Master ENS ICFP - First Year 2020/2021
Relativistic Quantum Mechanics and Introduction to Quantum Field Theory
Mid Term Homework

Marco Biroli

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1 Some operator identities : 6 points

1. We have that:

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{A^n}{n!} B \sum_{n=0}^{+\infty} \frac{(-A)^n}{n!} = \sum_{n,m=0}^{+\infty} \frac{A^n B (-A)^m}{n!m!} = \sum_{n,m=0}^{+\infty} (-1)^m \frac{A^n B A^m}{n!m!} = \sum_{n=0}^{+\infty} \frac{A^n}{n!} \sum_{m=0}^{+\infty} (-1)^m \frac{BA^m}{m!}$$

...

2. ...

3. We have that:

$$[F, G^\dagger] = [\sum_j f_j a_j, \sum_j g_j^\star a_j^\dagger] = \sum_{j,k=0}^{+\infty} f_j g_k^\star [a_j, a_k^\dagger] = \sum_{j,k=0}^{+\infty} f_j g_k^\star \delta_{jk} = \sum_{j=0}^{+\infty} f_j g_j^\star$$

Furthermore we have that $[F, G^\dagger] \propto \text{Id}$ and therefore we trivially have that $[F, [F, G^\dagger]] = [G^\dagger, [F, G^\dagger]] = 0$. Now applying question 2 we have that:

$$e^{G^\dagger} e^F = e^{-\frac{1}{2}\sum_j f_j g_j^\star} e^{G^\dagger + F} \Rightarrow e^{\frac{1}{2}\sum_j f_j g_j^\star} e^F = \underbrace{e^{\frac{1}{2}\sum_j f_j g_j^\star} e^{-\frac{1}{2}\sum_j f_j g_j^\star}}_{=e^A e^{-A}} e^{G^\dagger + F}$$

Now from Question 1 we have that for any A (trivially $[A, \text{Id}] = 0$) we get:

$$e^A \text{Id} e^{-A} = \text{Id} + \sum_{n=1}^{+\infty} \frac{1}{n!} \cdot 0 = \text{Id}$$

Hence the above formula simplifies to:

$$e^{F + G^\dagger} = e^{\frac{1}{2}\sum_j f_j g_j^\star} e^{G^\dagger} e^F$$

4. Similarly as before let $F = \int \mathrm{d}^3\mathbf{q} f(\mathbf{q}) a(\mathbf{q})$ and $G = \int \mathrm{d}^3\mathbf{q} h(\mathbf{q})^\dagger a(\mathbf{q})$. Then we have that:

$$[F, G^\dagger] = [\int \mathrm{d}^3\mathbf{q} f(\mathbf{q}) a(\mathbf{q}), \int \mathrm{d}^3\mathbf{q} h(\mathbf{q}) a^\dagger(\mathbf{q})] = \int \mathrm{d}^3\mathbf{q} f(\mathbf{q}) h(\mathbf{q}) [a(\mathbf{q}), a^\dagger(\mathbf{q})] = \int \mathrm{d}^3\mathbf{q} f(\mathbf{q}) h(\mathbf{q})$$

A similar direct application of 2 gives the desired result.

2 An example of an asymptotic series

We have that:

$$f(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}x e^{-x^2 - gx^4} \quad \text{hence} \quad |f(g)| < \int_{-\infty}^{+\infty} \mathrm{d}x |e^{-x^2 - gx^4}| = \int_{-\infty}^{+\infty} \mathrm{d}x e^{-x^2 - x^4 \operatorname{Re} g} < \int_{-\infty}^{+\infty} \mathrm{d}x e^{-x^4 \operatorname{Re} g}$$

Hence as long as $\operatorname{Re} g > 0$ this is obviously well defined from the last term and if $\operatorname{Re} g = 0$ this is obviously well defined from the before last term.

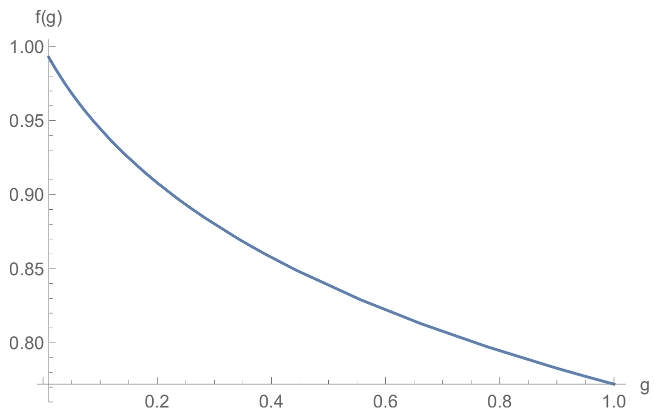

Figure 1: Plot of the numerical values of $f(g)$ for $g \in [0.01, 1]$.

1. This integral admits an exact solution given by:

$$f(g) = \frac{e^{\frac{1}{8g}} K_{\frac{1}{4}}(\frac{1}{8g})}{2\sqrt{\pi g}} \delta_{\mathrm{Re}\, g>0} + \delta_{\mathrm{Re}\, g=0} \quad \text{where} \quad K_n(z) \quad \text{is the modified Bessel function of the second kind.}$$

The plot of the numerical values for $g \in [0.01, 1]$ is given in Figure 1. Then $f(g)$ decreases monotonically when $g > 0$ increases since:

$$\frac{\mathrm{d}}{\mathrm{d}g} f(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}x (-x^4) e^{-x^2 - gx^4} = \frac{-1}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{+\infty} x^4 e^{-x^2-gx^4}}_{>0 \quad \text{when} \quad g\in\mathbb{R}^+} < 0$$

2. We have that:

$$e^{-gx^4} = \sum_{n=0}^{+\infty} \frac{(-gx^4)^n}{n!}$$

And plugging this in the expression of f and inverting the sum and the integral gives:

$$\frac{1}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \frac{(-g)^n}{n!} \int_{-\infty}^{+\infty} \mathrm{d}x \; x^{4n} e^{-x^2}$$

We notice the integral ressembles strongly the gamma function hence we change variables by taking $u = x^2$ ($\mathrm{d}u = 2\sqrt{u}\mathrm{d}x$) and we get:

$$2^{-1} \int_{-\infty}^{+\infty} \mathrm{d}u u^{2n-\frac{1}{2}} e^{-u} = 2^{-1}\Gamma(2n + \frac{1}{2}) = 2^{-4n}\sqrt{\pi}\frac{\Gamma(4n)}{\Gamma(2n)} \quad \text{from the Legendre duplication formula.}$$

Hence plugging it back up top we obtain:

$$\tilde{f}(g) = \sum_{n=0}^{+\infty} \left(\frac{(-1)^n (4n)!}{n! 2^{4n} (2n)!} \right) g^n$$

Notice that the terms f_n are monotonically increasing in norm and diverge hence the sum does not converge absolutely and $R = 0$ and it also does not converge conditionally. The order of magnitude of the first few terms is ...

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3 A relation between Dirac spinors

1. Remember that up to a re-writing we have that:

$$\omega_{ij} = \varepsilon_{ijk}\theta^k \quad \text{and} \quad \omega^{k0} = \nu^k \quad \text{and} \quad \omega_{\mu\nu} = 0 \quad \text{otherwise.}$$

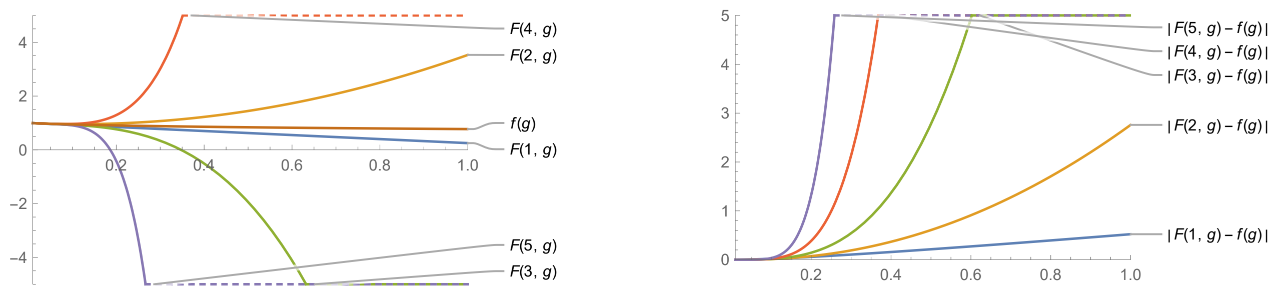

Figure 2: Series approximations of $f(g)$ and their errors for $g \in [0.01, 1]$.

Where the θ^k represent the rotations in the 3 spatial dimensions and the ν^k represent the boosts along the the three spatial directions. Hence the representation of $L(\mathbf{p})$ trivially has that $\theta^k = 0$. Then we have that:

$$i\gamma^0 D(L(\mathbf{p}))i\gamma^0 = i\gamma^0 \exp\left(\frac{1}{4}\omega_{\mu\nu}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\right)i\gamma^0$$

Now in order to simplify we need to bring the product with the gamma matrices inside. To do so we need to remember two properties. Firstly we have that $(i\gamma^0) = (i\gamma^0)^{-1}$ and secondly we have that: $Pe^A P^{-1} = e^{PAP^{-1}}$. Hence applying this formula here we obtain that:

$$i\gamma^0 D(L(\mathbf{p}))i\gamma^0 = \exp\left(\frac{\omega_{\mu\nu}}{4} i\gamma^0(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)i\gamma^0\right) = \exp\left(\frac{\omega_{\mu\nu}}{4}(i\gamma^0\gamma^\mu i\gamma^0 i\gamma^0\gamma^\nu i\gamma^0 - i\gamma^0\gamma^\nu \imath\gamma^0 i\gamma^0\gamma^\mu i\gamma^0)\right)$$

Now we use the formula from the course: $i\gamma^0\gamma^\mu i\gamma^0 = P^\mu_\nu \gamma^\nu$ where P^μ_ν is the parity operator defined in (3.33). Hence this means that the terms in the exponential simplify to:

$$i\gamma^0 D(L(\mathbf{p}))i\gamma^0 = \exp\left(\frac{\omega_{\mu\nu}}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(-1)^{1-\delta^\mu_0}(-1)^{1-\delta^\nu_0}\right) = \exp\left(\frac{\omega_{\mu\nu}(-1)^{\delta^\mu_0+\delta^\nu_0-\delta^\mu_0\delta^\nu_0}}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\right)$$

Hence now we take $\omega'_{\mu\nu} = \omega_{\mu\nu}(-1)^{\delta^\mu_0+\delta^\nu_0-\delta^\mu_0\delta^\nu_0}$, now since $\omega_{00} = 0$ we can discard the term $\delta^\mu_0\delta^\nu_0$ which means that we are left with:

$$\omega'_{\mu\nu} = \omega_{\mu\nu}(-1)^{\delta^\mu_0+\delta^\nu_0} \Leftrightarrow \omega'_{ij} = \omega_{ij} \wedge \omega'_{k0} = -\omega_{k0} \Rightarrow \theta'^k = \theta^k \wedge \nu'^k = -\nu^k$$

Hence we have that:

$$i\gamma^0 D(L(\mathbf{p}))i\gamma^0 = D(L(-\mathbf{p}))$$

4 Some traces of products of γ-matrices

We have that:

$$\operatorname{tr}\gamma_\mu\gamma_\nu = \operatorname{tr}\{\gamma_\mu, \gamma_\nu\} - \gamma_\nu\gamma_\mu = 2\operatorname{tr}\eta_{\mu\nu}I_4 - \operatorname{tr}\gamma_\nu\gamma_\mu = 2\operatorname{tr}\eta_{\mu\nu}I_4 - \operatorname{tr}\gamma_\mu\gamma_\nu$$

Hence adding on both side we obtain the desired equality:

$$\operatorname{tr}\gamma_\mu\gamma_\nu = \eta_{\mu\nu}\operatorname{tr}I_4 = 4\eta_{\mu\nu}$$

Similarly we have that:

$$\operatorname{tr}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = \ldots$$

5 Energy levels of a relativistic charged spin-0 particle in a harmonic electrostatic potential

1. From these relation we have that:

$$\begin{aligned} X^2|n\rangle &= \frac{1}{2m\Omega}\left(a^2 + (a^\dagger)^2 + \{a, a^\dagger\}\right)|n\rangle \\ &= \frac{1}{2m\Omega}(\sqrt{n}\sqrt{n-1}\,|n-2\rangle + \sqrt{n+1}\sqrt{n+2}\,|n+2\rangle + (n+1)\,|n\rangle + n\,|n\rangle) \end{aligned}$$

Hence we get that:

$$\langle n|X^4|n\rangle = (\langle n|X^2)(X^2|n\rangle) = \frac{1}{(2m\Omega)^2}(n^2 - n + n^2 + 3n + 2 + n + 1 + n) = \frac{2n^2 + 3n + 2}{(2m\Omega)^2}$$

2. We have that:

$$(D_\mu D^\mu + m^2)\Phi = 0 \quad \text{where} \quad D^\mu = \partial^\mu - iqA^\mu$$

Then taking $\Phi = e^{-iEt}\phi$ and plugging it in we get that:

...

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6 The axial current

7 Supersymmetry