ENS M1 General Relativity - Homework problems

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November 6, 2020

Contents

The 2nd, 3rd and 4th questions are interlinked. I'd advise that you tackle them in order.

1 Divergence and Laplacian

1. Write out an expression for the divergence of a vector $\vec{\nabla} \cdot \vec{V} = V^\mu{}_{;\mu}$ using the connection (Christoffel symbols) and use this, along with the symmetry of the metric, to show that $\vec{\nabla} \cdot \vec{V} = V^\mu{}_{,\mu} + \frac{1}{2} g^{\mu\nu} g_{\mu\nu,\gamma} V^\gamma$.

2. According to linear algebra the inverse of the metric $g_{\mu\nu}$ is $g^{\mu\nu} = g^{-1} c^{\mu\nu}$ where g is the determinant of the matrix $g_{\mu\nu}$ and where $c^{\mu\nu}$ is its cofactor matrix. Is $c^{\mu\nu}$ a tensor? Hint: how does g transform?

3. Show that $\partial g / \partial g_{\mu\nu} = c^{\mu\nu}$. Hint: does the $(\mu, \nu)^{\text{th}}$ component of the cofactor matrix actually contain the component $g_{\mu\nu}$?

4. Show that $g^{\mu\nu} g_{\mu\nu,\gamma} = \partial_\gamma \log g$. Hint: apply the chain rule to $\partial_\gamma g(g_{\mu\nu})$ and use the result of the previous section.

5. Thus obtain the convenient expression for the divergence:
   - $\vec{\nabla} \cdot \vec{V} = \frac{1}{\sqrt{|g|}} \partial_\gamma (\sqrt{|g|} V^\gamma)$.

6. Use this to obtain the expression for the Laplacian $\nabla^2 f \equiv \vec{\nabla} \cdot \vec{\nabla} f$ of a scalar f:
   - $\nabla^2 f = \frac{1}{\sqrt{|g|}} \partial_\gamma (\sqrt{|g|} g^{\gamma\mu} f_{,\mu})$.

7. Use this to show that the divergence in spherical coordinates (for which the line element is $dl^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$) can be written as
   - $\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} (\ldots) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} (\ldots) + \frac{1}{r^2 \sin^2\theta} (\ldots)$
   - where you should provide the expressions to replace the dots

8. And to show that the divergence in cylindrical coordinates (for which the line element is $dl^2 = dz^2 + dr^2 + r^2 d\phi^2$) is
   - $\nabla^2 f = f_{,zz} + f_{,rr} + f_{,r}/r + f_{,\phi\phi}/r^2$

2 Rotating coordinate frame

Consider an inertial coordinate system t, x, y, z in flat space time and convert the x, y values to r and ϕ by $r = \sqrt{x^2 + y^2}$ and $\phi = \operatorname{atan} y/x$. This gives a non-rotating cylindrical coordinate system, in which we denote the spatial coordinates by primed values: t, z', r', ϕ'. The line element in these coordinates is

$$ds^2 = -c^2 dt^2 + (dz')^2 + (dr')^2 + (r')^2 (d\phi')^2. \tag{1}$$

Now consider a transformation to a different system that is rotating about the z-axis: $z = z'$, $r = r'$, $\phi = \phi' - \Omega t$. Note that, for positive angular frequency Ω, this is a system whose origin ($\phi = 0$) is moving to higher ϕ' values as time proceeds. This coordinate system has nasty properties at $r > c/\Omega$, for which $r\Omega > c$, so we will restrict attention to $r \ll c/\Omega$.

1. Calculate the line element (and write out the corresponding matrix $g_{\mu\nu}$) in the rotating (un-primed) coordinate system.

2. compute the inverse $g^{\mu\nu}$ of the metric

3. Now look at this in rectilinear coordinates. The (x, y) coordinates are related to the (x', y') coordinates by

   - $\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{R}(\Omega t) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \Omega t & \sin \Omega t \\ -\sin \Omega t & \cos \Omega t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
   - use this to show that
   - $ds^2 = -(1 - (x^2 + y^2)\Omega^2)dt^2 + 2\Omega(ydxdt - xdydt) + dx^2 + dy^2 + dz^2$

4. Write out the matrix $h_{\mu\nu}$ defined such that the line element above is $ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu$ and calculate, to first order in the metric perturbations $|h_{\mu\nu}|$, which we assume here to be small, the connection coefficients ${\Gamma^i}_{00}$ and ${\Gamma^i}_{0j}$ where i and j can be either x or y.

5. Show thereby that the equations of motion, to first order in $|h_{\mu\nu}|$, and for non-relativistic test particles, are (with dots denoting derivative with respect to proper time)

   - $\ddot{x} = \Omega^2 x - \Omega \dot{y}$
     $\ddot{y} = \Omega^2 y + \Omega \dot{x}$

   and discuss the physical meaning of the two terms in each equation. In particular

   - what force would be needed to keep a particle of mass m at fixed x, y?
   - if an observer at x, $y = 0$, 0 were to fire some test particles in the x, y plane how would they move (for times much less than one rotation period)

These results are not particularly profound. They will be used below when we interpret the 'frame-dragging' for a rotating massive cylinder.

3 Frame dragging by a moving rod

Consider first a uniform density infinitely long cylindrical bar at rest. So it is stationary (it's actually static, which is more restrictive) and therefore we can obtain stationary solutions in weak field gravity $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where the perturbation $h_{\mu\nu}$ (and therefore also the trace-reversed perturbation $\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$) is only a function of position $h_{\mu\nu} = h_{\mu\nu}(\mathbf{x})$. The field equation is then $\nabla^2 \overline{h}_{\mu\nu} = -16\pi\kappa T_{\mu\nu}$ where $\kappa = G_{\mathrm{N}}/c^4$.

Let the mass density be ρ interior to $r = R$ and zero outside and let's neglect any stress in the pencil, so $T_{\mu\nu} = \rho c^2 \delta^0_\mu \delta^0_\nu$. So the solution for the trace reversed metric is $\overline{h}_{\mu\nu} = -4\Phi\delta^0_\mu\delta^0_\nu$ where $\nabla^2\Phi = 4\pi(G_{\mathrm{N}}/c^2)\rho$ (i.e. Φ is the Newtonian potential divided by c^2).

1. Use the expression for the Laplacian in cylindrical coordinates from question 1 to obtain a solution for $\Phi(r)$ with boundary conditions $\Phi(R) = 0$ and $\Phi_{,r}(r = 0) = 0$.

2. Show that $h_{\mu\nu}(r) = -2\Phi(r)\delta_{\mu\nu}$. Note that the x^μ coordinate system here is the normal rectilinear ct, x, y, z) system; the cylindrical (z, r, ϕ) system above was used only for convenience in calculating the potential.

3. The geometry of a 2-dimensional hypersurface $t =$ constant $z =$ constant is $ds^2 = (1-2\Phi)(dx^2+dy^2) = (1-2\Phi)(dr^2 + r^2 d\phi^2)$ where $x = r\cos\phi$ and $y = r\sin\phi$ and where ϕ ranges from zero to 2π. Show that, for $r > R$, the transformation $r' = (1+r\Phi_{,r})(1-\Phi)r$ and $\phi' = (1-\Phi_{,r}/r)\phi$ renders the geometry locally flat: $ds^2 = dr'^2 + r'^2 d\phi'^2$ but conical (i.e. the range of ϕ' is now from zero to $2\pi - \epsilon$). Show that the 'deficit angle' is $\epsilon = 4G_{\mathrm{N}}\mu/c^2$ where μ is the mass per unit length of the rod.

4. Now transform this to obtain the metric $h_{\alpha'\beta'}$ in the frame of a 'lab-frame' observer (the primed frame) with respect to whom the cylinder is moving in the $+z = x^3$ direction (remember our coordinates being (ct, x, y, z)) at speed v. Proceed as follows:
   (a) write down the Lorentz boost matrix $\Lambda^{\alpha'}{}_\alpha$ that effects the transformation $x^{\alpha'} = \Lambda^{\alpha'}{}_\alpha x^\alpha$.
   (b) check that you got the sign of v correct by applying this to the coordinates of the world-line of a particle at rest in the rod
   (c) use the transformation properties of $t_{\alpha\beta}x^\alpha x^\beta$ to show how a covariant tensor like $t_{\alpha\beta}$ transforms
   (d) apply this to the metric perturbation $h_{\alpha\beta}$

5. Show that for small boost velocity $v \ll c$ the metric in the lab-frame is
   - $$h_{\alpha'\beta'} = -2\Phi(r')\begin{bmatrix} 1 & & -2v/c \\ & \mathbf{I} & \\ -2v/c & & 1 \end{bmatrix}.$$

   where $\mathbf{I}$ the 2 by 2 identity matrix.

6. Dropping the primes on lab-frame coordinates here:
   (a) obtain the equation for $\ddot{z} = d^2z/d\tau^2$ for a particle fired by a lab-frame observer sitting at $x^i = (x, y, z) = (x > R, 0, 0)$ with velocity $\dot{x}^i = (\dot{x}, 0, 0)$ (i.e. towards or away from the axis of the moving cylinder). The right hand side should be expressed in terms of the potential Φ and the velocities of the rod and particle.
   (b) Describe and sketch the trajectory, as seen by the lab-frame observer, for a particle he fires towards the axis of the cylinder.
   (c) Does the apparent force on the particle bear any similarity to a) a centripetal force b) a coriolis forces or c) a magnetic force on a moving charge?

4 Frame-dragging inside a rotating cylinder

In weak-field gravity, and for stationary systems (for which $\Box = \nabla^2$), each component of $\Phi_{\mu\nu} \equiv -\overline{h}_{\mu\nu}/4$ obeys Poisson's equation with source term $T_{\mu\nu}$: i.e. $\nabla^2 \Phi_{\mu\nu} = 4\pi\kappa T_{\mu\nu}$. The equations of motions of test particles – the geodesic equation – depend only on spatial derivatives of $h_{\mu\nu} = \overline{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overline{h}$ and therefore only on derivatives of $\Phi_{\mu\nu}(\mathbf{x})$. That means one can calculate the components of the connection – and hence obtain equations of motion – by summing the effect of elements of the source just as in Newtonian gravity. I.e. using $\boldsymbol{\nabla}\Phi_{\mu\nu}(\mathbf{x}) = \kappa \int d^3x' T_{\mu\nu}(\mathbf{x}')(\mathbf{x} - \mathbf{x}')/|\mathbf{x} - \mathbf{x}'|^3$.

Here, you will apply this to a thin-walled massive cylinder of radius R that is rotating about the z-axis with angular velocity Ω, for which, neglecting the effect of stresses in the material of the cylinder, the stress-tensor is $T_{\mu\nu} = \rho c^2 \begin{bmatrix} 1 & \mathbf{v}/c \\ \mathbf{v}/c & 0 \end{bmatrix}$ where $\mathbf{v}$ is the 3-velocity of the material and where we will assume $|\mathbf{v}| \ll c$.

1. Appeal to Gauss's law to argue obtain the potential gradient $\boldsymbol{\nabla}\Phi$ a distance r from a needle of linear mass density $d\mu$.

2. Show thereby that one can replace the 3-dimensional integral above by

   - $\boldsymbol{\nabla}\Phi_{\mu\nu}(\mathbf{x}) = 2\kappa \int d^2x T_{\mu\nu}(\mathbf{x}')(\mathbf{x} - \mathbf{x}')/|\mathbf{x} - \mathbf{x}'|^2$

   where $\mathbf{x} \to (x, y)$. Or equivalently as a sum over needle-like source elements

   - $\boldsymbol{\nabla}\Phi_{\mu\nu}(\mathbf{x}) = \sum d\boldsymbol{\nabla}\Phi_{\mu\nu} = 2\kappa \sum dT_{\mu\nu}(\mathbf{x}')(\mathbf{x} - \mathbf{x}')/|\mathbf{x} - \mathbf{x}'|^2$

3. Using these, obtain expressions for $\Phi_{0x,y}$ and $\Phi_{0y,x}$ at a point on the axis (i.e. $x = y = 0$) as an integral over azimuthal angle $\phi = \operatorname{atan} y/x$ and in terms of μ the mass per unit length of the cylinder, and Ω. Hint: the result should be independent of the radius of the cylinder.

4. The gradients of all of the other potential components vanish at the origin by symmetry. If we take as boundary conditions that all of the $\Phi_{\mu\nu}$ vanish on the axis, therefore have, taking a first order expansion in position

   - $g_{\mu\nu} = \begin{bmatrix} -1 & h_{0x} & h_{0y} & \\ h_{0x} & 1 & & \\ h_{0y} & & 1 & \\ & & & 1 \end{bmatrix}$

   (a) Provide expressions for the two non-vanishing metric perturbation components here and compare with the corresponding terms in the metric for a rotating coordinate system in question 2.

   (b) Describe how these cause deflection of paths of particles thrown from the origin, and compare what is found here to what was found in question 3.

   (c) Why should we think this is a real physical effect, rather than just a coordinate artefact? After all, one could make a transformation to a relatively rotating frame to annul any effect locally. Hint: consider how distant stars would appear to the oberver.

Einstein felt that this kind of 'frame-dragging' largely vindicated 'Mach's principle' which he found highly influential in the development of his thinking about gravity. This principle is based on the observation that, if we feel ourselves not to be rotating – i.e. we are in an inertial frame – then distant stars appear stationary (and vice versa). The inference drawn from this is that somehow the state of motion of distant matter in the universe determines the inertial frame here.