HW2 - Probability

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September 28, 2020

1 Change of variables

1. From the change of variable theorem we know that:

$$f_{U,V}(u,v) = f_{X,Y}(uv, v(1-u))|J|^{-1}$$

Where:

$$J = \begin{vmatrix} \frac{y}{(x+y)^2} & -\frac{x}{(x+y)^2} \\ 1 & 1 \end{vmatrix} = \left| \frac{1}{x+y} \right| = |v^{-1}|$$

Then replacing in the definition and using the fact that X and Y are independent and hence we can split the joint law we get that:

$$\begin{aligned} f_{U,V}(u,v) &= \frac{uv^{k-1}}{(k-1)!} e^{-uv} 1_{\mathbb{R}^+}(uv) \frac{v^{k-1}(1-u)^{k-1}}{(k-1)!} e^{-v(1-u)} 1_{\mathbb{R}^+}(v(1-u))|v| \\ &= \left(\frac{u^{k-1}(1-u)^{k-1}}{(k-1)!} \right) \left(\frac{v^{2k-2}|v|e^{-v}}{(k-1)!} \right) 1_{\mathbb{R}^+}(uv) 1_{\mathbb{R}^+}(v(1-u)) \end{aligned}$$

Now notice that:

$$\begin{cases} uv \geq 0 \\ v(1-u) \geq 0 \end{cases} \Leftrightarrow \begin{cases} u, v \geq 0 \vee u, v \leq 0 \\ v, (1-u) \geq 0 \vee v, (1-u) \leq 0 \end{cases} \Leftrightarrow u, v \geq 0$$

Hence we can rewrite the above as:

$$f_{U,V}(u,v) = \left(\frac{u^{k-1}(1-u)^{k-1}}{(k-1)!} 1_{\mathbf{R}^+}(u) \right) \left(\frac{v^{2k-1}e^{-v}}{(k-1)!} 1_{\mathbf{R}^+}(v) \right)$$

We can already see that U, V are independent. The only thing left to compute is the normalization coefficient of at least one of the two laws. The law of V normalizes obviously to $\Gamma(2k)$ from the definition of the Γ function. Hence we can rewrite the above as:

$$f_{U,V}(u,v) = \left(\frac{u^{k-1}(1-u)^{k-1}\Gamma(2k)}{(k-1)!^2} 1_{\mathbf{R}^+}(u) \right) \left(\frac{v^{2k-1}e^{-v}}{\Gamma(2k)} 1_{\mathbf{R}^+}(v) \right) = f_U(u) f_V(v)$$

2. An immediate computation gives:

$$E[X] = \int_{\mathbb{R}} t f(t) \mathrm{d}t = \int_{\mathbb{R}^+} \frac{t^k}{(k-1)!} e^{-t} \mathrm{d}t = \frac{\Gamma(k+1)}{\Gamma(k)} = k$$

Then X, Y are identically distributed hence $E[Y] = E[X]$ and therefore:

$$E[V] = E[X+Y] = E[X] + E[Y] = 2k$$

Then we have that $X = UV$ and using the fact that U, V are independent we get:

$$E[X] = E[U]E[V] \Rightarrow E[U] = \frac{1}{2}$$

2 Order statistics

1. Let $(\Omega_i, \mathcal{F}_i, P_i)$ be the probability space of X_i then define the product probability space as $(\Omega, \mathcal{F}, P)$ and X as $(X_1, \cdots, X_n)$. Let $(\Omega, \mathcal{F}, P)$ also be the probability space of T. Then we define:

$$\begin{aligned} X_T : \Omega &\longrightarrow \mathbb{R} \\ \mathbf{x} &\longmapsto X(\mathbf{x})_{T(\mathbf{x})} \end{aligned}$$

   Then let $B \in \mathcal{B}(\mathbb{R})$ then we have that:

$$\{\mathbf{x} \in \Omega : X_T(\mathbf{x}) \in B\} \subset \bigotimes_{i \in [\![1,n]\!]} \{x_i \in \Omega_i : X_i(x_i) \in B\} \in \mathcal{F}$$

   Where the belonging to $\mathcal{F}$ follows from the definition of the product σ-algebra.

2. In order to define $(X_{(1)}, \cdots, X_{(n)})$ properly we consider it as an r.v. on the space $(\Omega, \mathcal{F}, P)$ defined as:

$$\begin{aligned} (X_{(1)}, \cdots, X_{(n)}) : \Omega &\longrightarrow \mathbb{R}^n \\ \mathbf{x} &\mapsto \sigma\mathbf{x}(X(\mathbf{x})) \end{aligned}$$

   Where $\sigma_{\mathbf{x}}$ is the permutation that put $X(\mathbf{x})$ in increasing order. Since we have a finite list of real numbers we know from the constructions of the real numbers that such a $\sigma_{\mathbf{x}}$. Furthermore adding as a constraint that in case of parity the smaller index goes first then $\sigma_{\mathbf{x}}$ is also unique for every $\mathbf{x}$. Then we have that σ is a random variable defined as:

$$\begin{aligned} \sigma : \Omega &\longrightarrow \mathfrak{S}_n \\ \mathbf{x} &\longmapsto \sigma_{\mathbf{x}} \end{aligned}$$

   We furthermore have that σ is injective and therefore measurable. Hence σ is a well-defined random variable.

3. From the previous question for shorthand we write $(X_{(1)}, \cdots, X_{(n)}) = \sigma(X)$ as an abuse of notation for:

$$(X_{(1)}, \cdots, X_{(n)})(\mathbf{x}) = \sigma_{\mathbf{x}}(X(\mathbf{x}))$$

   Then notice that:

$$f_{\sigma(X)}(\mathbf{x})\mathrm{d}\mathbf{x} = \sum_{\mu \in \mathfrak{S}_n} f_X(\mu^{-1}(\mathbf{x}))\mathrm{d}\mathbf{x} = \sum_{\mu \in \mathfrak{S}_n} f_X(\mu(\mathbf{x}))\mathrm{d}\mathbf{x} = \sum_{\mu \in \mathfrak{S}_n} \prod_{i=1}^{n} f_{X_i}(\mu(\mathbf{x})_i)\mathrm{d}\mathbf{x}$$

   Where on the last equality we used that the X_i are independent. Then since the X_i are identically distributed we have that $\forall i$, $f_{X_i} = f_{X_1}$. Now since the product commutes we have that the terms inside the sum are all equal up to a permutation of the terms, hence:

$$\sum_{\mu \in \mathfrak{S}_n} \prod_{i=1}^{n} f_{X_i}(\mu(\mathbf{x})_i)\mathrm{d}\mathbf{x} = \sum_{\mu \in \mathfrak{S}_n} \left(\prod_{i=1}^{n} f_{X_1}(x_i)\mathrm{d}x_i \right) = n! \left(\prod_{i=1}^{n} f_{X_1}(x_i)\mathrm{d}x_i \right) = n! f_X(\mathbf{x}')1_{\mathbf{x}'=\mu(\mathbf{x})}\mathrm{d}\mathbf{x}'$$

   Where we are free to chose any $\mu \in \mathfrak{S}_n$ since the terms in the product commute. If we fix ourselves with the choice $\mu = \sigma$ we get:

$$\sum_{\mu \in \mathfrak{S}_n} \prod_{i=1}^{n} f_{X_i}(\mu(\mathbf{x})_i)\mathrm{d}\mathbf{x} = n! f_X(\sigma(\mathbf{x}))\mathrm{d}\mathbf{x} = n! f_X(\mathbf{x}')1_{\mathbf{x}'=\sigma(\mathbf{x}')}\mathrm{d}\mathbf{x}'$$

   Call μ the function that maps $X_1, \cdots, X_n$ to $X_1, \cdots, X_n - X_{n-1}$. Then plugging this in the definition of the expectancy we get:

$$\begin{aligned} E[\varphi(\mu(\sigma(X)))] &= \int_{\mathbf{x} \in \Omega} \varphi(\mu(\sigma(X(\mathbf{x}))))f_{\mu(\sigma(X))}(\mu(\mathbf{x}))\mathrm{d}\mathbf{x} = n! \int_{\mathbf{x} \in \Omega} \varphi(\mu(\sigma(X(\mathbf{x}))))f_{\mu(X)}(\mu(\mathbf{x}'))1_{\mu(\mathbf{x}')=\sigma(\mu(\mathbf{x}'))}\mathrm{d}\mathbf{x} \\ &= n! \int_{\mathbf{x} \in \Omega} \varphi(\mu(X(\mathbf{x}')))f_{\mu(X)}(\mu(\mathbf{x}'))1_{\mu(\mathbf{x}')=\mu(\sigma(\mathbf{x}'))}\mathrm{d}\mathbf{x}' = n!\mathbb{E}[\varphi(\mu(X))1_\sigma] \text{ where } 1_\sigma(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} = \sigma(\mathbf{x}) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

4. From the previous exercise and the fact that the $X_i - X_{i-1}$ are independent we immediately get that the $X_{(i)} - X_{(i-1)}$ are independent. Then we have that $X_{(1)} = \min_i X_i$ hence:

$$F_{X_{(1)}}(x) = 1 - \prod_{i=1}^{n} P(X_i > x) = 1 - \prod_{i=1}^{n} e^{-\alpha x} = 1 - e^{-\alpha n x}$$

So $X_{(1)}$ follows an exponential law of parameter $n\alpha$. Now consider $X_{(i+1)} - X_{(i)}$. This can be re-written as:

$$X_{(i+1)} - X_{(i)} = \min_{i \in [\![1,n]\!], X_i > X_{(i)}} X_i - X_{(i)}$$

However notice that:

$$P(X_i = x + y | X_i > x) = \frac{P(X_i = x + y \cap X_i > x)}{P(X_i > x)} = \frac{\alpha e^{-\alpha(x+y)}}{e^{-\alpha x}} = \alpha e^{-\alpha y} = P(X_i = y)$$

Hence we get that:

$$X_{(i+1)} - X_{(i)} = \min_{i \in [\![1,n-i]\!]} X_i \sim \mathrm{Exp}(\alpha(n-i))$$

5. It is well known that the expectancy of an exponential random variable of parameter α is given by $\frac{1}{\alpha}$. Hence from the previous question we have that:

$$\mathbb{E}[X_{(i+1)} - X_{(i)}] = \frac{1}{\alpha(n-i)} \quad \text{and} \quad \mathbb{E}[X_{(1)}] = \frac{1}{\alpha n}$$

Denote by $u_i = \mathbb{E}[X_{(i)}]$ then we have that:

$$u_1 = \frac{1}{\alpha n} \quad \text{and} \quad u_{i+1} = u_i + \frac{1}{\alpha(n-i)} = \sum_{\ell=0}^{i} \frac{1}{\alpha(n-\ell)}$$

6. Notice that:

$$f_{X_{(k)}} = f_{X_{(1)}+(X_{(2)}-X_{(1)})+\cdots+(X_{(k)}-X_{(k-1)})} = f_{X_{(1)}} \star f_{X_{(2)}-X_{(1)}} \star \cdots \star f_{X_{(k)}-X_{(k-1)}}$$

Or in other words if we denote by (Y_j) independent exponential random variables of parameter α we have that:

$$X_{(k)} = \sum_{i=1}^{k-1} X_{(i)} - X_{(i-1)} 1_{i>1} = \sum_{i=1}^{k} \frac{Y_i}{n-i+1}$$

7. In general we have that:

$$F_{X_{(k)}}(x) = P\left(\max_{i \in \mathcal{I}} X_i < x \wedge \min_{i \in [\![1,n]\!] \setminus \mathcal{I}} X_i > x \big| |\mathcal{I}| \geq k\right) = \sum_{i=k}^{n} \binom{n}{i} F_{X_1}(x)^i (1 - F_{X_1}(x))^{n-i}$$

Where this comes simply from choosing which i elements will be smaller than x, and the fact that we need at least k elements to be smaller than x. Now simply taking the derivative with respect to x of the previous result we get that:

$$f_{X_{(k)}}(x) = \frac{\mathrm{d}}{\mathrm{d}x} \sum_{i=k}^{n} \binom{n}{i} F_{X_1}(x)^i (1 - F_{X_1}(x))^{n-i}$$

Now we denote:

$$s_i = \frac{\mathrm{d}}{\mathrm{d}x} F_{X_1}(x)^i (1 - F_{X_1}(x))^{n-i} = f_{X_1}(x) i F_{X_1}(x)^{i-1} (1 - F_{X_1}(x))^{n-i} - f_{X_1}(x)(n-i) F_{X_1}(x)^i (1 - F_{X_1}(x))^{n-i-1} = \ell_i - r_i$$

And notice that $\ell_{i+1} = r_i \frac{i+1}{n-i}$. Hence we get:

$$f_{X_{(k)}}(x) - \binom{n}{k} k f_{X_1}(x) F_{X_1}(x)^{k-1} (1 - F_{X_1}(x))^{n-k} = \sum_{i=k}^{n-1} \binom{n}{i+1} (i+1) \ell_{i+1} - \binom{n}{i} r_i = 0$$

We therefore get:

$$f_{X_{(k)}}(x) = \binom{n}{k} k f_{X_1}(x) F_{X_1}(x)^{k-1} (1 - F_{X_1}(x))^{n-k}$$