Twenty problems in probability

This section is a selection of famous probability puzzles, job interview questions (most high-tech companies ask their applicants math questions) and math competition problems. Some problems are easy, some are very hard, but each is interesting in some way. Almost all problems I have heard from other people or found elsewhere. I am acknowledging the source, or a partial source, in square brackets, but it is not necessarily the original source.

You should be reminded that all random choices (unless otherwise specified) are such that all possibilities are equally likely, and different choices within the same context are by default independent. Recall also that an *even bet* on the amount x on an event means a correct guess wins you x, while an incorrect guess means loss of the same amount.

1. [P. Winkler] One hundred people line up to board an airplane. Each has a boarding pass with assigned seat. However, the first person to board has lost his boarding pass and takes a random seat. After that, each person takes the assigned seat if it is unoccupied, and one of unoccupied seats at random otherwise. What is the probability that the last person to board gets to sit in his assigned seat?

2. [D. Knuth] Mr. Smith works on the 13th floor of a 15 floor building. The only elevator moves continuously through floors $1, 2, \dots, 15, 14, \dots, 2, 1, 2, \dots$, except that it stops on a floor on which the button has been pressed. Assume that time spent loading and unloading passengers is very small compared to the travelling time.

Mr. Smith complains that at 5pm, when he wants to go home, the elevator almost always goes up when it stops on his floor. What is the explanation?

Now assume that the building has n elevators, which move independently. Compute the proportion of time the first elevator on Mr. Smith's floor moves up.

3. [D. Barsky] *NCAA basketball pool.* There are 64 teams who play single elimination tournament, hence 6 rounds, and you have to predict all the winners in all 63 games. Your score is then computed as follows: 32 points for correctly predicting the final winner, 16 points for each correct finalist, and so on, down to 1 point for every correctly predicted winner for the first round. (The maximum number of points you can get is thus 192.) Knowing nothing about any team, you flip fair coins to decide every one of your 63 bets. Compute the expected number of points.

4. [E. Berlekamp] *Betting on the World Series.* You are a broker; your job is to accommodate your client's wishes without placing any of your personal capital at risk. Your client wishes to place an even $1,000 bet on the outcome of the World Series, which is a baseball contest decided in favor of whichever of two teams first wins 4 games. That is, the client deposits his $1,000 with you in advance of the series. At the end of the series he must receive from you either $2,000 if his team wins, or nothing if his team loses. No market exists for bets on the entire world

series. However, you can place even bets, in any amount, on each game individually. What is your strategy for placing bets on the individual games in order to achieve the cumulative result demanded by your client?

5. From *New York Times*, Science Times, D5, April 10, 2001:

> "Three players enter a room and a red or blue hat is placed on each person's head. The color of each hat is determined by [an independent] coin toss. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats [but not their own], the players must simultaneously guess the color of their own hats or pass. The puzzle is to find a group strategy that maximizes the probability that at least one person guesses correctly and no-one guesses incorrectly."

The naive strategy would be for the group to agree that one person should guess and the others pass. This would have probability 1/2 of success. Find a strategy with a greater chance for success. (The solution is given in the article.)

For a different problem, allow every one of n people to place an even bet on the color of his hat. The bet can either be on red or on blue and the amount of each bet is arbitrary. The group wins if their combined wins are strictly greater than their losses. Find, with proof, a strategy with maximal winning probability.

6. [L. Snell] Somebody chooses two nonnegative integers X and Y and secretly writes them on two sheets of paper. The distrubution of (X, Y) is unknown to you, but you do know that X and Y are different with probability 1. You choose one of the sheets at random, and observe the number on it. Call this random number W and the other number, still unknown to you, Z. Your task is to guess whether W is bigger than Z or not. You have access to a random number generator, i.e., you can generate independent uniform (on $[0, 1]$) random variables at will, so your strategy could be random.

Exhibit a stategy for which the probability of being correct is $1/2 + \epsilon$, for some $\epsilon > 0$. This ϵ may depend on the distribution of (X, Y), but your strategy of course can not.

7. A person's birthday occurs on a day i with probability p_i, where $i = 1, \ldots, n$. (Of course, $p_1 + \cdots + p_n = 1$.) Assume independent assignment of birthdays among different people. In a room with k people, let $P_k = P_k(p_1, \ldots, p_n)$ be the probability that no two persons share a birthday. Show that this probability is maximized when all birthdays are equally likely: $p_i = 1/n$ for all i.

8. [Putnam Exam] Two real numbers X and Y are chosen at random in the interval $(0, 1)$. Compute the probability that the closest integer to X/Y is even. Express the answer in the form $r + s\pi$, where r and s are rational numbers.

9. [L. Snell] Start with n strings, which of course have $2n$ ends. Then randomly pair the ends and tie together each pair. (Therefore you join each of the n randomly chosen pairs.) Let L be the number of resulting loops. Compute $E(L)$.

10. [Putnam Exam] Assume C and D are chosen at random from $\{1, \ldots, n\}$. Let p_n be the probability that $C + D$ is a perfect square. Compute $\lim_{n\to\infty}(\sqrt{n} \cdot p_n)$. Express the result in the form $(a\sqrt{b} + c)/d$, where a, b, c, d are integers.

11. [D. Griffeath] Let $\alpha \in [0, 1]$ be an arbitrary number, rational or irrational. The only randomizing device is an unfair coin, with probability $p \in (0, 1)$ of heads. Design a game between Alice and Bob so that Alice's winning probability is exactly α. The game of course has to end in a finite number of tosses with probability 1.

12. [Putnam Exam] Let $(X_1, \ldots, X_n)$ be a random vector from the set $\{(x_1, \ldots, x_n) : 0 < x_1 < \cdots < x_n < 1\}$. Also let f be a continuous function on $[0, 1]$. Set $X_0 = 0$. Let R be the Riemann sum

$$R = \sum_{i=0}^{n-1} f(X_{i+1})(X_{i+1} - X_i).$$

Show that $ER = \int_0^1 f(t)P(t)\,dt$, where $P(t)$ is a polynomial of degree n, independent of f, with $0 \le P(t) \le 1$ for $t \in [0, 1]$.

13. [R. Stanley] You have $n > 1$ numbers $0, 1, \ldots, n-1$ arranged on a circle. A random walker starts at 0 and at each step moves at random to one of its two nearest neighbors. For each i, compute the probability p_i that, when the walker is at i for the first time, all other points have been previously visited, i.e., that i is the last new point. For example, $p_0 = 0$.

14. [R. Stanley] Choose $X_1, \ldots, X_n$ from $[0, 1]$. Let p_n be the probability that $X_i + X_{i+1} \le 1$ for all $i = 1, \ldots, n-1$. Prove that $\lim_{n\to\infty} p_n^{1/n}$ exists and compute it.

15. [Putnam Exam] Each of the triples (r_i, s_i, t_i), $i = 1, \ldots, n$, is a randomly chosen permutation of (1, 2, 3). Compute the three sums $\sum_{i=1}^n r_i$, $\sum_{i=1}^n s_i$, and $\sum_{i=1}^n t_i$, and label them (not necessarily in order) A, B, C so that $A \le B \le C$. Let a_n be the probability that $A = B = C$ and let b_n be the probability that $B = A + 1$ and $C = B + 1$. Show that for every $n \ge 1$, either $4a_n \le b_n$ or $4a_{n+1} \le b_{n+1}$.

16. [Putnam Exam] Four points are chosen on the unit sphere. What is the probability that the origin lies inside the tetrahedron determined by the four points?

17. [Putnam Exam] An $m \times n$ checkerboard is colored randomly: each square is randomly painted white or black. We say that two squares, p and q, are in the same *connected monochromatic*

component (or *component*, in short) if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of components is greater than $mn/8$ and smaller than $(m+2)(n+2)/6$.

18. Choose, at random, three points on the circle $x^2 + y^2 = 1$. Interpret them as cuts that divide the circle into three arcs. Compute the expected length of the arc that contains the point $(1, 0)$.

Remark. Here is a "solution." Let L_1, L_2, L_3 be the lengths of the three arcs. Then $L_1 + L_2 + L_2 = 2\pi$ and by symmetry $E(L_1) = E(L_2) = E(L_3)$, so the answer is $E(L_1) = 2\pi/3$. Explain why this is wrong.

19. You are in possession of n pairs of socks (hence a total of $2n$ socks) ranging in shades of grey, labeled from 1 (white) to n (black). Take the socks blindly from a drawer and pair them at random. What is the probability that they are paired so that the colors of any pair differ by at most 1? You have to give an explicit formula, which may include factorials.

20. [P. Winkler] Choose two random numbers from $[0, 1]$ and let them be the endpoints of a random interval. Repeat this n times. What is the porbability that there is an interval which intersects all others.