Math 141 Tutorial 11 Solutions

Main problems

1. Find the values of p for which these series are convergent.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ (b) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n[\ln(\ln n)]^p}$ (c) $\displaystyle\sum_{n=1}^{\infty} n(1+n^2)^p$

Solutions:

(a) We'll use the integral test to determine the values of p for which the series converges. Let

$$f(x) = \frac{1}{x(\ln x)^p},$$

then for any p we have that $f(x)$ is continuous and positive for $x \geq 2$. It remains to show that f is decreasing. Indeed, we have

$$f'(x) = \frac{-1}{(x(\ln x)^p)^2}\left((\ln x)^p + xp(\ln x)^{p-1}\frac{1}{x}\right) = -\frac{(\ln x)^p + p(\ln x)^{p-1}}{(x(\ln x)^p)^2} \leq 0$$

for all $x \geq 2$. Hence, we can apply the integral test for convergence. Using the substitution $u = \ln x$, we get

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\mathrm{d}x = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u^p}\mathrm{d}u.$$

We'll consider three cases:

(i) $p < 1$,
(ii) $p = 1$,
(iii) and $p > 1$.

Case (i): If $p < 1$, we get

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\mathrm{d}x = \lim_{b\to\infty} \frac{u^{1-p}}{1-p}\bigg|_{\ln 2}^{\ln b} = \lim_{b\to\infty} \frac{(\ln b)^{1-p} - (\ln 2)^{1-p}}{1-p} = \infty$$

since $1 - p > 0$, and so $(\ln b)^{1-p} \to \infty$.
Case (ii): If $p = 1$, we get

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\mathrm{d}x = \lim_{b\to\infty} \ln x\bigg|_{\ln 2}^{\ln b} = \lim_{b\to\infty} \ln(\ln b) - \ln(\ln 2) = \infty.$$

Case (iii): If $p > 1$, we get

$$\int_2^\infty \frac{1}{x(\ln x)^p} \mathrm{d}x = \lim_{b\to\infty} \frac{u^{1-p}}{1-p}\Big|_{\ln 2}^{\ln b} = \lim_{b\to\infty} \frac{(\ln b)^{1-p} - (\ln 2)^{1-p}}{1-p} = -\frac{(\ln 2)^{1-p}}{1-p} < \infty.$$

since $1 - p < 0$, and so $(\ln b)^{1-p} \to 0$. Hence the series converges for $p \in (1, \infty)$.

(b) We'll use a similar approach to (a). Let

$$f(x) = \frac{1}{x \ln x[\ln(\ln x)]^p},$$

then for any p we have that $f(x)$ is continuous and positive for $x \geq 2$. Moreover, looking at f' we see that

$$\begin{aligned} f'(x) &= -\frac{\ln x[\ln(\ln x)]^p + x\frac{1}{x}[\ln(\ln x)]^p + x \ln x p[\ln(\ln x)]^{p-1}\frac{1}{x \ln x}}{\left(x \ln x[\ln(\ln x)]^p\right)^2}, \\ &= -\frac{\ln x[\ln(\ln x)]^p + [\ln(\ln x)]^p + p[\ln(\ln x)]^{p-1}}{\left(x \ln x[\ln(\ln x)]^p\right)^2}, \\ &\leq 0 \end{aligned}$$

for all $x \geq 2$. So, once again we can apply the integral test for convergence and the substitution $u = \ln(\ln x)$ to obtain

$$\int_2^\infty \frac{1}{x \ln x[\ln(\ln x)]^p} \mathrm{d}x = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u^p} \mathrm{d}u,$$

which only converges for $p \in (1, \infty)$, by the exact same analysis as in question (a). Hence, the series converges for $p \in (1, \infty)$.

(c) In this problem, instead of directly trying to apply the integral test, first we show that for $p \geq -1/2$, the series diverges by the divergence test. We do this in two cases:

(i) $p > 0$,

(ii) and $p >= -1/2$.

The case $p \geq 0$, it should be clear that

$$\lim_{n\to\infty} n(1 + n^2)^p = \infty,$$

so we'll focus on case (ii). Suppose $0 > p > -1/2$, then we get

$$\begin{aligned} \lim_{n\to\infty} n(1 + n^2)^p &= \lim_{n\to\infty} \frac{n}{(1 + n^2)^{-p}}, \\ &= \lim_{n\to\infty} \frac{n}{n^{-2p}(\frac{1}{n^2} + 1)^{-p}}, \\ &= \lim_{n\to\infty} \frac{n^{1+2p}}{(\frac{1}{n^2} + 1)^{-p}}, \end{aligned}$$

but since $0 > p > -1/2$, it follows that $1 + 2p \geq 0$ and so

$$\lim_{n\to\infty} n(1+n^2)^p = \lim_{n\to\infty} \frac{n}{(1+n^2)^{-p}} = \lim_{n\to\infty} \frac{n^{1+2p}}{(\frac{1}{n^2}+1)^{-p}} = \begin{cases} 1 & \text{if } p = -1/2, \\ \infty & \text{if } 0 > p > -1/2. \end{cases}.$$

Either way, by the test for divergence we have that the series diverges.

Next, we show that the integral convergence test to investigate what happens for $p < -1/2$. Let $f(x) = x(1+x^2)^p$, then $f(x)$ is positive and continuous for $x \geq 1$. Once again, we look at f' to see whether f is decreasing. We have

$$f'(x) = (1+x^2)^p + xp(1+x^2)^{p-1}2x = (1+x^2)^p + 2px^2(1+x^2)^{p-1},$$

so to determine whether $f'(x) \leq 0$, after multiplying (remember $p < 0$ right now) both sides of the inequality by $(1+x^2)^{p-1}$ is equivalent to whether

$$1 + x^2 + 2px^2 = 1 + (1+2p)x^2 \leq 0.$$

But $1 + 2p < 0$, so we observe that for x sufficiently large we'll get $f'(x) \leq 0$. Therefore we can apply the integral test for convergence and the substitution $u = 1 + x^2$ to find

$$\int_1^\infty x(1+x^2)^p \mathrm{d}x = \lim_{b\to\infty} \int_2^{1+b^2} \frac{1}{2} u^p \mathrm{d}u.$$

For clarity, we'll write $q = -p > 0$ (this isn't a u-sub) so the above becomes

$$\int_1^\infty x(1+x^2)^p \mathrm{d}x = \lim_{b\to\infty} \int_2^{1+b^2} \frac{1}{2u^q} \mathrm{d}u,$$

and by the same analysis as before, we can conclude that this integral only converges for $q > 1$. Since $q = -p$, that means that the integral only converges for $p < -1$, and likewise the series in question only converges for $p \in (-\infty, -1)$.

2. Using the direct comparison test, determine whether the following series are convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{4+2^n}$

Solutions: (Disclaimer - these are not the only correct comparisons that could be used).

(a) We'll compare the terms of the series with

$$b_n = \frac{1}{2n^2}.$$

Since

$$\frac{n}{2n^3+1} \leq \frac{n}{2n^3} = \frac{1}{2n^2}$$

and $\sum_{n=1}^{\infty} b_n$ converges, the series converges.

(b) We'll compare with $b_n = \frac{1}{n^2}$ to see that the series converges. We have

$$\frac{\cos^2 n}{n^2+1} \leq \frac{1}{n^2+1} \leq \frac{1}{n^2},$$

so by the DCT, the series converges.

(c) Compare with $b_n = \frac{1}{n}$ to see that the series diverges. We have

$$\frac{2+(-1)^n}{n} \geq \frac{2+(-1)}{n} = \frac{1}{n},$$

so by the DCT, the series diverges.

(d) We'll compare the terms of the series with

$$b_n = \frac{1}{5}\left(\frac{3}{2}\right)^n.$$

Since

$$\frac{3^n}{4+2^n} \geq \frac{3^n}{4\cdot 2^n + 2^n} = \frac{3^n}{52^n},$$

and the series $\sum_{n=1}^{\infty} b_n$ diverges, the series diverges.

3. Using the limit comparison test, determine whether the following series are convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}e^n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

Solutions: (Disclaimer - these are not the only correct comparisons that could be used).

(a) Compare with $b_n = 2\frac{4^n}{6^n}$ to see that the series converges. We have

$$\lim_{n\to\infty} \frac{\left(\frac{n+4^n}{n+6^n}\right)}{\left(2\frac{4^n}{6^n}\right)} = \lim_{n\to\infty} \frac{n+4^n}{n+6^n}\frac{6^n}{2\cdot 4^n} = \lim_{n\to\infty} \frac{1}{2}\frac{\frac{n}{4^n}+1}{\frac{n}{6^n}+1}.$$

To tackle this limit, we'll use the following (much more general) result. For any $k \geq 0$ and $r > 1$,

$$\lim_{n\to\infty} \frac{n^k}{r^n} = 0,$$

which can be proven by applying L'Hôpital's rule k times (or $\lceil k \rceil$ times, if k isn't an integer). Armed with this fact, we get that

$$\lim_{n\to\infty} \frac{\left(\frac{n+4^n}{n+6^n}\right)}{\left(2\frac{4^n}{6^n}\right)} = \frac{1}{2},$$

so the series converges.

(b) Compare with $b_n = 2e^{-n}$ to see that the series converges.

(c) Compare with $b_n = e^{-n}$ to see that the seris converges.

(d) Compare with

$$b_n = \frac{1}{n(n-1)},$$

to see that the series converges. Moreover, we can compare b_n with $c_n = \frac{1}{n^2}$ to see that the series $\sum_{n=1}^{\infty} b_n$ converges.

4. Determine whether the following series are convergent or divergent.

(a) $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$

(b) $\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$

(c) $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$

(d) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

(e) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

(f) $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n}$

Solutions: (Disclaimer - these are not the only correct comparisons that could be used).

(a) Compare with $b_n = \frac{1}{\sqrt{n}}$ to see that the series diverges by the DCT/LCT.

(b) Compare with $b_n = \frac{1}{n}$ to see that the series diverges by the DCT/LCT.

(c) Compare with $b_n = \frac{1}{n^{7/3}}$ to see that the series converges by the DCT/LCT.

(d) We'll perform the test for divergence to see that the series diverges. Observe that we have

$$\frac{n^n}{n!} = \frac{n \cdots n}{n(n-1)\cdots 1} = \frac{n}{n}\frac{n}{(n-1)}\cdots\frac{n}{1} \geq 1$$

for all $n \geq 1$. Hence by taking limits on both sides, it follows that

$$\lim_{n\to\infty} \frac{n^n}{n!} \geq \lim_{n\to\infty} 1 = 1,$$

and by the test for divergence the series diverges.

(e) We'll compare with $b_n = \frac{1}{n}$ to see that the series diverges. Recall that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1,$$

so we get

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1,$$

so by the LCT the series diverges.

(f) Compare with $b_n = \frac{1}{n}$ to see that the series diverges by the DCT/LCT.

Practice Problems

1. Determine whether the following series are convergent or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^2}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(\ln n)^n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{5^n}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n + 4^n}$

   (f) $\displaystyle\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$

   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{n \sin n}{n^2 + 1}$

   (h) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(2n+7)^3}$