Lecture hours 8-10

4

Definition (Linear Transformations). We define a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ as a function with two properties:

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ (we say T preserves vector addition)
2. $T(\vec{x}) = cT(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$ and $c \in \mathbb{R}$ (we say T preserves scalar multiplication)

Definition (Matrix Multiplication – the column view). Suppose now that A is an $m \times n$ matrix (so A has m rows and n columns, it might be a coefficient matrix for a system of m equations and n unknowns). Also, let $\vec{x}$ be a vector with n entries. We define the vector $A\vec{x}$ as follows: If $\vec{c}_k$ is the kth column of A, then

$$A\vec{x} = \begin{bmatrix} | & | & & | \\ \vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_3 \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_n\vec{c}_n.$$

That is, the product of a matrix and a vector is a linear combination of the columns of the matrix.

Problem 16 (Basis for a subspace)**.** Suppose that V is the set of all vectors $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$ such that

$$\begin{cases} x_1 + 2x_2 - 2x_3 + 2x_4 - x_5 = 0, \\ x_1 + 2x_2 - x_3 + 3x_4 - 2x_5 = 0, \\ 2x_1 + 4x_2 - 7x_3 + x_4 + x_5 = 0. \end{cases} \tag{4.1}$$

a) Explain why V is a subspace of $\mathbb{R}^5$.

b) Find a basis for V.

Solution 16 (Basis for a subspace)

a) Note V is the set of all solutions to the homogeneous system (4.1).

Since $\vec{0}$ is a solution to (4.1), V is non-empty.

Take any vector $(x_1, x_2, x_3, x_4, x_5)$ in V. For any real number λ we have

$$\begin{aligned} \lambda x_1 + 2\lambda x_2 - 2\lambda x_3 + 2\lambda x_4 - \lambda x_5 &= \lambda(x_1 + 2x_2 - 2x_3 + 2x_4 - x_5) = 0, \\ \lambda x_1 + 2\lambda x_2 - \lambda x_3 + 3\lambda x_4 - 2\lambda x_5 &= \lambda(x_1 + 2x_2 - x_3 + 3x_4 - 2x_5) = 0, \\ 2\lambda x_1 + 4\lambda x_2 - 7\lambda x_3 + \lambda x_4 + \lambda x_5 &= \lambda(2x_1 + 4x_2 - 7x_3 + x_4 + x_5) = 0. \end{aligned}$$

That is, V is closed under scalar multiplication. We can show in a similar way that V is closed under addition.

b) We will show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -4 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \textit{and} \quad \vec{v}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

is a basis for V.

1) The set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ spans V:

The rref for system (4.1) is given by

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 4 & -3 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

which means we can write all solutions to (4.1) as

$$(-2s-4t+3u, s, u-t, t, u) = s\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix} + t\begin{bmatrix}-4\\0\\-1\\1\\0\end{bmatrix} + u\begin{bmatrix}3\\0\\1\\0\\1\end{bmatrix} = s\vec{v}_1 + t\vec{v}_2 + u\vec{v}_3, \quad s, t, u \in \mathbb{R}.$$

In other words $V = span\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

2) The vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent: Using Gaussian elimination we can see that

$$rref\left[\begin{array}{ccc|c}-2 & -4 & 3 & 0\\1 & 0 & 0 & 0\\0 & -1 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{array}\right] = \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{array}\right].$$

That is, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set of vectors.

Problem 17 (Linear Transformations)**.** Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^l$ are two linear transformations. Define a transformation $F : \mathbb{R}^n \to \mathbb{R}^l$ by

$$F(\vec{v}) = S\left(T\left(\vec{v}\right)\right).$$

a) Explain why F is a linear transformation.

Now suppose that $n = 3$, $m = 2$, and $l = 3$, and T is induced by the matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 2 \end{bmatrix},$$

and S is induced by the matrix

$$\begin{bmatrix} -2 & 0 \\ 0 & 1 \\ 1 & 3 \end{bmatrix}.$$

b) Find $F(\vec{x})$ for any $\vec{x} \in \mathbb{R}^3$.

Hint : Write $F(\vec{x})$ in terms of $F\left(\vec{e}_1\right), F\left(\vec{e}_2\right)$ and $F\left(\vec{e}_3\right)$.

Solution 17 (Linear Transformations)

a) We need to determine whether T preserves addition and scalar multiplication. Preservation of scaling for F follows by combining preservation of scaling for both T and S:

$$F(c\vec{v}) = S(T(c\vec{v})) = S(cT(\vec{v})) = cS(T(\vec{v})) = cF(\vec{v}).$$

Preservation of addition is similar.

b) The answer is

$$F\left(\vec{x}\right) = \begin{bmatrix} -2 & 0 & -4 \\ 0 & -1 & 2 \\ 1 & -3 & 8 \end{bmatrix} \vec{x}.$$

Problem 18 (Linear Transformations). Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. In this case, the domain and codomain are the same, so we can repeat T. The linear transformation obtained by repeating T k times is denoted T^k.

a) Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with T not equal to the zero transformation[1], but T^2 equal to the zero transformation.

b) Find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ with neither T nor T^2 equal to the zero transformation, but T^3 equal to the zero transformation.

Solution 18 (Linear Transformations)

a) The transformation induced by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

b) The transformation induced by $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Problem 19 (Geometrical Linear Transformations). In each part below, you are given a matrix A. Describe the effect of the linear transformation $T(\vec{x}) = A\vec{x}$ in words.

a) $A = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$.

b) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, for some $\theta \in \mathbb{R}$.

c) $A = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$, for some $\theta \in \mathbb{R}$.

Solution 19 (Geometrical Linear Transformations) Draw what happens to the standard basis vectors after applying the linear transformation T.

a) Rotation of $\pi/4$ radians counter-clockwise.

b) Rotation by θ counter-clockwise.

c) Rotation around the y axis by θ counter-clockwise in the xz-plane.

[1] the *zero transformation* is the linear transformation that sends every vector to the zero vector in the codomain

Problem 20 (Geometrical Linear Transformations). Find all possible values of $\lambda \in \mathbb{R}$ for which there is a non-zero vector $\vec{v} \in \mathbb{R}^2$ with

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \vec{v} = \lambda \vec{v}.$$

Let A be a 2×2 matrix. What does the equation $A\vec{v} = \lambda\vec{v}$ mean geometrically? Can you think of a matrix A for which there are no non-zero vectors with this property?

Solution 20 (Geometrical Linear Transformations) We need to find all values of λ for which the equation

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

is consistent. This is a system of two linear equations for v_1, v_2, so can be written in augmented matrix form as

$$\left[\begin{array}{cc|c} 1-\lambda & -1 & 0 \\ 0 & 2-\lambda & 0 \end{array}\right].$$

There are 3 possible cases:

- If $\lambda = 1$ we have

  $$rref \left[\begin{array}{cc|c} 1-\lambda & -1 & 0 \\ 0 & 2-\lambda & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

  Then, $\vec{v} = (t, 0)$ for $t \in \mathbb{R}$.

- If $\lambda = 2$ we have

  $$rref \left[\begin{array}{cc|c} 1-\lambda & -1 & 0 \\ 0 & 2-\lambda & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

  Then, $\vec{v} = (t, -t)$ for $t \in \mathbb{R}$.

- If $\lambda \neq 1, 2$ we have

  $$rref \left[\begin{array}{cc|c} 1-\lambda & -1 & 0 \\ 0 & 2-\lambda & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right].$$

  Then, $\vec{v} = \vec{0}$.

In conclusion, the only way to get a non-zero $\vec{v}$ is if $\lambda = 1$ or $\lambda = 2$.

Geometrically, the equation $A\vec{v} = \lambda\vec{v}$ means that the linear transformation induced by A scales $\vec{v}$ with scale factor λ. Therefore, if A is a rotation in two dimensions (see Problem 19 b) for an explicit example) then there can be no vectors $\vec{v}$ with $A\vec{v} = \lambda\vec{v}$.