

RQM and QFT: Midterm Homework

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Exercise 1: Some operator identities

- a) To verify the identity is better to compute the derivatives of the function $F(t)$ which are, in general:

$$\left. \frac{d^n F(t)}{dt^n} \right|_0 = e^{tA} [A, \dots [A, [A, B]] \dots] e^{-tB} \Big|_0 = [A, \dots [A, [A, B]] \dots]$$

This relation can be proven simply by induction. For $n = 1$ we have the identity:

$$\left. \frac{dF(t)}{dt} \right|_0 = e^{tA} [A, B] e^{-tB} \Big|_0 = [A, B]$$

and the inductive step can be verified as follows:

$$\begin{aligned} \left. \frac{d^n F(t)}{dt^n} \right|_0 &= \left. \frac{d}{dt} \frac{d^{n-1} F(t)}{dt^{n-1}} \right|_0 = \left. \frac{d}{dt} (e^{tA} [A, \dots [A, [A, B]] \dots] e^{-tB}) \right|_0 = \\ &= e^{tA} A [A, \dots [A, [A, B]] \dots] e^{-tB} - e^{tA} [A, \dots [A, [A, B]] \dots] B e^{-tB} \Big|_0 = \\ &= e^{tA} [A, [A, \dots [A, [A, B]] \dots]] e^{-tB} \Big|_0 = e^{tA} [A, [A, \dots [A, [A, B]] \dots]] e^{-tB} \end{aligned}$$

Then one can express $F(t)$ as his Taylor expansion:

$$F(t) = B + \sum_{n=1}^{\infty} \frac{1}{n!} [A, \dots [A, [A, B]] \dots] t^n$$

and for $t = 1$ it is verified the given expression:

$$e^A B e^{-A} = B + \sum_{n=1}^{\infty} \frac{1}{n!} [A, \dots [A, [A, B]] \dots] \quad (\clubsuit)$$

- b) In the following the commutator between A and B will be indicated as $C = [A, B]$. For proving the equivalence it is useful to consider the following functions for \mathbb{R} to the operator space on which A , B and C live.

$$F(t) = e^{tA} e^{tB} \quad G(t) = e^{tA+tB} e^{\frac{t^2}{2}C} \quad D(t) = e^{\frac{t^2}{2}C+tA+tB}$$

By considering the derivatives w.r.t. t of each one of these operators we obtain:

$$\begin{aligned}
\frac{d}{dt}F(t) &= e^{tA}Ae^{tB} + e^{tA}e^{tB}B = e^{tA}e^{tB}e^{-tB}Ae^{tB} + e^{tA}e^{tB}B = && \text{from } (\clubsuit) \\
&= e^{tA}e^{tB}(A + t[-B, A] + B) = e^{tA}e^{tB}(A + B + tC) = \\
&= F(t)(A + B + tC) \\
\frac{d}{dt}G(t) &= e^{tA+tB}(A + B)e^{\frac{t^2}{2}C} + e^{tA+tB}e^{\frac{t^2}{2}C}tC = \\
&= e^{tA+tB}e^{\frac{t^2}{2}C}(A + B + tC) = G(t)(A + B + tC) \\
\frac{d}{dt}D(t) &= e^{\frac{t^2}{2}C+tA+tB}(tC + A + B) = D(t)(A + B + tC)
\end{aligned}$$

Another thing that one should notice is that:

$$F(0) = \mathbb{1} \quad G(0) = \mathbb{1} \quad D(0) = \mathbb{1}$$

where $\mathbb{1}$ stands for the identity operator on the image of these function. These three functions satisfy the same first order differential equation with the same first initial condition so by the Cauchy–Kowalevski theorem we have the unicity of the solution so the three functions are equal for every t . In particular for $t = 1$ the relation states:

$$e^A e^B = e^{A+B} e^{\frac{1}{2}C} = e^{\frac{1}{2}C+A+B} \quad (\spadesuit)$$

c) The first important relation to consider is:

$$[F, G^\dagger] = -[G^\dagger, F] = \sum_{i,j} f_i g_j^* [a_i, a_j^\dagger] = \sum_{i,j} f_i g_j^* \delta_{i,j} = \sum_i f_i g_i^*$$

The result (\spadesuit) allows us to write for G^\dagger and F :

$$e^{G^\dagger} e^F = e^{\frac{1}{2}[G^\dagger, F]} e^{G^\dagger+F} = e^{-\frac{1}{2} \sum_i f_i g_i^*} e^{G^\dagger+F}$$

then by inverting the exponential of a number we obtain:

$$e^{\frac{1}{2} \sum_i f_i g_i^*} e^{G^\dagger} e^F = e^{G^\dagger} e^F$$

d) One can define the following operators as integrals of the operators depending on a parameter:

$$F = \int d^3q f(q) a(q) \quad G = \int d^3q h(q) a^\dagger(q)$$

then one can evaluate the commutator between F and G :

$$\begin{aligned}
[F, G] &= \left[\int d^3q f(q) a(q), \int d^3q' h(q') a^\dagger(q') \right] = \int d^3q \int d^3q' f(q) h(q') [a(q), a^\dagger(q')] = \\
&= \int d^3q \int d^3q' f(q) h(q') \delta(q - q') = \int d^3q f(q) h(q)
\end{aligned}$$

then once again one can apply the relation (\spadesuit) to find the desired result.

Exercise 2: An example of an asymptotic series

- a) The value of the integral is monotonically decreasing for $g > 0$ since we have that:

$$\forall x \in \mathbb{R} \quad \alpha < \beta \implies e^{-\alpha x^4} < e^{-\beta x^4}$$

the values of the integral for different g s are:

g	0.01	0.1	1
$f(g)$	0.992	0.944	0.772

- b) If one expand the exponential and interchanges the sum and the integral one obtain:

$$\tilde{f}(g) = \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n! \sqrt{\pi}} \int_{-\infty}^{\infty} x^{4n} e^{-x^2} dx \right) g^n = \sum_{n=0}^{\infty} \frac{(-1)^n (4n-1)!!}{n! 2^{2n}} g^n$$

where the last passage is justified because one can rewrite the gaussian integral with the Gamma function:

$$\int_{-\infty}^{\infty} x^{2n} e^{-x^2} dx = 2 \int_0^{\infty} x^{2n} e^{-x^2} dx = \int_0^{\infty} t^{n-\frac{1}{2}} e^{-t} dt = \Gamma\left(n + \frac{1}{2}\right) = \frac{(2n-1)!!}{2^n} \sqrt{\pi}$$

where the change of variable is $t = x^2$. And by recalling the identity about double factorials $(2n-1)!! = 2^n n!$ one has that the asymptotic series is:

$$\tilde{f}(g) = \sum_{n=0}^{\infty} f_n g^n = \sum_{n=0}^{\infty} \left((-1)^n \frac{(4n)!}{2^{4n} (n)! (2n)!} \right) g^n$$

A good estimation of the first five coefficients of the sum is obtained by the means of Stirling's formula:

$$(-1)^n \frac{(4n)!}{2^{4n} (n)! (2n)!} \simeq (-1)^n \frac{2^{2n}}{\sqrt{\pi n}} \left(\frac{n}{e}\right)^n$$

and then the approximated values are:

k	0	1	2	3	4	5
f_k	0	-0.830	3.46	-28	3.39×10^2	-5.44×10^3

- c) The behaviour of the sum as N grows is typical of an asymptotic series. The value of the sum for a fixed value of g at first starts to approach the exact value but then it grows away from the exact value. For bigger values of g the number of terms after which the sum grows exponentially decreases. A graphical confront between the sum of the series and the value of the function has been made in Figure 1.

Exercise 3: A relation between Dirac spinors

- a) First it is necessary to recall the definition of the tensor $\omega_{\mu\nu}$ as:

$$\omega_{ij} = \epsilon_{ijk} \theta^k \quad \omega^{k0} = \xi^k \quad \omega_{\mu\nu} = 0 \text{ otherwise}$$

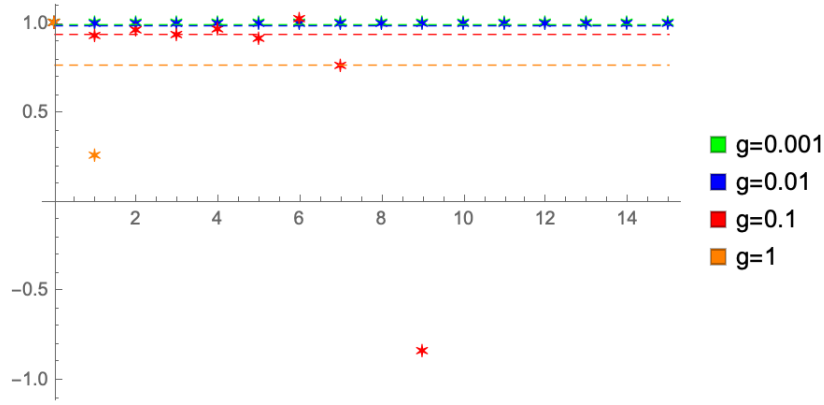


Figure 1: Confront between $f(g)$ and $\tilde{f}(g)$.

where the first six ω coefficients are connected to spatial rotations while the last non vanishing three ω coefficients are connected to boost. One can start from writing the definition of $D(L(\vec{p}))$ inside the RHS of the given expression:

$$\begin{aligned} i\gamma^0(D(L(\vec{p})))i\gamma^0 &= i\gamma^0 \left(\sum_{n \geq 0} \frac{1}{n!} \left(\frac{i}{4} \omega_{\mu\nu} \right)^n (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)^n \right) i\gamma^0 = \\ &= \sum_{n \geq 0} \frac{1}{n!} \left(\frac{i}{4} \omega_{\mu\nu} \right)^n i\gamma^0 (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)^n i\gamma^0 \end{aligned}$$

In the following the analysis will be made for a generic term with index n . By inserting the identity operator and recalling that $(i\gamma^0)^2 = \mathbb{1}$ one gets the following:

$$\begin{aligned} i\gamma^0 (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)^n i\gamma^0 &= (i\gamma^0 \gamma^\mu \gamma^\nu i\gamma^0 - i\gamma^0 \gamma^\nu \gamma^\mu i\gamma^0)^n = (i\gamma^0 \gamma^\mu i\gamma^0 i\gamma^0 \gamma^\nu i\gamma^0 - i\gamma^0 \gamma^\nu i\gamma^0 i\gamma^0 \gamma^\mu i\gamma^0)^n \\ &= \left((-1)^2 (\gamma^\mu)^\dagger (\gamma^\nu)^\dagger - (-1)^2 (\gamma^\nu)^\dagger (\gamma^\mu)^\dagger \right)^n = \left((\gamma^\mu)^\dagger (\gamma^\nu)^\dagger - (\gamma^\nu)^\dagger (\gamma^\mu)^\dagger \right)^n \end{aligned}$$

Now one can check that for the Dirac and the Weyl representation the following relations hold:

$$(\gamma^i)^\dagger = \gamma^i \quad (\gamma^0)^\dagger = -\gamma^0$$

We see that the operator for each term gets a factor of $(-1)^n$ every time one of the two matrices γ^μ or γ^ν is γ^0 . Both matrices can't be simultaneously equal to γ^0 since ω_{00} is identically equal to zero.

In the end this sum can be expressed in terms of a ω' with the opposite the boost components of ω , which, once exponentiated, correspond to $D(L(-\vec{p}))$.

- b) To prove the relation it is useful to start from the relation at spatial moment equal to zero. The definition of the spinors $u(\vec{p}, \sigma)$ and $v(\vec{p}, \sigma)$ is the following. Give the Dirac equation:

$$(\gamma^\mu \partial_\mu + m)\psi_{\pm, \vec{p}, \sigma} = 0 \quad (\bullet)$$

where the subscripts on ϕ denote the "internal quantum numbers" of the state. We have that the free particle should satisfy the mass-energy relation and so for every moment \vec{p} there are two different solutions associated and for every one of the previous there is the spinor degree of freedom σ . By doing the rewriting of the solution as:

$$\psi_{+, \vec{p}, \sigma} = u(\vec{p}, \sigma) e^{ip_\mu x^\mu} \quad \psi_{-, \vec{p}, \sigma} = v(\vec{p}, \sigma) e^{-ip_\mu x^\mu}$$

the two spinors should satisfy the following equations

$$(i\gamma^\mu p_\mu + m)u(\vec{\mathbf{p}}, \sigma) = 0 \quad (-i\gamma^\mu p_\mu + m)v(\vec{\mathbf{p}}, \sigma) = 0$$

Considering first the particle at rest solution the equations become:

$$\begin{aligned} (-i\gamma^0 + \mathbb{1})u(\vec{\mathbf{0}}, \sigma) &= 0 \iff i\gamma^0 u(\vec{\mathbf{0}}, \sigma) = \mathbb{1}u(\vec{\mathbf{0}}, \sigma) \\ (i\gamma^0 + \mathbb{1})v(\vec{\mathbf{0}}, \sigma) &= 0 \iff i\gamma^0 v(\vec{\mathbf{0}}, \sigma) = -\mathbb{1}v(\vec{\mathbf{0}}, \sigma) \end{aligned}$$

since $p_0 = -p^0 = -m$. Now to obtain the finite spatial momentum solution we can perform a boost in a reference frame where the particle has a momentum $\vec{\mathbf{p}}$. This is done with the action of the relative boost operator which is an operator inside the representation of the Lorentz group¹.

The change of frame of reference is, from the fact that ψ is a scalar function:

$$\psi_{\pm, \vec{\mathbf{p}}, \sigma}(x) = D(L(\vec{\mathbf{p}})) \psi_{\pm, \vec{\mathbf{0}}, \sigma}(L(\vec{\mathbf{p}})^{-1}x)$$

The only dependance of ψ from the coordinates is in the exponent. The actual calculation is simplified from the fact that the transformation goes from the rest frame with a particle momentum whose only non zero component is $p_0 = -m$. So we have for the pull-back:

$$\pm ip_0 \left(L(\vec{\mathbf{p}})^{-1} \right)_\nu^0 x^\nu = \pm ip_0 L(\vec{\mathbf{p}})_\nu^0 x^\nu = \pm (L(\vec{\mathbf{p}})p)_\mu x^\mu$$

so the exponent will remain coherent with the previous decomposition of ψ . In the end the ψ function can be written as:

$$\psi_{+, \vec{\mathbf{p}}, \sigma} = e^{+p_\mu x^\mu} D(L(\vec{\mathbf{p}}))u(\vec{\mathbf{0}}, \sigma) \quad \psi_{-, \vec{\mathbf{p}}, \sigma} = e^{-p_\mu x^\mu} D(L(\vec{\mathbf{p}}))v(\vec{\mathbf{0}}, \sigma)$$

Then to obtain the desired result one can start from:

$$\begin{aligned} i\gamma^0 u(\vec{\mathbf{p}}, \sigma) &= i\gamma^0 D(L(\vec{\mathbf{p}}))u(\vec{\mathbf{0}}, \sigma) = i\gamma^0 D(L(\vec{\mathbf{p}}))i\gamma^0 i\gamma^0 u(\vec{\mathbf{0}}, \sigma) = D(L(-\vec{\mathbf{p}}))u(\vec{\mathbf{0}}, \sigma) = u(-\vec{\mathbf{p}}, \sigma) \\ i\gamma^0 v(\vec{\mathbf{p}}, \sigma) &= i\gamma^0 D(L(\vec{\mathbf{p}}))v(\vec{\mathbf{0}}, \sigma) = i\gamma^0 D(L(\vec{\mathbf{p}}))i\gamma^0 i\gamma^0 v(\vec{\mathbf{0}}, \sigma) = -D(L(-\vec{\mathbf{p}}))v(\vec{\mathbf{0}}, \sigma) = -v(-\vec{\mathbf{p}}, \sigma) \end{aligned}$$

Exercise 4: Some traces of products of γ -matrices

By considering the trace of the anti commutator one has:

$$\begin{aligned} 2 \text{Tr } \gamma_\mu \gamma_\nu &= \text{Tr } (\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = \text{Tr } (\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = \text{Tr } (\eta_{\mu\rho} \eta_{\nu\sigma} (\gamma^\rho \gamma^\sigma + \gamma^\sigma \gamma^\rho)) = \\ &= \eta_{\mu\rho} \eta_{\nu\sigma} \text{Tr } (\gamma^\rho \gamma^\sigma + \gamma^\sigma \gamma^\rho) = \eta_{\mu\rho} \eta_{\nu\sigma} \text{Tr } \{ \gamma^\rho, \gamma^\sigma \} = 2\eta_{\mu\rho} \eta_{\nu\sigma} \eta^{\rho\sigma} \text{Tr } \mathbb{1} \end{aligned}$$

and recognising that the $\eta_{\mu\rho} \eta_{\nu\sigma} \eta^{\rho\sigma} = \eta_{\mu\nu}$ this becomes the first relation to prove.

One result that will be useful in the following deduction is that the same commutation relations hold for the γ 's with lower indices.

$$\{ \gamma_\mu, \gamma_\nu \} = \{ \eta_{\mu\rho} \gamma^\rho, \eta_{\nu\sigma} \gamma^\sigma \} = \eta_{\mu\rho} \eta_{\nu\sigma} \{ \gamma^\rho, \gamma^\sigma \} = 2\eta_{\mu\rho} \eta_{\nu\sigma} \eta^{\rho\sigma} \mathbb{1} = 2\eta_{\mu\nu} \mathbb{1}$$

Then by remembering the commutation relation of the γ 's one has:

$$\begin{aligned} \text{Tr } (\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) &= \text{Tr } (\gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\mu) = \\ &= \text{Tr } (-\gamma_\nu \gamma_\rho \gamma_\mu \gamma_\sigma + 2\eta_{\mu\sigma} \gamma_\nu \gamma_\rho) = \\ &= \text{Tr } (\gamma_\nu \gamma_\mu \gamma_\rho \gamma_\sigma - 2\eta_{\mu\rho} \gamma_\nu \gamma_\sigma \mathbb{1} + 2\eta_{\mu\sigma} \gamma_\nu \gamma_\rho \mathbb{1}) = \\ &= \text{Tr } (-\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma + 2\eta_{\mu\nu} \gamma_\rho \gamma_\sigma \mathbb{1} - 2\eta_{\mu\rho} \gamma_\nu \gamma_\sigma \mathbb{1} + 2\eta_{\mu\sigma} \gamma_\nu \gamma_\rho \mathbb{1}) \end{aligned}$$

¹In this case the particular representation we are working on is $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$

From which one has:

$$\text{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = \text{Tr}(\eta_{\mu\nu} \gamma_\rho \gamma_\sigma \mathbb{1} - \eta_{\mu\rho} \gamma_\nu \gamma_\sigma \mathbb{1} + \eta_{\mu\sigma} \gamma_\nu \gamma_\rho \mathbb{1}) = 4\eta_{\mu\nu} \eta_{\rho\sigma} - 4\eta_{\mu\rho} \eta_{\nu\sigma} + 4\eta_{\mu\sigma} \eta_{\nu\rho}$$

The last result that will be proven is that the trace of an odd collection of γ matrices is always zero. To prove this result one should first verify a result given in the lecture notes, that:

$$\{\gamma_5, \gamma^\mu\} = -i(\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^\mu + \gamma^\mu \gamma^0 \gamma^1 \gamma^2 \gamma^3) = -i((-1)^3 \gamma^\mu \gamma^0 \gamma^1 \gamma^2 \gamma^3 + \gamma^\mu \gamma^0 \gamma^1 \gamma^2 \gamma^3) = 0 \quad (\star)$$

since for every two pair of different indices $\mu \neq \nu$ the anti commutator is $\{\gamma^\mu, \gamma^\nu\} = 0$.

Then:

$$\begin{aligned} \text{Tr}(\gamma_{\mu_1} \dots \gamma_{\mu_{2n+1}}) &= \text{Tr}(\gamma_5 \gamma_5 \gamma_{\mu_1} \dots \gamma_{\mu_{2n+1}}) = \\ &= \text{Tr}(\gamma_5 \gamma_{\mu_1} \dots \gamma_{\mu_{2n+1}} \gamma_5) = \\ &= \text{Tr}((-1)^{2n+1} \gamma_5 \gamma_5 \gamma_{\mu_1} \dots \gamma_{\mu_{2n+1}}) = \\ &= (-1) \text{Tr}(\gamma_{\mu_1} \dots \gamma_{\mu_{2n+1}}) \end{aligned}$$

where the factor $(-1)^{2n+1}$ appears for the same reason that it has appeared in calculating the commutator between γ_5 and γ_μ . Since the Trace we where looking for is equal to its inverse then it is necessarily equal to zero.

Exercise 5: Energy levels of a relativistic charged spin-0 particle in a harmonic electrostatic potential

a) One should always remember that the operators a and a^\dagger don't commute. So for $n \geq 2$:

$$\begin{aligned} X^2 |n\rangle &= \frac{1}{2m\Omega} (a^2 + (a^\dagger)^2 + aa^\dagger + a^\dagger a) |n\rangle = \\ &= \frac{1}{2m\Omega} (\sqrt{n(n-1)} |n-2\rangle + \sqrt{(n+1)(n+2)} |n+2\rangle + (2n+1) |n\rangle) \end{aligned}$$

while for the expected value of X^4 :

$$\langle n | X^4 | n \rangle = \frac{1}{(2\pi\Omega)^2} 3(2n^2 + 2n + 1)$$

which can also be computed by the means of Wick's theorem.

b) The free Klein Gordon equation is the following:

$$(-D^\mu D_\mu + m^2)\Phi = 0$$

where the operators are the "covariant derivatives" with the four potential $D_\mu = \partial_\mu - iqA_\mu$. Then the Klein Gordon equation in the presence of a electrostatic field becomes:

$$\begin{aligned} &(-(\partial^t - i\frac{m}{2}\omega^2 x^2)(\partial_t + i\frac{m}{2}\omega^2 x^2) - \nabla^2 + m^2)\Phi = 0 \\ &((\partial_t + i\frac{m}{2}\omega^2 x^2)(\partial_t + i\frac{m}{2}\omega^2 x^2) - \nabla^2 + m^2)\Phi = 0 \\ &\partial_t^2 \Phi + im\omega^2 x^2 \partial_t \Phi - \frac{m^2}{4} \omega^4 x^4 \Phi - \nabla^2 \Phi + m^2 \Phi = 0 \end{aligned}$$

Because A_0 is stationary. This general equation can be simplified with the given ansatz as follows:

$$-\nabla^2 \phi + \left(m\omega^2 E x^2 - \frac{m^2 \omega^4}{4} x^4 - E^2 + m^2 \right) \phi = 0$$

where $\phi = \phi(x, y, z)$.

- c) In the previous equation the coefficients of the term with ϕ depend only on the spatial direction x . To simplify the expression one could say that in the other spatial directions the wave function is the same as the one for a free particle so the following ansatz can be justified:

$$\phi(x, y, z) = e^{ip_z z} e^{ip_y y} u(x)$$

Plugging it inside the equation gives:

$$\frac{d^2 u(x)}{dx^2} + \left(m\omega^2 E x^2 - \frac{m^2 \omega^4}{4} x^4 - E^2 + m^2 + p_z^2 + p_y^2 \right) u(x) = 0$$

To simplify the equation even more one can also redefine the "zero" of the energy to be exactly the value of the constant term and obtain:

$$\frac{d^2 u}{dx^2} + (\alpha x^2 + \beta x^4) u = 0 \quad \alpha = m\omega^2 E \quad \beta = -\frac{m^2 \omega^4}{4}$$

The fact that β means that no matter how small it is the state that one will find with perturbation theory will never be an exact eigenstate of the hamiltonian but only a metastable state. This mean that no matter how small the parameter β the series never converges. By forgetting all of this one can do perturbation theory anyway and find that the un-perturbed energy levels of an harmonic oscillators are:

$$\epsilon_n = \alpha \left(n + \frac{1}{2} \right)$$

and to calculate the first order correction to the energy one needs to find firstly the eigenfunctions, which for the n -th level are:

$$u_n(x) = \frac{\sqrt[4]{\alpha}}{\sqrt[4]{\pi}} \frac{1}{\sqrt{2^n n!}} H_n(\sqrt{\alpha} x) e^{-\frac{1}{2}\alpha x^2}$$

Then the correction is given by:

$$\epsilon_n^1 = \frac{1}{\sqrt{\pi}} \frac{1}{2^n n!} \frac{\beta}{\alpha^2} \int_{-\infty}^{\infty} t^4 H_n^2(t) e^{-t^2} dt$$

This criterion should be justified if $|\beta| \ll |\alpha|$ which means that $m \ll E$.

d)

Exercise 6: The axial current

- a) To fix notations the axial transformation acts as:

$$\psi \mapsto \chi = e^{i\epsilon\gamma_5} \psi$$

From the relation (\star) one has that:

$$e^{i\epsilon\gamma_5} \gamma^\mu = \left(\sum_{n \geq 0} \frac{(i\epsilon)^n}{n!} (\gamma_5)^n \right) \gamma^\mu = \gamma^\mu \left(\sum_{n \geq 0} \frac{(i\epsilon)^n}{n!} (-1)^n (\gamma_5)^n \right) = \gamma^\mu e^{-i\epsilon\gamma_5}$$

Another intermediate result, useful for later is that:

$$(e^{i\epsilon\gamma_5})^\dagger = \left(\sum_{n \geq 0} \frac{(i\epsilon)^n}{n!} (\gamma_5)^n \right)^\dagger = \sum_{n \geq 0} \frac{(-i\epsilon)^n}{n!} (\gamma_5^\dagger)^n = \sum_{n \geq 0} \frac{(-i\epsilon)^n}{n!} (\gamma_5)^n = e^{-i\epsilon\gamma_5}$$

Then after the symmetry $\bar{\psi}$ becomes:

$$\bar{\chi} = \chi^\dagger i\gamma^0 = \psi^\dagger e^{-i\epsilon\gamma_5} i\gamma^0 = \psi^\dagger i\gamma^0 e^{i\epsilon\gamma_5} = \bar{\psi} e^{i\epsilon\gamma_5}$$

b) One has that:

$$\begin{aligned}
S[\chi] &= \int d^4x \bar{\chi}(-\not{\partial} + iq\not{A}(x) + m)\chi = \int d^4x \bar{\psi}e^{i\epsilon\gamma^5}(-\not{\partial} + iq\not{A}(x) + m)e^{i\epsilon\gamma^5}\psi = \\
&= \int d^4x \bar{\psi}(-\not{\partial} + iq\not{A} + me^{i\epsilon\gamma^5}e^{i\epsilon\gamma^5})\psi = \int d^4x \bar{\psi}(-\not{\partial} + iq\not{A} + me^{2i\epsilon\gamma^5})\psi = \\
&= S[\psi] + m \int d^4x \bar{\psi}(e^{2i\epsilon\gamma^5} - 1)\psi
\end{aligned}$$

The two actions are invariant under axial transformation if and only if $m = 0$.

For the case $m = 0$ we have that the axial transformation is a symmetry of both the Action and the Lagrangian so there is an associated conserved current by Noether's theorem.

$$J_5^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \frac{\delta \psi}{\delta \epsilon} = -i\bar{\psi}\gamma^\mu\gamma^5\psi$$

c) The two Dirac eqs. concerning ψ are:

$$(\gamma^\mu \partial_\mu^R - iq\gamma^\mu A_\mu + m)\psi = 0$$

to obtain the Dirac equation for $\bar{\psi}$ one should take the hermitian conjugate of the previous and multiply from the right with $i\gamma^0$ to obtain:

$$\begin{aligned}
0 &= \psi^\dagger(\gamma^\mu \partial_\mu^R - iq\gamma^\mu A_\mu + m)^\dagger i\gamma^0 = \psi^\dagger i\gamma^0 i\gamma^0 \left(\partial_\mu^L (\gamma^\mu)^\dagger + iqA_\mu (\gamma^\mu)^\dagger + m \right) i\gamma^0 = \\
&= \bar{\psi}(\partial_\mu^L \bar{\gamma}^\mu + iqA_\mu \bar{\gamma}^\mu + m) = \bar{\psi}(-\gamma^\mu \partial_\mu^L - iq\gamma^\mu A_\mu + m)
\end{aligned}$$

where the partial derivative operator acts on the right. The equations of motions could have been found from the Lagrangian instead. From the given Lagrangian is easy to find the Dirac equation correspondent to the Dirac adjoint:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) - \frac{\partial \mathcal{L}}{\partial \psi} = -(\partial_\mu \bar{\psi})\gamma^\mu - iq\bar{\psi}\gamma^\mu A_\mu + m\bar{\psi}\mathbb{1} = \bar{\psi}(-\gamma^\mu \partial_\mu^L - iq\gamma^\mu A_\mu + m) = 0$$

while to find the equation for the actual Dirac spinor it is not completely clear what is the meaning of the derivative of the Lagrangian w.r.t. $\partial_\mu \bar{\psi}$. To make it more clear one can integrate by parts the lagrangian and suppose that the boundary conditions permit to eliminate the non-integral term to obtain an equivalent lagrangian:

$$\mathcal{L} = \bar{\psi}(-\gamma^\mu \partial_\mu^R + iq\gamma^\mu A_\mu - m)\psi \sim \bar{\psi}(\gamma^\mu \partial_\mu^L + iq\gamma^\mu A_\mu - m)\psi$$

It is important to notice the change of sign of the first term. Then the Euler-Lagrange equation become:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \right) - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = (\gamma^\mu \partial_\mu^R - iq\gamma^\mu A_\mu + m)\psi = 0$$

which is the same result as above. Now to check the explicit derivative of the axial current:

$$\begin{aligned}
\partial_\mu J_5^\mu &= -i\partial_\mu (\bar{\psi}\gamma^\mu\gamma^5\psi) = -i(\partial_\mu \bar{\psi})\gamma^\mu\gamma^5\psi + i\bar{\psi}\gamma^5\gamma^\mu(\partial_\mu \psi) = \\
&= -q\bar{\psi}\gamma^\mu\gamma^5 A_\mu\psi - im\bar{\psi}\gamma^5\psi - q\bar{\psi}\gamma^5\gamma^\mu A_\mu\psi - im\bar{\psi}\gamma^5\psi = \\
&= -q\bar{\psi}\gamma^\mu\gamma^5 A_\mu\psi - im\bar{\psi}\gamma^5\psi + q\bar{\psi}\gamma^\mu\gamma^5 A_\mu\psi - im\bar{\psi}\gamma^5\psi = -2im\bar{\psi}\gamma^5\psi
\end{aligned}$$

Exercise 7: Supersymmetry

a) The Lagrangian defined in the problem is:

$$\mathcal{L} = -\bar{\psi}(\gamma^\mu \partial_\mu + m)\psi - \left(\partial_\mu \phi^\dagger\right) \partial^\mu \phi - m^2 \phi^\dagger \phi$$

The symmetry defined in the problem acts as follows:

$$\begin{cases} \psi \longrightarrow \psi + \delta\psi = \psi + (\gamma^\mu \partial_\mu - m)\phi\epsilon \\ \phi \longrightarrow \phi + \delta\phi = \phi + \bar{\epsilon}\psi \end{cases}$$

Then by looking at the action of these transformation and by the definition of adjoint one has that:

$$\begin{cases} \bar{\psi} \longrightarrow (\psi + (\gamma^\mu \partial_\mu - m)\phi\epsilon)^\dagger i\gamma^0 = \bar{\psi} - \bar{\epsilon}(\gamma^\mu \partial_\mu + m)\phi^\dagger \\ \phi^\dagger \longrightarrow (\phi + \bar{\epsilon}\psi)^\dagger = \phi^\dagger + \bar{\psi}\epsilon \end{cases}$$

and one can find the from the second term in both expression the infinitesimal variation for the last two quantities.

b) The verification that this change is actually a symmetry of the action comes from substituting the above found relationships in the action.

Of course one will find the previous action and additional terms that should be proven to be zero. The additional terms, to be intended as integrated are the following.

$$\begin{aligned} & -\bar{\epsilon} \left(\gamma^\mu \partial_\mu \phi^\dagger \right) (\gamma^\mu \partial_\mu \psi) - \bar{\epsilon} \left(\gamma^\mu \partial_\mu \phi^\dagger \right) m\psi - \bar{\epsilon} m \phi^\dagger (\gamma^\mu \partial_\mu \psi) - \bar{\epsilon} m^2 \phi^\dagger \phi \\ & - \bar{\psi} \gamma^\mu \partial_\mu (\gamma^\mu \partial_\mu (\phi\epsilon)) + m\bar{\psi} \gamma^\mu \partial_\mu (\phi\epsilon) - m\bar{\psi} \gamma^\mu \partial_\mu (\phi\epsilon) + m^2 \bar{\psi} \phi\epsilon \\ & - \bar{\epsilon} \left(\gamma^\mu \partial_\mu \phi^\dagger + m\phi^\dagger \right) [\gamma^\mu \partial_\mu (\gamma^\mu \partial_\mu (\phi\epsilon)) - m\gamma^\mu \partial_\mu (\phi\epsilon) + m\gamma^\mu \partial_\mu (\phi\epsilon) - m^2 \phi\epsilon] \\ & - (\partial_\mu \phi^\dagger) (\partial^\mu \bar{\epsilon}\psi) - (\partial_\mu \bar{\psi}) \partial^\mu \phi - (\partial_\mu \bar{\psi}) (\partial^\mu \bar{\epsilon}\psi) + m^2 \phi^\dagger \bar{\epsilon}\psi - m^2 \bar{\psi}\epsilon\phi - m^2 \bar{\psi}\epsilon\bar{\epsilon}\psi \end{aligned}$$

Since ϵ is as infinitesimal spinor one only needs to prove that the first order in ϵ is zero, so the remaining terms are:

$$\begin{aligned} & -\bar{\epsilon} \left(\gamma^\mu \partial_\mu \phi^\dagger \right) (\gamma^\mu \partial_\mu \psi) - \bar{\epsilon} \left(\gamma^\mu \partial_\mu \phi^\dagger \right) m\psi - \bar{\epsilon} m \phi^\dagger (\gamma^\mu \partial_\mu \psi) - \bar{\epsilon} m^2 \phi^\dagger \phi \\ & - \bar{\psi} \gamma^\mu \partial_\mu (\gamma^\mu \partial_\mu (\phi\epsilon)) + m\bar{\psi} \gamma^\mu \partial_\mu (\phi\epsilon) - m\bar{\psi} \gamma^\mu \partial_\mu (\phi\epsilon) + m^2 \bar{\psi} \phi\epsilon \\ & - (\partial_\mu \phi^\dagger) (\partial^\mu \bar{\epsilon}\psi) - (\partial_\mu \bar{\psi}) \partial^\mu \phi + m^2 \phi^\dagger \bar{\epsilon}\psi - m^2 \bar{\psi}\epsilon\phi \end{aligned}$$

Where the terms that cancels are marked similarly. One can rewrite some of the terms and obtain:

$$\begin{aligned} & = -\bar{\epsilon} \left(\gamma^\mu \partial_\mu \phi^\dagger \right) (\gamma^\mu \partial_\mu \psi) - \bar{\epsilon} \left(\gamma^\mu \partial_\mu \phi^\dagger \right) m\psi - \bar{\epsilon} m \phi^\dagger (\gamma^\mu \partial_\mu \psi) - \bar{\psi} \gamma^\mu \partial_\mu (\gamma^\mu \partial_\mu \phi) \epsilon + \\ & \quad - \bar{\epsilon} \left(\partial_\mu \phi^\dagger \right) (\partial^\mu \psi) - (\partial_\mu \bar{\psi}) \partial^\mu \phi\epsilon \\ & = -\bar{\epsilon} \left(\gamma^\mu \partial_\mu \phi^\dagger \right) (\gamma^\mu \partial_\mu \psi) - m\bar{\epsilon} \gamma^\mu \partial_\mu (\phi^\dagger \psi) - \bar{\psi} \gamma^\mu \partial_\mu (\gamma^\mu \partial_\mu \phi) \epsilon - \bar{\epsilon} \left(\partial_\mu \phi^\dagger \right) (\partial^\mu \psi) - (\partial_\mu \bar{\psi}) \partial^\mu \phi\epsilon \\ & = -\bar{\epsilon} \left(\gamma^\mu \partial_\mu \phi^\dagger \right) (\gamma^\mu \partial_\mu \psi) - m\bar{\epsilon} \gamma^\mu \partial_\mu (\phi^\dagger \psi) - \bar{\psi} \gamma^\mu \partial_\mu (\gamma^\mu \partial_\mu \phi) \epsilon + \\ & \quad - \bar{\epsilon} \left(\gamma^\mu \partial_\mu \phi^\dagger \right) (\gamma^\mu \partial_\mu \psi) - (\gamma^\mu \partial_\mu \bar{\psi}) (\gamma^\mu \partial_\mu \phi) \epsilon \\ & = -m\bar{\epsilon} \gamma^\mu \partial_\mu (\phi^\dagger \psi) - \gamma^\mu (\bar{\psi} \gamma^\mu \partial_\mu \phi) \epsilon \end{aligned}$$

where the most important relation used is that $\partial^\mu = \gamma^\mu \gamma^\mu \partial_\mu$. The only terms remaining are terms that can be directly integrated and yield a null result. So in the end the two actions are the same.