Lecture hours 9-11

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This Tutorial is a review for the midterm.

Definitions

Definition (Subspace - Span version)**.** A subspace of $\mathbb{R}^n$ is a set of vectors in $\mathbb{R}^n$ that can be described as a span of vectors.

Definition (Subspace - Standard version)**.** A subspace of $\mathbb{R}^n$ subset V of $\mathbb{R}^n$ with the following properties:

(i) V is a non-empty set.

(ii) If $\vec{u}$ is in V , $\vec{u}$ is also in V for any scalar $k \in \mathbb{R}$ (We say V is closed under scalar multiplication.)

(iii) If $\vec{u}$ and $\vec{w}$ are in V, their sum $\vec{u} + \vec{w}$ is also in V. (We say V is closed under addition.)

Definition (Basis)**.** The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are a basis of a subspace V if they span V and are linearly independent. In other words, a basis of a subspace V is the minimal set of vectors needed to span all of V.

Definition (Dimension of a subspace)**.** The dimension of the subspace V is the number of vectors in a basis of V.

Definition (Linear Transformations)**.** We define a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ as a function with two properties:

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ (we say T preserves vector addition)
2. $T(\vec{x}) = cT(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$ and $c \in \mathbb{R}$ (we say T preserves scalar multiplication)

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Problem 21 (Subspace)**.** Suppose that V is the set of all solutions of the homogeneous system

$$\begin{cases} x_1 + 2x_2 - 2x_3 + 2x_4 - x_5 = 0, \\ x_1 + 2x_2 - x_3 + 3x_4 - 2x_5 = 0, \\ 2x_1 + 4x_2 - 7x_3 + x_4 + x_5 = 0. \end{cases} \tag{5.1}$$

Show that V is a subspace of $\mathbb{R}^5$.

Solution 21 (Subspace)

- A proof using the span version of the definition of subspace:

  The rref for system (5.1) is given by

  $$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 4 & -3 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

  which means we can write all solutions of (5.1) as

  $$(-2s-4t+3u, s, u-t, t, u) = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -4 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = s\vec{v}_1 + t\vec{v}_2 + u\vec{v}_3, \quad s, t, u \in \mathbb{R}.$$

  In other words V can be be described as the span of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

- A proof using the standard definition of subspace:

  See Problem 16 part a).

Problem 22 (Basis of a subspace)**.** Let U be a subspace of $\mathbb{R}^5$ defined by $U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2, x_3 = 7x_4\}$. Find a basis for U.

Solution 22 (Basis of a subspace)

a) Note that U is the set of all vectors in $\mathbb{R}^5$ such that

$$\begin{bmatrix} 3x_2 \\ x_2 \\ 7x_4 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 7 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad x_2, x_4, x_5 \in \mathbb{R}.$$

Define

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 7 \\ 1 \\ 0 \end{bmatrix} \quad and \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus $U = span\,\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

b) By Gaussian elimination we can see that

$$rref \left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

That is, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set of vectors.

Problem 23 (Basis of a subspace). Suppose $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis of $\mathbb{R}^4$. Prove that

$$\{\vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3, \vec{v}_3 + \vec{v}_4, \vec{v}_4\}$$

is also a basis of $\mathbb{R}^4$.

Solution 23 (Basis of a subspace)

a) The set $\{\vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3, \vec{v}_3 + \vec{v}_4, \vec{v}_4\}$ spans $\mathbb{R}^4$.

Take any vector $\vec{v} \in \mathbb{R}^4$. Since $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \in \mathbb{R}^4$ is a basis of $\mathbb{R}^4$, there are $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that

$$v = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4.$$

Note that

$$\begin{aligned}
\vec{v} &= c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 \\
&= c_1\vec{v}_1 + \vec{0} + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 \\
&= c_1\vec{v}_1 + (c_1\vec{v}_2 - c_1\vec{v}_2) + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 \\
&= c_1(\vec{v}_1 + \vec{v}_2) + (c_2 - c_1)\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4.
\end{aligned}$$

We can use the same strategy to obtain the terms $\vec{v}_2 + \vec{v}_3$ and $\vec{v}_3 + \vec{v}_4$ in the expression for $\vec{v}$:

$$\vec{v} = c_1(\vec{v}_1+\vec{v}_2)+(c_2-c_1)(\vec{v}_2+\vec{v}_3)+(c_3-c_2+c_1)(\vec{v}_3+\vec{v}_4)+(c_4-c_3+c_2-c_1)\vec{v}_4.$$

b) The set $\{\vec{v}_1 + \vec{v}_2, \vec{v}_2 + \vec{v}_3, \vec{v}_3 + \vec{v}_4, \vec{v}_4\}$ is linearly independent.

Suppose there are $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that

$$c_1(\vec{v}_1 + \vec{v}_2) + c_2(\vec{v}_2 + \vec{v}_3) + c_3(\vec{v}_3 + \vec{v}_4) + c_4\vec{v}_4 = \vec{0}. \tag{5.2}$$

We need to show $c_1, c_2, c_3, c_4 = 0$.

Note we can rewrite equation (5.2) as follows

$$c_1\vec{v}_1 + (c_1 + c_2)\vec{v}_2 + (c_2 + c_3)\vec{v}_3 + (c_3 + c_4)\vec{v}_4 = \vec{0}.$$

Since $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly independent vectors, we have $c_1, c_2, c_3, c_4 = 0$.

Problem 24 (Linear Transformations). Find $a, b \in \mathbb{R}$ such that $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + axyz)$$

is a linear transformation.

Solution 24 (Linear Transformations)

If we want T to preserve vector addition, b has to be zero.

And if we want T to preserve scalar multiplication, we need a to be zero.

Thus T is a linear transformation for $a = b = 0$. This is a statement you must justify using the definition of linear Transformation.

Problem 25 (Linear Transformations). Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let $\vec{v_1}, \ldots, \vec{v_k}$ be vectors in R^n. True or false? If false, give a counter-example. If true, explain why.

a) If the vectors $T(\vec{v_1}), \ldots, T(\vec{v_k})$ are linear independent, then $\vec{v_1}, \ldots, \vec{v_k}$ are also linear independent.

b) If the vectors $\vec{v_1}, \ldots, v_k$ are linear independent, then $T(\vec{v_1}), \ldots, T(\vec{v_k})$ are also linear independent.

Solution 25 (Linear Transformations)

a) This is true. Suppose there are $c_1, \ldots c_k \in \mathbb{R}$ such that

$$c_1\vec{v_1} + \cdots + c_k\vec{v_k} = \vec{0}.$$

We need to show that $c_1 = \cdots = c_k = 0$.

Since T is a linear transformation we have:

$$T\left(c_1\vec{v_1} + \cdots + c_k\vec{v_k}\right) = c_1T(\vec{v_1}) + \cdots + c_kT(\vec{v_k}),$$

and

$$T(\vec{0}) = \vec{0}.$$

Thus

$$c_1T(\vec{v_1}) + \cdots + c_kT(\vec{v_k}) = \vec{0}.$$

By definition of linear independence follows that $c_1 = \cdots = c_k = 0$.

b) This is false. Let $n = 2$ and define T as

$$T(x, y) = (x, 0).$$

By definition, the standard basis $\{\vec{e_1}, \vec{e_2}\}$ is a set of linearly independent vectors. However, $\{T(\vec{e_1}), T(\vec{e_2})\} = \{\vec{e_1}, \vec{0}\}$ is not a linearly independent set of vectors. Recall that any set of vectors that contains $\vec{0}$ is not linearly independent.