TD-Probability

Marco Biroli

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Chapter 1

TD 1

1.1 A strategic choice.

Let $X \in \{0,1\}^3$ (resp. Y) be the random variable corresponding to the results of the matches using the first strategy (resp. the second strategy). Then we have that (let $D = \{(1,1,1),(1,1,0),(0,1,1)\}$):

$$P(X \in D) = a^2 b + ab(1-a) + (1-a)ba = ab(2-a)$$

Similarly:

$$P(Y \in D) = b^2 a + ba(1-b) + (1-b)ab = ba(2-b)$$

Then since $a > b$ we have that $P(X \in D) < P(Y \in D)$, hence the winning strategy is BAB.

1.2 Derangements

1.2.1

Let E be a finite set and $A, B \subseteq E$. We denote by 1_A the indicator function of A and $\bar{A}$ the complement of A. Then we have that:

$$1_{\bar{A}} = 1 - 1_A \quad \text{and} \quad 1_{A \cap B} = 1_A \cdot 1_B \quad \text{and} \quad 1_{A \cup B} = 1_A + 1_B - 1_{A \cap B}$$

1.2.2

We will prove this by induction on n. The base case $n = 1$ as well as $n = 2$ are trivially satisfied. Now assume that this is satisfied for n then we have that (using the induction hypothesis for $n = 2$):

$$\text{card}\left(\bigcup_{i=1}^{n} A_i \bigcup A_{n+1}\right) = \text{card}\left(\bigcup_{i=1}^{n} A_i\right) + \text{card}(A_{n+1}) - \text{card}\left(\left(\bigcup_{i=1}^{n} A_i\right)\bigcap A_{n+1}\right)$$

Now we develop the last term into:

$$\left(\bigcup_{i=1}^{n} A_i\right)\bigcap A_{n+1} = \bigcup_{i=1}^{n}\left(A_i \bigcap A_{n+1}\right)$$

Now applying the induction hypothesis gives the desired result.

1.2.3

Let A_i be the set of permutations that fixes point i. Then from the inclusion-exclusion principle we have:

$$D_n = n! - \text{card}\left(\bigcup_{i=1}^{n} A_i\right) = n! - \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(n-k)! = n!\sum_{k=2}^{n}\frac{(-1)^k}{k!}$$

1.2.4

The probability that no one gets their jacket corresponds to the probability of having a derangement in other words:

$$p_n = \frac{D_n}{n!} = \sum_{k=2}^{n}\frac{(-1)^k}{k!} \xrightarrow{n \to \infty} \frac{1}{e}$$

1.2.5

We have that:

$$D_{n,l} = \binom{n}{l} D_{n-l} = \binom{n}{l}(n-l)! \sum_{k=2}^{n-l} \frac{(-1)^k}{k!} = \frac{n!}{l!} \sum_{k=2}^{n-l} \frac{(-1)^k}{k!}$$

Hence the probability that eaxctly l people leave with their jackets is:

$$p_l = \frac{D_{n,l}}{n!} = \frac{1}{l!} \sum_{k=2}^{n-l} \frac{(-1)^k}{k!}$$

1.2.6

The probability that a given person gets back their jacket is $p_s = \frac{1}{n}$. The probability that at least one person gets back their jacket is:

$$p_a = 1 - p_n = 1 - \sum_{k=2}^{n} \frac{(-1)^k}{k!}$$

Notice that $p_s < p_a$.

1.3   Balls in bins

1.3.1

(a) If all the balls are distinguishable then we have $\Omega = [\![1,n]\!]^r$ is the set of tuples where each element corresponds to where the i-th ball has been sent to. Then $\mathcal{F} = \mathcal{P}(\Omega)$ and since each event is sampled uniformly at random we have that:

$$\forall \omega \in \Omega, P(\omega) = \frac{1}{|\Omega|} = \frac{1}{n^r}$$

Then the probability of $(r_1, \cdots, r_n)$ is given by:

$$P[(r_1, \cdots, r_n)] = P[\{\omega \in \Omega : \forall i \in [\![1,n]\!]\, \#\{b \in \Omega : b = i\} = r_i\}] = \frac{1}{n^r} \binom{r}{r_1, r_2, \ldots, r_n}$$

(b) Now we have that $\Omega = \{(r_i \in \mathbb{N} : i \in [\![1,n]\!]) : \sum_{i=1}^{n} r_i = r\}$. Again we have that $\mathcal{F} = \mathcal{P}(\Omega)$. Then we have that:

$$P[(r_1, \ldots, r_n)] = \frac{1}{|\Omega|} = \frac{1}{\binom{r+n-1}{n-1}}$$

(c) Now we have that $\Omega = \{s \in \{0,1\}^n : \sum_{i=1}^{n} s_i = r\}$ corresponding to the tuple indicating if each state is occupied or not. Once again $\mathcal{F} = \mathcal{P}(\Omega)$. Now the probability is given by:

$$P[(r_1, \ldots, r_n)] = \frac{1}{|\Omega|} = \frac{1}{\binom{n}{r}}$$

1.3.2

The probability that at least two have the same birthday is the 1 minus the probability that none of them share a birthday. The probability that none of them share a birthday is given by $\frac{r!}{n^r}\binom{n}{r}$. Hence the probability that at least two people share a birthday is given by: $1 - \frac{r!}{n^r}\binom{n}{r}$.

1.3.3

Days are bins, accidents are distinguishable balls hence the probability is given by:

$$\frac{\binom{r}{n}n^{r-n}}{n^r} = n^{-n}\binom{r}{n}$$

1.3.4

Chapter 2

TD2

2.1 Symmetric Random Walk

Consider a balanced coin drawn n times. Denote $X_1, \cdots, X_n$ the results and S_k the partial sums.

2.1.1

The law of S_k is given by:

$$p_{n,r} = P(S_n = r) = \frac{1}{2^n} \binom{n}{\frac{n+r}{2}}$$

2.1.2

The number of paths from $(0,0)$ to $(2n+2, 0)$ never zero are equal to the number of paths from $(1,1)$ to $(2n+1, 1)$ which always stay above or equal to the line $y = 1$. Hence rescaling the y-axis by a factor 1 we get a bijection in between the strictly positive walks from $(0,0)$ to $(2n+2, 0)$ with the positive or zero walks from $(0,0)$ to $(2n, 0)$. Hence for symmetric random walks the number of random walks going from $(0,0)$ to $(2n+2, 0)$ never touching the axis is twice as much as the number of walks from $(0,0)$ to $(2n, 0)$ being always positive or 0. Furhtemore there are 4 times more walks going from 0 to $2n+2$ which therefore gives the desired result.

2.1.3

We now that the end of the random walk is going to be given by $a - b$. Now the number of possible only positive walks is given by the number of walks from $(1,1)$ to $(a + b, a - b)$ minus the number of walks from $(1, -1)$ to $(a + b, a - b)$ by the reflexion principle. Hence we get that:

$$p = p_{a+b-1,a-b-1} - p_{a+b-1,a-b+1} = \frac{1}{2^n} \frac{a-b}{a+b} \binom{a+b}{a}$$

2.1.4

a)

Up to a re-scaling of the y-axis we have the equivalent problem of computing the number of paths that go from $(0, -r)$ to $(n, k - r)$ and which touch the x-axis at least once. Now notice that from the reflexion principle this is equal to the number of paths from $(0, r)$ to $(n, k - r)$ and up to a second shifting this is equal to the number of paths from $(0, 0)$ to $(n, k - 2r)$. Hence the desired probability is given by $p_{n,k-2r} = p_{n,2r-k}$.

b)

Then for any r we have that:

$$P(\max\{S_1, \cdots, S_n\} = r) = \sum_{k=-\infty}^{+\infty} P(S_n = k, \max\{S_1, \cdots, S_n\} = r)$$

$$= \sum_{k=-\infty}^{+\infty} (P(S_n = k, \max\{S_1, \cdots, S_n\} \geq r) - P(S_n = k, \max\{S_1, \cdots, S_n\} > r))$$

$$= \sum_{k=-\infty}^{+\infty} (P(S_n = k, \max\{S_1, \cdots, S_n\} \geq r) - P(S_n = k, \max\{S_1, \cdots, S_n\} \geq r+1))$$

$$= \sum_{k=-\infty}^{+\infty} (p_{n,k-2r} - p_{n,k-2r-2}) = p_{n,r} + p_{n,r+1}$$

c)

This can be re-written as:

$$P(S_n = 0, S_1 < 0, \cdots, S_{n-1} < 0, S_0 = -r) = P(S_n = -r, S_1 < 0, \cdots, S_{n-1} < 0, S_0 = 0)$$

Now again notice that this corresponds to a symmetric re-writing of the problem 3. Hence we get immediately that:

$$\hat{p}_{n,r} = P(S_n = r, S_1 < r, \cdots, S_{n-1} < r) = \frac{1}{2^n} \frac{r}{n} \binom{n}{\frac{n+r}{2}} = \frac{r}{n} p_{n,r}$$

d)

2.2 Geometric and negative-binomial laws.

2.2.1

We want to compute:

$$P(T_1 = t + 1)$$

For such a thing to be the case we need to have thrown tails successively t times and heads the last time. Hence the probability is given by:

$$P(T_1 - 1 = t) = P(T_1 = t + 1) = (1-p)^t p = \mathcal{G}(p)$$

The expectancy of $\mathcal{G}(p)$ is given by:

$$\mathbb{E}[\mathcal{G}(p)] = \sum_{t=0}^{+\infty} (1-p)^t pt = p \cdot \frac{1-p}{p^2} = \frac{1-p}{p}$$

The variance of $\mathcal{G}(p)$ is given by:

$$\mathrm{Var}[\mathcal{G}(p)]^2 = \frac{p^2 - 3p + 2}{p^2} - \frac{1 - 2p + p^2}{p^2} = \frac{1-p}{p^2}$$

2.2.2

A geometric law corresponds to something not happening t times and then happening at the $t+1$ time. Then the infimum of two geometric laws corresponds to two things not happening t times and one of them happening at the $t+1$ time. The probability of which is given as follows:

$$P[(\inf(S_1, S_2) = s] = (1-p)^{2(s-1)}(p^2 + 2p(1-p)) = (1-p)^{2(s-1)}(2p - p^2) = (1-p)^{2(s-1)}(1 - (1-p)^2)$$

Hence the infimum is a geometric variable with law $\mathcal{G}(1 - (1-p)^2)$

2.2.3

We want to compute $P(T_m - m = k) =$.