зад. 1

2	7	6
9	5	1
4	3	8

1. $\mathbb{E}\left[Bin\left(5, \frac{1}{10}\right)\right] = \frac{1}{2}$

2. $\mathbb{E}\left[Ge\left(1/10^5\right)\right] = 10^5$

3. $X =$ #симулации до 1-ви успех $\sim Ge(1/10^5)$. $\mathbb{E}X = 10^5 =: 1/p$

$DX = \frac{1-p}{p^2}$

Търсим n: $\mathbb{P}(X > n) \leq 1\%$. Грубо, с Чебишов:

$$\mathbb{P}(X - \mathbb{E}X > \varepsilon) \leq \mathbb{P}(|X - \mathbb{E}X| > \varepsilon) \leq \frac{DX}{\varepsilon^2} = 1\%,$$

т.е. $\varepsilon = 10\sqrt{DX}$ и можем да изберем

$$n = \lceil \mathbb{E}X + 10\sqrt{DX} \rceil = 1099995.$$

За втората част:

1. $\mathbb{E}\left[1_{\{1\text{-та е позната}\}} + \ldots + 1_{\{5\text{-тата е позната}\}}\right]$

$= 5\,\mathbb{E}\,1_{\{1\text{-та е позната}\}} = 5 \cdot \frac{1}{5} = 1.$

2. $\mathbb{E}\left(Ge\left(1/5!\right)\right) = 5! = 120.$

3. Както по-горе, можем да изберем

$n = \lceil \mathbb{E}Y + 10\sqrt{DY} \rceil = 1315$

за $Y \sim Ge(1/5!)$

зад. 2 Нека Z_i = падналото се число от i-тия зар.

$$Cor(Z_1+\dots+Z_{n-1}, Z_2+\dots+Z_n)$$

$$= \frac{\mathbb{E}\left[(Z_1+\dots+Z_{n-1})(Z_2+\dots+Z_n)\right] - \left[\mathbb{E}(Z_1+\dots+Z_{n-1})\right]^2}{D(Z_1+\dots+Z_{n-1})}$$

$$= \frac{(n-2)\,\mathbb{E}Z_1^2 + \left[\cancel{(n-1)^2} - (n-2)\right](\mathbb{E}Z_1)^2 - \cancel{(n-1)^2(\mathbb{E}Z_1)^2}}{(n-1)\left[\mathbb{E}Z_1^2 - (\mathbb{E}Z_1)^2\right]}$$

$$= \frac{n-2}{n-1}$$

Резултатът не зависи от начина по който задаваме заровете, стига да са независими и с равни (крайни) очаквания и дисперсии.

Зад. 3 Нека X_i = събрани пари през i-тия ден

При n доброволци, X_i = сума на $100n$ нез. $10\cdot Ber\left(\frac{1}{25}\right)$

и следов. $EX_i = 40n$ и $DX_i = n\cdot 100^2\cdot\frac{1}{25}\cdot\frac{24}{25}$

Търсим n:

$$99\% \le \mathbb{P}\left(X_1+\dots+X_{100} \ge 10^6\right)$$

$$= \mathbb{P}\left(\frac{X_1+\dots+X_{100}-4000n}{\sqrt{n\cdot 100^2\cdot\frac{24}{25^2}\cdot 100}} \ge \frac{10^6-4000n}{\sqrt{n\cdot 100^2\cdot\frac{24}{25^2}\cdot 100}}\right)$$

$$\overset{ЦГТ}{\approx} \mathbb{P}\left(N(0,1) \ge \frac{10^6-4000n}{80\sqrt{6n}}\right)$$

$$\Rightarrow \frac{10^6-4000n}{80\sqrt{6n}} \le \Phi^{-1}(0.01) \approx 2.33$$

$$\Rightarrow n \ge 248,20 \quad \text{т.е.}$$

$\boxed{n = 249 \text{ е удачно}}$ 250 наум... . За II част:

$$\mathbb{P}\left(\frac{X_1+\dots+X_{99}-40\cdot 99\cdot 249}{\sqrt{n\cdot 100^2\cdot\frac{24}{25^2}\cdot 99}} \ge \frac{10^6-40\cdot 99\cdot 249}{72\sqrt{1826}}\right) \overset{ЦГТ}{\approx} \mathbb{P}\left(N(0,1) \ge 4,53\right) \approx 0$$

Зад. 4 1) $1 = c\int_0^\infty\int_0^\infty x e^{-\frac{x+y}{2}}\,dy\,dx$

$$= c\int_0^\infty x e^{-\frac{x}{2}}\cdot 2\,dx$$

$$z=\frac{x}{2} \quad = 8c\int_0^\infty z e^{-z}\,dz$$

$$\int x e^{-x}\,dx = -xe^{-x} - e^{-x}$$

$$= 8c \Rightarrow c = 1/8$$

2) $P(X>1, Y>1) = c\int_1^\infty\int_1^\infty x e^{-\frac{x+y}{2}}\,dy\,dx$

$$= c\int_1^\infty x e^{-\frac{x}{2}}\, 2\cdot\left[e^{-\frac{y}{2}}\right]_1^\infty dx$$

$$= 2c\cdot e^{-\frac{1}{2}}\cdot 4\left[-\frac{x}{2}e^{-\frac{x}{2}} - e^{-\frac{x}{2}}\right]_1^\infty$$

$$= e^{-\frac{1}{2}}\cdot\frac{3}{2}e^{-\frac{1}{2}} = \frac{3}{2e}$$

3) $E\left[\frac{Y}{X}\right] = c\int_0^\infty\int_0^\infty \frac{y}{x}\, x e^{-\frac{x+y}{2}}\,dx\,dy$

$$= c\int_0^\infty\int_0^\infty y e^{-\frac{x+y}{2}}\,dx\,dy = 1$$

, т.к. $c\int_0^\infty\int_0^\infty x e^{-\frac{x+y}{2}}\,dx\,dy = 1$

4) $f_Y(y) = \int_0^\infty c x e^{-\frac{x+y}{2}}\,dx = e^{-\frac{y}{2}}/2$

$$\Rightarrow f_{X|Y}(x,1) = \frac{x e^{-y/2} \cdot \frac{y}{8} e^{-\frac{x}{2}}}{e^{-\frac{1}{2}}/2} = x e^{-x/2}/4$$

и $\mathbb{E}[X \mid Y=1] = \int_0^\infty x^2 e^{-x/2}/4 \, dx$

$$= 4.$$

$$f_{X,Y}(x,y) = c \cdot x e^{-\frac{x}{2}} \cdot e^{-\frac{y}{2}} \quad \Rightarrow \text{незав.}$$