Math 141 Tutorial 4 Solutions

Main problems

1. In this exercise, you will practice using the Fundamental Theorem of Calculus (Form 2). For every integral below, do the following:
   - find an antiderivative (i.e. primitive) of the integrand,
   - evaluate the given integral by applying the FTC if possible. If the FTC does not apply, explain why.

(a) $\displaystyle\int_0^1 \left(3x^2 + \sqrt{x} - 2\right) \mathrm{d}x$ (c) $\displaystyle\int_0^\pi \sec^2(x) \,\mathrm{d}x$ (e) $\displaystyle\int_{-1}^2 \left((x+1)^2 + \frac{1}{x}\right) \mathrm{d}x$

(b) $\displaystyle\int_0^{3\pi/2} (\sin(x) + \cos(x)) \,\mathrm{d}x$ (d) $\displaystyle\int_0^{\frac{1}{\pi}} \frac{1}{1+x^2} \,\mathrm{d}x$ (f) $\displaystyle\int_1^4 (3^x + 1) \,\mathrm{d}x.$

Solutions:

(a) We have

$$\int_0^1 \left(3x^2 + \sqrt{x} - 2\right) \mathrm{d}x = \int_0^1 \left(3x^2 + x^{1/2} - 2\right) \mathrm{d}x.$$

Notice that

$$F(x) = x^3 + \frac{x^{3/2}}{3/2} - 2x = x^2 + \frac{2x^{3/2}}{3} - 2x$$

is an antiderivative of $3x^2 + \sqrt{x} - 2$. Therefore, since $3x^2 + \sqrt{x} - 2$ is continuous on $[0,1]$, the Fundamental Theorem tells us that

$$\int_0^1 \left(3x^2 + \sqrt{x} - 2\right) \mathrm{d}x = F(1) - F(0) = \left(1 + \frac{2}{3} - 2\right) - 0 = -\frac{1}{3}.$$

(b) Recall that $(\sin x)' = \cos(x)$ and $(\cos x)' = -\sin x$. Thus, $-\cos x$ is an antiderivative of $\sin x$ and $\sin x$ is an antiderivative of $\cos x$. In particular,

$$(-\cos x + \sin x)' = (-\cos x)' + (\sin x)' = \sin x + \cos x.$$

Thus, $(-\cos x + \sin x)$ is an antiderivative of the everywhere continuous function $(\sin x + \cos x)$. It follows from the FTC that

$$\begin{aligned}
\int_0^{3\pi/2} (\sin(x) + \cos(x))\, \mathrm{d}x &= -\cos x + \sin x \Big|_{x=0}^{x=\frac{3\pi}{2}} \\
&= \left(-\cos\frac{3\pi}{2} + \sin\frac{3\pi}{2}\right) - (-\cos 0 + \sin 0) \\
&= (-1) - (-1) \\
&= 0.
\end{aligned}$$

(c) Recall that $\sec^2(x)$ is the derivative of $\tan(x)$. However, the fundamental theorem does **not** apply here because $\sec^2(x)$ has an infinite discontinuity at $x = \pi/2$.

(d) Recall that $(\arctan(x))' = \frac{1}{1+x^2}$, with the latter being continuous on all of $\mathbb{R}$. Therefore, the Fundamental Theorem of Calculus applies:

$$\int_0^{1/\pi} \frac{1}{1+x^2}\, \mathrm{d}x = \arctan x \Big|_{x=0}^{x=\frac{1}{\pi}} = \arctan\left(\frac{1}{\pi}\right) - \arctan(0) = \arctan\left(\frac{1}{\pi}\right).$$

(e) An antiderivative for $(x+1)^2 + \frac{1}{x}$ is

$$\frac{1}{3}(x+1)^3 + \ln|x|\,, \qquad (x \neq 0)$$

Notice that we can also expand the integrand to $x^2 + 2x + 1 + \frac{1}{x}$ before finding an antiderivative.

The integrand is discontinuous at $x = 0$ (it is not defined at 0 and has a vertical asymptote there), and therefore the fundamental theorem of calculus does not apply on the interval $[-1, 2]$.

(f) The function

$$F(x) = \frac{3^x}{\ln(3)} + x$$

is an antiderivative of $3^x + 1$. Thus, by the FTC,

$$\begin{aligned}
\int_1^4 (3^x + 1)\, \mathrm{d}x &= \frac{3^x}{\ln(3)} + x \Big|_{x=1}^{x=4} \\
&= \frac{3^4}{\ln 3} + 4 - \frac{3^1}{\ln(3)} - 1 \\
&= \frac{3^4 - 3}{\ln 3} + 3 \\
&= \frac{78 + 3\ln(3)}{\ln(3)}.
\end{aligned}$$

2. Consider a particle moving along a line such that, at any time t, the instantaneous velocity of this particle is given by

$$v(t) = t^2 - 2t - 3, \quad (m/s).$$

(a) Express the displacement of the particle from times $t = 2$ to $t = 4$ using an integral and evaluate this integral.

(b) Express the distance traveled by the particle from times $t = 2$ to $t = 4$ using an integral and evaluate this integral.

Briefly explain the difference between the integrals obtained in (a)-(b). How does this relate to the total area under the curve of a sign-changing function?

Solution:

(a) Integrating the velocity of the particle from time $t = 2$ to $t = 4$ gives the total change in position (i.e. the displacement) over this period of time:

$$\begin{aligned}
\int_2^4 (t^2 - 2t - 3)\,\mathrm{d}t &= \frac{t^3}{3} - t^2 - 3t\Big|_{t=2}^{t=4} \\
&= \left(\frac{4^3}{3} - 4^2 - 3(4)\right) - \left(\frac{2^3}{3} - 2^2 - 3(2)\right) \\
&= \left(\frac{64}{3} - 16 - 12\right) - \left(\frac{8}{3} - 4 - 6\right) \\
&= \frac{56}{3} - 18 \\
&= \frac{56 - 54}{3} \\
&= \frac{2}{3} \quad \text{(metres)}.
\end{aligned}$$

(b) To obtain the total *distance* traveled by the particle from time $t = 2$ to time $t = 4$, we instead want to integrate the speed function, which is the absolute value of the velocity function. Thus, the total distance travelled is

$$\int_2^4 \left|t^2 - 2t - 3\right|\,\mathrm{d}t.$$

However, the absolute value function makes this tricky to compute. To circumvent this difficulty, we shall break the integral into two portions: one where the integrand is positive and another where the integrand is negative. After graphing the function (or, alternatively, by noting that $t^2 - 2t - 3 = (t-3)(t+1)$, one can find the zeroes of the given function), we see that $t^2 - 2t - 3 \leq 0$ for $0 \leq t \leq 3$, and that $t^2 - 2t - 3 \geq 0$ for $t \geq 3$. Thus, we split our integral as follows:

$$\begin{aligned}
\int_2^4 \left|t^2 - 2t - 3\right|\,\mathrm{d}t &= \int_2^3 \left|t^2 - 2t - 3\right|\,\mathrm{d}t + \int_3^4 \left|t^2 - 2t - 3\right|\,\mathrm{d}t \\
&= \int_2^3 -\left(t^2 - 2t - 3\right)\,\mathrm{d}t + \int_3^4 \left(t^2 - 2t - 3\right)\,\mathrm{d}t.
\end{aligned}$$

Now, each of these integrals can be evaluated using the Fundamental Theorem of Calculus. For instance, the first integral yields

$$\begin{aligned}
\int_2^3 -\left(t^2 - 2t - 3\right)\,\mathrm{d}t &= \int_2^3 \left(3 + 2t - t^2\right)\,\mathrm{d}t \\
&= 3t + t^2 - \frac{t^3}{3}\bigg|_{t=2}^{t=3} \\
&= 3(3) + 3^2 - \frac{3^3}{3} - \left(3(2) + 2^2 - \frac{2^3}{3}\right) \\
&= 9 - \left(10 - \frac{8}{3}\right) \\
&= 9 - \frac{22}{3} \\
&= \frac{27 - 22}{3} \\
&= \frac{5}{3}.
\end{aligned}$$

In a similar way, we use the FTC to compute the second integral:

$$\int_3^4 \left(t^2 - 2t - 3\right)\,\mathrm{d}t = \frac{7}{3}.$$

Therefore, the total distance traveled by the particle is

$$\int_2^4 \left|t^2 - 2t - 3\right|\,\mathrm{d}t = \frac{5}{3} + \frac{7}{3} = \frac{12}{3} = 4 \quad \text{(metres)}.$$

3. Using the Fundamental Theorem of Calculus (FTC), evaluate the following:

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_1^{x+4}\left(\frac{1}{t}+2\right)\,\mathrm{d}t$

(b) $\displaystyle\int_1^{x+4}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{t}+2\right]\right)\,\mathrm{d}t$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_{\sin(x)}^{2\pi}\cos(t)\,dt$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_{\sin(x)}^{x^2+1} e^{\sqrt[3]{t}}\,\mathrm{d}t.$

Solution:

(a) Define

$$F(x)=\int_1^{x}\left(\frac{1}{t}+2\right)\,\mathrm{d}t$$

and note that $F'(x)=\frac{1}{x}+2$ by the fundamental theorem. Therefore, by the Chain Rule,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_1^{x+4}\left(\frac{1}{t}+2\right)\,dt=\frac{\mathrm{d}}{\mathrm{d}x}F(x+4)&=F'(x+4)\frac{\mathrm{d}}{\mathrm{d}x}(x+4)\\&=F'(x+4)\\&=\frac{1}{x+4}+2.\end{aligned}$$

(b) Here we use the second part of the Fundamental Theorem:

$$\begin{aligned}\int_1^{x+4}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{t}+2\right]\right)\,\mathrm{d}t&=\frac{1}{t}+2\Big|_{t=1}^{t=x+4}\\&=\frac{1}{x+4}-1.\end{aligned}$$

(c) As in (a), we define a function $F(x)$ as

$$F(x)=\int_x^{2\pi}\cos(t)\,\mathrm{d}t=-\int_{2\pi}^{x}\cos(t)\,\mathrm{d}t$$

By the Fundamental Theorem, F is differentiable with derivative

$$F'(x)=-\cos(x).$$

But then, the Chain Rule implies that

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{\sin(x)}^{2\pi}\cos(t)\,dt=\frac{\mathrm{d}}{\mathrm{d}x}F(\sin(x))=F'(\sin(x))\cos x=-\cos\left(\sin(x)\right)\cos x.$$

(d) Consider the function

$$F(x)=\int_0^{x} e^{\sqrt[3]{t}}\,\mathrm{d}t$$

By the Fundamental Theorem, $F'(x) = e^{\sqrt[3]{x}}$. Now, observe that

$$\begin{aligned}
\int_{\sin(x)}^{x^2+1} e^{\sqrt[3]{t}}\,\mathrm{d}t &= \int_0^{x^2+1} e^{\sqrt[3]{t}}\,\mathrm{d}t - \int_0^{\sin x} e^{\sqrt[3]{t}}\,\mathrm{d}t \\
&= F(x^2+1) - F(\sin x).
\end{aligned}$$

Using once more the chain rule, this implies that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\int_{\sin(x)}^{x^2+1} e^{\sqrt[3]{t}}\,\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}\big(F(x^2+1) - F(\sin x)\big) \\
&= F'(x^2+1)\frac{\mathrm{d}}{\mathrm{d}x}\big(x^2+1\big) - F'(\sin x)\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) \\
&= 2xe^{\sqrt[3]{x^2+1}} - \cos x e^{\sqrt[3]{\sin x}}.
\end{aligned}$$

4. Compute the following integrals using substitution:

(a) $\displaystyle\int_0^2 2e^{3s+5}\,\mathrm{d}s$

(b) $\displaystyle\int 2x\cos\left((x-1)(x+1)\right)\,\mathrm{d}x$

(c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{1+\sin(x)}\,\mathrm{d}x$

(d) $\displaystyle\int \frac{2x^3+1}{\left(x^4+2x\right)^3}\,\mathrm{d}x$

(e) $\displaystyle\int_0^1 \sqrt{2t+1}\,dt$

(f) $\displaystyle\int e^{\cos(x)}\sin(x)\,dx$

(g) $\displaystyle\int_0^1 t^4(1+t^5)^{10}\,dt$

Solution:

(a) We define $u := 3s+5$ so that $\mathrm{d}u = 3\mathrm{d}s$. Now, $u(0) = 3(0)+5 = 5$ and $u(3) = 3(2)+5 = 11$. Our integral therefore becomes

$$\begin{aligned}\int_0^2 2e^{3s+5}\,\mathrm{d}s = \int_5^{11} 2e^u\frac{\mathrm{d}u}{3} &= \frac{2}{3}\int_5^{11} e^u\,\mathrm{d}u \\ &= \frac{2}{3}e^u\Big|_{u=5}^{u=11} \\ &= \frac{2(e^{11}-e^5)}{3}.\end{aligned}$$

(b) Notice that this integral is precisely

$$\int 2x\cos\left((x-1)(x+1)\right)\,\mathrm{d}x = \int 2x\cos\left(x^2-1\right)\mathrm{d}x.$$

Take $u := x^2-1$ so that $\mathrm{d}u = 2x\mathrm{d}x$. Thus,

$$\begin{aligned}\int 2x\cos\left((x-1)(x+1)\right)\mathrm{d}x &= \int 2x\cos\left(x^2-1\right)\mathrm{d}x \\ &= \int \cos(u)\,\mathrm{d}u \\ &= \sin(u)+C \\ &= \sin(x^2-1)+C.\end{aligned}$$

(c) Put $u := 1+\sin(x)$ and note that $\mathrm{d}u = \cos(x)\mathrm{d}x$. Moreover, $u(0) = 1$ and $u(\pi/2) = 2$. Therefore, our given integral becomes

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{1+\sin(x)}\,\mathrm{d}x &= \int_1^2 \frac{\mathrm{d}u}{u} \\ &= \ln(2)-\ln(1) \\ &= \ln 2.\end{aligned}$$

(d) Define $u := x^4 + 2x$ so that $\mathrm{d}u = (4x^3+2)\mathrm{d}x = 2(2x^3+1)\mathrm{d}x$. Thus, our given integral becmomes

$$\begin{aligned}\int \frac{2x^3+1}{\left(x^4+2x\right)^3}\,\mathrm{d}x &= \frac{1}{2}\int \frac{\mathrm{d}u}{u^3}\\ &= -\frac{1}{4u^2}+C\\ &= -\frac{1}{4\left(x^4+2x\right)^2}+C.\end{aligned}$$

(e) Put $u := 2t+1$, then $\mathrm{d}u = 2\,\mathrm{d}t$. Moreover, $u(0) = 1$ and $u(1) = 3$. Therefore, our given integral becomes

$$\begin{aligned}\int_0^1 \sqrt{2t+1}\,\mathrm{d}t &= \int_1^3 \frac{\sqrt{u}}{2}\,\mathrm{d}u,\\ &= \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_1^3,\\ &= \sqrt{3}-\frac{1}{3}.\end{aligned}$$

(f) Set $u := \cos(x)$, then $\mathrm{d}u = -\sin(x)\,\mathrm{d}x$. Hence,

$$\begin{aligned}\int e^{\cos(x)}\sin(x)\ \mathrm{d}x &= -\int e^u\,\mathrm{d}u,\\ &= -e^u+C,\\ &= -e^{\cos(x)}+C.\end{aligned}$$

(g) Put $u := 1+t^5$, then $\mathrm{d}u = 5t^4\,\mathrm{d}t$ and rearranging this gives us $t^4\,\mathrm{d}t = \mathrm{d}u/5$. Moreover, $u(0) = 1$ and $u(1) = 2$. Thus, our given integral becomes

$$\begin{aligned}\int_0^1 t^4(1+t^5)^{10}\ \mathrm{d}t &= \frac{1}{5}\int_1^2 u^10\,\mathrm{d}u,\\ &= \frac{1}{5}\cdot\frac{u^{11}}{11}\Big|_1^2,\\ &= \frac{2^{11}-1}{55}.\end{aligned}$$

5. Suppose that f is a continuous function on $\mathbb{R}$. Prove that $\displaystyle\int_a^b f(x+c)\,\mathrm{d}x = \int_{a+c}^{b+c} f(x)\,\mathrm{d}x$ for any $c \in \mathbb{R}$.

Solution:

Fix $c \in \mathbb{R}$ and consider the function $g(x) := x + c$ on $\mathbb{R}$. This function is differentiable for all x with *continuous* derivative $g'(x) = 1$. Hence, our given integral can be written as

$$\int_a^b f(x+c)\,\mathrm{d}x = \int_a^b f(g(x))\,\mathrm{d}x = \int_a^b f(g(x))\,\underbrace{g'(x)}_{=1}\,\mathrm{d}x.$$

By substitution, this is the same as

$$\begin{aligned}\int_a^b f(x+c)\,\mathrm{d}x &= \int_a^b f(g(x))g'(x)\,\mathrm{d}x \\ &= \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u \\ &= \int_{a+c}^{b+c} f(u)\,\mathrm{d}u.\end{aligned}$$

Challenge Problem

6. Compute
$$I = \int_0^{\pi} \frac{x\sin(x)}{1+\cos^2(x)}\,dx$$

Hint: make the substitution $u = \pi - x$ and with trig identities write down a simple definite integral for $2I$