Math 140 **Tutorial 6**

February 10 - 13

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1. For each of the following functions find the first, second and third derivatives ($\frac{df}{dx}, \frac{d^2f}{dx^2}, \frac{d^3f}{dx^3}$):

   (a) $f(x) = \dfrac{x^4}{24} + \dfrac{x^3}{6} + \dfrac{x^2}{2} + x + 1$

   (b) $f(x) = \sin x$

   (c) $f(x) = \cos x$

Solution

(a)

$$\frac{df}{dx} = \frac{x^3}{6} + \frac{x^2}{2} + x + 1 \tag{1}$$
$$\frac{d^2f}{dx^2} = \frac{x^2}{2} + x + 1 \tag{2}$$
$$\frac{d^3f}{dx^3} = x + 1 \tag{3}$$

(b)

$$\frac{df}{dx} = \cos x \tag{4}$$
$$\frac{d^2f}{dx^2} = -\sin x \tag{5}$$
$$\frac{d^3f}{dx^3} = -\cos x \tag{6}$$

(c)

$$\frac{df}{dx} = -\sin x \tag{7}$$
$$\frac{d^2f}{dx^2} = -\cos x \tag{8}$$
$$\frac{d^3f}{dx^3} = \sin x \tag{9}$$

2. Evaluate the following derivatives:

(a) $\frac{d}{dx}\left(ae^x + \frac{b}{x} + \frac{c}{x^2}\right)$

(b) $\frac{d}{dx}\left(\sqrt{x} + \frac{1}{\sqrt[3]{x}}\right)^2$

(c) $\frac{d}{dx}\left(\frac{x - x^2}{\sqrt{x}}\right)$

(d) $\frac{d}{dx}\left((3x^3 + 2)(7x + 2)\right)$

(e) $\frac{d}{dx}\left(\sec x e^x\right)$

(f) $\frac{d^2}{dx^2}\left(x^4 e^x\right)$

(g) $\frac{d}{dx}\left(x \sec x \tan x\right)$

(h) $\frac{d}{dx}\left(\frac{ax + b}{cx + d}\right)$

(i) $\frac{d}{dx}\left(\frac{1 - \sec x}{\cot x}\right)$

(j) $\frac{d}{dx}\left(\frac{x^3 e^x + 1}{2x + e^x}\right)$

Solution

(a) We can differentiate term by term, remember a, b, c are just constants, and remember we can write $\frac{1}{x} = x^{-1}$, $\frac{1}{x^2} = x^{-2}$. So the derivative is $ae^x - \frac{b}{x^2} - \frac{2c}{x^3}$

(b) We can expand out the brackets to get

$$\frac{d}{dx}\left(x + 2x^{1/6} + x^{-2/3}\right) = 1 + \frac{1}{3}x^{-5/6} - \frac{2}{3}x^{-5/6}$$

(c) Again, we can simplify this expression to get

$$\frac{d}{dx}\left(x^{1/2} - x^{3/2}\right) = \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{1/2}$$

(d) We can expand the brackets, or we can also apply the product rule with $u = (3x^2 + 2), v = (7x + 2)$. We know the derivative is equal to

$$u\frac{dv}{dx} + v\frac{du}{dx} = (3x^2 + 2) \cdot 7 + (7x + 2) \cdot (6x)$$

(e) Here we apply the product rule with $u = \sec x, v = e^x$ and remember that $\frac{d}{dx} \sec x = \sec x \tan x$ so we get

$$\sec x \tan x e^x + \sec x e^x$$

(f) First we will compute the first derivative using the product rule with $u = x^4, v = e^x$, then we get

$$4x^3 e^x + x^4 e^x$$

we need to differentiate this again to get the second derivative. We will apply the product rule to each of the terms, which gives the final answer

$$12x^2 e^x + 4x^3 e^x + 4x^3 e^x + x^4 e^x$$

(g) Here we will start by applying the product rule with $u = x, v = \sec x \tan x$ to get that the derivative is

$$\sec x \tan x + x \frac{d}{dx} (\sec x \tan x)$$

we can apply the product rule again to differentiate $\sec x \tan x$ with $u = \sec x, v = \tan x$ which gives the final answer

$$\sec x \tan x + x \left(\sec x \tan^2 x + \sec^3 x \right)$$

(h) We apply the quotient rule with $u = ax + b, v = cx + d$. Remember the quotient rule is

$$\frac{d}{dx} \left(\frac{u(x)}{v(x)} \right) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}$$

So we get

$$\frac{a(cx + d) - c(ax + b)}{(cx + d)^2} = \frac{ad - bc}{(cx + d)^2}$$

(i) Here again we apply the quotient rule with $u = 1 - \sec x, v = \cot x$ and remember that $\frac{d}{dx} \cot x = -\csc^2 x$ we get

$$\frac{\sec x \tan x \cot x + \csc^2 x(1 - \sec x)}{\cot^2 x}$$

(j) Apply the quotient rule with $u = x^3 e^x + 1, v = 2x + e^x$ gives

$$\frac{\frac{d}{dx} (x^3 e^x) (2x + e^x) - (x^3 e^x + 1)(2 + e^x)}{(2x + e^x)^2}$$

The final thing we need is the derivative of $x^3 e^x$ in the numerator, which we can calculate using the product rule and is equal to $3x^2 e^x + x^3 e^x$ therefore the final answer isn

$$\frac{(3x^2 e^x + x^3 e^x)(2x + e^x) - (x^3 e^x + 1)(2 + e^x)}{(2x + e^x)^2}$$

3. Find the equations of the tangent line of the curves at the given point:

   (a) $y = xe^x + x^3$ at $(0,0)$
   (b) $y = \sin x + \cos x$ at $(0,1)$
   (c) $y = e^x \cos x + \sin x$ at $(0,1)$

Solution

(a) First we evaluate the derivative

$$\frac{d}{dx}\left(xe^x + x^3\right) = e^x + xe^x + 3x^2$$

so at 0 the slope is $y'(0) = 1$. The line we are looking for has slope 1 and goes through the origin so has equation $y = x$

(b) First we evaluate the derivative

$$\frac{d}{dx}(\sin x + \cos x) = \cos x - \sin x$$

so at 0 the slope is $y'(0) = 1$. We are looking for a line with slope 1 and goes through the point $(0,1)$. That is $y = x + 1$.

(c) First we evaluate the derivative

$$\frac{d}{dx}(e^x \cos x + \sin x) = e^x \cos x - e^x \sin x + \cos x$$

so the slope at 0 is $y'(0) = 2$. We're looking for a line with slope 2 going through the point $(0,1)$. That is $y = 2x + 1$

4. (a) Where does $f(x) = e^x \cos x$ have a horizontal tangent line?
   (b) Where does $f(x) = \frac{x^2+1}{x+1}$ have a tangent line parallel to the line $2y = x - 3$?

Solution

(a) We differentiate

$$f'(x) = e^x \cos x - e^x \sin x$$

The tangent line is horizontal means it has slope 0. So we want to solve the equation $f'(x) = 0$. So this means we want to solve

$$e^x \cos x - e^x \sin x = 0$$

Remember e^x is never 0 so we can divide by it to get $\sin x = \cos x$. This happens when $x = \pi/4 + n\pi$ for any integer n.

(b) Differentiate by using the quotient rule

$$f'(x) = \frac{2x(x+1) - (x^2+1)}{(x+1)^2} = \frac{x^2+2x-1}{(x+1)^2}$$

To be parallel to the line $2y = x - 3$ means it has the same slope, which is $\frac{1}{2}$, so we want to solve the equation

$$\frac{x^2+2x-1}{(x+1)^2} = \frac{1}{2}$$

Moving things around we get

$$2x^2 + 4x - 2 = x^2 + 2x + 1$$

which is the same as

$$x^2 + 2x - 3 = 0$$

which has solutions $x = 1, x = -2$

5. Let

$$f(x) = \begin{cases} x^2, & x \leq 2 \\ mx + b, & x > 2. \end{cases}$$

Find the values of m and b that make f differentiable everywhere.

Solution

Because $x^2, mx + b$ are differentiable, $f(x)$ is differentiable away from $x = 2$, we have to make sure it is differentiable at $x = 2$. The derivative of x^2 is $2x$ so at 2 it is 4. The derivative of $mx + b$ at 2 is m. So we have to make sure the two derivatives agree, which happens when $m = 2$.

6. Suppose that f, g, h are differentiable functions. Prove that

$$(fgh)' = f'gh + fg'h + fgh'$$

What can you say about the derivative of a product of n differentiable functions?

Solution

We are going to apply the chain rule with $u = (fg), v = h$ we get

$$(fgh)' = (fg)'h + (fg)h'$$

Now apply the chain rule for $(fg)'$ gives the final answer

$$f'gh + fg'h + fgh'$$

7. Find the derivative of the function. Do not simplify.

(a) $y = (x^2 + 7x + 2)^{20}$

(b) $y = x\,(4x + 1)^{100}$

(c) $y = \cos\left(\sqrt{\sin x}\right)$

(d) $y = \tan^2(\sin(3x + \ln x))$

(e) $f(x) = \ln(e^{-x} + xe^{-x})$

(f) $h(t) = t \ln\left(\frac{1}{t}\right)$

(g) $f(x) = x^x$

(h) $y = e^{e^x}$

Solution

(a) We want to apply the chain rule. We will let $f(x) = x^{20}$ and $g(x) = x^2 + 7x + 2$. We know that our function is $f(g(x))$ and its derivative is

$$f'(g(x))g'(x)$$

In our case $f'(x) = 20x^{19}$ and $g'(x) = 2x + 7$. Therefore the derivative is

$$20(x^2 + 7x + 2)^{19} \cdot (2x + 7)$$

(b) First we apply the product rule. The derivative is

$$(4x + 1)^{100} + x \cdot \frac{d}{dx}(4x + 1)^{100}$$

So we want to evaluate $\frac{d}{dx}(4x + 1)^{100}$ for which we will use the chain rule with $f(x) = x^100$, and $g(x) = 4x + 1$. The chain rule tells us that the derivative is

$$100(4x + 1)^{99} \cdot 4$$

So the final answer is

$$(4x + 1)^{100} + 400x(4x + 1)^{99}$$

(c) Here we want to apply the chain rule with $f(x) = \cos x$ and $g(x) = \sqrt{\sin x}$ the chain rule says that the derivative of $f(g(x))$ is $f'(g(x))g'(x)$. We know that $f'(x) = -\sin x$, and $g'(x)$ is the derivative

$$\frac{d}{dx}\sqrt{\sin x}$$

To evaluate $g'(x)$ we need to apply the chain rule once more. We will use $u(x) = \sqrt{x}$ and $v(x) = \sin x$ and we want to differentiate $u(v(x))$. So

$$g'(x) = \frac{d}{dx}\sqrt{\sin x} = \cos x \frac{1}{2\sqrt{\sin x}}$$

Putting it all together we see that the final answer is

$$-\sin\left(\sqrt{\sin x}\right) \cdot \frac{\cos x}{2\sqrt{\sin x}}$$

(d) Similar to the previous question, we will have to apply the chain rule several times in this question. We start by applying the chain rule to the functions $f(x) = \tan^2 x$, $g(x) = \sin(3x + \ln x)$. We know that the final answer will be

$$f'(g(x))g'(x)$$

so we need to differentiate $f(x)$ and $g(x)$. Lets start with $f(x) = (\tan x)^2$. We will use the chain rule to differentiate it with $s(x) = x^2$ and $t(x) = \tan x$. We know that $s'(x) = 2x$ and $t'(x) = \sec^2 x$. Therefore

$$f'(x) = 2(\tan x)\sec^2 x$$

Now lets differentiate $g(x)$. Once more we will apply the chain rule, this time with $u(x) = \sin x$ and $v(x) = 3x + \ln x$. We know that $u'(x) = \cos x$ and $v'(x) = 3 + \frac{1}{x}$. Therefore

$$g'(x) = \cos(3x + \ln x) \cdot \left(3 + \frac{1}{x}\right)$$

So we can conclude the the following messy looking final answer

$$2\tan\left(\sin(3x + \ln x)\right)\sec^2\left(\sin(3x + \ln x)\right) \cdot \cos(3x + \ln x) \cdot \left(3 + \frac{1}{x}\right)$$

(e) We will apply the chain rule with $u(x) = \ln x, v(x) = e^{-x} + xe^{-x}$. We know $u'(x) = \frac{1}{x}$ and $v'(x) = -e^{-x} + e^{-x} - xe^{-x}$, where we used the product rule to find $v'(x)$. Therefore, by the chain rule

$$f'(x) = \frac{1}{e^{-x} + xe^{-x}} \cdot (-xe^{-x})$$

(f) By first applying the product rule we can see that

$$h'(t) = \ln\left(\frac{1}{t}\right) + t\frac{d}{dt}\ln\left(\frac{1}{t}\right)$$

To evaluate $\frac{d}{dt}ln(1/t)$ we can apply the chain rule with $f(t) = \ln t$ and $g(t) = \frac{1}{t}$. We know $f'(t) = \frac{1}{t}$ and $g'(t) = -\frac{1}{t^2}$. So

$$\frac{d}{dt} = t \cdot (-\frac{1}{t^2}) = -\frac{1}{t}$$

Therefore

$$h'(t) = \ln\left(\frac{1}{t}\right) - 1$$

Another way to do this is to note that $\ln(1/t) = -\ln(t)$.

(g) To do this, we first write

$$x^x = e^{\ln(x^x)} = e^{x \ln x}$$

Now we can use the chain rule with $u(x) = e^x$ and $v(x) = x \ln x$. We know $u'(x) = e^x$ and $v'(x) = \ln x + 1$. Therore

$$f'(x) = e^{x \ln x} \cdot (\ln x + 1) = x^x(\ln x + 1)$$

(h) Here we apply the chain rule with $u(x) = e^x$ and $v(x) = e^x$, then $u'(x) = e^x$ and $v'(x) = e^x$. So

$$y'(x) = e^{e^x} \cdot e^x$$