Full Solutions
MATH110 December 2012

December 4, 2014

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Educational Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

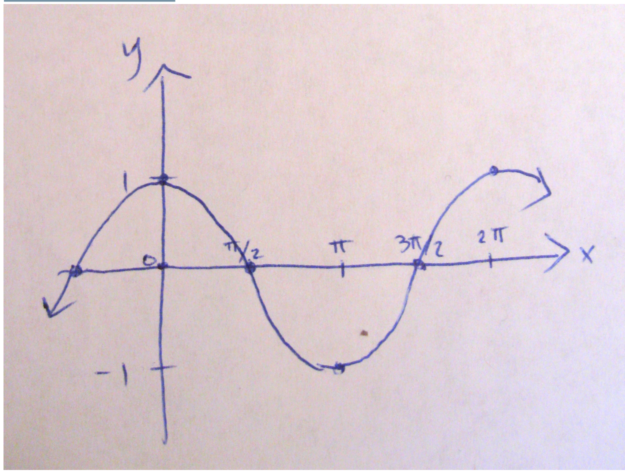
1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
    - Re-do related homework and webwork questions.
    - The Math Exam Resources offers mini video lectures on each topic.
    - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
    - Do as many exam simulations as possible.

    Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Educational Resources.

Question 1 (a)

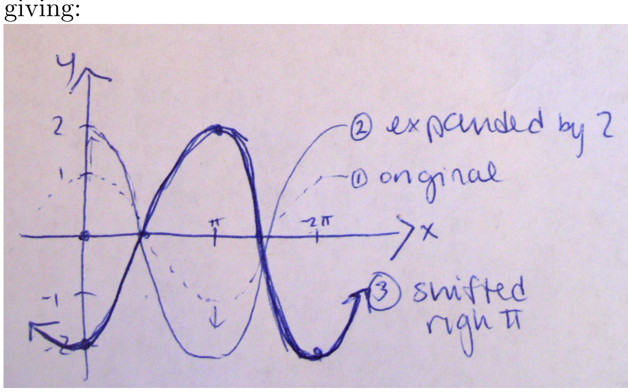
SOLUTION 1. The graph of $f(x) = \cos x$ looks like this:



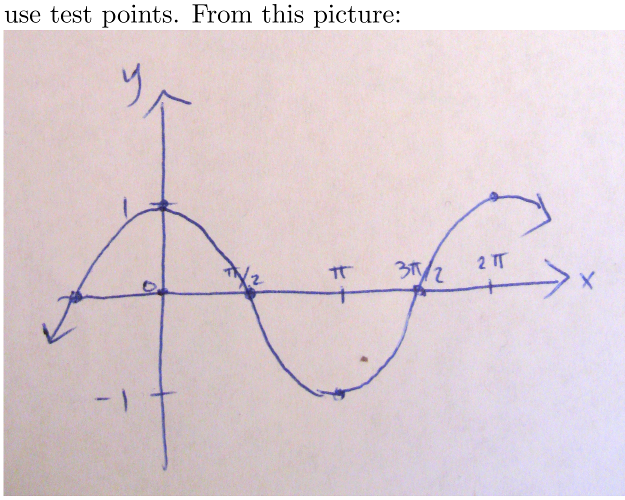
The rules for shifting functions tell us that

- Multiplying the function by 2 increases its amplitude (stretch the height of the graph) by 2,
- subtracting by π on the inside of the function shifts the graph right by π units,

giving:



SOLUTION 2. If you've forgotten the rules used in the first solution (or never learned them), you can also use test points. From this picture:



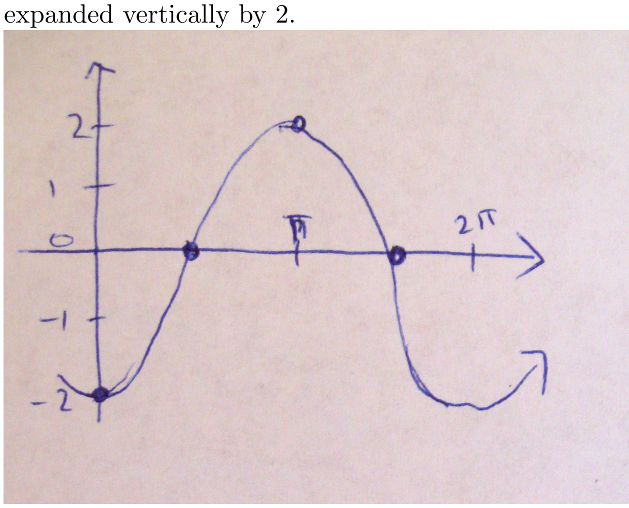
you have the following points, placed in a table of values:

x-value	y-value
0	1
$\pi/2$	0
π	-1
$3\pi/2$	0

Plugging these same x-values into $f(x) = 2\cos(x - \pi)$ gives:

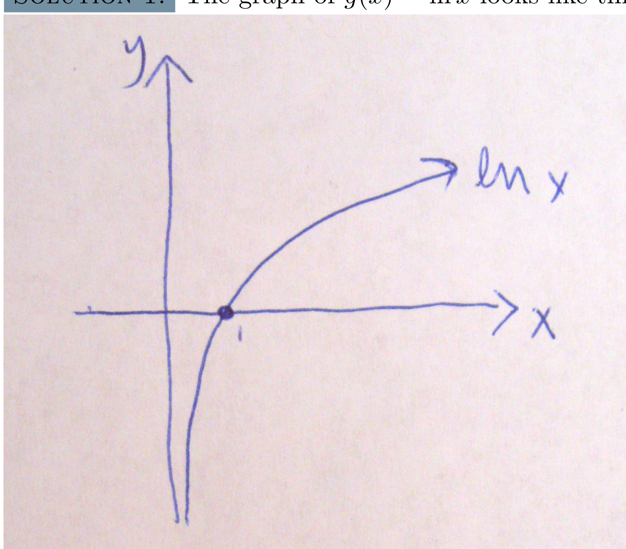
x-value	calculations	y-value
0	$2\cos(0 - \pi) = 2\cos(-\pi) = 2(-1)$	-2
$\pi/2$	$2\cos(\pi/2 - \pi) = 2\cos(-\pi/2) = 2(0)$	0
π	$2\cos(\pi - \pi) = 2\cos(0) = 2(1)$	2
$3\pi/2$	$2\cos(3\pi/2 - \pi) = 2\cos(\pi/2) = 2(0)$	0

Drawing a picture based on these points, it should be clear that the graph has been shifted right by π and expanded vertically by 2.



Question 1 (b)

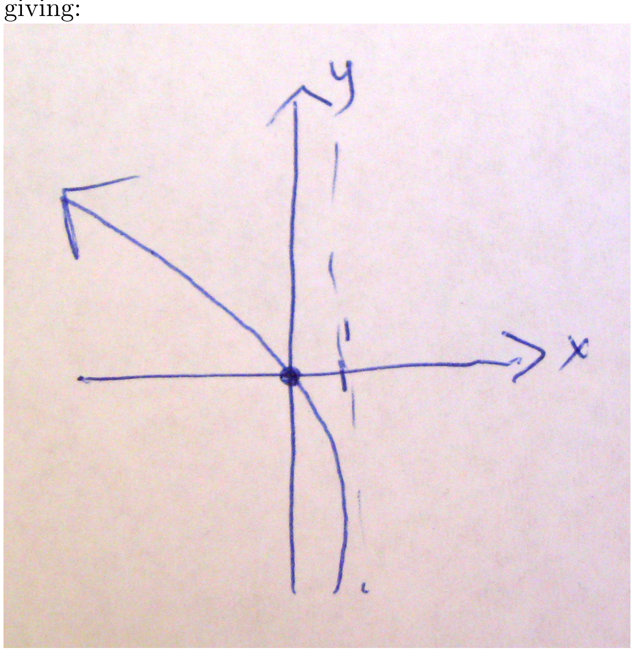
SOLUTION 1. The graph of $g(x) = \ln x$ looks like this:



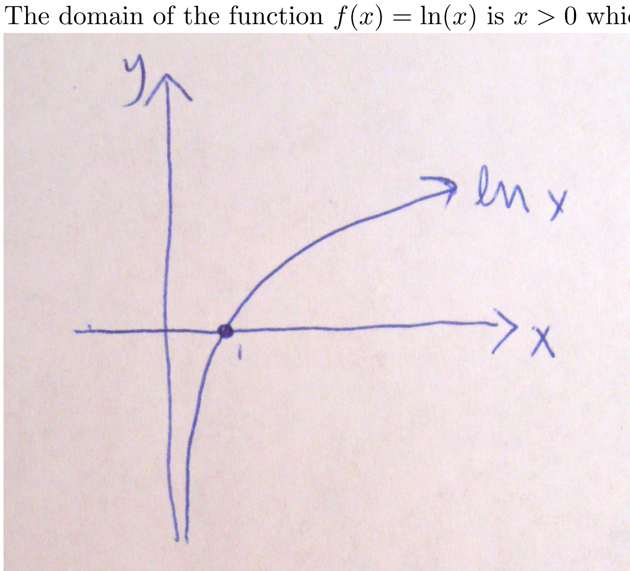
Using function rules for the graph of $g(x) = \ln(1 - x)$

- the negative in front of the x means the graph has been reflected over the y-axis and
- *(1-x)* inside the function means that it is shifted right 1

giving:



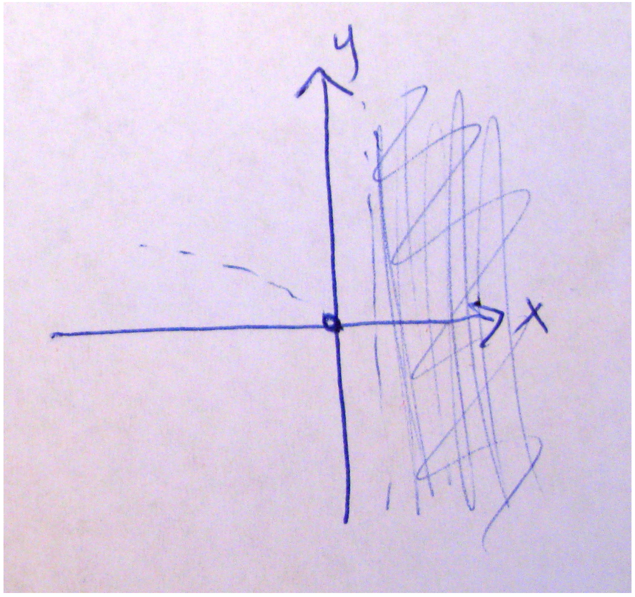
Solution 2. If you've forgotten the rules used in the first solution (or never learned them), you can also facts about the domain of $\ln(x)$ to determine where the graph should go.
The domain of the function $f(x) = \ln(x)$ is $x > 0$ which can be seen in the graph below.



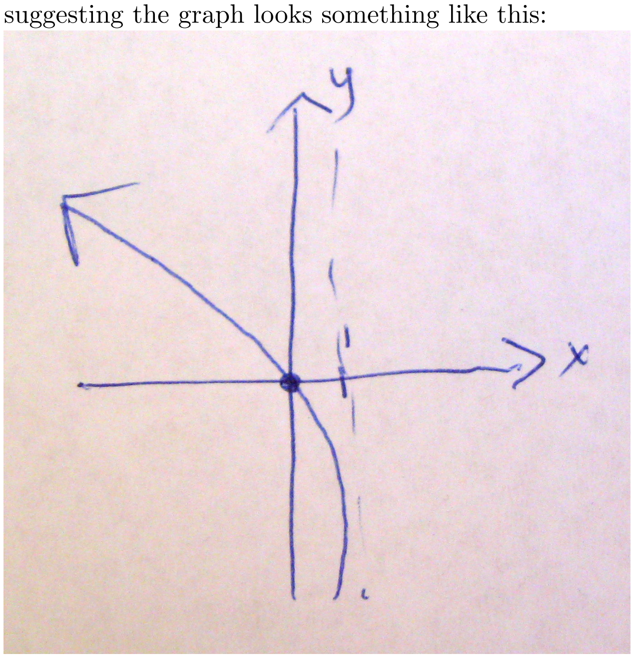
The graph of $g(x) = \ln(1 - x)$ will have a similar shape, but has a different domain. Let's test some values to see where the domain of this function is.

x-value		y-value
-2	$\ln(1-(-2)) = \ln(3)$	$\ln(3)$
-1	$\ln(1-(-2)) = \ln(2)$	$\ln(2)$
0	$\ln(1-0) = \ln(1)$	0
1	$\ln(1-1) = \ln(0)$	undefined
2	$\ln(1-2) = \ln(-1)$	undefined

So clearly this function is defined on the domain $x < 1$,



suggesting the graph looks something like this:



Question 2 (a)

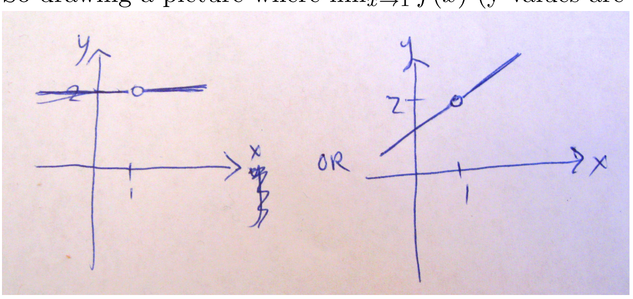
SOLUTION. One common function where $\lim_{x\to 0} f(x)$ doesn't exist is the rational function $f(x) = \frac{1}{x}$. The same is true for its negative, $g(x) = -\frac{1}{x}$. Both have a limit of ∞ or $-\infty$ at 0.
However, when these two functions are added together, we get:
$\lim_{x\to 0} \frac{1}{x} + (-\frac{1}{x}) = \lim_{x\to 0} 0 = 0$
Which is a perfectly satisfactory limit. Thus, these two functions are a counterexample to this statement, proving it false.
To see an interactive illustration of this solution, see this page.

Question 2 (b)

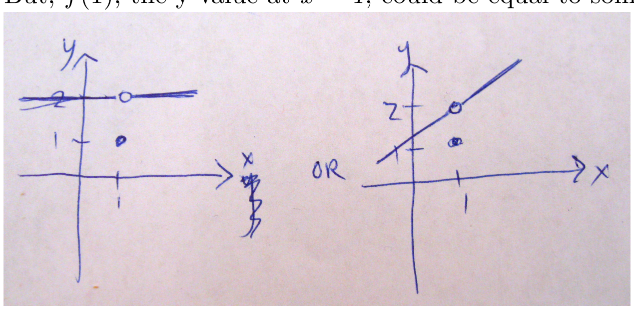
Solution. As suggested in the hint, we will simplify the limit on the left hand side of the equal sign. If we just plug in *2* we get 0/0, so we will have to do some simplifying. Because the numerator and denominator are polynomials, factoring is a good first step.
The denominator is a difference of squares, so we can write:
$\lim_{x\to 2} \frac{x-2}{4-x^2} = \lim_{x\to 2} \frac{2-x}{(2-x)(2+x)}$
Canceling the common terms, and then plugging in the 2 gives:
$\lim_{x\to 2} \frac{1}{2+x} = \frac{1}{2+2} = \frac{1}{4}$
Which matches the right hand side of the original equation. Thus the statement is true.

Question 2 (c)

Solution. Remember that a limit describes the y-values of a graph **around**, but not **at** the x-value, whereas $f(x)$ gives the y-value of a graph **at** x.
So drawing a picture where $\lim_{x\to 1} f(x)$ (y-values are equal to 2 near $x = 1$) could look like this:



But, $f(1)$, the y-value at $x = 1$, could be equal to something else, like 1:

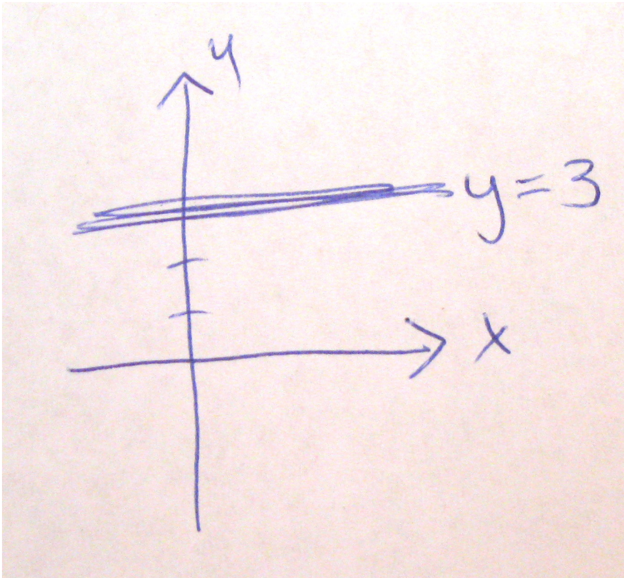


Because the conclusion of the statement is not necessarily true, the statement is false.

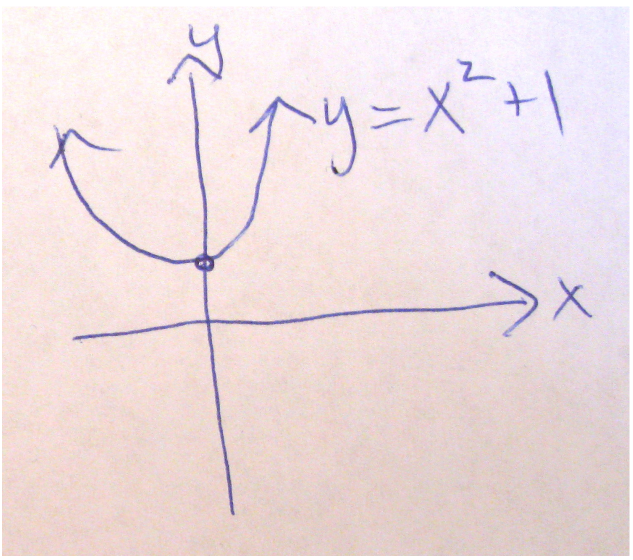
Question 2 (d)

Solution. Polynomials are functions like lines, parabolas, cubics, etc. The best examples to consider are horizontal lines and parabolas.

- No horizontal line crosses the x-axis.



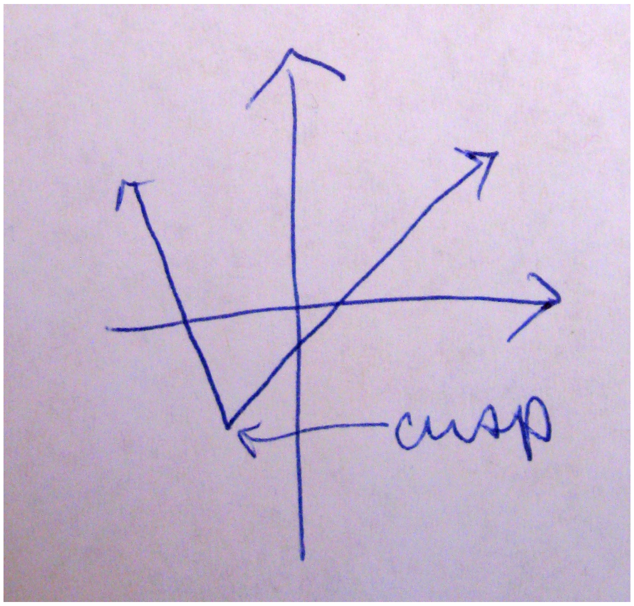
- The graph of a parabola can be shifted so that it doesn't cross the x-axis.

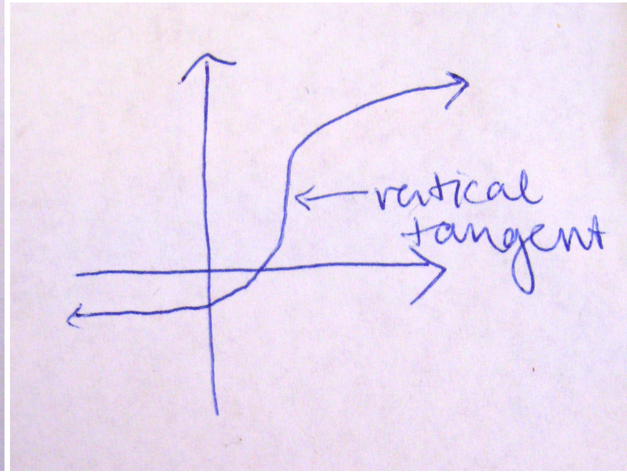


Either of these would be a counter-example, so the statement is false.

Question 2 (e)

SOLUTION. A function is non-differentiable if it has a vertical tangent or a sharp corner (cusp). A function is continuous if it can be drawn in one go without lifting your pencil. It is possible to draw a continuous function with one (or multiple!) of the non-differentiable properties just listed:





So a function can be continuous, without necessarily being differentiable. Thus the statement is false.

Question 2 (f)

SOLUTION. Horizontal tangent lines occur when the derivative equals zero. The derivative of y is
$y' = \frac{1}{2\sqrt{x}}$
However, this derivative is never zero, because for all positive values of x (the domain of the derivative), y' will be a positive number, not zero.
Thus $y = \sqrt{x}$ has no horizontal tangent lines, so the statement is true.

Question 3 (a)

SOLUTION. We apply the product rule to $f(x)$ to get:
$f'(x) = g'(x) * x^{1/2} + g(x) * (1/2)(x^{-1/2})$
Now we simply evaluate at $x = 9$, using the other information given in the problem to fill in *g(9)* and *g'(9)*.
Recall also that $x^{1/2} = \sqrt{x}$.

$$\begin{aligned} f'(9) &= g'(9) * (9)^{1/2} + g(9) * (1/2)(9^{-1/2}) \\ &= 4 * 3 + 2 * (1/2)(1/3) \\ &= 12 + 1/3 \end{aligned}$$

So the value of $f'(9)$ is 37/3.

Question 3 (b)

SOLUTION. You could use the quotient rule on each term of this function, but it will be easier to re-write it as follows:
$f(x) = a + bx^{-1} + cx^{-2} + dx^{-3}$
Remembering that a, b, c and d are constants, we can simply use the power rule, which gives:
$f'(x) = b(-x^{-2}) + c(-2x^{-3}) + d(-3x^{-4})$
You can leave the answer like this. If you wanted to simplify, it would be written:

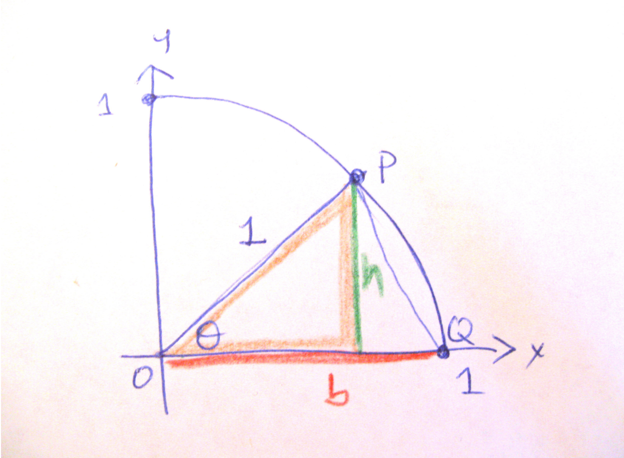
$f'(x) = \frac{-b}{x^2} + \frac{-2c}{x^3} + \frac{-3d}{x^4}$

Question 3 (c)

SOLUTION. To begin calculating this derivative, apply the quotient rule. So
$f'(x) = \frac{(e^{3x})' * (3x) - e^{3x} * (3x)'}{(3x)^2}$
The derivative of *3x* is simple to calculate, but in order to calculate $(e^{3x})'$ we will need to use the chain rule as follows:
$(e^{3x})' = e^{3x} * (3x)' = e^{3x} * 3$
Plugging this back into the quotient rule, we get:
$f'(x) = \frac{(e^{3x} * 3) * (3x) - e^{3x} * (3)}{(3x)^2}$
You can leave this answer as is.

Question 3 (d)

SOLUTION. First we need to find a formula for $f(\theta)$, the area of the triangle. The area of a triangle is $(b)(h)/2$. See the labeled picture below:



The base of the triangle is a radius of the circle, which is equal to 1. We can find the height by a trig ratio. Using the blue right triangle, we can see that $\sin\theta = h/1$. So $h = \sin\theta$. Plugging these into our area formula, we get:
$f(\theta) = (1)(\sin\theta)/2 = (\sin\theta)/2$
It is now easy to find the derivative,
$f'(\theta) = (\cos\theta)/2$
completing the question.

Question 4

SOLUTION. As suggested by the hint, we will solve for zero on one side of the equation, giving:
$-x^3 + x^2 + 2x - \ln(x) - 1 = 0$
We now need to show that there is a number c in the interval *(1,2)* that makes the left side equal to zero. This should remind you of the Intermediate Value Theorem (IVT). The statement of the IVT says

> If f(x) is continuous on [a,b] and for some number L, f(a) < L < f(b), then there exists c in (a,b) such that f(c) = L." In our scenario, the left side of the equation is *f(x)*, *a = 1*, *b=2*, and"L = 0'.

In order to use the IVT, we need to check that the function is continuous on the interval *[1,2]*. As each term of the function is continuous on the interval, the whole function is also continuous on the interval.

Now we compute *f(1)* and *f(2)*.
$f(1) = -1 + 1 + 2 - \ln(1) - 1 = 1 > 0$
$f(2) = -8 + 4 + 4 - \ln(2) - 1 < 0$
Because *f(1) < 0 < f(2)*, we can now use the IVT to conclude that there exists a number c in the interval *(1,2)* such that *f(c) = 0*.

Question 5

SOLUTION. Using the limit defnition of the derivative we have
$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(1+\frac{1}{(x+h)^2})-(1+\frac{1}{x^2})}{h}$
To simplify, we cancel the ones in the numerator, find a common denominator for the fractions, and simplify:

$$\begin{aligned}
&= \lim_{h\to 0} \frac{\frac{x^2}{(x^2)(x+h)^2} - \frac{(x+h)^2}{x^2(x+h)^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x^2}{(x^2)(x+h)^2} - \frac{x^2+2xh+h^2}{x^2(x+h)^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{-2xh-h^2}{(x^2)(x+h)^2}}{h} \\
&= \lim_{h\to 0} \frac{h(-2x-h)}{h(x^2)(x+h)^2} \\
&= \lim_{h\to 0} \frac{(-2x-h)}{(x^2)(x+h)^2} \\
&= \frac{-2x}{x^4}
\end{aligned}$$

So $f'(x) = \dfrac{-2}{x^3}$

Question 6

SOLUTION. In order for $f(x)$ to be differentiable, the derivative at any point must be equal from both the left and right. Since $f(x)$ is clearly differentiable at all points in $(-\infty, \infty)$ other than $x = 1$, our point of interest is $x = 1$.
The derivative of $f(x)$ at $x = 1$ from the left will be
$\left(\sqrt{x - \frac{3}{4}}\right)' = \frac{1}{2\sqrt{x-\frac{3}{4}}}$
which when evaluated at $x = 1$ gives $\frac{1}{2\sqrt{1/4}} = 1$
The derivative of $f(x)$ at $x = 1$ from the right will be
$(ax + b)' = a$
To be differentiable, these must be equal, so
$a = 1$
This gives us the value for a. Now we also know that in order for the function to be differentiable, it must also be continuous. In particular, this means both the right and left hand limits at $x = 1$ must be equal. So we have:

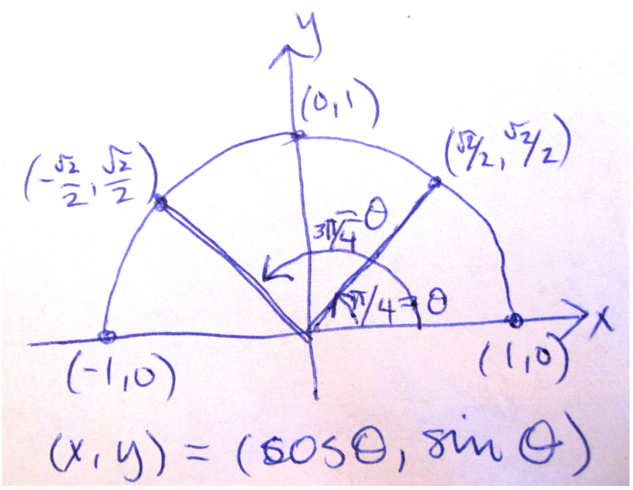
$$
\begin{aligned}
\lim_{x\to 1^-} f(x) &= \lim_{x\to 1^+} f(x)\\
\lim_{x\to 1^-} \sqrt{x-\frac{3}{4}} &= \lim_{x\to 1^+} 1x+b\\
\sqrt{1/4} &= 1+b\\
\frac{-1}{2} &= b
\end{aligned}
$$

So $f(x)$ will be differentiable when $a = 1$ and $b = -1/2$

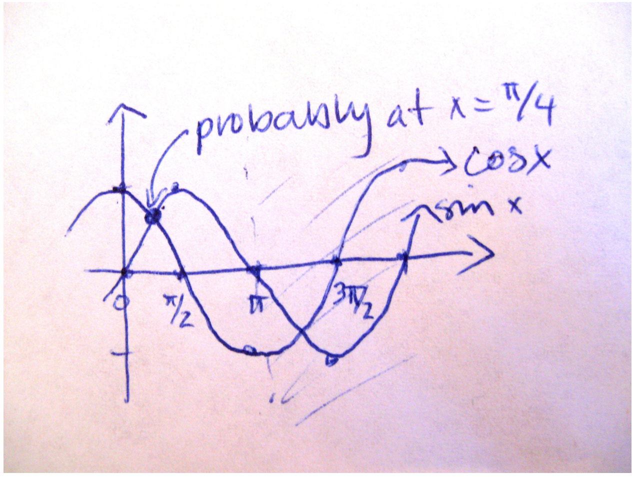
Question 7 (a)

SOLUTION. The equation $f(x) = 0$ can be written as
$\sin x - \cos x = 0$
or
$\sin x = \cos x$
Now it remains to find where $\sin x$ is equal to $\cos x$ in the interval $[0, \pi)$. There are multiple ways to do this.

Unit Circle The coordinates of points on the unit circle are given by $\sin\theta$ and $\cos\theta$, so finding where $\sin\theta = \cos\theta$ is the same as finding where the x and y coordinates of a point on the unit circle are the same. This occurs when the angle is equal to $\pi/4$ radians.

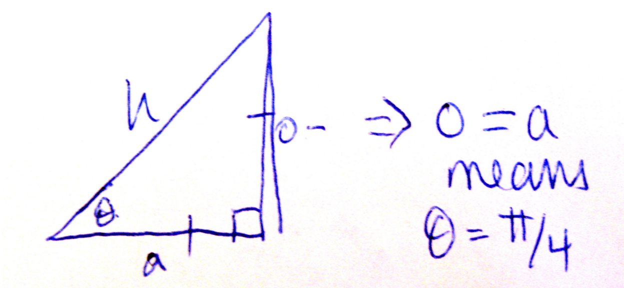


Drawing the graph If you draw the graphs of $\sin x$ and $\cos x$ very carefully, you can make an educated guess about where the two functions are equal. However, this method will never be as accurate as knowing key values of $\sin x$ and $\cos x$ from the unit circle.



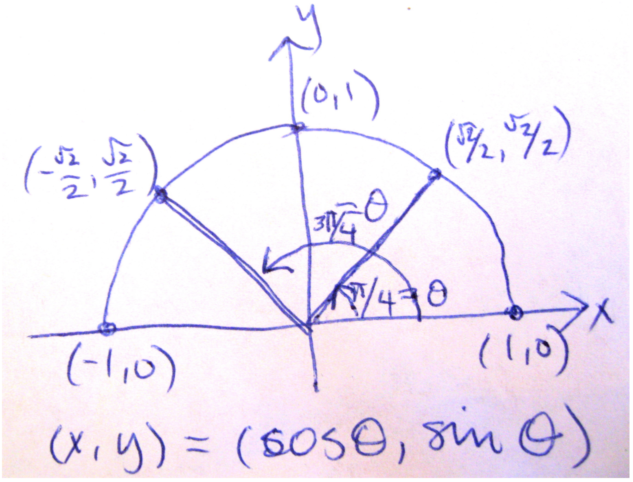
Trig ratios One last method is writing out $\sin x$ and $\cos x$ as trig ratios, where, given a right triangle with angle x, $\sin x = o/h$ and $\cos x = a/h$. Setting this equal gives:

$o/h = a/h$, or $o = a$. The opposite and adjacent sides of a right triangle are equal when the right triangle is isoceles, meaning each angle is 45 degrees or $x = \pi/4$.



Question 7 (b)

SOLUTION. This question is similar to the previous one, except we are using the derivative:
$f'(x) = \cos x - (-\sin x) = \cos x + \sin x$
Setting this equal to zero and moving a term over gives
$\sin x = -\cos x$ (or $\cos x = -\sin x$)
This means that $\sin x$ and $\cos x$ must have equal values but opposite signs. On the unit circle, $\cos\theta$ corresponds to x-values and $\sin\theta$ corresponds to y-values, so the opposite signs place us in quadrant II of the coordinate plane (the restriction of $[0, \pi)$ rules out quadrant IV). The angle where the values of $\sin\theta$ and $\cos\theta$ are equal in this quadrant is $3\pi/4$, so $x = 3\pi/4$.



Question 8

SOLUTION. Consider a point P=(a,b) on the curve where b=y(a). The tangent line through this point will go through (a,b) and have slope y'(a) where y' is the derivative. Therefore the tangent line L(x) will satisfy

$$L(x) = y'(a)(x - a) + y(a).$$

We know that y=1/x and so

$$y(a) = \frac{1}{a}.$$

Next we calculate the derivative:

$$y' = -\frac{1}{x^2}$$

and plugging in the point x=a we get

$$y'(a) = -\frac{1}{a^2}.$$

Up to this point we have enough information to compute the tangent line for any point x=a on the curve. However, we are looking for the two points where the slope is parallel (therefore equal) to -100, the slope of the line given in the question. This gives the equation:

$$-\frac{1}{a^2} = -100$$

Solving for a, we get

$$a = \pm\frac{1}{10}.$$

Therefore, we need to find the equation of the tangent line where a = *-1/10* and a = *1/10*.

- When a = *-1/10*,

$$y(a) = y(-1/10) = \frac{1}{-1/10} = -10$$

and the slope is $m = -100$ (because the lines are parallel). Therefore, the equation of the tangent line is

$$L(x) = -100(x - (-1/10)) + (-10)$$

This is a sufficient answer, but if you simplify to *y=mx + b* form, you find instead that *L(x) = -100x -20*.

- When a = *1/10*,

$$y(a) = y(-1/10) = \frac{1}{1/10} = 10,$$

with slope $m = -100$. Therefore, the equation of the tangent line is

$$L(x) = -100(x - 1/10) + 10.$$

This is a sufficient answer, but if you simplify to *y=mx + b* form, you find *L(x) = -100x +20*

Question 9 (a)

SOLUTION. The second fact that must be proven is:
Let k be an integer. If the claim is true for $n = k$ (that is, $2^1 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$), then the statement must be true for $n = k + 1$ (meaning $2^1 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{k+2} - 1$).
This fact basically states that if the claim holds for some integer, it is also true for the consecutive integer.

Question 9 (b)

SOLUTION. We start with Fact 1 - we know that the claim is true for $n = 1$. Then, using Fact 2, since the claim is true for $n = 1$, we can say that it is also true for the next integer, $n = 2$. We can then use Fact 2 again; since we just said the claim is true for $n = 2$, it is also true for the next integer, $n = 3$. We can continue this process until we get to $n = 6$, which is the value of n shown in the statement of the question. (Note that you can prove the above statement is true simply by doing the calculations - but this does not use the inductive reasoning steps described in part a).

Question 9 (c)

SOLUTION. As suggested in the hint, we can assume
"For n = k, $2^1 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$." **(Equation 1)**
We want to then prove that this claim also holds for $n = k + 1$, meaning that we eventually want to show that
$2^1 + 2^2 + \cdots + 2^{k+1} = 2^{k+2} - 1$ **(Equation 2)**
using the previous statement **(1)** as our starting point. So let's start with this equation:

$$2^1 + 2^2 + \cdots + 2^k = 2^{k+1} - 1.$$

And in an attempt to reach equation **(2)**, we add 2^{k+1} to both sides.

$$2^1 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}.$$

Now the left side of the equation matches **(2)**, but what about the right hand side? Let us combine the terms to see that

$$\begin{aligned} 2^1 + 2^2 + \cdots + 2^k + 2^{k+1} &= 2^{k+1} + 2^{k+1} - 1 \\ &= 2(2^{k+1}) - 1 \\ &= 2^{k+1+1} - 1 \\ &= 2^{k+2} - 1 \end{aligned}$$

.
This is exactly the equation what we were trying to show. Hence we have proved that if the claim holds for $n = k$, it also holds for $n = k + 1$, proving Fact 2.

Question 10

SOLUTION. The question is asking about change in area, so we will start with the formula for area of a circle

$$A = \pi r^2.$$

Because the question involves rates, we will be implicitly differentiating with respect to time. This gives:
$\frac{dA}{dt} = \pi(2r)\frac{dr}{dt}$
Where dA/dt is the rate at which the area of the circle is changing, and dr/dt the rate at which the radius is changing. Plugging in the information from the statement of the question, we have:
$\frac{dA}{dt} = \pi(2 * 100)(900) = 180,000\pi$
Thus, the rate at which the area of the circle is increasing is $180,000\pi$ km/h.

Good Luck for your exams!