ENS M1 General Relativity - Midterm Problems Solutions

Matteo Vilucchio

November 13, 2020

Exercise 1: Divergence and Laplacian

1. From the definition:

$$\boldsymbol{\nabla}\cdot\vec{\mathbf{V}} = \nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\gamma}V^\gamma = \partial_\mu V^\mu + \frac{1}{2}g^{\mu\nu}(\partial_\mu g_{\nu\gamma} + \partial_\gamma g_{\nu\mu} - \partial_\nu g_{\mu\gamma})V^\gamma$$
$$= \partial_\mu V^\mu + \frac{1}{2}g^{\mu\nu}\partial_\gamma g_{\mu\nu}V^\gamma$$

2. (CHECK NEEDED) The metric transforms as a $(0,2)$ tensor and its inverse transforms as a $(2,0)$ tensor. Since the determinant of the matrix is a scalar conserved by any Lorentz transformation one can definitely say that the cofactor matrix transforms as a tensor.

3. The determinant of a 4×4 matrix can be written as:

$$g = \epsilon^{\mu\nu\gamma\rho}g_{0\mu}g_{1\nu}g_{2\gamma}g_{3\sigma}$$

and then by differentiating one gets the actual definition of a minor:

$$\frac{\partial g}{\partial g_{\rho\sigma}} = \epsilon^{\sigma\nu\gamma\delta}g_{\tau\nu}g_{\eta\gamma}g_{\pi\delta} = c^{\rho\sigma} \quad \tau,\eta,\pi\neq\rho$$

The indices τ, η and π are not summed over.

4. By applying the chain rule to the following:

$$\partial_\gamma g = \frac{\partial g}{\partial g_{\mu\nu}}\partial_\gamma g_{\mu\nu} = g\ g^{\mu\nu}\ \partial_\gamma g_{\mu\nu}$$

then by looking at the previous equation one can see that the only solution is the natural logarithm. So one obtains that:

$$g^{\mu\nu}\partial_\gamma g_{\mu\nu} = \partial_\gamma(\ln|g|)$$

5. The important notion to realise to make this point is that:

$$\frac{1}{\sqrt{|g|}}\partial_\gamma\left(\sqrt{|g|}V^\gamma\right) = \frac{1}{\sqrt{|g|}}\partial_\gamma\left(\sqrt{|g|}\right)V^\gamma + \frac{\sqrt{|g|}}{\sqrt{|g|}}\partial_\gamma V^\gamma = \frac{1}{2}\frac{1}{|g|}\partial_\gamma g\ V^\gamma + \partial_\gamma V^\gamma = \frac{1}{2}\partial_\gamma(\ln|g|) + \partial_\gamma V^\gamma$$

where the last part is exactly what has already been obtained in point 1. In the end one can read that:

$$\boldsymbol{\nabla}\cdot\vec{\mathbf{V}} = \frac{1}{\sqrt{|g|}}\partial_\gamma\left(\sqrt{|g|}V^\gamma\right)$$

6. This exercice is about writing the laplacian in a handy way. One can do it in two different ways:

$$\nabla^2 f = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} f = \nabla_\mu \nabla^\mu (f) = \nabla^\mu \nabla_\mu (f)$$

to apply the previous formula one has to choose the first proposition since the formula we have derived only applies to vectors. By recalling that the covariant derivative of a scalar function is just the partial derivative one can obtain:

$$\nabla^2 f = \frac{1}{\sqrt{|g|}} \partial_\gamma \left(\sqrt{|g|} \partial^\gamma f \right) = \frac{1}{\sqrt{|g|}} \partial_\gamma \left(\sqrt{|g|} g^{\gamma\mu} \partial_\mu f \right)$$

7. In spherical coordinates the matrix associated with the metric is:

$$[g_{\mu\nu}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{bmatrix}$$

where a choice of basis has been made. The choice of this basis is the standard one w.r.t. the cartesian basis. From this the determinant reads as:

$$g = r^4 \sin^2\theta$$

This determinant is non zero for $r > 0$ and for $\theta \in]0, \pi[$. In this region one can find the inverse matrix which is:

$$[g^{\mu\nu}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{bmatrix}$$

then by applying the formulas obtained in point 6 one can directly have the expressions for the laplacian of a scalar function in spherical coordinates.

$$\begin{aligned} \nabla^2 f &= \frac{1}{r^2 \sin\theta} \partial_\gamma \left(r^2 \sin\theta g^{\gamma\mu} \partial_\mu f \right) = \\ &= \frac{1}{r^2 \sin\theta} \partial_r \left(r^2 \sin\theta g^{rr} \partial_r f \right) + \frac{1}{r^2 \sin\theta} \partial_r \left(r^2 \sin\theta g^{\theta\theta} \partial_\theta f \right) + \frac{1}{r^2 \sin\theta} \partial_\phi \left(r^2 \sin\theta g^{\phi\phi} \partial_\phi f \right) = \\ &= \frac{1}{r^2} \partial_r \left(r^2 \, \partial_r f \right) + \frac{1}{r^2 \sin\theta} \partial_\theta (\sin\theta \, \partial_\theta f) + \frac{1}{r^2 \sin^2\theta} \partial^2_\phi f \end{aligned}$$

8. In cylindrical coordinates one has that the metric and its inverse are:

$$[g_{\mu\nu}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [g^{\mu\nu}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad g = r^2$$

and the range of validity for the existence of the inverse matrix is $r > 0$. Then the laplacian in cylindrical coordinates becomes:

$$\nabla^2 f = \frac{1}{r} \partial_r (r \partial_r f) + \frac{1}{r^2} \partial^2_\phi f + \partial^2_z f$$

Exercise 2: Rotating coordinate frame

1. Intuitively one can say that the change in frame will "mix up" the angular and temporal coordinates. By looking specifically at the change of variables one has that:

$$\begin{cases} z = z' \\ r = r' \\ \phi = \phi' - \Omega t \\ t = t' \end{cases} \iff \begin{cases} \mathrm{d}z = \mathrm{d}z' \\ \mathrm{d}r = \mathrm{d}r' \\ \mathrm{d}\phi = \mathrm{d}\phi' - \Omega \, \mathrm{d}t \\ \mathrm{d}t = \mathrm{d}t' \end{cases}$$

then by substituting the differentials inside the line element one obtains the expression for the invariant in the other frame of reference:

$$ds^2 = \left(r^2\Omega^2 - c^2\right) \mathrm{d}t^2 + 2\Omega r^2 \,\mathrm{d}t \,\mathrm{d}\phi + r^2 \,\mathrm{d}\phi^2 + \mathrm{d}r^2 + \mathrm{d}z^2$$

and in matrix form the metric becomes:

$$[g_{\mu\nu}] = \begin{bmatrix} r^2\Omega^2 - c^2 & 0 & \Omega r^2 & 0 \\ 0 & 1 & 0 & 0 \\ \Omega r^2 & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. The inverse of this matrix becomes:

$$[g^{\mu\nu}] = \begin{bmatrix} -\frac{1}{c^2} & 0 & \frac{\Omega}{c^2} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\Omega}{c^2} & 0 & \frac{1}{r^2} - \frac{\Omega^2}{c^2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. The line element in the primed frame in cartesian coordinates is:

$$ds'^2 = -c^2 \,\mathrm{d}t'^2 + \mathrm{d}x'^2 + \mathrm{d}y'^2 + \mathrm{d}z'^2$$

and with the change of variable becomes:

$$a$$

4.

5.

Exercise 3: Frame dragging by a moving rod

1.

2.

3.

4.

5.

6.

Exercise 4: Frame-dragging inside a rotating cylinder

1.

2.

3.

4.