TD-Probability

Marco Biroli

September 7, 2020

Chapter 1

TD 1

1.1 A strategic choice.

Let $X \in \{0,1\}^3$ (resp. Y) be the random variable corresponding to the results of the matches using the first strategy (resp. the second strategy). Then we have that (let $D = \{(1,1,1),(1,1,0),(0,1,1)\}$):

$$P(X \in D) = a^2 b + ab(1-a) + (1-a)ba = ab(2-a)$$

Similarly:

$$P(Y \in D) = b^2 a + ba(1-b) + (1-b)ab = ba(2-b)$$

Then since $a > b$ we have that $P(X \in D) < P(Y \in D)$, hence the winning strategy is BAB.

1.2 Derangements

1.2.1

Let E be a finite set and $A, B \subseteq E$. We denote by 1_A the indicator function of A and $\bar{A}$ the complement of A. Then we have that:

$$1_{\bar{A}} = 1 - 1_A \quad \text{and} \quad 1_{A \cap B} = 1_A \cdot 1_B \quad \text{and} \quad 1_{A \cup B} = 1_A + 1_B - 1_{A \cap B}$$

1.2.2

We will prove this by induction on n. The base case $n = 1$ as well as $n = 2$ are trivially satisfied. Now assume that this is satisfied for n then we have that (using the induction hypothesis for $n = 2$):

$$\text{card}\left(\bigcup_{i=1}^{n} A_i \bigcup A_{n+1}\right) = \text{card}\left(\bigcup_{i=1}^{n} A_i\right) + \text{card}(A_{n+1}) - \text{card}\left(\left(\bigcup_{i=1}^{n} A_i\right) \bigcap A_{n+1}\right)$$

Now we develop the last term into:

$$\left(\bigcup_{i=1}^{n} A_i\right) \bigcap A_{n+1} = \bigcup_{i=1}^{n} \left(A_i \bigcap A_{n+1}\right)$$

Now applying the induction hypothesis gives the desired result.

1.2.3

Let A_i be the set of permutations that fixes point i. Then from the inclusion-exclusion principle we have:

$$D_n = n! - \text{card}\left(\bigcup_{i=1}^{n} A_i\right) = n! - \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)! = n! \sum_{k=2}^{n} \frac{(-1)^k}{k!}$$

1.2.4

The probability that no one gets their jacket corresponds to the probability of having a derangement in other words:

$$p_n = \frac{D_n}{n!} = \sum_{k=2}^{n} \frac{(-1)^k}{k!} \overset{n \to \infty}{\longrightarrow} \frac{1}{e}$$

1.2.5

We have that:

$$D_{n,l} = \binom{n}{l} D_{n-l} = \binom{n}{l}(n-l)! \sum_{k=2}^{n-l} \frac{(-1)^k}{k!} = \frac{n!}{l!} \sum_{k=2}^{n-l} \frac{(-1)^k}{k!}$$

Hence the probability that eaxctly l people leave with their jackets is:

$$p_l = \frac{D_{n,l}}{n!} = \frac{1}{l!} \sum_{k=2}^{n-l} \frac{(-1)^k}{k!}$$

1.2.6

The probability that a given person gets back their jacket is $p_s = \frac{1}{n}$. The probability that at least one person gets back their jacket is:

$$p_a = 1 - p_n = 1 - \sum_{k=2}^{n} \frac{(-1)^k}{k!}$$

Notice that $p_s < p_a$.

1.3 Balls in bins

1.3.1

(a) If all the balls are distinguishable then we have $\Omega = [\![1, n]\!]^r$ is the set of tuples where each element corresponds to where the i-th ball has been sent to. Then $\mathcal{F} = \mathcal{P}(\Omega)$ and since each event is sampled uniformly at random we have that:

$$\forall \omega \in \Omega, P(\omega) = \frac{1}{|\Omega|} = \frac{1}{n^r}$$

Then the probability of $(r_1, \cdots, r_n)$ is given by:

$$P[(r_1, \cdots, r_n)] = P[\{\omega \in \Omega : \forall i \in [\![1, n]\!] \, \#\{b \in \Omega : b = i\} = r_i\}] = \frac{1}{n^r} \binom{r}{r_1, r_2, \ldots, r_n}$$

(b) Now we have that $\Omega = \{(r_i \in \mathbb{N} : i \in [\![1, n]\!]) : \sum_{i=1}^n r_i = r\}$. Again we have that $\mathcal{F} = \mathcal{P}(\Omega)$. Then we have that:

$$P[(r_1, \ldots, r_n)] = \frac{1}{|\Omega|} = \frac{1}{\binom{r+n-1}{n-1}}$$

(c) Now we have that $\Omega = \{s \in \{0, 1\}^n : \sum_{i=1}^n s_i = r\}$ corresponding to the tuple indicating if each state is occupied or not. Once again $\mathcal{F} = \mathcal{P}(\Omega)$. Now the probability is given by:

$$P[(r_1, \ldots, r_n)] = \frac{1}{|\Omega|} = \frac{1}{\binom{n}{r}}$$

1.3.2

The probability that at least two have the same birthday is the 1 minus the probability that none of them share a birthday. The probability that none of them share a birthday is given by $\frac{r!}{n^r}\binom{n}{r}$. Hence the probability that at least two people share a birthday is given by: $1 - \frac{r!}{n^r}\binom{n}{r}$.

1.3.3

Days are bins, accidents are distinguishable balls hence the probability is given by:

$$\frac{\binom{r}{n} n^{r-n}}{n^r} = n^{-n} \binom{r}{n}$$

1.3.4

Chapter 2

TD2

2.1 Symmetric Random Walk

Consider a balanced coin drawn n times. Denote $X_1, \cdots, X_n$ the results and S_k the partial sums.

2.1.1

The law of S_k is given by:

$$p_{n,r} = P(S_n = r) = \frac{1}{2^n}\binom{n}{\frac{n+r}{2}}$$

2.1.2

The number of paths from $(0,0)$ to $(2n+2,0)$ never zero are equal to the number of paths from $(1,1)$ to $(2n+1,1)$ which always stay above or equal to the line $y=1$. Hence rescaling the y-axis by a factor 1 we get a bijection in between the strictly positive walks from $(0,0)$ to $(2n+2,0)$ with the positive or zero walks from $(0,0)$ to $(2n,0)$. Hence for symmetric random walks the number of random walks going from $(0,0)$ to $(2n+2,0)$ never touching the axis is twice as much as the number of walks from $(0,0)$ to $(2n,0)$ being always positive or 0. Furhtemore there are 4 times more walks going from 0 to $2n+2$ which therefore gives the desired result.

2.1.3

We now that the end of the random walk is going to be given by $a-b$. Now the number of possible only positive walks is given by the number of walks from $(1,1)$ to $(a+b, a-b)$ minus the number of walks from $(1,-1)$ to $(a+b, a-b)$ by the reflexion principle. Hence we get that:

$$p = p_{a+b-1,a-b-1} - p_{a+b-1,a-b+1} = \frac{1}{2^n}\frac{a-b}{a+b}\binom{a+b}{a}$$

2.1.4

a)

Up to a re-scaling of the y-axis we have the equivalent problem of computing the number of paths that go from $(0,-r)$ to $(n, k-r)$ and which touch the x-axis at least once. Now notice that from the reflexion principle this is equal to the number of paths from $(0,r)$ to $(n,k-r)$ and up to a second shifting this is equal to the number of paths from $(0,0)$ to $(n, k-2r)$. Hence the desired probability is given by $p_{n,k-2r} = p_{n,2r-k}$.

b)

Then for any r we have that:

$$\begin{aligned}
P(\max\{S_1,\cdots,S_n\}=r) &= \sum_{k=-\infty}^{+\infty} P(S_n=k,\max\{S_1,\cdots,S_n\}=r)\\
&= \sum_{k=-\infty}^{+\infty} \left(P(S_n=k,\max\{S_1,\cdots,S_n\}\geq r)-P(S_n=k,\max\{S_1,\cdots,S_n\}>r)\right)\\
&= \sum_{k=-\infty}^{+\infty} \left(P(S_n=k,\max\{S_1,\cdots,S_n\}\geq r)-P(S_n=k,\max\{S_1,\cdots,S_n\}\geq r+1)\right)\\
&= \sum_{k=-\infty}^{+\infty} \left(p_{n,k-2r}-p_{n,k-2r-2}\right) = p_{n,r}+p_{n,r+1}
\end{aligned}$$

c)

This can be re-written as:

$$P(S_n=0,S_1<0,\cdots,S_{n-1}<0,S_0=-r)=P(S_n=-r,S_1<0,\cdots,S_{n-1}<0,S_0=0)$$

Now again notice that this corresponds to a symmetric re-writing of the problem 3. Hence we get immediately that:

$$\hat{p}_{n,r}=P(S_n=r,S_1<r,\cdots,S_{n-1}<r)=\frac{1}{2^n}\frac{r}{n}\binom{n}{\frac{n+r}{2}}=\frac{r}{n}p_{n,r}$$

d)

2.2 Geometric and negative-binomial laws.

2.2.1

We want to compute:

$$P(T_1=t+1)$$

For such a thing to be the case we need to have thrown tails successively t times and heads the last time. Hence the probability is given by:

$$P(T_1-1=t)=P(T_1=t+1)=(1-p)^t p=\mathcal{G}(p)$$

The expectancy of $\mathcal{G}(p)$ is given by:

$$\mathbb{E}[\mathcal{G}(p)]=\sum_{t=0}^{+\infty}(1-p)^t pt=p\cdot\frac{1-p}{p^2}=\frac{1-p}{p}$$

The variance of $\mathcal{G}(p)$ is given by:

$$\mathrm{Var}[\mathcal{G}(p)]^2=\frac{p^2-3p+2}{p^2}-\frac{1-2p+p^2}{p^2}=\frac{1-p}{p^2}$$

2.2.2

A geometric law corresponds to something not happening t times and then happening at the $t+1$ time. Then the infimum of two geometric laws corresponds to two things not happening t times and one of them happening at the $t+1$ time. The probability of which is given as follows:

$$P[(\inf(S_1,S_2)=s]=(1-p)^{2(s-1)}(p^2+2p(1-p))=(1-p)^{2(s-1)}(2p-p^2)=(1-p)^{2(s-1)}(1-(1-p)^2)$$

Hence the infimum is a geometric variable with law $\mathcal{G}(1-(1-p)^2)$

2.2.3

We want to compute $P(T_m-m=k)=$.