TDs - QFT

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Chapter 1

TD1

1.1 Matrix Groups

1.2 The relationship between $SO(3)$ and $SU(2)$.

1.3 Representations of $SU(2)$.

1. An immediate computation yields the desired result
2. Let $|a\rangle$ an eigenvector of $\hat{\mathbf{J}}^2$ then:

$$\hat{\mathbf{J}}^2 |a\rangle = a |a\rangle \Rightarrow \langle a| \hat{\mathbf{J}}^2 |a\rangle = a \langle a|a\rangle \Rightarrow ||\hat{\mathbf{J}} |a\rangle ||^2 = a|| |a\rangle || \Rightarrow a > 0$$

   We propose as a writing for them $j(j+1)$ notice that:

$$j(j+1) = x \Leftrightarrow j^2 + j - x = 0 \Rightarrow j = \frac{-j + \sqrt{j^2 + 4x}}{2}$$

   Hence the writing as $j(j+1)$ is not restrictive and covers all of $\mathbb{R}^+$.

3. Let $|v\rangle$ an eigenvector of $\hat{\mathbf{J}}^2$ and $\hat{\mathbf{J}_3}$ with eigenvalues $j(j+1)$ and m. Then:

$$\hat{\mathbf{J}}^2 \hat{\mathbf{J}_+} |v\rangle = \hat{\mathbf{J}_+} \hat{\mathbf{J}}^2 |v\rangle = j(j+1)\hat{\mathbf{J}_+} |v\rangle$$

   Since the operator $\hat{\mathbf{J}}^2$ commutes with the $\hat{\mathbf{J}_i}$. Then:

$$\hat{\mathbf{J}_3}\hat{\mathbf{J}_+} |v\rangle = (\hat{\mathbf{J}_+}\hat{\mathbf{J}_3} + [\hat{\mathbf{J}_3}, \hat{\mathbf{J}_+}]) |v\rangle = (m\hat{\mathbf{J}_+} + i\hat{\mathbf{J}_2} + 1\hat{\mathbf{J}_1}) |v\rangle = (m+1)\hat{\mathbf{J}_+} |v\rangle$$

   Identically for $\hat{\mathbf{J}_-}$ we obtain the same thing but with $m-1$ as the eigenvalue for $\hat{\mathbf{J}_3}$.

4. Assume that there is no such vector than the ladder operator would span an infinite family of eigenvectors of $\hat{\mathbf{J}_3}$ and $\hat{\mathbf{J}_+}$ and hence V would be infinite dimensional.
5. We have that:

$$\hat{\mathbf{J}_-}\hat{\mathbf{J}_+} = \hat{\mathbf{J}_1}^2 - i[\hat{\mathbf{J}_1}, \hat{\mathbf{J}_2}] + \hat{\mathbf{J}_2}^2 = \hat{\mathbf{J}}^2 - \hat{\mathbf{J}_3}^2 + \hat{\mathbf{J}_3}$$

   Then applying this for $|v_0\rangle$ we get:

$$\hat{\mathbf{J}_-}\hat{\mathbf{J}_+} |v_0\rangle = 0 = (j(j+1) - m_0^2 + m_0) |v_0\rangle \Rightarrow j(j+1) = m_0(m_0+1)$$

6. An identical argument tells us that successive application of the lowering ladder operator must lead to a vanishing state. Then from definition we have that:

$$|w_0\rangle = (\hat{\mathbf{J}_-})^k |v_0\rangle \Rightarrow m_0' = m_0 - k$$

7. Similarly as before we get the exact same result but with a minus sign.
8. We then have the system:

$$\begin{cases} j(j+1) = m_0(m_0+1) \\ j(j+1) = (m_0-k)(m_0-k-1) \end{cases} \Rightarrow \begin{cases} j(j+1) = m_0(m_0+1) \\ k^2 + k = 2m_0(1+k) \end{cases} \Rightarrow \begin{cases} j = \frac{k}{2} \\ \frac{k}{2} = m_0 \end{cases}$$

9. We have that $\hat{\mathbf{J}_+}$ sends $|j,m\rangle$ to $|j,m+1\rangle$ and similarly $\hat{\mathbf{J}_-}$ sends $|j,m\rangle$ to $|j,m-1\rangle$. Then we get that:

$$\hat{\mathbf{J}_+}|j,m\rangle = x|j,m+1\rangle \Rightarrow \langle j,m|\hat{\mathbf{J}_-}\hat{\mathbf{J}_+}|j,m\rangle = |x|^2 = j(j+1)-m(m+1)$$

Hence we obtain:

$$x = \sqrt{j(j+1)-m(m+1)}$$

Then we have that:

$$\hat{\mathbf{J}}_{\mathbf{1}}|j,m\rangle = \frac{\hat{\mathbf{J}_+}+\hat{\mathbf{J}_-}}{2}|j,m\rangle = \frac{x}{2}\left(|j,m+1\rangle + |j,m-1\rangle\right)$$

Similarly:

$$\hat{\mathbf{J}}_{\mathbf{2}}|j,m\rangle = \frac{\hat{\mathbf{J}_+}-\hat{\mathbf{J}_-}}{2i}|j,m\rangle = \frac{x}{2i}\left(|j,m+1\rangle - |j,m-1\rangle\right)$$

10. Since $\hat{\mathbf{J}}^2$ commutes with the $\hat{\mathbf{J}}_{\mathbf{i}}$ we know that the eigenspaces of $\hat{\mathbf{J}}^2$ are sub-representations of $SU(2)$. We now restrict ourselves to one eigenspace, call it $\tilde{V}_j$ corresponding to the eigenvalue $j(j+1)$. As said previously there must be at least one eigenvector of $\hat{\mathbf{J}}^2$ and $\hat{\mathbf{J}}_{\mathbf{3}}$ which is killed by $\hat{\mathbf{J}_+}$ call it $|j,j,1\rangle$. Then from this eigenvector we can build $|j,m,1\rangle = \hat{\mathbf{J}_-}^{j-m}|j,j,1\rangle$. Which is an irreducible subspace of $\tilde{V}_j$. Then we can write $\tilde{V}_j = V_j^1 \oplus \tilde{V}_j'$. We can then repeat the process on $\tilde{V}_j'$ until we spanned the whole space. Then we have:

$$V = V_0^1 \oplus \cdots \oplus V_0^{n_0} \oplus V_{1/2}^1 \oplus \cdots \oplus V_{1/2}^{n_{1/2}} \oplus \cdots$$

11. We have that $\vec{L} = \vec{R}\wedge\vec{P}$ where $\vec{R}$ and $\vec{P}$ are operators on $L^2(\mathbb{R}^3)$ where $[R_j, P_k] = i\delta_{jk}$. Then we have that $[L_a, L_b] = i\varepsilon_{abc}L_c$. Then the space we describe is $V : \{\psi : S^2 \to \mathbb{C}\}$ and the spherical harmonic decomposition tells us that:

$$\psi(\theta,\varphi) = \sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{\ell} a_{\ell,m}Y_\ell^m(\theta,\varphi)$$

Furthermore we have that:

$$\vec{L}^2 = Y_\ell^m = \ell(\ell+1)Y_\ell^m \quad \text{and} \quad L_3Y_\ell^m = mY_\ell^m$$

Hence the subspace $V_\ell = \mathrm{Span}(Y_\ell^{-\ell},\cdots,Y_\ell^{\ell})$ is stable under rotation and $V = V_0\oplus V_1\oplus V_2\oplus\cdots$

12. We have:

$$e^{2i\pi\hat{\mathbf{J}}_{\mathbf{3}}}|j,m\rangle = e^{2i\pi m}|j,m\rangle$$

Now if j is an integer we have that $m\in\mathbb{Z}$ and hence $e^{2i\pi\hat{\mathbf{J}}_{\mathbf{3}}} = \mathrm{Id}$. However if j is a half integer then m is also a half integer and hence $e^{2i\pi\hat{\mathbf{J}}_{\mathbf{3}}} = -\mathrm{Id}$.

13. In QM for example we usually consider the wavefunctions of one particle with no spin we will use the space $L^2(\mathbb{R}^3,\mathbb{C})$ however now if we introduce spin we will consider $L^2(\mathbb{R}^3,\mathbb{C})\otimes\mathbb{C}^2$ or similarly if we consider two particles we need to consider $L^2(\mathbb{R}^3,\mathbb{C})\otimes L^2(\mathbb{R}^3,\mathbb{C})$. Then we know also that:

$$V_{j_1}\otimes V_{j_2} = V_{|j_1-j_2|}\oplus V_{|j_1-j_2|+1}\oplus\cdots\oplus V_{j_1+j_2}$$

Chapter 2

TD2

2.1 Properties of time-like vectors.

1. Let $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{C}_+$. Then $\mathbf{A} \cdot \mathbf{B} < 0$.
2. Let $\mathbf{A}, \mathbf{B} \in \mathcal{C}_+$ and $\mu, \nu \in \mathbb{R}^+$ then $(\mu\mathbf{A} + \nu\mathbf{B})^2 = \mu^2\mathbf{A}^2 + 2\mu\nu\mathbf{A} \cdot \mathbf{B} + \nu^2\mathbf{B}^2 < 0$. Hence $(\mathbf{A} + \mathbf{B}) \in \mathcal{C}_+$.
3. A special Lorentz transformation is an isometry of the Minkowski space hence $\mathcal{C}_+$ is stable under it.
4. We have that:
$$a^i - \beta^i a^0 = 0 \Rightarrow \beta^i = \frac{a^i}{a^0}$$
5. Suppose by induction that this is true for n the base cases being trivial. Then for $n+1$ note that C_+ is stable under addition so any case can be reduced to the base case $n = 2$. We prove this case here:
$$\sqrt{-(\mathbf{A} + \mathbf{B})^2} = \sqrt{-(\mathbf{A}' + \mathbf{B}')^2} = \sqrt{{a^0}^2 + 2a^0b^0 + {b^0}^2 - \vec{a}^2} \geq a_0 + b_0 - ||\vec{a}|| \geq \sqrt{-\mathbf{A}^2} + \sqrt{-\mathbf{B}^2}$$

2.2 Applications to 4-momenta

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