Math 141 Tutorial 3

Main problems

1. Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous functions and let $c \in \mathbb{R}$ be such that $a < c < b$. Given that

$$\int_a^c f(x)\,\mathrm{d}x = 4, \quad \int_a^b f(x)\,\mathrm{d}x = -2, \quad \int_a^c g(x)\,\mathrm{d}x = -1, \quad \int_c^b g(x)\,\mathrm{d}x = 3$$

determine the value of each of the following integrals.

(a) $\displaystyle\int_a^c (f(x) + 2g(x))\,\mathrm{d}x$

(b) $\displaystyle\int_c^b f(x)\,\mathrm{d}x$

(c) $\displaystyle\int_a^b (2f(x) - 5g(x))\,\mathrm{d}x$

2. Given that

$$\int_0^\pi \sin(x)\,\mathrm{d}x = 2 \quad \text{and} \quad \int_0^\pi \sin^2(x)\,\mathrm{d}x = \frac{\pi}{2},$$

and

$$\int_{-\pi}^0 \sin(x)\,\mathrm{d}x = -2 \quad \text{and} \quad \int_{-\pi}^0 \sin^2(x)\,\mathrm{d}x = \frac{\pi}{2},$$

determine the value of each of the following integrals.

(a) $\displaystyle\int_0^\pi \left(2\sin^2(x) - \pi\sin(x)\right)\,\mathrm{d}x$

(b) $\displaystyle\int_0^\pi \cos^2(x)\,\mathrm{d}x$

(c) $\displaystyle\int_{-\pi}^\pi \sin(x)\left(\sin(x) + 1\right)\,\mathrm{d}x$

3. For each function f below, find constants m and M such that

$$m \leq \int_a^b f(x)dx \leq M.$$

   (a) $f(x) = x^3 + 1$ with $a = 0$ and $b = 2$.

   (b) $f(x) = \ln(x^2 + 4x + 14)$ with $a = -4$ and $b = 2$.

4. Consider the function F given by

$$F(x) = \int_0^x \cos(t)e^{t^2}\, dt.$$

   Find *where* the local maxima and minima of $F(x)$ on $(0, 2\pi)$ occur. (Do not try and evaluate F at these points!)

Challenge Problems

5. Compute the following limits

   (a) $\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\right)$

   (b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2\pi}{n} \sin\left(\frac{2\pi i}{n}\right)$

   Hint: how do these limits relate to Riemann sums?

6. Let

$$g(x) = \int_0^{h(x)} \frac{1}{\sqrt{1+t^4}}\,\mathrm{d}t, \qquad h(x) = \int_0^{\cos(x)} \left(1+\sin(s^2)\right)\,\mathrm{d}s.$$

   What's the value of $g'(\frac{\pi}{2})$?