зад. 13 $\quad X,\ f_X \ ;\ f_{g(X)}\ ?$

$$\mathbb{P}(g(X) \le t) \overset{g\uparrow}{=} \mathbb{P}(X \le g^{-1}(t)) = F_X(g^{-1}(t))$$

$$f_{g(X)}(t) = F_X'(g^{-1}(t))\,|g^{-1}(t)'| = \underbrace{f_X(g^{-1}(t))\,|g^{-1}(t)'|}$$

$$X_1, X_2 \to X_1/(X_1+X_2)$$

$$f_{X_1,X_2}(x_1,x_2) \to \underbrace{f_{Y_1,Y_2}(y_1}_{??}, y_2)$$

$$\begin{cases} Y_1 = X_1 X_2 \\ Y_2 = X_1 \end{cases} \Rightarrow \begin{cases} x_1 = Y_2 \\ x_2 = \dfrac{Y_1}{Y_2} \end{cases}$$

$$f_{X_1,X_2}\left(\underline{y_2}, \underline{\frac{y_1}{y_2}}\right) \cdot |\ldots|$$

$$|J| = \begin{vmatrix} \dfrac{\partial X_1}{\partial Y_1} & \dfrac{\partial X_1}{\partial Y_2} \\ \dfrac{\partial X_2}{\partial Y_1} & \dfrac{\partial X_2}{\partial Y_2} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \dfrac{1}{Y_2} & -\dfrac{Y_1}{Y_2^2} \end{vmatrix} = -\frac{1}{Y_2}$$

$X_1, X_2 \sim Exp(\lambda)$ нез. $\qquad Y = \frac{X_1}{X_1+X_2} \qquad f_Y(y) - ?$

$$\left| \begin{array}{l} Y = \frac{X_1}{X_1+X_2} \\ Z = X_1 \end{array} \right. \Rightarrow \left| \begin{array}{l} X_1 = Z \\ X_2 = \frac{Z - ZY}{Y} = \frac{Z}{Y} - Z \end{array} \right.$$

$$* \; f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \lambda e^{-\lambda x_1} \lambda e^{-\lambda x_2} = \lambda^2 e^{-\lambda(x_1+x_2)} 1_{\{x_1,x_2>0\}}$$

$$\begin{vmatrix} 0 & 1 \\ -\frac{Z}{Y^2} & \frac{1}{Y} - 1 \end{vmatrix} = \frac{Z}{Y^2}$$

$$f_{Y,Z}(y,z) = f_{X_1,X_2}\left(z, \frac{z}{y} - z\right) \cdot \left| \frac{z}{y^2} \right|$$

$$= \lambda^2 e^{-\lambda\left(\frac{z}{y}\right)} \cdot \frac{z}{y^2} \quad \text{за } y \in (0,1)$$

$$z \in (0, \infty)$$

$$f_Y(y) = \int_0^\infty \lambda^2 e^{-\frac{\lambda z}{y}} \frac{z}{y^2} \, dz \overset{\frac{\lambda z}{y} = t}{=} \int_0^\infty \cancel{\lambda^2} e^{-t} \frac{t \cdot \cancel{y}}{\cancel{\lambda}} \cdot \frac{1}{\cancel{y^2}} \cdot d\frac{t\cancel{y}}{\cancel{\lambda}}$$

$= \int_0^\infty e^{-t} \cdot t \, dt = 1$, т.е. $f_X(y) = 1$ за $y \in (0,1)$

$Y \sim Unif(0,1)$!

$X_1, X_2 \sim Exp(\lambda)$ $\dfrac{X_1}{X_1+X_2}$ е разп. pivotal quantity

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$X_1 \sim N(\mu_1, \sigma_1^2)$ $X_2 \sim N(\mu_2, \sigma_2^2)$ нез.

Искаме на $a_1 X_1 + a_2 X_2 \sim N(a_1\mu_1 + a_2\mu_2, a_1^2\sigma_1^2 + a_2^2\sigma_2^2)$

$Y = a_1 X_1 + a_2 X_2$

$Z = X_1$

$$\Rightarrow \left| \begin{array}{l} X_1 = Z \\ X_2 = \dfrac{Y - a_1 X_1}{a_2} = \dfrac{Y - a_1 Z}{a_2} \end{array} \right. \Rightarrow J = \begin{vmatrix} 0 & 1 \\ \frac{1}{a_2} & \dots \end{vmatrix} = -\frac{1}{a_2}$$

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{\sqrt{2\pi a_1^2 \sigma_1^2}} e^{-\frac{(x-a_1\mu_1)^2}{2a_1^2\sigma_1^2}} \cdots \qquad X_1 \sim N(a_1\mu_1, a_1^2\sigma_1^2)$$

$$f_{Y,Z}(y,z) = f_{X_1,X_2}\left(z, \frac{y-a_1 z}{a_2}\right) \cdot |J| \quad \cdots$$

$$X \sim N(\mu, \sigma^2) \qquad f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

~~зад~~. 14 $X_1, X_2 \sim U(0,1)$ $\quad Y = X_1 + X_2$ $\quad f_Y(y) = ?$

$$* \left| \begin{array}{l} Y = X_1 + X_2 \\ Z = X_1 \end{array} \right. \Rightarrow \left| \begin{array}{l} X_1 = Z \\ X_2 = Y - Z \end{array} \right. \qquad J = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$$

$$\Rightarrow f_{Y,Z}(y,z) = f_{X,Y}(z, y-z) = f_X(z) \cdot f_Y(y-z)$$

$$\Rightarrow f_Y(y) = \int_{-\infty}^{\infty} f_X(z) f_Y(y-z)\, dz$$

$$f_{X_1+X_2}(y) = \int_{-\infty}^{\infty} f_{X_1}(z) f_{X_2}(y-z)\, dz$$

$$P(X_1 + X_2 = k) = \sum P(X_1 = m) P(X_2 = k-m)$$

В случая: $f_{X_1+X_2}(x) = \int_{-\infty}^{\infty} \underbrace{1_{\{z\in(0,1)\}}}_{f_{X_1}(z)} \cdot \underbrace{1_{\{x-z\in(0,1)\}}}_{f_{X_2}(x-z)}\, dz$

$0 \le x-z \le 1$

$x \ge z \ge x-1$

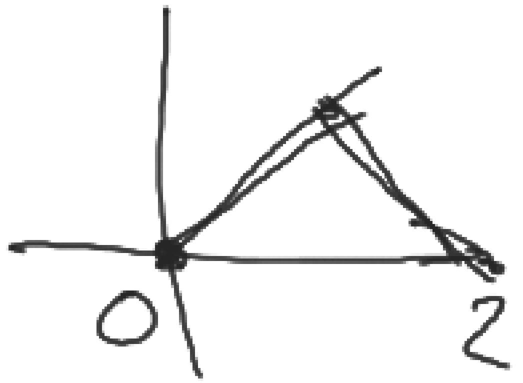
$0 \le z \le 1$

Ако $x\in(0,1]$, то $z\in(0,x]$

$x\in[1,2)$, то $z\in(x-1,1]$

$\Rightarrow f_{X_1+X_2}(x) = \int_0^x 1\, dt = x$ за $x\in[0,1]$

$f_{X_1+X_2}(x) = \int_{x-1}^1 1\, dt = 2-x$ за $x\in[1,2]$



зад 15* $X_1, X_2, \dots, X_n$ са iid $F_{X_1}(x)$, $f_{X_1} = F'_{X_1}$

$P(X_1 < x)$

Какво е р-то на max?

$Y = \max\{X_1, \dots, X_n\}$

$$F_Y(t) = P(\max\{X_1, \dots, X_n\} < t) = P(X_1 < t, X_2 < t, \dots, X_n < t)$$

$$\overset{\text{нез.}}{=} P(X_1 < t) \cdots P(X_n < t) \overset{\text{ед. р-ни}}{=} (P(X_1 < t))^n = F_{X_1}(t)^n$$

$Z = \min\{X_1, \dots, X_n\}$

$$F_Z(t) = P(Z < t) = 1 - P(Z \geq t) = 1 - P(X_1 \geq t, X_2 \geq t, \dots, X_n \geq t)$$

$$\overset{\text{нез.}}{=} 1 - P(X_1 \geq t) \cdots P(X_n \geq t) \overset{\text{ед. р}}{=} 1 - P(X_1 \geq t)^n$$

$$= 1 - (1 - F_{X_1}(t))^n$$

зад. 21 $A \sim N(3,2)$ 5 ед.

$B \sim N(3,3)$

$C \sim N(1,10)$

$a_1 = 5$

$5N(3,2) \stackrel{d}{=} N(15,50)$

а) Всяко разпр. $a_1 \cdot A + (5-a_1)B$

$$\stackrel{d}{=} a_1 N(3,2) + (5-a_1) N(3,3)$$

$$= N(3a_1, 2a_1^2) + N((5-a_1)\cdot 3, 3(5-a_1)^2)$$

$$= N(15, 2a_1^2 + 3(5-a_1)^2)$$

б) a_1 ? $2a_1^2 + 3(5-a_1)^2$ да е мин

$= 5a_1^2 - 30a_1 + 75$ мин се достига в $a_1 = \frac{30}{10} = 3$

т.е. оптималното порт. $\sim N(15, 2\cdot 9 + 3\cdot 4) = N(15,30)$

B) $5D \overset{d}{=} 5N(-2, 20) \overset{d}{=} N(-10, 500)$

$$\mathbb{P}(N_1(-10,500) > N_2(15,30))$$

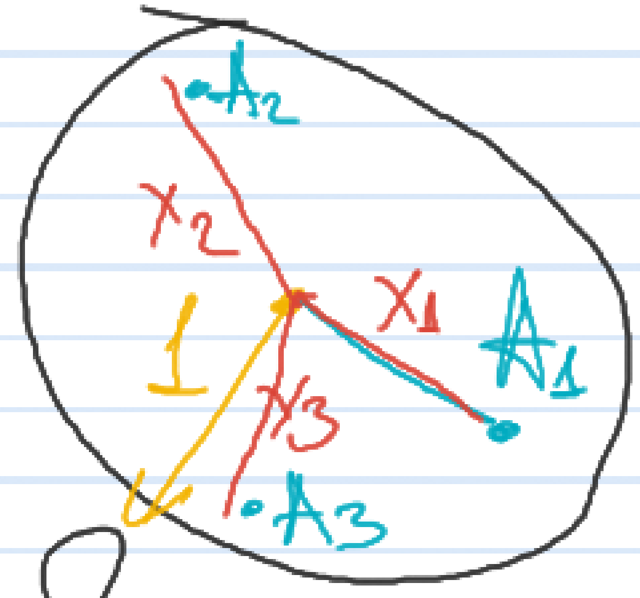
$$\mathbb{P}(N_1(-10,500) - N_2(15,30) > 0)$$

$$= \mathbb{P}(N(-25, 530) > 0)$$

$$= \mathbb{P}(N(0, 530) > 25) = \mathbb{P}\left(N(0,1) > \frac{25}{\sqrt{530}}\right)$$

$$= 1 - \Phi\left(\frac{25}{\sqrt{530}}\right)$$

зад. 19



$f_{X_1}(x) = ?$

$\mathbb{P}(X_1 \le t) = \frac{\pi \cdot t^2}{\pi \cdot 1^2} = t^2$

за $t \in (0,1)$

$\Rightarrow f_{X_1}(t) = 2t \cdot \mathbb{1}_{\{t \in [0,1]\}}$

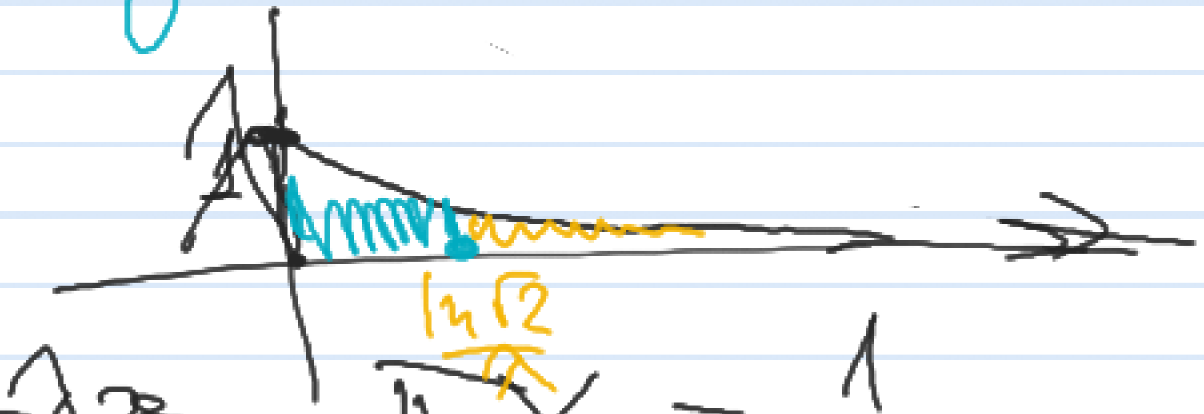
$\mathbb{E}X_1 = \int_0^1 t \cdot 2t \, dt = \frac{2}{3}$

$Z = \min(X_1, X_2, X_3)$

$\mathbb{E}Z = ?$

Мода на X = ст-та x, за която $f_X(x)$ е max

Медианата = x : $\mathbb{P}(X < x) = \frac{1}{2}$



$f(x) = \lambda e^{-\lambda x}$

$\mathbb{E}X = \frac{1}{\lambda}$

мода = 0

медиана = $\frac{\ln 2}{\lambda}$

$e^{-\lambda t} = \frac{1}{2}$

$\lambda t = \ln 2 \quad t = \frac{\ln 2}{\lambda}$

$$\mathbb{P}(Z \le t) = 1 - \mathbb{P}(Z > t) = 1 - \mathbb{P}(X_1 > t)^3$$

$t \in (0,1)$

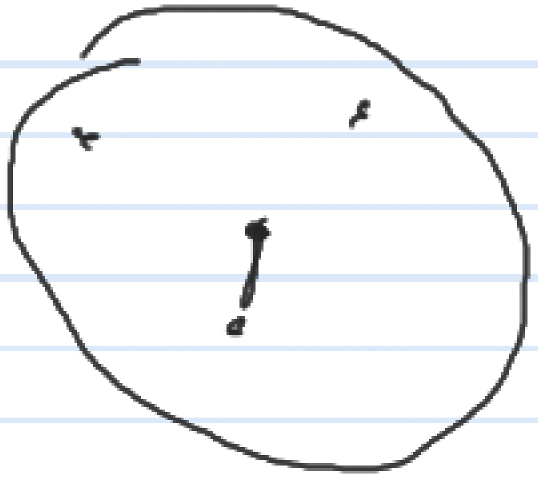
$$= 1 - (1-t^2)^3$$

$$\Rightarrow f_Z(t) = 3(1-t^2)^2 \cdot 2t$$

и $$\mathbb{E}Z = \int_0^1 t \cdot 3(1-t^2)^2 \cdot 2t \, dt$$

$$= \int_0^1 6t^6 - 12t^4 + 6t^2 \, dt$$

$$= 6\frac{1}{7} - 12\frac{1}{5} + 6\frac{1}{3} = \frac{16}{35}$$



до $\max(X_1, X_2, X_3)$ аналогично

$X_{(1)}$ – най-малкото

$X_{(2)}$ – средното

$X_{(3)}$ – най-голямото

$\mathbb{P}(X_1 \leq t)$

$\mathbb{P}(X_{(1)} \leq t) = 1 - \mathbb{P}(X_1 > t, \dots, X_n > t)$
$= 1 - (1 - F_{X_1}(t))^n$

$$\mathbb{P}(X_{(2)} \leq t) = \mathbb{P}(X_1 \leq t, X_2 \leq t, X_3 > t)$$
$$+ \mathbb{P}(X_1 \leq t, X_2 > t, X_3 \leq t)$$
$$+ \mathbb{P}(X_1 > t, X_2 \leq t, X_3 \leq t) + \mathbb{P}(X_1 \leq t, X_2 \leq t, X_3 \leq t)$$
$$= 3\, \mathbb{P}(X_1 \leq t)^2 \cdot \mathbb{P}(X_1 > t) +$$
$$= 3 F_{X_1}(t)^2 (1 - F_{X_1}(t)) + F^3_{X_1}(t)$$

Анал. $X_1, \dots, X_n$ $\quad \mathbb{P}(X_{(k)} \leq t) = \sum_{j=k}^{n} \binom{n}{j} F_{X_1}(t)^j (1 - F_{X_1}(t))^{n-j}$.