Full Solutions
MATH104 December 2013

April 4, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. Using the quotient rule, we see that
$f'(x) = \dfrac{\cos(x)(e^x + 7) - e^x \sin(x)}{(e^x + 7)^2}$
Since we do not need to simplify, this is an acceptable answer.

Question 1 (b)

SOLUTION. Following the hint, we need to solve
$a = f(-1) = \lim_{x\to -1} f(x) = \lim_{x\to -1} \dfrac{x^2 + 3x + 2}{x + 1}$.
Solving the limit yields
$\lim_{x\to -1} \dfrac{x^2 + 3x + 2}{x + 1} = \lim_{x\to -1} \dfrac{(x + 1)(x + 2)}{x + 1} = \lim_{x\to -1}(x + 2) = 1.$
and so $a = 1$.

Question 1 (c)

SOLUTION. Following the hint, we have

$$\begin{aligned}
\lim_{x\to\infty} \frac{9x^2 - 6x + 8}{-3x^3 + 4} &= \lim_{x\to\infty} \frac{x^2(9 - 6/x + 8/x^2)}{x^3(-3 + 4/x^3)} \\
&= \lim_{x\to\infty} \frac{1}{x} \cdot \frac{9 - \frac{6}{x} + \frac{8}{x^2}}{-3 + \frac{4}{x^3}} \\
&= \lim_{x\to\infty} \frac{1}{x} \cdot \lim_{x\to\infty} \frac{9 - \frac{6}{x} + \frac{8}{x^2}}{-3 + \frac{4}{x^3}} \\
&= (0) \cdot \left(\frac{9}{-3}\right) = 0
\end{aligned}$$

Thus the answer is 0. The splitting of the limit of a product into a product of limits is valid since the product is finite.

Question 1 (d)

SOLUTION 1. Using the hint, we find the derivative at the given point. Applying the chain rule we find

$$\begin{aligned}
y &= \ln\left(x^2\right) \\
\frac{dy}{dx} &= \frac{1}{x^2} \cdot 2x \\
\frac{dy}{dx} &= \frac{2}{x} \\
\left.\frac{dy}{dx}\right|_{x=e^2} &= \frac{2}{e^2}
\end{aligned}$$

SOLUTION 2. Using the hint, we find the derivative at the given point. In the first step below, we use our logarithm rules to see that $\ln(x^2) = 2\ln(x)$. Hence

$$
\begin{aligned}
y &= \ln(x^2) \\
y &= 2\ln(x) \\
\frac{dy}{dx} &= \frac{2}{x} \\
\left.\frac{dy}{dx}\right|_{x=e^2} &= \frac{2}{e^2}
\end{aligned}
$$

Question 1 (e)

SOLUTION. Following the hint, we begin differentiating.

$$
\begin{aligned}
y &= x^3 + 3x^2 + 2 \\
\frac{dy}{dx} &= 3x^2 + 6x \\
\frac{d^2y}{dx^2} &= 6x + 6
\end{aligned}
$$

Solving for $\frac{d^2y}{dx^2} = 0$ gives $0 = 6x + 6$ and so $x = -1$. A quick check shows that the sign of the second derivative changes at this point and so we know that $x = -1$ is indeed an inflection point. Plugging $x = -1$ into the first derivative gives us the slope of the tangent line at the inflection point:
$\left.\frac{dy}{dx}\right|_{x=-1} = 3(-1)^2 + 6(-1) = 3 - 6 = -3$.

Question 1 (f)

SOLUTION. Using implicit differentiation with respect to x, we see that

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(xy + x^2 + 1) \\
&= (1 \cdot y + x \cdot \frac{d}{dx}y) + 2x + 0 \\
&= y + x\frac{dy}{dx} + 2x \\
&= \frac{y + 2x}{1 - x}
\end{aligned}
$$

and substituting in the point $(-1, 1)$, we have

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{y + 2x}{1 - x} \\
&= \frac{1 + 2(-1)}{1 - (-1)} = \frac{-1}{2}
\end{aligned}
$$

We use the slope-point formula to determine that the equation of the tangent line at $(-1, 1)$ is given by

$$
\begin{aligned}
y-1 &= \frac{-1}{2}(x-(-1)) \\
y-1 &= \frac{-1}{2}x-\frac{1}{2} \\
y &= \frac{-1}{2}x+\frac{1}{2}
\end{aligned}
$$

Question 1 (g)

SOLUTION. Taking the derivative of this function yields
$f'(x) = 1 - \dfrac{1}{x^2}$
The derivative is undefined at $x = 0$ (as is the original function) and setting the derivative to 0 also gives

$$
\begin{aligned}
\frac{1}{x^2} &= 1 \\
x &= \pm 1
\end{aligned}
$$

Thus f has three critical points and four intervals to check:

- On $(-\infty, -1)$ the derivative is positive and the function is increasing.
- On $(-1, 0)$ the derivative is negative and the function is decreasing.
- On $(0, 1)$ the derivative is negative and the function is decreasing.
- On $(1, \infty)$ the derivative is positive and the function is increasing.

Therefore the function is increasing on $(-\infty, -1) \cup (1, \infty)$.

Question 1 (h)

SOLUTION. For continuously compounded interest, we have the formula $A = Pe^{rt}$ where A is the amount, P is the principal, r is the rate, and t is the time.
We are given that $P = 10000, A = 100000, t = 2$. Plugging these in yields

$$
\begin{aligned}
100000 &= 10000e^{2r} \\
10 &= e^{2r} \\
\ln 10 &= 2r \\
r &= \frac{\ln 10}{2}
\end{aligned}
$$

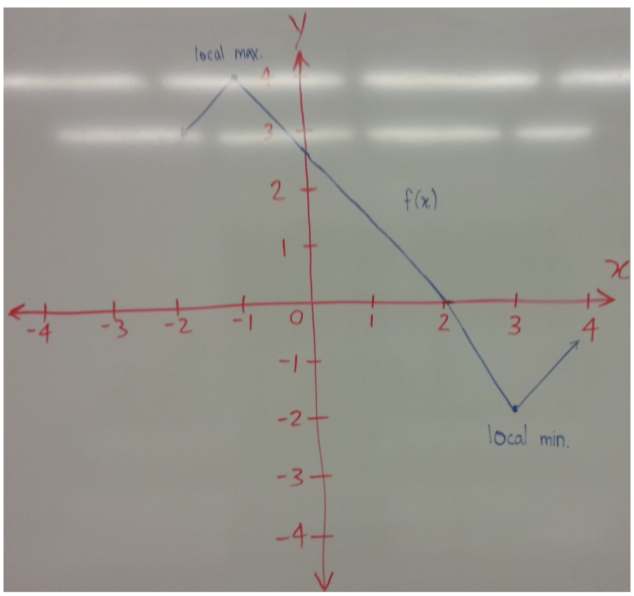
Question 1 (i)

SOLUTION. To solve this question, we begin by finding the equation of the tangent line to the function $f(x) = \ln(x)$. Differentiating gives us $f'(x) = \frac{1}{x}$ and plugging in $a = 1$ we obtain the y value of $y = \ln(1) = 0$. Using the point slope formula, we have at the point $(1, 0)$

$$\frac{y-0}{x-1} = 1$$
Solving gives
$y = x - 1$
Thus, the approximation we seek is given by
$y = 1.25 - 1 = 0.25$.

Question 1 (j)

SOLUTION. The correct answer is (E).
To show that E is correct, we can use the intermediate value theorem twice. In order for f to be a continuous function that connects $(-1,4)$ and $(3,-2)$, we know that it must have some value at $x = 0$ by the fact that the function is continuous for all real values of x. Thus, it crosses the y axis. As these two points live both above and below the x axis, we see that $f(-1) \geq 0 \geq f(3)$ and since f is continuous, the intermediate value theorem states that there is some value c in the interval $(-1,3)$ such that $f(c) = 0$. Thus the answer is (E).
Note. For the curious minded, the graph below gives a counter example to all of the other 4 conditions.

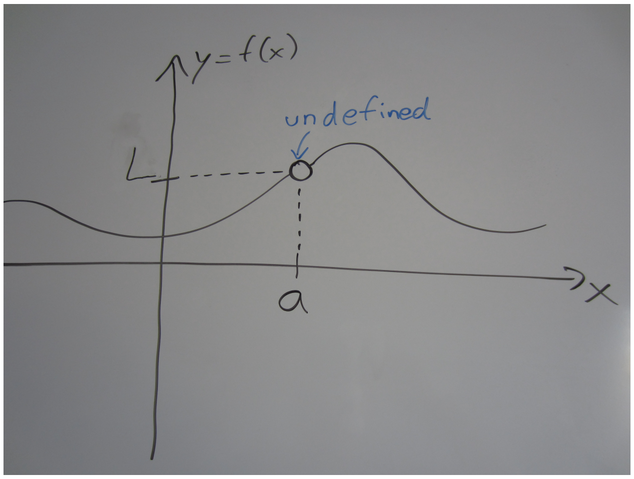


Question 1 (k)

SOLUTION. We are given that $f(x) = x^2 + e^{-2x}$. Taking the derivative yields
$f'(x) = 2x - 2e^{-2x}$.
Since this exists everywhere, our function is differentiable hence continuous thus (C) is not the correct answer.
Factoring gives
$f'(x) = 2(x - e^{-2x})$
When we plug in $x = 0$, we see that the $f'(0) = 2(0 - e^{-2(0)}) = -2$. As this is negative, we know that $f(x)$ is decreasing at $x = 0$. **Thus the correct answer is (B).**

Question 1 (l)

SOLUTION. None of these have to be true! The fact that the limit exists does not imply anything about the value at that point. In fact, $f(x)$ need not even be defined at a. Therefore, the following is a counter example



Hence, the correct answer is (E).

Question 1 (m)

SOLUTION. To see this, notice that the equation of the line using the point slope formula is given by
$\frac{y-2}{x-3} = f'(3) = 5.$
Isolating gives
$y - 2 = 5(x - 3)$
and simplifying one last time
$y = 5x - 13.$
We are using this line to approximate $f(x)$. Thus, if we set $y = 0$ and solve, we can get an approximation to the root of $f(x)$. Hence, set $0 = 5x - 13$ which gives $x = 13/5 = 2.6$. Thus the correct answer is (C).

Question 1 (n)

SOLUTION. The area of a triangle is given by

$$A = \frac{bh}{2}.$$

To determine the rate of change of area of the triangle, we evaluate A' by differentiating A with respect to t, recognizing that both b and h are functions of time.

$$A' = \frac{1}{2}\left(bh' + hb'\right).$$

Since b' = 3 and h' = -3, A' becomes

$$A' = \frac{3}{2}\left(h - b\right).$$

By inspecting this equation we can determine the answer. Clearly A is not always increasing nor always decreasing nor constant since the both the sign and the value of A' depend on the values of b and h. So (A), (B), and (E) are not true. If A is decreasing, then A' < 0 and so

$$0 > \frac{3}{2}\left(h - b\right) \quad \Rightarrow \quad b > h.$$

Thus, the correct answer is (D).

Question 2 (a)

SOLUTION. Setting the numerator and the denominator of the first derivative to 0 gives
$-8x = 0$ and so $x = 0$
$(x^2-4)^2 = 0$ and so $0 = x^2 - 4 = (x-2)(x+2)$ giving solutions $x = \pm 2$.

Question 2 (b)

SOLUTION. Setting the second derivative's numerator to zero yields $8(3x^2+4) = 0$ which does not have a solution. Setting the denominator to zero yields $(x^2-4)^3 = 0$ which has the zeros $x = \pm 2$. Hence $f''(x)$ is never zeros and $f''(x)$ does not exist for $x \pm 2$.

Question 2 (c)

SOLUTION. We proceed as suggested by the hint.

On the interval $(-\infty, -2)$, we see that the derivative at the point $x = -10$ is given by $f'(-10) = \dfrac{-8(-10)}{((-10)^2-4)^2} > 0$ and so the function is increasing on this interval.

On the interval $(-2, 0)$, we see that the derivative at the point $x = -1$ is given by $f'(-1) = \dfrac{-8(-1)}{((-1)^2-4)^2} > 0$ and so the function is increasing on this interval.

On the interval $(0, 2)$, we see that the derivative at the point $x = 1$ is given by $f'(1) = \dfrac{-8(1)}{((1)^2-4)^2} < 0$ and so the function is decreasing on this interval.

On the interval $(2, \infty)$, we see that the derivative at the point $x = 10$ is given by $f'(10) = \dfrac{-8(10)}{((10)^2-4)^2} < 0$ and so the function is decreasing on this interval.

Question 2 (d)

SOLUTION 1. We proceed as suggested by the hint.
On the interval $(-\infty, -2)$, we see that the second derivative at the point $x = -10$ is given by $f''(-10) = \dfrac{8(3(-10)^2+4)}{((-10)^2-4)^3} > 0$ and so the function is concave up on this interval.

On the interval $(-2, 2)$, we see that the second derivative at the point $x = 0$ is given by $f''(0) = \dfrac{8(3(0)^2+4)}{((0)^2-4)^3} < 0$ and so the function is concave down on this interval.
On the interval $(2, \infty)$, we see that the second derivative at the point $x = 10$ is given by $f''(10) = \dfrac{8(3(10)^2+4)}{((10)^2-4)^3} > 0$ and so the function is concave up on this interval.
To summarize, $f(x)$ is concave up on $(-\infty, -2) \cup (2, \infty)$ and concave down on $(-2, 2)$.

SOLUTION 2. We can also argue a bit more intelligently as follows.
Look at the second derivative given by
$f''(x) = \frac{8(3x^2+4)}{(x^2-4)^3}$
Notice that the numerator of the second derivative is always positive and so the sign of the second derivative is given by the behaviour of the denominator, namely the polynomial
$(x^2-4)^3$

Since this polynomial is to the power of 3, the sign of the cube of the polynomial is the same as the sign of the polynomial itself, given by
$x^2 - 4$
Next, this polynomial is an upwards facing parabola with roots at ± 2. Thus, it is negative between $-2 < x < 2$ and is positive on $(-\infty, -2) \cup (2, \infty)$.
Hence, $f(x)$ is concave up on $(-\infty, -2) \cup (2, \infty)$ and concave down on $(-2, 2)$.

Question 2 (e)

Solution. Using parts (a) and parts (c) we can see that the extrema can only occur possible at $x = 0$, $x = \pm 2$. Since the latter two points are not in the domain, we need only to check the point $x = 0$ for extrema. Looking at parts (c), we see to the immediate left of 0, the function is increasing and to its immediate right the function is decreasing. Thus this point is a local maximum. The coordinates are given by $(0, 0)$ found by plugging in 0 into the original function.

Question 2 (f)

Solution. Points where the numerator goes to infinity or the denominator goes to zero are candidates for where a fraction might have a vertical asymptote. In our case the denominator goes to zero for $x = \pm 2$. To evaluate the corresponding limits, plug in a value that is close to the limiting value on either side:

$$\lim_{x \to -2^-} \frac{x^2}{x^2 - 4} = \infty.$$

The sign is positive since the numerator approaches 4 and plugging in a number slightly smaller than -2 yields a very small, positive denominator. Similar considerations yield

$$\lim_{x \to -2^+} \frac{x^2}{x^2 - 4} = -\infty$$
$$\lim_{x \to 2^-} \frac{x^2}{x^2 - 4} = -\infty$$
$$\lim_{x \to 2^+} \frac{x^2}{x^2 - 4} = \infty$$

Thus there are vertical asymptotes at $x = \pm 2$.
For the horizontal asymptotes, we check the limit of the function as x goes to positive and negative infinity. We can do this by factoring out the largest power of the numerator and the denominator.

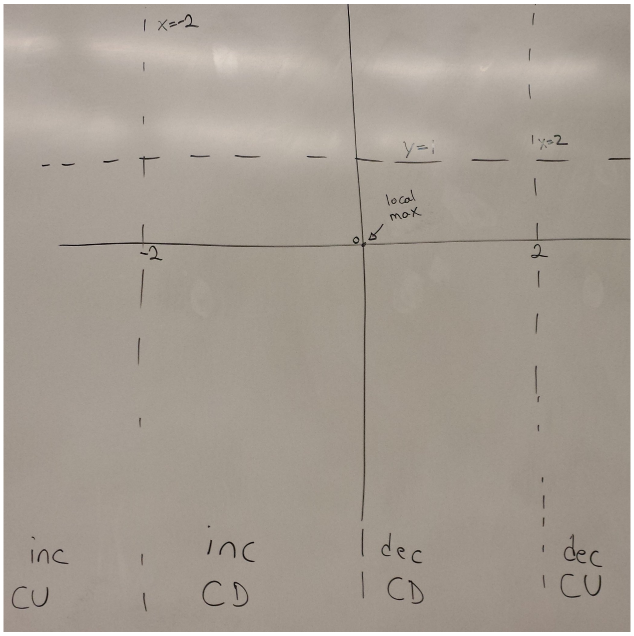
$$\lim_{x \to -\infty} \frac{x^2}{x^2 - 4} = \lim_{x \to -\infty} \frac{x^2}{x^2(1 - 4/x^2)} = \lim_{x \to -\infty} \frac{1}{1 - 4/x^2} = 1$$

$$\lim_{x \to \infty} \frac{x^2}{x^2 - 4} = \lim_{x \to \infty} \frac{x^2}{x^2(1 - 4/x^2)} = \lim_{x \to \infty} \frac{1}{1 - 4/x^2} = 1$$

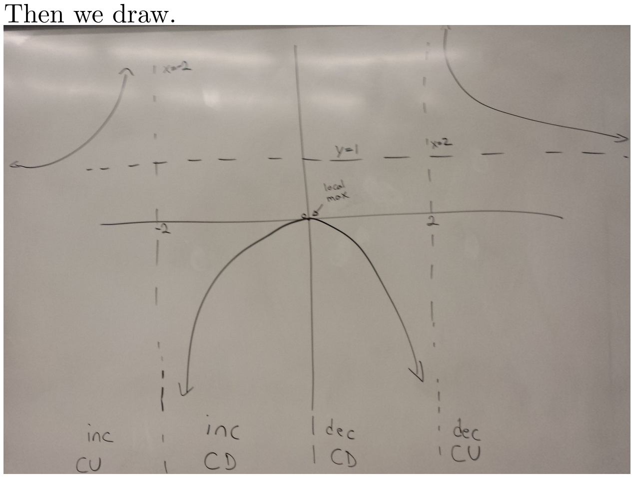
This tells us that we have horizontal asymptotes in both directions, at $y = 1$.

Question 2 (g)

Solution. We draw where our function is increasing, decreasing, concave up, concave down as well as the only local extrema we found on a graph. We also include the asymptotes.



Then we draw.



Question 3

SOLUTION. To do this question, we start by letting h_1 and h_2 denote the height of the water level in the smaller and larger cylinder respectively. We also let $r_1 = 5$, $r_2 = 8$ be the radii of the tanks and V_1, V_2 be the respective volumes of water within each tank.
Since we are told that each tank is being filled at the same rate, this implies that

$$\frac{dV_1}{dt} = \frac{dV_2}{dt}$$

(i.e. The tanks have an equal rate of change of volume of water per unit time). Since the radius of the tank is not changing with time, but the height of the water level is changing with time, we can differentiate the expression for volume of a cylinder with respect to time giving us

$$\frac{dV_1}{dt} = \pi r_1^2 \frac{dh_1}{dt}, \quad \frac{dV_2}{dt} = \pi r_2^2 \frac{dh_2}{dt}.$$

We want to find how fast the height of water in the larger tank (i.e. tank 2) is changing, hence we are looking for the value of dh_2/dt. Using the above equations we can write

$$\pi r_1^2 \frac{dh_1}{dt} = \pi r_2^2 \frac{dh_2}{dt}.$$

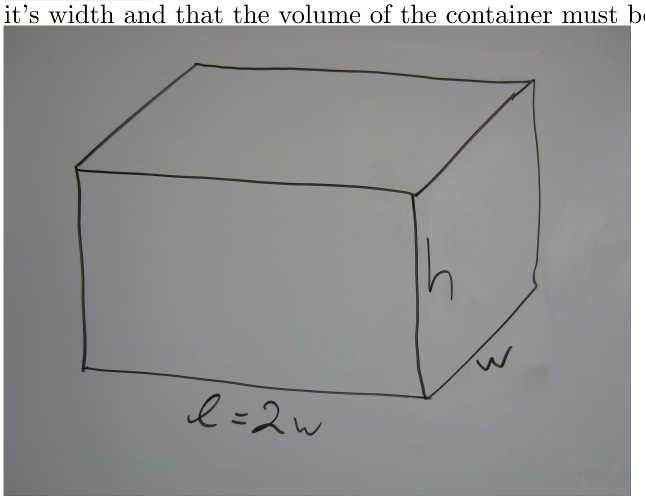
Isolating for dh_2/dt gives

$$\frac{dh_2}{dt} = \frac{r_1^2}{r_2^2}\frac{dh_1}{dt} = \frac{5^2}{8^2}(0.5) = \frac{25}{128}\frac{\text{m}}{\text{min}}.$$

Thus, the water level in the larger tank is rising at a rate of (25/128) metres per minute.

Question 4

SOLUTION. We wish to *minimize* the **cost**, C, subject to the constraint that the length of the base is twice it's width and that the volume of the container must be $V = 8\text{m}^3$.



The cost C can be expressed as the sum of the costs for all the sides. Recall that this is an open-top box (i.e., no top). If we let l, w, and h denote the length, width, and height of the box in metres, respectively, then:

$$\begin{aligned} C &= C_{\text{sides}} + C_{\text{bottom}} \\ &= 6(2hw + 2hl) + 4.5(lw) \end{aligned}$$

since material for the base costs \$4.50 per square metre, and material for the sides costs \$6 per square metre.
We are also given the constraint

$$V = lwh = 8$$

Using the fact that the length is twice the width (i.e., $l = 2w$), the cost function and volume constraint are expressed as

$$C = 36hw + 9w^2, \quad V = 2w^2h = 8$$

To reduce this into a single variable problem, we use the constraint to express the *height* of the box *in terms of the width*:

$$h = \frac{4}{w^2},$$

and substitute this into the cost function C, making it a function of w only:

$$C(w) = \frac{144}{w} + 9w^2$$

To optimize C, we determine first determine its critical points by differentiating with respect to w and setting the derivative equal to zero:

$$\begin{aligned} C'(w) &= -\frac{144}{w^2} + 18w \\ 0 &= -\frac{144}{w^2} + 18w \\ 144 &= 18w^3 \\ 8 &= w^3 \\ w &= 2 \end{aligned}$$

Therefore $w = 2$ is a critical point. To determine whether or not it is the point that minimizes C, we use the **second derivative test**. The second derivative of C with respect to w is

$$C''(w) = \frac{288}{w^3} + 18$$

which is clearly greater than zero when $w = 2$. Thus, by the second derivative test, C has a local minimum at $w = 2$ (Moreover, it is the *global minimum* since there are no other critical points.) Substituting $w = 2$ into the cost function, we determine that the minimum cost is

$$C(2) = \frac{144}{2} + 9(2)^2 = 72 + 36 = \$108$$

Question 5

SOLUTION. To use the elasticity of demand, we evaluate ϵ at $p = 4$. If $\epsilon < -1$, then demand is elastic and the price should be lowered to increase revenue. Otherwise, if $\epsilon > -1$, then demand is inelastic and the price should be raised.

A quick calculation using the given price-demand relationship shows that if $p = 4$, then either $q = 4200$ or $q = 1800$. Hence, we will have to consider both scenarios.

To evaluate $\frac{dq}{dp}$, we opt to use the chain rule on the price-demand relationship. Differentiating it with respect to p, gives

$$1 = 2\left(\frac{q-3000}{600}\right)\frac{1}{600}\frac{dq}{dp} \quad \Rightarrow \quad \frac{dq}{dp} = 300\left(\frac{600}{q-3000}\right)$$

Putting together all the pieces to evaluate ϵ at $p = 4$, we get

$$\begin{aligned} \epsilon &= \frac{p}{q}\frac{dq}{dp} \\ &= \frac{1}{q}\left(\frac{q-3000}{600}\right)^2 300\left(\frac{600}{q-3000}\right) \\ &= \frac{300}{q}\left(\frac{q-3000}{600}\right) \end{aligned}$$

Depending on the value of q, the value of ϵ will be different.
If $q = 4200$, then $\epsilon = 3/21$.
If $q = 1800$, then $\epsilon = -6/18$.
Thus, in either case, the price should be raised in order to increase revenue.
Note: The case where $q = 4200$ is may seem somewhat counter-intuitive since the elasticity of demand relationship suggests here that raising the price actually increases demand (in stark contrast to the general rule that increasing price decreases demand). However, this phenomenon is not actually very unusual. You may want to ponder what this means... (e.g. Do expensive handbags become more or less desirable to wealthy shoppers if the price drops?)

Question 6 (a)

SOLUTION. Using the formula for linear approximation to a function $f(x)$ near $x = a = 8$, we have

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= \sqrt[3]{8} + \frac{1}{3(8)^{2/3}}(x-8) \\ &= 2 + \frac{1}{12}(x-8) \end{aligned}$$

Since 7 is somewhat close to 8, we can estimate $\sqrt[3]{7}$ by evaluating L(7). Thus,

$$L(7) = 2 - \frac{1}{12} = \frac{23}{12} \quad \Rightarrow \quad \sqrt[3]{7} \approx \frac{23}{12}.$$

Question 6 (b)

SOLUTION. Our function we are considering is $f(x) = x^{1/3}$. Taking derivatives gives

$$\begin{aligned} f'(x) &= \frac{x^{-2/3}}{3} \\ f''(x) &= \frac{-2x^{-5/3}}{9} \end{aligned}$$

Since the linear approximation is centred about $a = 8$ we are looking for the maximum value of $|f''(x)| = \frac{2}{9}x^{-5/3}$ on the interval $[7, 8]$. Since $x^{-5/3}$ is decreasing, the maximum value is attained at $x = 7$, that is, $M = |f''(7)|$. Overall we find that

$$|\text{Error}| \leq \frac{1}{2}M(x-a)^2 = \frac{1}{2} \cdot \frac{2}{9}7^{-5/3}(7-8)^2 = \frac{7^{-5/3}}{9}.$$

Question 6 (c)

SOLUTION. Since the second derivative $f''(x) = \frac{-2x^{-5/3}}{9}$ is negative on $(0, \infty)$, the function $f(x)$ is concave down there. Thus, the tangent line is an over estimate of the function and the true value must lie in the interval

$$\left[\frac{23}{12} - \frac{7^{-5/3}}{9}, \frac{23}{12}\right]$$

Good Luck for your exams!