Lecture hours 19-20

9

Definitions and Theorems

Definition (Coordinates of a vector).

If we have a basis of $\mathbb{R}^n$, $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ and $\vec{x} \in \mathbb{R}^n$, so $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$, for some $c_1, \ldots c_n \in \mathbb{R}$. We define the coordinates of $\vec{x}$ in the basis $\mathcal{B}$ as

$$[\vec{x}]_{\mathcal{B}} \stackrel{\text{def}}{=} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Definition (Change of basis matrix).

Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis of $\mathbb{R}^n$. We call the matrix $S = [\vec{v}_1, \ldots, \vec{v}_n]$ the change of basis matrix of basis $\mathcal{B}$.

For any $\vec{x} \in \mathbb{R}^n$ we have

$$\vec{x} = S\,[\vec{x}]_{\mathcal{B}}\,.$$

So, if we have to find the coordinates $[\vec{x}]_{\mathcal{B}}$ we compute

$$[\vec{x}]_{\mathcal{B}} = S^{-1}\vec{x}.$$

Note that

- If $\mathcal{B} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ (the standard basis), we have $[\vec{x}]_{\mathcal{B}} = \vec{x}$.

Problem 37 (Coordinates)**.** Let $\mathfrak{B}$ be the basis of $\mathbb{R}^4$ given by

$$\mathfrak{B} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Find the coordinates for vector

$$\vec{v} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

in the basis $\mathfrak{B}$.

Problem 38 (Coordinates)**.** In this problem we will be working with the following bases of $\mathbb{R}^3$:

$$\mathfrak{B} = \left\{ \begin{bmatrix} -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 9 \end{bmatrix} \right\}, \quad E = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \mathfrak{D} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

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1. Find the change of basis matrix from basis $\mathfrak{B}$ to the standard basis E.
2. Find the change of basis matrix from the standard basis E to basis $\mathfrak{D}$.
3. Find the change of basis matrix from basis $\mathfrak{B}$ to basis $\mathfrak{D}$.

Problem 39 (Coordinates for linear transformations)**.** In this problem we will be working with the same bases of $\mathbb{R}^3$ as in Problem 38:

$$\mathfrak{B} = \left\{ \begin{bmatrix} -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 9 \end{bmatrix} \right\}, \quad E = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \mathfrak{D} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\vec{v} = A\vec{v}$$

where

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Find the matrix $[T]_{\mathfrak{B}}^{\mathfrak{D}}$.

Remember that the matrix $[T]_{\mathfrak{B}}^{\mathfrak{D}}$ satisfies:

$$[T\vec{v}]_{\mathfrak{D}} = [T]_{\mathfrak{B}}^{\mathfrak{D}}[\vec{v}]_{\mathfrak{B}}.$$

for any $\vec{v} \in \mathbb{R}^3$.

Problem 40 (Elementary matrices)**.** Write the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 2 \\ -1 & 1 & 1 \end{bmatrix}$$

as a product of elementary row matrices. Use your expression to find A^{-1}.