$$f_X(x) \geq 0$$

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

$$\mathbb{P}(X \in [a, b]) = \int_a^b f_X(x)\, dx$$

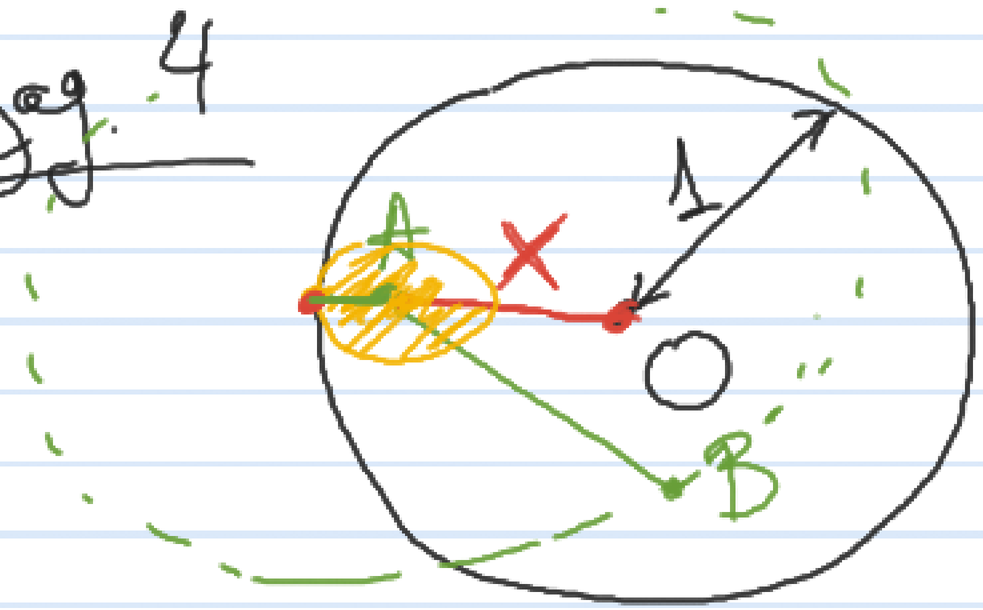
$$F_X(x) = \mathbb{P}(X < x)$$

$$F_X'(x) = f_X(x)$$

$$\mathbb{E}X = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx \quad \leftarrow$$

$$= \int_{-\infty}^{\infty} x\, dF_X(x)$$

зад. 4



$$\mathbb{P}(k(A, AB) \subset k(O, 1)) = ?$$

$\Sigma p_k \cdot [\quad]$

При фиксирано A, $B \in$ вътре в кръга $k(A, 1-OA)$

$$\mathbb{P}(B \in k(A, 1-x)) = \frac{\pi (1-x)^2}{\pi \cdot 1^2} = (1-x)^2$$

Ако знаем $f_X(x)$, отг. би бил

$$\int_0^1 (1-x)^2 \underbrace{f_X(x)}\, dx$$

$$\mathbb{E} g(X) = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

$$f_X(x) = F'_X(x)$$

$$F_X(x) = \mathbb{P}(X < x) = \mathbb{P}(X \in k(0,x)) = \frac{\pi \cdot x^2}{\pi \cdot 1^2} = \underline{\underline{x^2}}$$

за $x \in [0,1]$

$$\Rightarrow f_X(x) = 2x \cdot \mathbb{1}_{\{x \in [0,1]\}}$$

Заместване: $\int_0^1 (1-x)^2 \cdot 2x \, dx$

$$= 2\int_0^1 x - 2x^2 + x^3 \, dx = 2\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$$

$$= 2\left(\frac{6-8+3}{12}\right) = \underline{\underline{\frac{1}{6}}}.$$

$$\mathbb{P}(K_1 \subset K_2) = \mathbb{E}\,\mathbb{1}_{\{K_1 \subset K_2\}}$$

X

$$= \mathbb{E}\Big(\underbrace{\mathbb{E}\big(\mathbb{1}_{\{K_1 \subset K_2\}} \mid X\big)}_{(1-X)^2}\Big) = \underline{\underline{\mathbb{E}\big[\mathbb{P}(K_1 \subset K_2 \mid X)\big]}}$$

$$= \mathbb{E}(1-X)^2$$

$$\underline{\mathbb{E}X = \mathbb{E}\big(\mathbb{E}(X \mid Y)\big)} \quad \text{(tower property)}$$

зад. 5

$X_1 \sim Exp(\lambda_1)$ $\quad X_2 \sim Exp(\lambda_2)$

$EX_1 = 8$ мин $\Rightarrow \lambda_1 = 1/8$ так

$EX_2 = 5$

$A = \{$ чака < 4 мин $\}$

$P(I \mid A) = ?$

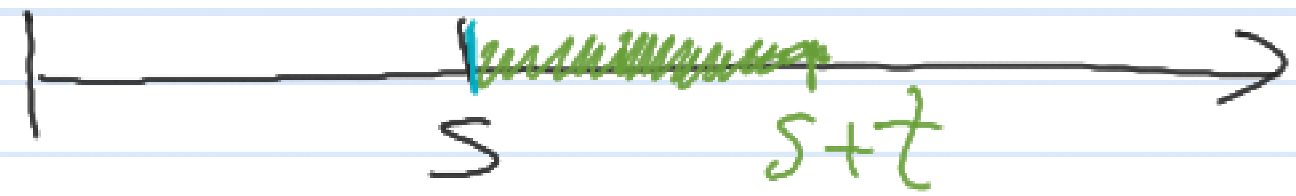
I – избира опашка 1

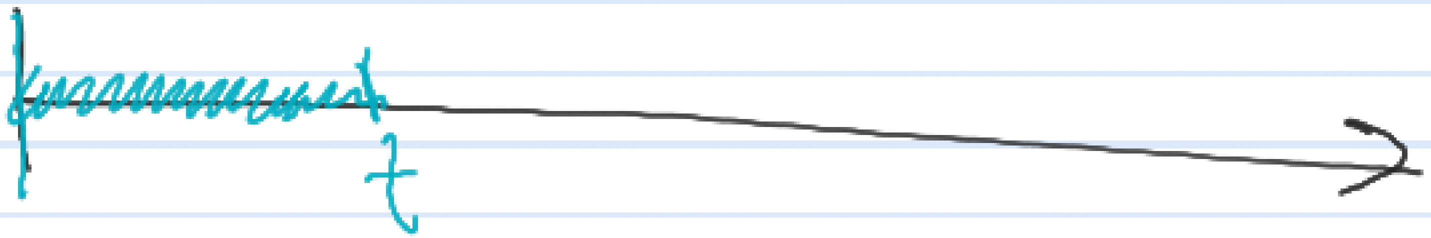
II — " — 2

$X \sim Exp(\lambda)$, ако $f_X(x) = \lambda e^{-\lambda x} 1_{\{x>0\}}$

Липса на памет

$$P(X > s+t \mid X > s) = P(X > t)$$





$$\frac{P(X>s+t,\ X>s)}{P(X>s)} = \frac{P(X>s+t)}{P(X>s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t}$$

$$P(X>s) = \int_s^\infty \lambda e^{-\lambda x}\, dx \underset{y=\lambda x}{=} \int_{s\lambda}^\infty e^{-y}\, dy = \left[-e^{-y}\right]_{s\lambda}^\infty = e^{-\lambda s}$$

$F_X(x) = P(X < x) = 1 - e^{-\lambda x}$

за $x > 0$

и 0 иначе

$\mathbb{E}X = \frac{1}{\lambda}$

$\mathbb{D}X = \frac{1}{\lambda^2}$

$$\mathbb{E}X = \int_0^\infty x \cdot \underbrace{\lambda e^{-\lambda x}}_{f_X(x)} \, dx$$

$y = \lambda x$

$$= \int_0^\infty \frac{y}{\lambda} e^{-y} \, dy$$

$$= \frac{1}{\lambda} \int_0^\infty y e^{-y} \, dy$$

$$= \frac{1}{\lambda} \int_0^\infty -y \, de^{-y}$$

$$= \frac{1}{\lambda} \left(\left[-y e^{-y} \right]_0^\infty - \int_0^\infty e^{-y} \, d(-y) \right)$$

$$= \frac{1}{\lambda} \int_0^\infty e^{-y} \, dy = \frac{1}{\lambda} \left[-e^{-y} \right]_0^\infty$$

$$= \frac{1}{\lambda}.$$

$$P(I \mid A) = \frac{P(I \cap A)}{P(A)}$$

$$P(A) = P(A \mid I)P(I) + P(A \mid II)P(II)$$

$$= P(X_1 < 4) \cdot \frac{1}{2} + P(X_2 < 4) \cdot \frac{1}{2}$$

$$= \left(1 - e^{-\frac{1}{8} \cdot 4} + 1 - e^{-\frac{1}{5} \cdot 4}\right) \cdot \frac{1}{2}$$

$X_1 \sim Exp(1/8)$

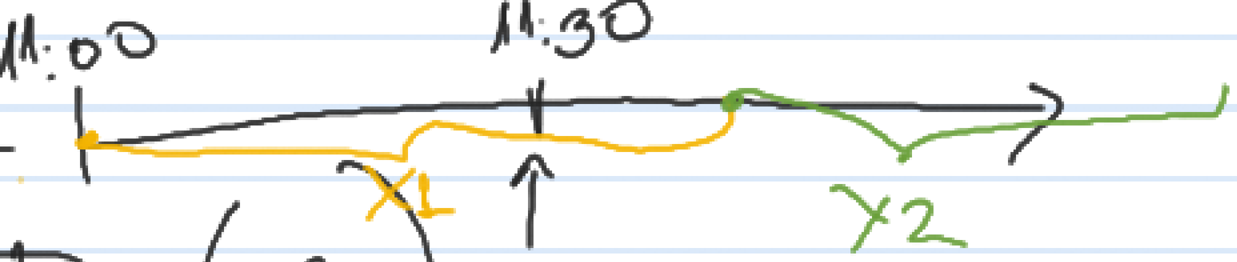
$X_2 \sim Exp(1/5)$

$F_X(x) = 1 - e^{-\lambda x}$

$$P(I \cap A) = P(A \mid I) \cdot P(I) = \left(1 - e^{-\frac{1}{8} \cdot 4}\right) \cdot \frac{1}{2}$$

Отг: $\dfrac{1 - e^{-1/2}}{2 - e^{-1/2} - e^{-4/5}}$

зад. 6



$X \sim Exp(2)$

X_i – „времето за обсл. на i"

Y – „времето, което 2-рият е бил в поликл."

$\mathbb{E}Y = ?$

$$Y = \mathbb{1}_{\{X_1 < 1/2\}} \cdot X_2 + \mathbb{1}_{\{X_1 \geq \frac{1}{2}\}}\left(X_2 + \left(X_1 - \frac{1}{2}\right)\right)$$

$$= X_2 + \left(X_1 - \frac{1}{2}\right)\mathbb{1}_{\{X_1 \geq \frac{1}{2}\}}$$

$$\mathbb{E}Y = \mathbb{E}X_2 + \mathbb{E}\left(X_1 - \frac{1}{2}\right)\mathbb{1}_{\{X_1 \geq \frac{1}{2}\}}$$

$$= \frac{1}{2} + \int_{\frac{1}{2}}^{\infty}\left(x - \frac{1}{2}\right) 2 \cdot e^{-2x}\, dx$$

зад. 7 $X \sim Exp(\lambda)$

Каква е плътността на $Y =$

а) $-X$

б) $2X - 1$

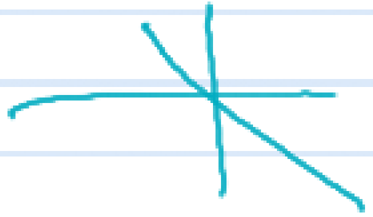
в) $\sqrt{X}$

$f_X(x)$ ✓

$\mathbb{P}(X < x)$

$f_g(x) = ?$

$\mathbb{P}(Y < x)$ чрез $\mathbb{P}(X < x)$



а) $Y = -X$

$$\mathbb{P}(Y < y) = \mathbb{P}(-X < y)$$

$$= \mathbb{P}(X > -y)$$

$$\rightarrow \mathbb{P}(X > s) = \begin{cases} e^{-\lambda s}, & s \geq 0 \\ 1, & s < 0 \end{cases}$$

$$= 1 \cdot \mathbb{1}_{\{-y < 0\}} + e^{\lambda y} \mathbb{1}_{\{-y \geq 0\}}$$

$$\Rightarrow f_Y(y) = \frac{d}{dy} \mathbb{P}(Y < y)$$

$$= \frac{d}{dy}\left(1 \cdot \mathbb{1}_{\{y > 0\}} + e^{\lambda y} \mathbb{1}_{\{-y \geq 0\}}\right)$$

$$= \lambda e^{\lambda y} \mathbb{1}_{\{y \leq 0\}}$$

Πυκνότητα για $Y=-X$

$$e \quad f_Y(y) = \lambda e^{\lambda y} \mathbb{1}_{\{y \le 0\}}$$

$$\mathbb{P}(Y<y) = \mathbb{P}(X>-y) = \begin{cases} e^{\lambda y}, & y \le 0 \\ 1, & y>0 \end{cases}$$

$$\Rightarrow f_Y(y) = \begin{cases} \lambda e^{\lambda y}, & y \le 0 \\ 0, & y>0 \end{cases}$$

$$Y = 2X-1$$

$$\mathbb{P}(2X-1<y)$$

$$= \mathbb{P}\left(X < \frac{y+1}{2}\right)$$

$$= F_X\left(\frac{y+1}{2}\right)$$

$$* \; \mathbb{P}(X<s) = \begin{cases} 1-e^{-\lambda s}, & s \ge 0 \\ 0, & s<0 \end{cases}$$

$$F_X(s)$$

$$\Rightarrow f_Y(y) = \lambda e^{-\lambda\left(\frac{y+1}{2}\right)} \cdot \frac{1}{2} \cdot \mathbb{1}_{\left\{\frac{y+1}{2} \ge 0\right\}}$$

$$= \frac{1}{2}\lambda e^{-\lambda\left(\frac{y+1}{2}\right)} \mathbb{1}_{\{y \ge -1\}}$$

$$f_{g(X)}(x) = ?$$

$$\mathbb{P}(g(X) < x) = \mathbb{P}(X < g^{-1}(x))$$

(g е обратима)

g е ↑

$$= F_X(g^{-1}(x))$$

$$\Rightarrow f_{g(X)}(x) = \frac{d}{dx} F_{g(X)}(x) = \frac{d}{dx}\left(F_X(g^{-1}(x))\right)$$

$$= F_X'(g^{-1}(x)) \cdot g^{-1}(x)'$$

$$= f_X(g^{-1}(x))\, g^{-1}(x)'$$

g е $\uparrow$ или $\downarrow$

$$f_{g(X)}(x) = f_X(g^{-1}(x)) \cdot |g^{-1}(x)'| .$$