Math 141 Tutorial 5 Solutions

Main problems

1. Compute the following integrals using integration by parts (IBP)

(a) $\displaystyle\int_0^{\ln(2)} se^s \, \mathrm{d}s$

(b) $\displaystyle\int x\cosh(x)\, \mathrm{d}x$

(c) $\displaystyle\int_0^1 \arctan(x)\, \mathrm{d}x$

(d) $\displaystyle\int_1^e \ln(x^8)\, \mathrm{d}x$

(e) $\displaystyle\int x\sec^2 x\, \mathrm{d}x$

(f) $\displaystyle\int \arcsin(x)\, \mathrm{d}x$

(g) $\displaystyle\int \frac{\ln x}{x^2}\, \mathrm{d}x$

Solution:

(a) Here we take $u = s$ and $\mathrm{d}v = e^s\,\mathrm{d}s$. Then, $\mathrm{d}u = \mathrm{d}s$ and $v = e^s$. Thus,

$$\begin{aligned}\int_0^{\ln(2)} se^s\,\mathrm{d}s &= se^s\Big|_{s=0}^{s=\ln 2} - \int_0^{\ln 2} e^s\,\mathrm{d}s \\ &= e^{\ln 2}\ln 2 - \left(e^{\ln 2} - e^0\right) \\ &= 2\ln 2 - 2 + 1 \\ &= 2\ln 2 - 1.\end{aligned}$$

(b) Take $u = x$ and $\mathrm{d}v = \cosh x\,\mathrm{d}x$. Then, $\mathrm{d}u = \mathrm{d}x$ and $v = \sinh x$. Consequently,

$$\begin{aligned}\int x\cosh x\,\mathrm{d}x &= x\sinh x - \int \sinh x\,\mathrm{d}x \\ &= x\sinh x - \cosh x + C \\ &= x\sinh x - \cosh x + C.\end{aligned}$$

(c) As a first step, We choose $u = \arctan x$ and $\mathrm{d}v = \mathrm{d}x$. This gives

$$\mathrm{d}u = \frac{1}{1+x^2}\,\mathrm{d}x \quad \text{and} \quad v = x.$$

Thus, our given integral becomes

$$\begin{aligned}\int_0^1 \arctan(x)\,\mathrm{d}x &= x\arctan(x)\Big|_{x=0}^{x=1} - \int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x \\ &= \arctan(1) - \int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x \\ &= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x.\end{aligned}$$

To handle this second integral that has appeared, we will make the substitution $t = x^2+1$. Then, $t(0) = 1$ and $t(1) = 2$. Furthermore, $\mathrm{d}t = 2x\,\mathrm{d}x$ so that

$$\begin{aligned}\int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x &= \int_1^2 \frac{\mathrm{d}u/2}{u} \\ &= \frac{1}{2}\int_1^2 \frac{\mathrm{d}u}{u} \\ &= \frac{1}{2}\left(\ln u\Big|_{u=1}^{u=2}\right) \\ &= \frac{\ln 2}{2}.\end{aligned}$$

So, to summarize:

$$\begin{aligned}\int_0^1 \arctan(x)\,\mathrm{d}x &= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x \\ &= \frac{\pi}{4} - \frac{\ln 2}{2}.\end{aligned}$$

(d) Notice that $\ln x^8 = 8\ln x$. Thus,

$$\begin{aligned}\int_1^e \ln(x^8)\,\mathrm{d}x &= 8\int_1^e \ln x\,\mathrm{d}x \\ &= 8\left[x\ln x\Big|_{x=1}^{x=e} - \int_1^e \frac{x}{x}\,\mathrm{d}x\right] \\ &= 8\left[e\ln e - 1\ln 1 - \int_1^e \mathrm{d}x\right] \\ &= 8\left[e\ln e - e + 1\right] \\ &= 8.\end{aligned}$$

(e) We proceed using IBP. Take $u := x$ and $\mathrm{d}v = \sec^2 x\,\mathrm{d}x$. Then, $\mathrm{d}u = \mathrm{d}x$ and $v = \tan x$. Thus,

$$\begin{aligned}\int x\sec^2 x\,\mathrm{d}x &= x\tan x - \int \tan x\,\mathrm{d}x \\ &= x\tan x - \ln|\sec x| + C.\end{aligned}$$

(f) We proceed via IBP. First, choose $u := \arcsin x$ and $\mathrm{d}v = \mathrm{d}x$. This gives

$$\mathrm{d}u = \frac{\mathrm{d}x}{\sqrt{1-x^2}} \quad \text{and} \quad v = x.$$

Consequently,

$$\int \arcsin(x)\,\mathrm{d}x = x \arcsin x - \int \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}}.$$

Now, the integral $-\int \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}}$ is handled with the substitution $t = 1 - x^2$ (which has $\mathrm{d}t = -2x\,\mathrm{d}x$):

$$\begin{aligned}\int \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}} &= -\frac{1}{2}\int \frac{\mathrm{d}x}{\sqrt{u}} \\ &= -\sqrt{u} + C \\ &= -\sqrt{1-x^2} + C.\end{aligned}$$

To summarize, we have found that

$$\begin{aligned}\int \arcsin(x)\,\mathrm{d}x &= x \arcsin x - \int \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}} \\ &= x \arcsin x + \sqrt{1-x^2} - C \\ &= x \arcsin x + \sqrt{1-x^2} + C',\end{aligned}$$

where C' also denotes an arbitrary constant.

(g) Here, we instead proceed using IBP. Take $u = \ln x$ and $\mathrm{d}v = \frac{1}{x^2}\,\mathrm{d}x$. This gives

$$\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x \quad \text{and} \quad v = -\frac{1}{x}.$$

Thus,

$$\begin{aligned}\int \frac{\ln x}{x^2}\,\mathrm{d}x &= -\frac{\ln x}{x} - \int \frac{1}{x}\left(-\frac{1}{x}\right)\mathrm{d}x \\ &= -\frac{\ln x}{x} + \int \frac{1}{x^2}\,\mathrm{d}x \\ &= -\frac{\ln x}{x} - \frac{1}{x} + C \\ &= -\frac{1+\ln x}{x} + C.\end{aligned}$$

2. (a) Using integration by parts, prove the following reduction formula:

$$\int (\ln x)^n \; \mathrm{d}x = x\,(\ln x)^n - n \int (\ln x)^{n-1} \; \mathrm{d}x.$$

(b) Using your result from (a), determine

$$\int_1^e (\ln x)^3 \; \mathrm{d}x.$$

Solution:

(a) Taking $u = (\ln x)^n$ and $\mathrm{d}v = \mathrm{d}x$ gives us that

$$\mathrm{d}u = \frac{n(\ln x)^{n-1}}{x}\,\mathrm{d}x$$

and we choose $v = x$. Therefore, an integration by parts gives

$$\begin{aligned} \int (\ln x)^n \; \mathrm{d}x &= x\,(\ln x)^n - \int x \cdot \frac{n(\ln x)^{n-1}}{x}\,\mathrm{d}x \\ &= x\,(\ln x)^n - n \int (\ln x)^{n-1}\,\mathrm{d}x. \end{aligned}$$

(b) We use the reduction formula from (a) twice:

$$\begin{aligned} \int (\ln x)^3 \; \mathrm{d}x &= x\,(\ln x)^3 - 3 \int (\ln x)^2 \; \mathrm{d}x \\ &= x\,(\ln x)^3 - 3x\,(\ln x)^2 + 6 \int \ln x \,\mathrm{d}x \\ &= x\,(\ln x)^3 - 3x\,(\ln x)^2 + 6x \ln x - 6x + C. \end{aligned}$$

Then, we use this antiderivative to evaluate

$$\begin{aligned} \int_1^e (\ln x)^3 \; \mathrm{d}x &= x\,(\ln x)^3 - 3x\,(\ln x)^2 + 6x \ln x - 6x \Big|_1^e \\ &= (e - 3e + 6e - 6e) - (0 - 0 + 0 - 6) \\ &= -2e + 6 \\ &= 2(3 - e). \end{aligned}$$

3. Compute the following trigonometric integrals.

(a) $\displaystyle\int_0^{\pi/2} \sin^8(x)\cos^5(x)\,\mathrm{d}x$

(b) $\displaystyle\int \sin^5(x)\,\mathrm{d}x$

(c) $\displaystyle\int_{-\pi/4}^{0} \tan^3(x)\sec^4(x)\,\mathrm{d}x$

(d) $\displaystyle\int \sin^2(x)\cos^4(x)\,\mathrm{d}x$

(e) $\displaystyle\int \tan^3(x)\sec(x)\,\mathrm{d}x$

(f) $\displaystyle\int_0^{\pi/10} \cos^4(5x)\,\mathrm{d}x$

Solution:

(a) With the substitution $u := \sin(x)$, we obtain

$$\begin{aligned}\int_0^{\pi/2} \sin^8(x)\cos^5(x)\,\mathrm{d}x &= \int_0^{\pi/2} \sin^8(x)\left(\cos^2(x)\right)^2\cos(x)\,\mathrm{d}x\\ &= \int_0^{\pi/2} \sin^8(x)\left(1-\sin^2(x)\right)^2 \underbrace{\cos(x)\,\mathrm{d}x}_{\mathrm{d}u}\\ &= \int_0^1 u^8(1-u^2)^2\,\mathrm{d}u\end{aligned}$$

Then, we can evaluate this last integral after distributing:

$$\int_0^1 u^8(1-u^2)^2\,\mathrm{d}u = \int_0^1 (u^8 - 2u^{10} + u^{12})\,\mathrm{d}u = \frac{1}{9} - \frac{2}{11} + \frac{1}{13} = \frac{8}{1287}.$$

(b) We wish to use the substitution $u := \cos(x)$. To this end, we write

$$\int \sin^5(x)\,\mathrm{d}x = \int \left(\sin^2(x)\right)^2\sin(x)\,\mathrm{d}x = \int \left(1-\cos^2(x)\right)^2 \underbrace{\sin(x)\,\mathrm{d}x}_{-\mathrm{d}u}$$

Substituting then yields

$$\begin{aligned}\int \sin^5(x)\,\mathrm{d}x = -\int (1-u^2)^2\,\mathrm{d}u &= -\int \left(1-2u^2+u^4\right)\,\mathrm{d}u\\ &= -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + C\\ &= -\left(\cos(x) - \frac{2}{3}\cos^3(x) + \frac{1}{5}\cos^5(x)\right) + C\end{aligned}$$

(c) In order to utilize the substitution $u := \tan(x)$, we write

$$\begin{aligned}\int_{-\pi/4}^{0} \tan^3(x)\sec^4(x)\,\mathrm{d}x &= \int_{-\pi/4}^{0} \tan^3(x)\sec^2(x)\sec^2(x)\,\mathrm{d}x\\ &= \int_{-\pi/4}^{0} \tan^3(x)(1+\tan^2(x))\underbrace{\sec^2(x)\,\mathrm{d}x}_{\mathrm{d}u}\end{aligned}$$

We can now substitute to obtain

$$\begin{aligned}\int_{-\pi/4}^{0} \tan^3(x)\sec^4(x)\,\mathrm{d}x = \int_{-1}^{0} u^3(1+u^2)\mathrm{d}u &= \int_{-1}^{0} \left(u^3+u^5\right)\mathrm{d}u \\ &= -\frac{1}{4}-\frac{1}{6} = -\frac{5}{12}\end{aligned}$$

(d) We begin by using trigonometric identities to simplify the integrand. We have

$$\begin{aligned}\sin^2(x)\cos^4(x) &= (\sin(x)\cos(x))^2\cos^2(x) \\ &= \left(\frac{1}{2}\sin(2x)\right)^2\left(\frac{1+\cos(2x)}{2}\right) \\ &= \frac{1}{8}\sin^2(2x)\left(1+\cos(2x)\right) \\ &= \frac{1}{8}\left(\frac{1-\cos(4x)}{2}\right)\left(1+\cos(2x)\right) \\ &= \frac{1}{16}\left(1+\cos(2x)-\cos(4x)-\cos(4x)\cos(2x)\right) \\ &= \frac{1}{16}\left(1+\cos(2x)-\cos(4x)-\frac{\cos(2x)+\cos(6x)}{2}\right).\end{aligned}$$

Hence

$$\begin{aligned}\int \sin^2(x)\cos^4(x)\,\mathrm{d}x &= \frac{1}{16}\int\left(1+\cos(2x)-\cos(4x)-\frac{\cos(2x)+\cos(6x)}{2}\right)\mathrm{d}x \\ &= \frac{1}{32}\int\left(2+\cos(2x)-2\cos(4x)-\cos(6x)\right)\mathrm{d}x \\ &= \frac{1}{32}\left(2x+\frac{1}{2}\sin(2x)-\frac{1}{2}\sin(4x)-\frac{1}{6}\sin(6x)\right)+C \\ &= \frac{1}{192}\left(12x+3\sin(2x)-3\sin(4x)-\sin(6x)\right)\end{aligned}$$

(e) With the substitution $u := \sec(x)$, we obtain

$$\begin{aligned}\int \tan^3(x)\sec(x)\,\mathrm{d}x &= \int \tan^2(x)\tan(x)\sec(x)\,\mathrm{d}x \\ &= \int (\sec^2 - 1)\underbrace{\tan(x)\sec(x)\,\mathrm{d}x}_{\mathrm{d}u} \\ &= \int \left(u^2-1\right)\mathrm{d}u \\ &= \frac{1}{3}u^3 - u + C \\ &= \frac{1}{3}\sec^3(x)-\sec(x)+C\end{aligned}$$

(f) We begin with the simple substitution $u = 5x$ to simplify the integral:

$$\int_0^{\pi/10} \cos^4(5x)\,\mathrm{d}x = \frac{1}{5}\int_0^{\pi/2} \cos^4(u)\,\mathrm{d}u.$$

Then, we use trigonometric identities to simplify the integrand:

$$\begin{aligned}
\cos^4(u) = \left(\cos^2(u)\right)^2 &= \left(\frac{\cos(2u)+1}{2}\right)^2 \\
&= \frac{1}{4}\left(\cos^2(2u) + 2\cos(2u) + 1\right) \\
&= \frac{1}{4}\left(\frac{\cos(4u)+1}{2} + 2\cos(2u) + 1\right) \\
&= \frac{1}{8}\left(\cos(4u) + 4\cos(2u) + 3\right).
\end{aligned}$$

We conclude that

$$\begin{aligned}
\int_0^{\pi/10} \cos^4(5x)\,\mathrm{d}x = \frac{1}{5}\int_0^{\pi/2} \cos^4(u)\,\mathrm{d}u &= \frac{1}{40}\int_0^{\pi/2} \left(\cos(4u) + 4\cos(2u) + 3\right)\,\mathrm{d}u \\
&= \frac{1}{40}\left[\frac{1}{4}\sin(4u) + 2\sin(2u) + 3u\right]_0^{\pi/2} \\
&= \frac{1}{40}\left[3 \cdot \frac{\pi}{2}\right] = \frac{3\pi}{80}.
\end{aligned}$$

4. Compute the integrals below using Trigonometric Substitution

(a) $\displaystyle\int \frac{\sqrt{2-x^2}}{x^2}\,\mathrm{d}x$

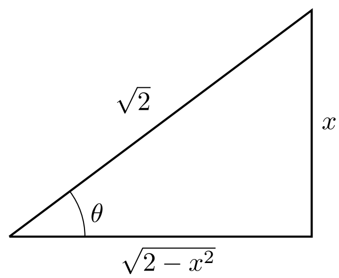
(b) $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,\mathrm{d}x$

(c) $\displaystyle\int \sqrt{7+6x-x^2}\,\mathrm{d}x$

(d) $\displaystyle\int_0^a x^2\sqrt{a^2-x^2}\,\mathrm{d}x$

Solution:

(a) The trigonometric substitution we will use is:



$$
\begin{aligned}
\sqrt{2}\sin\theta &= x\\
\sqrt{2}\cos\theta\,\mathrm{d}\theta &= \mathrm{d}x\\
\sqrt{2}\cos\theta &= \sqrt{2-x^2}\\
\cot\theta &= \frac{\sqrt{2-x^2}}{x}
\end{aligned}
$$

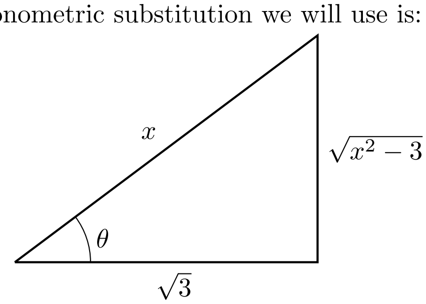
With this, we have

$$
\begin{aligned}
\int \frac{\sqrt{2-x^2}}{x^2}\,\mathrm{d}x &= \int \frac{\sqrt{2}\cos\theta}{\left(\sqrt{2}\sin\theta\right)^2}\sqrt{2}\cos\theta\,\mathrm{d}\theta\\
&= \int \frac{\cos^2\theta}{\sin^2\theta}\,\mathrm{d}\theta\\
&= \int \frac{1-\sin^2\theta}{\sin^2\theta}\,\mathrm{d}\theta\\
&= \int \left(\csc^2\theta - 1\right)\,\mathrm{d}\theta\\
&= -\cot\theta - \theta + C\\
&= -\frac{\sqrt{2-x^2}}{x} - \arcsin\left(\frac{x}{\sqrt{2}}\right) + C.
\end{aligned}
$$

Note: if we instead use the substitution $x = \sqrt{2}\cos\theta$, then we obtain an equivalent solution:

$$\int \frac{\sqrt{2-x^2}}{x^2}\,\mathrm{d}x = -\frac{\sqrt{2-x^2}}{x} + \arccos\left(\frac{x}{\sqrt{2}}\right) + \widetilde{C}.$$

(b) The trigonometric substitution we will use is:



$$
\begin{aligned}
\sqrt{3}\sec\theta &= x\\
\sqrt{3}\sec\theta\tan\theta\,\mathrm{d}\theta &= \mathrm{d}x\\
\sqrt{3}\tan\theta &= \sqrt{x^2-3}
\end{aligned}
$$

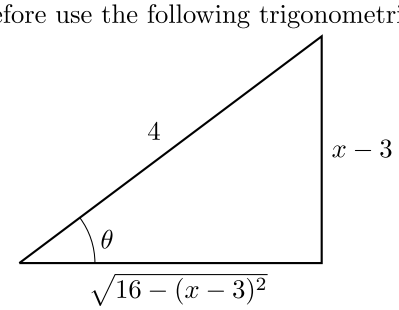
When $x = \sqrt{3}$ we have $\sec\theta = 1$ so $\theta = \operatorname{arcsec}(1) = 0$. Similarly, when $x = 2$ we have $\sec\theta = \frac{2}{\sqrt{3}}$ so $\theta = \pi/6$. With this, we have

$$\begin{aligned}
\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,\mathrm{d}x &= \int_0^{\pi/6} \frac{\sqrt{3}\tan\theta}{\sqrt{3}\sec\theta}\sqrt{3}\sec\theta\tan\theta\,\mathrm{d}\theta \\
&= \sqrt{3}\int_0^{\pi/6} \tan^2\theta\,\mathrm{d}\theta \\
&= \sqrt{3}\int_0^{\pi/6} \left(\sec^2\theta - 1\right)\,\mathrm{d}\theta \\
&= \sqrt{3}\Big[\tan\theta - \theta\Big]_0^{\pi/6} \\
&= \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right) = 1 - \frac{\sqrt{3}\pi}{6}.
\end{aligned}$$

(c) As a first step, we observe that

$$\sqrt{7+6x-x^2} = \sqrt{16-(x-3)^2}$$

We therefore use the following trigonometric substitution:

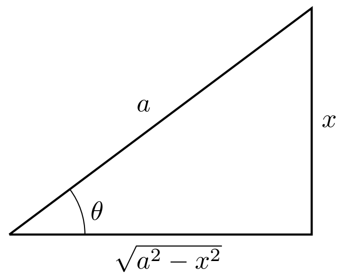


$$\begin{aligned}
4\sin\theta &= x-3 \\
4\cos\theta\,\mathrm{d}\theta &= \mathrm{d}x \\
4\cos\theta &= \sqrt{16-(x-3)^2}
\end{aligned}$$

We then have

$$\begin{aligned}
\int \sqrt{7+6x-x^2}\,\mathrm{d}x &= \int \sqrt{16-(x-3)^2}\,\mathrm{d}x \\
&= \int 4^2\cos^2\theta\,\mathrm{d}\theta \\
&= 8\int \left(1+\cos(2\theta)\right)\,\mathrm{d}\theta \\
&= 8\left(\theta + \frac{1}{2}\sin(2\theta)\right) \\
&= 8\left(\arcsin\left(\frac{x-3}{4}\right) + \sin(\theta)\cos(\theta)\right) \\
&= 8\left(\arcsin\left(\frac{x-3}{4}\right) + \frac{(x-3)\sqrt{16-(x-3)^2}}{4^2}\right) \\
&= 8\arcsin\left(\frac{x-3}{4}\right) + \frac{1}{2}(x-3)\sqrt{7+6x-x^2}
\end{aligned}$$

(d) For this last problem, we use the following trigonometric substitution:



$$a \sin\theta = x$$
$$a\cos\theta \, \mathrm{d}\theta = \mathrm{d}x$$
$$a\cos\theta = \sqrt{a^2 - x^2}$$

With this,

$$\begin{aligned}
\int_0^a x^2 \sqrt{a^2 - x^2} \, \mathrm{d}x &= a^4 \int_{\arcsin(0)}^{\arcsin(1)} \sin^2\theta \cos^2\theta \, \mathrm{d}\theta \\
&= a^4 \int_0^{\pi/2} \left(\frac{\sin(2x)}{2} \right)^2 \mathrm{d}\theta \\
&= \frac{a^4}{4} \int_0^{\pi/2} \frac{1 - \cos(4x)}{2} \, \mathrm{d}\theta \\
&= \frac{a^4}{8} \left[\theta - \frac{1}{4} \sin(4x) \right]_0^{\pi/2} \\
&= \frac{a^4}{16} \pi.
\end{aligned}$$

Practice Problems

5. Evaluate the following integrals using a method of your choice.

(a) $\displaystyle\int x\sec^2(x)\,\mathrm{d}x$

(b) $\displaystyle\int_0^{\sqrt{\pi}} x^3\cos(x^2)\,\mathrm{d}x$

(c) $\displaystyle\int x\sin^3(x)\cos^3(x)\,\mathrm{d}x$

(d) $\displaystyle\int \sin(ax)\cos(bx)\,\mathrm{d}x, (a, b \neq 0, a \neq \pm b)$

(e) $\displaystyle\int_0^1 \frac{x}{x^4+1}\,\mathrm{d}x$

(f) $\displaystyle\int_1^e \frac{\ln x}{x}\,\mathrm{d}x$

(g) $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\,\mathrm{d}x$

(h) $\displaystyle\int \frac{-3x}{\sqrt{x^2-16}}\,\mathrm{d}x$

(i) $\displaystyle\int_0^{3/10} \frac{x^2}{\sqrt{9-25x^2}}\,\mathrm{d}x$

(j) $\displaystyle\int \frac{4x^5}{(2x^2-3)^{\frac{3}{2}}}\,\mathrm{d}x$

(k) $\displaystyle\int_0^{\pi/3} \frac{\sin(t)\cos(t)}{\sqrt{1+\cos^2(t)}}\,\mathrm{d}t$

(l) $\displaystyle\int \tan^2(x)\,\mathrm{d}x$

(m) $\displaystyle\int \frac{\sin^2\left(\frac{1}{x}\right)}{x^2}\,\mathrm{d}x$

(n) $\displaystyle\int \left(\frac{\ln x}{x}\right)^2\,\mathrm{d}x$

Solution:

(a) We begin by integrating by parts with $u = x$ and $v = \tan(x)$ so that $\mathrm{d}u = \mathrm{d}x$ and $\mathrm{d}v = \sec^2(x)\mathrm{d}x$. This yields

$$\int x\sec^2(x)\,\mathrm{d}x = x\tan(x) - \int \tan(x)\,\mathrm{d}x = x\tan(x) - \ln|\sec x| + C.$$

Alternatively, we can write

$$\int x\sec^2(x)\,\mathrm{d}x = x\tan(x) + \ln|\cos x| + C.$$

(b) We begin with the substitution $z = x^2$ to simplify our integral:

$$\int_0^{\sqrt{\pi}} x^3\cos(x^2)\,\mathrm{d}x = \frac{1}{2}\int_0^{\pi} z\cos(z)\,\mathrm{d}z.$$

Integrating by parts with $u = z$ and $v = \sin(z)$ so that $\mathrm{d}u = \mathrm{d}z$ and $\mathrm{d}v = \cos(z)\mathrm{d}z$ then yields

$$\frac{1}{2}\int_0^{\pi} z\cos(z)\,\mathrm{d}z = \frac{1}{2}\left(\underbrace{z\sin(z)\Big|_0^{\pi}}_{0} - \int_0^{\pi}\sin(z)\,\mathrm{d}z\right) = \frac{1}{2}\cos(z)\Big|_0^{\pi} = -1.$$

(c) We begin by using trigonometric identities:

$$
\begin{aligned}
\sin^3(x)\cos^3(x) = (\sin(x)\cos(x))^3 &= \left(\frac{\sin(2x)}{2}\right)^3 \\
&= \frac{1}{8}\sin(2x)\sin^2(2x) \\
&= \frac{1}{8}\sin(2x)\frac{1-\cos(4x)}{2} \\
&= \frac{1}{16}\left(\sin(2x) - \sin(2x)\cos(4x)\right) \\
&= \frac{1}{16}\left(\sin(2x) - \frac{\sin(6x)-\sin(2x)}{2}\right) \\
&= \frac{1}{32}\left(3\sin(2x) - \sin(6x)\right)
\end{aligned}
$$

Thus,

$$\int x\sin^3(x)\cos^3(x)\,\mathrm{d}x = \frac{3}{32}\int x\sin(2x)\,\mathrm{d}x - \frac{1}{32}\int x\sin(6x)\,\mathrm{d}x.$$

Now, we can integrate each term with integration by parts. Indeed,

$$
\begin{aligned}
\int \underbrace{x}_{u}\underbrace{\sin(2x)\,\mathrm{d}x}_{\mathrm{d}v} &= -\frac{1}{2}x\cos(2x) + \frac{1}{2}\int \cos(2x)\,\mathrm{d}x \\
&= -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + C_1.
\end{aligned}
$$

Similarly, we find

$$
\begin{aligned}
\int \underbrace{x}_{u}\underbrace{\sin(6x)\,\mathrm{d}x}_{\mathrm{d}v} &= -\frac{1}{6}x\cos(6x) + \frac{1}{6}\int \cos(6x)\,\mathrm{d}x \\
&= -\frac{1}{6}x\cos(6x) + \frac{1}{36}\sin(6x) + C_2.
\end{aligned}
$$

Combining these results, we conclude that

$$
\begin{aligned}
&\int x\sin^3(x)\cos^3(x)\,\mathrm{d}x \\
&= \frac{3}{32}\left(-\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x)\right) - \frac{1}{32}\left(-\frac{1}{6}x\cos(6x) + \frac{1}{36}\sin(6x)\right) + C \\
&= \frac{27\sin(2x) - \sin(6x) - 54x\cos(2x) + 6x\cos(6x)}{1152} + C
\end{aligned}
$$

(d) This problem can be solved using sin and cos product identities. However, we will use integration by parts:

$$\int \underbrace{\sin(ax)}_{u}\underbrace{\cos(bx)\,\mathrm{d}x}_{\mathrm{d}v} = \frac{1}{b}\sin(ax)\sin(bx) - \frac{a}{b}\int \cos(ax)\sin(bx)\mathrm{d}x.$$

Similarly, integrating the second term by parts yields

$$\int \underbrace{\cos(ax)}_{u} \underbrace{\sin(bx)\,\mathrm{d}x}_{\mathrm{d}v} = -\frac{1}{b}\cos(ax)\cos(bx) - \frac{a}{b}\int \sin(ax)\cos(bx)\mathrm{d}x.$$

Plugging this into our initial inequality we find that

$$\begin{aligned}
&\int \sin(ax)\cos(bx)\,\mathrm{d}x \\
&= \frac{1}{b}\sin(ax)\sin(bx) - \frac{a}{b}\left(-\frac{1}{b}\cos(ax)\cos(bx) - \frac{a}{b}\int \sin(ax)\cos(bx)\mathrm{d}x\right) \\
&= \frac{1}{b}\sin(ax)\sin(bx) + \frac{a}{b^2}\cos(ax)\cos(bx) + \frac{a^2}{b^2}\int \sin(ax)\cos(bx)\mathrm{d}x.
\end{aligned}$$

Subtracting $\frac{a^2}{b^2}\int \sin(ax)\cos(bx)\mathrm{d}x$ on either side,

$$\left(1 - \frac{a^2}{b^2}\right)\int \sin(ax)\cos(bx)\,\mathrm{d}x = \frac{1}{b}\sin(ax)\sin(bx) + \frac{a}{b^2}\cos(ax)\cos(bx) + C.$$

Equivalently, we have

$$\frac{b^2 - a^2}{b^2}\int \sin(ax)\cos(bx)\,\mathrm{d}x = \frac{1}{b}\sin(ax)\sin(bx) + \frac{a}{b^2}\cos(ax)\cos(bx) + C.$$

Since $a \neq \pm b$, we can multiply either side by $\frac{b^2}{b^2 - a^2}$ to obtain our final result:

$$\int \sin(ax)\cos(bx)\,\mathrm{d}x = \frac{b}{b^2 - a^2}\sin(ax)\sin(bx) + \frac{a}{b^2 - a^2}\cos(ax)\cos(bx) + \widetilde{C}.$$

(e) We first rewrite the integral as follows:

$$\int_0^1 \frac{x}{x^4 + 1}\,\mathrm{d}x = \int_0^1 \frac{x}{(x^2)^2 + 1}\,\mathrm{d}x.$$

Now, we make the substitution $u = x^2$ with $\mathrm{d}u = 2x\,\mathrm{d}x$. Since $u(0) = 0$ and $u(1) = 1$, we have

$$\begin{aligned}
\int_0^1 \frac{x}{x^4 + 1}\,\mathrm{d}x &= \int_0^1 \frac{x}{(x^2)^2 + 1}\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 \frac{\mathrm{d}u}{u^2 + 1} \\
&= \frac{1}{2}\arctan u\Big|_0^1 \\
&= \frac{\arctan(1) - \arctan(0)}{2} \\
&= \frac{\pi}{8}.
\end{aligned}$$

(f) We use the substitution $u := \ln x$. This yields $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$ along with $u(1) = 0$ and $u(e) = 1$. Consequently,

$$\int_1^e \frac{\ln x}{x}\,\mathrm{d}x = \int_0^1 u\,\mathrm{d}u = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}.$$

(g) As a first step, observe that

$$\int \frac{1}{\sqrt{1-4x^2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{1-(2x)^2}}\,\mathrm{d}x.$$

Making the substitution $u = 2x$ with $\mathrm{d}u = 2\mathrm{d}x$ transforms this integral into the following recognizable form:

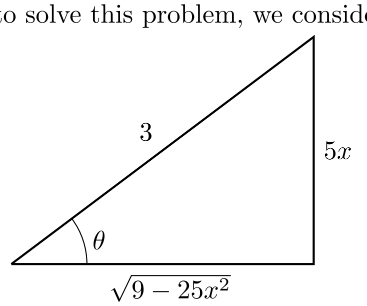
$$\begin{aligned}\int \frac{1}{\sqrt{1-4x^2}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{1-(2x)^2}}\,\mathrm{d}x\\ &= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u\\ &= \frac{1}{2}\arcsin u + C\\ &= \frac{1}{2}\arcsin(2x) + C.\end{aligned}$$

(h) Instead of solving this problem with trigonometric substitution, we consider instead the simpler substitution $u = x^2 - 16$. Then

$$\begin{aligned}\int \frac{-3x}{\sqrt{x^2-16}}\,\mathrm{d}x = \frac{1}{2}\int \frac{-3}{\sqrt{u}}\,\mathrm{d}u &= \frac{-3}{2}\int u^{-1/2}\,\mathrm{d}u\\ &= \frac{-3}{2}\frac{u^{1/2}}{1/2} + C = -3\sqrt{x^2-16} + C\end{aligned}$$

(i) *Note: before the correction, the upper bound was $3/5$ instead of $3/10$. However, since the integrand is not defined at $x = 3/5$ the fundamental theorem of calculus does not apply. You will see these types of integrals later on in the semester.*

In order to solve this problem, we consider the following trigonometric substitution:



$$\begin{aligned}&\frac{3}{5}\sin\theta = x\\ &\frac{3}{5}\cos\theta\,\mathrm{d}\theta = \mathrm{d}x\\ &3\cos\theta = \sqrt{9-25x^2}\end{aligned}$$

We then compute

$$\begin{aligned}
\int_0^{3/10} \frac{x^2}{\sqrt{9-25x^2}}\,\mathrm{d}x &= \int_{\arcsin(0)}^{\arcsin(1/2)} \frac{\left(\frac{3}{5}\sin\theta\right)^2}{3\cos\theta}\,\frac{3}{5}\cos\theta\,\mathrm{d}\theta \\
&= \frac{9}{125}\int_0^{\pi/6} \sin^2\theta\,\mathrm{d}\theta \\
&= \frac{9}{250}\int_0^{\pi/6} (1-\cos(2\theta))\,\mathrm{d}\theta \\
&= \frac{9}{250}\left[1-\frac{1}{2}\sin(2\theta)\right]_0^{\pi/6} \\
&= \frac{9}{250}\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right) = \frac{3}{1000}\left(2\pi-3\sqrt{3}\right)
\end{aligned}$$

(j) We consider the substitution $u = 2x^2-3$ so that $\mathrm{d}u = 4x\mathrm{d}x$. Then

$$\begin{aligned}
\int \frac{4x^5}{(2x^2-3)^{\frac{3}{2}}}\,\mathrm{d}x &= \int \frac{x^4}{(2x^2-3)^{\frac{3}{2}}}\,4x\mathrm{d}x \\
&= \int \frac{\frac{1}{4}(u+3)^2}{(u)^{\frac{3}{2}}}\,\mathrm{d}u \\
&= \frac{1}{4}\int \frac{u^2+6u+9}{(u)^{\frac{3}{2}}}\,\mathrm{d}u \\
&= \frac{1}{4}\int u^{1/2}+6u^{-1/2}+9u^{-3/2}\,\mathrm{d}u \\
&= \frac{1}{4}\left(\frac{2}{3}u^{3/2}+12u^{1/2}-18u^{-1/2}\right)+C \\
&= \frac{1}{6}\left((2x^2-3)^{3/2}+18(2x^2-3)^{1/2}-27(2x^2-3)^{-1/2}\right)+C
\end{aligned}$$

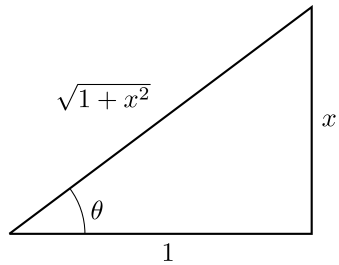
Multiplying and dividing by $\sqrt{2x^2-3}$, we obtain a nicer expression:

$$\begin{aligned}
\int \frac{4x^5}{(2x^2-3)^{\frac{3}{2}}}\,\mathrm{d}x &= \frac{1}{6}\left(\frac{(2x^2-3)^2+18(2x^2-3)-27}{\sqrt{2x^2-3}}\right)+C \\
&= \frac{1}{6}\left(\frac{4x^4+24x^2-72}{\sqrt{2x^2-3}}\right)+C \\
&= \frac{2}{3}\left(\frac{x^4+6x^2-18}{\sqrt{2x^2-3}}\right)+C
\end{aligned}$$

(k) We begin with the substitution $x = \cos(t)$:

$$\int_0^{\pi/3} \frac{\sin(t)\cos(t)}{\sqrt{1+\cos^2(t)}}\,\mathrm{d}t = -\int_1^{1/2} \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x = \int_{1/2}^{1} \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x.$$

We can then solve the integral with the substitution $u = 1 + x^2$. However, in order to provide another example with trigonometric substitution, we consider instead the substitution below.



$$\tan\theta = x$$
$$\sec^2\theta \, \mathrm{d}\theta = \mathrm{d}x$$
$$\sec\theta = \sqrt{1+x^2}$$

We then have

$$\begin{aligned}
\int_0^{\pi/3} \frac{\sin(t)\cos(t)}{\sqrt{1+\cos^2(t)}} \, \mathrm{d}t &= \int_{1/2}^{1} \frac{x}{\sqrt{1+x^2}} \, \mathrm{d}x \\
&= \int_{\arctan(1/2)}^{\arctan(1)} \frac{\tan\theta}{\sec\theta} \sec^2\theta \, \mathrm{d}\theta \\
&= \int_{\arctan(/1/2)}^{\pi/4} \sec\theta \tan\theta \, \mathrm{d}\theta \\
&= \sec\theta \Big|_{\arctan(/1/2)}^{\pi/4} \\
&= \sqrt{2} - \sec(\arctan(1/2)) \\
&= \sqrt{2} - \frac{\sqrt{5}}{2}.
\end{aligned}$$

To find $\sec(\arctan(1/2))$*, you can use trigonometric identities. For instance, noting that* $\sec^2(\arctan(1/2)) = \tan^2(\arctan(1/2)) + 1 = \left(\frac{1}{2}\right)^2 + 1 = \frac{5}{4}$ *we can deduce* $\sec(\arctan(1/2)) = \sqrt{5}/2$.

(l) Recall that $\tan^2(x) = \sec^2(x) - 1$ and

$$\frac{\mathrm{d}}{\mathrm{d}x} \tan(x) = \sec^2(x),$$

so that gives

$$\int \tan^2(x) dx = \int \sec^2(x) - 1 dx = \tan(x) - x + C.$$

(m) We proceed first by substitution, set $u := x^{-1}$, then $\mathrm{d}u = -x^{-2}$, which gives us

$$\int \frac{\sin^2\left(\frac{1}{x}\right)}{x^2} dx = -\int \sin^2(u) \mathrm{d}u.$$

We've already seen the integral of $\sin^2(u)$ in class, so using that result we end with

$$\int \frac{\sin^2\left(\frac{1}{x}\right)}{x^2} dx = -\int \sin^2(u) \mathrm{d}u = \frac{u}{2} - \frac{1}{4}\cos(2u) + C = \frac{1}{2x} - \frac{1}{4}\cos\left(\frac{2}{x}\right) + C.$$

(n) First we perform a substitution, then perform integration by parts twice. Start with setting $t := \ln(x)$, then $x = e^t$ and $\mathrm{d}t = x^{-1}\mathrm{d}x$, and

$$\int \left(\frac{\ln(x)}{x}\right)^2 dx = \int t^2 e^{-t} dt.$$

Integration by parts once, with $u = t^2$ and $\mathrm{d}v = e^{-t}\mathrm{d}t$ gives

$$\int t^2 e^{-t} dt = -t^2 e^{-t} + 2 \int t e^{-t} \mathrm{d}t.$$

Finally, the second integration parts, this time with $u = t$ and $\mathrm{d}v = e^{-t}\mathrm{d}t$ gives

$$\int t e^{-t} \mathrm{d}t = -t e^{-t} + \int e^{-t} \mathrm{d}t = -t e^{-t} - e^{-t}.$$

Putting it all together, we end up with

$$\int \left(\frac{\ln(x)}{x}\right)^2 dx = -t^2 e^{-t} + 2\left(-t e^{-t} - e^{-t}\right) = -\frac{1}{x}\left((\ln(x))^2 + 2\ln(x) + 1\right).$$

Challenge Problems

6. Prove that the following equation is correct for any continuously differentiable functions $f(x), g(x)$ and $h(x)$:

$$\int_a^b f'(x)g(x)h(x)\,dx = f(x)g(x)h(x)\Big|_a^b - \int_a^b f(x)g'(x)h(x)\,dx - \int_a^b f(x)g(x)h'(x)\,dx$$

7. Evaluate

$$\int_{-\pi}^{\pi} \arctan\left(\pi^x\right)\,\mathrm{d}x.$$

Hint: consider using the substitution $u := -x$. You might need the identity

$$\arctan(1/s) = \operatorname{arccot}(s) = \frac{\pi}{2} - \arctan(s),$$

where the first equality is valid for $s > 0$.

8. Compute the following integral with the appropriate method(s)

$$\int \frac{x\ln(x)}{\sqrt{x^2-1}}\,\mathrm{d}x$$

Hint: start with integration by parts.

9. (a) Using trigonometric substitution show that

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2+a^2}\right) + C.$$

(b) Use the hyperbolic substitution $x = a \cdot \sinh(t)$ to show that

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \operatorname{arcsinh}\left(\frac{x}{a}\right) + C$$

where $a > 0$ is a constant.

(c) Using part (a), provide an expression for $\operatorname{arcsinh}\left(\frac{x}{a}\right)$ in terms of the logarithm function.

Recall: $\cosh^2(\phi) = 1 + \sinh^2(\phi)$ and $\cosh(x) > 0$ for all $x \in \mathbb{R}$.