TD1 Introduction to Dynamical Systems, ENS

Dynamics and Modelling, 2020

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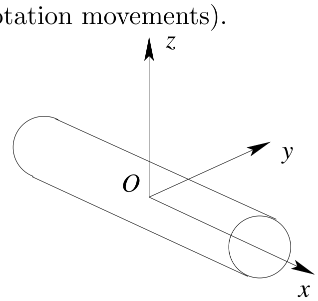
1 The spinning tube, an optical illusion

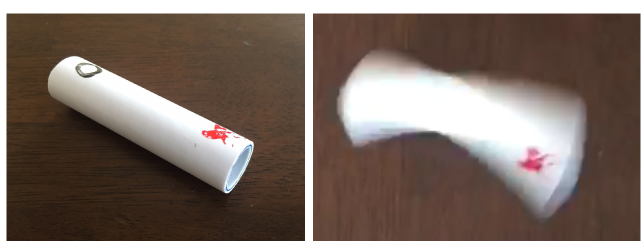
Let us consider the problem of the rotation of a tube of length L and radius R on a table, rotating around two axes. One of them along the axis of symmetry of the tube Os, the second, normal to the table Oz.

A surprising phenomenon occurs when the cylinder is rotating. While both ends are marked using different colours, only one of them is visible. In addition, this marker seems to appear exactly every quarter turn of the cylinder around the Oz axis...

This is a remarkably reproducible observation!

We will note ω_z the rotation around the axis Oz and ω_s the rotation along the axis Os , the axis of symmetry of the cylinder. We will attach the $Oxyz$ reference frame to the center of the cylinder (in order to retain only rotation movements).





1. We will assume the cylinder initially aligned with the Ox axis, and markers turned upwards. Calculate the trajectory of both markers associated with the ω_s rotation in the case of $\omega_z = 0$.

2. By combining it with the ω_z rotation, we can deduct trajectories for marker for ω_s and $omega_z$ fixed.

3. Graphically represent these trajectories with `Python`. Express these trajectories in the form of two autonomous dynamical systems (one for each marker).

4. This dynamical system is complicated (non-linear and high-order). However, the information we are interested in is only the moment when markers are visible. Represent, with `Python` the place *seen from above* of markers when they are on top of the cylinder, $z = R$ (this is a Poincaré cross-section!).

5. Under what condition will we see the marker exatly four times per turn? Can you interpret in mechanical terms why this condition is realised? (indication: $L = 8\,R\,$.)

6. Represent the velocity associated with each marker at those moments. Why do we see only one of the two markers?

2 Beam buckling and critical slowing down

Simplified modeling of beam buckling is being considered. X measures the maximum amplitude of the movement and μ is a parameter relateded to the force exerted on the beam. For a certain value of this parameter, the rectilinear beam ceases to be stable, and the beam buckles either up or down.

This problem can be modeled by a low order dynamical model

$$\dot{X} = \mu\, X - X^3\,.$$

1. Identify the fixed points of this system depending on the value of μ.

2. Study their stability (at fixed μ).

3. Graphically represent the phase portrait of this problem, using `Python`, for $\mu = -10\,$, $\mu = 0\,$, $\mu = 10\,$.

4. Graphically represent the potential of this problem, using `Python`, for $\mu = -10\,$, $\mu = 0\,$, $\mu = 10\,$.

5. Let us now consider the vicinity of instability, for simplicity, we will assume $\mu = 0$. Show that $X \to 0$ when $t \to +\infty\,$, but that the decay is not exponential in this case.