Midterm homework problems

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1 Divergence and Laplacian

1. We have the definition of Christoffel symbols:

$$\Gamma^k_{ij} = \frac{\partial \mathbf{e}_i}{\partial x^j} \cdot \mathbf{e}^k$$

Then we have that:

$$\boldsymbol{\nabla} \cdot \mathbf{V} = \partial_i (V^j \mathbf{e_j})^i = \frac{\partial V^i}{\partial x^i} + \Gamma^i_{ij} V^j = V^i_{,i} + \frac{1}{2} g^{im}(g_{mi,j} + g_{mj,i} - g_{ij,m}) V^j = V^i{}_{,i} + \frac{1}{2} g^{\mu\nu} g_{\mu\nu,\gamma} V^\gamma$$

2. Since the determinant is not an invariant scalar the cofactor matrix does not transform as a tensor.

3. We have that:

$$g = \sum_\nu g_{\mu\nu} c^{\mu\nu} \text{ hence } \frac{\partial g}{\partial g_{\mu\nu}} = \frac{\partial}{\partial g_{\mu\nu}} \sum_{\nu'} g_{\mu\nu'} c^{\mu\nu'} = c^{\mu\nu}$$

4. Following the hint we have that:

$$\partial_\gamma g = \frac{\partial g}{\partial g_{\mu\nu}} \partial_\gamma g_{\mu\nu}$$

Now using the previous questions we can replace this by:

$$g g^{\mu\nu} g_{\mu\nu,\gamma}$$

Then up to refactorizing we obtain:

$$\frac{\partial_\gamma g}{g} = g^{\mu\nu} g_{\mu\nu,\gamma}$$

Which using the definition of the logarithm gives:

$$\partial_\gamma \log |g| = g^{\mu\nu} g_{\mu\nu,\gamma}$$

5. We start from the end and we differentiate to obtain:

$$\frac{1}{\sqrt{|g|}} \partial_\gamma (\sqrt{|g|} V^\gamma) = V^\gamma{}_{,\gamma} + \frac{1}{\sqrt{|g|}} V^\gamma \frac{1}{2\sqrt{|g|}} \partial_\gamma |g| = V^\mu{}_{,\mu} + \frac{1}{2} V^\gamma \frac{\partial_\gamma |g|}{|g|} = V^\mu{}_{,\mu} + \frac{1}{2} V^\gamma \log |g|$$

Now using question 4 we re-obtain the formula of question 1 and this concludes the proof.

6. Using the above formula by replacing: $V_\gamma = f_{,\gamma}$ (hence $V^\gamma = g^{\gamma\mu} f_{,\mu}$) we obtain:

$$\nabla^2 f = \frac{1}{\sqrt{|g|}} \partial_\gamma (\sqrt{|g|} f^{,\gamma}) = \frac{1}{\sqrt{|g|}} \partial_\gamma (\sqrt{|g|} g^{\gamma\mu} f_{,\mu})$$

7. In spherical coordinates we have that:

$$[g_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

Then in order to apply the previous formula we need to compute g and $[g^{\mu\nu}]$. We have quite simply:

$$g = r^4 \sin^2\theta \text{ and } g^{\mu\mu} = \frac{1}{g_{\mu\mu}} \text{ and } g^{\mu\nu} = 0 \text{ otherwise.}$$

Plugging this in the previous formula we obtain:

$$\nabla^2 f = \frac{1}{r^2 \sin\theta}\partial_\gamma(r^2 \sin\theta g^{\gamma\mu} f_{,\mu}) = \frac{1}{r^2 \sin\theta}\left(\partial_r(r^2 \sin\theta f_{,r}) + \partial_\theta(\sin\theta f_{,\theta}) + \partial_\varphi(\frac{1}{\sin\theta} f_{,\varphi})\right)$$

Now simplifying the derivatives gives:

$$\begin{aligned}\nabla^2 f &= \frac{1}{r^2 \sin\theta}\left(\sin\theta \partial_r(r^2 f_{,r}) + \partial_\theta(\sin\theta f_{,\theta}) + \frac{1}{\sin\theta}\partial_\varphi f_{,\varphi}\right) \\ &= \frac{1}{r^2}\partial_r(r^2 f_{,r}) + \frac{1}{r^2 \sin\theta}\partial_\theta(\sin\theta f_{,\theta}) + \frac{1}{r^2 \sin^2\theta}\partial_\varphi f_{,\varphi}\end{aligned}$$

8. Repeating an identical argument but using:

$$[g_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{pmatrix}$$

Gives us immediately that:

$$\nabla^2 f = \frac{1}{r}\partial_\gamma(r g^{\gamma\mu} f_{,\mu}) = r^{-1}(\partial_z(r f_{,z} + \partial_r(r f_{,r}) + \partial_\phi(r^{-1} f_{,\phi})) = f_{,zz} + r^{-1} f_{,r} + f_{,rr} + r^{-2} f_{,\phi\phi}$$

2 Rotating coordinate frame.

1. We have that:

$$t = t \quad \text{and} \quad z = z' \quad \text{and} \quad r = r' \quad \text{and} \quad \phi = \phi' - \Omega t$$

Hence we immediately get that:

$$\mathrm{d}t = \mathrm{d}t \quad \text{and} \quad \mathrm{d}z = \mathrm{d}z' \quad \text{and} \quad \mathrm{d}r = \mathrm{d}r' \quad \text{and} \quad \mathrm{d}\phi = \mathrm{d}\phi' - t\mathrm{d}\Omega - \Omega\mathrm{d}t = \mathrm{d}\phi' - \Omega\mathrm{d}t$$

Where in the last equality we add the assumption that we place ourselves in a rotating frame at constant angular velocity. Now plugging this in the expression for a line element we obtain:

$$\begin{aligned}\mathrm{d}s^2 &= -c^2\mathrm{d}t^2 + (\mathrm{d}z')^2 + (\mathrm{d}r')^2 + (r')^2(\mathrm{d}\phi')^2 = -c^2\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\phi + \Omega\mathrm{d}t)^2 \\ &= -c^2\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + r^2\Omega^2\mathrm{d}t^2 + 2r^2\Omega\mathrm{d}\phi\mathrm{d}t \\ &= (r^2\Omega^2 - c^2)\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + 2r^2\Omega\mathrm{d}t\mathrm{d}\phi\end{aligned}$$

Hence we also get:

$$[g_{\mu\nu}] = \begin{pmatrix} (r^2\Omega^2 - c^2) & 0 & 0 & r^2\Omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ r^2\Omega & 0 & 0 & r^2 \end{pmatrix}$$

2. The inverse can be immediately obtained through it's cofactor formulation and gives:

$$[g^{\mu\nu}] = \begin{pmatrix} -\frac{1}{c^2} & 0 & 0 & \frac{\Omega}{c^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\Omega}{c^2} & 0 & 0 & \frac{c^2 - r^2\Omega^2}{c^2 r^2} \end{pmatrix} \quad \text{and} \quad g = -c^2 r^2$$

3. The line element is given by:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

Now as seen in exercise 1 we have that:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + \mathrm{d}z^2$$

Now we are making the change $\theta' = \theta + \Omega t$ hence we obtain:

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\Omega^2\mathrm{d}t^2 + r^2\Omega\mathrm{d}\theta\mathrm{d}t + \mathrm{d}z^2$$

Now replacing back with:

$$\mathrm{d}r = \frac{2\mathrm{d}x + 2\mathrm{d}y}{2\sqrt{x^2 + y^2}} \quad \text{and} \quad \mathrm{d}\theta = \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2 + y^2}$$

We obtain:

$$\mathrm{d}s^2 = -(c^2 - (x^2 + y^2)\Omega^2)\mathrm{d}t^2 + \Omega(y\mathrm{d}x\mathrm{d}t - x\mathrm{d}y\mathrm{d}t) + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

4. We have that:

$$\begin{pmatrix} -1+h_{00} & h_{01} & h_{02} & h_{03} \\ h_{10} & 1+h_{11} & h_{12} & h_{13} \\ h_{20} & h_{21} & 1+h_{22} & h_{23} \\ h_{30} & h_{31} & h_{32} & 1+h_{33} \end{pmatrix} = \begin{pmatrix} -(1-(x^2+y^2)\Omega^2) & \Omega y & -\Omega x & 0 \\ \Omega y & 1 & 0 & 0 \\ -\Omega x & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence we get:

$$[h_{\mu\nu}] = \begin{pmatrix} (x^2+y^2)\Omega^2 & \Omega y/2 & -\Omega x/2 & 0 \\ \Omega y/2 & 0 & 0 & 0 \\ -\Omega x/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then the Christoffel's symbols are given by:

$$[\Gamma^x_{\mu\nu}] = \begin{pmatrix} -\Omega^2 x & 0 & \Omega/2 & 0 \\ 0 & 0 & 0 & 0 \\ \Omega/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad [\Gamma^y_{\mu\nu}] = \begin{pmatrix} -\Omega^2 y & -\Omega/2 & 0 & 0 \\ -\Omega/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

5. Then the equations of motions are given by the equation for a geodesic:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\nu\gamma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} = 0$$

Which when we plug in the values of the Christoffel's symbols we get:

$$\begin{cases} \ddot{x} = \Omega(\Omega x - \dot{y}) \\ \ddot{y} = \Omega(\Omega y + \dot{x}) \end{cases}$$

3 Frame dragging by a moving rod.

1. We have the equation:

$$\nabla^2 \Phi = 4\pi(G_N/c^2)\rho\Theta(R-r)$$

From symmetry arguments we know already that Φ will only be a function of r. Hence we have that by plugging the expression of the laplacian found in part 1 we obtain:

$$\Phi_{,zz} + \Phi_{,rr} + \Phi_{,r}/r + \Phi_{,\phi\phi}/r^2 = \Phi_{,rr} + \Phi_{,r}/r = 4\pi(G_N/c^2)\rho$$

This can be re-written as:

$$\partial_r(r\Phi_{,r}) = 4\pi(G_N/c^2)\rho r \Rightarrow r\Phi_{,r} = 4\pi(G_N/c^2)\rho r^2/2 + c$$

Which gives immediately through integration a solution of the form:

$$\Phi(r) = \pi(G_N/c^2)\rho r^2 + c\log(r) + c'$$

Now the condition $\Phi_{,r}(0) = 0$ ensures that $c = 0$ and the condition $\Phi(R) = 0$ ensures that $c' = -\pi(G_N/c^2)\rho R^2$ hence the final solution is given by:

$$\Phi(r) = \pi(G_N/c^2)\rho(r^2 - R^2)$$

2. We have that:

$$\overline{h_{\mu\nu}} = -4\Phi\delta^0_\mu\delta^0_\nu$$

Hence we get that:

$$h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h = -4\Phi\delta^0_\mu\delta^0_\nu$$

Now we have that:

$$\eta_{\mu\nu}(\overline{h}_{\mu\nu}) = h - \frac{1}{2}Tr(\eta)h = h - 2h = -h$$

Which when plugged in the previous equation gives immediately that:

$$-h = 4\Phi \Leftrightarrow h = -4\Phi$$

Hence we obtain that:

$$h_{\mu\nu}(r) = -2\Phi\delta_{\mu\nu}$$

3. We have that:
$$\mathrm{d}s^2 = (1-2\Phi)(\mathrm{d}r^2 + r^2\mathrm{d}\phi^2)$$
Now we apply the following transformations:
$$r' = (1 + r\Phi_{,r})(1-\Phi)r \quad \text{and} \quad \phi' = (1 - \Phi_{,r}/r)\phi$$
Now inserting the expression of Φ that we found above we get that:
$$r' = r(\frac{1}{2} - \alpha/4)(2 - \alpha\log(r/R))$$
Then inserting the differentials and linearizing to first order gives the desired result.

4. (a) We have that:
$$[\Lambda^\mu_\nu] = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$
   (b) Consider a particle at rest then we have that:
$$\mathbf{x} = (ct \quad 0 \quad 0 \quad 0)^T$$
   Applying the tensor above we get:
$$[\Lambda^\mu_\nu x^\nu] = \begin{pmatrix} ct\gamma \\ 0 \\ 0 \\ \beta\gamma ct \end{pmatrix}$$
   (c) We have that $t_{\alpha\beta}x^\alpha x^\beta$ is a scalar hence it is invariant under a Lorentz transformation. Therefore we can write:
$$t_{\alpha\beta}x^\alpha x^\beta = t_{\alpha'\beta'}\Lambda^{\alpha'}_{\alpha}x^\alpha\Lambda^{\beta'}_{\beta}x^\beta$$
   Hence we immediately get that:
$$t_{\alpha'\beta'} = (\Lambda^{-1})^{\alpha'}_{\alpha}(\Lambda^{-1})^{\beta'}_{\beta}t_{\alpha\beta}$$
   Where the inverse can be easily found since $L(\beta)^{-1} = L(-\beta)$.
   (d) Then the perturbation to the metric is expressed as :
$$[h_{\alpha'\beta'}] = 4\pi\frac{G_N}{c^2}\rho\log\left(\frac{r}{R}\right)\begin{pmatrix} \frac{c^2+v^2}{c^2-v^2} & 0 & 0 & -\frac{2cv}{c^2-v^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{2cv}{c^2-v^2} & 0 & 0 & \frac{c^2+v^2}{c^2-v^2} \end{pmatrix}$$

5. A first order Taylor expansion in β quickly yields:
$$[h_{\alpha'\beta'}] = 4\pi\frac{G_N}{c^2}\rho\log\left(\frac{r}{R}\right)\begin{pmatrix} 1 & 0 & 0 & -2\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2\beta & 0 & 0 & 1 \end{pmatrix}$$

6. (a) We use the geodesic equation:
$$-\frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} = \Gamma^z_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$$
   Then:
$$\Gamma^\nu_{\mu\lambda} = \frac{1}{2}\eta^{\nu\gamma}(\partial_\mu h_{\gamma\lambda} + \partial_\lambda h_{\gamma\mu} - \partial_\gamma h_{\lambda\mu})$$

We start by computing the Christoffel symbols:

$$\Gamma^z_{00} = \frac{1}{2}\eta^{z\gamma}(\partial_0 h_{\gamma 0} + \partial_0 h_{\gamma 0} - \partial_\gamma h_{00}) = \partial_0 h_{z0} - \frac{1}{2}\partial_z h_{00} = \partial_0 4\Phi(r)v/c + \partial_z\Phi(r) = 0$$
$$\Gamma^z_{01} = \frac{1}{2}\eta^{z\gamma}(\partial_0 h_{\gamma 1} + \partial_1 h_{\gamma 0} - \partial_\gamma h_{10}) = \frac{1}{2}(\partial_0 h_{z1} + \partial_1 h_{z0} - \partial_z h_{10}) = \frac{1}{2}(0 + \partial_x 4\Phi(r)v/c + 0) = 2v/c\,\partial_x\Phi(r)$$
$$\Gamma^z_{02} = \frac{1}{2}(\partial_0 h_{z2} + \partial_2 h_{z0} - \partial_z h_{20}) = 2v/c\,\partial_y\Phi(r)$$
$$\Gamma^z_{03} = \frac{1}{2}(\partial_0 h_{z3} + \partial_3 h_{z3} - \partial_z h_{30}) = \frac{1}{2}(-2\partial_z\Phi(r) - 4v/c\partial_z\Phi(r)) = -(1-2v/c)\partial_z\Phi(r) = 0$$
$$\Gamma^z_{11} = \frac{1}{2}(\partial_1 h_{z1} + \partial_1 h_{z1} - \partial_z h_{11}) = \partial_z\Phi(r) = 0$$
$$\Gamma^z_{12} = \frac{1}{2}(\partial_1 h_{z2} + \partial_2 h_{z1} - \partial_z h_{21}) = 0$$
$$\Gamma^z_{13} = \frac{1}{2}(\partial_1 h_{z3} + \partial_3 h_{z1} - \partial_z h_{31}) = -\partial_x\Phi(r)$$
$$\Gamma^z_{22} = \frac{1}{2}(\partial_2 h_{z2} + \partial_2 h_{z2} - \partial_z h_{22}) = \partial_z\Phi(r) = 0$$
$$\Gamma^z_{23} = \frac{1}{2}(\partial_2 h_{z3} + \partial_3 h_{z2} - \partial_z h_{32}) = -\partial_y\Phi(r)$$
$$\Gamma^z_{33} = \frac{1}{2}(\partial_3 h_{z3} + \partial_3 h_{z3} - \partial_z h_{33}) = 0$$

Hence the geodesic equation becomes:

$$-\ddot{z} = \Gamma^z_{01}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x}{\mathrm{d}\tau} + \Gamma^z_{02}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}y}{\mathrm{d}\tau} + \Gamma^z_{13}\frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}z}{\mathrm{d}\tau} + \Gamma^z_{23}\frac{\mathrm{d}y}{\mathrm{d}\tau}\frac{\mathrm{d}z}{\mathrm{d}\tau}$$

Plugging in the values we get:

$$-\ddot{z} = 2v/c(\partial_x\Phi(r)\gamma\dot{x} + \partial_y\Phi(r)\dot{t}\dot{y}) - \dot{z}(\dot{x}\partial_x\Phi(r) + \dot{y}\partial_y\Phi(r)) = (\dot{\mathbf{x}}\cdot\boldsymbol{\nabla}\Phi)(\frac{2v}{c}\gamma - \dot{z}) = \dot{x}\partial_x\Phi(\frac{2v}{c}\dot{t} - \dot{z})$$

Where in the last equality we used the fact that $\dot{y} = y = 0$ and $\partial_z\Phi = 0$. Now a similar derivation for $\Gamma^t_{\mu\nu}$ and $\Gamma^x_{\mu\nu}$ yields:

$$-\ddot{t} = \frac{2v}{c}\dot{z}\dot{x}\partial_x\Phi \quad\text{and}\quad -\ddot{x} = \partial_x\Phi(\dot{t}^2 - \dot{x}^2 + \dot{z}^2)$$

Hence the final system of equation gives:

$$\begin{cases} -\ddot{z} = \dot{x}\partial_x\Phi\left(\frac{2v}{c}\dot{t} - \dot{z}\right) \\ -\ddot{t} = \frac{2v}{c}\dot{z}\dot{x}\partial_x\Phi \\ -\ddot{x} = \partial_x\Phi(\dot{t}^2 - \dot{x}^2 + \dot{z}^2) \end{cases}$$

Now we can also replace $\partial_x\Phi$ by its value: $\frac{2G_N\pi\rho}{c^2x}$ (outside of the cylinder) which gives:

$$\begin{cases} -\ddot{z} = \dot{x}\frac{2G_N\pi\rho}{c^2x}\left(\frac{2v}{c}\dot{t} - \dot{z}\right) \\ -\ddot{t} = \frac{2v}{c}\dot{z}\dot{x}\frac{2G_N\pi\rho}{c^2x} \\ -\ddot{x} = \frac{2G_N\pi\rho}{c^2x}(\dot{t}^2 - \dot{x}^2 + \dot{z}^2) \end{cases}$$

Which when solved numerically gives the figures plotted below.

4 Frame-dragging inside a rotating cylinder.

1. Consider V to be a cylinder of height H and radius r around the origin. Then from Gauss's law we have that:
$$2\pi rH\boldsymbol{\nabla}\Phi(r) = \oint_{\partial V}\boldsymbol{\nabla}\Phi\mathrm{d}\mathbf{S} = \int_V \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\Phi\mathrm{d}V = \int_V \frac{4\pi G_N}{c^2}\rho(\mathbf{x})\mathrm{d}V = \frac{4\pi G_N}{c^2}\mathrm{d}\mu H$$
Hence up to re-writing we obtain:
$$\boldsymbol{\nabla}\Phi(r) = \frac{2G_N}{rc^2}\mathrm{d}\mu\hat{\mathbf{r}}$$

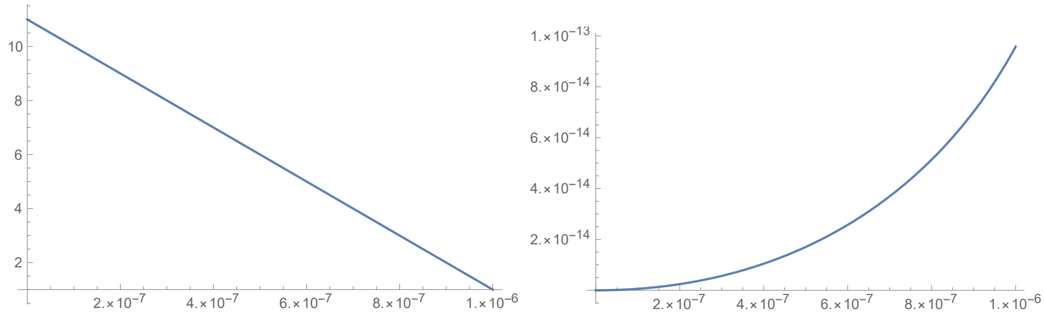

Figure 1: Plot of $x(\tau)$ on the left and $z(\tau)$ on the right.

2. Notice that we can re-write this as:

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\Phi_{\mu\nu}) = 4\pi\kappa T_{\mu\nu} \Leftrightarrow \mathbf{L}\cdot\mathbf{u} = f$$

Now the Green function for a needle pointing along the z-axis at position $\mathbf{x}' = (x, y)$ is given by:

$$\boldsymbol{\nabla}\cdot\mathbf{G}(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}') \Rightarrow \mathbf{G}(\mathbf{x},\mathbf{x}') = \frac{1}{2\pi}\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^2}$$

Hence we have that the solution $\mathbf{u} = \boldsymbol{\nabla}\Phi_{\mu\nu}$ to the equation $\mathbf{L}\cdot\mathbf{u} = f$ is given by:

$$\boldsymbol{\nabla}\Phi_{\mu\nu} = \mathbf{u} = \int G(\mathbf{x},\mathbf{x}')f(\mathbf{x}')\mathrm{d}^2\mathbf{x}' = \int \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^2}2\kappa T_{\mu\nu}(\mathbf{x}')\mathrm{d}^2\mathbf{x}'$$

3. We have that (in cylindrical coordinates):

$$[T_{\mu\nu}] = \rho c^2\begin{bmatrix}1 & \mathbf{v}/c\\ \mathbf{v}/c & 1\end{bmatrix} = \rho c^2\begin{bmatrix}1 & 0 & \Omega R/c & 0\\ 0 & 0 & 0 & 0\\ \Omega R/c & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Then to get it in cartesian coordinates we simply have to compute:

$$[T_{\mu\nu}] = \rho(\mathbf{r})c^2\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & \Omega R/c & 0\\ 0 & 0 & 0 & 0\\ \Omega R/c & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & \sin\theta & 0\\ 0 & -\sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$
$$= \rho(\mathbf{r})c\Omega R\begin{bmatrix}\frac{c}{R\Omega} & -\sin\theta & \cos\theta & 0\\ -\sin\theta & 0 & 0 & 0\\ \cos\theta & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Now since we are dealing with a thin-walled cylinder we have that:

$$\rho(\mathbf{r}) = \frac{\rho}{2\pi R}\delta(r-R)$$

Hence replacing up top we obtain:

$$[T_{\mu\nu}] = \frac{c\rho\Omega\delta(r-R)}{2\pi}\begin{bmatrix}\frac{c}{R\Omega} & -\sin\theta & \cos\theta & 0\\ -\sin\theta & 0 & 0 & 0\\ \cos\theta & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Then using the equation of the previous question we obtain:

$$\boldsymbol{\nabla}\Phi_{01}(\mathbf{0}) = 2\kappa\int \mathrm{d}^2\mathbf{x}' T_{01}(\mathbf{x}')\frac{\mathbf{0}-\mathbf{x}'}{|\mathbf{0}-\mathbf{x}'|^2} = -2\kappa\int \mathrm{d}^2\mathbf{x}'\frac{c\rho\Omega\delta(x'-R)}{2\pi}\sin\theta\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^2}$$
$$= \frac{2\kappa c\rho\Omega}{2\pi R}\int_0^{2\pi} R\sin\theta\mathrm{d}\theta = \frac{\kappa c\rho\Omega}{\pi}\int_0^{2\pi}\begin{pmatrix}\sin\theta\cos\theta\\ \sin^2\theta\end{pmatrix}\mathrm{d}\theta = \kappa c\rho\Omega\hat{\mathbf{y}}$$

Similarly we have that:

$$\boldsymbol{\nabla}\Phi_{02}(\mathbf{0}) = 2\kappa\int \mathrm{d}^2\mathbf{x}'\frac{c\rho\Omega\delta(x'-R)}{2\pi}\cos\theta\frac{\hat{\mathbf{r}}}{|\mathbf{x}'|^2} = -\frac{\kappa c\rho\Omega}{\pi}\int_0^{2\pi}\begin{pmatrix}\cos^2\theta\\ \sin\theta\cos\theta\end{pmatrix}\mathrm{d}\theta = -\kappa c\rho\Omega\hat{\mathbf{x}}$$

4. (a) We have that:

$$\Phi_{\mu\nu} = -\overline{h_{\mu\nu}}/4$$

From the previous question we also know that:

$$[\Phi_{\mu\nu}(\mathbf{x})] = \kappa c\rho\Omega \begin{bmatrix} 0 & y+\ell & -x+\ell' & 0 \\ y+\ell & 0 & 0 & 0 \\ -x+\ell' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then for $[\Phi(\mathbf{0})] = \mathbf{0}$ we simply need to take $\ell = \ell' = 0$. Then we have that:

$$[\overline{h_{\mu\nu}}(\mathbf{x})] = 4\kappa c\rho\Omega \begin{bmatrix} 0 & -y & x & 0 \\ -y & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then since $\overline{h}_{\mu\nu}$ is traceless we also have that $[h_{\mu\nu}] = [\overline{h}_{\mu\nu}]$.

(b) We have the following expression for the Christoffell symbols, derived similarly as in the 3rd part:

$$\Gamma^t_{\mu\nu} = 0 \text{ and } \Gamma^x_{02} = -4c\kappa\rho\Omega \text{ and } \Gamma^y_{01} = 4c\kappa\rho\Omega \text{ and } \Gamma^z_{\mu\nu} = 0$$

Where the unmentioned values of Γ^x and Γ^y are zero (apart from the symmetric values). This gives the following equations:

$$\begin{cases} \ddot{x} - 8c\kappa\rho\Omega\dot{y} = 0 \\ \ddot{y} + 8c\kappa\rho\Omega\dot{x} = 0 \end{cases}$$

Now for simplicity we call $\alpha = 8c\kappa\rho\Omega$ then we have that the solutions are given by:

$$x(\tau) = \frac{v_y - v_y\cos(t\alpha) + v_x\sin(t\alpha)}{\alpha} \text{ and } y(t) = \frac{v_x\cos(t\alpha) - v_x + v_y\sin(t\alpha)}{\alpha}$$

Where (v_x, v_y) are the initial velocities of the particle. We can see from the above equations that these are the parametric equations for a circle of center $c = (\frac{v_y}{\alpha}, -\frac{v_x}{\alpha})$ and of radius:

$$r = \frac{\sqrt{v_x^2 + v_y^2}\sin(4ct\kappa\rho\Omega)}{8c\kappa\rho\Omega} = \frac{\sin(t\alpha/2)}{\alpha}\sqrt{v_x^2 + v_y^2}$$

We hence will obtain a periodic opening and closing orbit around c.

(c) Consider a second still object at an arbitrary position in space. Then if we change to a rotating frame the cylinder might have stopped spinning however the other object will have started spinning.