Full Solutions
MATH110 April 2013

April 16, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

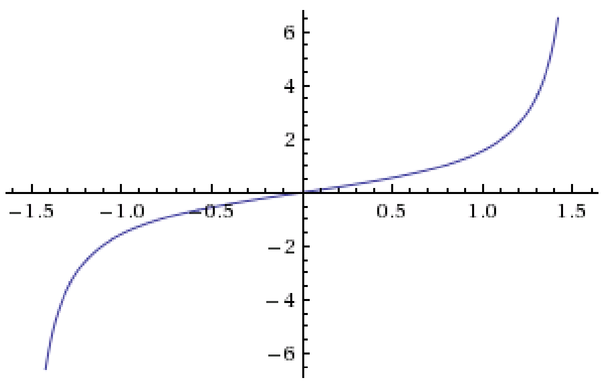
Question 1 (a)

SOLUTION. This statement is **true**.
If f has no tangent line that crosses the x-axis, then all tangent lines must have slope 0. Hence $f' = 0$ at every point, and thus the polynomial f is necessarily constant.
This contradicts the assumption we started with. So the statement is **true**.

Question 1 (b)

Easiness: 3.0/5

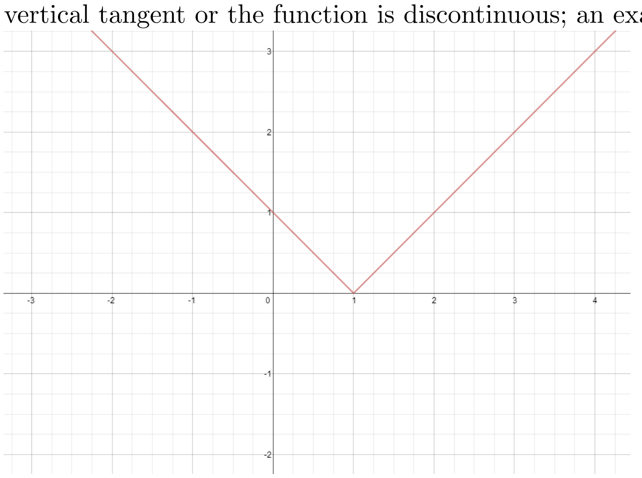
SOLUTION 1. This statement is **false**.
Consider, for example, the graph of $f(x) = \tan(x)$ for $-\pi/2 < x < \pi/2$. Nowhere do we have $f'(x) = \sec(x) = 0$, but still the curve switches from concave down to concave up.



SOLUTION 2. The statement is **false**.
Two more examples are $f(x) = \sin(x)$, $0 < x < \pi$, and $f(x) = \cos(x)$, $-\pi/2 < x < \pi/2$. The argument is the same as in solution 1.

Question 1 (c)

SOLUTION. The statement is **false**.
A minimum always occurs at a critical point - however, there are two possible reasons for a critical point: either $f'(x) = 0$ **or** $f'(x)$ does not exist. In this second case, the derivative is not equal to zero, but there can still be a local minimum. For example, the derivative does not exist if there is a cusp (sharp corner), a vertical tangent or the function is discontinuous; an example of a cusp is given below:



In this case there is a minimum at $x = 1$ but $f'(x)$ does not exist, so it can't be equal to zero. So we found a counter example and hence the statement is **false**.

Question 1 (d)

SOLUTION. This statement is **false**.
As a counterexample, consider $f(x) = 1/x$. It has a vertical asymptote at $x = 0$, but its reciprocal $1/f = x$ has no vertical asymptotes.

Question 2 (a)

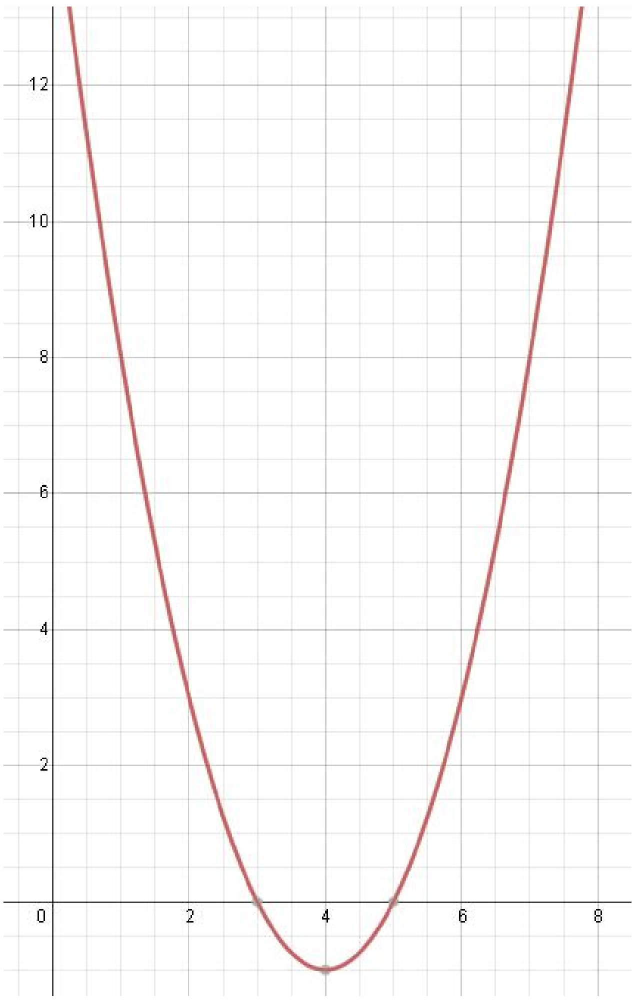
SOLUTION. A simple-like parabola with a zero at three will work. The other zero one can be anywhere, for no particular reason lets make it at 5. So the function

$$f(x) = (x - 3)(x - 5) = x^2 - 8x + 15$$

satisfies all three conditions.
To double check,

$$\begin{aligned} f(3) &= (3 - 3)(3 - 5) = 0 \\ f'(3) &= 2(3) - 8 = -2 < 0 \\ f''(3) &= 2 > 0 \end{aligned}$$

Let's see the picture



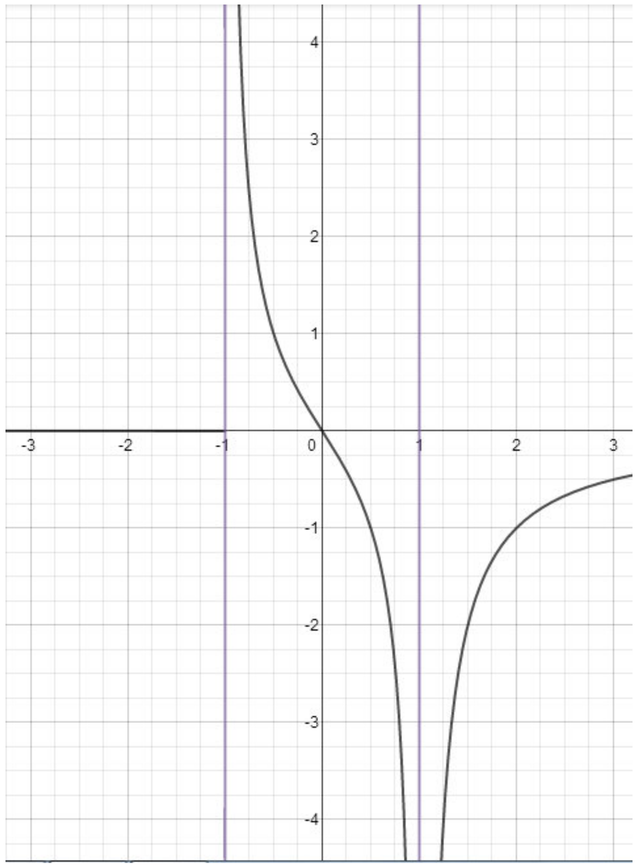
Note: Many other functions are possible as well, it does not even have to be a second order polynomial.

Question 2 (b)

SOLUTION. See the second hint for how each limit translates into a feature of the graph.

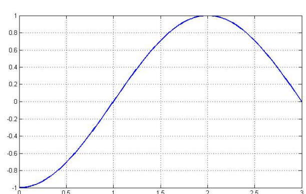
1. Your graph can do whatever it wants to the left of $x = -1$, so long as it approaches the point *(-1,0)*. The simplest (yet somewhat boring) way to achieve this is by setting the function to *0* for $x < -1$.
2. Between $x = -1$ and $x = 1$ the function has to fall from ∞ to $-\infty$.
3. To the right of $x = 1$ the function has to come back from $-\infty$ and then can do whatever you want it to do.

There is a lot of freedom in how your graph may look, but as long as you follow the key points above your graph will be correct. Below is our example *(the purple lines represent asymptotes)*:



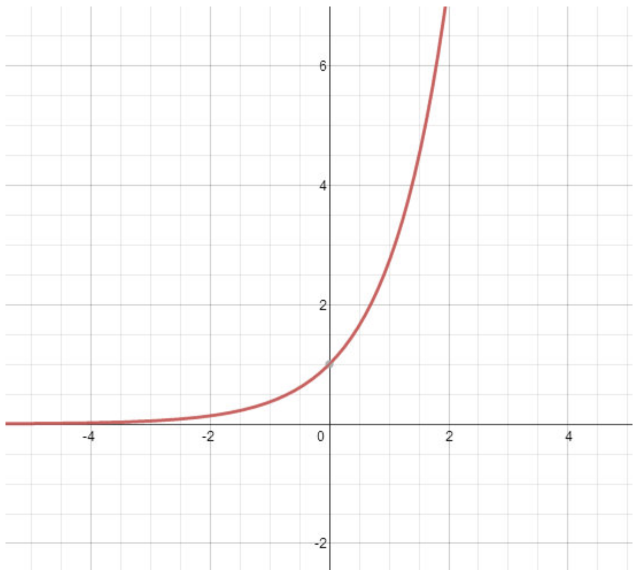
Question 2 (c)

SOLUTION. Your drawing should have a function that changes concavity at $x = 1$. Critical points are where we find a minimum, maximum, or a "corner" where the derivative does not exist. Your graph should have one of these features at $x = 2$.
In our example below the function switches from concave up to concave down at $x = 1$ and has a maximum at $x = 2$.



Question 2 (d)

SOLUTION. Any exponential function of the form $f(x) = a^x$, where a is a real number, will satisfy the equation above because $f'(x) = \ln(a)a^x = kf(x)$ for $k = ln(a)$. One example would be our favorite $f(x) = e^x$. Here $k = ln(e) = 1$ and a graph is shown below.



Question 3 (a)

SOLUTION. The function has x-intercepts whenever
$0 = \frac{\sin(x)}{e^x}$
This occurs whenever $\sin(x) = 0$ on the interval $[0, \pi]$. This occurs when the x value is 0 or π.

Question 3 (b)

SOLUTION. The functions $\sin(x)$ and e^x are both continuous everywhere. Further, the function in the denominator, e^x, is never 0. Hence $f(x)$ is continuous everywhere and thus there are no vertical asymptotes.

Question 3 (c)

SOLUTION. Using the quotient rule, we find:

$$\begin{aligned} f'(x) &= \frac{(\cos x)(e^x) - (\sin x)(e^x)}{(e^x)^2} \\ &= \frac{e^x(\cos x - \sin x)}{(e^x)^2} \\ &= \frac{\cos x - \sin x}{e^x} \end{aligned}$$

Setting $f'(x) = 0$, we can clear the denominator and get:

$$\cos x - \sin x = 0$$

or, equivalently,

$$\cos x = \sin x$$

This equation has infinitely many solutions, but only one which is in the interval *[0,π]*. The trig ratios $\cos x$ and $\sin x$ are equal on a 45-45-90 degree triangle, so the angle in the interval *[0,π]* that solves the above equation is 45 degrees, or in radian measure, $x = \pi/4$.
Next, we check to see if there are any critical points where $f'(x)$ is undefined. The derivative would be undefined where the denominator is equal to zero, but because e^x is always positive, we don't need to worry about this here.
Thus we have one critical point at $x = \pi/4$. Performing the first derivative test, we get:

	$[0, \pi/4)$	$x = \pi/4$	$(\pi/4, \pi]$
$f'(x)$	+	0	-
$f(x)$	increasing	max	decreasing

Hence *f(x)* has a maximum at $x = \pi/4$.

Question 3 (d)

SOLUTION. We know from the previous part of the question that

$$f'(x) = \frac{\cos x - \sin x}{e^x}$$

We take the derivative of $f'(x)$ to find:

$$f''(x) = \frac{(-\sin x - \cos x)e^x - (e^x)(\cos x - \sin x)}{(e^x)^2} \qquad = \frac{e^x(-\sin x - \cos x - \cos x + \sin x)}{(e^x)^2} = \frac{-2\cos x}{e^x}$$

Setting $f''(x)$ equal to zero to solve for possible inflection points gives:

$$0 = \frac{-2\cos x}{e^x}$$

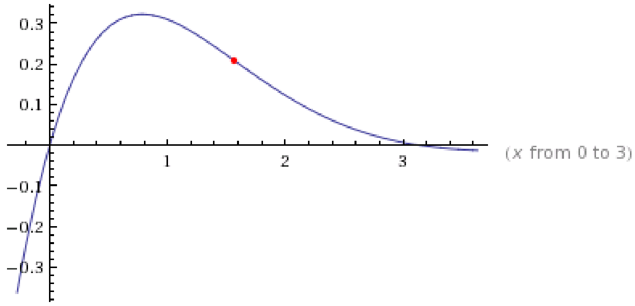
or, equivalently,

$$\cos x = 0$$

Using the unit circle we see that $x = \pi/2$ is the only solution in $[0, \pi]$ of the equation $\cos(x) = 0$. This is our only possible point of inflection, because $f''(x)$ is continuous everywhere. Testing the values of $f''\left(\frac{\pi}{2}\right)$ with the second derivative test gives:
:{| class="wikitable" |- | || $[0, \pi/2)$ || $x = \pi/2$ || $(\pi/2, \pi]$ |- | $f''(x)$|| - || 0 || + |- | $f(x)$ || concave down || inflection point || concave up |}
Hence $f(x)$ has an inflection point at $x = \pi/2$.

Question 3 (e)

SOLUTION. The function's value is zero at each end of the interval *[0,π]*, we have found a maximum at *π/4* and an inflection point at *π/2* so your sketch should look something like the plotted function here.



Make sure you visibly identify the maximum and inflection point on your sketch of the function.

Question 4

SOLUTION. Using implicit differentiation, we solve for the derivative of y in both curves

$$\begin{aligned} xy &= a \\ y + xy' &= 0 \\ y' &= \frac{-y}{x} \end{aligned}$$

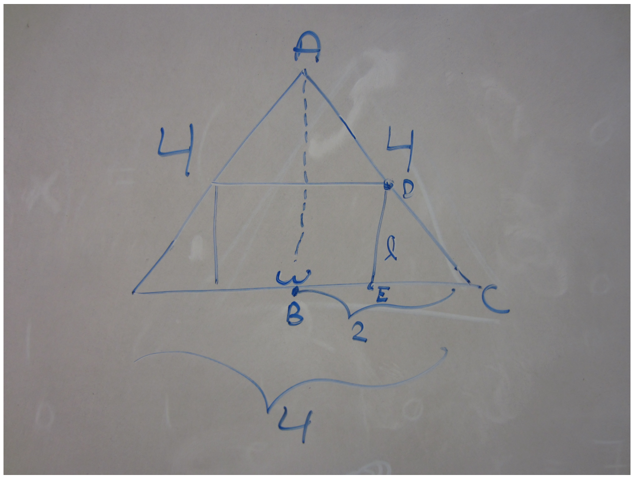
and for the second curve,

$$\begin{aligned} x^2 - y^2 &= b \\ 2x - 2yy' &= 0 \\ 2(x - yy') &= 0 \\ x - yy' &= 0 \\ y' &= \frac{x}{y} \end{aligned}$$

These two slopes are negative reciprocals of each other. Hence, if the curves intersect each other, they do at a right angle.

Question 5

SOLUTION. Let's start with drawing a sketch.



The area of the rectangle with width w and length l is given by

$Area = wl$

The constrains are $0 \leq w \leq 4$, $0 \leq l \leq h$, where h is the height of the surrounding triangle. However, w and l are not independent: a larger width results in a shorter length and vice versa. Hence, the next step is to express the formula for the Area with respect to just one variable. We achieve this by observing the similar triangles ABC and DEC to find

$$\frac{h}{2} = \frac{l}{2 - w/2}$$

or, equivalently,

$$w = 4\left(1 - \frac{l}{h}\right)$$

Plugging this into the formula for the Area of the rectangle we find

$$\text{Area}(l) = 4\left(1 - \frac{l}{h}\right)l = 4\left(l - \frac{l^2}{h}\right)$$

Now, that we expressed the Area as a function of just one variable, we can go about maximizing it as usual: Set the derivative to zero and solve for the variable l:

$$\text{Area}'(l) = 4\left(1 - \frac{2l}{h}\right)$$

Setting this expression to zero we obtain $l = h/2$ as critical point. Since the area vanishes at the endpoints, $Area(0) = 0$ and $Area(h) = 0$, the critical point $l = h/2$ is the maximum we are looking for.
Hence, the dimensions of the rectangle that maximizes the area are

$$w = 4\left(1 - \frac{\frac{l}{2}}{h}\right) = 2, \quad l = \frac{h}{2}$$

Question 6

SOLUTION. First we find an formula for the quantity we're minimizing: the slope of the tangent line. This formula is the derivative of the function:

$$f'(x) = 3 - \sqrt{3}\sin x - \cos x$$

This is the formula we are minimizing, which we will now call $g(x) = 3 - \sqrt{3}\sin x - \cos x$. Following our usual procedure for finding maxima/minima, we take the derivative of $g(x)$, set it equal to zero, and solve for the critical points.

$$g'(x) = -\sqrt{3}\cos x + \sin x$$

Setting equal to zero and solving gives:

$$\sqrt{3}\cos x = \sin x$$

So we are looking for an angle where $\sin(x)$ is equal to $\sqrt{3}\cos(x)$. There is no "formula" to solve this equation; instead you have to use trig ratios and special triangles, or consult the unit circle. On a 30-60-90 degree triangle, or at the angles 30 (in radians: $x = \pi/6$) and 60 (in radians: $x = \pi/3$) on the unit circle, $\sin(x)$ and $\cos(x)$ are related in a ratio of $1 : \sqrt{3}$ so these angles are our candidates. At these values we have

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \quad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \quad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

Hence $x = \pi/3$ is the solution of equation and hence our critical point. We check if this point is indeed a minimum by performing the first derivative test:

	$(0, \pi/3)$	$x = \pi/3$	$(\pi/3, \pi/2)$
$g'(x)$	-	0	+
$g(x)$	decreasing	min	increasing

Hence the tangent line *g(x)* has the smallest slope when $x = \pi/3$.
Note that we do not need to check the endpoints of the interval, because neither is included in the domain.

Question 7

SOLUTION. Since we are interested in the rate of change of the volume of the sphere while being given information about the rate of change of its radius we are facing a typical related rates problem.
There are two variables here at stake: the radius of the sphere, r, and its volume, V. Now there isn't ONE sphere, since the sphere is expanding with time. This means that actually, both the radius and the volume are actually functions of time that we can write $r(t)$ and $V(t)$.
We know the relationship that exists between the volume and the radius and a sphere, which at any time t is expressed by the equation

$$V(t) = \frac{4}{3}\pi(r(t))^3$$

Since we are interested in the relationship that links the rates of change with respect of time for both the radius and the volume of the sphere, it makes sense to differentiate this equation on both sides with respect to time. We obtain:

$$\begin{aligned} V'(t) &= \frac{4}{3}\pi \cdot 3(r(t))^2 \cdot r'(t) \\ &= 4\pi(r(t))^2 \cdot r'(t) \end{aligned}$$

Now the problem tells us that there is a specific time, let's call it t_0, at which we know both the radius of the sphere and its instantaneous rate of change. In other words, if we use the data given in the statement of the problem, we have:

- $r(t_0) = 3 \cdot 10^{13}$
- $r'(t_0) = 7 \cdot 10^6$

And we are interested in obtaining the rate of change of volume at that moment, that is $V'(t_0)$. So we can use the equation we obtained above at the time t_0, substitute the available information and obtain that:

$$\begin{aligned} V'(t_0) &= 4\pi(r(t_0))^2 \cdot r'(t_0) \\ &= 4\pi(3 \cdot 10^{13})^2 \cdot 7 \cdot 10^6 \\ &= 4\pi \cdot 9 \cdot 10^{26} \cdot 7 \cdot 10^6 \\ &= 4 \cdot 7 \cdot 9 \cdot \pi \cdot 10^{32} \\ &= 252\pi \cdot 10^{32} \end{aligned}$$

Since 252π is roughly equal to 800, we can say that the Bubble Nebula is expanding at a rate of approximatively $8(10^{34})$ km^3/h.
Fun Fact: For info, the volume of the Earth is roughly 10^{12} km^3 and the Sun's volume is roughly 10^{18} km^3.

Question 8

SOLUTION. The question is asking us for the rate at which strain is increasing. Since strain is indicated by the variable ϵ, this means we are seeking $\frac{d\epsilon}{dt}$.
To find $\frac{d\epsilon}{dt}$ we differentiate the given formula with respect to t. Remember that a, b and n are constants, while ϵ and σ are variables that depend on t. So $\epsilon(t) = a\sigma(t) + b(\sigma(t))^n$ becomes

$$\frac{d\epsilon}{dt}(t) = a\frac{d\sigma}{dt}(t) + bn(\sigma(t))^{n-1}\frac{d\sigma}{dt}(t)$$

We were asked to find $\frac{d\epsilon}{dt}$ where $\sigma(t) = \sigma_0$, so we can plug that value into the expression above. Finally, recall we are also told earlier in the problem that the stress is increasing at a constant rate R. In the expression above, the rate of stress increasing is given by $\frac{d\sigma}{dt}(t)$, so we replace $\frac{d\sigma}{dt}(t)$ with R. This gives our final answer of:

$$\frac{d\epsilon}{dt} = aR + bn\sigma_0^{n-1}R$$

Question 9 (a)

Easiness: 4.0/5

SOLUTION. The linear approximation formula is

$$f(x) \approx L(x) = f'(a)(x-a) + f(a)$$

For some value a that is near the value to be approximated. In our case, $f(x) = e^x$ and a sensible choice for a would be *0* as it is easy to calculate $f(0) = e^0$ and $f'(0) = e^0$. Plugging this into our formula for *L(x)* we have:

$$L(x) = e^0(x-0) + e^0$$

$$L(x) = x + 1$$

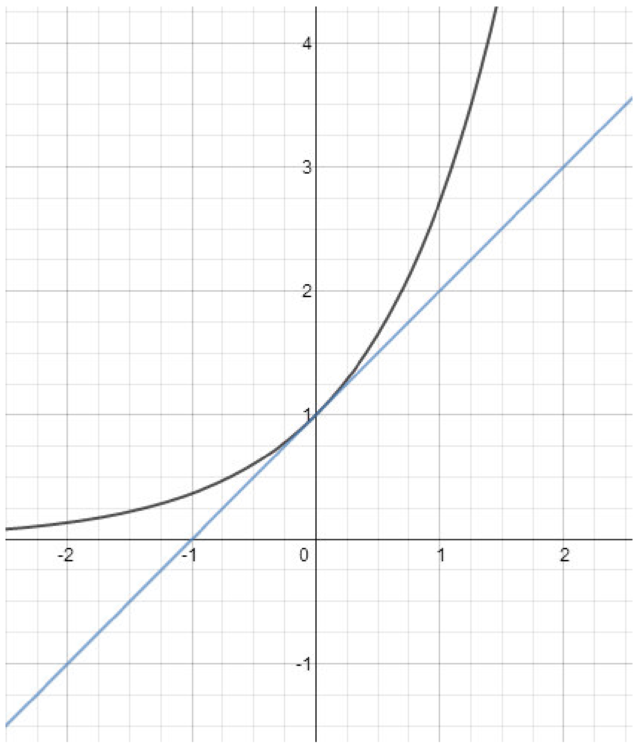
To approximate $e^{0.1}$, we simply plug in $x = 0.1$ to get

$$L(0.1) = 0.1 + 1 = 1.1$$

Question 9 (b)

Easiness: 5.0/5

SOLUTION. The function $f(x) = e^x$ is concave up, so all tangent lines, including our linear approximation, will lie below the graph.
The purple graph is *f(x)*, the blue line is the linear approximation *L(x) = x+1* from part (a). Thus our linear approximation is an underestimate.



Good Luck for your exams!