# Chapter 1

# TD1

## 1.1 Eulerian/Lagrangian cinematics.

We consider the following velocity field: $\mathbf{v} = (\alpha x, -\alpha y, 0)$.

### 1.1.1

To check whether it is compressible we compute:

$$\boldsymbol{\nabla}\mathbf{v} = (\alpha, -\alpha, 0) \neq \vec{0}$$

To check whether it is irrotational we compute:

$$\omega = \boldsymbol{\nabla} \times \mathbf{v} = 0$$

### 1.1.2

The streamlines follow the following:

$$\mathrm{d}\mathbf{l} \times \mathbf{v} = 0$$

### 1.1.3

### 1.1.4

### 1.1.5

We now consider the following flow:

$$\mathbf{v} = \begin{pmatrix} v_0 e^{kz} \sin(\omega t - kx) \\ 0 \\ v_0 e^{kz} \cos(\omega t - kx) \end{pmatrix}$$

#### a)

We have that:

$$\boldsymbol{\nabla}\mathbf{u} = v_0 e^{kz} \left(-k\cos(\omega t - kx) + k\cos(\omega t - kx)\right) = 0$$

And:

$$\boldsymbol{\nabla} \times \mathbf{u} = \mathbf{e_y} \left(v_0 e^{kz} \left(k\sin(\omega t - kx) - (-1)(-k)\sin(\omega t - kx)\right)\right) = 0$$

The streamlines then respect:

$$\mathrm{d}\mathbf{l} \times \mathbf{v} = 0 \Leftrightarrow \mathrm{d}z v_0 e^{kz} \sin(\omega t - kx) - \mathrm{d}x v_0 e^{kz} \cos(\omega t - kx) = 0$$

Which simplifies:

$$\mathrm{d}z \sin(\omega t - kx) - \mathrm{d}x \cos(\omega t - kx) = 0 \Leftrightarrow z = \int \tan^{-1}(\omega t - kx)\mathrm{d}x = -\frac{\log(|\sin(kx - t\omega)|)}{k} + c$$

#### b)

We have the following: $v_0 \ll \frac{\omega}{k}$.

### 1.1.6

Streaklines follow the following PDE:

$$\begin{cases} \frac{\partial x_p}{\partial t} = v_0 e^{kz}\sin(\omega t - kx) \\ \frac{\partial z_p}{\partial t} = v_0 e^{kz}\cos(\omega t - kx) \end{cases}$$

Hence at the first order we can write:

$$x_p(t) = x_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \cdots \quad \& \quad z_p(t) = z_0 + \varepsilon Z_1 + \varepsilon^2 Z_2 + \cdots$$

Then the velocity field can be written as:

$$\mathbf{v}(x_p(t), z_p(t), t) = \underbrace{\mathbf{v}(x_0, z_0, t)}_{\sim\varepsilon} + \underbrace{\frac{\partial \mathbf{v}}{\partial x}(x - x_0) + \frac{\partial \mathbf{v}}{\partial z}(z - z_0)}_{\sim\varepsilon^2}$$

Then at the first order we get:

$$\begin{cases} \varepsilon\dot{X}_1 = v_0 e^{kz_0}\sin(\omega t - kx_0) \\ \varepsilon\dot{Z}_1 = v_0 e^{kz_0}\cos(\omega t - kx_0) \end{cases} \Leftrightarrow \begin{cases} x_1 = -\frac{v_0}{\omega} e^{kz_0}\cos(\omega t - kx_0) \\ z_1 = \frac{v_0}{\omega} e^{kz_0}\sin(\omega t - kx_0) \end{cases}$$

So we observe a circular trajectory. Now at second order we get:

$$\begin{cases} \dot{x_2} = v_0 e^{kz_0} k(-x_1\cos(\omega t - kx_0) + z_1\sin(\omega t - kx_0)) \\ \dot{z_2} = v_0 e^{kz_0} k(x_1\sin(\omega t - kx_0) + z_1\cos(\omega t - kx_0)) \end{cases} \Leftrightarrow \begin{cases} \dot{x_2} = v_0^2 e^{2kz_0}\frac{k}{\omega} \\ \dot{z_2} = 0 \end{cases}$$

We therefore obtain Stokes' drift.

## 1.2 Dimensional Analysis.

### 1.2.1

#### a)

The parameters of our problem are: $\omega\,[s^{-1}], k\,[m^{-1}], g\,[m.s^{-2}], h\,[m]$

#### b)

By dimensional analysis we guess:

$$\omega(k) \propto \sqrt{gk}$$

#### c)

We can use the above formula only if the height is negligible. Otherwise we would have to add a corrective term:

$$\omega(k) \propto \sqrt{gk} f(k \cdot h)$$

Now making asymptotic analysis we know that:

$$\lim_{h\to+\infty} f(kh) = 1 \quad \& \quad \lim_{h\to 0} f(kh) = 0$$

From physical intuition we also expect the system to respond linearly with $kh$ close to 0 hence a good candidate for our general dispersion relation would be something of the form:

$$\omega(k) \propto \sqrt{gk}\tanh(\alpha kh)$$

#### d)

Similarly when the only force we are considering is surface tension we then have the following parameters:

$$\omega\,[s^{-1}], k\,[m^{-1}], \gamma\,[N.m^{-1}], h\,[m], \rho\,[\text{kg}.m^{-3}]$$

We then can guess something of the form:

$$\Pi_1 = kh, \quad \Pi_2 = \frac{\rho\omega^2 k^3}{\gamma}$$

Pi's theorem then gives:

$$\Pi_2 = f(\Pi_1) \Leftrightarrow \omega^2 = \frac{\gamma}{\rho k^3} f(kh)$$

So in deep water we guess:

$$\omega^2 = \frac{\gamma}{\rho k^3}$$

### 1.2.2

The parameters of the problem are:

$$R\,[m], t\,[s], E[m.s^{-2}]$$