École Polytechnique Fédérale de Lausanne

Report

Introduction to Quantitative Finance

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Contents

Chapter 1

Introduction

This paper presents basic ideas on quantitative finance and mathematical modeling of financial markets. It can be considered as an introduction on mathematical finance. The paper is divided into three chapters. The first one presents the modern portfolio theory developed by Harry Max Markowitz in 1952. It is a mathematical framework for analyzing and constructing portfolios of assets. The second chapter discusses risk. It presents possible risk measures and provides general axioms for this measures to be coherent as defined by Philippe Artzner in his 1998 article. The last chapter introduces options and the most common pricing models. It also describes some hedging strategies using option contracts.

The financial part of this paper should be easy to understand with some basic comprehension of financial markets but the derivation of the formulas and models requires a background knowledge of linear algebra, functional analysis, probabilities and statistics.

Chapter 2

Portfolio Theory

2.1 Portfolio Analysis

A portfolio analysis starts with information concerning individual securities and its purpose is finding portfolios which best meet the objectives of the investor. Using past performance of individual securities, the analysis gives portfolios which already performed well in the past. It is possible to use the beliefs of the analyst regarding the future performances to choose securities for better and worse portfolios.

Uncertainty is a salient feature of security investment since it is difficult (even impossible) to model and predict some economic and non-economic influences (level of market, prosperity, success of a particular security, ...) even if every information is available. It is best to express a judgment concerning the potentialities and weaknesses of securities.

Another salient feature is the correlation between security returns. One important but rather trivial observation is that diversification eliminates risk if security returns weren't correlated and can't be used if all securities were perfectly correlated (moved up and down together). It is generally expected that securities of the same industry are highly correlated. To reduce risk it is necessary to avoid a portfolio with highly correlated securities.

It is important to settle on some particular measure of the variability of return on portfolio. "Largest Loss" is a possible measure but a bad one. [1, p. 17]. A better measure is standard deviation which is determined using :

1. the standard deviation of each security.

2. the correlation between each pair of securities.

3. the amount invested in each security

As a remark, using standard deviation assumes the returns follow a normal distribution. However, returns in the financial markets are skewed away from the average because of a large number of surprising drops or spikes in prices. Additionally, the standard deviation assumes that price movements in either direction are equally risky.

If a portfolio is said "efficient" then it is impossible to obtain greater average return without incurring greater standard deviation and/or obtain smaller standard deviation without giving up return on the average. Another type of input for portfolio analysis is "probability beliefs" of experts which is expressed as statements about the future of a security based on various economical data (past performance, prospects for the economy and the market, new developments in the industry, ...).

2.2 Definitions

Definition 1. Expected Return : It is a measure of the center of the distribution of the random variable that is the return. It is calculated using the following formula:

$$E(R) = \sum_{i=1}^{n} R_i P_i$$

Where
R_i is the return of scenario i
P_i is the probability for the return R_i in scenario i
n is the number of scenarios

Definition 2. Dispersion : Dispersion is a statistical term that describes the size of the distribution of values expected for a particular variable. Dispersion can be measured by several different statistics, such as range, variance, and standard deviation. In finance and investing, dispersion usually refers to the range of possible returns on an investment, but it can also be used to measure the risk inherent in a particular security or investment portfolio. It is often interpreted as a measure of the degree of uncertainty, and thus, risk, associated with a particular security or investment portfolio [2].

Definition 3. Standard Deviation : The standard deviation is a statistic that measures the dispersion of a data set relative to its mean and is calculated as the square root of the variance.

$$s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2}$$

Where
x_i = Value of the i-th point in the data set
$\overline{x}$ = The mean value of the data set
N = The number of points in the data set

Definition 4. Sharpe Ratio : In finance, the Sharpe ratio measures the performance of a portfolio compared to a risk-free asset, after adjusting for its risk. It represents the average return earned in excess of the risk-free rate per unit of volatility or total risk [2]. It is defined as

$$S_p = \frac{R_p - R_f}{\sigma_p}$$

Where
R_p is the return of the portfolio
R_f is the return of the risk-free asset
σ_p is the standard deviation of the portfolio

Definition 5. Beta : A beta coefficient is a measure of the volatility, or systematic risk, of an individual stock in comparison to the unsystematic risk of the entire market. In statistical terms, beta represents the slope of the line through a regression of data points from an individual stock's returns against those of the market [2].

$$\beta_e = \frac{Covariance(R_e, R_m)}{Variance(R_m)}$$

Where
R_e is the return on an individual stock
R_m is the return on the overall market

2.3 Example : Two securities portfolio

Suppose 1\$ is invested in two securities denoted as $s1$ and $s2$. Let ω be the amount in \$ invested in security 1. Let μ_1 and μ_2 be the average returns and σ_1 and σ_2 be the standard deviation of security 1 and 2 respectively. Let $\sigma_{12} = cov(s1, s2)$. One can show that

$$E(Return) = \omega \cdot \mu_1 + (1 - \omega) \cdot \mu_2$$

and

$$Var(Return) = \omega^2 \cdot \sigma_1^2 + (1 - \omega)^2 \cdot \sigma_2^2 + 2 \cdot \omega \cdot (1 - \omega) \cdot \sigma_{12}$$

The investor can find the portfolio minimizing $Var(Return)$

$$\min_{0 \leq \omega \leq 1} Var(Return)$$

Note that $Var(Return)$ is a polynomial of second degree in ω and verifies

$$\frac{\partial Var}{\partial \omega} = 2 \cdot \{(\sigma_1^2 + \sigma_2^2 - 2 \cdot \sigma_{12}) \cdot \omega + \sigma_{12} - \sigma_2^2\}$$

$$\frac{\partial^2 Var}{\partial \omega^2} = 2 \cdot \{\sigma_1^2 + \sigma_2^2 - 2 \cdot \sigma_{12}\}$$

Two cases are possible :

1. If $\sigma_1^2 + \sigma_2^2 - 2 \cdot \sigma_{12} > 0$ it has a minimum in $\omega_0 = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2 \cdot \sigma_{12}}$.

2. If $\sigma_1^2 + \sigma_2^2 - 2 \cdot \sigma_{12} = 0$ it is minimized in $\omega = 0$ or $\omega = 1$. Note that it happens when the two securities are perfectly correlated and the minimum is the smaller σ_i.

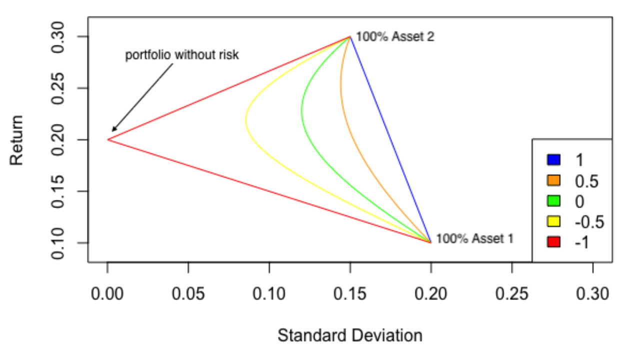

Figure 2.1: risk and return of a two-assets portfolio as function of the correlation

2.4 Markowitz Mean Variance Analysis

Suppose n risky assets are given. Let

- $\mu = (\mu_1, ..., \mu_n)^T$ be the vector of expected returns.
- Σ be the covariance matrix for the returns on the assets in the portfolio.
- ω be a vector of portfolio weights such that $\sum_i \omega_i = 1$. The weights can be negative, which means investors can short a security.
- μ_0 be a target mean return or the portfolio.
- $\mathbf{1}$ is the vector of ones and has the dimension of μ.

In this case

$$E(\omega) = \mu^T \omega \text{ and } Var(\omega) = \omega^T \Sigma \omega$$

This section explains the efficient frontier that was first formulated by Harry Markowitz [1] in 1952.

2.4.1 Minimum Variance Portfolio

Assuming that Σ is non singular and $\mathbf{1}^T \Sigma^{-1} \mathbf{1} \neq 0$, it is possible to find a portfolio with the smallest variance. This problem can be written as

$$\min_{\omega} \frac{1}{2} \omega^T \Sigma \omega \text{ subject to } \mathbf{1}^T \omega = 1 \tag{2.1}$$

Using the Lagrange multiplier gives the optimal portfolio $\omega_{min} = \frac{1}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}} \Sigma^{-1} \mathbf{1}$ which minimizes the variance. Note that

$$E(\omega_{min}) = \frac{\mathbf{1}^T \Sigma^{-1} \mu}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}} \text{ and } Var(\omega_{min}) = \frac{1}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}$$

2.4.2 Risk Minimization

For a given choice of target mean return, choose the portfolio which minimizes the variance. Assuming there are no risk-free assets and Σ is non singular, this problem [3] can be written as

$$\min_{\omega} \frac{1}{2} \omega^T \Sigma \omega \text{ subject to } \mu^T \omega = \mu_0 \text{ and } \mathbf{1}^T \omega = 1 \tag{2.2}$$

One can check if the minimum variance portfolio has the required mean return and if not then compute another feasible solution. Using Lagrange multipliers for the convex minimization problem subject to linear constraints, one can show that

$$\omega_0 = \lambda_1 \Sigma^{-1}\mathbf{1} + \lambda_2 \Sigma^{-1}\mu$$

and

$$\begin{pmatrix} \mathbf{1}^T\Sigma^{-1}\mathbf{1} & \mathbf{1}^T\Sigma^{-1}\mu \\ \mathbf{1}^T\Sigma^{-1}\mu & \mu^T\Sigma^{-1}\mu \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \mu_0 \end{pmatrix}$$

Also the variance of the optimal portfolio is parabolic in the mean and can be computed as

$$Var(\omega_0) = \begin{pmatrix} 1 \\ \mu_0 \end{pmatrix}^T \begin{pmatrix} \mathbf{1}^T\Sigma^{-1}\mathbf{1} & \mathbf{1}^T\Sigma^{-1}\mu \\ \mathbf{1}^T\Sigma^{-1}\mu & \mu^T\Sigma^{-1}\mu \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \mu_0 \end{pmatrix}$$

The efficient frontier is defined by the set

$$\{(\sqrt{Var(\omega_0)}, \mu_0) : \omega_0 \text{ is optimal}\}$$

Let $M = \begin{pmatrix} \mathbf{1}^T\Sigma^{-1}\mathbf{1} & \mathbf{1}^T\Sigma^{-1}\mu \\ \mathbf{1}^T\Sigma^{-1}\mu & \mu^T\Sigma^{-1}\mu \end{pmatrix}$ and $\delta = det(M)$. It is easy to see that $M = \begin{pmatrix} \mathbf{1} & \mu \end{pmatrix}^T \Sigma^{-1} \begin{pmatrix} \mathbf{1} & \mu \end{pmatrix}$ and M is positive definite since Σ is a positive definite covariance matrix. In that case, $\delta > 0$ if and only if $\mathbf{1}$ and μ are linearly independent.

1. If $\delta = 0$ then $\mu = \tau\mathbf{1}$ for some $\tau \in \mathbb{R}$. If $\frac{\mu_0}{\tau} \neq 1$ then the problem is unfeasible. Otherwise, ω_{min} (2.1) solves the problem which would have been detected as mentioned before.

2. If $\delta > 0$ then :

   $$\lambda_1 = -\mu^T v \text{ and } \lambda_2 = \mathbf{1}^T v$$

   Where

   $$v = \frac{1}{\delta}\Sigma^{-1}(\mu_0\mathbf{1} - \mu)$$

   Plugging this values gives the optimal solution

   $$\omega_0 = (1-\alpha)\omega_{min} + \alpha\omega_{mk}$$

   Where

   $$\omega_{min} = \frac{1}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}\Sigma^{-1}\mathbf{1}\ ,\ \omega_{mk} = \frac{\Sigma^{-1}\mu}{\mathbf{1}^T\Sigma^{-1}\mu} \text{ and } \alpha = \frac{\mu_0 - \mu^T\omega_{min}}{\mu^T(\omega_{mk} - \omega_{min})}$$

Observe that the optimal solution is a linear combination of two possible portfolios : The minimum variance portfolio and the "market weights" portfolio.

2.4.3 Return-Risk Balance

In practice, one would like to have a better understanding of the return-risk trade-off since the goal is to both maximize return while minimizing risk. An alternative strategy is to try to balance these two objectives in a single objective function. For a given risk tolerance $\lambda > 0$, this problem can be written as

$$\min_{\omega} \frac{1}{2}\omega^T \Sigma \omega - \lambda \mu^T \omega \text{ subject to } \mathbf{1}^T \omega = 1 \tag{2.3}$$

Using the same strategy as before, one can show that

$$\omega_\lambda = (1 - \alpha_\lambda)\omega_{min} + \alpha_\lambda \omega_{mk}$$

Where ω_{min} and ω_{mk} are defined in the previous section and

$$\alpha_\lambda = \lambda \mathbf{1}^T \Sigma^{-1} \mu$$

This portfolio gives an expected return and a risk defined by

$$E(\omega_\lambda) = E(\omega_{min}) + \lambda \frac{\delta}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}$$

Where $\delta > 0$ is the same as the previous section.

For $\lambda = 0$, the optimal solution is the same as 2.1 and has the same mean return which is expected. If $\lambda \to \infty$, $E(\omega_\lambda) \to \infty$. Thus, in order to understand the risk-return trade-off, one can analyse the curve

$$\{(\sqrt{Var(\omega_\lambda)}, E(\omega_\lambda)) : \lambda \in [0, \infty]\}$$

In the context of Markowitz mean-variance portfolio theory [1], this is called the efficiency curve or efficient frontier. Any portfolio associated with a point on the efficient frontier is called an efficient portfolio. Every possible combination of risky assets, can be plotted in the risk-expected return space, and the collection of all such possible portfolios defines a region in this space. The upper boundary of this region is precisely the efficient frontier. According to Markowitz, only two mutual funds are required so that any investor can achieve their desired balance between return and risk. This is called the "two mutual fund separation theorem" and will be explained in the next section. As a remark, in the last two optimization problems (2.2 and 2.3) the idea of combining two portfolios already appeared and gave an intuition for this theorem.

2.4.4 Two Mutual Fund Theorem

A mutual fund is a type of financial vehicle made up of a pool of money collected from many investors to invest in securities like stocks, bonds, money market instruments, and other assets [2].

The theorem states that any portfolio on the efficient frontier can be generated by holding a combination of any two given portfolios on the frontier. This can be applied if there is at most one risk-free asset. The two possibilities are presented separately :

1. If there are no risk-free assets : Suppose ω^1 and ω^2 are two efficient portfolios solutions of the problem 2.2 for the expected returns μ_0^1 and μ_0^2 respectively. Then for every α, $\alpha\omega^1 + (1-\alpha)\omega^2$ is the optimal solution of the problem 2.2 with the expected return $\alpha\mu_0^1 + (1-\alpha)\mu_0^2$.

2. If there is exactly one risk-free asset, the first fund can be chosen to be a very simple fund containing only the risk-free asset and the other fund contains no risk-free assets. This strategy seems reasonable since the risk-free assets tend to have lower expected returns than the risky ones so the investor should invest in the risk-free asset to keep the risk down but at the same time invest in the risky ones to increase the expected return. The effect of risk-free assets will be discussed in the next section.

The idea behind this theorem is to treat the two distinct portfolios as two "mutual funds" and then obtain any desired investment performance by investing in these two assets only. In practice, as seen before, the minimum variance portfolio (2.1) is generally used as one of two funds.

2.4.5 The Effect of a Risk-Free Asset

It was assumed in the previous sections that all the securities were risky and their covariance matrix was non singular. However, in reality there exists assets whose variance is so low that it is considered as zero, for example treasury bills [2]. In addition, they are uncorrelated with the other assets. The optimization problem 2.2 can be reformulated as

$$\min_{\omega} \frac{1}{2}\omega^T \Sigma \omega \text{ subject to } \mu^T \omega + \mu_f \omega_f = \mu_0 \text{ and } \mathbf{1}^T \omega + \omega_f = 1 \tag{2.4}$$

Where ω_f is the the weight to be assigned to the risk-free asset and μ_f its expected return. Since it is always possible to achieve μ_f return, one can assume that $\mu_0 \geq \mu_f$. In addition, Σ is still supposed non singular. Using the same method as before, one can show that

$$\omega_0 = \lambda \Sigma^{-1}(\mu - \mu_f \mathbf{1})$$

and

$$\begin{pmatrix} 1 & \mathbf{1}^T \Sigma^{-1}(\mu - \mu_f \mathbf{1}) \\ \mu_f & \mu^T \Sigma^{-1}(\mu - \mu_f \mathbf{1}) \end{pmatrix} \begin{pmatrix} \omega_f \\ \lambda \end{pmatrix} = \begin{pmatrix} 1 \\ \mu_0 \end{pmatrix}$$

Two cases are considered : If $\mu = \mu_f \mathbf{1}$ then the optimal solution is simply $\omega_f = 1$ and $\omega = \mathbf{0}$ where $\mathbf{0}$ is the vector of zeros. This makes perfect sense because the investor could achieve the required return without any risk by investing only in the risk-free asset. Otherwise, solving this system gives the solution

$$\begin{pmatrix} \omega_f \\ \lambda \end{pmatrix} = \begin{pmatrix} 1 - (\mu_0 - \mu_f) \frac{\mathbf{1}^T \Sigma^{-1}(\mu - \mu_f \mathbf{1})}{(\mu - \mu_f \mathbf{1})^T \Sigma^{-1}(\mu - \mu_f \mathbf{1})} \\ \frac{\mu - \mu_f}{(\mu - \mu_f \mathbf{1})^T \Sigma^{-1}(\mu - \mu_f \mathbf{1})} \end{pmatrix}$$

Plugging this into ω_0 gives

$$\begin{pmatrix} \omega_f \\ \omega_0 \end{pmatrix} = (1 - \alpha)\omega_F + \alpha \omega_M$$

Where

$$\omega_F = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} \ , \ \omega_M = \begin{pmatrix} 1 - \mathbf{1}^T \Sigma^{-1}(\mu - \mu_f \mathbf{1}) \\ \Sigma^{-1}(\mu - \mu_f \mathbf{1}) \end{pmatrix}$$

and

$$\alpha = \frac{\mu_0 - \mu_f}{(\mu - \mu_f \mathbf{1})^T \Sigma^{-1}(\mu - \mu_f \mathbf{1})}$$

Hence, the optimal portfolio is a linear combination of two efficient portfolios. As a remark, ω_M represents the fund described in the previous theorem. The fact that the efficient frontier becomes a straight line will be discussed in the next section.

It is remarkable that every solution to the Markowitz problem can be represented as a linear combination of only two portfolios.

2.5 The Capital Asset Pricing Model

2.5.1 Capital Market Line

In this section, the same notations as 2.4 are used. Also, suppose that one risk-free asset is available having the expected return μ_f.
The Sharpe ratio can be used to evaluate a portfolio's past performance where actual returns are used in the formula. It can also help explain whether a portfolio's excess returns are due to smart investment decisions or a result of too much risk.
As seen in 2.4.3 and 2.4.5, adding diversification should increase the Sharpe ratio compared to similar portfolios with a lower level of diversification.
Using the previous notations, the Sharpe ratio can be defined as

$$S_\omega = \frac{(\mu - \mu_f \mathbf{1})^T \omega}{\sqrt{\omega^T \Sigma \omega}}$$

After plotting the efficient frontier curve, one can plot the tangent line to the curve drawn from the point representing the risk-free portfolio. This line is called the Capital Market Line (CML) [4, p. 5]. It is easy to see that the slope of this line is the Sharpe ratio of the tangency portfolio. This portfolio is composed entirely of the risky assets and is an optimal solution of the problem 2.2. It is expected that the Sharpe ratio is lower than the slope of the CML and the closer the Sharpe ratio to the slope of CML, the better the performance of the fund in terms of return against risk.

First, one can find the portfolio having the largest Sharpe ratio. This problem can be described as

$$\max_{\omega} \frac{(\mu - \mu_f \mathbf{1})^T \omega}{\sqrt{\omega^T \Sigma \omega}} \text{ subject to } \mathbf{1}^T \omega = 1 \tag{2.5}$$

Using the Lagrange multiplier gives the optimal portfolio

$$\omega_0 = \frac{\Sigma^{-1}(\mu - \mu_f \mathbf{1})}{\mathbf{1}^T \Sigma^{-1}(\mu - \mu_f \mathbf{1})}$$

This portfolio can be expressed in terms of the minimal variance portfolio and the "market weights" portfolio as seen in 2.4.2 such that

$$\omega_0 = (1 - \alpha)\omega_{min} + \alpha \omega_{mk}$$

Where

$$\omega_{min} = \frac{1}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}\Sigma^{-1}\mathbf{1} \ , \ \omega_{mk} = \frac{\Sigma^{-1}\mu}{\mathbf{1}^T\Sigma^{-1}\mu} \text{ and } \alpha = \frac{\mathbf{1}^T\Sigma^{-1}\mu}{\mathbf{1}^T\Sigma^{-1}(\mu - \mu_f\mathbf{1})}$$

This portfolio is efficient and represents the optimal solution for the problem 2.2 with the target mean return $\mu_0 = \frac{\mu^T\Sigma^{-1}(\mu-\mu_f\mathbf{1})}{\mathbf{1}^T\Sigma^{-1}(\mu-\mu_f\mathbf{1})}$. As a very interesting observation, this portfolio represents the part containing only the risky assets of the "fund" portfolio ω_M (2.4.5). Finally, as explained before and proved now, ω_0 is the tangency portfolio and is called the Market Portfolio. This result implies that the slope of the CML is the largest Sharpe ratio.

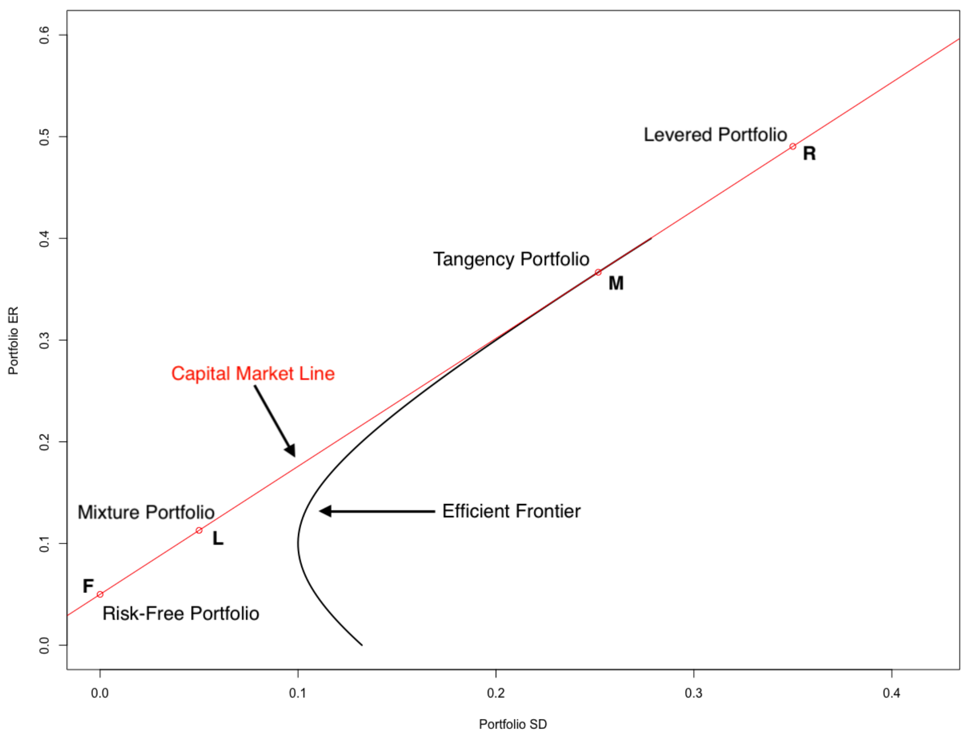

Figure 2.2: Efficient Frontier and Capital Market Line

The CML results from the combination of the market portfolio **M** and the risk-free portfolio **F**. All points along the CML have superior risk-return profiles to any portfolio on the efficient frontier, with the exception of the market portfolio. On one hand, the portfolios between the risk-free asset and the tangency portfolio are portfolios composed of risk-free assets and the tangency portfolio (For example the point **L**). On the other hand, the portfolios on the CML above and to the right of the tangency portfolio are generated by borrowing at the risk-free rate and investing the proceeds into the tangency portfolio. They are called levered [2] portfolios of risky-assets (For example the point **R**).

To summarize, as a consequence of the two mutual fund theorem 2.4.4, the efficient frontier when one risk-free asset is available becomes a straight line and every portfolio ω on that line verifies

$$E(\omega) = \mu_f + \frac{E(\omega_M) - \mu_f}{\sigma_M}\sigma_\omega$$

Where $\sigma_M = \sqrt{Var(\omega_M)}$ and $\sigma_\omega = \sqrt{Var(\omega)}$
In practice, all investors should choose the market portfolio then, depending on their appetite of risk, combine this risky investment with borrowing or lending at the risk-free rate.

2.5.2 Security Market Line

Suppose that one additional risky asset is available denoted i. One can be interested in the impact of adding this asset to the market portfolio ω_M, in particular does it improve the market's portfolio risk/return characteristics. Let $\omega_\alpha = (1-\alpha)\omega_M + \alpha\omega_i$ a portfolio consisting of the market portfolio and the asset i. Then the expected return and variance of this portfolio are

$$R_\alpha = (1-\alpha)R_M + \alpha\mu_i$$

$$\sigma_\alpha^2 = (1-\alpha)^2\sigma_M^2 + \alpha^2\sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} \text{ where } \sigma_{iM} = cov(R_M, R_i)$$

Therefore,

$$\frac{dR_\alpha}{d\sigma_\alpha} = \frac{dR_\alpha/d\alpha}{d\sigma_\alpha/d\alpha} = \frac{\sigma_\alpha(\mu_i - R_M)}{(\alpha-1)\sigma_M^2 + \alpha\sigma_i^2 + (1-2\alpha)\sigma_{iM}}$$

Using the fact that the Standard Deviation - Return curve of this combinations must lie entirely below the capital market line and tangent at the market portfolio ($\alpha = 0$)

$$\frac{R_M - \mu_f}{\sigma_M} = \left.\frac{dR_\alpha}{d\sigma_\alpha}\right|_{\alpha=0}$$

This easily gives the formula for the CAPM [5]

$$R_i = \mu_f + \beta_i(R_M - \mu_f) \tag{2.6}$$

The beta measures the asset sensitivity to a movement in the overall market.

As a representation of the CAPM, one can define the Security Market Line (SML)

$$E(\beta_i) = \mu_f + \beta_i(R_M - \mu_f) \tag{2.7}$$

then plot the risky assets as points in the graph. The assets above the line are undervalued because for a given amount of risk (beta), they yield a higher return. The assets below the line are overvalued because for a given amount of risk, they yield a lower return. While the SML can be a valuable tool in equity evaluation and comparison, it should not be used in isolation, as the expected return of an investment over the risk-free rate of return is not the sole consideration when making investment choices.

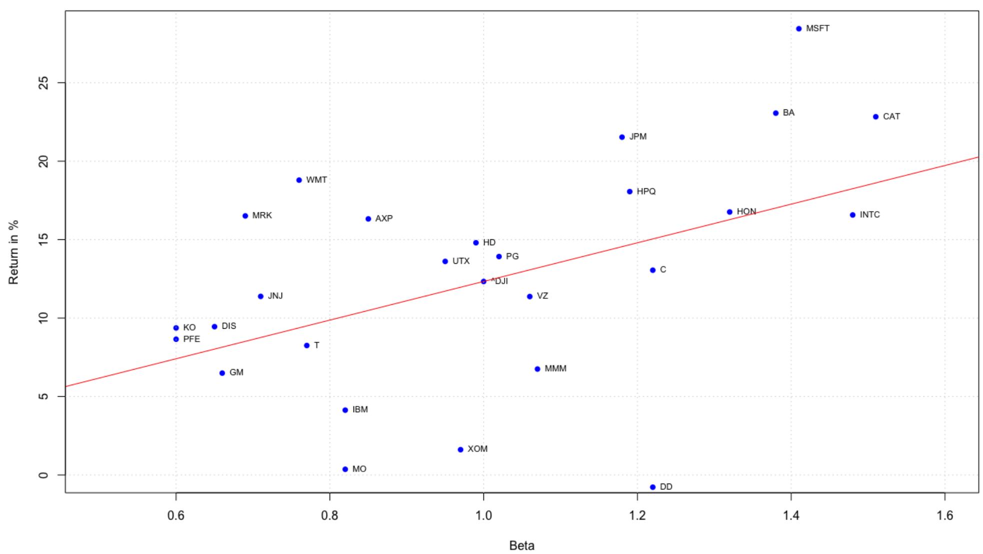

Figure 2.3: An estimation of the CAPM and the SML (red) for the Dow Jones Industrial Average over the period 2016-2020 for monthly data.

2.5.3 Asset Pricing and Required Return

One can compute the required expected return using the Security Market Line then compare with an independent estimation of the expected return based on either fundamental or technical analysis techniques. For example, suppose there exists a stock with no dividends and a beta compared to the market of 1.3 which means this stock is riskier than the market. Also, assume the investor estimates the market to rise by 8% and the risk-free rate is 3%. Then the expected return of the stock based on the CAPM is

$$9.5\% = 3\% + 1.3 \cdot (8\% - 3\%)$$

Assuming that the CAPM is correct, an investor can conclude if an asset is undervalued or overvalued but it can also suggest an error in estimating its price. Another way to use this model is to justify a return estimation based on the past performance of a stock and its peers. It can also be used to compare a portfolio with other investors to see why it didn't perform as well and find which holdings are not on the SML.

The CAPM can be used to price an asset. In order to see how this is done consider an asset i that is purchased for the price P and later sold at the price Q. Suppose Q is a random variable with mean μ_Q. In this case, $R_i = \frac{Q-P}{P}$. Using the CAPM formula 2.6, one can find the relation

$$P = \frac{\mu_Q}{1 + \mu_f + \beta_i(R_M - \mu_f)}$$

Note that after simple calculation

$$\beta_i = \frac{Cov(R_i, R_M)}{\sigma_M^2} = \frac{Cov(Q, R_M)}{P\sigma_M^2}$$

Therefore,

$$P = \frac{1}{1 + \mu_f}\left[\mu_Q - \frac{Cov(Q, R_M)(R_M - \mu_f)}{\sigma_M^2}\right] \tag{2.8}$$

The formula 2.8 is called the certainty equivalent pricing formula. Note that the certainty equivalent pricing formula demonstrates that the purchase price P is a linear function of the sale price Q. This formula does not require knowledge of the asset's beta but does require the $Cov(Q, R_M)$ and σ_M^2.

A positive aspect of this model is its simplicity and the fact that it takes account of the time value of money based on the theory of time preference and the risk premium which reflects the extra return investors demand because they want to be compensated for the risk that the cash flow might not materialize after all. However, it depends on many assumptions (2.4) that have been shown not to hold in reality.

2.5.4 Limitations

Actually, some of issues have been explained in the introduction 2.1 in particular the assumptions behind using standard deviation as a measure of risk is that the returns follow a normal distribution but price movements in both directions are not equally risky. Besides that, CAPM suppose that the risk-free rate is stable during the investment period which is not the case : for example the U.S. treasury bonds changed over a 10 year holding period. This can make some stocks go from undervalued to overvalued, and vice versa. Another limitation is that the market portfolio is just a theoretical portfolio that can't be purchased and is substituted by a major stock index in general : This was done to plot the figure 2.3.
The most serious critique of the CAPM is the assumption that future cash flows can be estimated for the discounting process. If an investor could estimate the future return of a stock with a high level of accuracy, the CAPM would not be necessary.
Some other assumptions can be criticized such that :

- All investors are rational and risk averse.
- Investing cannot influence prices.
- All information is available at the same time to all investors.
- No transaction or taxation costs.
- Lending and borrowing at risk-free rate is always possible.

In reality, normal people tend to have fragmented and no optimized portfolios. Many economists believe that the Capital Asset Pricing Model is fundamentally flawed and irrational. Some critics say it is possible to get identical discount rates for different amounts of risk. Besides that, the model should include other possible investments than stocks and bonds for example art, watches or real estate.

2.6 Portfolio Simulation

In this final section, the theory and models discussed previously will be tested using real stocks data. One can try to use computer simulation to have an intuition about possible limitations due to the conditions imposed in the beginning (2.4).

2.6.1 Markowitz Efficient Frontier

The chosen ticker list contains the following 10 stocks : AAPL, MA, V, MSFT, BAC, JPM, NFLX, SCI and AAXN. Some of the companies are highly correlated like Apple and Microsoft or Visa and MasterCard. It should be interesting to see the effect of these correlations on the plot of the efficient frontier.

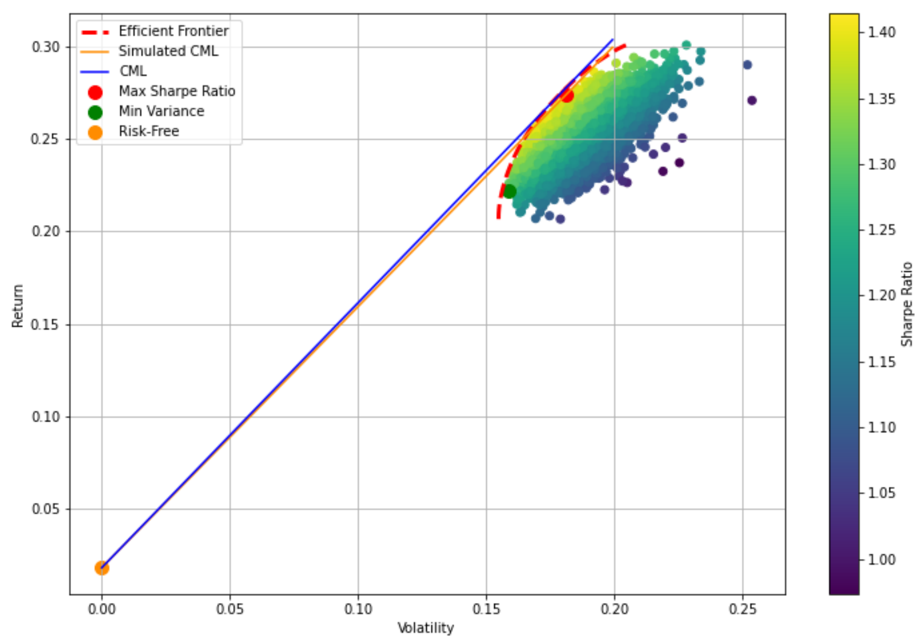

Figure 2.4: Efficient Frontier and CML for the period 1.1.2016 - 1.1.2020

To obtain the following figure, 20000 random portfolios are simulated and the performance of each one is plotted as a point on the Standard Deviation - Return graph using a color graduation depending on the Sharpe ratio.

The efficient frontier is computed using Sequential Least Squares Programming for the minimization problem 2.4.2. In this part, short positions are not allowed which implies that the weights are in $[0, 1]$ and the sum of the weights is 1.
The simulated Critical Market Line is just the line that passes through the Risk-Free only portfolio and the simulated portfolio having the maximum Sharpe ratio. The other CML is the line that starts at the Risk-Free only portfolio and has the maximum Sharpe ratio as its slope (2.5.1) : the maximization is done using the formula of the Sharpe ratio (2.5) without using the simulated portfolios.

As a remark, the period 2016 to 2020 is chosen for its strong economic growth where the returns of the stocks are positive. Trying to plot the efficient frontier during a recession is actually hard because the conditions in the simulation of portfolios and in the minimization function need to be changed : Negative weights are essential but this implies further questioning. What are the bounds imposed for short positions? How to replace the constraint of the sum of weights is 1? Is it allowed to only have negative weights? For those reasons, it won't be done in this section.

Chapter 3

Risk Analysis

3.1 Definition

Risk often refers to the chance an outcome or investment's actual gains will differ from an expected outcome or return. In general, financial theory classifies investment risks affecting asset values into two categories: systematic risk and unsystematic risk. Systematic risks, also known as market risks, are risks that can affect an entire economic market overall or a large percentage of the total market. Unsystematic risk, also known as specific risk or idiosyncratic risk, is a category of risk that only affects an industry or a particular company.

3.2 Coherent Measures Of Risk

In this section, the publication [6] by Philippe Artzner and Freddy Delbaen will be examined. The paper studies both market and non-market risks and presents methods for measuring these risks. The completeness of markets [7] is not assumed.

3.2.1 Definitions and Notations

The risk is defined as the random variable "future net worth" interpreted as possible future values of positions or portfolios currently held. The measure of risk of a position will be whether its future value belongs or does not belong to the subset of acceptable risks (decided by a supervisor). It

depends on a specified "reference instrument" which is an available risk-free asset such as default free zero coupon bonds. To be coherent, a measure of risk need to satisfy four axioms. To present these axioms, some notations are necessary.
Let Ω be the set of states of nature, and assume it is finite. It contains the possible outcomes of a strategy. The final net worth of each state is represented by the random variable X. Let $\mathcal{G}$ be the set of all risks

$$\mathcal{G} = \{f : \Omega \to \mathbb{R} \mid f \text{ function}\} \simeq \mathbb{R}^{card(\Omega)}$$

A measure of risk ρ is a mapping $\mathcal{G}$ from to $\mathbb{R}$

$$\rho : \mathcal{G} \to \mathbb{R}$$

The number $\rho(X)$ is the quantity of cash that needs to be added to the risky position and the reference instrument (if positive) or withdrawn (if negative) to proceed with with the investor plans.
As a remark, this definition allows for negative risks.
The risk measure ρ is coherent if it satisfies the following axioms :

1. Translation Invariance :

$$\forall X \in \mathcal{G} \quad \forall \alpha \in \mathbb{R} \quad \rho(X + \alpha \cdot r) = \rho(X) - \alpha$$

   Where r is the total rate of return on a reference instrument. This means that adding (resp. subtracting) the sure initial amount α to the initial position and investing it in the reference instrument, simply decreases (resp. increases) the risk measure by α.

2. Subadditivity :

$$\forall X_1, X_2 \in \mathcal{G} \quad \rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$$

   This means that a merger does not create extra risk.

3. Positive Homogeneity :

$$\forall X \in \mathcal{G} \text{ and } \lambda \geq 0 \quad \rho(\lambda X) = \lambda \rho(X)$$

   This means that position size influences the risk. The most common example is that it is harder and takes longer to liquidate larger positions.

4. Monotonicity :

$$\forall X, Y \in \mathcal{G} \text{ with } X \leq Y \quad \rho(Y) \leq \rho(X)$$

Previously, the term "acceptable risks" have been used without explaining its actual meaning. The paper defines acceptance sets as the sets of acceptable future net worth. It is supposed that all the possible outcomes of a strategy are known but their probabilities could be unknown or not subject to common agreement. Also, it is assumed that the markets are liquid.
First, the cone of non-negative elements in $\mathcal{G}$ shall be denoted by L_+ and its negative by L_-. Second, let $\mathcal{A}_j : j \in J$ be a set of final net worths, expressed in some currency, which, in its corresponding country, are accepted by regulator/supervisor j and

$$\mathcal{A} = \bigcap_{j \in J} \mathcal{A}_j$$

The acceptance set $\mathcal{A}$ satisfies these axioms :

1. $L_+ \subseteq \mathcal{A}$
2. $\mathcal{A} \cap L_{--} = \varnothing$ where $L_{--} = \{X \mid \forall \omega \in \Omega, X(\omega) < 0\}$
3. $\mathcal{A}$ is convex.
4. $\mathcal{A}$ is a positively homogeneous cone.

The rationale for Axioms 1 and 2 is that a final net worth which is always non-negative does not require extra capital, while a net worth which is always strictly negative certainly does. The third axiom reflects risk aversion on the part of the supervisor or regulator.

3.2.2 Correspondence

On one hand, there is a correspondence between acceptance sets and measures of risk.
Given a total rate of return r on a reference instrument, the risk measure associated to the acceptance set $\mathcal{A}$ is $\rho_{\mathcal{A},r} : \mathcal{G} \to \mathbb{R}$ such that

$$\rho_{\mathcal{A},r}(X) = \inf\{\alpha \mid X + \alpha \cdot r \in \mathcal{A}\}$$

The acceptance set associated to a risk measure ρ is

$$\mathcal{A}_\rho = \{X \in \mathcal{G} \mid \rho(X) \leq 0\}$$

On the second hand, there is a correspondence between the axioms on acceptance sets and the axioms on measures of risks :

- If a set satisfies the four axioms then its corresponding risk measure is coherent.
- If a risk measure is coherent then its corresponding acceptance set is closed and satisfies the four axioms.

3.2.3 Example : Value at Risk

Value at Risk is widely used by corporate treasures and fund managers since it was pioneered in 1993 by JPMorgan. It is defined as : Given $\alpha \in]0,1[$ and a reference instrument total rate of return r, the VaR at level α of the final net worth X with distribution $\mathbb{P}$ is

$$VaR_\alpha(X) = -\inf\{x \mid \mathbb{P}[X \leq x \cdot r] > \alpha\}$$

This represents the negative of the quantile q_α^+ of X/r.

While it satisfies the translation invariance, positive homogeneity and monotonicity, it doesn't satisfy subadditivity. Therefore, VaR is not a coherent risk measure. It can increase when two independent projects are merged and that creates severe aggregation problems : The use of value at risk does not encourage diversification. This comes from the fact that, in general, knowing the q-quantile of two correlated random variables X and Y doesn't give much information about the q-quantile of their sum.

Chapter 4

Option Pricing Theory

Option Pricing Theory uses available variables to theoretically provide an estimation of an option's fair value. These variables could be stock price, exercise price, volatility, interest rate and time until expiration for example. The primary goal of option pricing theory is to calculate the probability that an option will be exercised or the strike price is achieved. Some commonly used models to value options are Black-Scholes, binomial option pricing, and Monte-Carlo simulation. Options are usually used for hedging which can be viewed as an insurance for investments. This chapter also presents some investing strategies that can be used after evaluating option contracts.

4.1 Definitions

Definition 6. Options : Options are financial instruments that are derivatives based on the value of underlying securities such as stocks. An options contract offers the buyer the opportunity to buy or sell, depending on the type of contract they hold, the underlying asset.

- Call options allow the holder to buy the asset at a stated price within a specific time frame.
- Put options allow the holder to sell the asset at a stated price within a specific time frame.

Each option contract will have a specific expiration date by which the holder must exercise their option. The stated price on an option is known as the strike price [2].

Definition 7. Futures : Futures are derivative financial contracts that obligate the parties to transact an asset at a predetermined future date and price. Here, the buyer must purchase or the seller must sell the underlying asset at the set price, regardless of the current market price at the expiration date. Futures can be used for hedging or trade speculation [2].

Definition 8. Hedge : A hedge is an investment to reduce the risk of adverse price movements in an asset. The most common way of hedging in the investment world is through derivatives. Derivatives are securities that move in correspondence to one or more underlying assets. They include options, swaps, futures and forward contracts. The underlying assets can be stocks, bonds, commodities, currencies, indices or interest rates. [2].

4.2 Examples : Option Strategies

1. Purchase the underlying stock and simultaneously write sell a call option on those same shares (Covered Call) : This is a very popular strategy because it generates income and reduces some risk of being long stock alone.

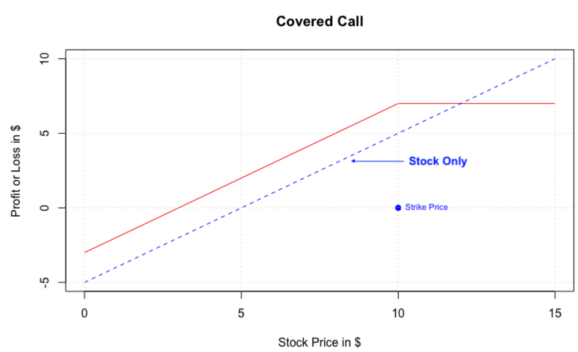

Figure 4.1: Covered Call

2. Buy a stock and a put (Married Put) : The reason an investor would use this strategy is simply to protect their downside risk when holding a stock. This strategy functions just like an insurance policy, and establishes a price floor should the stock's price fall sharply.

Figure 4.2: Married Put

3. Buy a call with one strike price and sell a call with another (Bull Call Spread) : It is used when an investor is bullish on the underlying asset and expects a moderate rise in the price of the asset. It limits the risk compared to a simple call option but reduces the gains.

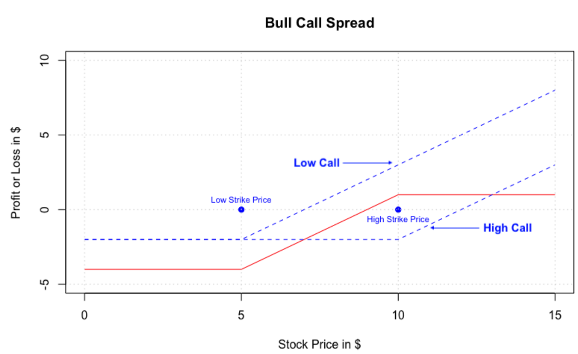

Figure 4.3: Bull Call Spread

4.3 Geometric Brownian Motion

It is tempting to suggest a generalized Wiener process for a stock price but it fails to capture a key aspect : The expected percentage return required by investors from a stock is independent of the stock's price. Clearly, the assumption of constant expected drift rate is inappropriate and needs to be replaced by the assumption that the expected return is constant. Another reasonable assumption is that the variability of the percentage return in a short period of time is the same regardless of the stock price. These assumptions leads to the equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{4.1}$$

Where
S_t is the stock price at time t
μ is the expected rate of return
σ is the volatility of the stock price
W_t is a standard Wiener process

The model of stock price behavior satisfying this Stochastic Differential Equation (SDE) is known as geometric Brownian motion [8].
Let $G(S,t) = \ln S_t$, the Taylor series expansion of G gives

$$\Delta G = \frac{\partial G}{\partial S}\Delta S + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\Delta S^2 + \cdots$$

Using the properties of the Brownian motion W_t, the equation 4.1 can become discrete such as

$$\Delta S = \mu S \Delta t + \sigma S \epsilon \sqrt{\Delta t} \text{ where } \epsilon \sim \mathcal{N}(0,1)$$

The term ΔS^2 is important because it contains the term Δt and shouldn't be simplified yet

$$\Delta S^2 = \sigma^2 S^2 \epsilon^2 \Delta t + \text{terms of higher order in } \Delta t$$

Keeping the terms of order ΔS and Δt, replacing ϵ^2 by its expectation and taking the limits as ΔS and Δt tend to zero gives

$$dG = \frac{\partial G}{\partial S}dS + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2 dt$$

Finally, replacing dS using 4.1 gives the process

$$dG = (\mu - \frac{\sigma^2}{2})t + \sigma dW_t$$

This means that G follows a Wiener Process of drift $\mu - \frac{\sigma^2}{2}$ and a variance σ^2. In that case, for a future time T and an arbitrary initial value G_0

$$G_T - G_0 = (\mu - \frac{\sigma^2}{2})T + \sigma^2 W_T$$

Therefore

$$\ln \frac{S_T}{S_0} \sim \mathcal{N}((\mu - \frac{\sigma^2}{2})T, \sigma^2 T) \tag{4.2}$$

and

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right) \tag{4.3}$$

Before presenting some properties of the geometric Brownian motion, one can simulate this process and compare with the corresponding Brownian motion as in the following figure 4.4.

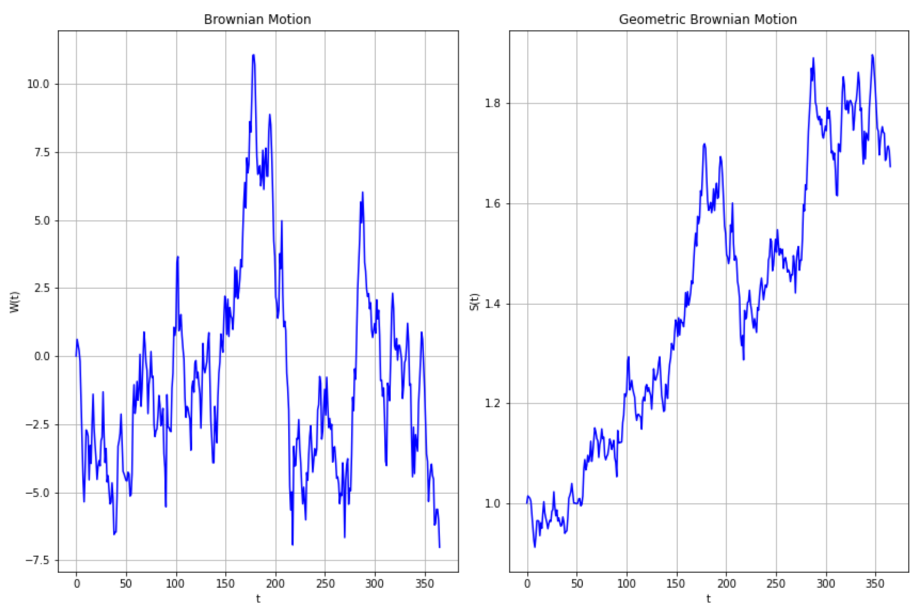

Figure 4.4: Brownian Motion and Geometric Brownian Motion

The equation 4.2 implies that for every $t \geq 0$, S_t/S_0 follows a log-normal distribution and have the following properties

$$E(S_t/S_0) = e^{\mu t} \text{ and } Var(S_t/S_0) = (e^{\frac{\sigma^2 t}{2}} - 1)e^{2\mu t}$$

It follows that

$$\ln S_T \sim \mathcal{N}(\ln S_0 + (\mu - \frac{\sigma^2}{2})T, \sigma^2 T)$$

and the continuously compounded rate of return per annum earned on a stock between times 0 and T, $r_T = \frac{1}{T}\ln(S_T/S_0)$, satisfies

$$r_T \sim \mathcal{N}(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}) \tag{4.4}$$

As T increases, the standard deviation of r_T declines. It means that the same average return for longer periods is more certain than shorter ones. Also, the expected return μ required by investors is not the expected continuously compounded return on the stock and the volatility should be estimated from historical data.

As an important remark, the geometric Brownian motion model used for stock's price has the Markov property since the future values of S_t only depends on the present value. Under this process, the return to the holder of the stock in a small period of time is normally distributed and the returns in two non overlapping periods are independent.

The following figure represents simulations of Apple stock price for the month of December 2019 using historical data from January 2016 to November 2019 to estimate the parameters of the geometric Brownian motion model. Under the assumption that the stock price follows a geometric Brownian motion, S_t/S_0 has to follow a log-normal distribution. Supposing that the dependence of the log returns $R_t = \ln(S_t/S_{t-1})$ is quite weak, one can assume that

$$R_1, \cdots, R_n \overset{\text{i.i.d}}{\sim} \mathcal{N}(\mu - \frac{\sigma^2}{2}, \sigma^2)$$

Therefore, the estimators of μ and σ^2 are given by

$$\hat{\mu} = \overline{R} + \frac{\hat{\sigma}^2}{2} \text{ and } \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(R_i - \overline{R})^2$$

Where $\overline{R}$ is the mean value of the returns.

Using the equation 4.1 derived previously where S_0 takes the last given stock price, μ and σ their estimators and W_t as a simulated Standard Wiener process gives a prediction for the future stock prices.

$$W_t = W_{t-1} + \epsilon_t \text{ such that } \epsilon_t \overset{\text{i.i.d}}{\sim} \mathcal{N}(0,1) \text{ and } W_0 = 0$$

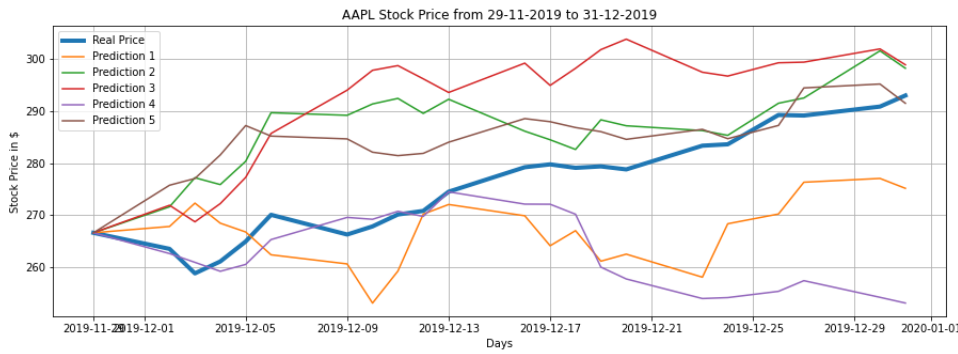

Figure 4.5: AAPL Stock Price Prediction December 2019

This model is very important in financial mathematics, especially as a critical assumption of the Black-Scholes model which is covered later.

4.4 Example : American Call Option Valuation

Suppose that the stock price S follows a geometric Brownian motion as defined in the previous section. In that case

$$Y_t = \frac{S_t}{S_0} \sim Lognormal((\mu - \frac{\sigma^2}{2})t, \sigma^2 t)$$

If there are no transaction fees or costs and for a strike price K, the value of a call option on the asset S at the time t is

$$C_t = \max(0, S_t - K)$$

One can try to find the expected value of C_t. First, it is easy to see that the density of C_t is

$$f_{C_t}(x) = \frac{1}{S_0} f_{Y_t}(\frac{x+K}{S_0})$$

Hence

$$\begin{aligned}
E(C_t) &= \int_0^\infty x f_{C_t}(x)\,dx \\
&= \int_{K/S_0}^\infty (S_0 y - K) f_{Y_t}(y)\,dy \quad \text{Changing the variable to } y = \frac{x+K}{S_0} \\
&= S_0 \int_{K/S_0}^\infty y f_{Y_t}(y)\,dy - K \int_{K/S_0}^\infty f_{Y_t}(y) \\
&= S_0\, g_{Y_t}(K/S_0) - K\,[F_{Y_t}(y)]_{K/S_0}^\infty
\end{aligned}$$

Where g_{Y_t} is the partial expectation of Y_t. The derivation of g for a Lognormal distribution won't be presented in this paper but isn't too complicated [9]. This finally gives

$$E(C_t) = S_0\, e^{\mu t} \cdot N(d_1) - K \cdot N(d_2) \tag{4.5}$$

Where
$N(x)$ is the cumulative probability distribution function for a standardized normal distribution

$$d_1 = \frac{\ln(S_0/K) + (\mu + \sigma^2/2)t}{\sigma\sqrt{t}}$$

and

$$d_2 = \frac{\ln(S_0/K) + (\mu - \sigma^2/2)t}{\sigma\sqrt{t}} = d_1 - \sigma\sqrt{t}$$

Now suppose that the world is risk-neutral, hence the expected rate of return μ is replaced by the risk-free rate r. Applying the discount at the risk-free rate on the equation 4.5 gives a similar formula to the Black-Scholes pricing model discussed later. In practical terms, for a maturity time T, the option value at the time of purchase $t = 0$ is

$$C_0 = E(C_T) = S_0 \cdot N(d_1) - K\, e^{-rT} \cdot N(d_2) \tag{4.6}$$

4.5 Option Pricing Models

4.5.1 Binomial Option Pricing Model

The binomial option pricing model [10, Ch. 11] is a numerical method for the valuation of options. It addresses cases where the Black Scholes model doesn't have a closed form such as the valuation of American put options that can be exercised at any time before maturity. The model traces the evolution of the option's underlying variables and uses it to compute the option value. In this section, the studied variable will be the stock price and the model will be the Cox, Ross and Rubinstein (CRR) method. It is implemented by following three steps.

First, constructing a binomial lattice (tree) with discrete time steps where each node represents a possible price for the underlying asset.

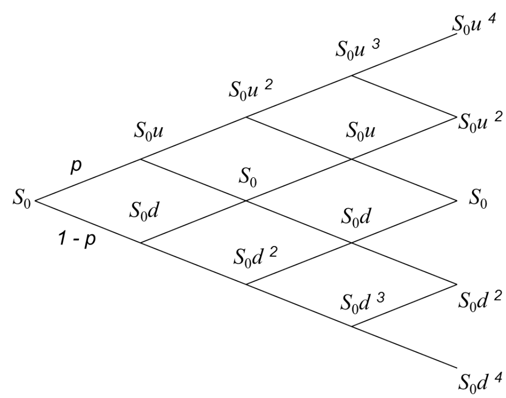

Figure 4.6: Binomial Price Tree

At each step, the stock price goes up by a factor of $u \geq 1$ with a probability p or down by a factor $0 < d \leq 1$ with a probability $1 - p$. To reduce the number of nodes and decrease the computation time of the tree, it was supposed that $d = 1/u$.

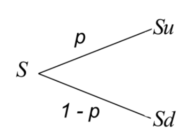

Figure 4.7: One Step Binomial Price Tree

Second, computing the option value at each final node which represents the expiration time T. This is is easily done by computing its intrinsic value.

Finally, working backward through the tree to compute the option value in each node until the initial node. Supposing the world is risk-neutral and there are no arbitrage opportunities, it is possible to evaluate the option at each step. In that case, the expected return from the traded asset is the risk-free interest rate and the value of an option is its expected value discounted at the risk-free rate. Before that, the values for u and p need to be specified. Matching the expected return for a period Δt (which corresponds to a time step in the tree) of the asset with the tree gives the condition

$$Se^{r\Delta t} = pSu + (1-p)Sd$$

Again, matching the variance gives

$$\sigma^2 \Delta t = pu^2 + (1-p)d^2 - e^{2r\Delta t}$$

Using the fact that $d = 1/u$ gives the solution

$$u = e^{\sigma\sqrt{\Delta t}} \quad d = e^{-\sigma\sqrt{\Delta t}} \quad p = \frac{e^{r\Delta t} - d}{u - d}$$

As a remark, these calculations used the fact that the stock price follow a geometric Brownian motion and that there are no dividends paid. Also, u and d satisfy the equations if the terms of order higher than Δt are ignored when approximating the exponential function.

Since a risk-neutral world was assumed, the value of an option at each node can be calculated as its expected value after Δt discounted by the risk-free interest rate r. For example, in the figure 4.7, let f be the option value at the node S and f_u, f_d at the nodes Su, Sd respectively. Therefore

$$f = (pf_u + (1-p)f_d) \cdot e^{-r\Delta t}$$

The computed value f in each node is called the "binomial value". If the option is American, it is necessary to check at each node to see whether early exercise is preferable to holding the option for a further time period Δt. In practical terms, for each node the option value takes the maximum between f and the exercise price at that node. For example : Suppose a stock is trading at 50$ with a volatility of 20%, the risk-free rate is 5% and the strike price is 52$ for 2 years. This means that $S_0 = 50$, $K = 52$, $r = 0.05$, $T = 2$ and $\sigma = 0.2$. If the option can't be exercised before maturity, Black-Scholes gives a put price of 4.1078$ but the Binomial model with possible early exercise gives 4.8601$.

In the case of European options without dividends, the binomial model value converges on the Black–Scholes formula value as the number of time steps increases. For example : Suppose a stock is trading at 42$ with a volatility of 20%, the risk-free rate is 10% and the strike price is 40$ for 6 months. This means that $S_0 = 42$, $K = 40$, $r = 0.1$, $T = 0.5$ and $\sigma = 0.2$. The Black-Scholes model (discussed in the next section) gives a call price of 4.7594$. The figure 4.8 shows the convergence of the binomial model when Δt converges to 0.

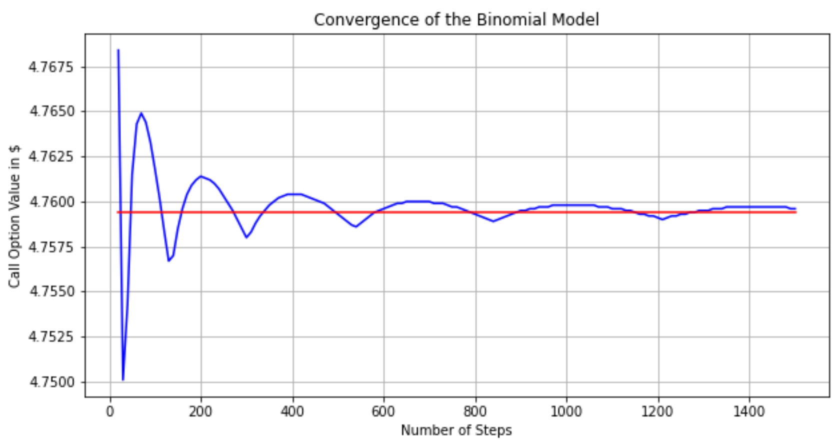

Figure 4.8: Binomial Model Convergence

4.5.2 Black-Scholes Model

Suppose that the market consists of at least one risky asset S and one risk-free asset. Besides that, these assumptions are required to develop the Black-Scholes equation [11] :

- The risk-free rate is constant.
- The stock price follows a geometric Brownian motion with constant drift and volatility.
- There are no dividends from the stocks.

- There is no arbitrage opportunity.
- Ability to borrow and lend any amount, even fractional, of cash at the risk-free rate.
- Ability to buy and sell any amount, even fractional, of the stocks.
- There are no transaction fees or costs.
- There is a derivative security (call or put option) also trading in this market.

In addition, let :

- S_t be the price of the underlying asset at time t.
- $V(S_t, t)$ be the price of the option as a function of the underlying asset S at time t.
- $C(S_t, t)$ be the price of the call option on the asset S.
- $P(S_t, t)$ be the price of the put option on the asset S.
- K be the strike price.
- r be the annualized risk-free interest rate.
- μ be the annualized drift of the stock S.
- σ be the standard deviation of the log return of the stock S.

The Black-Scholes equation is a partial differential equation describing the price evolution of call and put option under the assumptions stated before. The equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{4.7}$$

The derivation of this equation and the pricing model won't be presented in this paper but are clearly explained in John Hull's Options, Futures, and Other Derivatives [10, Ch. 13].
To have a concrete interpretation, it can be written as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S}$$

First, important tools for interpretation are presented. Greek letters are used in finance to represent the sensitivity of the price of derivatives to underlying parameters and appears in the equation 4.7 :
Δ represents the rate of change between the option's price and a 1$ change in the underlying asset's price. In other words, the price sensitivity.

$$\Delta = \frac{\partial V}{\partial S}$$

Θ represents the rate of change between the option price and time. In other words, the time sensitivity, sometimes known as an option's time decay. It indicates the amount an option's price would decrease as the time to expiration decreases.

$$\Theta = \frac{\partial V}{\partial t}$$

Γ represents the rate of change between an option's Δ and the underlying asset's price. This is called second-order price sensitivity. It is used to determine how stable an option's Δ is. From the viewpoint of the option issuer, it is the cost of hedging the option.

$$\Gamma = \frac{\partial^2 V}{\partial S^2}$$

In the equation 4.7, the left term consists of the time and price sensitivity. The right term consists the riskless return from a long position in the derivative and a short position consisting of Δ shares of the underlying asset. Therefore, the Black-Scholes equation says that the portfolio represented by the right side is riskless and its return, over any infinitesimal time interval, can be expressed as the sum of Θ and a term incorporating Γ. It states that over any infinitesimal time interval the loss from Θ and the gain from the Γ term compensate each other. The key element in the Black-Scholes analysis is that the return from the riskless portfolio in any very short period of time must be the risk-free interest rate.

Once the Black–Scholes PDE, with boundary and terminal conditions, is derived for a derivative, it can be solved numerically using standard methods of numerical analysis. However, for European call and put options, an exact formula for the solution can be derived. The difference with American options is that an European option may only be exercised on expiry.

The formulas are :

$$C(S_t, t) = S_t \cdot N(d_1) - Ke^{-r(T-t)} \cdot N(d_2) \tag{4.8}$$

and

$$P(S_t, t) = Ke^{-r(T-t)} \cdot N(-d_2) - S_t \cdot N(-d_1) \tag{4.9}$$

Where
T is the time to maturity of the option
$N(x)$ is the cumulative probability distribution function for a standardized normal distribution

$$d_1 = \frac{\ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{(T-t)}}$$

and

$$d_2 = \frac{\ln(S_t/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{(T-t)}} = d_1 - \sigma\sqrt{(T-t)}$$

To provide an interpretation of the equation 4.8, it can be written as

$$C(S_t, t) = e^{-r(T-t)}[S_t \cdot e^{r(T-t)} \cdot N(d_1) - K \cdot N(d_2)]$$

The expression $K \cdot N(d_2)$ is the strike price times the probability that the strike price will be paid (When the option is exercised). The expression $S_t \cdot e^{r(T-t)} \cdot N(d_1)$ is the expected value in a risk-neutral world of a variable that is equal to S_t if $S_t > K$ and zero otherwise. As a remark, when using this formula, time is normally measured as the number of trading days left in the life of the option divided by the number of trading days in 1 year. Also, the interest rate could be stochastic depending on the time and not fixed as mentioned before.

Finally, here is an example : Suppose a stock is trading at 42$ with a volatility of 20%, the risk-free rate is 10% and the strike price is 40$ for 6 months. This means that $S_0 = 42$, $K = 40$, $r = 0.1$, $T = 0.5$ and $\sigma = 0.2$. This gives a call price of 4.76$ and a put price of 0.81$. Ignoring the time value of money, the stock price has to rise by 2.76$ for the purchaser of the call to break even. Similarly, the stock price has to fall by 2.81$ for the purchaser of the put to break even.

For the case of American options, since it can be executed at any time before expiration the Black-Scholes equation becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \leq 0 \tag{4.10}$$
$$\text{subject to } V(S_t, t) \geq H(S_t) \text{ and } V(S_T, T) \geq H(S_T)$$

This problem is related to the optimal stopping problem of finding the time to execute the option. It has been shown that it is never optimal to execute a call option before the expiration date : Even if the investor thinks it is overpriced, he should sell the option than exercising it [10, Ch. 9.5]. For the case of a put option, it can be optimal to execute it early [10, Ch. 9.6]. In general this inequality does not have a closed form solution but there exists a numerical analysis method to approximate the value of an American call option with one dividend.

Speaking of dividends, it was assumed in the beginning that the option doesn't pay any. It is possible to modify the Black-Scholes model to take account of the dividends. Assume that the amount and timing of dividends during the life of an option can be predicted with certainty. For European options, the stock price needs to be reduced by the present value of all the dividends during the life of the option, the discounting being done from the ex-dividend rates at the risk-free rate and then use the Black-Scholes formulas 4.8 and 4.9.

Chapter 5

Conclusion

In conclusion, mathematics can be a powerful tool for understanding financial markets. First, we learned how to classify portfolios in terms of risk-return balance and how to make them optimal and efficient. Second, we derived a more general procedure for measuring risk taking account of the markets and the investor requirements. Finally, we introduced option contracts and the most common pricing models for stocks and derivatives. However, it is very important to remember that all of the mathematics used to derive this theories are based on non-trivial assumptions that in reality seem to be not satisfied. Nevertheless, the theory developed in this paper can be improved even with weaker assumptions.

I wish I had more time to go deeper in this subject but this semester project needs to be submitted. This project was my first introduction to quantitative finance. I always enjoyed learning about financial markets, reading stock news and even investing but I never had a deep understanding of finance. This is why I really enjoyed working on this project and I want to thank my professor and supervisor Dr. Stephan Morgenthaler for giving me this opportunity.

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