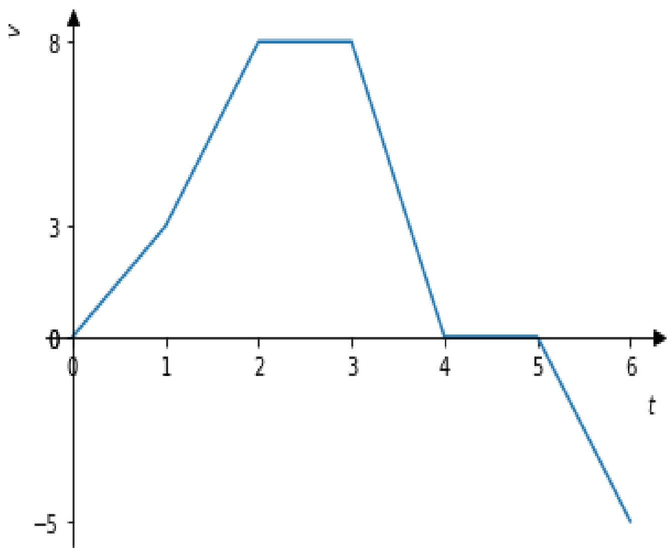
1. The graph below shows the velocity v in metres per second of a particle at time t seconds.



(a) Describe the motion of this particle. When is it moving forwards? When is it accelerating? When is it decelerating?

(b) Express the velocity $v(t)$ as a function of t.

(c) Sketch a graph of the acceleration of this particle.

(d) Express the acceleration $a(t)$ of the particle as a function of t

(e) Sketch a graph for the position of this particle.

(f) Express the position $s(t)$ as a function of t

Solution

(a) The particle starts accelerating forward at a constant rate between $t = 0$ and $t = 1$, it then increases its acceleration until $t = 2$. Between $t = 2$ and $t = 3$, it moves forward at a constant speed, and then begins to decelerate and reaches a stop at $t = 4$ and stays still until $t = 5$, at which point it starts accelerating backwards.

(b) We will show the first few line segments here. For $0 \le t \le 1$ we need to find the equation of the line going through the points $(0,0)$ and $(1,3)$. This is $v = 3t$, therefore $v(t) = 3t$. Then for $1 \le t \le 2$ we need to find the equation of the line through $(1,3)$ and $(2,8)$. This is $v = 5t - 2$. We keep going like this to get the final answer

$$v(t) = \begin{cases} 3t & 0 \le t \le 1 \\ 5t - 2 & 1 \le t \le 2 \\ 8 & 2 \le t \le 3 \\ -8t + 32 & 3 \le t \le 4 \\ 0 & 4 \le t \le 5 \\ -5t + 25 & 5 \le t \le 6 \end{cases}$$

(c)

(d) We can find the acceleration by differentiating the velocity. So we get

$$a(t) = \begin{cases} 3 & 0 \le t \le 1 \\ 5 & 1 \le t \le 2 \\ 0 & 2 \le t \le 3 \\ -8 & 3 \le t \le 4 \\ 0 & 4 \le t \le 5 \\ -5 & 5 \le t \le 6 \end{cases}$$

(e)

(f) Remember that velocity is the rate of change of position. This means that

$$\frac{d}{dt} s(t) = v(t)$$

So we need to find a function whose derivative is $v(t)$. This is the antiderivative of $v(t)$. For the interval $0 \le t \le 1$ we need to find some function whose derivative is $3t$. Such a function is exactly of the form $\frac{3}{2}t^2 + C$ where C is some constant. We want our particle to start at distance 0. So that means $s(0) = 0$, so we need to choose $C = 0$. This means that between 0 and 1 the distance is

$$s(t) = \frac{3}{2}t^2$$

So at $t = 1$, the position of the particle is $\frac{3}{2}$. Between 1 and 2 we want the derivative to be $5t - 2$ and we also want $s(1) = \frac{3}{2}$. The antiderivative of $5t - 2$ is $\frac{5}{2}t^2 - 2t + C$ where C is some constant. By plugging in $t = 1$, we see that C has to equal 1. So between 1 and 2 the position is given by

$$s(t) = \frac{5}{2}t^2 - 2t + 1$$

So at $t = 2$ the position is 7. We keep going with the same method. The final answer should be.

$$s(t) = \begin{cases} \frac{3}{2}t^2 & 0 \leq t \leq 1 \\ \frac{5}{2}t^2 - 2t + 1 & 1 \leq t \leq 2 \\ 8t - 9 & 2 \leq t \leq 3 \\ -4t^2 + 32t - 45 & 3 \leq t \leq 4 \\ 19 & 4 \leq t \leq 5 \\ -\frac{5}{2}t^2 + 25t - 43.5 & 5 \leq t \leq 6 \end{cases}$$

2. Suppose we have a circle inscribed in a square.

   (a) If the perimeter of the square is increasing at a rate of 10 m/s, at what rate is the circumference of the circle increasing?

   (b) If the area of the circle is increasing at a rate of 10 m^2/s, at what rate is the diagonal of the square increasing when the area of the circle is π?

Solution

(a) If the side length of the square is s. Then the radius r of the circle is $s/2$. Now the perimeter P of the square is $4s$, and the circumference C of the circle is $2\pi r = \pi \cdot s$. Therefore we can see that $C = \frac{\pi}{4}P$. Now if we differentiate with respect to time t on both sides we get

$$\frac{dC}{dt} = \frac{\pi}{4}\frac{dP}{dt}$$

If the perimeter of the square is increasing at a rate of $10m/s$, then $\frac{dP}{dt} = 10$, thefefore $\frac{dC}{dt} = 2.5\pi$

(b) The area of the circle A, is equal to $A = \pi r^2 = \frac{\pi}{4}s^2$. The length of the diagonal of the square D is equal to $D = \sqrt{2}s$. We want to write D in terms of A. Notice that $\frac{\pi}{8}D^2 = A$, therefore

$$D = \frac{\sqrt{\pi}}{2\sqrt{2}}\sqrt{A}$$

So by differentiating both sides with respect to t and using the chain rule we get

$$\frac{dD}{dt} = \frac{\sqrt{\pi}}{2\sqrt{2}}\frac{1}{2\sqrt{A}} \cdot \frac{dA}{dt}$$

The question wants us to evaluate this when $A = \pi$ and $\frac{dA}{dt} = 10$. So we get the final answer is

$$\frac{5}{2\sqrt{2}}$$

3. If \$1500 is borrowed at 8% interest, find the amounts due at the end of 5 years if the interest is compounded:

   (a) annually;

   (b) monthly;

   (c) daily;

   (d) continuously.

Solution

(a) The initial amount is 1500. For each year we multiply it by 1.08. So for 5 years this will be

$$1.08^5 \times 1500 = 2203.9921152$$

(b) Now instead of multiplying by 1.08 for every year, we will multiply by

$$\left(1 + \frac{0.08}{12}\right)$$

for every month. This means that the final amount due after 5 years is

$$\left(1 + \frac{0.08}{12}\right)^{12\times 5} \times 1500 = 2234.76856245$$

(c) Here we multiply by $\left(1 + \frac{0.08}{365}\right)$ for every day, so the final amount due will be

$$\left(1 + \frac{0.08}{365}\right)^{365\times 5} \times 1500 = 2237.63897036$$

(d) Now we remember that the limit

$$\lim_{n\to\infty}\left(1 + \frac{0.08}{n}\right)^{n} = e^{0.08}$$

Therefore, when the interest is compounded continuously, the final amount due is

$$\left(e^{0.08}\right)^5 \times 1500 = 2237.73704646$$

4. Recall that the rate of cooling of an object is proportional to the difference in temperature between the object and its surrounding.

   A freshly brewed cup of coffee has temperature 95°C in a 20°C room. When its temperature is 70°C, it is cooling at a rate of 1°C per minute. When does this occur?

Solution

Since the rate of cooling of an object is proportional to the difference in temperature between the object and its surrounding, if we let T denote the temperature of the cup and t denote the time, then

$$\frac{dT}{dt} = k(T - 20)$$

where k is some constant. Now let $y = T - 20$, then $\frac{dy}{dt} = \frac{dT}{dt}$. We can then write the equation above as

$$\frac{dy}{dt} = ky$$

This means that

$$y = y_0 e^{kt}$$

where y_0 is the initial value for y. From the problem we can deduce that $y_0 = 95 - 20 = 75$. Therefore,

$$y = 75e^{kt}$$

and

$$y' = 75ke^{kt}$$

The problem tells us that at a specific time t_0, we have $y(t_0) = 70 - 20 = 50$ and $y'(t_0) = -1$. Therefore

$$50 = 75e^{kt_0}$$

and

$$-1 = 75ke^{kt_0}$$

Dividing the two equations above, we get that $k = -\frac{1}{50}$. When we plug that back into our first equation we see that

$$50 = 75e^{-t_0/50}$$

We can rearrange this equation and take ln of both sides to find

$$\ln(2/3) = -\frac{t_0}{50}$$

Therefore,

$$t_0 = -50\ln(2/3) = 20.2732554054$$

5. Two cars start moving from the same point. One travels south at 60 km/h and the other travel west at 25 km/h. At what rate is the distance between the cars increasing two hours later?

Solution

The coordinates of the first care are $(0, -60t)$, and the coordinates of the second one are $(-25t, 0)$. So, the distance between the cars is

$$\sqrt{25^2t^2 + 60^2t^2} = t\sqrt{25^2 + 60^2} = 65t$$

So the distance between the cars is increasing at a constant rate of 65 km/h.

6. A particle is moving along the curve $y = \sqrt{x}$. As the particle passes through the point (4,2), its x-coordinate increases at a rate of 3 units/s. How fast is its y-coordinate changing at this instant? *Hint: think of the coordinates as functions of time, $y = y(t)$ and $x = x(t)$; what can you say about the coordinates of the particle at time t?.*

Solution

We can write y, x as functions of t. So

$$y(t) = \sqrt{x(t)}$$

Differentiating both sides with respect to t and using the chain rule we find

$$y'(t) = x'(t)\frac{1}{2\sqrt{x(t)}}$$

We want to find y' when $x = 4, y = 2$ and $x' = 3$. So plugging these all in we find

$$y' = \frac{3}{2\sqrt{4}} = \frac{3}{4}\text{units/s}$$