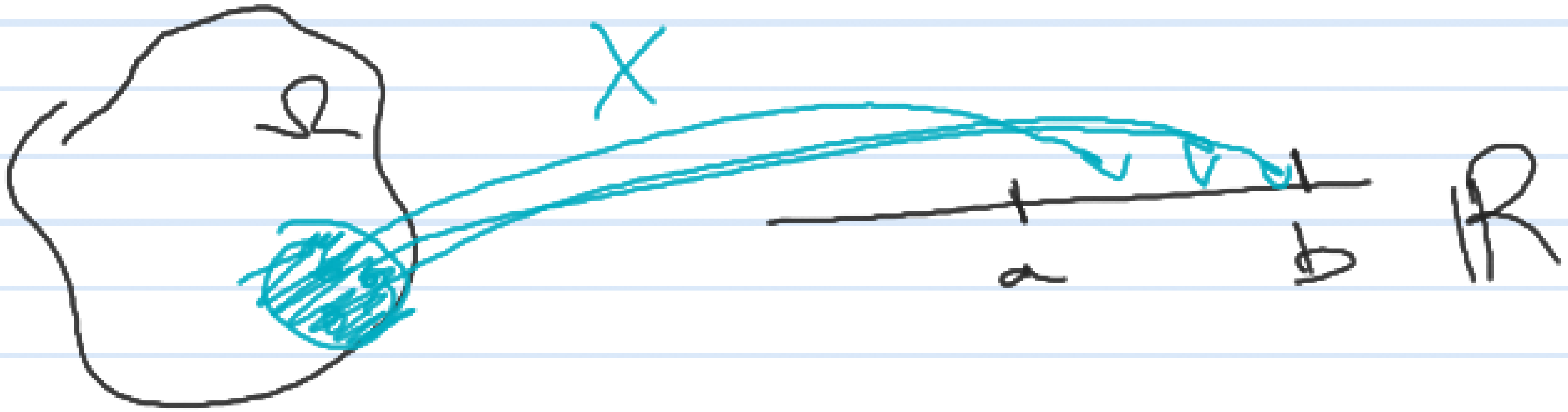
Непр. случ. вел. $(\Omega, \mathcal{F}, \mathbb{P})$

$$X : \Omega \to \mathbb{R}$$

$$X^{-1}((a,b)) \in \mathcal{F}$$

$$\mathbb{P}(X \in (a,b)) = \dots$$



Ако X има плътност – е непр.

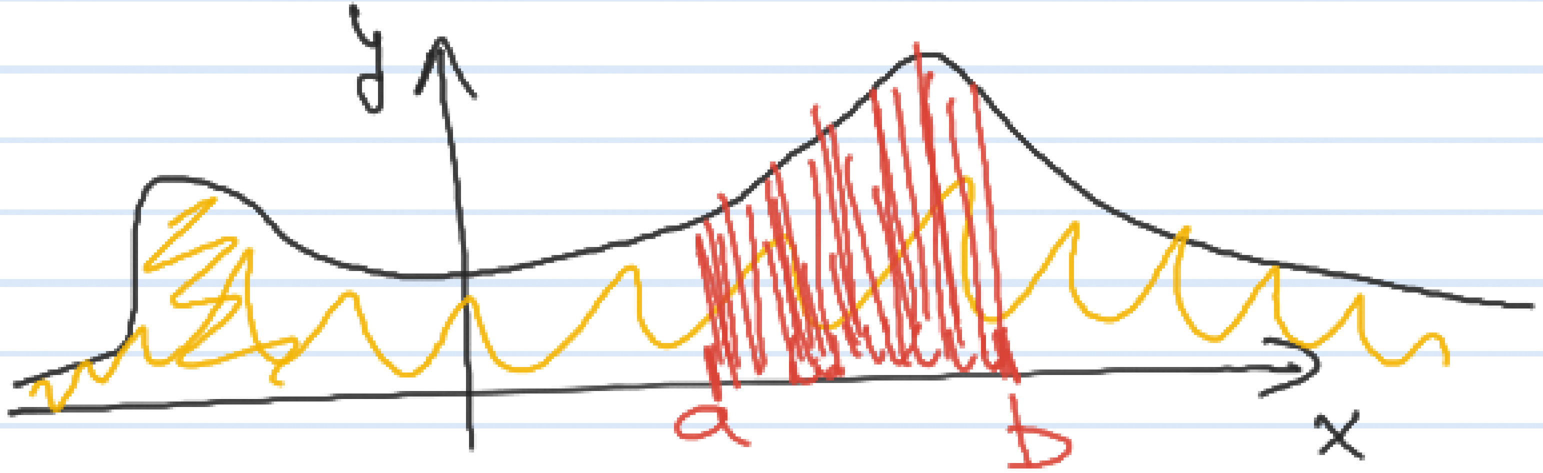
дискр.

X	x_1	x_2	x_3	
	p_1	p_2	p_3	...

$$\sum p_i = 1$$

непр.

pdf $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$



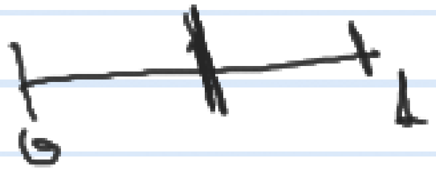
$$P(X \in [a, b]) = P(X \in (a, b))$$

$$= \dots = \int_a^b f_X(x)\,dx$$

cdf $F_X(x) = P(X < x)$

Ако има плътност $P(X = y) = \int_y^y f_X(t)\,dt = 0$

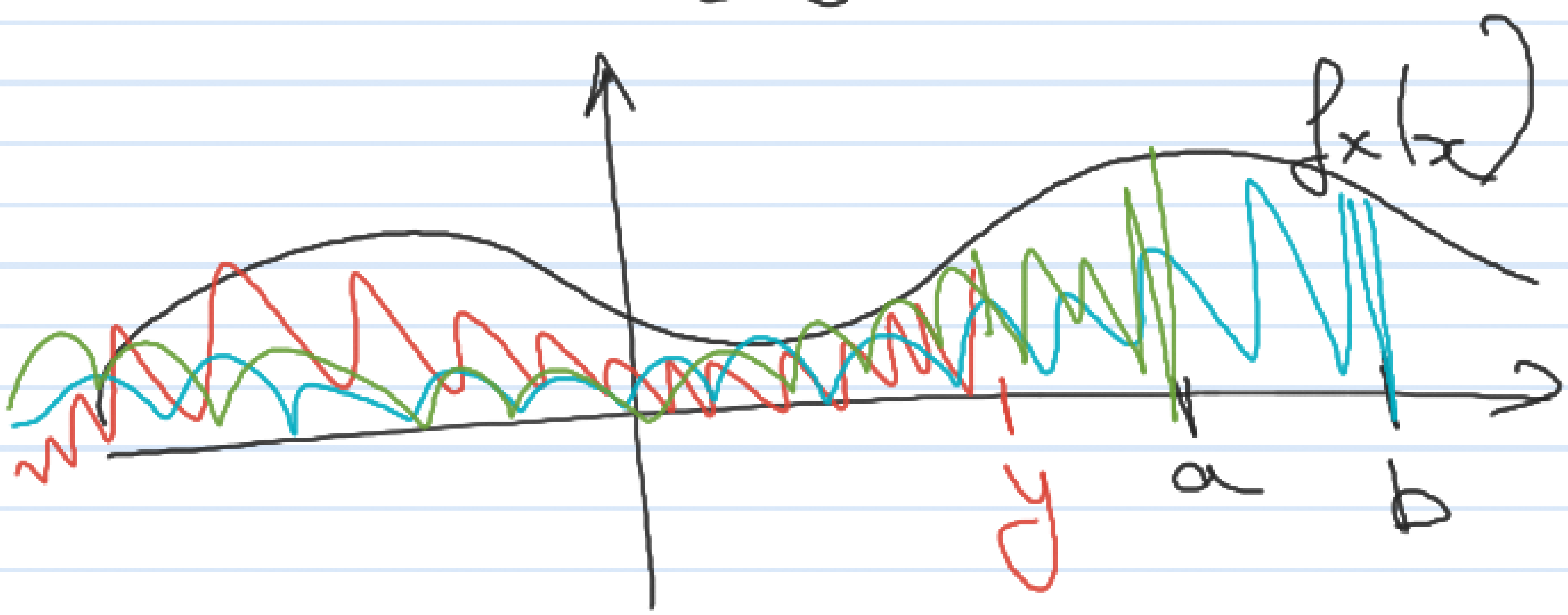
има плътност $\Leftrightarrow$ F е непрек. + диф нещо



$$P\left(X = \frac{1}{2}\right) = \frac{\left|\left[\frac{1}{2}, \frac{1}{2}\right]\right|}{1} = 0$$

$$\boxed{F'_X(x) = f_X(x)}$$

$$F_X(x) = \mathbb{P}(X < x)$$
$$= \int_{-\infty}^{x} f_X(x)\,dx$$



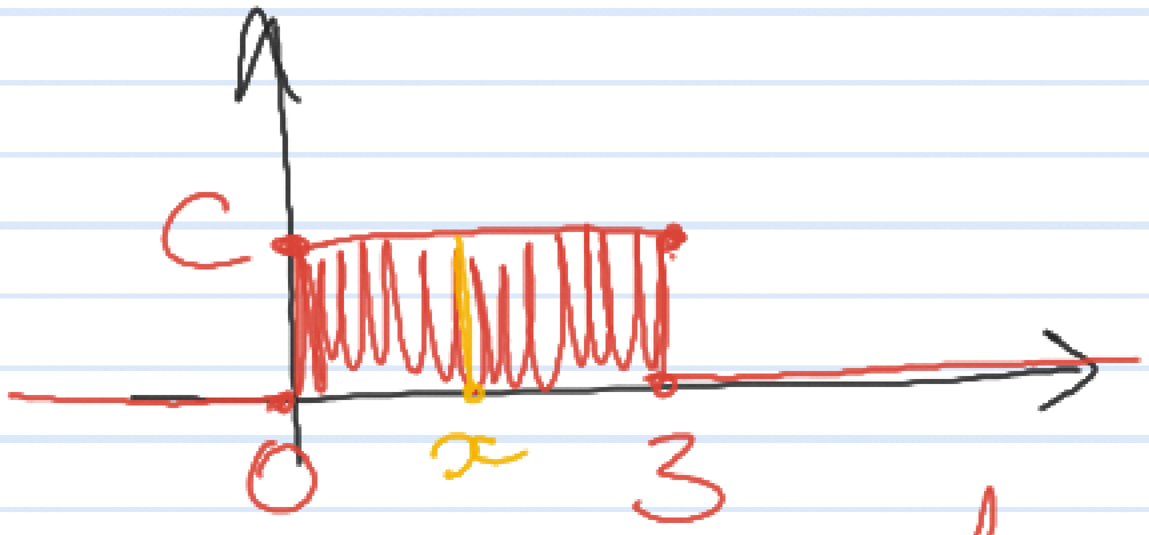
$F_X(y)$

$$\mathbb{P}(X \in (a, b)) = \int_a^b f_X(x)\,dx$$
$$= \underline{F(b)} - \underline{F(a)}$$

$X \sim U(a,b)$

$\sim Unif(a,b)$ – равномерно в/у $[a,b]$

Каква е плътността на $X \sim U[0,3]$?



$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\int_0^3 c\,dx = 1$$

$$3c = 1 \Rightarrow c = \frac{1}{3}$$

$$f_X(x) = \frac{1}{3} \cdot \mathbb{1}_{\{x \in [0,3]\}} = \begin{cases} \frac{1}{3}, & x \in [0,3] \\ 0, & \text{иначе}. \end{cases}$$

$X \sim U(a,b)$

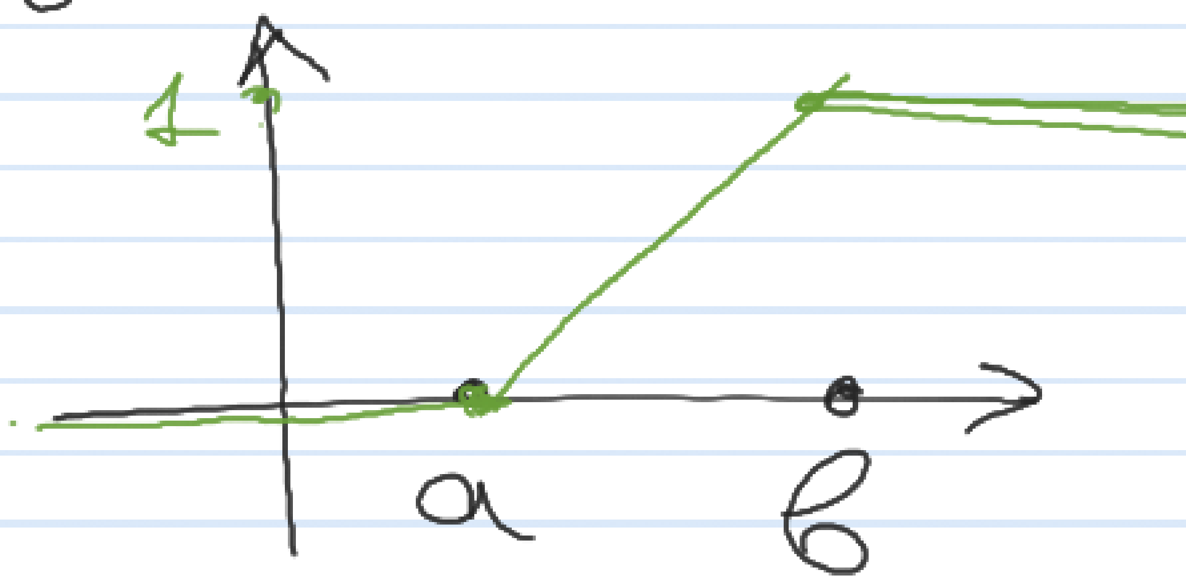
$$f_X(x) = \frac{1}{b-a} \mathbb{1}_{\{x \in [a,b]\}}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \mathbb{P}(X < x)$$

Ако $x \le a$: е 0

$x \ge b$: е 1

$$\int_0^x \frac{1}{b-a}\,dt = \frac{x}{b-a}$$



$$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$\mathbb{E}X^2 = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$\mathbb{E}g(X) = \int_{-\infty}^{\infty} g(y) \cdot f_X(y)\,dy$$

Law of the unconscious statistician

$$\mathbb{E}\,U(a,b) = \frac{b-a}{2}$$

$$D\left[U(a,b)\right] = \frac{(b-a)^2}{12}$$

$$\mathbb{E}X^2 = \sum x^2 \cdot \mathbb{P}(X=x)$$

zag. 1 $f_X(x) = c(x^2+2x)\mathbb{1}_{\{x\in[0,1]\}} = \{ \dots$

1. $\int_{-\infty}^{\infty} f_X(x) = 1 = \int_0^1 c(x^2+2x)\,dx = c\left[\frac{x^3}{3} + 2\frac{x^2}{2}\right]_0^1$

$$= c\left(\frac{1}{3}+1\right) \Rightarrow \boxed{c = \frac{3}{4}}$$

2. $\mathbb{E}X, \mathbb{D}X$

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x\cdot c\cdot(x^2+2x)\,dx$$

$$= c\left[\frac{x^4}{4} + 2\frac{x^3}{3}\right]_0^1 = c\cdot\frac{11}{12} = \frac{3}{4}\cdot\frac{11}{12} = \frac{11}{16}$$

$DX = \mathbb{E}X^2 - (\mathbb{E}X)^2$

$\mathbb{E}X^2 = \int_0^1 x^2 \cdot f_X(x)\,dx = \dots$ аналог. сметки

3) $\mathbb{P}(X < \mathbb{E}X) \overset{1.}{=} \mathbb{P}\left(X < \frac{11}{16}\right)$

$= \int_{\cancel{-\infty}\ 0}^{11/16} f_X(x)\,dx$

$\mathbb{P}\left(X < \frac{11}{16}\right) = F_X\left(\frac{11}{16}\right)$

$F_X(y) = ?$

$$F_X(y) = \begin{cases} 0 & y < 0 \\ c\left(\frac{y^3}{3} + y^2\right) & \text{за } y \in [0,1] \\ 1 & , y > 1 \end{cases}$$

$\mathbb{P}(X < y) = \int_0^y c \cdot (x^2 + 2x)\,dx$

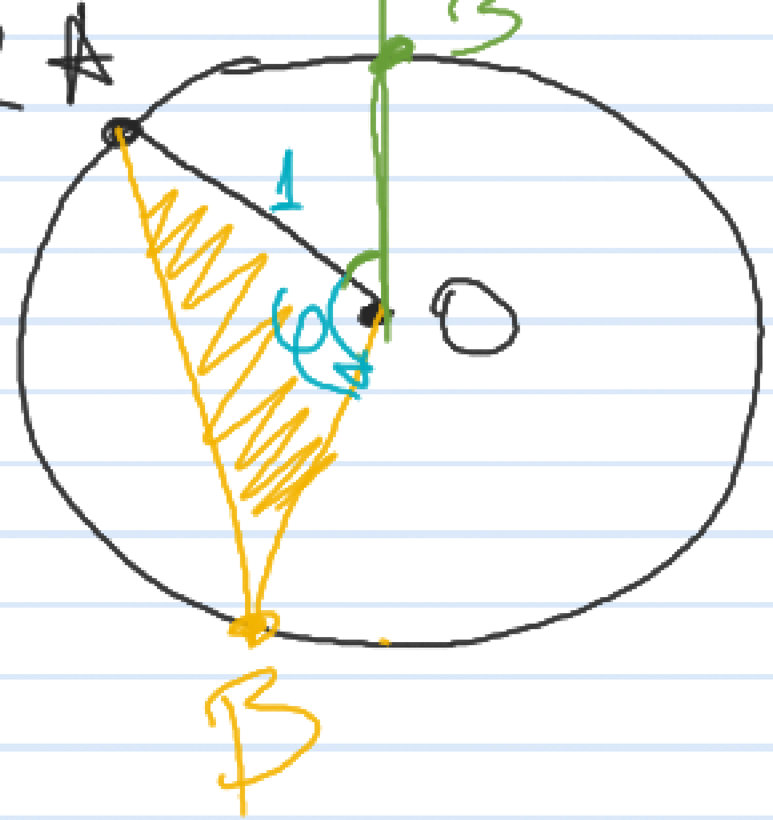
$= c \cdot \left[\frac{x^3}{3} + 2\frac{x^2}{2}\right]_0^y$

4) $\mathbb{E}(X^2 + 3X)$

$\xrightarrow{\text{I}} \mathbb{E}X^2 + 3\mathbb{E}X = \dots$ вече сметнато в 2.

$\xrightarrow{\text{II}} \int_0^1 (y^2 + 3y) f_X(y)\,dy$

Зад. 2



$\mathbb{E} S_{AOB} = ?$

I $\hookrightarrow$ $\varphi \sim U(0, \pi)$

II $\hookrightarrow$ $\varphi \sim U[0, 2\pi]$

$$\mathbb{E}X = \int x f_X(x)\, dx$$

$$\mathbb{E} \sin X = \int \sin x\, f_X(x)\, dx$$

I) $S_{AOB} = \frac{\sin\varphi}{2}$

$$\mathbb{E} S_{AOB} = \mathbb{E}\frac{\sin\varphi}{2} = \int_0^{\pi} \frac{\sin x}{2} \cdot \frac{1}{\pi}\, dx = \frac{1}{2\pi}\left[-\cos x\right]_0^{\pi} = \frac{1}{\pi}$$

II) $S_{AOB} = \frac{\sin\varphi}{2} \mathbb{1}_{\{\varphi \le \pi\}} + \frac{\sin(2\pi - \varphi)}{2} \mathbb{1}_{\{\varphi > \pi\}}$

$$\mathbb{E} S_{AOB} = \int_0^{\pi} \frac{\sin y}{2} \cdot \frac{1}{2\pi}\, dy + \int_{\pi}^{2\pi} \frac{\sin(2\pi - y)}{2} \cdot \frac{1}{2\pi}\, dy$$

$$= \dots = \frac{1}{\pi}$$

Зад. 3 $X \sim U(0,7)$ – „време до счупване"

→ на 5-та година се сменя

→ или преди това, ако се счупи

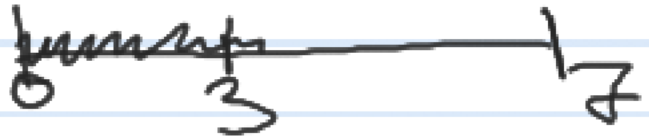
$Y =$ „време до смяна"

$F_Y(y), \mathbb{E}Y, \mathbb{D}Y = ?$

Ако са продадени 1000, колко средно ще се подменят преди 5-тата год.?

$\mathbb{P}(Y<3) = \mathbb{P}(X<3) = \frac{3}{7}$

Ако $Y \sim U_{nf}[0,5]$: $\quad \mathbb{P}(Y<3) = \frac{3}{5}$ – не



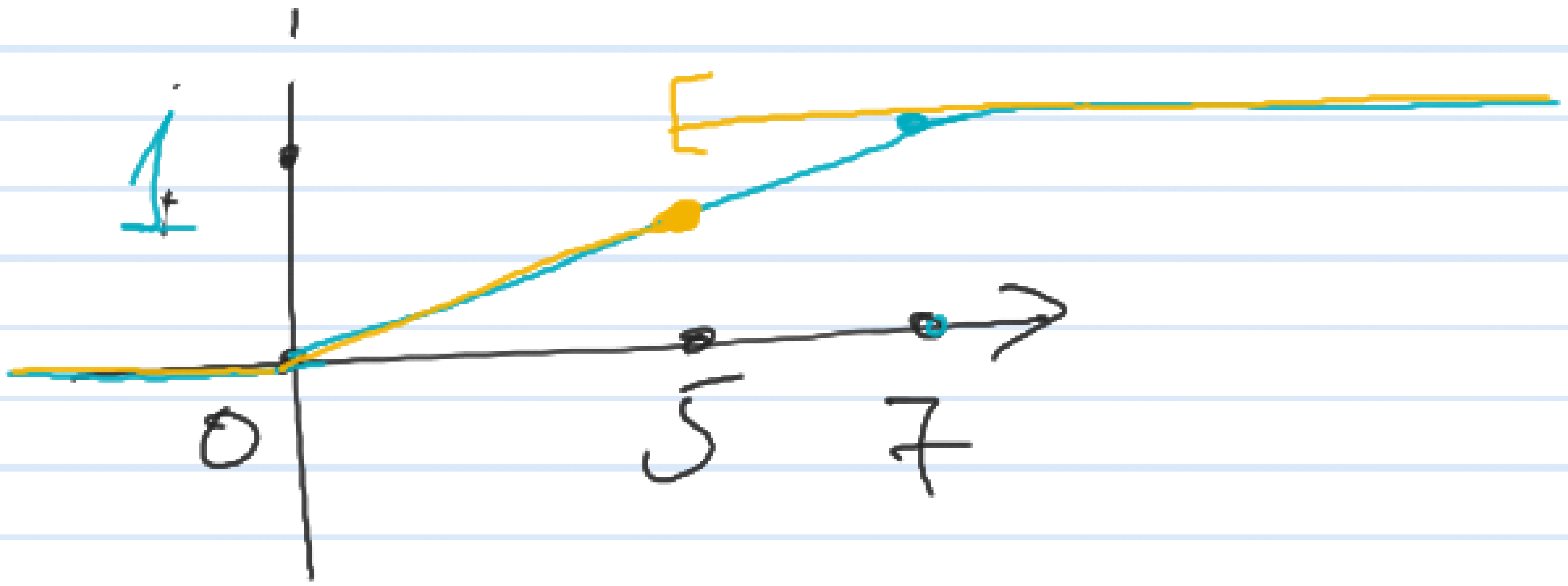
1) Y е в $[0,5]$

$$F_Y(y) = \mathbb{P}(Y<y) = \begin{cases} 0 & , y \le 0 \\ \frac{y}{7} & , y \in (0,5] \\ 1 & , y > 5 \end{cases}$$

Ако $y<5$:

$$\mathbb{P}(Y<y) = \mathbb{P}(\cancel{X}<y) = \frac{y}{7}$$

$$= \int_0^y \frac{1}{7}\,dx = \frac{y}{7}$$



$F_X(x) = P(U(0,7) < x)$

$= \int_0^x \frac{1}{7}\,dy = \frac{x}{7}$

$F_Y(y) = P(Y<y) = \begin{cases} 0 & \dots \\ \frac{y}{7} & \text{за } y\in[0,5] \\ 1 & y>5 \end{cases}$

Ако има плътност $P(Y=y) = 0$

В случая:

$P(Y=y) = P(X \geq 5) = \frac{2}{7}$

Y няма плътност!, т.к.

- $F_Y(y)$ е прекъсната
- $P(Y=5) = \frac{2}{7} \neq 0$

$\mathbb{E}Y$?

$\mathbb{E}X = \sum \mathbb{E}(X|A_i)\cdot P(A_i)$

$$Y = X\mathbb{1}_{\{X\le 5\}} + 5\,\mathbb{1}_{\{X>5\}}$$

$$\mathbb{E}Y = \mathbb{E}\,X\mathbb{1}_{\{X\le 5\}} + 5\cdot\mathbb{1}_{\{X>5\}}$$

$$= \mathbb{E}X\mathbb{1}_{\{X\le 5\}} + 5\cdot\mathbb{E}\,\mathbb{1}_{\{X>5\}}$$

$$= \int_0^5 x\cdot \underbrace{f_X(x)}_{\frac{1}{7}}\,dx + 5\int_5^7 \underbrace{f_X(x)}_{\frac{1}{7}}\,dx$$

$$\mathbb{E}\,\mathbb{1}_A = \mathbb{P}(A)$$

$$\mathbb{E}X\mathbb{1}_A = \int_A x f_X(x)\,dx$$

$$= \frac{25}{14} + 5\cdot\frac{2}{7}$$

$$= \frac{45}{14}$$

$$DY = \mathbb{E}Y^2 - (\mathbb{E}Y)^2$$

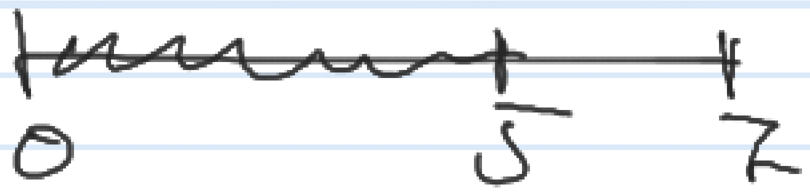
$$\mathbb{E}Y^2 = \mathbb{E}X^2\mathbb{1}_{\{X\le 5\}} + \mathbb{E}\,5^2\cdot\mathbb{1}_{\{X>5\}}$$

$$= \int_0^5 x^2\cdot\frac{1}{7}\,dx + \int_5^7 25\cdot\frac{1}{7}\,dx$$

$$= \frac{125}{21} + \frac{50}{7} = \frac{275}{21}$$

$$X_i = \begin{cases} 1, & \text{ако апарат } i \text{ е счупен преди 5-та год.} \\ 0, & \text{иначе} \end{cases}$$

$$X_i \sim Ber(\mathbb{P}(X<5)) \qquad , p = \mathbb{P}(X<5) = \frac{5}{7}$$



$$\mathbb{E}X_i = p$$

$$\#\text{счупени} = X_1 + X_2 + \dots + X_{1000} \sim Bin(1000, p)$$

$$\mathbb{E}\#\text{счупени} = \mathbb{E}(X_1 + \dots + X_n)$$

$$= n \cdot \mathbb{E}X_1 \qquad , n = 1000$$