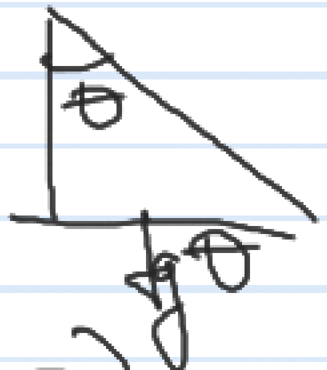
Zad. 7

$\Theta \sim U\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$f_X(x) = ?\ ,\ \mathbb{E}X = ?$

R-e:



$tg : \left(-\frac{\pi}{2}; \frac{\pi}{2}\right) \to \mathbb{R}$

$arctg : \mathbb{R} \to \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$= F(x) - F(a)$$

$$\mathbb{P}(X \le t) = \mathbb{P}(tg\,\Theta \le t)$$

$$= \mathbb{P}(\Theta \le arctg\, t)$$

$$= \int_{-\frac{\pi}{2}}^{arctg\, t} \frac{1}{\pi}\,dx$$

$$= \left(arctg\, t + \frac{\pi}{2}\right) / \pi$$

$$= \frac{1}{2} + arctg\, t / \pi = F_X(t)$$

$$f_X(t) = \frac{1}{\pi} \cdot \frac{1}{1+t^2}$$

$\mathbb{E}X = \int x f_X(x)\,dx$

$$\mathbb{E}X = \frac{1}{\pi}\int_{-\infty}^{\infty} x \cdot \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{\pi \cdot 2}\int_{\infty}^{\infty} \frac{1}{1+x^2}\, d\left(x^2+1\right) \qquad \left(d(x^2+1) = 2x\,dx \right)$$

$$= \frac{1}{2\pi} \ln\left(1+x^2\right)\Big|_{-\infty}^{\infty}$$ – не съществува!

$$* \int_{0}^{\infty} \frac{1}{x}\,dx = \infty \qquad \int_{\infty}^{\infty} x \cdot \frac{1}{1+x^2}\,dx \approx \int x \cdot \frac{1}{x^2}\,dx$$

$$\approx \int \frac{1}{x}\,dx$$

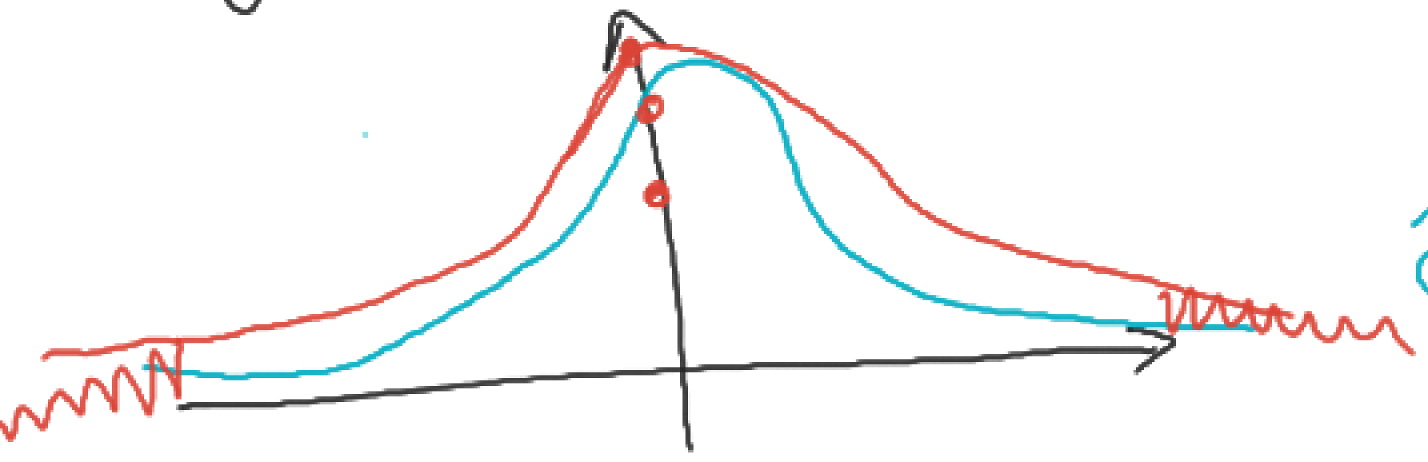
$X \sim Cauchy(0,1)$

$X \sim Cauchy(x_0, \gamma)$ $\qquad \mathbb{P}(X \leq x) = \frac{1}{\pi}\operatorname{arctg}\left(\frac{x - x_0}{\gamma}\right) + \frac{1}{2}$

X няма очакване!

$\mathbb{E}X^n < \infty$, то $\mathbb{E}X^k < \infty$ за $0 \le k < n$ $(k \in \mathbb{R})$

Щом X няма очакване, то X няма и дисперсия.



$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$f_Y(y) = \frac{1}{\pi} \cdot \frac{1}{1+y^2} \qquad \mathbb{E}Y = \int y f_Y(y)\,dy$$

$\mathbb{E}X^n$ – n-ти момент

$$\frac{c\cdot\frac{1}{n}}{\frac{1}{n}} \to c$$

зад. 9 $\mathbb{P}(\text{ези}) = \underbrace{3/4}_{p}$, $\underbrace{2000}_{n}$ пъти

$\mathbb{P}(\text{между } 1475 \text{ и } 1535 \text{ ези}) = ?$

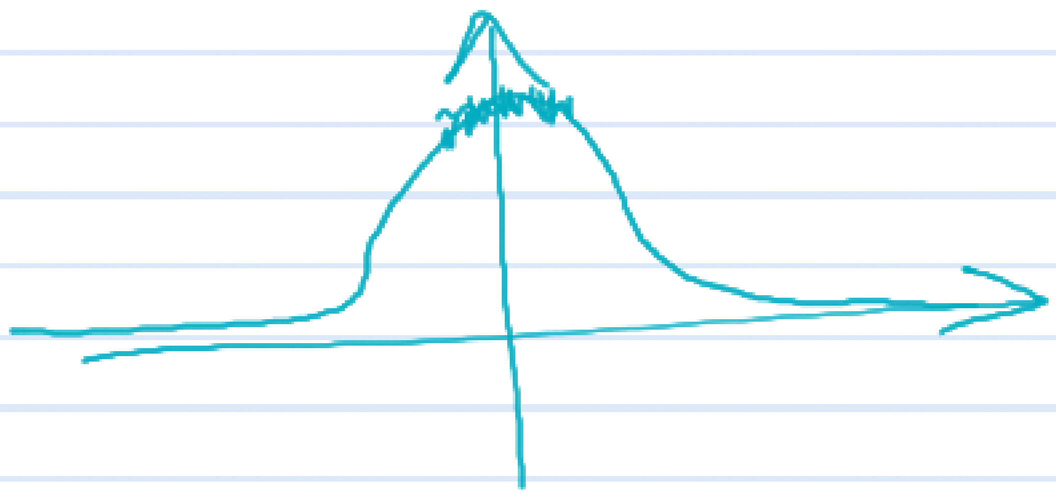
Р-е: $X_i = \begin{cases} 1, & \text{ези на } i\text{-то} \\ 0, & \text{иначе} \end{cases}$, $X_i \sim Ber(p)$

$$\mathbb{P}(1475 \le X_1 + \dots + X_n \le 1535)$$

ЦГТ: Нека $X_1, X_2, \dots$ нез. и едн. разпр. (iid) сл. вел. и $\mu = \mathbb{E}X_1$, $\sigma^2 = \mathbb{D}X_1 < \infty$. Тогава

$$\frac{\frac{X_1 + \dots + X_n}{n} - \mu}{\sigma/\sqrt{n}} \xrightarrow[n\to\infty]{d} N(0,1)$$

$$\frac{X_1 + \dots + X_n - n\mu}{\sigma\sqrt{n}}$$



$X \sim N(\mu, \sigma^2)$, ако

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$X_1 \sim N(\mu_1, \sigma_1^2)$

$X_2 \sim N(\mu_2, \sigma_2^2)$

1. $cX_1 \sim N(c\cdot\mu_1, c^2\sigma_1^2)$
2. $c + X_1 \sim N(c+\mu_1, \sigma_1^2)$
3. $X_1 \perp X_2$

$X_1 + X_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

$$^* \mathbb{P}(1475 \le X_1 + \dots + X_n \le 1535)$$

$$= \mathbb{P}\left(\frac{1475 - \mu n}{\sigma\sqrt{n}} \le \frac{X_1 + \dots + X_n - \mu n}{\sigma\sqrt{n}} \le \frac{1535 - \mu n}{\sigma\sqrt{n}}\right)$$

$\mu = \frac{3}{4}$, $n = 2000$, $\sigma^2 = \frac{3}{16}$

$$= \mathbb{P}\left(\frac{-25}{\frac{\sqrt{3}}{4}\cdot\sqrt{2000}} \le \quad \le \frac{35}{\frac{\sqrt{3}}{4}\cdot\sqrt{2000}}\right)$$

$n \ge 30$, $DX_1 < \infty$

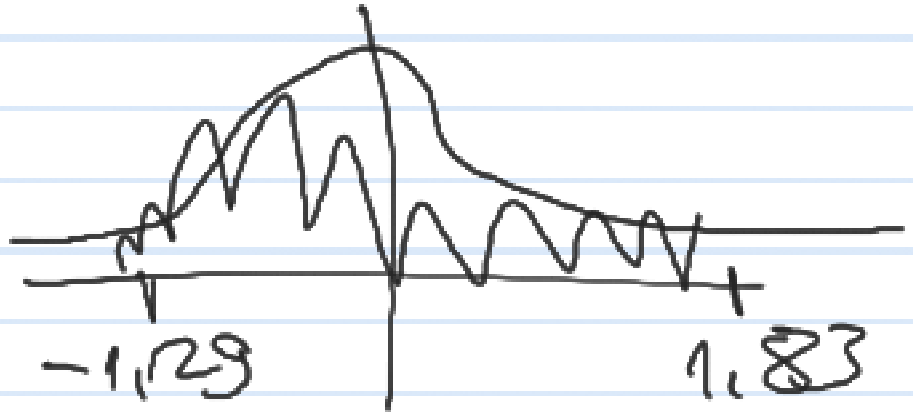
$$\approx \mathbb{P}(-1{,}29 \le N(0,1) \le 1{,}83)$$

$$= \int_{-1{,}29}^{1{,}83} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

$\int e^{-x^2/2} dx$ – няма явен вид

$$P(N(0,1) \leq x) = F_{N(0,1)}(x) = \Phi(x)$$

$$P(-1{,}29 \leq N(0,1) \leq 1{,}83)$$



$$= \Phi(1{,}83) - \Phi(-1{,}29)$$

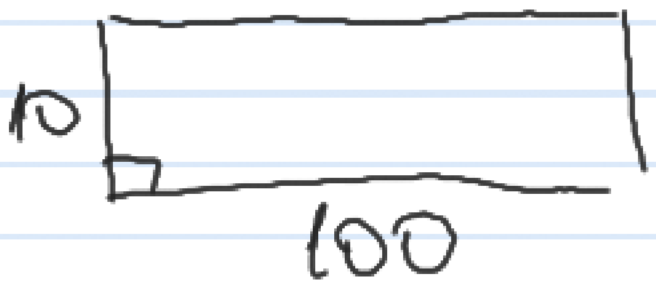
$$= 0{,}9664 - (1 - 0{,}9015)$$

$$= 0{,}9664 - (0{,}0985)$$ = zucko...

$$X \sim N(\mu, \sigma^2)$$

$$\Rightarrow X - \mu \sim N(0, \sigma^2) \quad \rightarrow \quad \underbrace{\frac{X-\mu}{\sigma}}_{z\text{-score}} \sim N(0,1)$$

зад. 20



X_i = време на живот на диод i $\sim Exp(1/10)$

$\mathbb{E}\,Exp(\lambda) = \frac{1}{\lambda}$

$\mathbb{P}($да трябва да има ремонт след 3 години$)$

$= \mathbb{P}($да работят $\leq 75\%$ на 3-тата година$)$

$$Y_i = \begin{cases} 1, & \text{ако диод } i \text{ работи на 3-тата година} \\ 0, & \text{ако не работи} \end{cases}$$

$\mathbb{P}(Y_i = 1) = \mathbb{P}(X_i \geq 3) = e^{-\frac{3}{10}}$

$\mathbb{P}(Exp(\lambda) > t) = e^{-\lambda t}$

$Y_i \sim Ber(e^{-3/10})$, $\mathbb{E} Y_i = e^{-3/10} = p$

$DY_i = p(1-p)$

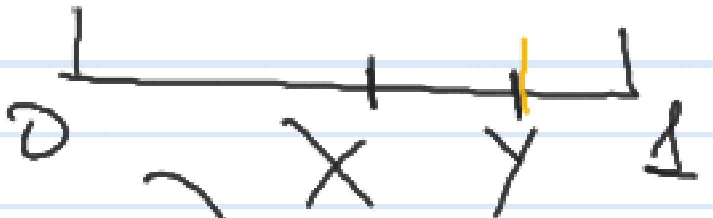
$$\mathbb{P}(Y_1 + \dots + Y_{1000} \leq 750)$$

$$= \mathbb{P}\left(\frac{Y_1 + \dots + Y_{1000} - 1000 \cdot e^{-3/10}}{\sqrt{e^{-3/10} e^{-9/10}}\,\sqrt{1000}} \leq \frac{750 - 1000 e^{-3/10}}{\dots} \right)$$

$n > 30$
$\sigma_{Y_1} < \infty \Rightarrow$ ЦПТ

$$\approx \mathbb{P}(N(0,1) \leq 0{,}66) = \Phi(0{,}66) \approx 74{,}54\%$$

зад. 11



$$f_{X,Y}(x,y) = \begin{cases} cxy, & 0 \le x \le y \le 1 \\ 0, & \text{иначе} \end{cases}$$

$$= cxy\,\mathbb{1}_{\{0 \le x \le y \le 1\}}$$

1. $c = ?$

$$\Rightarrow \int_0^1 \left[\int_x^1 cxy\,dy\right] dx$$

$$= \int_0^1 cx \left[\frac{y^2}{2}\right]_x^1 dx = \frac{c}{2}\int_0^1 x(1-x^2)\,dx$$

= сметка

$$\Rightarrow \int_0^1 \int_0^y cxy\,dx\,dy = c\int_0^1 y\left[\frac{x^2}{2}\right]_0^y dy$$

$$= \frac{c}{2}\int_0^1 y^3\,dy = \dots$$

$$* \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

$$\mathbb{P}(X \in (a,b)) = \int_a^b f_X(x)\,dx$$

$$\mathbb{P}((X,Y) \in A) = \iint_A f_{X,Y}(x,y)\,dx$$

$$\mathbb{E}XY = \int_{-\infty}\int_{-\infty} xy\,f_{X,Y}(x,y)\,dx\,dy$$

$$F_{X,Y}(x,y) = \mathbb{P}(X \le x, Y \le y)$$

$$\frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} = \frac{\partial^2 F_{X,Y}(x,y)}{\partial y \partial x} = f_{X,Y}(x,y)$$

2. Маргинални плътности и очаквания ?

$f_X(x)$; $\mathbb{E}X$ . . . ?

$$f_X(x) = \int_x^1 cxy\,dy = cx\left[\frac{y^2}{2}\right]\Bigg|_x^1$$

$x \in [0,1]$

$$= \frac{c}{2}x(1-x^2)$$

$$= \int_{-\infty}^{\infty} cxy\,\mathbb{1}\{0 \le x \le y \le 1\}\,dy$$

$\int_x^1 cxy\,dy$; $\mathbb{E}X = \int_0^1 x \cdot f_X(x)\,dx$

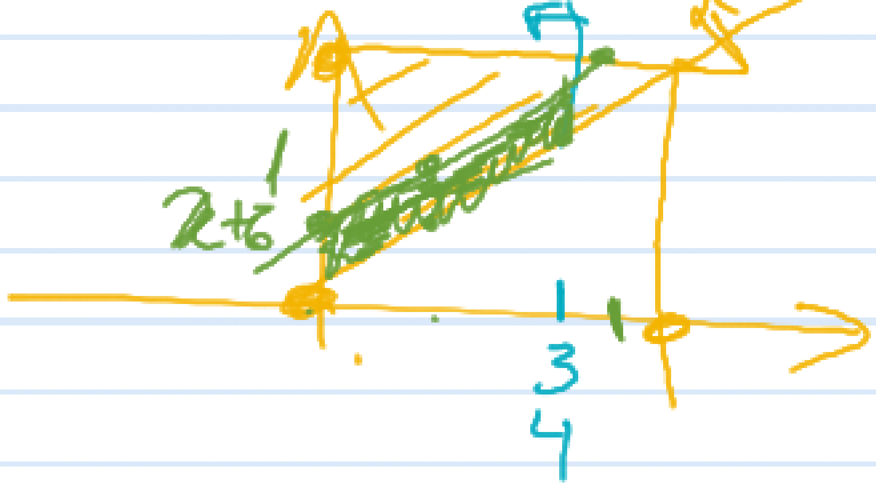
$= \dots$

Y \ X	x_1	x_2
y_1	$\circ$	
y_2	$\circ$	
	$\circ$	

$$\mathbb{P}(X = x_1) = \sum \mathbb{P}(X = x_1, Y = y_j)$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

3. $\mathbb{P}\left(X < \frac{3}{4}, Y - X < \frac{1}{6}\right)$



$$= \mathbb{P}\left(X < \frac{3}{4}, \underline{Y < X + \frac{1}{6}}\right)$$

$$= \int_0^{3/4} \int_0^{x+1/6} cxy \, \mathbb{1}_{\{0 \le x \le y \le 1\}} \, dy \, dx$$

$$= \int_0^{3/4} \int_x^{x+\frac{1}{6}} cxy \, dy \, dx$$

$$= c\int_0^{3/4} x \left[\frac{y^2}{2}\right]_x^{x+\frac{1}{6}} dx = \frac{c}{2} \int_0^{3/4} x\left(\left(x+\frac{1}{6}\right)^2 - x^2\right) dx$$

$$= \dots$$

5. $\mathbb{P}\left(Y - X < \frac{1}{3}\right)$

$Y < X + \frac{1}{3}$

$0 \le x \le \frac{2}{3}$
$x \le y \le x + \frac{1}{3}$

$\frac{2}{3} \le x \le 1$
$x \le y \le 1$

$$= \int_0^{\frac{2}{3}} \int_x^{x+\frac{1}{3}} \cdots + \int_{\frac{2}{3}}^{1} \int_x^{1} \cdots$$

$$= \int_0^1 \int_x^{\min\{x+\frac{1}{3}, 1\}} \cdots$$

$$\mathbb{P}(X = x \mid Y = y) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)}$$

4. $\mathbb{E}\left(Y - X \mid X = \frac{1}{4}\right) = \mathbb{E}\left(Y - \frac{1}{4} \mid X = \frac{1}{4}\right)$

$$= \int_{-\infty} \left(y - \frac{1}{4}\right) f_{Y|X}\left(y \mid \frac{1}{4}\right) dy \quad ; \quad f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$