Math 140 **Tutorial 4**

Jan 27 - 30

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1. By calculating the limit
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
find the derivatives of the following functions at $x = a$:

   (a) $f(x) = \frac{1}{x}$ for $a \neq 0$.

   (b) $f(x) = \sqrt{x}$ for $a \neq 0$.

   (c) $f(x) = \frac{1}{\sqrt{x}}$ for $a \neq 0$.

   (d) $f(x) = x^3 - 3x + 5$.

   (e) $f(x) = x^{1/4}$.

   (f) * $f(x) = \sin(x^2)$.

   [Hint: For (f) you can use that as h approaches 0, $\sin h \approx h$ and $\cos h \approx 1 - h^2$]

GROUP WORK 1, SECTION 2.7

Follow that Car

The distance travelled by a car is given by $d(t) = 8\left(t^3 - 6t^2 + 12t\right)$, where d is in miles and t is in hours.

1. Draw a graph of $d(t)$ from $t = 0$ to $t = 3$.

2. Does the car ever stop?

3. What is the average velocity over [1, 3]? over [1.5, 2.5]? over [1.9, 2.1]?

4. Estimate the instantaneous velocity at $t = 2$. Give a physical interpretation of your answer.

1. We know that $d(t) = 8(t^3 - 6t^2 + 12t)$

now factorize $= 8t(\underbrace{t^2 - 6t + 12})$

$b^2 - 4ac < 0$ so has no roots

$d(0) = 0$

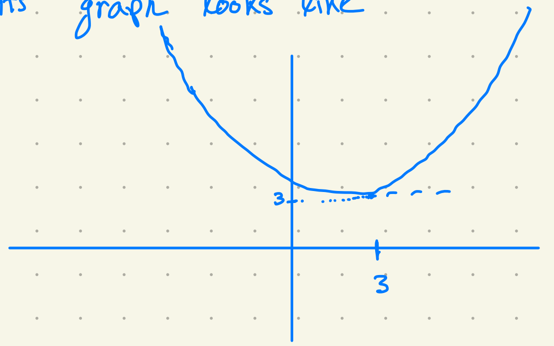
$d(1) = 52$

$d(2) = 64$

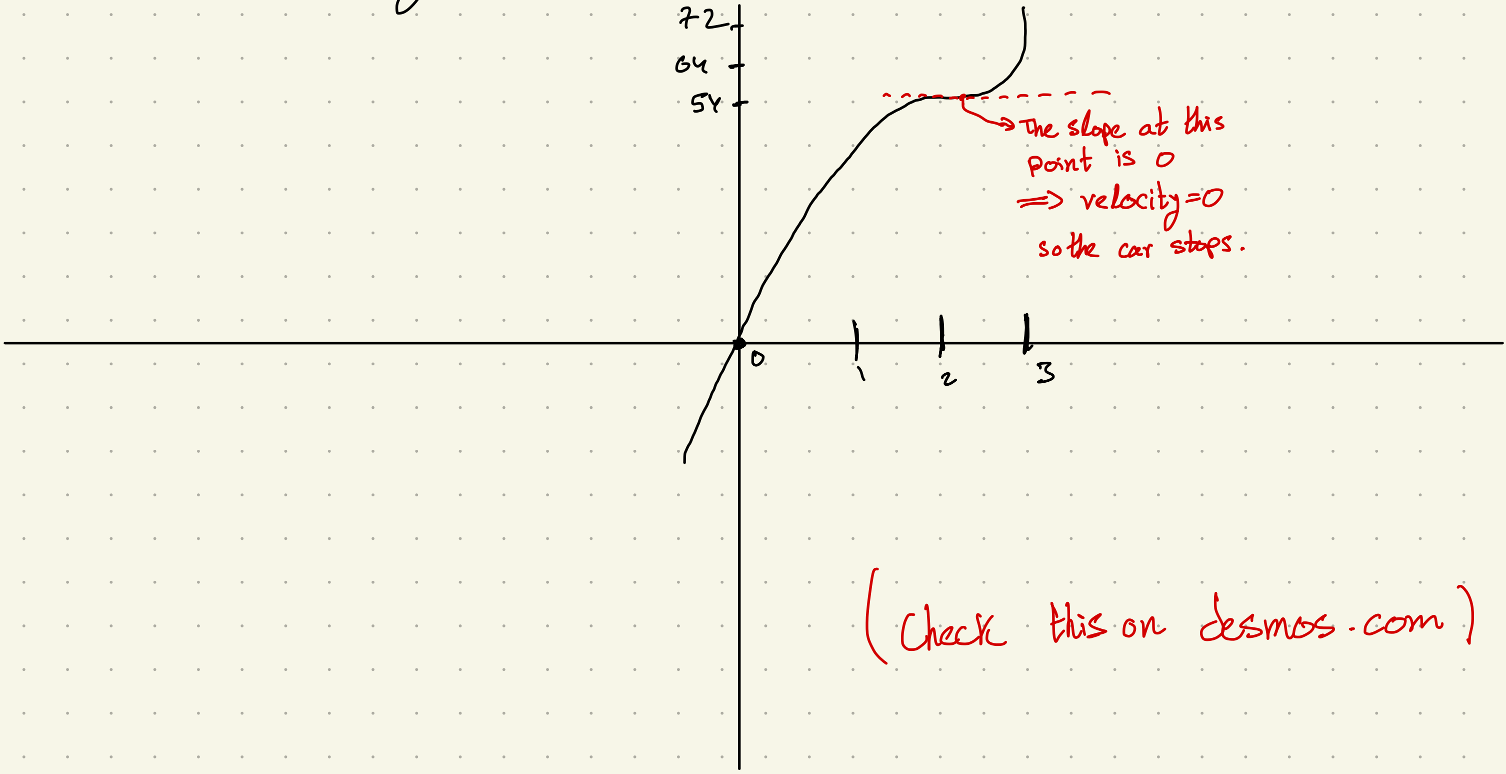
$d(3) = 72$

$t^2 - 6t + 12 \overset{\text{Complete the square}}{=} (t-3)^2 + 3$

So its graph looks like



So the graph of $8t(t^2 - 6t + 12)$ should look like



The slope at this point is 0
$\Rightarrow$ velocity $= 0$
so the car stops.

(Check this on desmos.com)

3. average velocity $= \dfrac{\Delta \text{distance}}{\Delta \text{time}}$

- on $[1,3]$: Δ distance $= d(3) - d(1) = 20$

  Δ time $= 3 - 1 = 2$

  average velocity $= \frac{20}{2} = 10$ miles/hour

- On $[1.5, 2.5]$: Δ distance $= d(2.5) - d(1.5) = 2$

  Δ time $= 2.5 - 1.5 = 1$

  average velocity $= \frac{2}{1} = 2$ miles/hour

- On $[1.9, 2.1]$: Δ distance $= d(21) - d(1.9) = 0.016$

  Δ time $= 21 - 1.9 = 0.2$

  average velocity $= \frac{0.016}{0.2} = 0.8$ miles/hour

Looks like as we get closer to $t=2$, the average velocity approaches 0. So the instantaneous velocity at $t=2$ is 0 miles/hour. That means the car stops momentarily.

GROUP WORK 3, SECTION 2.7

Connect the Dots

A company does a study on the effect of production value p of an advertisement on its consumer approval rating A. After interviewing eight focus groups, they come up with the following data:

Production Value	Consumer Approval
\$1000	32%
\$2000	33%
\$3000	46%
\$3500	55%
\$3600	61%
\$3800	65%
\$4000	69%
\$5000	70%

Assume that $A(p)$ gives the consumer approval percentage as a function of p.

1. Estimate $A'(\$3500)$. Is this likely to be an overestimate or an underestimate?

We can estimate $A'(3500) \cong \frac{A(3600) - A(3500)}{3600 - 3500} = \frac{61-55}{100} = 0.07$

It looks like around 3500, A is increasing at a decreasing rate and $3600 > 3500$ so this is likely an underestimate.

2. Interpret your answer to Problem 1 in real terms. What does your estimate of $A'(\$3500)$ tell you?

It means roughly that every \$1 of extra production value will increase consumer approval by 0.07%

3. What are the units of $A'(p)$?

It is %/\$

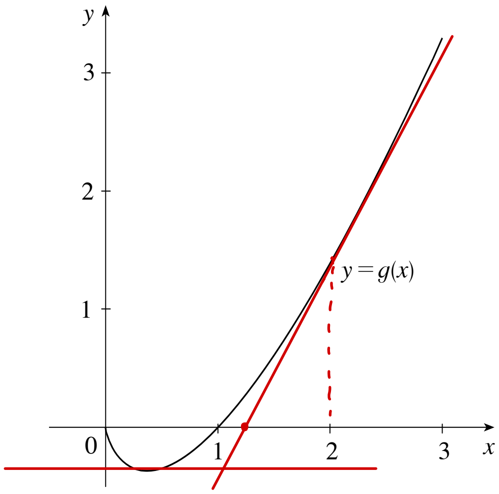
4. Estimate $A'(\$3550)$. Is your estimate better or worse than your estimate of $A'(\$3500)$? Why?

Based on the information given, our best estimate is still 0.07 and we expect it to give a slightly better estimate since $A'(3500)$ was likely an underestimate.

GROUP WORK 1, SECTION 2.8

Tangent Lines and the Derivative Function

The following is a graph of $g(x) = x \ln x$.



It is a fact that the derivative of this function is $g'(x) = \ln x + 1$.

1. Sketch the line tangent to $g(x)$ at $x = 2$ on the graph above.

2. Find an equation of the tangent line at $x = 2$.

$g'(2) = \ln(2) + 1$ and $g(2) = 2\ln(2)$. So the line has slope $\ln(2)+1$ and passes through the point $(2, 2\ln(2))$. So it has equation

$$y = (\ln(2)+1)x - 2.$$

3. Now sketch the line tangent to $g(x)$ at $x = \frac{1}{e} \approx 0.368$.

4. Find an equation of the tangent line at $x = \frac{1}{e}$.

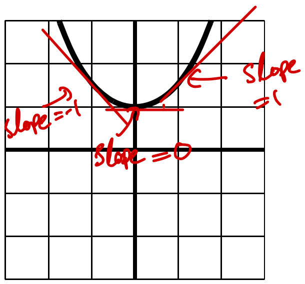
Note: $\ln(\frac{1}{e}) = -\ln(e) = -1.$

This line has slope $g'(\frac{1}{e}) = \ln(\frac{1}{e}) + 1 = 0$ and passes through $(\frac{1}{e}, g(\frac{1}{e})) = (\frac{1}{e}, \frac{-1}{e})$ so it has equation $y = \frac{-1}{e}$.

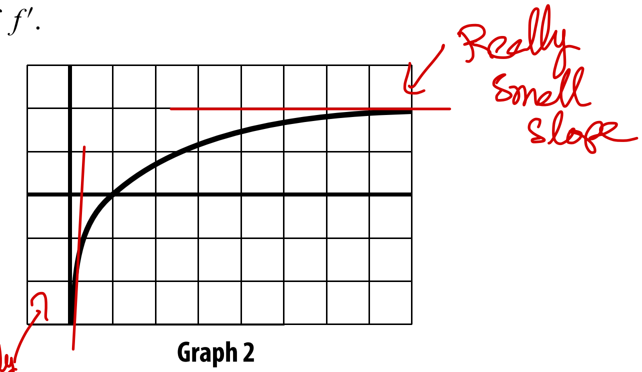
GROUP WORK 3, SECTION 2.8

The Derivative Function

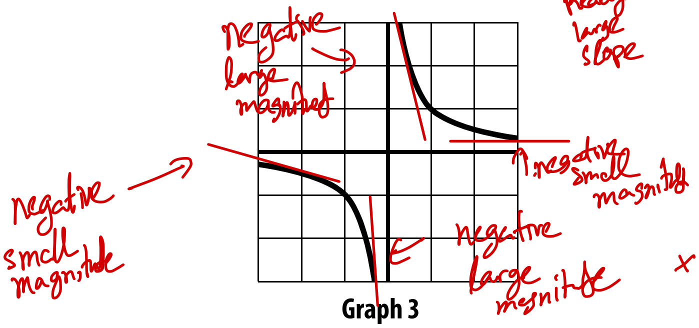
The graphs of several functions f are shown below. For each function, estimate the slope of the graph of f at various points. From your estimates, sketch graphs of f'.



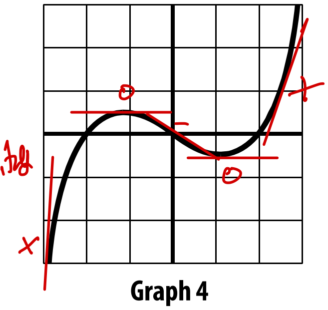
Graph 1



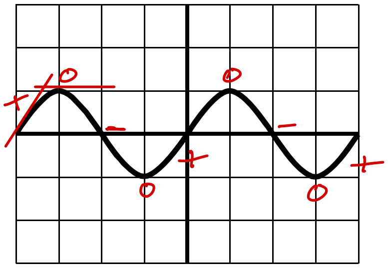
Graph 2



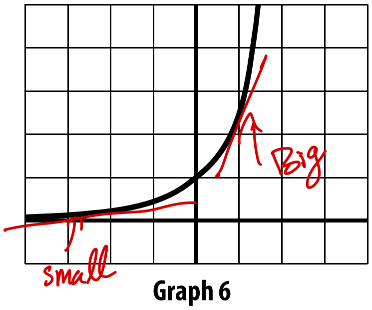
Graph 3



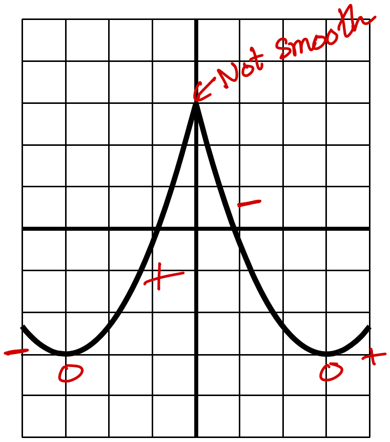
Graph 4



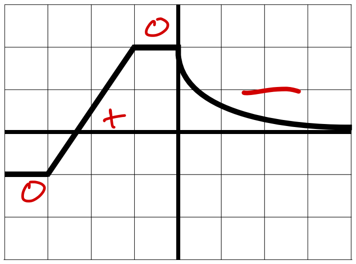
Graph 5



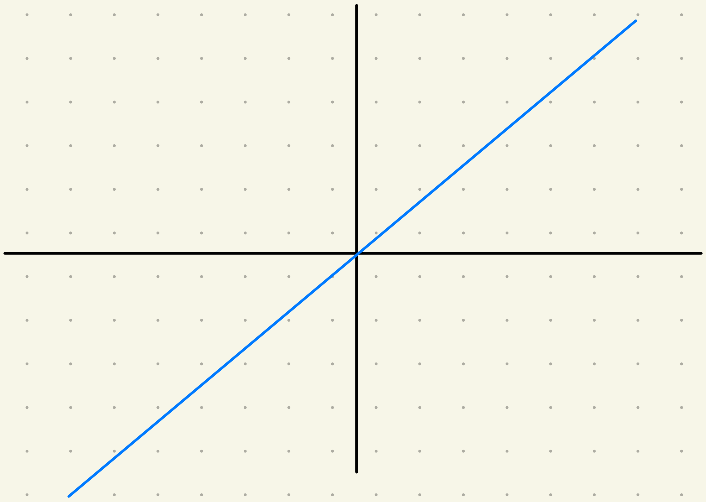
Graph 6



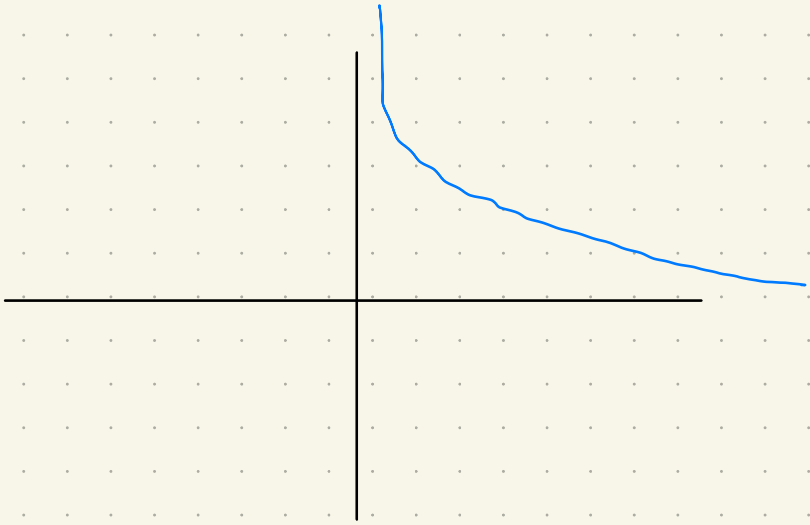
Graph 7



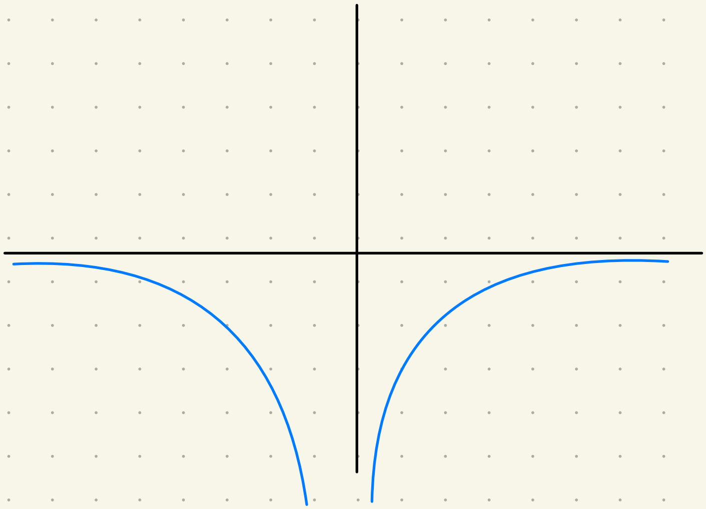
Graph 8



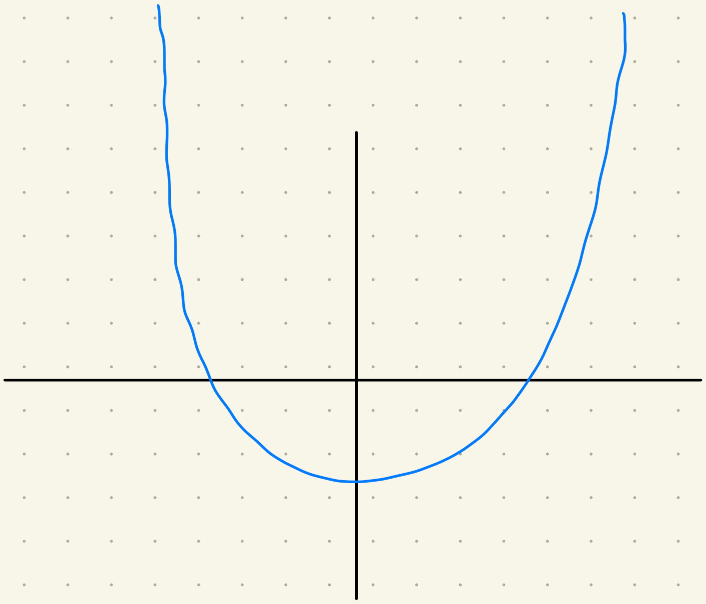
Graph 1



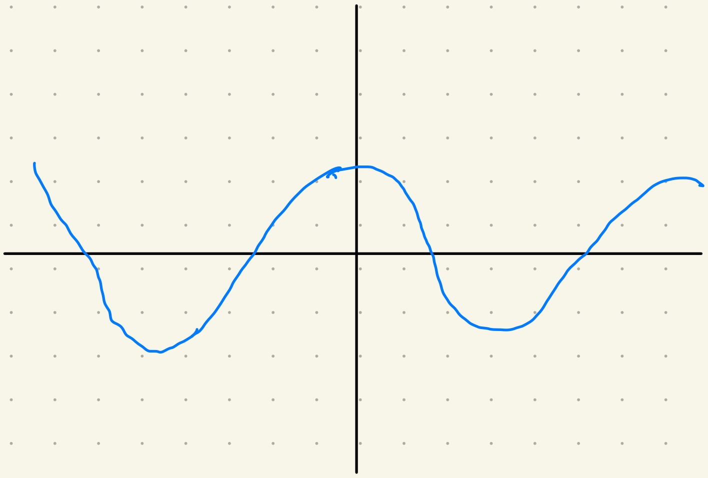
Graph 2



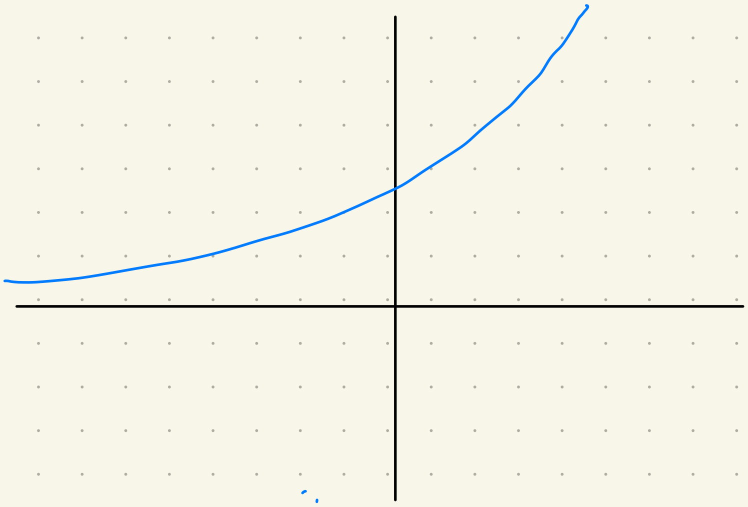
Graph 3



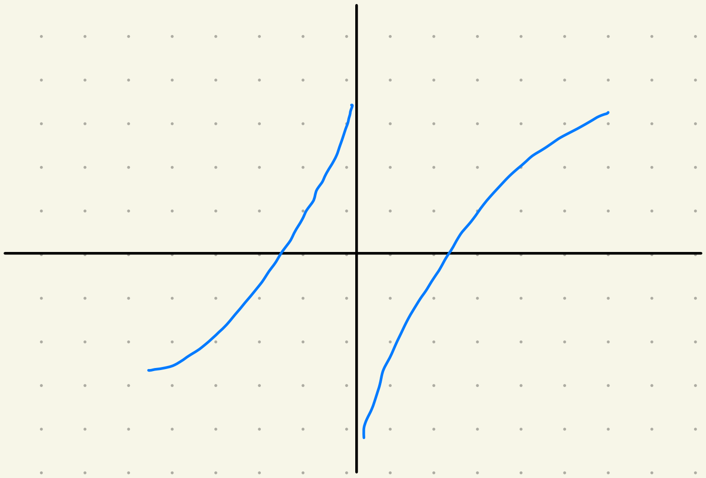
Graph 4



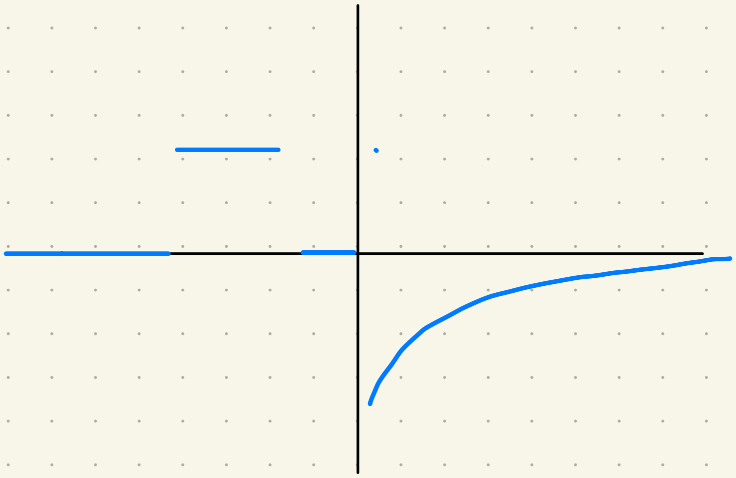
Graph 5



Graph 6



Graph 7



Graph 8

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1. By calculating the limit
$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$
find the derivatives of the following functions at $x=a$:

   (a) $f(x)=\dfrac{1}{x}$ for $a\neq 0$.

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   (d) $f(x)=x^3-3x+5$.

   (e) $f(x)=x^{1/3}$.

   (f) $f(x)=\sin(x^2)$.

   $\big[$Hint: For (f) you can use that as h approaches 0, $\sin h\approx h$ and $\cos h\approx 1-h^2$ $\big]$

Solution

(a)
$$\lim_{h\to 0}\frac{\frac{1}{a+h}-\frac{1}{a}}{h}=\lim_{h\to 0}\frac{-h}{ha(a+h)}=\lim_{h\to 0}\frac{-1}{a(a+h)}=\frac{-1}{a^2}$$

(b)
$$\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h}$$

multiply top and bottom by $\sqrt{a+h}+\sqrt{a}$ gives
$$\lim_{h\to 0}\frac{h}{h(\sqrt{a+h}+\sqrt{a})}=\lim_{h\to 0}\frac{1}{\sqrt{a+h}+\sqrt{a}}=\frac{1}{2\sqrt{a}}$$

(c)
$$\lim_{h\to 0}\frac{\frac{1}{\sqrt{a+h}}-\frac{1}{\sqrt{a}}}{h}=\lim_{h\to 0}\frac{\sqrt{a}-\sqrt{a+h}}{h\sqrt{a+h}\sqrt{a}}$$

again, multiply top and bottom by $\sqrt{a} + \sqrt{a+h}$ gives

$$\lim_{h \to 0} \frac{-h}{h\sqrt{a+h}\sqrt{a}(\sqrt{a+h} + \sqrt{a}) = } \frac{-1}{2\sqrt{a}^3}$$

(d)

$$\lim_{h \to 0} \frac{(a+h)^3 - 3(a+h) + 5 - a^3 + 3a - 5}{h} = \lim_{h \to 0} \frac{3a^2h + 3ah^2 + h^3 - 3h}{h} =$$

$$\lim_{h \to 0} 3a^2 + 3ah + h^2 - 3 = 3a^2 - 3$$

(e)

$$\lim_{h \to 0} \frac{(a+h)^{1/4} - a^{1/4}}{h}$$

Now multiply top and bottom by $(a+h)^{2/3} + (a+h)^{1/3}a^{1/3} + a^{2/3}$. This gives,

$$\lim_{h \to 0} \frac{h}{h((a+h)^{2/3} + (a+h)^{1/3}a^{1/3} + a^{2/3})} = \lim_{h \to 0} \frac{1}{(a+h)^{1/3}a^{1/3} + a^{2/3}} =$$

$$\frac{1}{3a^{2/3}}$$

(f)

$$\lim_{h \to 0} \frac{\sin\left(a^2 + 2ah + h^2\right) - \sin(a^2)}{h}$$

then we use the formula $\sin(x+y) = \sin x \cos y + \cos x \sin y$ with $x = a^2$, $y = 2ah + h^2$.

$$\lim_{h \to 0} \sin(a^2) \left[\frac{\cos(h(2a+h) - 1}{h} \right] + \cos(a^2) \left[\frac{\sin(h(2a+h)}{h} \right]$$

now applying the hint and expanding this is equal to

$$2a\cos(a^2)$$