Full Solutions
MATH102 December 2013

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question A 01

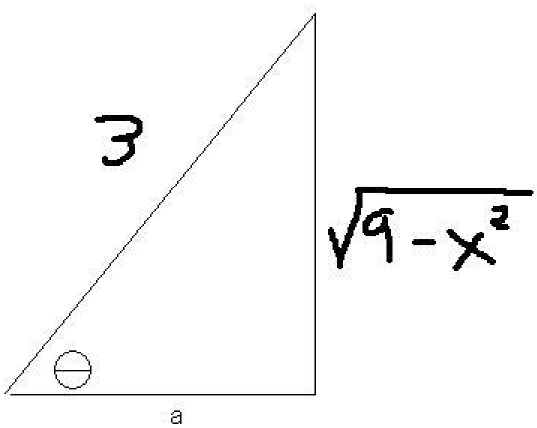
SOLUTION. f has an **inflection point** wherever f'' *changes sign*. Thus we begin by looking for the zeroes of f'':

$$\begin{aligned} f(x) &= \sin(x) + ax^2 \\ f'(x) &= \cos(x) + 2ax \\ f''(x) &= -\sin(x) + 2a \end{aligned}$$

Solving for a, we find that $a = \frac{1}{2}\sin(x)$. Since $-1 \le \sin(x) \le 1$, we have $-\frac{1}{2} \le a \le \frac{1}{2}$.
But by the same inequalities, $0 \le f''(x) \le 2$ when $a = \frac{1}{2}$. Similarly, $-2 \le f''(x) \le 0$ when $a = -\frac{1}{2}$. In these cases, f has no inflection points since f'' does not *change sign*.
Therefore $-\frac{1}{2} < a < \frac{1}{2}$. This is equivalent to answer (D).

Question A 02

SOLUTION. Let $\theta = \arcsin\left(\frac{\sqrt{9-x^2}}{3}\right)$. Then $\sin(\theta) = \frac{\sqrt{9-x^2}}{3}$. This gives the following triangle:



To solve for the last side, we simply use the Pythagorean theorem

$$a^2 = 3^2 - \left(\sqrt{9-x^2}\right)^2 = 9 - 9 + x^2 = x^2$$

and so $a = x$ (we only need to consider the positive root since a has to be positive).
Hence, taking the cosine of theta in our triangle above, we see that $\cos(\theta) = \cos\left(\arcsin\left(\frac{\sqrt{9-x^2}}{3}\right)\right) = \frac{a}{3} = \frac{x}{3}$, which is answer (B).

Question A 03

SOLUTION. We start at the point $(0, 1)$, where the slope of the tangent line at this point is
$\frac{dy}{dx} = 2 - y^2 = 2 - [y(0)]^2 = 2 - 1 = 1$
By applying **Euler's method** with a step size of $\Delta t = 0.1$, our next point would be at the value $y(0.1) = y(0) + \left.\frac{dy}{dx}\right|_{x=0}(0.1 - 0) = 1 + 0.1 = 1.1$, which is answer (E).

Question A 04

SOLUTION. Noting that $f(x_1)=0$, we solve for x_1 using the equation of the tangent line to f at $(x_0, f(x_0))$:

$$\begin{aligned} y - y_0 &= m(x-x_0) \\ f(x_1) - f(x_0) &= f'(x_0)(x_1 - x_0) \\ 0 - f(x_0) &= f'(x_0)(x_1 - x_0) \\ x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \end{aligned}$$

which is answer (E).

Question A 05

SOLUTION. Taking the second derivative of our function gives

$$\begin{aligned} y &= \cos(x) \\ y' &= -\sin(x) \\ y'' &= -\cos(x) \end{aligned}$$

To the immediate left of $x = \pi/2$, the second derivative is negative and to its immediate right it is positive. So we get an *overestimate* when $x < \pi/2$ and an *underestimate* when $x > \pi/2$. Thus the answer is (D).

Question A 06

SOLUTION. We note that $y=-1$ and $y=1$ appear to be **equilibrium solutions**, since the slope field is zero at these values. Thus $y=-1$ and $y=1$ are constant solutions to the differential equation in question, i.e., $y=-1,\ 1 \implies \frac{dy}{dt}=0$.
By the zero product property, it follows that $\frac{dy}{dt}$ contains the factors $(y-1)(y+1)$. Therefore we eliminate (A), (D), and (E). We now choose a point, say $(0,0)$. The slope field is clearly decreasing at that point and so $\frac{dy}{dt}<0$. Plugging $y=0$ into both (B) and (C) only yields a negative value with (B).

Question A 07

SOLUTION 1. Substituting $t = 2000$ into the given possibilities gives

(a) $N = 60 + 120\sin\left(\frac{2\pi}{11}(t-2000)+\frac{\pi}{2}\right) = 60 + 120\sin\left(\frac{\pi}{2}\right) = 180$

(b) $N = 60 + 60\sin\left(\frac{11}{2\pi}(t+2000)\right) = 60 + 60\sin\left(\frac{22000}{\pi}\right)$

(c) $N = 60 + 60\cos\left(\frac{11}{2\pi}(t+2000)\right) = 60 + 60\cos\left(\frac{22000}{\pi}\right)$

(d) $N = 60 + 60\sin\left(\frac{2\pi}{11}(t-2000)\right) = 60 + 60\sin(0) = 60$

(e) $N = 60 + 60\cos\left(\frac{2\pi}{11}(t-2000)\right) = 60 + 60\cos(0) = 120$

Therefore, the answer is (E).

SOLUTION 2. A more methodical method of approaching the problem is as follows:
We know that the minimum of the function is 0 and the maximum is 120, which means that we are looking for a trigonometric function of the form

$$60 + 60\sin(f(t))$$

or

$$60 + 60\cos(f(t))$$

for some function $f(t)$.
Since the period is 11 years, we see that

$$f(t) = \frac{2\pi}{11} \cdot g(t)$$

for some other function $g(t)$.
Lastly, we notice that a peak was detected in 2000. If we used the cosine function, we would have

$$g(t) = t - 2000$$

since the peak of the cosine function is at 0 (up to integer multiples of 2π). If we used the sine function, we would have

$$f(t) = \frac{2\pi}{11} \cdot g(t) + \frac{\pi}{2}$$

and

$$g(t) = t - 2000$$

since the peak of the sine function occurs at $\frac{\pi}{2}$ up to integer multiples of 2π.
Thus, the possible answers are:

$$N = 60 + 60\sin\left(\frac{2\pi}{11}(t - 2000) + \frac{\pi}{2}\right)$$

or

$$N = 60 + 60\cos\left(\frac{2\pi}{11}(t - 2000)\right)$$

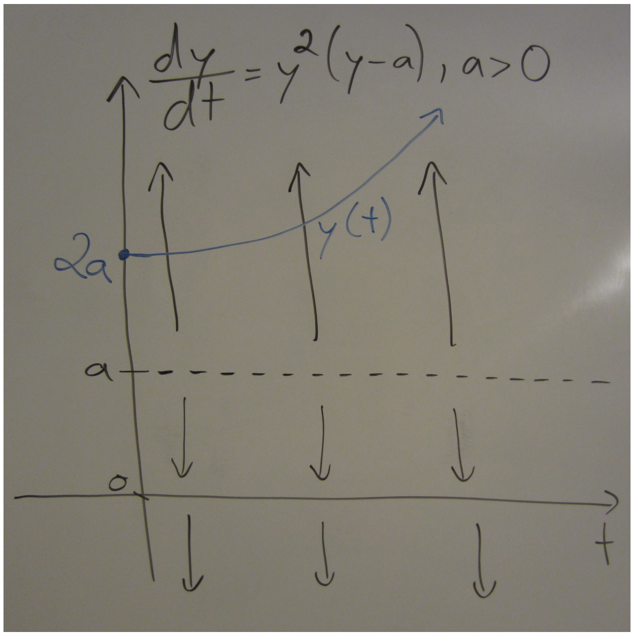
The latter is a possible choice in the question, and so the answer is (E).

Question A 08

SOLUTION. To start with, we calculate the **steady states**, i.e.:

$$\frac{dy}{dt} = 0$$
$$y^2(y - a) = 0$$

Thus the steady states are $y = 0$, $y = a$.
The initial condition is $y(0) = 2a$, $a > 0$, which is above $y = a$. Since $\left.\frac{dy}{dt}\right|_{t=0} = (y(0))^2(y(0) - a) = (2a)^2(2a - a) > 0$, the solution is *increasing* away from the steady state $y = a$ to infinity:



In other words, $\lim_{t\to\infty} y(t) = \infty$, which is answer (B).

Question B 01

Solution. Differentiate both sides of $\tan y = x$ with respect to x. (Don't forget to apply the chain rule!)

$$\begin{aligned}\frac{d}{dx}\tan y &= \frac{d}{dx}x\\ \sec^2 y\cdot\frac{dy}{dx} &= 1\\ \frac{dy}{dx} &= \frac{1}{\sec^2 y}.\end{aligned}$$

Now apply the trig identity $\sec^2 y = 1+\tan^2 y$ to get

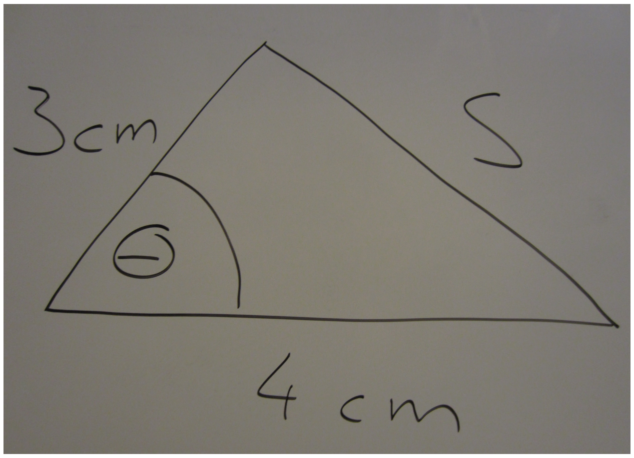
$$\frac{dy}{dx} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

after noting that $\tan y = x \implies \tan^2 y = x^2$.

Question B 02

Solution. Following the hint, we recall the **cosine law**:

$$\begin{aligned}a^2 &= b^2+c^2-2bc\cdot\cos\theta\\ S^2 &= 3^2+4^2-2(3)(4)\cdot\cos\theta\\ S^2 &= 25-24\cos\theta\end{aligned}$$



Differentiating implicitly with respect to time,

$$\begin{aligned}\frac{d}{dt}(S^2) &= \frac{d}{dt}(25 - 24\cos\theta)\\ 2S\cdot\frac{dS}{dt} &= 24\sin\theta\cdot\frac{d\theta}{dt}\\ \frac{dS}{dt} &= \frac{12}{S}\sin\theta\cdot\frac{d\theta}{dt}\end{aligned}$$

At the time in question, we are given that $\theta = \frac{\pi}{2}$, $\frac{d\theta}{dt} = 1$. From the cosine law above, we also have $S = 5$, when $\theta = \frac{\pi}{2}$. Therefore

$$\begin{aligned}\frac{dS}{dt} &= \frac{12}{S}\sin\theta\cdot\frac{d\theta}{dt}\\ &= \frac{12}{5}\sin\left(\frac{\pi}{2}\right)\cdot 1\\ &= \frac{12}{5}\text{ cm/s}\end{aligned}$$

Question B 03

SOLUTION. Following the hint, we have

$$f'(x) = 5x^4 - 5x^3 = 5x^3(x-1)$$

The zeroes of f' are $x = 0, 1$.
On $(-\infty, 0)$ (say at $x = -10$), $f'(x) > 0$ and so the function is increasing on this interval.
On $(0, 1)$ (say at $x = 1/2$), $f'(x) < 0$ and so the function is decreasing on this interval.
On $(1, \infty)$ (say at $x = 10$), $f'(x) > 0$ and so the function is increasing on this interval.
Therefore we have a **local maximum** at $x = 0$ and a **local minimum** at$x = 1$.

Question B 04

SOLUTION. We can rewrite the differential equation as

$$y'' = -4y = -2^2 \cdot y$$

Since

$$\frac{d^2}{dx^2}\sin(\omega x) = -\omega^2 \sin(x)$$

and

$$\frac{d^2}{dx^2}\cos(\omega x) = -\omega^2 \cos(x)$$

we see that $\sin(\omega x)$ and $\cos(\omega x)$ are solutions to the equation, with $\omega = 2$. The solution of the equation must also satisfy the initial condition $y(0) = 1$, which is only the case for $y = \cos(2x)$. Hence the solution is $y = \cos(2x)$.

Question B 05

SOLUTION 1. Recall that the definition of the derivative of a function f at x is

$$f'(x) := \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Following hint 1, we see that if $x = 3$ and $f(x) = \sqrt{x}$, then

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{\sqrt{3+h} - \sqrt{3}}{h} = L \\ L &= \frac{1}{2\sqrt{3}} \end{aligned}$$

SOLUTION 2. The slower solution uses hint 2 and multiplies the numerator and the denominator by the conjugate of the numerator

$$\begin{aligned} L &= \lim_{h\to 0} \frac{\sqrt{3+h} - \sqrt{3}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{3+h} - \sqrt{3}}{h} \cdot \frac{\sqrt{3+h} + \sqrt{3}}{\sqrt{3+h} + \sqrt{3}} \\ &= \lim_{h\to 0} \frac{3+h-3}{h(\sqrt{3+h} + \sqrt{3})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{3+h} + \sqrt{3})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{3+h} + \sqrt{3}} \\ &= \frac{1}{\sqrt{3} + \sqrt{3}} \\ &= \frac{1}{2\sqrt{3}} \end{aligned}$$

Question B 06

SOLUTION. Polynomials are continuous, so the only point where we are concerned with the continuity of f is $x = 1$. It remains to find a value a such that $\lim_{x\to 1^-} f(x) = f(1) = \lim_{x\to 1^+} f(x)$.

- By the definition of $f(x)$ we have $f(1) = a - 1$.
- The right-sided limit is $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} a - x = a - 1$.
- The left-sided limit is $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1$.

Hence we need to find a value a such that $a - 1 = 1$. It follows that the value a that makes $f(x)$ continuous on $(-\infty, \infty)$ is $a = 2$.

Question B 07

SOLUTION. The solution of the differential equation $\frac{dy}{dt} = ky$ is

$$y(t) = y(0) \cdot e^{kt}$$

where in this case y is the number of viral particles, t is the time in days, and k is some constant.
We are given that $y(0) = 1000$, $k = 0.05$. Thus we have $y(t) = 1000e^{0.05t}$.
Solving for t when $y(t) = 350000$ yields:

$$\begin{aligned} 350000 &= 1000e^{0.05t} \\ 350 &= e^{0.05t} \\ \ln 350 &= 0.05t \\ t &= \frac{\ln 350}{0.05} \text{ days} \end{aligned}$$

Question B 08

SOLUTION. (i)

$$\lim_{x\to 0^+} e^{1/x} = \lim_{h\to\infty} e^h = \infty$$

(ii)

$$\lim_{x\to\infty} e^{1/x} = \lim_{h\to 0} e^h = e^0 = 1$$

(iii)

$$\lim_{x\to 0^+} \frac{1}{ax + e^{1/x}} = \lim_{h\to\infty} \frac{1}{\frac{a}{h} + e^h} = 0$$

(iv)

$$\lim_{x\to\infty} \frac{1}{ax+e^{1/x}} = \lim_{h\to 0^+} \frac{1}{\frac{a}{h}+e^h} = 0$$

Question C 01

SOLUTION. The volume of a cone is given by

$$V = \frac{1}{3}\pi r^2 h$$

By the physical application of the problem, we can assume that the base radius of the cone changes as the height increases. First, we should express the radius of the cone as a function of its height.
By similar triangles, and the fact that $r = 1$ when $h = 10$, we find that the following relation holds:

$$\frac{r}{h} = \frac{1}{10} \qquad \Longleftrightarrow \qquad r = \frac{h}{10}$$

Thus the volume of the shell can be expressed entirely in terms of h:

$$\begin{aligned} V &= \frac{1}{3}\pi\left(\frac{h}{10}\right)^2 h \\ &= \frac{\pi}{300}h^3 \end{aligned}$$

The rate of change in the volume of the cone can be computed by differentiating V with respect to t, giving us

$$\frac{dV}{dt} = \frac{\pi}{100}h^2 \cdot \frac{dh}{dt}$$

Since we are told that the height of the cone changes at a constant rate of 0.1 cm/year, the rate of change of in the cone's volume when $h = 10$ cm and $r = 1$ cm is given by

$$\begin{aligned} \frac{dV}{dt} &= \frac{\pi}{100}(10\text{cm})^2 \cdot 0.1\frac{\text{cm}}{\text{year}} \\ &= \frac{\pi}{10}\frac{\text{cm}^3}{\text{year}} \end{aligned}$$

Question C 02 (a)

SOLUTION. The constant I has a positive effect on the rate of change of the fish population (i.e., as I increases, $\frac{dF}{dt}$ increases). It is also independent of the number of fish or fishermen.
Hence, the constant I represents the ***rate at which the company adds fish to the lake***.

Question C 02 (b)

SOLUTION. The term containing α tends to make the value of F decrease since it has a negative sign

preceding it. Because of this, and the fact that this term depends on both the *population of fish* and the *number of fishermen*, we conclude that this term models the interaction between fishermen and the fish. The interaction of fishermen and fish results in fewer fish because they get caught by the fishermen.
Thus the constant α represents the ***rate at which a single fisherman catches fish***. The larger the value of α (i.e., the more effective a fisherman is at catching fish), the faster F decreases.

Question C 02 (c)

SOLUTION.

- In the steady state, the derivative is equal to 0. Hence we solve the equation

$I - \alpha NF$ for F and obtain
$F_* = \frac{I}{N\alpha}$

- The solution F of the equation is

$F(t) = \frac{I}{\alpha N}(1 - e^{-\alpha N t})$
In order to find t_*, where $F(t_*) = \frac{F_*}{2}$, we set
$\frac{F_*}{2} = \frac{I}{\alpha N}(1 - e^{-\alpha N t_*})$
and use that $\frac{F_*}{2} = \frac{I}{2\alpha N}$ to find

$$\frac{I}{2\alpha N} = \frac{I}{\alpha N}(1 - e^{-\alpha N t_*}) \quad \Rightarrow \quad \frac{1}{2} = e^{-\alpha N t_*} \quad \Rightarrow \quad \ln(\frac{1}{2}) = -\alpha N t_*$$

Now we solve for $t_* = -\frac{\ln(\frac{1}{2})}{\alpha N} = -\frac{\ln(1)-\ln(2)}{\alpha N} = -\frac{0-\ln(2)}{\alpha N} = \frac{\ln(2)}{\alpha N}$

Question C 02 (d)

SOLUTION. Applying $N = \frac{F}{3}$, the equation becomes $\frac{dF}{dt} = I - \frac{\alpha}{3}F^2$
For $t \to \infty$, the solution F reaches its steady state F_*. Hence, we again set the left-hand side equal zero and solve for F_*.

$$\begin{aligned} 0 &= I - \frac{\alpha}{3}F_*^2 \\ \Rightarrow \quad F_*^2 &= \frac{3I}{\alpha} \\ \Rightarrow \quad F_* &= \sqrt{\frac{3I}{\alpha}} \end{aligned}$$

We only consider the positive root since the amount of fish can not be negative.

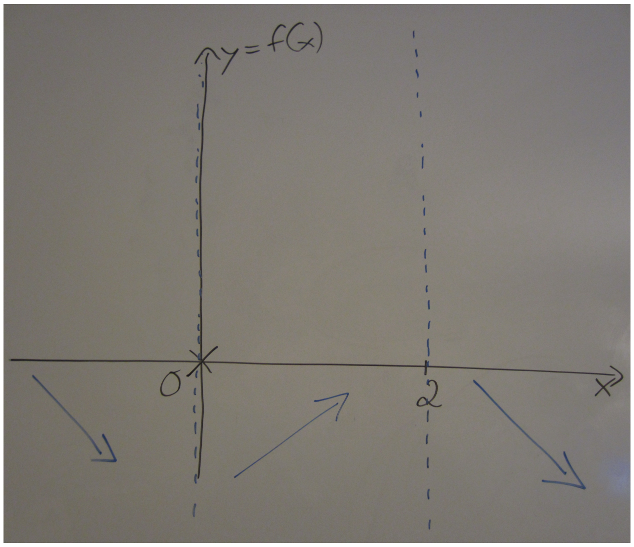
Question C 03

SOLUTION. We begin by finding the zeroes of the original function,
$0 = x^2 e^{-x}$
and since $e^{-x} > 0$ always, only $x = 0$ is a root. But we note that f is always greater or equal to zero.
Next we take the derivative and see that
$f'(x) = 2xe^{-x} - x^2 e^{-x} = xe^{-x}(2 - x)$
Finding the roots gives us $x = 0, 2$. Making a sign chart, we see that

1. The function is decreasing between $(-\infty, 0)$

2. The function is increasing between $(0, 2)$
3. The function is decreasing between $(2, \infty)$

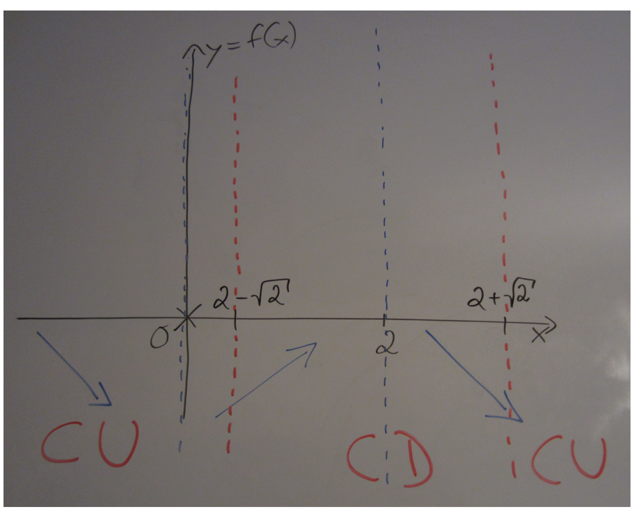
Graphically, we have



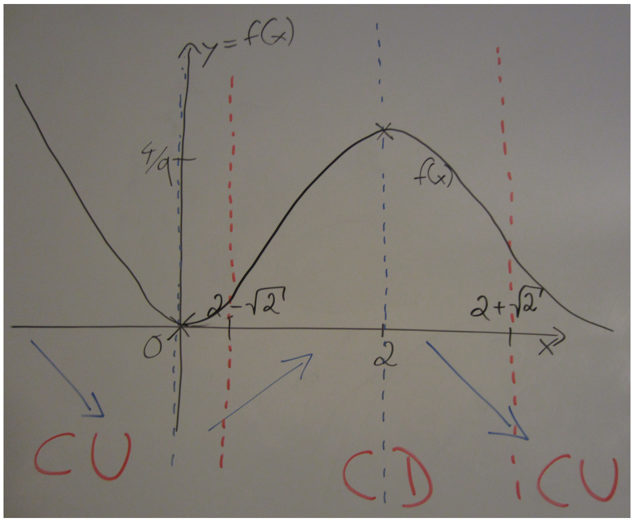
and thus we have that $x = 0$ is a local minimum and $x = 2$ is a local maximum. The coordinates are given by $(0, 0)$ and $(2, 4e^{-2})$ (note that $4e^{-2}$ is roughly $4/9$).
Lastly taking the second derivative, we have that
$f''(x) = 2(e^{-x} - xe^{-x}) - (2xe^{-x} - x^2e^{-x}) = e^{-x}(2 - 2x - 2x + x^2) = e^{-x}(2 - 4x + x^2)$
The zeroes of the second derivative are given by $x^2 - 4x + 2 = 0$.
Using the quadratic formula gives us the roots
$x = \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(2)}}{2} = \frac{4\pm\sqrt{8}}{2} = \frac{4\pm2\sqrt{2}}{2} = 2 \pm \sqrt{2}$
Making a sign chart, we see that

1. The function is concave up between $(-\infty, 2 - \sqrt{2})$
2. The function is concave down between $(2 - \sqrt{2}, 2 + \sqrt{2})$
3. The function is concave up between $(2 + \sqrt{2}, \infty)$

Graphically we have



Hence, we have that $2 \pm \sqrt{2}$ are inflection points. Combining all this information, we have the following. First plot all known points of interest and label where the function is increasing, decreasing, concave up and concave down. Then connect the dots in a consistent way. Don't forget that $\lim_{x\to\infty} x^2 e^{-x} = 0$ was given to us and we can use this information.



Question C 04

SOLUTION. First we compute the lengths denoted in the diagram in terms of x.
Since DC = 400, we have that the length of BC is $400 - x$. As AD = 300, we can use the Pythagorean theorem on triangle ADB to see that the length of AB is equal to $\sqrt{300^2 + x^2}$.
Let k be the energy required to swim one metre. We seek to minimize the energy function given by

$$E(x) = 2k \cdot \sqrt{90000 + x^2} + k \cdot (400 - x)$$

which is equivalent to minimizing

$$E(x) = 2\sqrt{90000 + x^2} + 400 - x$$

since k is a positive constant.
To optimize the energy required, we first find the critical points of E by differentiating, setting the derivative to 0, and solving for x:

$$
\begin{aligned}
E(x) &= 2\sqrt{90000+x^2}+400-x \\
E'(x) &= 2\cdot\frac{1}{2\sqrt{90000+x^2}}\cdot\frac{d}{dx}(90000+x^2)+0-1 \\
&= \frac{2x}{\sqrt{90000+x^2}}-1
\end{aligned}
$$

$$
\begin{aligned}
0 &= \frac{2x}{\sqrt{90000+x^2}}-1 \\
\sqrt{90000+x^2} &= 2x \\
90000+x^2 &= 4x^2 \\
90000 &= 3x^2 \\
30000 &= x^2 \\
x &= \pm\sqrt{30000} = \pm 100\sqrt{3}
\end{aligned}
$$

Clearly $x \geq 0$ and the relevant critical point is $x = 100\sqrt{3}$. Further, $x \leq 400$. Thus, the maximum of our function occurs at either $x = 0$, $100\sqrt{3}$, 400. We check these values in our energy function.

$$
\begin{aligned}
E(0) &= 2\sqrt{90000+0^2}+400-0 = 2\cdot 300+400 = 1000 \\
E(400) &= 2\sqrt{90000+400^2}+400-400 = 2\cdot 500 = 1000 \\
E(100\sqrt{3}) &= 2\sqrt{90000+(100\sqrt{3})^2}+400-100\sqrt{3} \\
&= 2\sqrt{120000}+400-100\sqrt{3} \\
&= 200\sqrt{12}+400-100\sqrt{3} \\
&= 400\sqrt{3}+400-100\sqrt{3} \\
&= 300\sqrt{3}+400 \\
&= 100(3\sqrt{3}+4) < 100(3(2)+4) = 1000
\end{aligned}
$$

Therefore, the penguin's energy consumption is minimized when $x = 100\sqrt{3}$.

Good Luck for your exams!