# Math 141 Tutorial 3 Solutions

## Main problems

1. Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous functions and let $c \in \mathbb{R}$ be such that $a < c < b$. Given that

$$\int_a^c f(x)\,\mathrm{d}x = 4, \quad \int_a^b f(x)\,\mathrm{d}x = -2, \quad \int_a^c g(x)\,\mathrm{d}x = -1, \quad \int_c^b g(x)\,\mathrm{d}x = 3$$

determine the value of each of the following integrals.

(a) $\displaystyle\int_a^c (f(x) + 2g(x))\,\mathrm{d}x$

(b) $\displaystyle\int_c^b f(x)\,\mathrm{d}x$

(c) $\displaystyle\int_a^b (2f(x) - 5g(x))\,\mathrm{d}x$

Solution:

Note that $f$ and $g$ are continuous on $[a, b]$ and thus continuous on $[a, c]$ and $[c, b]$. This ensures that $f$ and $g$ are Riemann integrable on these intervals and guarantees the existence of their respective integrals on $[a, c]$ and $[c, b]$.

(a) Using the properties shown in the last problem, we have

$$\begin{aligned}
\int_a^c (f(x) + 2g(x))\,\mathrm{d}x &= \int_a^c f(x)\,\mathrm{d}x + \int_a^c 2g(x)\,\mathrm{d}x \\
&= \int_a^c f(x)\,\mathrm{d}x + 2\int_a^c g(x)\,\mathrm{d}x \\
&= 4 + 2 \cdot (-1) \\
&= 2.
\end{aligned}$$

(b) By additivity, we have

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

Or, equivalently,

$$\int_c^b f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x - \int_a^c f(x)\,\mathrm{d}x = -2 - 4 = -6.$$

(c) First, we note that

$$\int_a^b g(x)\,\mathrm{d}x = \int_a^c g(x)\,\mathrm{d}x + \int_c^b g(x)\,\mathrm{d}x = -1 + 3 = 2.$$

Hence, by the properties shown in the last question,

$$\begin{aligned}
\int_a^b (2f(x) - 5g(x))\,\mathrm{d}x &= \int_a^b 2f(x)\,\mathrm{d}x + \int_a^b (-5g(x))\,\mathrm{d}x \\
&= 2\int_a^b f(x)\,\mathrm{d}x - 5\int_a^b g(x)\,\mathrm{d}x \\
&= 2(-2) - 5(2) \\
&= -14.
\end{aligned}$$

2. Given that

$$\int_0^\pi \sin(x)\,\mathrm{d}x = 2 \quad \text{and} \quad \int_0^\pi \sin^2(x)\,\mathrm{d}x = \frac{\pi}{2},$$

and

$$\int_{-\pi}^0 \sin(x)\,\mathrm{d}x = -2 \quad \text{and} \quad \int_{-\pi}^0 \sin^2(x)\,\mathrm{d}x = \frac{\pi}{2},$$

determine the value of each of the following integrals.

(a) $\displaystyle\int_0^\pi \left(2\sin^2(x) - \pi\sin(x)\right)\,\mathrm{d}x$

(b) $\displaystyle\int_0^\pi \cos^2(x)\,\mathrm{d}x$

(c) $\displaystyle\int_{-\pi}^\pi \sin(x)\left(\sin(x) + 1\right)\,\mathrm{d}x$

Solution:

Note that all functions here are continuous on $\mathbb{R}$ and therefore Riemann integrable on any interval of the form $[a, b] \subset \mathbb{R}$.

(a) We have

$$\begin{aligned}
\int_0^\pi \left(2\sin^2(x) - \pi\sin(x)\right)\,\mathrm{d}x &= 2\int_0^\pi \sin^2(x)\,\mathrm{d}x - \pi\int_0^\pi \sin(x)\,\mathrm{d}x \\
&= 2\left(\frac{\pi}{2}\right) - \pi(2) \\
&= -\pi.
\end{aligned}$$

(b) Recall that $\cos^2(x) = 1 - \sin^2(x)$ for all $x \in \mathbb{R}$. Thus,

$$\begin{aligned}
\int_0^\pi \cos^2(x)\,\mathrm{d}x = \int_0^\pi \left(1 - \sin^2(x)\right)\,\mathrm{d}x &= \int_0^\pi \mathrm{d}x - \int_0^\pi \sin^2(x)\,\mathrm{d}x \\
&= \pi - \frac{\pi}{2} \\
&= \frac{\pi}{2}.
\end{aligned}$$

(c) Linearity of the integral tells us that

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin(x)\,(\sin(x)+1)\,\mathrm{d}x &= \int_{-\pi}^{\pi} \left(\sin^2(x)+\sin(x)\right)\,\mathrm{d}x \\
&= \int_{-\pi}^{\pi} \sin^2(x)\,\mathrm{d}x + \int_{-\pi}^{\pi} \sin(x)\,\mathrm{d}x \\
&= \int_{-\pi}^{0} \sin^2(x)\,\mathrm{d}x + \int_{0}^{\pi} \sin^2(x)\,\mathrm{d}x + \int_{-\pi}^{0} \sin(x)\,\mathrm{d}x + \int_{0}^{\pi} \sin(x)\,\mathrm{d}x \\
&= \frac{\pi}{2} + \frac{\pi}{2} - 2 + 2 \\
&= \pi.
\end{aligned}$$

3. Using the comparison principle, for each function $f$ below find constants $m$ and $M$ such that

$$m \leq \int_a^b f(x)dx \leq M.$$

(a) $f(x) = x^3 + 1$ with $a = 0$ and $b = 2$.

(b) $f(x) = \ln(x^2 + 4x + 14)$ with $a = -4$ and $b = 2$.

Solution:

For each question, we'll find constants $A$ and $B$ such that $A <= f(x) <= B$ for all $x \in [a, b]$, then using the comparison principle combined with the linearity properties of the integral to get $m = A(b-a)$ and $M = B(b-a)$:

$$A(b-a) = A\int_a^b 1\;dx \leq \int_a^b f(x) \leq B\int_a^b 1\;dx = B(b-a).$$

(a) We start by trying to find the local minimum and maximum of $f(x) = x^3 + 1$ on $[0, 2]$. Differentiating $f(x)$, we get

$$f'(x) = 3x^2,$$

and solving for $x$ such that $f'(x) = 0$ gives $f'(x) = 3x^2 = 0$ at $x = 0$. We can't conclude whether $x = 0$ is a local min/max just yet because

$$f''(0) = 6(0) = 0,$$

i.e., $x = 0$ is an inflection point. Next, we'll check $f(x)$ at the endpoints of $[0, 2]$:

$$\begin{aligned}
f(0) &= 0^3 + 1 = 1 \\
f(2) &= 2^3 + 1 = 9.
\end{aligned}$$

Since we know $f(x)$ is an increasing function on $[0, 2]$ ($f'(x) \geq 0$ for all $x \in [0, 2]$), we can conclude that

$$1 \leq f(x) \leq 9,$$

for all $x \in [0,2]$. Hence by the work above, we get

$$2 = 1 \cdot 2 \leq \int_0^2 3x^2 + 1 \, dx \leq 9 \cdot 2 = 18.$$

That is, $m = 2$ and $M = 18$.

(b) We approach this problem similarly to the previous one. Differentiating $f(x)$, we get

$$f'(x) = \frac{2x+4}{x^2+4x+14}.$$

Solving for $f'(x) = 0$, we get $x = -2$. Next, we compute $f(-4)$, $f(-2)$, and $f(2)$ to find the min/max of $f(x)$ on $[-4, 2]$. In doing so, we see that

$$f(-4) \approx 2.6391 \qquad f(-2) \approx 2.3026 \qquad f(2) \approx 3.2581.$$

Hence by rounding down/up, we have $2.3 \leq f(x) \leq 3.3$ and so

$$13.8 =\leq 2.3 \int_{-4}^2 1 \, dx \leq \int_{-4}^2 \ln(x^2+4x+14) \, dx \leq 3.3 \int_{-4}^2 1 \, dx = 19.8,$$

i.e., we could have $m = 13.8$ and $M = 19.8$.

4. Consider the function $F$ given by

$$F(x) = \int_0^x \cos(t) e^{t^2} \, \mathrm{d}t.$$

Find *where* the local maxima and minima of $F(x)$ on $(0, 2\pi)$ occur. (Do not try and evaluate $F$ at these points!)

Solution:

By the fundamental theorem of calculus, $F$ is differentiable with derivative

$$F'(x) = \cos(x) e^{x^2}.$$

Recalling techniques from Calculus 1, we begin by searching for $x \in (0, 2\pi)$ such that $F'(x) = 0$. Since we know what $F'(x)$ is, this is the same as looking for $x \in (0, 2\pi)$ such that

$$\cos(x) e^{x^2} = 0.$$

Since $e^{x^2} > 0$ for all $x \in \mathbb{R}$, the above can only occur when $\cos(x) = 0$. Since we are restricting ourselves to $x \in (0, 2\pi)$, this leaves only the following possibilities:

$$x = \frac{\pi}{2} \quad \text{or} \quad x = \frac{3\pi}{2}.$$

Using the second derivative test with $F''(x) = 2xe^{x^2}\cos(x) - \sin(x)e^{x^2}$, we have

$$F''\left(\frac{\pi}{2}\right) < 0 \quad \text{and} \quad F''\left(\frac{3\pi}{2}\right) > 0.$$

The relative min is thus at $\frac{3\pi}{2}$ and the relative max is at $\frac{\pi}{2}$.

## Challenge Problems

5. Compute the following limits

   (a) $\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\right)$

   (b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2\pi}{n} \sin\left(\frac{2\pi i}{n}\right)$

   *Hint: how do these limits relate to Riemann sums?*

6. Let
$$g(x) = \int_0^{h(x)} \frac{1}{\sqrt{1+t^4}}\,\mathrm{d}t, \qquad h(x) = \int_0^{\cos(x)} \big(1+\sin(s^2)\big)\,\mathrm{d}s.$$
   What's the value of $g'(\frac{\pi}{2})$?