Lecture hours 30-32

14

This tutorial is a review for Unit 3.

Problem 56.

a) Find a 2×2 matrix such that $A^2 = A$ and compute its eigenvalues.

b) If A is an $n \times n$ matrix such that $A^2 = A$. What can you say about the eigenvalues of A?

Solution 56 a) Take

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then $A^2 = A$ and the eigenvalues are 0 and 1.

b) Suppose λ is an eigenvalue of A. That is, there exists $\vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$ such that

$$A\vec{x} = \lambda\vec{x}.$$

Observe that

$$A^2\vec{x} = A\vec{x} = \lambda\vec{x}.$$

But,

$$A^2\vec{x} = A(A\vec{x}) = A(\lambda\vec{x}) = \lambda^2\vec{x}.$$

Then $\lambda\vec{x} = \lambda^2\vec{x}$. Since $\vec{x} \neq \vec{0}$, it must be that $\lambda(\lambda - 1) = 0$. Thus λ is either 0 or 1.

Problem 57. Find:

a) A diagonalizable matrix that is not invertible.

b) An invertible matrix that is not diogonalizable.

c) A non invertible matrix that is not diagonalizable.

Solution 57 a) Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $det(A) = 0$, A is not invertible and it is already a diagonal matrix.

b) Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Since $det(A) \neq 0$, A is invertible. But it has one eigenvalue $\lambda = 1$ with algebraic multiplicity of 2 and geometric multiplicity of 1, so it cannot be diagonalizable.

c) Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Since $det(A) = 0$, A is not invertible. t has one eigenvalue $\lambda = 0$ with algebraic multiplicity of 2 and geometric multiplicity of 1, so it cannot be diagonalizable.

Problem 58. Let A be the 3×3 matrix given by

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}.$$

a) Find the eigenvalues of A.

b) Find a basis for each one of the eigenspaces of A.

c) Find a diagonal matrix D and an invertible matrix S such that A = SDS^{-1}.

d) For any $n \times n$ matrices X and Y it holds that

$$det(XY) = det(X)det(Y) \quad trace(XY) = trace(YX)$$

Can you use this fact to find det(A) and trance(A)? How?

Solution 58 a) The characteristic polynomial of A is given by $det(A - \lambda I) = (2 - \lambda)^2(1 - \lambda)$. Thus, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$.

b) Remember that the eigenspace of λ_1 is $E_{\lambda_1} = ker(A - \lambda_1 I)$. We can use Gaussian elimination to find a basis for the kernel of a matrix. In this case, a basis of E_{λ_1} is given by

$$\vec{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

and basis of E_{λ_2} is given by

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Since the geometric and algebraic multiplicities for each eigenvalue are equal, A is diagonalizable.

c) Let

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

and $S = [\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3]$.

d) By part c) we have

$$det(A) = det(SDS^{-1}) = det(S)det(D)det(S^{-1}) = det(D) = 4$$

and

$$trace(A) = trace(SDS^{-1}) = trace(DS^{-1}S) = trace(D) = 5.$$