DM Quantum Mechanics 2019

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December 26, 2019

Problem 1

1)

The evolution operator is

$$\hat{U}(t,0) = e^{-\frac{i}{\hbar}\hat{H}t} = \sum_{k=0}^{\infty} \frac{(-\frac{i}{\hbar}t)^k}{k!}\hat{H}^k$$

2)

We have that since the Hamiltonian $\hat{H} = \frac{\hat{P}^2}{2m} + U(\hat{X}) + V(\hat{X}) = \hat{H}_0 + V(\hat{X})$ is time independent then:

$$|\psi(\tau)\rangle = \hat{U}(\tau,0)|\psi_0\rangle = \sum_{k=0}^{\infty} \frac{(\frac{-i}{\hbar}\tau)^k}{k!}(\hat{H}_0+\hat{V})^k|\psi_0\rangle \underset{\text{ordre 1}}{\approx} |\psi_0\rangle - \frac{i}{\hbar}\tau(\hat{H}_0+V_0\cos\left(q\hat{X}\right))|\psi_0\rangle = (1-\frac{i}{\hbar}\tau(E_0+V_0\cos\left(q\hat{X}\right))|\psi_0\rangle$$

3)

In position representation we have

$$\langle\mathbf{r_0}|\mathbf{p_0}\rangle = (2\pi\hbar)^{-3/2}e^{\frac{i}{\hbar}\mathbf{p_0}\cdot\mathbf{r_0}}$$

Now in the basis $|\mathbf{p}\rangle$ we have

$$\cos\left(q\hat{X}\right)|\mathbf{p_0}\rangle = \int d^3\mathbf{p}\,|\mathbf{p}\rangle\langle\mathbf{p}|\cos\left(q\hat{X}\right)|\mathbf{p_0}\rangle$$

4)

Problem 3

1)

Since all the operators involving the z-component are not composed with operators including the x,y-components, then we can write $\hat{H} = \hat{H}_{x,y}\otimes I + I\otimes\hat{H}_z$. Therefore if we call $|\psi_\perp\rangle$(reps. $|\psi_z\rangle$) the eigenstates of $\hat{H}_z$ (resp. $\hat{H}_{x,y}$) with energy E_z (resp. $E_\perp$), then the eigenstate $|\psi\rangle = |\psi_\perp\rangle\otimes|\psi_z\rangle$ satisfies

$$\hat{H}|\psi\rangle = (\hat{H}_{x,y}\otimes I)|\psi\rangle + (I\otimes\hat{H}_z)|\psi\rangle = E_\perp|\psi\rangle + E_z|\psi\rangle = (E_\perp + E_z)|\psi\rangle$$

Hence $|\psi\rangle$ is an eigenstate of $\hat{H}$ with eigenvalues $E_z + E_\perp$.

2)

Since the operators including the component x are not composed with those involving the component y, we can write that $\hat{H}_{x,y} = \hat{H}_x\otimes I + I\otimes\hat{H}_y$, with $\hat{H}_x$ and $\hat{H}_y$ the Hamiltonians for two harmonic oscillators of pulsations $\omega_\perp$. Hence the spectrum of $\hat{H}_x$ (resp. $\hat{H}_y$) is given by $E_{n_x} = \hbar\omega_\perp(n_x + 1/2)$ (reps. $E_{n_y} = \hbar\omega_\perp(n_y + 1/2)$). If we denote by $|\psi_{n_x}\rangle$ and $|\psi_{n_y}\rangle$ the two respective eigenstates, then we get that

$$\hat{H}_{x,y}|\psi_{n_x}\rangle\otimes|\psi_{n_y}\rangle = (\hat{H}_x\otimes I)|\psi_{n_x}\rangle\otimes|\psi_{n_y}\rangle + (I\otimes\hat{H}_y)|\psi_{n_x}\rangle\otimes|\psi_{n_y}\rangle = (E_{n_x} + E_{n_y})|\psi_{n_x}\rangle\otimes|\psi_{n_y}\rangle$$

i.e. the eigenstates of $\hat{H}_{x,y}$ are the $|\psi_{n_x}\rangle\otimes|\psi_{n_y}\rangle$, with eigenvalues $E_n = (E_{n_x} + E_{n_y}) = \hbar\omega_\perp(n_x + n_y + 1) = \hbar\omega_\perp(n+1)$. Then for a given n (which we recall being a non-negative integer) the degeneracy of the eigenspace associated to E_n is $n+1$ (since we can take $n_x \in [0,n]$ with $n_x \in \mathbb{N}$ and n_y is automatically chosen).

3)

We have that in any isolated system the total angular momentum and the Hamiltonian commute (which is equivalent to the conservation of the total angular momentum). Hence we have that

$$[\hat{H}, \hat{L}_z\otimes I] = 0$$

However, since we have that $\hat{L}_z$ acts on the same Hilbert space as $\hat{H}_{x,y}$, i.e. we have that $\hat{H} = \hat{H}_{x,y}\otimes I$, then it directly follows that

$$[\hat{H}_{x,y}\otimes I, \hat{L}_z\otimes I] = 0 \Rightarrow [\hat{H}_{x,y}, \hat{L}_z] = 0$$

4)

a)

We have that

$$\hat{a}_x = \frac{\hat{x}\sqrt{\frac{m\omega_\perp}{\hbar}} + i\hat{p}_x\frac{1}{\sqrt{m\hbar\omega_\perp}}}{\sqrt{2}} = \frac{\frac{\hat{x}}{a_0} + i\frac{\hat{p}_x}{p_0}}{\sqrt{2}}$$

$$\hat{a}_y = \frac{\hat{y}\sqrt{\frac{m\omega_\perp}{\hbar}} + i\hat{p}_y\frac{1}{\sqrt{m\hbar\omega_\perp}}}{\sqrt{2}} = \frac{\frac{\hat{y}}{a_0} + i\frac{\hat{p}_y}{p_0}}{\sqrt{2}}$$

$$\hat{a}_{x,y} = \hat{a}_x \otimes \hat{a}_y$$

b)

From $[\hat{a}_x, \hat{a}_x^\dagger] = 1$ (and same for y), we get that

$$[\hat{a}_{x,y}, \hat{a}_{x,y}^\dagger] = [\hat{a}_x \otimes \hat{a}_y, \hat{a}_x^\dagger \otimes \hat{a}_y^\dagger] = 1$$

c)

From $\hat{x} = \frac{a_0}{\sqrt{2}}(\hat{a}_x + \hat{a}_x^\dagger)$ and $\hat{p}_x = \frac{p_0}{i\sqrt{2}}(\hat{a}_x - \hat{a}_x^\dagger)$ (and similarly for y) we get that

$$\begin{aligned}
\hat{L}_z &= \frac{a_0}{\sqrt{2}}(\hat{a}_x + \hat{a}_x^\dagger) \otimes \frac{p_0}{i\sqrt{2}}(\hat{a}_y - \hat{a}_y^\dagger) - \frac{a_0}{\sqrt{2}}(\hat{a}_y + \hat{a}_y^\dagger) \otimes \frac{p_0}{i\sqrt{2}}(\hat{a}_x - \hat{a}_x^\dagger) \\
&= \frac{a_0 p_0}{2i}(\hat{a}_x \otimes \hat{a}_y - \hat{a}_x \otimes \hat{a}_y^\dagger + \hat{a}_x^\dagger \otimes \hat{a}_y - \hat{a}_x^\dagger \otimes \hat{a}_y^\dagger - \hat{a}_y \otimes \hat{a}_x + \hat{a}_y \otimes \hat{a}_x^\dagger - \hat{a}_y^\dagger \otimes \hat{a}_x + \hat{a}_y^\dagger \otimes \hat{a}_x^\dagger) \\
&= i\hbar(\hat{a}_x \otimes \hat{a}_y^\dagger - \hat{a}_x^\dagger \otimes \hat{a}_y)
\end{aligned}$$

where we simplified the terms $\hat{a}_x \otimes \hat{a}_y$ with $\hat{a}_y \otimes \hat{a}_x$ since both $\hat{x}, \hat{p}_x$ commute with $\hat{y}, \hat{p}_y$ (same for $\hat{a}_x^\dagger \otimes \hat{a}_y^\dagger$).

Similarly using that we already know tha Hamiltonian for a one dimensional harmonic oscillator we get

$$\hat{H}_{x,y} = \hat{H}_x \otimes I + I \otimes \hat{H}_y = \hbar\omega_\perp(\hat{N}_x \otimes I + \frac{1}{2}I \otimes I) + \hbar\omega_\perp(I \otimes \hat{N}_y + \frac{1}{2}I \otimes I) = \hbar\omega_\perp(\hat{a}_x^\dagger\hat{a}_x \otimes I + I \otimes \hat{a}_y^\dagger\hat{a}_y + I \otimes I)$$

d)

By using again the fact that $\hat{x}, \hat{p}_x$ commute with $\hat{y}, \hat{p}_y$ we get

$$[\hat{a}_+, \hat{a}_+^\dagger] = \frac{1}{2}[\hat{a}_x + i\hat{a}_y, \hat{a}_x^\dagger - i\hat{a}_y^\dagger] = \frac{1}{2}([\hat{a}_x, \hat{a}_x^\dagger] + [\hat{a}_y, \hat{a}_y^\dagger]) = 1$$

$$[\hat{a}_-, \hat{a}_-^\dagger] = \frac{1}{2}[\hat{a}_x - i\hat{a}_y, \hat{a}_x^\dagger + i\hat{a}_y^\dagger] = \frac{1}{2}([\hat{a}_x, \hat{a}_x^\dagger] + [\hat{a}_y, \hat{a}_y^\dagger]) = 1$$

$$[\hat{a}_+, \hat{a}_-^\dagger] = \frac{1}{2}[\hat{a}_x + i\hat{a}_y, \hat{a}_x^\dagger + i\hat{a}_y^\dagger] = \frac{1}{2}([\hat{a}_x, \hat{a}_x^\dagger] - [\hat{a}_y, \hat{a}_y^\dagger] = 0$$

$$[\hat{a}_-, \hat{a}_+^\dagger] = \frac{1}{2}[\hat{a}_x - i\hat{a}_y, \hat{a}_x^\dagger - i\hat{a}_y^\dagger] = \frac{1}{2}([\hat{a}_x, \hat{a}_x^\dagger] - [\hat{a}_y, \hat{a}_y^\dagger]) = 0$$

$$[\hat{a}_+, \hat{a}_-] = \frac{1}{2}[\hat{a}_x + i\hat{a}_y, \hat{a}_x - i\hat{a}_y] = 0 \Rightarrow [\hat{a}_+^\dagger, \hat{a}_-^\dagger] = 0$$

From now on I will omit the tensor products not to make the text too lengthy.

e)

If two operators commute then they are diagonalizable in the same basis. Here we have, by decomposing the commutator with the well known relations to decompose a commutator which contains a composition of operators

$$[\hat{N}_+, \hat{N}_-] = [\hat{a}_+^\dagger\hat{a}_+, \hat{a}_-^\dagger\hat{a}_-] = \hat{a}_+^\dagger[\hat{a}_+, \hat{a}_-^\dagger]\hat{a}_- + \hat{a}_+^\dagger\hat{a}_-^\dagger[\hat{a}_+, \hat{a}_-] + [\hat{a}_+^\dagger, \hat{a}_-^\dagger]\hat{a}_-\hat{a}_+ + \hat{a}_-^\dagger[\hat{a}_+^\dagger, \hat{a}_-]\hat{a}_+ = 0$$

Hence they are diagonalizable in the same basis. Now notice that from $[\hat{a}_x, \hat{a}_x^\dagger] = 1$ (and same for y) and from the fact that $\hat{a}_x$ commutes with $\hat{a}_y$, we have $[\hat{a}_\pm, \hat{a}_\pm^\dagger] = \frac{1}{2}[\hat{a}_x \pm i\hat{a}_y, \hat{a}_x^\dagger \mp i\hat{a}_y^\dagger] = 1$. Therefore if we denote by $|n_\sigma\rangle$ ($\sigma \in \{+,-\}$) an eigenstate of $\hat{N}_\sigma$ of eigenvalue n_σ, we have that:

$$\hat{N}_\sigma \hat{a}_\sigma |n_\sigma\rangle = \hat{a}_\sigma^\dagger \hat{a}_\sigma \hat{a}_\sigma |n_\sigma\rangle = (\hat{a}_\sigma \hat{a}_\sigma^\dagger \hat{a}_\sigma - 1) |n_\sigma\rangle = (n_\sigma - 1)\hat{a}_\sigma |n_\sigma\rangle$$

$$\hat{N}_\sigma \hat{a}_\sigma^\dagger |n_\sigma\rangle = \hat{a}_\sigma^\dagger \hat{a}_\sigma \hat{a}_\sigma^\dagger |n_\sigma\rangle = \hat{a}_\sigma^\dagger (\hat{a}_\sigma^\dagger \hat{a}_\sigma + 1) |n_\sigma\rangle = (n_\sigma + 1)\hat{a}_\sigma^\dagger |n_\sigma\rangle$$

which means that if $\hat{a}_\sigma |n_\sigma\rangle \neq 0$ then it is an eigenstate of $\hat{N}_\sigma$ of eigenvalue $(n_\sigma - 1)$ and that $\hat{a}_\sigma^\dagger |n_\sigma\rangle$ is an eigenstate of $\hat{N}_\sigma$ of eigenvalue $(n_\sigma + 1)$. Moreover we have that the eigenvalues of $\hat{N}_\sigma$ are non-negative, indeed we have that $0 \leq ||\hat{N}_\sigma |n_\sigma\rangle||^2 = \langle n_\sigma|\hat{a}_\sigma^\dagger \hat{a}_\sigma|n_\sigma\rangle = n_\sigma$. Therefore if any of the eigenvalues of $\hat{N}_\sigma$, say $s_\sigma > 0$, were not a natural integer, then we could apply $\hat{a}_\sigma$ on $|s_\sigma\rangle$ $\lceil s_\sigma \rceil$ times, getting then a negative eigenvalue, which we know being impossible. Hence the spectrum of $\hat{N}_\sigma$ belongs to $\mathbb{N}$.

f)

We have that $\hat{N}_- = \hat{a}_-^\dagger \hat{a}_- = \frac{1}{2}(\hat{a}_x^\dagger + i\hat{a}_y^\dagger)(\hat{a}_x - i\hat{a}_y) = \frac{1}{2}(\hat{a}_x^\dagger \hat{a}_x - i\hat{a}_x^\dagger \hat{a}_y + i\hat{a}_y^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y)$ and $\hat{N}_+ = \hat{a}_+^\dagger \hat{a}_+ = \frac{1}{2}(\hat{a}_x^\dagger - i\hat{a}_y^\dagger)(\hat{a}_x + i\hat{a}_y) = \frac{1}{2}(\hat{a}_x^\dagger \hat{a}_x + i\hat{a}_x^\dagger \hat{a}_y - i\hat{a}_y^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y)$. Hence notice that $\hat{N}_- - \hat{N}_+ = i(\hat{a}_x \hat{a}_y^\dagger - \hat{a}_x^\dagger \hat{a}_y)$, from which it directly follows that $\hat{L}_z = \hbar(\hat{N}_- - \hat{N}_+)$. Then we have that $\hat{N}_- + \hat{N}_+ = \hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y$ which implies that $\hat{H}_{x,y} = \hbar\omega(\hat{N}_- + \hat{N}_+ + I)$.

g)

First of all we study the action of $\hat{N}_\pm$ on $\hat{a}_\mp^\dagger |n_\pm\rangle$. Now, if we use the commutation relations found in question 4.d we have:

$$\hat{N}_+ \hat{a}_-^\dagger |n_+\rangle = \hat{a}_+^\dagger \hat{a}_+ \hat{a}_-^\dagger |n_+\rangle = \hat{a}_-^\dagger \hat{a}_+^\dagger \hat{a}_+ |n_+\rangle = n_+ \hat{a}_-^\dagger |n_+\rangle$$

and

$$\hat{N}_- \hat{a}_+^\dagger |n_-\rangle = \hat{a}_-^\dagger \hat{a}_- \hat{a}_+^\dagger |n_-\rangle = \hat{a}_+^\dagger \hat{a}_-^\dagger \hat{a}_- |n_-\rangle = n_- \hat{a}_+^\dagger |n_-\rangle$$

The eigenvalues of $\hat{H}_{x,y}$ are given by

5)

a)

If we denote by $|0\rangle$ the state for which $\hat{a}_+ |0\rangle = 0$ and $\hat{a}_- |0\rangle = 0$ then we have that