Lecture hours 19-20

9

Definitions and Theorems

Definition (Coordinates of a vector).

If we have a basis of \mathbb{R}^n , $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ and $\vec{x} \in \mathbb{R}^n$, so $\vec{x} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$, for some $c_1, \dots c_n \in \mathbb{R}$. We define the coordinates of \vec{x} in the basis \mathcal{B} as

$$[\vec{x}]_{\mathcal{B}} \stackrel{\mathrm{def}}{=} \left[egin{matrix} c_1 \ dots \ c_n \end{array} \right].$$

Definition (Change of basis matrix).

Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a basis of \mathbb{R}^n . We call the matrix $S = [\vec{v}_1, \dots, \vec{v}_n]$ the change of basis matrix of basis \mathcal{B} .

For any $\vec{x} \in \mathbb{R}^n$ we have

$$\vec{x} = S[\vec{x}]_{\mathcal{B}}$$
.

So, if we have to find the coordinates $[\vec{x}]_{\mathcal{B}}$ we compute

$$[\vec{x}]_{\mathcal{B}} = S^{-1}\vec{x}.$$

Note that

• If $\mathcal{B}=\{\vec{e}_1,\vec{e}_2,\ldots,\vec{e}_n\}$ (the standard basis), we have $[\vec{x}]_{\mathcal{B}}=\vec{x}.$

Problem 37 (Coordinates). Let \mathfrak{B} be the basis of \mathbb{R}^4 given by

$$\mathfrak{B} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Find the coordinates for vector

$$\vec{v} = \begin{bmatrix} 4\\3\\2\\1 \end{bmatrix}$$

in the basis B.

Solution 37 (Coordinates) Let $E=\{\vec{e}_1,\vec{e}_2,\vec{e}_3,\vec{e}_4\}$ be the standard basis of \mathbb{R}^4 . Define

$$S = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The change of basis matrix from the standard basis E to basis \mathfrak{B} is given by

$$S^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & -2 \end{bmatrix}$$

Thus

$$[\vec{v}]_{\mathfrak{B}} = S^{-1}[\vec{v}]_E = S^{-1}\vec{v} = \begin{bmatrix} 1\\1\\1\\-1 \end{bmatrix}.$$

Problem 38 (Coordinates). In this problem we will be working with the following bases of \mathbb{R}^3 :

$$\mathfrak{B} = \left\{ \begin{bmatrix} -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 9 \end{bmatrix} \right\}, \quad E = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \mathfrak{D} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

1. Find the change of basis matrix from basis \mathfrak{B} to the standard basis E.

2. Find the change of basis matrix from the standard basis E to basis \mathfrak{D} .

3. Find the change of basis matrix from basis \mathfrak{B} to basis \mathfrak{D} .

Solution 38 (Coordinates) Define

$$S = \begin{bmatrix} -3 & 1 & 5 \\ 2 & -1 & 4 \\ -3 & -1 & 9 \end{bmatrix}, \quad and \quad R = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 1 \\ 1 & -1 & 3 \end{bmatrix}$$

1. The change of basis matrix from basis \mathfrak{B} to basis E is given by S.

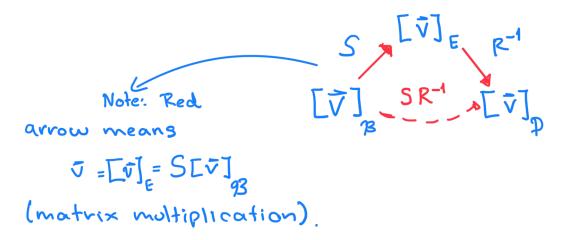
2. The change of basis matrix from basis E to basis \mathfrak{D} is given by

$$R^{-1} = \frac{1}{8} \begin{bmatrix} 7 & 2 & -3 \\ -5 & 2 & 1 \\ -4 & 0 & 4 \end{bmatrix}$$

3. The change of basis matrix from basis \mathfrak{B} to basis \mathfrak{D} is given by

$$R^{-1}S = \frac{1}{8} \begin{bmatrix} 7 & 2 & -3 \\ -5 & 2 & 1 \\ -4 & 0 & 4 \end{bmatrix} \begin{bmatrix} -3 & 1 & 5 \\ 2 & -1 & 4 \\ -3 & -1 & 9 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 2 & -1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

What are the columns of $R^{-1}S$ in terms of coordinates?



Problem 39 (Coordinates for linear transformations). In this problem we will be working with the same bases of \mathbb{R}^3 as in Problem 38:

$$\mathfrak{B} = \left\{ \begin{bmatrix} -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 9 \end{bmatrix} \right\}, \quad E = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad \mathfrak{D} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

. Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\vec{v} = A\vec{v}$$

where

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Find the matrix $[T]_{\mathfrak{B}}^{\mathfrak{D}}$.

Remember that the matrix $[T]_{\mathfrak{B}}^{\mathfrak{D}}$ satisfies:

$$[T\vec{v}]_{\mathfrak{D}} = [T]_{\mathfrak{B}}^{\mathfrak{D}}[\vec{v}]_{\mathfrak{B}}.$$

for any $\vec{v} \in \mathbb{R}^3$.

Solution 39 (Coordinates for linear transformations)

Define

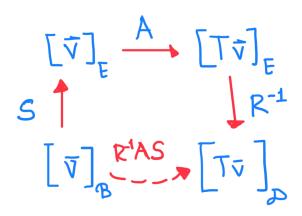
$$S = \begin{bmatrix} -3 & 1 & 5 \\ 2 & -1 & 4 \\ -3 & -1 & 9 \end{bmatrix}, \quad and \quad R = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 1 \\ 1 & -1 & 3 \end{bmatrix}$$

Take any $\vec{v} \in \mathbb{R}^3$.

To find $[T]_{\mathfrak{B}}^{\mathfrak{D}}$ take the following steps:

- 1. Since we start with $[\vec{v}]_{\mathfrak{B}}$, we first need to change to standard basis coordinates: $[\vec{v}]_E = S[\vec{v}]_{\mathfrak{B}}$.
- 2. We know A is the matrix representation of T in the standard basis coordinates. Thus $[T\vec{v}]_E = AS[\vec{v}]_{\mathfrak{B}}$.

3. Finally, we need to transform vector $AS\vec{v}$ from standard coordinates to \mathfrak{D} coordinates. That is $[T\vec{v}]_{\mathfrak{B}}=R^{-1}AS[\vec{v}]_{\mathfrak{B}}$. Therefore $[T]_{\mathfrak{B}}^{\mathfrak{D}}=R^{-1}AS$.



Problem 40 (Elementary matrices). Write the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 2 \\ -1 & 1 & 1 \end{bmatrix}$$

as a product of elementary row matrices. Use your expression to find A^{-1} .

Solution 40 (Elementary matrices) By Gauss-Jordan elimination, we have $E_5E_4E_3E_2E_1A=I$, where

$$E_1 = \left[egin{array}{cccc} 1 & 0 & 0 \ 0 & 1 & 0 \ 1 & 0 & 1 \end{array}
ight], \ E_2 = \left[egin{array}{cccc} 1 & 0 & 0 \ 0 & 1 & 2 \ 0 & 0 & 1 \end{array}
ight], \ E_3 = \left[egin{array}{cccc} 1 & 0 & 0 \ 0 & 0 & 1 \ 0 & 1 & 0 \end{array}
ight], \ E_4 = \left[egin{array}{cccc} 1 & 0 & 0 \ 0 & 1 & 0 \ 0 & 0 & 1/2 \end{array}
ight], \ E_5 = \left[egin{array}{cccc} 1 & 0 & 1 \ 0 & 1 & 0 \ 0 & 0 & 1 \end{array}
ight]. \ E_5 = \left[egin{array}{cccc} 0 & 1 & 0 \ 0 & 1 & 0 \ 0 & 0 & 1 \end{array}
ight].$$

 $A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1}$. We have $A^{-1} = E_5 E_4 E_3 E_2 E_1$.