HW2 - Probability

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1 Change of variables

1. From the change of variable theorem we know that:

$$f_{U,V}(u,v) = f_{X,Y}(uv, v(1-u))|J|^{-1}$$

Where:

$$J = \begin{vmatrix} \frac{y}{(x+y)^2} & -\frac{x}{(x+y)^2} \\ 1 & 1 \end{vmatrix} = \frac{1}{x+y} = v^{-1}$$

Then replacing in the definition and using the fact that X and Y are independent and hence we can split the joint law we get that:

$$\begin{aligned} f_{U,V}(u,v) &= \frac{uv^{k-1}}{(k-1)!} e^{-uv} 1_{\mathbb{R}^+}(uv) \frac{v^{k-1}(1-u)^{k-1}}{(k-1)!} e^{-v(1-u)} 1_{\mathbb{R}^+}(v(1-u))v \\ &= \frac{e^{-v}\sqrt{v^2}\left((1-u)uv^2\right)^{k-1}}{((k-1)!)^2} 1_{\mathbb{R}^+}(uv) 1_{\mathbb{R}^+}(v-uv) \end{aligned}$$

Then integrating for u on $\mathbb{R}$ gives:

$$f(v) = ...$$

2. ...

2 Order statistics

1. Let $(\Omega_i, \mathcal{F}_i, P_i)$ be the probability space of X_i then define the product probability space as $(\Omega, \mathcal{F}, P)$. Let $(\llbracket 1, n \rrbracket, \mathcal{P}(\llbracket 1, n \rrbracket), P')$ be the probability space of T. Then define the probability space $(\Omega', \mathcal{F}', P')$ as the product probability space of the two previous ones. Then we define:

$$\begin{aligned} X_T : \Omega \times \llbracket 1, n \rrbracket &\longrightarrow \mathbb{R} \\ (\mathbf{x}, i) &\longmapsto x_i \end{aligned}$$

Then let $B \in \mathcal{B}(\mathbb{R})$ then we have that:

$$\{(\omega, i) \in \Omega' : X_T(\omega, i) \in B\} \subset \bigotimes_{i \in \llbracket 1, n \rrbracket} \{\omega_i \in \Omega_i : \exists i, X_i(\omega) \in B\} \times \llbracket 1, n \rrbracket \in \mathcal{F}'$$

Where the belonging to $\mathcal{F}'$ follows from the definition of the product σ-algebra.

2. I think that $(X_{(1)}, \cdots, X_{(n)})$ is ill-defined since there exists no clear order relation on functions which might not even come from the same space. I assume that what was meant was that:

$$\forall \omega \in \Omega, \exists \sigma \in \mathfrak{S}_n, \sigma(X(\omega)) = \sigma\left((X_1(\omega_1), \cdots, X_n(\omega_n))\right) = (X_{\sigma(1)}(\omega_{\sigma(1)}), \cdots, X_{\sigma(n)}(\omega_{\sigma(n)})) \quad \text{is in increasing order.}$$

Then admitting the axiom of choice let $\omega \in \Omega$ then since we have a finite list of real numbers we know from the constructions of the real numbers we can order it. Then we define the permutation σ_ω as the one which sets them in the right order and in case of parity the smaller index goes first. Then we have that σ is a random variable defined as:

$$\begin{aligned} \sigma : \Omega &\longrightarrow \mathfrak{S}_n \\ \omega &\longmapsto \sigma_\omega \end{aligned}$$

We furthermore have that σ is injective and therefore measurable. Hence σ is a well-defined random variable.

3. From the previous question we have that:

$$\begin{aligned}
\mathbb{E}[\varphi(X_{(1)},\cdots,X_{(n)}-X_{(n-1)})] &= \mathbb{E}[\varphi(X_{\sigma(1)},\cdots,X_{\sigma(n)}-X_{\sigma(n-1)})] \\
&= \int_{\omega\in\Omega}\varphi(\sigma(X(\omega)))P'(\sigma(X(\omega)))\mathrm{d}\omega \\
&= \int_{\omega\in\Omega}\varphi(\sigma(X(\omega))P(X(\omega))\mathrm{d}\omega \\
&= \sum_{\sigma\in\mathfrak{S}_n}\int_{\omega\in\Omega}\varphi(X(\omega))P(X(\omega))1_{X_1\leq\cdots\leq X_n}\mathrm{d}\omega \\
&= n!\int_{\omega\in\Omega}\varphi(X(\omega))P(X(\omega))1_{X_1\leq\cdots\leq X_n}\mathrm{d}\omega \\
&= n!\,\mathbb{E}[\varphi(X(\omega))1_{X_1\leq\cdots\leq X_n}]
\end{aligned}$$

4. We write $(X_{(1)},\cdots,X_{(n)})=\sigma(X)$. Then notice that:

$$f_{\sigma(X)}(\mathbf{x})\mathrm{d}\mathbf{x}=\sum_{\mu\in\mathfrak{S}_n}f_X(\mu^{-1}(\mathbf{x}))\mathrm{d}\mathbf{x}=\sum_{\mu\in\mathfrak{S}_n}f_X(\mu(\mathbf{x}))\mathrm{d}\mathbf{x}=\sum_{\mu\in\mathfrak{S}_n}\prod_{i=1}^{n}f_{X_i}(\mu(\mathbf{x})_i)\mathrm{d}\mathbf{x}$$

Where one the last equality we used that the X_i are independent. Then since the X_i are identically distributed we have that $\forall i$, $f_{X_i}=f_{X_1}$. Now since the product commutes we have that the terms inside the sum are all equal up to a permutation of the terms, hence:

$$\sum_{\mu\in\mathfrak{S}_n}\prod_{i=1}^{n}f_{X_i}(\mu(\mathbf{x})_i)\mathrm{d}\mathbf{x}=\sum_{\mu\in\mathfrak{S}_n}\left(\prod_{i=1}^{n}f_{X_1}(x_i)\mathrm{d}x_i\right)=n!\left(\prod_{i=1}^{n}f_{X_1}(x_i)\mathrm{d}x_i\right)=n!f_X(\mathbf{x}')1_{\mathbf{x}'=\mu(\mathbf{x})}\mathrm{d}\mathbf{x}'$$

Where we are free to chose any $\mu\in\mathfrak{S}_n$ since the terms in the product commute. If we fix ourselves with the choice $\mu=\sigma$ we get:

$$\sum_{\mu\in\mathfrak{S}_n}\prod_{i=1}^{n}f_{X_i}(\mu(\mathbf{x})_i)\mathrm{d}\mathbf{x}=n!f_X(\sigma(\mathbf{x}))=n!f_X(\mathbf{x}')1_{\mathbf{x}'=\sigma(\mathbf{x})}\mathrm{d}\mathbf{x}$$

Then plugging this in the definition of the expectancy we get:

$$\begin{aligned}
E[\varphi(\sigma(X))] &= \int_{\mathbf{x}\in\Omega}\varphi(\sigma(X(\mathbf{x})))f_{\sigma(X)}(\mathbf{x})\mathrm{d}\mathbf{x}=n!\int_{\mathbf{x}\in\Omega}\varphi(\sigma(X(\mathbf{x})))f_X(\mathbf{x}')1_{\mathbf{x}'=\sigma(\mathbf{x})}\mathrm{d}\mathbf{x}=n!\int_{\mathbf{x}\in\sigma(\Omega)}\varphi(X(\mathbf{x}))f_X(\mathbf{x})\mathrm{d}\mathbf{x} \\
&= n!\mathbb{E}[\varphi(X)1_\sigma]\ \text{ where }\ 1_\sigma(\mathbf{x})=\begin{cases}1 & \text{if }\ \mathbf{x}=\sigma(\mathbf{x})\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$