Master ENS ICFP - First Year 2020/2021
Relativistic Quantum Mechanics and Introduction to Quantum Field Theory
Mid Term Homework

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1 Some operator identities : 6 points

1. Consider $F(t) = e^{tA}Be^{-tA}$ then consider the following inductive hypothesis:

$$\mathcal{H}_n : \text{"}\frac{\mathrm{d}^n F(t)}{\mathrm{d}t^n} = e^{tA} \underbrace{[A, [A, \cdots, [A, B] \cdots]}_{n \text{ commutators}} e^{-tA}\text{"}$$

The base case $n = 1$ is trivially satisfied:

$$\frac{\mathrm{d}F(t)}{\mathrm{d}t} = e^{tA}ABe^{-tA} + e^{tA}B(-A)e^{-tA} = e^{tA}(AB - BA)e^{-tA} = e^{tA}[A, B]e^{-tA}$$

Then suppose $\mathcal{H}_n$ for $n \in \mathbb{N}$. Then we have that:

$$\begin{aligned}
\frac{\mathrm{d}^{n+1}F(t)}{\mathrm{d}t^{n+1}} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}^n F(t)}{\mathrm{d}t^n} = \frac{\mathrm{d}}{\mathrm{d}t} e^{tA} \underbrace{[A, [A, \cdots, [A, B] \cdots]}_{n \text{ commutators}} e^{-tA} \\
&= e^{tA}A \underbrace{[A, [A, \cdots, [A, B] \cdots]}_{n \text{ commutators}} e^{-tA} + e^{tA} \underbrace{[A, [A, \cdots, [A, B] \cdots]}_{n \text{ commutators}} (-A)e^{-tA} \\
&= e^{tA}(A \underbrace{[A, [A, \cdots, [A, B] \cdots]}_{n \text{ commutators}} - \underbrace{[A, [A, \cdots, [A, B] \cdots]}_{n \text{ commutators}} A)e^{-tA} \\
&= e^{tA} \underbrace{[A, [A, \cdots, [A, B] \cdots]}_{n+1 \text{ commutators}} e^{-tA}
\end{aligned}$$

Hence if $\mathcal{H}_n$ is true for $n \in \mathbb{N}$ so is $\mathcal{H}_{n+1}$ then by induction we can conclude that $\mathcal{H}_n$ is true for all $n \in \mathbb{N}$. Hence we have that:

$$F(t) = F(0) + \sum_{n=1}^{+\infty} \frac{t^n}{n!} \frac{\mathrm{d}^n F(t)}{\mathrm{d}t^n}\Big|_{t=0} = B + \sum_{n=1}^{+\infty} \frac{t^n}{n!} \underbrace{[A, [A, \cdots, [A, B] \cdots]}_{n \text{ commutators}}$$

Hence:

$$e^A B e^{-A} = F(1) = B + \sum_{n=1}^{+\infty} \frac{1}{n!} \underbrace{[A, [A, \cdots, [A, B] \cdots]}_{n \text{ commutators}}$$

2. Let $G(t) = e^{tA}e^{tB}$. Then notice that:

$$\begin{aligned}
\frac{\mathrm{d}G(t)}{\mathrm{d}t} &= Ae^{tA}e^{tB} + e^{tA}e^{tB}B = Ae^{tA}e^{tB} + e^{tA}Be^{-tA}e^{tA}e^{tB} = Ae^{tA}e^{tB} + (B + t[A, B] + 0)e^{tA}e^{tB} \\
&= (A + B + t[A, B])e^{tA}e^{tB} = (A + B + t[A, B])G(t)
\end{aligned}$$

Therefore G is a solution to the differential equation:

$$\partial_t f(t) = (A + B + t[A, B])f(t)$$

Now notice furthermore that:

$$\partial_t e^{tA+tB+\frac{t^2}{2}[A,B]} = (A+B+t[A,B])e^{tA+tB+\frac{t^2}{2}[A,B]}$$

And identically:

$$\partial_t e^{\frac{t^2}{2}[A,B]}e^{tA+tB} = t[A,B]e^{\frac{t^2}{2}[A,B]}e^{tA+tB} + e^{\frac{t^2}{2}[A,B]}(A+B)e^{tA+tB} = (A+B+t[A,B])e^{\frac{t^2}{2}[A,B]}e^{tA+tB}$$

Hence all three functions are solution to the same differential equation and furthermore at $t=0$ they are all equal to the identity hence they must be equal for all t. Then taking $t=1$ gives the desired equalities.

3. We have that:

$$[F,G^\dagger] = [\sum_j f_j a_j, \sum_j g_j^\star a_j^\dagger] = \sum_{j,k=0}^{+\infty} f_j g_k^\star [a_j, a_k^\dagger] = \sum_{j,k=0}^{+\infty} f_j g_k^\star \delta_{jk} = \sum_{j=0}^{+\infty} f_j g_j^\star$$

Furthermore we have that $[F,G^\dagger] \propto \mathrm{Id}$ and therefore we trivially have that $[F,[F,G^\dagger]] = [G^\dagger,[F,G^\dagger]] = 0$. Now applying question 2 we have that:

$$e^{G^\dagger}e^F = e^{-\frac{1}{2}\sum_j f_j g_j^\star}e^{G^\dagger+F} \Rightarrow e^{\frac{1}{2}\sum_j f_j g_j^\star}e^F = \underbrace{e^{\frac{1}{2}\sum_j f_j g_j^\star}e^{-\frac{1}{2}\sum_j f_j g_j^\star}}_{=e^A e^{-A}}e^{G^\dagger+F}$$

Now from Question 1 we have that for any A (trivially $[A,\mathrm{Id}]=0$) we get:

$$e^A \mathrm{Id} e^{-A} = \mathrm{Id} + \sum_{n=1}^{+\infty}\frac{1}{n!}\cdot 0 = \mathrm{Id}$$

Hence the above formula simplifies to:

$$e^{F+G^\dagger} = e^{\frac{1}{2}\sum_j f_j g_j^\star}e^{G^\dagger}e^F$$

4. Similarly as before let $F = \int \mathrm{d}^3\mathbf{q} f(\mathbf{q})a(\mathbf{q})$ and $G = \int \mathrm{d}^3\mathbf{q} h(\mathbf{q})^\dagger a(\mathbf{q})$. Then we have that:

$$[F,G^\dagger] = [\int \mathrm{d}^3\mathbf{q} f(\mathbf{q})a(\mathbf{q}), \int \mathrm{d}^3\mathbf{q} h(\mathbf{q})a^\dagger(\mathbf{q})] = \int \mathrm{d}^3\mathbf{q} f(\mathbf{q})h(\mathbf{q})[a(\mathbf{q}),a^\dagger(\mathbf{q})] = \int \mathrm{d}^3\mathbf{q} f(\mathbf{q})h(\mathbf{q})$$

A similar direct application of 2 gives the desired result.

2 An example of an asymptotic series

We have that:

$$f(g) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\mathrm{d}x e^{-x^2-gx^4} \quad \text{hence} \quad |f(g)| < \int_{-\infty}^{+\infty}\mathrm{d}x|e^{-x^2-gx^4}| = \int_{-\infty}^{+\infty}\mathrm{d}x e^{-x^2-x^4\operatorname{Re} g} < \int_{-\infty}^{+\infty}\mathrm{d}x e^{-x^4\operatorname{Re} g}$$

Hence as long as $\operatorname{Re} g > 0$ this is obviously well defined from the last term and if $\operatorname{Re} g = 0$ this is obviously well defined from the before last term.

1. This integral admits an exact solution given by:

$$f(g) = \frac{e^{\frac{1}{8g}}K_{\frac{1}{4}}(\frac{1}{8g})}{2\sqrt{\pi g}}\delta_{\operatorname{Re} g>0} + \delta_{\operatorname{Re} g=0} \quad \text{where} \quad K_n(z) \quad \text{is the modified Bessel function of the second kind.}$$

The plot of the numerical values for $g \in [0.01, 1]$ is given in Figure 1. Then $f(g)$ decreases monotonically when $g > 0$ increases since:

$$\frac{\mathrm{d}}{\mathrm{d}g}f(g) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\mathrm{d}x(-x^4)e^{-x^2-gx^4} = \frac{-1}{\sqrt{\pi}}\underbrace{\int_{-\infty}^{+\infty}x^4 e^{-x^2-gx^4}}_{>0 \text{ when } g\in\mathbb{R}^+} < 0$$

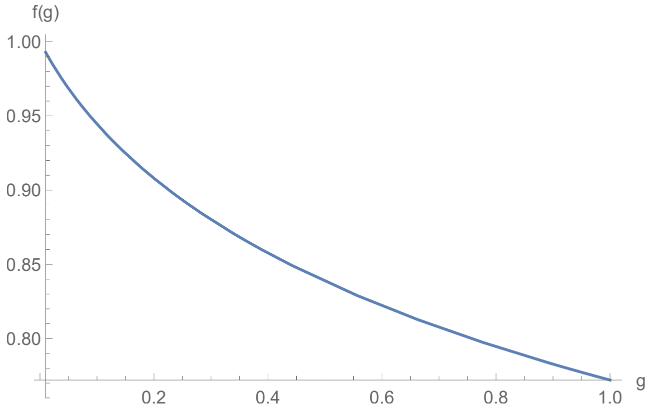

Figure 1: Plot of the numerical values of $f(g)$ for $g \in [0.01, 1]$.

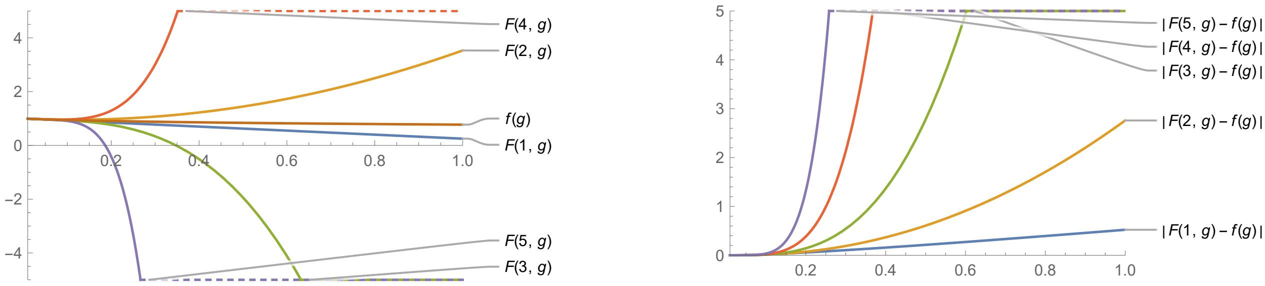

Figure 2: Series approximations of $f(g)$ and their errors for $g \in [0.01, 1]$.

2. We have that:

$$e^{-gx^4} = \sum_{n=0}^{+\infty} \frac{(-gx^4)^n}{n!}$$

And plugging this in the expression of f and inverting the sum and the integral gives:

$$\frac{1}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \frac{(-g)^n}{n!} \int_{-\infty}^{+\infty} \mathrm{d}x \; x^{4n} e^{-x^2}$$

We notice the integral ressembles strongly the gamma function hence we change variables by taking $u = x^2$ ($\mathrm{d}u = 2\sqrt{u}\mathrm{d}x$) and we get:

$$2^{-1} \int_{-\infty}^{+\infty} \mathrm{d}u u^{2n-\frac{1}{2}} e^{-u} = 2^{-1}\Gamma(2n + \frac{1}{2}) = 2^{-4n}\sqrt{\pi}\frac{\Gamma(4n)}{\Gamma(2n)} \quad \text{from the Legendre duplication formula.}$$

Hence plugging it back up top we obtain:

$$\tilde{f}(g) = \sum_{n=0}^{+\infty} \left(\frac{(-1)^n (4n)!}{n! 2^{4n} (2n)!} \right) g^n$$

Notice that the terms f_n are monotonically increasing in norm and diverge hence the sum does not converge absolutely and $R = 0$ and it also does not converge conditionally.

3. We can see that the speed of the divergence is very strongly related to the value of g if $g \ll 1$ the divergence is negligible but as soon as $g \sim 1$ the divergence is clearly apparent.

3 A relation between Dirac spinors

1. Remember that up to a re-writing we have that:

$$\omega_{ij} = \varepsilon_{ijk}\theta^k \quad \text{and} \quad \omega^{k0} = \nu^k \quad \text{and} \quad \omega_{\mu\nu} = 0 \quad \text{otherwise.}$$

Where the θ^k represent the rotations in the 3 spatial dimensions and the ν^k represent the boosts along the the three spatial directions. Hence the representation of $L(\mathbf{p})$ trivially has that $\theta^k = 0$. Then we have that:

$$i\gamma^0 D(L(\mathbf{p}))i\gamma^0 = i\gamma^0 \exp\left(\frac{1}{4}\omega_{\mu\nu}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\right)i\gamma^0$$

Now in order to simplify we need to bring the product with the gamma matrices inside. To do so we need to remember two properties. Firstly we have that $(i\gamma^0) = (i\gamma^0)^{-1}$ and secondly we have that: $Pe^AP^{-1} = e^{PAP^{-1}}$. Hence applying this formula here we obtain that:

$$i\gamma^0 D(L(\mathbf{p}))i\gamma^0 = \exp\left(\frac{\omega_{\mu\nu}}{4}i\gamma^0(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)i\gamma^0\right) = \exp\left(\frac{\omega_{\mu\nu}}{4}(i\gamma^0\gamma^\mu i\gamma^0 i\gamma^0\gamma^\nu i\gamma^0 - i\gamma^0\gamma^\nu \imath\gamma^0 i\gamma^0\gamma^\mu i\gamma^0)\right)$$

Now we use the formula from the course: $i\gamma^0\gamma^\mu i\gamma^0 = P^\mu_\nu\gamma^\nu$ where P^μ_ν is the parity operator defined in (3.33). Hence this means that the terms in the exponential simplify to:

$$i\gamma^0 D(L(\mathbf{p}))i\gamma^0 = \exp\left(\frac{\omega_{\mu\nu}}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(-1)^{1-\delta^\mu_0}(-1)^{1-\delta^\nu_0}\right) = \exp\left(\frac{\omega_{\mu\nu}(-1)^{\delta^\mu_0+\delta^\nu_0-\delta^\mu_0\delta^\nu_0}}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\right)$$

Hence now we take $\omega'_{\mu\nu} = \omega_{\mu\nu}(-1)^{\delta^\mu_0+\delta^\nu_0-\delta^\mu_0\delta^\nu_0}$, now since $\omega_{00} = 0$ we can discard the term $\delta^\mu_0\delta^\nu_0$ which means that we are left with:

$$\omega'_{\mu\nu} = \omega_{\mu\nu}(-1)^{\delta^\mu_0+\delta^\nu_0} \Leftrightarrow \omega'_{ij} = \omega_{ij} \wedge \omega'_{k0} = -\omega_{k0} \Rightarrow \theta'^k = \theta^k \wedge \nu'^k = -\nu^k$$

Hence we have that:

$$i\gamma^0 D(L(\mathbf{p}))i\gamma^0 = D(L(-\mathbf{p}))$$

2. We have that:

$$(ip_\mu\gamma^\mu + m)u(\mathbf{p},\sigma) = 0 \;\text{ and }\; (-ip_\mu\gamma^\mu + m)v(\mathbf{p},\sigma) = 0$$

Hence if we take $\mathbf{p} = 0$ and $p_0 = -p^0 = -m$ for a particle at rest the above simplifies to:

$$(-im\gamma^0 + m)u(\mathbf{0},\sigma) = 0 \;\text{ and }\; (im\gamma^0 + m)v(\mathbf{0},\sigma) = 0$$

Which up to dividing by m simplifies to:

$$(-i\gamma^0 + \mathbf{1})u(\mathbf{0},\sigma) = 0 \;\text{ and }\; (i\gamma^0 + \mathbf{1})v(\mathbf{0},\sigma) = 0$$

Hence up to a re-writing we have that:

$$i\gamma^0 u(\mathbf{0},\sigma) = u(\mathbf{0},\sigma) \;\text{ and }\; i\gamma^0 v(\mathbf{0},\sigma) = -v(\mathbf{0},\sigma)$$

Then from the definition of u, v and $D(L(\mathbf{p}))$ we have that $u(\mathbf{p},\sigma) = D(L(\mathbf{p}))u(\mathbf{0},\sigma)$ and identically for v. Hence we have that:

$$i\gamma^0 u(\mathbf{p},\sigma) = i\gamma^0 D(L(\mathbf{p}))u(\mathbf{0},\sigma) = i\gamma^0 D(L(\mathbf{p}))i\gamma^0 i\gamma^0 u(\mathbf{0},\sigma) = D(L(-\mathbf{p}))u(\mathbf{0},\sigma) = u(-\mathbf{p},\sigma)$$

And identically:

$$i\gamma^0 v(\mathbf{p},\sigma) = i\gamma^0 D(L(\mathbf{p}))v(\mathbf{0},\sigma) = i\gamma^0 D(L(\mathbf{p}))i\gamma^0 i\gamma^0 v(\mathbf{0},\sigma) = D(L(-\mathbf{p}))(-v(\mathbf{0},\sigma)) = -v(-\mathbf{p},\sigma)$$

4 Some traces of products of γ-matrices

We have that:

$$\operatorname{tr}\gamma_\mu\gamma_\nu = \operatorname{tr}\{\gamma_\mu,\gamma_\nu\} - \gamma_\nu\gamma_\mu = 2\operatorname{tr}\eta_{\mu\nu}I_4 - \operatorname{tr}\gamma_\nu\gamma_\mu = 2\operatorname{tr}\eta_{\mu\nu}I_4 - \operatorname{tr}\gamma_\mu\gamma_\nu$$

Hence adding on both side we obtain the desired equality:

$$\operatorname{tr}\gamma_\mu\gamma_\nu = \eta_{\mu\nu}\operatorname{tr}I_4 = 4\eta_{\mu\nu}$$

Similarly we have that:

$$\operatorname{tr}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = \operatorname{tr}\gamma_\mu\gamma_\nu(2\eta_{\rho\sigma}I_4 - \gamma_\sigma\gamma_\rho) = 2\eta_{\rho\sigma}\operatorname{tr}\gamma_\mu\gamma_\nu - \operatorname{tr}\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\rho = 2\eta_{\rho\sigma}4\eta_{\mu\nu} - \operatorname{tr}\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\rho$$

Now we can repeat the argument but instead of permuting the last two term we permute the two middle terms and then we permute the first two terms in such a way as to bring γ_σ to the front of the queue. Then we have that:

$$\operatorname{tr}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = 2\eta_{\rho\sigma}4\eta_{\mu\nu} - 2\eta_{\nu\sigma}4\eta_{\mu\rho} + 2\eta_{\mu\sigma}4\eta_{\nu\rho} - \operatorname{tr}\gamma_\sigma\gamma_\mu\gamma_\nu\gamma_\rho$$

Then from the cyclicity of the trace we can bring γ_σ to the back again. Hence we obtain:

$$\operatorname{tr}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = 4(\eta_{\rho\sigma}\eta_{\mu\nu} + \eta_{\nu\sigma}\eta_{\mu\rho} - \eta_{\mu\sigma}\eta_{\nu\rho})$$

Now for the last trace property of the γ-matrices. We have that:

$$\operatorname{tr}\gamma_{\mu_1}\cdots\gamma_{\mu_{2n+1}} = \operatorname{tr}\gamma_5\gamma_5\gamma_{\mu_1}\cdots\gamma_{\mu_{2n+1}} = (-1)^{2n+1}\operatorname{tr}\gamma_5\gamma_{\mu_1}\cdots\gamma_{\mu_{2n+1}}\gamma_5 = -\operatorname{tr}\gamma_{\mu_1}\cdots\gamma_{\mu_{2n+1}}$$

Where in the first equality we used that $\gamma_5^2 = 1$ and in the second that $\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$. From the above we can then conclude that:

$$\operatorname{tr}\gamma_{\mu_1}\cdots\gamma_{\mu_{2n+1}} = 0$$

5 Energy levels of a relativistic charged spin-0 particle in a harmonic electrostatic potential

1. From these relation we have that:

$$\begin{aligned} X^2\left|n\right\rangle &= \frac{1}{2m\Omega}\left(a^2 + (a^\dagger)^2 + \{a, a^\dagger\}\right)\left|n\right\rangle \\ &= \frac{1}{2m\Omega}(\sqrt{n}\sqrt{n-1}\left|n-2\right\rangle + \sqrt{n+1}\sqrt{n+2}\left|n+2\right\rangle + (n+1)\left|n\right\rangle + n\left|n\right\rangle) \end{aligned}$$

   Hence we get that:

$$\left\langle n\right| X^4\left|n\right\rangle = (\left\langle n\right| X^2)(X^2\left|n\right\rangle) = \frac{6}{(2m\Omega)^2}\left(n(n+1) + \frac{1}{2}\right)$$

2. We have that:

$$(-D_\mu D^\mu + m^2)\Phi = 0 \quad \text{where} \quad D^\mu = \partial^\mu - iqA^\mu$$

   Expanding this slightly gives:

$$\left((\partial_t + i\frac{m}{2}\omega^2x^2)^2 - \nabla^2 + m^2\right)\Phi = 0$$

   Now inserting the Ansatz we are given: $\Phi = e^{-iEt}\phi$ this gives:

$$(\partial_t + i\frac{m}{2}\omega^2x^2)^2e^{-iEt}\phi + e^{-iEt}(-\nabla^2 + m^2)\phi = 0$$

   Hence to simplify we compute:

$$e^{iEt}(\partial_t + i\frac{m}{2}\omega^2x^2)^2e^{-iEt} = e^{iEt}(\partial_t^2 + im\omega^2x^2\partial_t - \frac{m^2}{4}\omega^4x^4)e^{-iEt} = -E^2 + m\omega^2x^2E - \frac{m^2}{4}\omega^4x^4$$

   Putting everything together we obtain:

$$\left(E^2 - m\omega^2x^2E + \frac{m^2}{4}\omega^4x^4 + \nabla^2 - m^2\right)\phi = 0$$

3. We see from the above that the coefficients depend only on x hence the equation is invariant for translations in y and z. We therefore know that the solution can be written as:

$$\phi(x, y, z) = e^{ik_yy}e^{ik_zz}f(x)$$

   Plugging this in the equation on top we obtain:

$$\left(E^2 - m\omega^2x^2E + \frac{m^2}{4}\omega^4x^4 - k_y^2 - k_z^2 + \partial_x^2 - m^2\right)f(x) = 0$$

   Hence reorganizing the terms we obtain:

$$\left(-\frac{\partial_x^2}{2m} + \underbrace{\frac{\omega^2E}{2}}_{\alpha}x^2 - \underbrace{\frac{m}{8}\omega^4}_{\beta}x^4\right)f = \underbrace{\frac{1}{2m}\left(E^2 - k_y^2 - k_z^2 - m^2\right)}_{\varepsilon}f$$

Now if we consider the x^4 term as a perturbation the base energies are the ones of the harmonic oscillator which are given by:

$$\varepsilon_n = \Omega\left(n+\frac{1}{2}\right) = \omega\sqrt{\frac{E}{m}}\left(n+\frac{1}{2}\right)$$

Now the first order corrections are given by:

$$\varepsilon_n^1 = \langle n|\,\beta x^4\,|n\rangle = \frac{6\beta}{(2m\Omega)^2}\left(n(n+1)+\frac{1}{2}\right) = -\frac{\omega^4}{4m\Omega^2}\left(n(n+1)+\frac{1}{2}\right)$$

Where in our equivalence we have that:

$$\frac{m}{2}\Omega^2 = \alpha = \frac{\omega^2 E}{2} \Rightarrow \Omega^2 = \frac{\omega^2 E}{m}$$

Hence replacing above we obtain:

$$\varepsilon_n^1 = -\frac{\omega^2}{4E}\left(n(n+1)+\frac{1}{2}\right)$$

Now we know that our treatment makes sense only if $|\varepsilon_n^1| \ll |\varepsilon_n^0|$ or in other words:

$$\frac{\omega^2 n^2}{E} \ll \omega n\sqrt{\frac{E}{m}} \Leftrightarrow \omega^2 n^2 m \ll E^3$$

4. We have seen that (where $k^2 = k_y^2 + k_z^2$):

$$\varepsilon = \frac{1}{2m}(E^2 - k^2 - m^2)$$

Hence inverting the equation we obtain:

$$E = \sqrt{2m\varepsilon + k^2 + m^2}$$

Then $\varepsilon = \varepsilon_n^0 + \varepsilon_n^1$ hence ε is quantized by n. Furthermore if we assume the system bounded in y and z we will also retrieve a quantization n_y, n_z for k_y and k_z. In the non relativistic limit we have that $E = m + \eta + \frac{\xi}{m}$. Hence plugging it in we obtain:

$$E = m + \omega\left(n+\frac{1}{2}\right) + \frac{4k^2 - \omega^2(2n(1+n)+1)}{8m} + \mathcal{O}(\frac{1}{m^2})$$

6 The axial current

1. We know that $\{\gamma_5, \gamma^\mu\} = 0$ and hence $\gamma_5\gamma^\mu = -\gamma^\mu\gamma_5$. Then we have that:

$$e^{i\varepsilon\gamma_5}\gamma^\mu = \sum_{n\in\mathbb{N}}\frac{1}{n!}(i\varepsilon\gamma_5)^n\gamma^\mu = \sum_{n\in\mathbb{N}}\frac{1}{n!}(-1)^n\gamma^\mu(i\varepsilon\gamma_5)^n = \gamma^\mu\sum_{n\in\mathbb{N}}\frac{(-i\varepsilon\gamma_5)^n}{n!} = \gamma^\mu e^{-i\varepsilon\gamma_5}$$

Notice also trivially that $(e^{i\varepsilon\gamma_5})^\dagger = e^{-i\varepsilon\gamma_5}$ since $\gamma_5 = \gamma_5^\dagger$. Hence the axial transformation gives:

$$\overline{e^{i\varepsilon\gamma_5}\psi} = (e^{i\varepsilon\gamma_5}\psi)^\dagger i\gamma^0 = \psi^\dagger e^{-i\varepsilon\gamma_5} i\gamma^0 = \psi^\dagger i\gamma^0 e^{i\varepsilon\gamma_5} = \overline{\psi}e^{i\varepsilon\gamma_5}$$

2. If we replace ψ by the axial transformed $e^{i\varepsilon\gamma_5}\psi$ we also have to replace $\overline{\psi}$ by the axial transformed $\overline{\psi}e^{i\varepsilon\gamma_5}$. Hence we obtain:

$$S = \int \mathrm{d}^4x\overline{\psi}e^{i\varepsilon\gamma_5}(-\not{\partial} + iq\not{A} - m)e^{i\varepsilon\gamma_5}\psi$$

Now notice that:

$$e^{i\varepsilon\gamma_5}\not{a}e^{i\varepsilon\gamma_5} = e^{i\varepsilon\gamma_5}a_\mu\gamma^\mu e^{i\varepsilon\gamma_5} = (a_\mu e^{i\varepsilon\gamma_5} + [e^{i\varepsilon\gamma_5}, a_\mu])\gamma^\mu e^{i\varepsilon\gamma_5} = a_\mu + [e^{i\varepsilon\gamma_5}, a_\mu]\gamma^\mu e^{i\varepsilon\gamma_5}$$

Then since $[\partial_\mu, e^{i\varepsilon\gamma_5}] = 0$ and $[A_\mu, e^{i\varepsilon\gamma_5}] = 0$ we have that:

$$S = \int \mathrm{d}^4x\overline{\psi}(-\not{\partial} + iq\not{A} - me^{i2\varepsilon\gamma_5})\psi$$

Hence the action is left unchanged if and only if $m = 0$ or $\varepsilon = 0$. The second case corresponding to the trivial case of reducing the transformation to the identity can be discarded. Now assuming $m = 0$. The infinitesimal transformation corresponding to the axial transformation is given by:

$$\psi \longmapsto e^{i\varepsilon\gamma_5}\psi = (1 + i\varepsilon\gamma_5)\psi = \psi + i\varepsilon\gamma_5\psi$$

Now this transformation conserves both the action and the Lagrangian hence from the formula (4.53) of the notes we have that the corresponding conserved current is given by:

$$j_5^\mu = -\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi^a}\frac{\partial\psi^a}{\partial\varepsilon} = -i\overline{\psi}\gamma^\mu\gamma_5\psi$$

3. The Dirac equations are given by:

$$(\not\partial - iq\not A + m)\psi = 0 \text{ and } \overline{\psi}(-\overleftarrow{\not\partial} + iq\not A + m) = 0$$

Then we have that:

$$\partial_\mu j_5^\mu = \partial_\mu(-i\overline{\psi}\gamma^\mu\gamma_5\psi) = -\overline{\psi}(\not\partial + \overleftarrow{\not\partial})\gamma_5\psi = -\overline{\psi}(iq\not A - m - iq\not A - m)\gamma_5\psi = 2m\overline{\psi}\gamma_5\psi \propto m$$

7 Supersymmetry

1. The transformations are given by:

$$\psi \mapsto \psi + \delta\psi = \psi + (\not\partial - m)\phi\varepsilon \text{ and } \phi \mapsto \phi + \delta\phi = \phi + \overline{\varepsilon}\psi$$

Then we have that:

$$\phi^\dagger \mapsto (\phi + \delta\phi)^\dagger = \phi^\dagger + \psi^\dagger\overline{\varepsilon}^\dagger = \phi^\dagger + \psi^\dagger(\varepsilon^\dagger i\gamma^0)^\dagger = \phi^\dagger + \psi^\dagger(-(i\gamma^0)^\dagger\varepsilon) = \phi^\dagger + \overline{\psi}\varepsilon$$

And similarly:

$$\overline{\psi} \mapsto \overline{\psi + \delta\psi} = (\psi + (\not\partial - m)\phi\varepsilon)^\dagger i\gamma^0 = \overline{\psi} + \varepsilon^\dagger\phi^\dagger(\not\partial - m)^\dagger i\gamma^0 = \overline{\psi} - \overline{\varepsilon}\phi^\dagger(\overleftarrow{\not\partial} + m)$$

2. The action is given by:

$$S_i = \int \mathrm{d}^4x \left(-\overline{\psi}(\not\partial + m)\psi - (\partial_\mu\phi^\dagger)\partial^\mu\phi - m^2\phi^\dagger\phi\right)$$

Then after the transformation we get:

$$S_f = S_i + \delta S$$

Where:

$$\begin{aligned}\delta S = \int \mathrm{d}^4x\ &\overline{\varepsilon}\phi^\dagger(\overleftarrow{\not\partial} + m)(\not\partial + m)\psi + \overline{\varepsilon}\phi^\dagger(\overleftarrow{\not\partial} + m)(\not\partial + m)(\not\partial - m)\phi\varepsilon - \overline{\psi}(\not\partial + m)(\not\partial - m)\phi\varepsilon \\ &- (\partial_\mu\overline{\psi}\varepsilon)\partial^\mu\phi - (\partial_\mu\phi^\dagger)\partial^\mu\overline{\varepsilon}\psi - (\partial_\mu\overline{\psi}\varepsilon)\partial^\mu\overline{\varepsilon}\psi - m^2\overline{\psi}\varepsilon\phi - m^2\phi^\dagger\overline{\varepsilon}\psi - m^2\overline{\psi}\varepsilon\overline{\varepsilon}\psi\end{aligned}$$

Now we start doing a few simplifications. Firstly since ε is infinitesimal we neglect all second order terms. Which gives:

$$\begin{aligned}\delta S = \int \mathrm{d}^4x\ &\overline{\varepsilon}\phi^\dagger(\overleftarrow{\not\partial}\not\partial + \overleftarrow{\not\partial}m + m\not\partial + m^2)\psi - \overline{\psi}(\not\partial^2 - m^2)\phi\varepsilon \\ &- (\partial_\mu\overline{\psi}\varepsilon)\partial^\mu\phi - (\partial_\mu\phi^\dagger)\partial^\mu\overline{\varepsilon}\psi - m^2\overline{\psi}\varepsilon\phi - m^2\phi^\dagger\overline{\varepsilon}\psi\end{aligned}$$

Now developing and simplifying gives:

$$\begin{aligned}\delta S = \int \mathrm{d}^4x\ &\overline{\varepsilon}(\not\partial\phi^\dagger)\not\partial\psi + m\overline{\varepsilon}(\not\partial\phi^\dagger)\psi + m\overline{\varepsilon}\phi^\dagger\not\partial\psi + m^2\overline{\varepsilon}\phi^\dagger\psi - \overline{\psi}\not\partial^2\phi\varepsilon + m^2\overline{\psi}\phi\varepsilon \\ &- (\partial_\mu\overline{\psi})\partial^\mu\phi\varepsilon - \overline{\varepsilon}(\partial_\mu\phi^\dagger)\partial^\mu\psi - m^2\overline{\psi}\varepsilon\phi - m^2\phi^\dagger\overline{\varepsilon}\psi\end{aligned}$$

Now some terms simplify giving:

$$\delta S = \int \mathrm{d}^4x\ \overline{\varepsilon}(\not\partial\phi^\dagger)\not\partial\psi + m\overline{\varepsilon}\not\partial(\phi^\dagger\psi) - \overline{\psi}\not\partial^2\phi\varepsilon - (\partial_\mu\overline{\psi})(\partial^\mu\phi)\varepsilon - \overline{\varepsilon}(\partial_\mu\phi^\dagger)\partial^\mu\psi$$

Now notice that we can write:

$$(\partial\!\!\!/\phi^\dagger)(\partial\!\!\!/\psi) = (\gamma^\mu \partial_\mu \phi^\dagger)(\gamma^\mu \partial_\mu \psi) = (\partial_\mu \phi^\dagger)(\gamma^\mu \gamma^\mu \partial_\mu \psi) = (\partial_\mu \phi^\dagger)(\partial^\mu \psi)$$

Hence we can further simplify the expression to:

$$\delta S = \int \mathrm{d}^4 x \; m\bar{\varepsilon}\partial\!\!\!/(\phi^\dagger \psi) - \overline{\psi}(\partial\!\!\!/^2 \phi)\varepsilon - (\partial_\mu \overline{\psi})(\partial^\mu \phi)\varepsilon$$

Applying the same reasoning again on the $\partial\!\!\!/^2$ we can further simplify to:

$$\delta S = \int \mathrm{d}^4 x \; m\bar{\varepsilon}\partial\!\!\!/(\phi^\dagger \psi) - \overline{\psi}\partial_\mu \partial^\mu \phi\varepsilon - (\partial_\mu \overline{\psi})(\partial^\mu \phi)\varepsilon$$

Which we can factorize as:

$$\delta S = \int \mathrm{d}^4 x \; m\bar{\varepsilon}\partial\!\!\!/(\phi^\dagger \psi) - \partial_\mu(\overline{\psi}\partial^\mu \phi)\varepsilon$$

Now the integral is easy to compute:

$$\delta S = m\bar{\varepsilon}\gamma^\mu \left[\phi^\dagger \psi\right]_{\text{bounds}} - \left[\overline{\psi}\partial^\mu \phi\right]_{\text{bounds}} \varepsilon$$

Furthermore we know that all fields vanish outside of a compact region of space time and or decay to 0. Hence the above simplifies to 0 and we have proved that the action is conserved.