is:

$$p_l = \frac{D_{n,l}}{n!} = \frac{1}{l!} \sum_{k=2}^{n-l} \frac{(-1)^k}{k!}$$

1.2.6

The probability that a given person gets back their jacket is $p_s = \frac{1}{n}$. The probability that at least one person gets back their jacket is:

$$p_a = 1 - p_n = 1 - \sum_{k=2}^{n} \frac{(-1)^k}{k!}$$

Notice that $p_s < p_a$.

1.3 Balls in bins

1.3.1

(a) If all the balls are distinguishable then we have $\Omega = [\![1, n]\!]^r$ is the set of tuples where each element corresponds to where the i-th ball has been sent to. Then $\mathcal{F} = \mathcal{P}(\Omega)$ and since each event is sampled uniformly at random we have that:

$$\forall \omega \in \Omega, P(\omega) = \frac{1}{|\Omega|} = \frac{1}{n^r}$$

Then the probability of $(r_1, \cdots, r_n)$ is given by:

$$P[(r_1, \cdots, r_n)] = P[\{\omega \in \Omega : \forall i \in [\![1, n]\!] \, \#\{b \in \Omega : b = i\} = r_i\}] = \frac{1}{n^r} \binom{r}{r_1, r_2, \ldots, r_n}$$

(b) Now we have that $\Omega = \{(r_i \in \mathbb{N} : i \in [\![1, n]\!]) : \sum_{i=1}^n r_i = r\}$. Again we have that $\mathcal{F} = \mathcal{P}(\Omega)$. Then we have that:

$$P[(r_1, \ldots, r_n)] = \frac{1}{|\Omega|} = \frac{1}{\binom{r+n-1}{n-1}}$$

(c) Now we have that $\Omega = \{s \in \{0, 1\}^n : \sum_{i=1}^n s_i = r\}$ corresponding to the tuple indicating if each state is occupied or not. Once again $\mathcal{F} = \mathcal{P}(\Omega)$. Now the probability is given by:

$$P[(r_1, \ldots, r_n)] = \frac{1}{|\Omega|} = \frac{1}{\binom{n}{r}}$$

1.3.2

The probability that at least two have the same birthday is the 1 minus the probability that none of them share a birthday. The probability that none of them share a birthday is given by $\frac{r!}{n^r}\binom{n}{r}$. Hence the probability that at least two people share a birthday is given by: $1 - \frac{r!}{n^r}\binom{n}{r}$.

1.3.3

Days are bins, accidents are distinguishable balls hence the probability is given by:

$$\frac{\binom{r}{n} n^{r-n}}{n^r} = n^{-n} \binom{r}{n}$$

1.3.4