

Lecture hours 13-15

Definitions and Theorems

Definition (Kernel and Image of a linear transformation). Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation

- The kernel $\ker(T)$ is the set of vectors $\vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = 0$.
- The image of T is the set of all vectors $\vec{y} \in \mathbb{R}^m$ such that $T(\vec{x}) = \vec{y}$ for some $\vec{x} \in \mathbb{R}^n$.

Definition (Rank and Nullity of a linear transformation). Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation

- The rank of T is the dimension of the image of T , $\text{rank } T = \dim(\text{im } T)$.
- The nullity of T is the dimension of the kernel of T , $\text{nullity } T = \dim(\ker T)$.

Theorem (Rank Nullity Theorem).

- In terms of linear transformations:

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation

$$\text{rank } T + \text{nullity } T = n.$$

- In terms of matrices:

Let A be an $m \times n$ matrix

$$\dim(\text{im } A) + \dim(\ker A) = \text{number of columns of } A = n.$$

Problem 29 (Rank and Nullity). Let $\vec{v} \neq \vec{0}$ be the vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Define a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$T(\vec{x}) = \vec{v} \times \vec{x}.$$

- a) What is the nullity of T ?
- b) What is the rank of T ? Why?

Problem 30 (Rank Nullity Theorem). True or false? Justify your answer.

- a) If A is a 2×4 matrix with kernel of dimension 2, then the equation $A\vec{x} = \vec{e}_2$ is consistent.
- b) There is a 5×5 matrix A such that $\dim(\text{im}A) = \dim(\text{ker}A)$.

Problem 31 (Rank Nullity Theorem). Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ be the linear transformation defined by

$$T \left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) = \begin{bmatrix} a - b \\ c - d \end{bmatrix}.$$

Find the kernel, nullity, image and rank of T .

Problem 32 (Rank Nullity Theorem). Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the linear transformation defined by

$$T \left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Find the kernel, nullity, image and rank of T .