Topology in physics

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Chapter 1

Introduction

1.1 Brief introduction to topology

1.1.1 Geometry vs Topology.

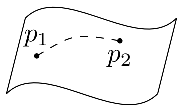


Figure 1.1: A simple example of a manifold and the complexity of the notion of distance on such an object.

Take Figure 1.1.1 for example. A geometrical study would try to indentify local details of the structure, the radius of curvature at each point on the boundary for example. In contrast topology is only interested in the global view of the object, for example does the object contain a hole?

Local details: differential geometry

Again we take the Figure 1.1.1 as reference. A common question would be for example what is the distance in between p_1 and p_2. However there is no obvious answer to such a question. One first has to define what is called a metric on the space, which is an object that will define distance in this space. The mathematical construction is written as $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$. Let's say now that we want to compute gradients of vectors. To do such a thing one must understand how vectors change in the space. Hence we need a notion of transport for vectors. One way to do so is what is called parallel transport. The idea is to drag a vector along a path in the space to another position whilst not changing the vector. Notice in Figure 1.1.1 however that parallel transport depends on the path used to transport the vector. This concept is called mistmatch under parallel transport.

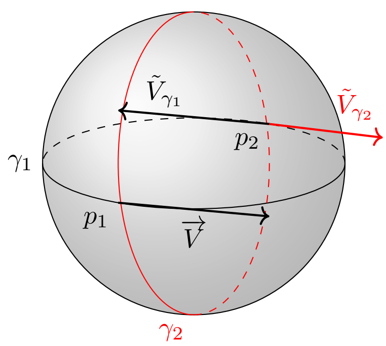


Figure 1.2: A simple example of mismatch under parallel transport. Notice that the vector $\vec{V}$ transported through γ_1 (resp. γ_2) leads to $\tilde{V}_{\gamma_1}$ (resp. $\tilde{V}_{\gamma_2}$) and however $\tilde{V}_{\gamma_1} \neq \tilde{V}_{\gamma_2}$.

Actually the way mathematician define curvature is by saying that a vector parallely transported around a loop will be mismatched with itself.

Riemann curvature tensor

Suppose we have a very big manifold. We are now interested at what happens very close to a certain point p as shown in Figure 1.1.1. Now at the lowest order in the perturbation we a vector $\overrightarrow{V}$ at p will be deformed from p to p_{end} as given by:

$$\tilde{V}^{\mu}_{\gamma_2}(p_{end}) - \tilde{V}^{\mu}_{\gamma_1}(p_{end}) \equiv V^{\alpha} R^{\mu}{}_{\alpha\lambda\nu} \varepsilon^{\lambda} \delta^{\nu}$$

The tensor R is the Reimann curvature tensor and is a local property of the manifold.

Curvature of surface

Notice now that the Reimann tensor has an upper and a lower index hence something intersting to do is to contract these indeces. In fact we have that:

$$R^{\mu}{}_{\alpha\lambda\nu} \rightarrow R^{\lambda}{}_{\alpha\lambda\nu} = [Ric]_{\mu\nu}$$

Which is called the Ricci tensor and is used in General Relativity. In order to introduce a scalar quantity of curvature one must use a notion of distance given by the $g^{\mu\nu}$ tensor. Which we get as:

$$[Ric]_{\mu\nu} \rightarrow \mathcal{R} = g^{\mu\nu}[Ric]_{\mu\nu}$$

Which also enters into Einstein's equations. Now in the special case of 2D surface we have that:

$$\mathcal{R} = 2\mathcal{K} \text{ where } K \text{ is the gaussian curvature of the surface.}$$

Hence in 2 dimensions we have an intuitive interpretation of the contraction of the Ricci tensor.

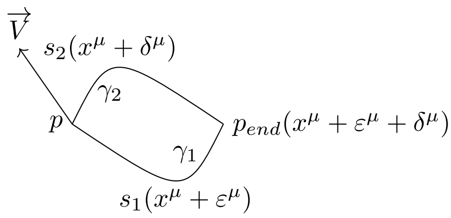


Figure 1.3: The graphical intuition behind the microscopic reasoning leading to the Riemann curvature tensor.

1.1.2 Global view: topology

The idea now with topology is to ignore local details and interest ourselves only with features that are robust against smooth perturbations. We are especially interested in classification of objects in term of their general properties. Characteristics that allow us to split objects into classes are called topological invariants and they are the quantization of the properties which are robust against smooth perturbations. An example of such an invariant for surfaces is the $\nu =$ "genus" which corresponds to the number of holes or handles. Notice that this is indeed a general property of the object and not a local one. It is however possible to connect geometrical properties to topological ones by integrating them over the manifold. This is done through the Gauss-Bonnet theorem (the following statement works only for surface without edges however it can be generalized to any manifold):

$$\underbrace{\frac{1}{2\pi}\int_{\mathcal{M}} \mathcal{K}\mathrm{d}s}_{\text{local gaussian curvature}} = \underbrace{2(1-\nu)}_{\text{global genus}}$$

Homotopy classes and winding numbers.

Let $\mathcal{M}$ be a general manifold. Now a path is defined as a map from $[0,1]$ to the manifold. A loop is a path for which $\gamma(0) = \gamma(1)$. The goal now is to look at all possible loops and characterize them. Now notice in Figure 1.1.2 that we can classify the paths as such: $\gamma_1 \equiv \gamma_3$ however $\gamma_1 \not\equiv \gamma_2 \not\equiv \gamma_4$ and $\gamma_2 \not\equiv \gamma_4$. We characterize the paths with the number of loops that they encircle and the number of handles that they encircle. Hence we divide groups with two integers $n_1, n_2 \in \mathbb{Z}$ where n_1 is the number of holes and n_2 the number of handles. Then we have that γ_1 and γ_3 are part of $[\alpha_{0,0}]$, γ_2 is part of $[\alpha_{1,0}]$ and γ_4 in $[\alpha_{0,1}]$. Then the set of all classes $\{[\alpha_{n_1,n_2}]\}$ is called the fundamental homotopy group of $\mathcal{M}$ denoted by $\Pi_1(\mathcal{M})$ which is robust under perturbation. Now let's take for example $R^2 \setminus (0,0)$. Then we define the winding number of a path as:

$$W[\gamma] = \frac{1}{2\pi}\int_I \frac{\mathrm{d}\theta(t)}{\mathrm{d}t}\mathrm{d}t = n \in \mathbb{Z} \text{ where } \theta(t) \text{is the angle spanned by } \gamma(t) \text{around the origin.}$$

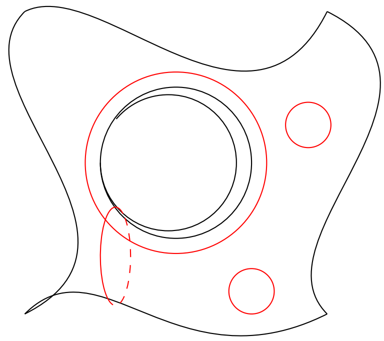
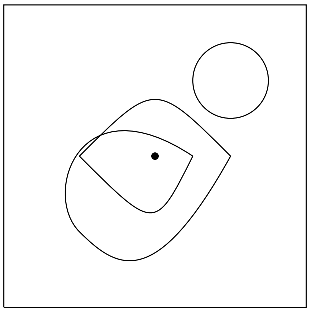


Figure 1.4: ..



Intuitevly W corresponds to the angle spanned by $\gamma(t)$ in units of 2π. Then we have that $\Pi_1(\mathbb{R}^2 \setminus (0,0)) = \mathbb{Z} \equiv \Pi_1(S^1) \equiv \Pi_1(\mathbb{C} \setminus \{0\}) \equiv \Pi_1(U(1))$

Winding number for complex maps.

Take a loop $z(t) : [0,1] \to \mathbb{C} \setminus \{0\}$ where $z(0) = z(1)$. Then introduce $x(t) = \operatorname{Re} z(t)$ and $y(t) = \operatorname{Im} z(t)$. Now define a new function:

$$u(t) = \frac{z(t)}{|z(t)|} = e^{i\theta(t)} \in U(1)$$

Notice that it is sufficient to characterize u in order to characterize z and $u(t) : [0,1] \to S^1$. Then the winding number of u can easily be computed as:

$$W = \frac{1}{2\pi} \int_I \mathrm{d}t \left(\frac{\mathrm{d}\theta}{\mathrm{d}t} \right) = \frac{1}{2\pi i} \int_I \mathrm{d}t u^* \left(\frac{\mathrm{d}u}{\mathrm{d}t} \right) \in \mathbb{Z}$$

Hence as expected we get that $\Pi_1(S^1) = \mathbb{Z}$.

Quantum Physics.

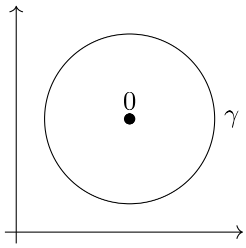
Take any wave-function $\psi(\mathbf{r})$ and we admit that ψ is well defined for $\mathbf{r} \in \mathbb{R}^2 \setminus \{0\}$ hence there can potentially be a singularity at the origin. We write the wave-function as: $\psi(\mathbf{r}) = \sqrt{\rho} e^{i\theta(\mathbf{r})}$. Then we define the loop γ as shown in Figure 1.1.2. Then the winding number which characterizes the winding of our wavefunction is given by:

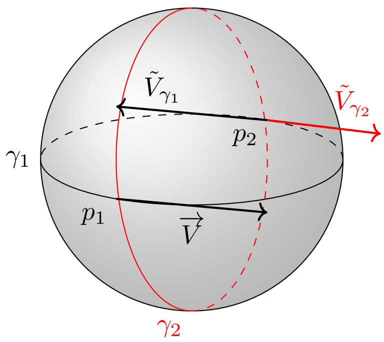
$$W = \frac{1}{2\pi} \oint_\gamma (\boldsymbol{\nabla}\theta) \cdot \mathrm{d}\mathbf{l} = \frac{1}{2\pi \rho i} \oint \psi^* (\boldsymbol{\nabla}\psi) \cdot \mathrm{d}\mathbf{l}$$

Notice however that the last term corresponds to the circulation of current of $\psi(r)$ around the origin then the quantization of the winding number immediately relates to the single-valuedness of the wave-function.

Quantized vortices in superfluids.

Take a conventional fluid rotation at an angular frequency $\boldsymbol{\Omega}$. Then the velocity field is given by $\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{r}$. Then the vorticity of the flow is given by $\boldsymbol{\nabla} \times \mathbf{v} = 2\boldsymbol{\Omega} = cst$. Now in the case of a superfluid the system as





a whole is described by a many-body wavefunction $\psi(\mathbf{r}) = \sqrt{\rho}e^{i\phi(\mathbf{r})}$. Then the velocity field related to this wave-function we get that:

$$\mathbf{v} = \frac{\hbar}{2mi\rho}\left[\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right] = \frac{\hbar}{m}\boldsymbol{\nabla}\phi$$

Then naively one might think that the vorticity is given by $\boldsymbol{\nabla}\times\mathbf{v}$ and since $\boldsymbol{\nabla}\times\boldsymbol{\nabla} = 0$ then the vorticity of a superfluid must vanish. However this is true if and only if the phase ϕ is a smooth function. However this is not always the case, it is possible that $\phi(\mathbf{r})$ presents a singularity line (resp. point) in 3D (resp. 2D) around which it changes by $2\pi n$ where $n \in \mathbb{Z}$ since ψ must be single valued. With such a phase it is possible to have a non-vanishing vorticity. An simple example in 2D is as follows. Let $\psi(x, y) \sim x + iy$ then the phase presents a singularity at the origin and the winding number of the wavefunction is 1. Generally if we compute the integral of the vorticity over a small region Σ:

$$\int_\Sigma \boldsymbol{\nabla}\times\mathbf{v}\mathrm{d}\mathbf{s} = \oint_{\mathrm{d}\varepsilon}\mathbf{v}\mathrm{d}\mathbf{l} = \frac{\hbar}{m}\oint_{\mathrm{d}\Sigma}\boldsymbol{\nabla}\phi\cdot\mathrm{d}\mathbf{l} = \frac{h}{m}n \text{ where } n \in \mathbb{Z}$$

Similarly the computation for the velocity field yields:

$$\oint_{\mathrm{d}\Sigma}\mathbf{v}\cdot\mathrm{d}\mathbf{l} = \frac{h}{m} \Rightarrow \mathbf{v} = \left(\frac{h}{m}\right)\frac{1}{r}\mathbf{1}_\phi$$

Notice that this corresponds to our intuition of vortices where we imagine and increasing and diverging velocity the closer we go to the center. This might seem unphysical however in reality it is not a problem since the density of the superfluid vanishes at the origin as well and hence cancels out the divergence of the velocity.

1.2 Topology in electromagnetism

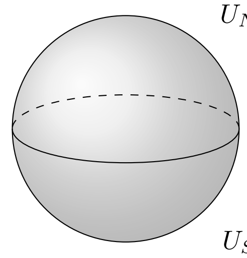
We start from Maxwell's equations, in particular $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ which tells us that we have no magnetic charges in nature. In 1931 Dirac wondered what would happen if there existed point like magnetic charges and hence we modified the equation to $\boldsymbol{\nabla}\cdot\mathbf{B} = 4\pi\rho_m = 4\pi g\delta^{(3)}(\mathbf{r})$. Then the magnetic field generated from such a charge would give $\mathbf{B} = g\frac{\mathbf{r}}{r^3}$ which is purely radial. We know interest ourselves to:

$$\Phi = \int_{S^2}\mathbf{B}\cdot\mathrm{d}\mathbf{S} = \int_V(\boldsymbol{\nabla}\cdot\mathbf{B})\mathrm{d}V = 4\pi g$$

Nothing special happened yet, this is just a reformulation of Gauss's theorem. We now introduce the gauge potential $\boldsymbol{\nabla}\times\mathbf{A} = \mathbf{B}$. Re-writing the above we get:

$$\Phi = \int_{S^2}(\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathrm{d}\mathbf{S} = \int_V\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A})\mathrm{d}V = 0$$

Hence such a construction cannot work. Our underlying false assumption here is that $\mathbf{A}$ was globally defined which is false if magnetic charges can exist. We put ourselves in spherical coordinates then we write $\mathbf{A} =$



$g(1-\cos\theta)\boldsymbol{\nabla}\varphi$. The reader can check that $\boldsymbol{\nabla}\times\mathbf{A} = g\frac{\mathbf{r}}{r^3} = \mathbf{B}$. So this is indeed the potential we are looking for and notice indeed that such a potential is not well-defined everywhere since $\boldsymbol{\nabla}\varphi = \frac{1}{r\sin\theta}\mathbf{1}_\varphi$ and $\mathbf{1}_\varphi$ is not well-defined along the z-axis. Now the prefactor cancels for $\theta = 0$ however it does not cancel for $\theta = \pi$ hence the potential is for sure ill-defined on the negative z-axis. However this is not the approach that we are going to follow. We will do what has been done 50 years later in 1975 by Wu and Yang.

Then we define $\mathbf{A}_N = \mathbf{A} = g(1-\cos\theta)\boldsymbol{\nabla}\varphi$ in the U_N region and similarly we define $\mathbf{A}_S = -g(1+\cos\theta)\boldsymbol{\nabla}\varphi$ on U_S. Then recomputing the integral we now get:

$$\Phi = \int_{S^2}\mathbf{B}\cdot\mathrm{d}\mathbf{S} = \int_{U_N}(\boldsymbol{\nabla}\times\mathbf{A}_N)\cdot\mathrm{d}\mathbf{S} + \int_{U_S}(\boldsymbol{\nabla}\times\mathbf{A}_S)\cdot\mathrm{d}S = \int_{\partial U_N=\partial U_S=\varepsilon}(\mathbf{A}_N-\mathbf{A}_S) = 2g\int_\varepsilon(\boldsymbol{\nabla}\varphi)\cdot\mathrm{d}\ell$$

Where in third equality we used Stokes theorem accounting for the orientation of integration along the boundary. Furthermore from the definition of the azimuthal angle we can easily compute the last integral and we get:

$$\Phi = \int_{S^2}\mathbf{B}\cdot\mathrm{d}\mathbf{S} = 4\pi g$$

Gauge transformation (on the equator).

Notice that we can rewrite the potential on one hemisphere as a Gauge transformation of the other:

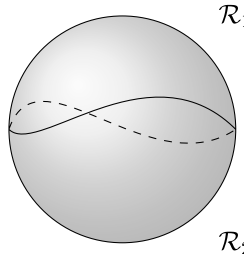
$$\mathbf{A}_N = \mathbf{A}_S + \boldsymbol{\nabla}(2g\varphi)$$

Now consider a particle coupled to the monopole field then Schrodinger's equation is given by:

$$\frac{1}{2m}[\mathbf{p}-q\mathbf{A}]^2\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

Then the Gauge transformation will force $\psi_S(\mathbf{r}) = \psi_N(\mathbf{r})e^{-i2g\varphi}$. Now physically we expect ψ_N and ψ_S to be compatible and $\psi_{N,S}$ on their overlap (the equator $\varepsilon = (\theta = \pi/2)$) must be single valued. Hence this forces that $2g\in\mathbb{Z}$ telling us that the monopole charge must be quantized. Now re-introducing the $\hbar, c, e$ constants by re-scaling units we get the real Dirac quantization law: $2g\left(\frac{\hbar}{ec}\right)\in\mathbb{Z}$.

Monopoles and winding numbers.



We now define $\mathbf{A}_1$ on $\mathcal{R}_1$ and $\mathbf{A}_2$ on $\mathcal{R}_2$ and we write $\partial\mathcal{R}_1 = \partial\mathcal{R}_2 = \partial\mathcal{R}$. Then on the boundary we have:

$$\text{For } \ell\in\partial\mathcal{R}\to U(1)\cong S^1 \quad \text{we get} \quad \begin{cases}\mathbf{A}_2 = \mathbf{A}_1 + \boldsymbol{\nabla}chi\\ \psi_2(\ell) = \psi_1(\ell)e^{i\chi(\ell)}\end{cases}$$

Now once again the single-valuedness of the wavefunction imposes that the map $e^{i\chi(\ell)}$ must loop on S^1. Hence the winding number must be quantized:

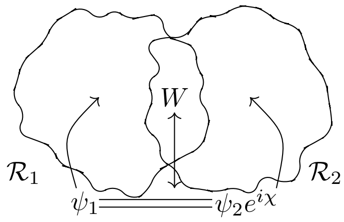
$$W = \frac{1}{2\pi}\int_{\partial R}(\boldsymbol{\nabla}\chi)\cdot\mathrm{d}\ell\in\mathbb{Z}$$

Then we get:

$$\Phi = \int_{S^2} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \int_{\mathcal{R}_1} \boldsymbol{\nabla} \times \mathbf{A}_1 \cdot \mathrm{d}\mathbf{S} + \int_{\mathcal{R}_2} \boldsymbol{\nabla} \times \mathbf{A}_2 \cdot \mathrm{d}\mathbf{S} = \oint_{\partial R} (\mathbf{A}_2 - \mathbf{A}_1) \cdot \mathrm{d}\ell = \oint_{\partial \mathcal{R}} (\boldsymbol{\nabla}\chi) \cdot \mathrm{d}\ell = 2\pi W$$

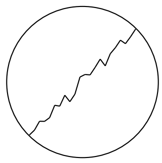
Hence necessarily we must have $2g = W$.

Conclusion.

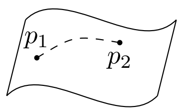


In conclusion if we have no monopole we can define $\mathbf{A}$ globally and $\psi(\mathbf{r})$ too. However as soon as we have a monopole $(2g)$ then we must define $\mathbf{A}$ locally over several regions covering the space, and the quantity that relates the wavefunctions over the gluing regions of the patches is set by the winding number.

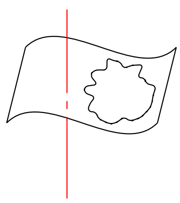
1.2.1 Fibre bundles in a nutshell.



We first define a Manifold: A manifold $\mathcal{M}$ is defined as an atlas of maps: $\{(U_i, \varphi_i)\}$. The U_i are an open



subset cover of $\mathcal{M}$ i.e. $\bigcup U_i = \mathcal{M}$. The φ_i are smooth maps that map $U_i \to \mathcal{R}_i \in \mathbb{R}^m$. We also have rules on the overlaps which impose smoothness: $U_i \cap U_j : \psi_{ij} = \varphi_i \circ \varphi_j$ must be smooth. So what are fibre bundles? A hand-wavy definition of fibre bundles consists in considering them as "a product of two manifolds". We could write is as $\mathcal{M} \times \mathcal{F} = \mathcal{E}$ where $\mathcal{M}$ is a base manifold, $\mathcal{F}$ is the fibre and $\mathcal{E}$ is the bundle. Hence by construction the product is locally trivial i.e. $\mathcal{E}\Big|_{U_i} \cong U_i \times \mathcal{F}$, however globally the product is not trivial $\mathcal{E} \ncong \mathcal{M} \times \mathcal{F}$.



Bundle examples.

A simple example is given by $\mathcal{M} = S^1$ and $\mathcal{F} = I = (a, b)$. Then we can construct the trivial bundle given by the unit cylinder of height $b - a$. However an example of a non-trivial bundle that we can build is, for example, a Mobius strip.

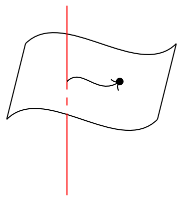
Section

We start by defining what a section is. A local section is a mpa $s_i : U_i \to \mathcal{E}$. This corresponds to selecting at every point of $\mathcal{M}$ choosing a point of the corresponding fiber. A global section is a map $s : \mathcal{M} \to \mathcal{E}$. If a global section can be defined then the fiber bundle is necessarily trivial.

Transition functions.

Suppose we have a local section $s_1 : U_1 \to \mathcal{E}$ and another local section $s_2 : U_2 \to \mathcal{E}$ and $U_1 \cap U_2 \neq \emptyset$. Then let $p \in U_1 \cap U_2$ we have that $s_1(p) = s_2(p)t_{21}(p)$ where we define t_{21} to be the transition function. Now if we have that $t_{ij}(p) = \mathrm{Id}_{\mathcal{F}}, \forall p \in \mathcal{M}$ then the fiber bundle is trivial because we can build a global section. Note that if $\mathcal{M} \cong \mathbb{R}^m$ then the fiber bundle will always be trivial. Hence we will almost always work either with $\mathcal{M} \cong S^m$ (the spheres) or $\mathcal{M} \cong \mathbb{I}^m$ (the tori).

Parallel transport.



Suppose we have a section $s_i(p) : U_i \to \mathcal{E}$ and path $\gamma(t) : [0,1] \to \mathcal{M}$ and a lifted curve $\tilde{\gamma}(t) : [0,1] \to \mathcal{E}$ corresponding to the generalization of the path to fiber space. Such that $\gamma(0) = p, \gamma(1) = p_1$, $\tilde{\gamma}(0) = s_i(p)$ and $\tilde{\gamma}(1) = s_i(p_1) \exp\Big(-\int_\gamma \mathcal{A}_\mu \mathrm{d}x^\mu\Big)$. Where $\mathcal{A}$ is called a connection and is a 1-form and is a local property (it is defined only over U_i). The connection has some constraints and most notoriously we have the notion of compatibility expressed as follows. On $U_i \cap U_j$ the connection must satisfy: $\mathcal{A}_j = t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij}^{-1}\mathrm{d}t_{ij}$.

Curvature

As we did before we use the notion of parallel transport to define curvature. Using the notion of parallel transport over a closed loop we can write:

$$\tilde{\gamma}(1) = s_i(p) = e^{-\oint_\gamma \mathcal{A}} = s_i(p)e^{-\int_\Sigma \mathcal{F}} \quad \text{where} \quad \mathcal{F} = \mathrm{d}\mathcal{A}$$

Since $\mathcal{A}$ is a 1-form we have that $\mathcal{F}$ is a 2-form and hence has to be written $\mathcal{F} = \frac{1}{2}\mathcal{F}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ where $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$ so long as we are looking at Abelian situations.

1.3 Berry's geometric phase

1.4 Topological Matter

1.5 Synthetic topological systems