$X \sim Ber(p)$ $\quad$ $\begin{array}{c|c} 0 & 1 \\ \hline 1-p & p \end{array}$ $\quad$ $\mathbb{E}X = p$, $\mathbb{D}X = p(1-p)$

$X \sim Bin(n,p)$ $\quad$ $\mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

$\mathbb{E}X = np$, $\mathbb{D}X = np(1-p)$

$X \sim Ge(p)$ $\quad$ $\mathbb{P}(X=k) = (1-p)^{k-1} \cdot p$ $\quad$ или $\quad$ $\mathbb{P}(X=k) = (1-p)^k p$

$\mathbb{E}X = 1/p$ $\quad$ $\mathbb{E}X = \sum k p (1-p)^k = p \cdot \frac{1-p}{p^2} = \frac{1-p}{p}$

$\mathbb{D}X = 1/p^2$

$X \sim Po(\lambda)$ $\quad$ $\mathbb{P}(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$

$\mathbb{E}X = \mathbb{D}X = 1/\lambda$

зад. 16 $A - 0{,}2$ → в-ст за попадение
$B - 0{,}3$

$A = P($A #ходове до да улучи, а B – не$) = ?$

$E[\#$ изстрели до 1-во попадение$] = ?$

Р-е: $A = \underbrace{0{,}2\cdot 0{,}7}_{p^{n-1}} + \underbrace{0{,}8\cdot 0{,}7}_{\ne 1-p}\cdot \underbrace{0{,}2\cdot 0{,}7}_{p}$
$+ \dots + 0{,}8 \cdot 0{,}2 \cdot 0{,}7 + \dots$

$= 0{,}7\cdot 0{,}2\left(1 + 0{,}8\cdot 0{,}7 + (0{,}8\cdot 0{,}7)^2 + \dots\right)$

$= 0{,}14\left(\dfrac{1}{1-0{,}56}\right)$

$= \dfrac{0{,}14}{0{,}44} = \dfrac{7}{22}$

$X \sim Ge(p) \qquad P(X=k) = p(1-p)^{k-1}$

$p = P($някой да удари на 1-вия ход$)$
$= 1 - P($и двамата да пропуснат$)$
$= 1 - 0{,}7\cdot 0{,}8 = 0{,}44$

$X =$ # ходове до 1-во попадение „успешния" $\sim Ge(p)$

$EX = 1/p$

$= \sum k\, p(1-p)^{k-1} = \dfrac{p}{1-p}\cdot\dfrac{1-p}{p^2} = \dfrac{1}{p}$

От на зад.: E # изстрели
$= E$ #ходове $\cdot 2 = 2E$ #ходове $= \dfrac{2}{p}$

$X \perp Y \overset{def}{\iff} P(X=k, Y=m) = P(X=k)P(Y=m)$ за всички k, m

Съвм. на X и Y

$$P(X=k, Y=m)$$

$X \backslash Y$	y_1	y_2	y_3		
x_1	p_{11}				$P(X=x_1)$
x_2				$\dots$	
$\vdots$					
	$P(Y=y_1)$				

$p_{ij} = P(X=x_i, Y=y_j)$

$$\mathbb{E}(XY) = \sum km \cdot P(X=k, Y=m)$$

$$Cov(X,Y) = \mathbb{E}\left[(X-\mathbb{E}X)(Y-\mathbb{E}Y)\right]$$

$$= \mathbb{E}XY - \mathbb{E}X\,\mathbb{E}Y$$

$$Cor(X,Y) = \frac{Cov(X,Y)}{\sqrt{DX}\cdot\sqrt{DY}} \in [-1,1]$$

ако $X \perp Y$, то $Cov(X,Y) = 0$

$$Cov(X,X) = DX$$

$$Cor(X,X) = \frac{DX}{\sqrt{DX}\sqrt{DX}} = 1$$

$$Cor(X,-X) = -1$$

зад. 18 $\xi, \eta \sim Ge(p)$

$\mathbb{P}(\xi = k) = \mathbb{P}(\eta = k) = (1-p)^k \cdot p \quad ; \xi \perp \eta$

$Z = \max(\xi, \eta)$

1) Разпр на Z (оч.акв. изиск.)

2) $\tau = (Z, \xi)$ (корел. им = ?)

$q := 1-p$

Р-е: 1)

$\mathbb{P}(Z=k) = \mathbb{P}(\max(\xi,\eta) = k)$

$= \mathbb{P}(\xi = k, \eta \le k) + \mathbb{P}(\xi < k, \eta = k)$

$= \mathbb{P}(\xi=k)\mathbb{P}(\eta \le k) + \mathbb{P}(\xi < k)\mathbb{P}(\eta = k)$

$= q^k \cdot p(p + qp + \dots + q^k p)$

$+ \underbrace{(p + qp + \dots + q^{k-1}p)}_{A_k} q^k p$

$= \underline{\underline{A_k \cdot q^k \cdot p \cdot 2 + q^{2k} p^2}}$

$A_k = p(1 + q + \dots + q^{k-1}) = p \cdot \frac{1-q^k}{p} = 1 - q^k$

, т.е. $\mathbb{P}(\zeta=k)=(1-q^k)q^k\cdot p\cdot 2+q^{2k}p^2$

- $X\sim Ge(p)$

$$\mathbb{P}(X\geq k)=(1-p)^{k+1}p+(1-p)^{k+2}\cdot p+\dots$$
$$=p\cdot q^{k+1}(1+q+q^2+\dots)$$
$$=q^{k+1}$$

2) $\mathbb{P}(\zeta=k,\eta=m)=?$

$k\in\mathbb{N}_0$
$m\leq k$

I) $m<k$

$$\mathbb{P}(\zeta=k,\eta=m)$$
$$=\mathbb{P}(\xi=k,\eta=m)\underset{\xi\perp\eta}{=}q^k\cdot p\cdot q^m\cdot p$$

II) $m=k$

$$\mathbb{P}(\zeta=k,\eta=k)$$
$$=\mathbb{P}(\xi\leq k,\eta=k)$$
$$=\mathbb{P}(\xi\leq k)\mathbb{P}(\eta=k)$$
$$=\underline{\underline{(1-q^{k+1})q^k\cdot p}}$$

D

зад. 19   3 бели, 2 черни $\eta > \xi$

ξ = номер на I бяла

$\eta = \begin{cases} \text{черна след I бяла, ако има такава} \\ 6, \text{ иначе} \end{cases}$

$P(\xi = k, \eta = m) = ?$

$P(\eta > 2 \mid \xi = 1)$ и $P(\eta = 3 \mid \xi < 3) = ?$

$P(\xi = 2, \eta = 3) = P(2б2) = \frac{2}{5} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{10}$

$P(\xi = 2, \eta = 4) = P(2бб2) = \frac{2}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2}$

$P(\xi = 2, \eta = 5) = P(2ббб2) = \frac{2 \cdot 3 \cdot 2 \cdot 1 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} =$

$P(\xi = 3, \eta = 6) = P(22б) = \frac{2 \cdot 1 \cdot 3}{5 \cdot 4 \cdot 3} = \frac{1}{10}$

Р-е:

$\xi \backslash \eta$	2	3	4	5	6	$P(\xi = \cdot)$
1	3/10	2/10	1/10	0	0	6/10
2	0	1/10	1/10	1/10	0	3/10
3	0	0	0	0	1/10	1/10
$P(\eta = \cdot)$	3/10	3/10	2/10	1/10	1/10	

6+6+4+6+8+10 +18

$P(\xi = 1, \eta = 2) = P(\text{бяла, черна}) = \frac{3}{5} \cdot \frac{2}{4}$

$P(\xi = 1, \eta = 3) = \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} = \frac{2}{10}$

$P(\xi = 1, \eta = 4) = P(ббб2) = \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \cdot \frac{2}{2} = \frac{1}{10}$

$$\mathbb{P}(\eta > 2 \mid \xi = 1) = \frac{\mathbb{P}(\eta > 2, \xi = 1)}{\mathbb{P}(\xi = 1)}$$

$$= \frac{3/10}{6/10} = \frac{1}{2}$$

$$\mathbb{P}(\eta = 3 \mid \xi < 3) = \frac{\mathbb{P}(\eta = 3, \xi < 3)}{\mathbb{P}(\xi < 3)}$$

$$= \frac{3/10}{9/10} = \frac{1}{3}$$

ξ	1	2	3
	6/10	3/10	1/10

η	2	3	4	5	6
	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

$$Cor(\xi, \eta) = ?$$

$$= \frac{\mathbb{E}\xi\eta - \mathbb{E}\xi\,\mathbb{E}\eta}{\sqrt{D\xi}\sqrt{D\eta}}$$

$$\mathbb{E}\xi = \frac{3}{2} \ ; D\xi = \frac{27}{10} - \frac{9}{4} = \frac{18}{40} = \frac{9}{20}$$

$$\mathbb{E}\eta = \frac{34}{10} = \frac{17}{5}$$

$$D\eta = \frac{12+27+32+25+36}{10} - \left(\frac{17}{5}\right)^2 = \cdots$$

$$\mathbb{E}\xi\eta = \sum km\,\mathbb{P}(\xi = k, \eta = m)$$

$$= \frac{58}{10}$$

зад. 20 – същото

зад. 21 $1, 2, 3, 4$ и 5 → избираме 3 от тях

X = „средното от тях"

Y = „най-малкото от тях"

1. съвм. р-е.
2. марг. р-е.
3. незав. ли са?
4. корел = ? EX, DX, EY, DY, EXY
5. разпр, очаквани числа на $Z = X - 2Y$? $X - 2Y$

Р-е: $P(X = k, Y = m)$

$k \in [2, 3, 4]$

$m \in k$

$1] 2 [3, 4, 5$; $\underbrace{1, 2}$, 3, $\underbrace{4, 5}$; $\underbrace{1, 2, 3}$, 4, $\underbrace{5}$

3 възм   4 възм   3 възм

Y\X	2	3	4	
1	3/10	2/10	1/10	6/10
2	0	2/10	1/10	3/10
3	0	0	1/10	1/10
	3/10	4/10	3/10	

$P(X = 2, Y = 1) = 3/10$

$P(X = 2)\, P(Y = 1) = 3/10 \cdot 6/10 \neq$ горното

⇒ не са нез.

Възможните с-ти за (X,Y) са $(2,1)$, $(3,1)$, $(4,1)$, $(3,2)$, $(4,2)$ и $(4,3)$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$\Rightarrow$ възм за $X-2Y$ $\qquad 0 \quad 1 \quad 2 \quad -1 \quad 0 \quad -2$

$X-2Y$	-2	-1	0	1	2
	$1/10$	$2/10$	$4/10$	$2/10$	$1/10$

$$\mathbb{E}(X-2Y) = -2\cdot\frac{1}{10} + (-1)\cdot\frac{2}{10} + \cdots$$

$$\mathbb{E}(X-2Y) = \mathbb{E}X - 2\mathbb{E}Y$$

$$\cancel{D(X-2Y) \stackrel{?}{=} DX - 2DY} \;?!$$

$D(cX) = c^2 DX$

Ако са незав.

$$D(X-2Y) = DX + D(-2Y) = DX + 4DY$$

По причин X и Y :

$D(X+Y) = DX + DY + 2\operatorname{cov}(X,Y)$

$D(X+Y) = \mathbb{E}(X+Y)^2 - (\mathbb{E}[X+Y])^2$

$= \mathbb{E}(X^2 + 2XY + Y^2) - (\mathbb{E}X + \mathbb{E}Y)^2$

$= \mathbb{E}X^2 + 2\mathbb{E}XY + \mathbb{E}Y^2 - (\mathbb{E}X)^2 - 2\mathbb{E}X\mathbb{E}Y - (\mathbb{E}Y)^2$

$= DX + DY + 2\operatorname{Cov}(X,Y)$

$D(X-2Y)$

Зад. 22* 3 пъти монета $0\ 0\ 0$

$\underline{\underline{X}}$ = # ези-та от 1-те две

$\underline{\underline{Y}}$ = # ези-та от посл. две

ХВ.

1) Съвм. р-е:

$Y \backslash X$	0	1	2	
0	1/8	1/8	0	2/8
1	1/8	2/8	1/8	4/8
2	0	1/8	1/8	2/8
	2/8	4/8	2/8	

$$\underbrace{\mathbb{E}[X \mid Y=k]}_{\text{число}} = \sum m \cdot \mathbb{P}(X=m \mid Y=k)$$

$\mathbb{P}(X=0, Y=0) = \mathbb{P}(TTT) = 1/8$

$\mathbb{P}(X=0, Y=1) = \mathbb{P}(TTE) = 1/8$

$\mathbb{P}(X=0, Y=2) = 0$

$\mathbb{P}(X=1, Y=0) = \mathbb{P}(TEE)$

$\mathbb{P}(X=1, Y=1) = \mathbb{P}(TET, ETE) = 2/8$

$\vdots$

$\mathbb{P}(X=\cdot \mid Y=\cdot)$

$$\mathbb{P}(X=k \mid Y=0) = \frac{\mathbb{P}(X=k, Y=0)}{\mathbb{P}(Y=0)} = \frac{1/8}{2/8}$$

$[X=\cdot \mid Y=0]$	0	1
	1/2	1/2

$$\mathbb{E}(X \mid Y=k) = \sum m \, \mathbb{P}(X=m \mid Y=k)$$

$\mathbb{E}(X \mid Y)$	$\mathbb{E}(X \mid Y=y_1)$	$\mathbb{E}(X \mid Y=y_2)$	$\dots$
	$\mathbb{P}(Y=y_1)$	$\mathbb{P}(Y=y_2)$	$\dots$

$g(Y)$

$\mathbb{E}(X \mid Y)$	$\mathbb{E}(X \mid Y=0) = 1/2$	$\mathbb{E}(X \mid Y=1) = 1$	$\mathbb{E}(X \mid Y=2) = 3/2$
	$2/8$	$4/8$	$2/8$

$\mathbb{E}(X \mid Y) =: Z$	$1/2$	1	$3/2$
	$1/4$	$2/4$	$1/4$

$\mathbb{E}(X \mid Y=0)$

$$= 0 \cdot \mathbb{P}(X=0 \mid Y=0) + 1 \cdot \mathbb{P}(X=1 \mid Y=0)$$

$$= 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$$

$\mathbb{E}(X \mid Y=1)$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{2}{4} + 2 \cdot \frac{1}{4}$$

$$= 1$$

$$\mathbb{E}(X \mid Y=2) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 2$$

$$\boxed{\mathbb{E}(\mathbb{E}(X \mid Y)) = \mathbb{E}X}$$

Зад. 4) 3 зара - n пъти

успех = сума да е 13 15 17

665
656
566

$p = \mathbb{P}(\text{успех}) = \frac{34}{6^3}$

а) $X \sim Bin(n, p)$

$\mathbb{P}(X > \frac{n}{2}) = \sum_{k=[\frac{n}{2}]+1}^{n} \mathbb{P}(X = k)$

б) $n = 14$ $\quad 7^{\text{ят}}$ успех

преди $5^{\text{я}}$ не успех

$HG(N, K, n) \to$ # изтеглени спец.

N – точки, K – спец., n – изтегления

$NB(r, p) \to$ # опити до k-ти успех

r – # неуспехи, p – успех

$\mathbb{P}(7^{\text{ят}}$ успех да е на $7, 8, 9, 10, 11)$

11: 4 неусп. 7 успх, $k \geq 7$

$\mathbb{P}(7^{\text{ят}}$ успех да е на k-то хв.$)$

$= \binom{k-1}{6} p^6 \cdot (1-p)^{k-7}$

6 V и k-7 X

$\underbrace{V\ V\ V\ V\ V\ V}_{k\ \text{хв}} \mid V$

$\mathbb{P}(\text{печалба} \cap \{3\text{-тото е } V\})$

$= \mathbb{P}(VXV, XVV, VVV)$

$= \left(\frac{2}{3}\right) \cdot \left(\frac{1}{3}\right)^2 \cdot 2 + \left(\frac{1}{3}\right)^3 = \frac{5}{27}$

От.: $\dfrac{5/27}{7/27}$