Dynamique et Modelisation

Alessandro Pacco

February 8, 2020

Contents

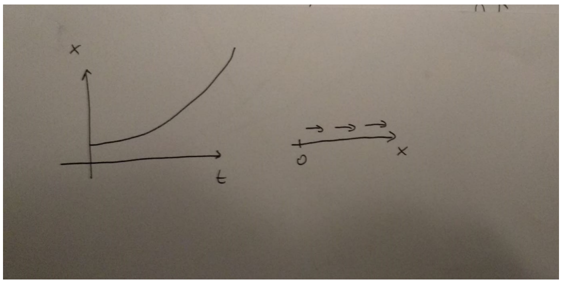
Chapter 1

Introduction to dynamical systems

Newton solved the 2-body problem in 1666.

Definition 1. *Given $f : \mathbb{R}^N \to \mathbb{R}^N$, a dynamical system is a Cauchy probelm of the form $\dot{x} = f(x)$ with $x(t_0) = x_0$, and where $x : \mathbb{R} \to \mathbb{R}^N$. We call N the number of degrees of freedom. $\mathbb{R}^N$ is called the phase space. For a given solution x, the flow of f is $\phi(x_0, t) = x(t)$. A trajectory is one of the curves of the flow $\phi(x_k, t)$ (i.e. we change the initial condition x_k for the solution).*

For example if we take a dynamical system of the form $\dot{x} = \mu x$, $\mu \in \mathbb{R}^*_+$, then the graphs are the following:



In general we can construct a solution to the Cauchy problem thanks to $x(t+\epsilon) = x(t) + \epsilon f(x)$. If for example we take $f(x) = \mu x$ then we have a linear dynamical system; if instead we decide to consider $f(x) = \mu x - x^2$ then we get a nonlinear dynamical system. For example if we take a harmonic oscillator then we get a differential equation of the form $m\ddot{x} + kx = 0$, with $\omega^2 = k/m$, which implies $\ddot{x} = -\omega^2 x$. Then we get a 2 order dynamical system (i.e. N=2) of the form

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -\omega^2 x_1 \end{cases}$$

Another example is the dumped harmonic oscillator:

$$\ddot{x} = -\gamma\dot{x} - \omega^2 x$$

which leads, with

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\omega^2 x_1 - \gamma x_2 \end{cases}$$

to the matrix problem

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

For example the dynamical system defined by

$$\ddot{x} = -\omega^2 x + \cos(\omega_F t)$$

has $N = 3$, with

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\omega^2 x_1 + \cos(\omega_F x_3) \\ \dot{x}_3 = 1 \end{cases}$$

The pendulum has an equation of the form

$$\ddot{x} + \frac{g}{l}\sin(x) = 0$$

which represents a dynamical system of order 2 which is not linear.
Now we take $f : \mathbb{R}^3 \to \mathbb{R}^3$, $x = (x_1, x_2, x_3)$. $\dot{x} = f(x)$, $f(x) = (-10x_1 + 10x_2, 28x_1 - x_2 - x_1x_3, -\frac{8x_3}{3} + x_1x_2)$ which was given by Lorentz in 1963.

Theorem 1. *(Cauchy-Lipschitz): Given* $\dot{x} = f(x)$*; if* f *is locally Lipschitz , i.e.* $\forall x_l \in \mathbb{R}^N$ *there exists a neighborhood* U *of* x_l *such that* $\forall (x, y) \in U^2$, $|f(x) - f(y)| \leq k|x - y|$*, then there exists a unique trajectory passing for any point* x *of the phase space.*

Example 1. *: We consider the example of the bucket of water*

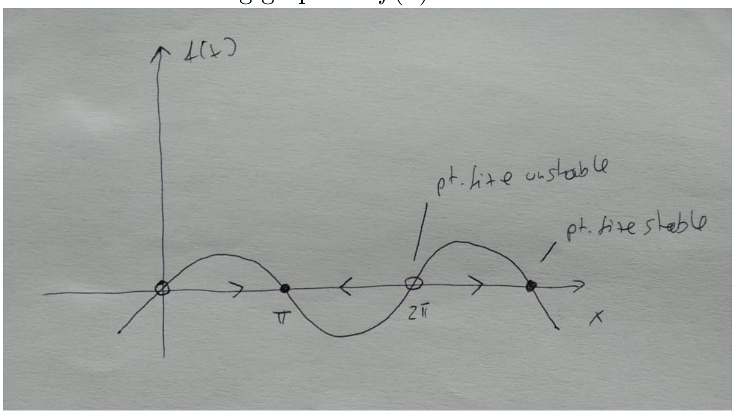
...

1.1 System of order 1 with flow on the $\mathbb{R}$ axis

Take $\dot{x} = \sin(x)$, $x(t_0) = x_0$. Then we have that

$$\int \frac{1}{\sin(x)} = \int dt \Rightarrow -\ln|\frac{1}{\sin(x)} + \cot(x)| = t + c,\ x_0 = \frac{\pi}{4}$$

we have the following graph for $f(x)$:



Another example is given by $\dot{x} = x^2 - 1$: INSERT GRAPH
Another example is given by $\dot{x} = 10x - x^3$, which give the follwing graph: Insert graph.
The last example is given by $\dot{x} = f(x)$, $f(x) = x - \cos(x)$: Insert graph

1.2 Potential Formulation

This technique consists basically in writing $f(x) = -\frac{dV}{dx}$. Then if we want to solve $\dot{x} = f(x) = -\frac{dV}{dx}$, we can write $\dot{v} = \frac{dV}{dx}\frac{dx}{dt} = -\left(\frac{dV}{dx}\right)^2$. Then we have to find the minimum of v (graph)

1.3 System of order 2: the pendulum

We have $Ml\ddot{\theta} = -Mg\sin\theta \Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0$ and if θ is small we linearize and write $\ddot{\theta} + \frac{g}{l}\theta = 0$. The solution is then given by $\theta(t) = \theta_0\cos(\omega t + \phi)$, with $\omega = \sqrt{\frac{g}{l}}$, $T = 2\pi\sqrt{\frac{l}{g}}$.
Now, if we consider $\ddot{x} + \sin x = 0$, with

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin x_1 \end{cases}$$

we obtain the following graph: graph

We can write the energy as $E = \frac{1}{2}\dot{\theta}^2 + \frac{g}{l}(1 - \cos\theta)$, $E(0, 0) = 0$. The following differential equation follows:

$$\frac{dE}{dt} = \dot{\theta}(\ddot{\theta} + \frac{g}{l}\sin\theta) = 0$$