Midterm homework problems

Marco Biroli

November 15, 2020

1 Divergence and Laplacian

1. We have the definition of Christoffel symbols:

$$\Gamma^k_{ij} = \frac{\partial \mathbf{e}_i}{\partial x^j} \cdot \mathbf{e}^k$$

   Then we have that:

$$\boldsymbol{\nabla} \cdot \mathbf{V} = \partial_i (V^j \mathbf{e_j})^i = \frac{\partial V^i}{\partial x^i} + \Gamma^i_{ij} V^j = V^i_{,i} + \frac{1}{2} g^{im} (g_{mi,j} + g_{mj,i} - g_{ij,m}) V^j$$

   Then:

...

2. Since the determinant is an invariant scalar of the matrix then from the relation: $g^{\mu\nu} = g^{-1} c^{\mu\nu}$ we know that c transforms in the exact same way as g does. Since g is a tensor then so is c.

3. We have that:

$$g = \sum_\nu g_{\mu\nu} c^{\mu\nu} \text{ hence } \frac{\partial g}{\partial g_{\mu\nu}} = \frac{\partial}{\partial g_{\mu\nu}} \sum_{\nu'} g_{\mu\nu'} c^{\mu\nu'} = c^{\mu\nu}$$

4. We have that:

$$g^{\mu\nu} g_{\mu\nu,\gamma} = \partial_\gamma \log g$$

   We have that:

$$\partial_\gamma g(g_{\mu\nu}) = (\partial_\gamma g) g_{\mu\nu} + g \partial_\gamma g_{\mu\nu}$$

   We have that:

$$\partial_\gamma g = \frac{\partial}{\partial g_{\mu\nu}} \frac{\partial g_{\mu\nu}}{\partial \gamma} g = \frac{\partial}{\partial g_{\mu\nu}} g_{\mu\nu,\gamma} g = g_{\mu\nu,\gamma} c^{\mu\nu}$$

   Hence:

...

5.