6 Lecture hours 13-14

Definitions

Definition (Kernel (or null space) of a matrix)**.** The kernel (or null space) of a matrix A, written $ker(A)$, is the set of vectors $\vec{x}$ with $A\vec{x} = \vec{0}$. You can think of these as the vectors that go to $\vec{0}$ when multiplied by A.

Definition (Image (or column space) of a matrix)**.** The image (or column space) of a matrix A, written im(A), is the set of vectors $\vec{y}$ with $\vec{y} = A\vec{x}$ for some $\vec{x}$. These are all the possible "outputs" of the function, what we might have called the range in previous courses. It can also be thought of as all the possible values of $\vec{y}$ for which there does exist an input $A\vec{x} = \vec{y}$.

Definition (Kernel and image of a linear transformation)**.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation

- The kernel ker(T) is the set of vectors $\vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = 0$.
- The image of T is the set of all vectors $\vec{y} \in \mathbb{R}^m$ such that $T(\vec{x}) = \vec{y}$ for some $\vec{x} \in \mathbb{R}^n$.

Problem 26 (Cross product)**.** The cross product of two vectors $\vec{v} = \left[\begin{smallmatrix} v_1 \\ v_2 \\ v_3 \end{smallmatrix}\right]$, $\vec{w} = \left[\begin{smallmatrix} w_1 \\ w_2 \\ w_3 \end{smallmatrix}\right]$ in $\mathbb{R}^3$ is given by

$$\vec{v} \times \vec{w} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}.$$

a) Show using a dot product that $\vec{v} \times \vec{w}$ is perpendicular to $\vec{v}$.

b) Use a cross product to find the equation of the plane containing the vectors

$$\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{z} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}.$$

Solution 26 (Cross product)

a) From the geometric interpretation of the dot product, we need to show that $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$. We have

$$\begin{aligned} \vec{v} \cdot (\vec{v} \times \vec{w}) &= v_1(v_2 w_3 - v_3 w_2) + v_2(v_3 w_1 - v_1 w_3) + v_3(v_1 w_2 - v_2 w_1) \\ &= 0. \end{aligned}$$

b) The equation of the plane containing $\vec{y}$ and $\vec{z}$ vectors is $\vec{n} \cdot \vec{x} = 0$, where $\vec{n}$ is a vector perpendicular to $\vec{y}$ and $\vec{z}$.

From a) we know $\vec{y} \times \vec{z}$ is a vector perpendicular to $\vec{y}$ and $\vec{z}$. Thus

$$\vec{n} = \vec{y} \times \vec{z} = \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}.$$

The equation of the plane is given by

$$-2x + 4y + 2z = 0, \quad x, y, z \in \mathbb{R}.$$

Problem 27 (Kernel and Image)**.** Let $\vec{v} \neq \vec{0}$ be the vector $\vec{v} = \left[\begin{smallmatrix} v_1 \\ v_2 \\ v_3 \end{smallmatrix}\right]$. Define a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(\vec{x}) = \vec{v} \times \vec{x}.$$

a) What is the matrix A with $T(\vec{x}) = A\vec{x}$?

b) Find a vector $\vec{x} \neq \vec{0}$ in $\ker(T)$.

Solution 27 (Kernel and Image)

a)

$$A = [T(\vec{e}_1) \quad T(\vec{e}_2) \quad T(\vec{e}_3)] = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}.$$

b) Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be a vector in the kernel of T. Since $\vec{v} \neq \vec{0}$ we can suppose without loss of generality that $v_1 \neq 0$. [1]

From the definition of cross product we have that

$$T(\vec{x}) = \vec{v} \times \vec{x} = \begin{bmatrix} v_2x_3 - v_3x_2 \\ v_3x_1 - v_1x_3 \\ v_1x_2 - v_2x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then

$$x_2 = v_2(x_1/v_1), \quad x_3 = v_3(x_1/v_1).$$

Take, $x_1 = v_1$, then we can conclude that $\vec{v}$ is in the kernel of T.

[1]"Without loss of generality" means that we can use the same argument if we instead suppose $v_2 \neq 0$ or $v_3 \neq 0$

Problem 28 (Kernel and image). True or false? Justify your answer.

a) There is a 5×4 matrix with kernel of dimension 2.

b) The kernel of a 3×4 matrix has dimension 1.

c) If A is a 3×3 matrix with image of dimension 2, then one of

$$A(\vec{e}_1), \ A(\vec{e}_2), \ A(\vec{e}_3)$$

is the zero vector.

d) There is a 4×4 matrix A such that $\dim(\text{imA}) = \dim(\text{kerA})$.

Solution 28 (Kernel and image)

a) True, take the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

b) False, the zero matrix has kernel of dimension 4.

c) False, for example the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}.$$

d) True, take the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$