Advanced Quantum Physics
Week 5

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Tensor product

When a quantum system is characterized by several degrees of freedom, associated to different quantum properties, one can ask what is the relevant Hilbert space to describe it. For example, the state of an electron will depend on its position in $\mathbb{R}^3$ and on its spin. Mathematically, the relevant object to describe the state of the electron is a *tensor product state* and it belongs to a tensor product of Hilbert spaces. We will review the construction and properties of these tensor product states below.

Tensor product of Hilbert spaces

Let $\mathcal{E}$ and $\mathcal{F}$ be two Hilbert spaces. The space $\mathcal{G}$ written as

$$\mathcal{G} = \mathcal{E} \otimes \mathcal{F}$$

is called the *tensor product* of $\mathcal{E}$ and $\mathcal{F}$ if it is possible to associate an element of $\mathcal{G}$ to any pair $|e\rangle \in \mathcal{E}$, $|f\rangle \in \mathcal{F}$. This element is the tensor product of $|e\rangle$ and $|f\rangle$

$$|e\rangle \otimes |f\rangle \in \mathcal{G}$$

The tensor product must satisfy the following properties

- Linearity with respect to multiplication

$$\big(\lambda|e\rangle\big) \otimes |f\rangle = \lambda\big(|e\rangle \otimes |f\rangle\big)$$
$$|e\rangle \otimes \big(\mu|f\rangle\big) = \mu\big(|e\rangle \otimes |f\rangle\big)$$

- Distributivity with vector addition

$$|e\rangle \otimes \big(|f_1\rangle + |f_2\rangle\big) = |e\rangle \otimes |f_1\rangle + |e\rangle \otimes |f_2\rangle$$
$$\big(|e_1\rangle + |e_2\rangle\big) \otimes |f\rangle = |e_1\rangle \otimes |f\rangle + |e_2\rangle \otimes |f\rangle$$

- If $\{|e_m\rangle\}$ is a basis of $\mathcal{E}$ and $\{|f_n\rangle\}$ is a basis of $\mathcal{F}$, then $\{|e_m\rangle \otimes |f_n\rangle\}$ must be a basis of $\mathcal{G}$. In other words, any element of $\mathcal{G}$ is uniquely written as

$$|\Psi\rangle = \sum_{m,n} c_{m,n}\, |e_m\rangle \otimes |f_n\rangle$$

Separable states and entangled states

Any state $|\Psi\rangle$ of a tensor product Hilbert space $\mathcal{G} = \mathcal{E} \otimes \mathcal{F}$ can be written as

$$|\Psi\rangle = \sum_{m,n} c_{m,n}\, |e_m\rangle \otimes |f_n\rangle,$$

where $\{|e_m\rangle\}$ is a basis of $\mathcal{E}$ and $\{|f_n\rangle\}$ is a basis of $\mathcal{F}$.

For some special states that are called *separable states* or *product states*, the expression above can be simplified and rewritten as just the product of two states in $\mathcal{E}$ and $\mathcal{F}$

$$|\Psi_{\text{separable}}\rangle = |e\rangle \otimes |f\rangle,$$

where

$$|e\rangle = \sum_m a_m |e_m\rangle$$
$$|f\rangle = \sum_n b_n |f_n\rangle$$

One can see that the condition for a state to be separable is that its coefficients $c_{m,n}$ in the basis $\{|e_m\rangle \otimes |f_n\rangle\}$ can be written as $c_{m,n} = a_m\, b_n$ for all m, n.

When a state cannot be written as just the product of two states in $\mathcal{E}$ and $\mathcal{F}$, it is called *entangled state*. We will see that separable and entangled states behave differently, especially after a measurement associated to an operator of just one of the two Hilbert space is performed.

Some properties of tensor products

Here are some properties and notations related to tensor products

- If $\mathcal{E}$ and $\mathcal{F}$ are finite-dimensional with respective dimensions $N_\mathcal{E}$ and $N_\mathcal{F}$, then $\mathcal{G} = \mathcal{E} \otimes \mathcal{F}$ has dimension $N_\mathcal{E} \times N_\mathcal{F}$.

- The bra $\langle\Psi|$ associated to the ket $|\Psi\rangle = |e\rangle \otimes |f\rangle$ is

$$\langle\Psi| = \langle e| \otimes \langle f|$$

  Note that by convention we do not change the order of the two states in the bra.

- The scalar product of two separable states $|\Psi\rangle = |e\rangle \otimes |f\rangle$ and $|\chi\rangle = |e'\rangle \otimes |f'\rangle$ is given by

$$\langle\chi|\Psi\rangle = \langle e'|e\rangle\, \langle f'|f\rangle$$

- It can sometimes be cumbersome to always write the full expression of tensor product states. When it does not introduce any ambiguity, we use the shortcut notations

$$|e\rangle \otimes |f\rangle = |e\rangle|f\rangle = |e, f\rangle$$

Tensor product of operators

We can now ask how operators act on a state of the tensor product Hilbert space $\mathcal{G} = \mathcal{E} \otimes \mathcal{F}$. They have the general expression

$$\hat{C}_{\mathcal{G}} = \hat{A}_{\mathcal{E}} \otimes \hat{B}_{\mathcal{F}},$$

where $\hat{A}_{\mathcal{E}}$ and $\hat{B}_{\mathcal{F}}$ are operators belonging to $\mathcal{E}$ and $\mathcal{F}$ respectively. Their action on a separable state $|\Psi\rangle = |e\rangle \otimes |f\rangle$ is

$$\hat{C}_{\mathcal{G}}|\Psi\rangle = \big(\hat{A}_{\mathcal{E}} \otimes \hat{B}_{\mathcal{F}}\big)\big(|e\rangle \otimes |f\rangle\big) = \big(\hat{A}_{\mathcal{E}}|e\rangle\big) \otimes \big(\hat{B}_{\mathcal{F}}|f\rangle\big)$$

This determines the action of $\hat{C}_{\mathcal{G}}$ on all the members of the basis $\{|e_m\rangle \otimes |f_n\rangle\}$ and therefore fully defines the action of the operator on any $|\Psi\rangle \in \mathcal{G}$.

An operator $\hat{A}_{\mathcal{E}}$ that would only act on the degrees of freedom associated to $\mathcal{E}$ can easily be promoted to an operator in $\mathcal{G}$ (and similarly for an operator $\hat{B}_{\mathcal{F}}$ in $\mathcal{F}$)

$$\hat{A}_{\mathcal{E}} \to \hat{A}_{\mathcal{E}} \otimes \mathbb{1}$$
$$\hat{B}_{\mathcal{F}} \to \mathbb{1} \otimes \hat{B}_{\mathcal{F}}$$

When it does not introduce any ambiguity, the operator $\hat{A}_{\mathcal{E}} \otimes \mathbb{1}$ (resp. $\mathbb{1} \otimes \hat{B}_{\mathcal{E}}$) is sometimes simply written $\hat{A}$ (resp. $\hat{B}$).

Two examples

Here are two examples to illustrate the use of tensor products.

- Particle in $\mathbb{R}^3$: If we consider a particle in a one-dimensional space, the relevant Hilbert space to describe it is $\mathcal{L}^2(\mathbb{R})$. A possible basis for this space are the Hermite functions $\{\phi_n\}$. If we now turn to a three-dimensional problem, the state of the system is a vector in $\mathcal{L}^2(\mathbb{R}^3)$. Any state in $\mathcal{L}^2(\mathbb{R}^3)$ can however be written as

$$\langle x, y, z|\Psi\rangle = \sum_{k,l,m} a_{k,l,m}\, \phi_k(x)\phi_l(y)\phi_m(z)$$

We see that the state can be expressed as a sum over separable states $|\phi_k\rangle \otimes |\phi_l\rangle \otimes |\phi_m\rangle$

$$|\Psi\rangle = \sum_{k,l,m} a_{k,l,m}\, |\phi_k\rangle \otimes |\phi_l\rangle \otimes |\phi_m\rangle$$

The state $|\Psi\rangle$ therefore belongs to the tensor product Hilbert space $\mathcal{L}^2(\mathbb{R}) \otimes \mathcal{L}^2(\mathbb{R}) \otimes \mathcal{L}^2(\mathbb{R})$ and we can identify $\mathcal{L}^2(\mathbb{R}^3) = \mathcal{L}^2(\mathbb{R}) \otimes \mathcal{L}^2(\mathbb{R}) \otimes \mathcal{L}^2(\mathbb{R})$.

- Hilbert space for an electron. The degrees of freedom associated with the quantum state of an electron are its three spatial coordinates x, y, z and its spin. The state of the electron is therefore a vector in $\mathcal{L}^2(\mathbb{R}^3) \otimes \mathcal{E}_{\text{spin}}$. The Hilbert space $\mathcal{E}_{\text{spin}}$ is the two-dimensional Hilbert space describing the spin state. The state of the electron can then be written as

$$|\Psi\rangle = \sum_k \sum_{\sigma=\pm} a_{k,\sigma} |\varphi_k\rangle \otimes |\sigma\rangle_z$$

where the $\{\varphi_k\}$ are basis functions for $\mathcal{L}^2(\mathbb{R}^3)$ and the eigenstates $|\pm\rangle_z$ of $\hat{S}_z$ are a basis for the two-dimensional Hilbert space $\mathcal{E}_{\text{spin}}$. The Hamiltonian for an electron in a magnetic field would be written

$$\begin{aligned}\mathcal{H} &= \frac{\hat{p}^2}{2m} \otimes \mathbb{1} - \gamma \left(B_x \otimes \hat{S}_x + B_y \otimes \hat{S}_y + B_z \otimes \hat{S}_z \right) \\ &= \frac{\hat{p}^2}{2m} \otimes \mathbb{1} - \gamma \left(\vec{B} \otimes \hat{\vec{S}} \right)\end{aligned}$$

Note that while the first term only acts on the space degrees of freedom, the second term couples the spin and the coordinates.

Addition of angular momentum

We have previously seen that, for some systems, it can be useful to organize the basis of its Hilbert space using eigenvectors of the angular momentum operators $\{\hat{J}^2, \hat{J}_z\}$. Let us now consider two systems (or a single system characterized by two different degrees of freedom). This could for example be an electron with its orbital and spin degrees of freedom, or the coupled system of the spin of the electron and the spin of the proton of a hydrogen atom. We suppose that the first (resp. second) system is described by a vector in the Hilbert space $\mathcal{E}_1$ (resp. $\mathcal{E}_2$). The Hilbert space relevant for the full system is the tensor product Hilbert space

$$\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$$

Our goal is to find a way to construct a basis of $\mathcal{E}$ based on the total angular momentum of the system and to relate it to the standard angular momentum basis of the two subsystems.

Total angular momentum basis

The total angular momentum for the combined system is

$$\hat{\vec{J}} = \hat{\vec{J}}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \hat{\vec{J}}_2 = \hat{\vec{J}}_1 + \hat{\vec{J}}_2,$$

where $\hat{J}_1$ and $\hat{J}_2$ are the angular momentum operators for the subsystems. It is easy to see that $\hat{\vec{J}}$ indeed satisfies the Lie algebra of an angular momentum

$$\left[\hat{J}_x, \hat{J}_y\right] = i\hbar \hat{J}_z \qquad \left[\hat{J}_y, \hat{J}_z\right] = i\hbar \hat{J}_x \qquad \left[\hat{J}_z, \hat{J}_x\right] = i\hbar \hat{J}_y$$

We therefore know that we can organize the basis of $\mathcal{E}$ using eigenvectors of $\{\hat{J}^2, \hat{J}_z\}$. But we can go one step further by observing that $\hat{J}_1^2$ and $\hat{J}_2^2$ commute with both $\hat{J}^2$ and $\hat{J}_z$ and also between themselves. In other words $\hat{J}_1^2, \hat{J}_2^2, \hat{J}^2, \hat{J}_z$ commute and it is possible to find a basis of $\mathcal{E}$ constructed with eigenvectors $|j_1, j_2; j, m\rangle$ of $\{\hat{J}_1^2, \hat{J}_2^2, \hat{J}^2, \hat{J}_z\}$

$$\begin{aligned}
\hat{J}_1^2 |j_1, j_2; j, m\rangle &= j_1(j_1+1)\hbar^2 |j_1, j_2; j, m\rangle \\
\hat{J}_2^2 |j_1, j_2; j, m\rangle &= j_2(j_2+1)\hbar^2 |j_1, j_2; j, m\rangle \\
\hat{J}^2 |j_1, j_2; j, m\rangle &= j(j+1)\hbar^2 |j_1, j_2; j, m\rangle \\
\hat{J}_z |j_1, j_2; j, m\rangle &= m\hbar |j_1, j_2; j, m\rangle
\end{aligned}$$

Relation to angular momentum basis of $\mathcal{E}_1$ and $\mathcal{E}_2$

Let us go back to the original subsystems. A possible basis for $\mathcal{E}_1$ is made of the eigenvectors $|j_1, m_1\rangle$ of $\{\hat{J}_1^2, \hat{J}_{1z}\}$. Similarly, the eigenvectors $|j_2, m_2\rangle$ of $\{\hat{J}_2^2, \hat{J}_{2z}\}$ are a basis for $\mathcal{E}_2$. This means that the tensor product states $\{|j_1, m_1\rangle \otimes |j_2, m_2\rangle\}$ are a basis of the tensor product Hilbert space $\mathcal{E}$. We have therefore identified two different basis for $\mathcal{E}$

- The basis $\{|j_1, j_2; j, m\rangle\}$. These states are eigenvectors of $\{\hat{J}_1^2, \hat{J}_2^2, \hat{J}^2, \hat{J}_z\}$.

- The basis $\{|j_1, m_1\rangle \otimes |j_2, m_2\rangle\}$. These states are eigenvectors of $\{\hat{J}_1^2, \hat{J}_2^2, \hat{J}_{1z}, \hat{J}_{2z}\}$.

In both cases, the states are eigenvectors of $\hat{J}_1^2$ and $\hat{J}_2^2$. We can therefore separate the Hilbert space $\mathcal{E}$ into subspaces that have fixed values for j_1 and j_2. The question is then, at fixed j_1 and j_2, how the states $|j_1, j_2; j, m\rangle$ are related to the states $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$?

Relating the basis at fixed j_1 and j_2

Let us focus on the subspace of the full Hilbert space associated with fixed values of j_1 and j_2. A possible basis for this subspace are the states $\{|j_1, m_1\rangle \otimes |j_2, m_2\rangle\}$, with $m_1 = -j_1, \ldots, j_1$ and $m_2 = -j_2, \ldots, j_2$. For simplicity, in the following, we will write these states as

$$|j_1, m_1; j_2, m_2\rangle \equiv |j_1, m_1\rangle \otimes |j_2, m_2\rangle$$

The dimension of the subspace is $(2j_1 - 1)(2j_2 - 1)$. Another basis for this subspace is given by the states $\{|j_1, j_2; j, m\rangle\}$. We want to understand what are the possible values for j, m and what is the expression for these states in terms of the $|j_1, m_1; j_2, m_2\rangle$. In other words, we want to find the coefficient to change basis

$$|j_1, j_2; j, m\rangle = \sum_{m_1, m_2} C^{j,m}_{j_1,m_1;j_2,m_2} |j_1, m_1; j_2, m_2\rangle$$
$$C^{j,m}_{j_1,m_1;j_2,m_2} = \langle j_1, m_1; j_2, m_2 | j_1, j_2; j, m\rangle$$

The coefficients $C^{j,m}_{j_1,m_1;j_2,m_2}$ are called Clebsch-Gordan coefficients. In mathematics, they are used a lot in representation theory of Lie groups. The Clebsch-Gordan coefficients are tabulated in many books and in general there is no need to recompute them. It is however a good exercise to learn how they are obtained.

A simple example: two spin-$1/2$

Let us first take the example of two spin-1/2 particles. We will only be interested in the spin degrees of freedom. The full Hilbert space is $\mathcal{E} = \mathcal{E}_1^{\text{spin}} \otimes \mathcal{E}_2^{\text{spin}}$, where $\mathcal{E}_{1,2}^{\text{spin}}$ are the two-dimensional Hilbert spaces describing a spin-1/2. The dimension of $\mathcal{E}$ is $4 = 2 \times 2$. One can construct a basis of tensor product states $|\sigma_1\rangle_z \otimes |\sigma_2\rangle_z$, where $|\sigma_1\rangle_z$ and $|\sigma_2\rangle_z$ are eigenstates of the $\hat{S}_{1z}$ and $\hat{S}_{2z}$ operators respectively. They are also eigenstates of $\hat{S}_1^2$ and $\hat{S}_2^2$ with eigenvalue $3\hbar^2/4$. We use the following shortcut notation for this basis

$$\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$$

We now want to find the common eigenstates of $\{\hat{S}_1^2, \hat{S}_2^2, \hat{S}^2, \hat{S}_z\}$, where $\hat{\vec{S}} = \hat{\vec{S}}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \hat{\vec{S}}_2$. We will write these eigenstates $|S, M\rangle$

$$\hat{S}^2 |S, M\rangle = S(S+1)\hbar^2$$
$$\hat{S}_z |S, M\rangle = M\hbar$$
$$\hat{S}_1^2 |S, M\rangle = \frac{3}{4}\hbar^2$$
$$\hat{S}_2^2 |S, M\rangle = \frac{3}{4}\hbar^2$$

We have that $\hat{S}_z = \hat{S}_{1z} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{S}_{2z}$. Therefore the four states $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$ are eigenvectors of $\hat{S}_z$ with eigenvalues $\hbar, 0, 0, -\hbar$ respectively and we see that M can only take three values $M = -1, 0, 1$. Let us call $\mathcal{E}_M$ the eigensubspace associated to the eigenvalue $M\hbar$ of the $\hat{S}_z$ operator. We have that $\dim \mathcal{E}_{-1} = \dim \mathcal{E}_1 = 1$ and $\dim \mathcal{E}_0 = 2$. Because $\mathcal{E}_1$ is one-dimensional, the state $|++\rangle$ must be both an eigenvector of $\hat{S}_z$ and of $\hat{S}^2$. We can indeed check that

$$\begin{aligned} \hat{S}^2|++\rangle &= \left(\hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2\right)|++\rangle = \left(\frac{1}{2}\big[\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+\big] + \hat{S}_z^2\right)|++\rangle \\ &= \frac{1}{2}\hat{S}_+\hbar\big(|+-\rangle + |-+\rangle\big) + \hbar^2|++\rangle = \frac{\hbar^2}{2}2|++\rangle + \hbar^2|++\rangle = 2\hbar^2|++\rangle \end{aligned}$$

The state $|++\rangle$ is an eigenvector of $\hat{S}^2$ with eigenvalue $2\hbar^2 = 1(1+1)\hbar^2$. We identify $|S=1, M=1\rangle = |++\rangle$. The same calculation for $|--\rangle$ shows that it is also an eigenstate of $\hat{S}^2$ with eigenvalue $2\hbar^2$ and we identify $|S=1, M=-1\rangle = |--\rangle$. We are now left with the two-dimensional subspace $\mathcal{E}_0$. The states $\{|+-\rangle, |-+\rangle\}$ are a basis of this subspace. These states are however not eigenstates of $\hat{S}^2$. Indeed

$$\begin{aligned} \hat{S}^2|+-\rangle &= \left(\frac{1}{2}\big[\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+\big] + \hat{S}_z^2\right)|+-\rangle = \hbar^2\big(|+-\rangle + |-+\rangle\big) \\ \hat{S}^2|-+\rangle &= \left(\frac{1}{2}\big[\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+\big] + \hat{S}_z^2\right)|-+\rangle = \hbar^2\big(|+-\rangle + |-+\rangle\big) \end{aligned}$$

From the equations above it is however easy to construct eigenstates of $\hat{S}^2$

$$\begin{aligned} &\hat{S}^2\big(|+-\rangle + |-+\rangle\big) = 2\hbar^2\big(|+-\rangle + |-+\rangle\big) \\ &\hat{S}^2\big(|+-\rangle - |-+\rangle\big) = 0 \end{aligned}$$

The first state corresponds to $S = 1$, while the second state has $S = 0$. To summarize we have decomposed the original Hilbert space into a three $S = 1$ states and an $S = 0$ state. We sometimes write $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$. We found the following expression for the states $|S, M\rangle$

$$\begin{aligned} |1,1\rangle &= |++\rangle \\ |1,0\rangle &= \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right) \\ |1,-1\rangle &= |--\rangle \\ |0,0\rangle &= \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) \end{aligned}$$

The basis of eigenvectors of $\{\hat{S}^2, \hat{S}_z\}$ has three states with $S = 1$ and $M = -1, 0, 1$. They form what is called a *spin triplet*. The fourth state has $S = 0$ and $M = 0$ and is called a *spin singlet*. We can observe that S can only take two values $S = 0, 1$. Let us note that the spin triplet is symmetric under the exchange of the two spins, while the singlet is antisymmetric.

Construction for generic j_1 and j_2

In the example above, it was easy to diagonalize $\hat{S}^2$ and find the expression of the $|S, M\rangle$ in the tensor product basis. Let us now see how the construction can be performed for generic values of j_1 and j_2. To simplify the notation, we will write the elements of the $\{\hat{J}_1^2, \hat{J}_2^2, \hat{J}\cdot\hat{J}_z\}$ and $\{\hat{J}_1^2, \hat{J}_2^2, \hat{J}_{1z}, \hat{J}_{2z}\}$ basis in the following way

$$
\begin{aligned}
|j, m\rangle &\equiv |j_1, j_2; j, m\rangle && \text{(with a comma!)} \\
|m1; m2\rangle &\equiv |j_1, m_1; j_2, m_2\rangle && \text{(with a semicolon!)}
\end{aligned}
$$

As a first step to find the expression of the $|j, m\rangle$ states, we can divide the Hilbert space in eigensubspaces of $\hat{J}_z$. The eigensubspace associated to the eigenvalue $m\hbar$ will be denotes $\mathcal{E}(m)$. Because $\hat{J}_z = \hat{J}_{1z} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{J}_{2z}$, the states $|m_1; m_2\rangle$ are also eigenstates of $\hat{J}_z$ with eigenvalues $m\hbar = (m_1 + m_2)\hbar$. The allowed values for m_1 and m_2 are $m_1 = -j_1, \ldots, j_1$ and $m_2 = -j_2, \ldots, j_2$. Let us also recall a general result that we have obtained earlier: the dimension of the eigensubspaces $\mathcal{E}_{j,m}$ associated to eigenvalues $j(j+1)\hbar^2$ of $\hat{J}^2$ and $m\hbar$ of $\hat{J}^2$ is independent of m.

The first tower of states with $j = j_1 + j_2$

We see that the largest possible value for m is $j_1 + j_2$. There is only one state with $m = j_1 + j_2$, namely $|j_1; j_2\rangle$. Therefore $\dim \mathcal{E}(j_1 + j_2) = 1$. The state $|j_1; j_2\rangle$ must then be an eigenstate of $\hat{J}^2$. One can check that

$$\hat{J}^2 |j_1; j_2\rangle = (j_1 + j_2)(j_1 + j_2 + 1)\hbar^2 |j_1; j_2\rangle$$

Quite naturally, the state $|j_1; j_2\rangle$ is an eigenvector of $\hat{J}^2$ with $j = j_1 + j_2$ and we can make the first identification

$$|j_1 + j_2, j_1 + j_2\rangle = |j_1; j_2\rangle$$

This is the only state in $\mathcal{E}_{j_1+j_2, j_1+j_2}$ and we thus have $\dim \mathcal{E}_{j_1+j_2,m} = 1$ for all values of m. By repeatedly acting on this state with $\hat{J}_-$ we will get all the states

$$|j_1 + j_2, j_1 + j_2\rangle \quad \ldots \quad |j_1 + j_2, -(j_1 + j_2)\rangle$$

This is a total of $2(j_1 + j_2 + 1)$ states.

The second tower of states with $j = (j_1 + j_2) - 1$

We now focus on the subspace $\mathcal{E}((j_1 + j_2) - 1)$. The dimension of this subspace is 2. We know one of the states of this subspace: $|j_1 + j_2, j_1 + j_2 - 1\rangle$ obtained from the procedure above. We can find another state $|\Psi\rangle$ with $m = j_1 + j_2 - 1$ and orthogonal to $|j_1 + j_2, j_1 + j_2 - 1\rangle$. This new state must be an eigenstate of $\hat{J}^2$. Because $m = j_1 + j_2 - 1$ we have that $j \geq j_1 + j_2 - 1$. But on the other hand j cannot be equal to $j_1 + j_2$, otherwise we would have $\dim \mathcal{E}_{j_1+j_2, j_1+j_2-1} = 2$ which is contradicting the result obtained above.

We conclude that $|\Psi\rangle$ is the state $|j_1+j_2-1, j_1+j_2-1\rangle$. Acting on this state with $\hat{J}_-$ we get all the states

$$|j_1+j_2-1, j_1+j_2-1\rangle \quad \ldots \quad |j_1+j_2-1, -(j_1+j_2-1)\rangle$$

This is a total of $2(j_1+j_2-1+1)$ states. Because there is only one state in $\mathcal{E}_{j_1+j_2-1,j_1+j_2-1}$ we conclude that $\dim \mathcal{E}_{j_1+j_2-1,m} = 1$ for all values of m.

More towers of states

We can continue this procedure by considering the next subspace $\mathcal{E}((j_1+j_2)-2)$ which has dimension 3. We have already constructed two states in this subspace. We can find a third state that is orthonormal to these two. For the same reasons as above, this state will be $|j_1+j_2-2, j_1+j_2-2\rangle$. Acting with $\hat{J}_-$ we get $2(j_1+j_2-2+1)$ states with $j = j_1+j_2-2$. We can continue this construction as long as $\dim \mathcal{E}(m-1) > \dim \mathcal{E}(m)$. It is easy to see that the last tower of states will be the one corresponding to $j = |j_1 - j_2|$ (see Fig. 1). All in all, we will have generated all the states with

$$j = (j_1+j_2), \ldots, |j_1-j_2|$$

The total number of constructed states is

$$\big[2(j_1+j_2)+1\big] + \big[2(j_1+j_2-1)+1\big] + \ldots + \big[2|j_1-j_2|+1\big] = (2j_1+1)(2j_2+1)$$

This number is the dimension of the full Hilbert space, as computed from the number of states in the tensor product state basis $\{|j_1, m_1\rangle \otimes |j_2, m_2\rangle\}$. We have therefore a confirmation that the states $|j, m\rangle$ that we have constructed are indeed a new basis for the Hilbert space.

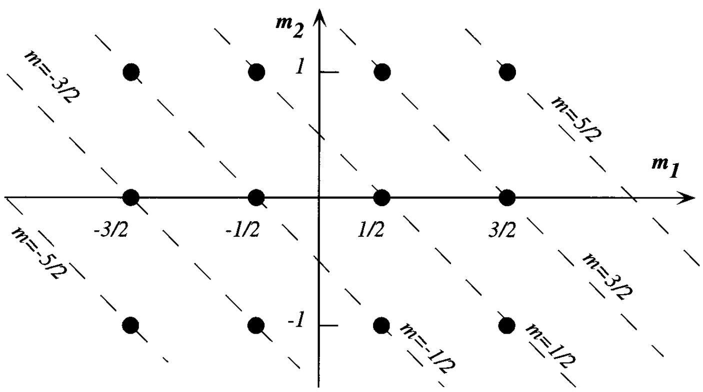

Figure 1: Representation of the tensor product basis (for $j_1 = 3/2$ and $j_2 = 1$). Every point is associated to a given $|m_1; m_2\rangle$ state. We can see that $\dim \mathcal{E}(5/2) = 1$. The dimension increases by 1 when we decrease m by 1, $\dim \mathcal{E}(3/2) = 2$. This is also true for $\dim \mathcal{E}(1/2) = 3$. When $m < |j_1 - j_2|$, the dimension does not increase anymore and the construction of towers of states stops at $j = |j_1 - j_2|$.

Summary

If we consider two systems characterized by angular momenta $\hat{J}_1$ and $\hat{J}_2$. We suppose that the systems both have well-defined values for j_1 and j_2. The Hilbert space for this combined system is for example spanned by $\{|j_1, m_1\rangle \otimes |j_2, m_2\rangle\}$ and has dimension $(2j_1+1)(2j_2+1)$. We can also construct a basis of common eigenvectors to $\{\hat{J}_1^2, \hat{J}_2^2, \hat{J}^2, \hat{J}_z\}$. These eigenvector can be expressed in the original tensor product basis

$$|j_1, j_2; j, m\rangle = \sum_{m_1, m_2} C^{j,m}_{j_1,m_1;j_2,m_2} |j_1, m_1; j_2, m_2\rangle$$

We have shown that j cannot take any value

$$j = (j_1 + j_2), \ldots, |j_1 + j_2|$$

Why is this construction useful? We could in principle work in the tensor product basis. However, many combined systems will not be invariant under the rotation of a single subsystem but rather under a full rotation of the combined system. The infinitesimal generators for these total rotations are the $\hat{J}$ operators and it makes sense to use these to organize the basis of the Hilbert space.

As an illustration, we can investigate what is known as the hyperfine structure of hydrogen. We will discuss this more later. In the hydrogen atom, the interaction between the spin of the proton and the spin of the electron leads to a small dipole-dipole interaction energy. It can be described by a term

$$\mathcal{H}_{\text{hyperfine}} = \frac{A}{\hbar^2} \hat{\vec{S}}_{\text{proton}} \cdot \hat{\vec{S}}_{\text{electron}},$$

where A is some constant with units of energy. This Hamiltonian does not commute with single rotations of the proton spin or single rotations of the electron spin. It does however commute with a combined rotation of the proton and electron spin. The corresponding generator is $\hat{\vec{S}} = \hat{\vec{S}}_{\text{proton}} + \hat{\vec{S}}_{\text{electron}}$. One can indeed shown that

$$\mathcal{H}_{\text{hyperfine}} = \frac{A}{2\hbar^2}\left[\hat{S}^2 - \hat{S}^2_{\text{proton}} - \hat{S}^2_{\text{electron}}\right] = \frac{A}{2\hbar^2}\left[\hat{S}^2 - \frac{3}{2}\hbar^2\right]$$

This makes it clear that $[\mathcal{H}_{\text{hyperfine}}, \hat{\vec{S}}] = 0$. One can think of this system as a combined system of two $j_1 = 1/2$, $j_2 = 1/2$ angular momenta. Because of the rotational invariance, we know that we can find common eigenvectors of $\{\mathcal{H}_{\text{hyperfine}}, \hat{S}^2, \hat{S}_z\}$. Our previous construction of states $|S, M\rangle$ from two spin-1/2, allows to immediately find the eigenvectors and their energy

$$|1,1\rangle, |1,0\rangle, |1,-1\rangle \rightarrow E_{\text{triplet}} = \frac{A}{2}\big(2 - \frac{3}{2}\big)$$
$$|0,0\rangle \rightarrow E_{\text{singlet}} = \frac{A}{2}\big(0 - \frac{3}{2}\big)$$

We see that the energy spectrum has three degenerate states from the spin triplet and one state from the spin singlet. The energy splitting between these states is A. In practice, A is a small energy, $A \simeq 5.87\,10^{-6}$eV. The possibility to excite transitions between these two hyperfine states is at the origin of the hydrogen maser and atomic clocks for other atoms.