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Lecture hours 8-10

Definition (Linear Transformations)**.** We define a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ as a function with two properties:

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ (we say T preserves vector addition)

2. $T(\vec{x}) = cT(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$ and $c \in \mathbb{R}$ (we say T preserves scalar multiplication)

Definition (Matrix Multiplication – the column view)**.** Suppose now that A is an $m \times n$ matrix (so A has m rows and n columns, it might be a coefficient matrix for a system of m equations and n unknowns). Also, let $\vec{x}$ be a vector with n entries. We define the vector $A\vec{x}$ as follows: If $\vec{c}_k$ is the kth column of A, then

$$A\vec{x} = \begin{bmatrix} | & | & & | \\ \vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_3 \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_n\vec{c}_n.$$

That is, the product of a matrix and a vector is a linear combination of the columns of the matrix.

Problem 16 (Basis for a subspace)**.** Suppose that V is the set of all vectors $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$ such that

$$\begin{cases} x_1 + 2x_2 - 2x_3 + 2x_4 - x_5 = 0, \\ x_1 + 2x_2 - x_3 + 3x_4 - 2x_5 = 0, \\ 2x_1 + 4x_2 - 7x_3 + x_4 + x_5 = 0. \end{cases} \tag{4.1}$$

a) Explain why V is a subspace of $\mathbb{R}^5$.

b) Find a basis for V.

Problem 17 (Linear Transformations)**.** Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^l$ are two linear transformations. Define a transformation $F : \mathbb{R}^n \to \mathbb{R}^l$ by

$$F(\vec{v}) = S\left(T\left(\vec{v}\right)\right).$$

a) Explain why F is a linear transformation.

Now suppose that $n = 3$, $m = 2$, and $l = 3$, and T is induced by the matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 2 \end{bmatrix},$$

and S is induced by the matrix

$$\begin{bmatrix} -2 & 0 \\ 0 & 1 \\ 1 & 3 \end{bmatrix}.$$

b) Find $F(\vec{x})$ for any $\vec{x} \in \mathbb{R}^3$.

Hint : Write $F(\vec{x})$ in terms of $F\left(\vec{e}_1\right), F\left(\vec{e}_2\right)$ and $F\left(\vec{e}_3\right)$.

Problem 18 (Linear Transformations)**.** Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. In this case, the domain and codomain are the same, so we can repeat T. The linear transformation obtained by repeating T k times is denoted T^k.

a) Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with T not equal to the zero transformation[1], but T^2 equal to the zero transformation.

b) Find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ with neither T nor T^2 equal to the zero transformation, but T^3 equal to the zero transformation.

Problem 19 (Geometrical Linear Transformations)**.** In each part below, you are given a matrix A. Describe the effect of the linear transformation $T(\vec{x}) = A\vec{x}$ in words.

a) $A = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$.

b) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, for some $\theta \in \mathbb{R}$.

c) $A = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$, for some $\theta \in \mathbb{R}$.

[1]the *zero transformation* is the linear transformation that sends every vector to the zero vector in the codomain

Problem 20 (Geometrical Linear Transformations). Find all possible values of $\lambda \in \mathbb{R}$ for which there is a non-zero vector $\vec{v} \in \mathbb{R}^2$ with

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \vec{v} = \lambda \vec{v}.$$

Let A be a 2×2 matrix. What does the equation $A\vec{v} = \lambda\vec{v}$ mean geometrically? Can you think of a matrix A for which there are no non-zero vectors with this property?