Math 140 **Tutorial 11**

March 24 - 27

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1. Sketch the graphs of a continuous functions on the interval $[1,5]$ and which satisfies the following properties

   (a) Absolute maximum at 1, absolute minimum at 3, local minimum at 2 and local maximum at 4.

   (b) Absolute maximum at 2, absolute minimum at 5, 4 is a critical number but there is no local maximum or minimum there.

2. Find the absolute maximum and absolute minimum values of f on the given interval.

   (a) $f(x) = x^3 - 6x + 5,\ [-2, 5]$

   (b) $f(x) = x + \dfrac{1}{x},\ [0.2, 4]$

   (c) $f(x) = x^a(1-x)^b,\ [0,1]$ (a and b are positive, real numbers)

   (d) $f(t) = 2\cos t + \sin 2t,\ [0, \pi/2]$

Solution

(a) First we find the critical points by solving the equation $f'(x) = 0$. This is

$$3x^2 - 6 = 0$$

Which has two solutions $\pm\sqrt{2}$. Since f is continuous, the absolute maximum and minimum will either be critical points or will be on the boundary of the interval. So we have to check all four points.

$$f(-2) = 9,\ f(-\sqrt{2}) = 10.657,\ f(\sqrt{2}) = -0.657,\ f(5) = 100$$

Therefore, the absolute maximum is 100 and the absolute minimum is -0.657

(b) Again we find the critical points by solving $f'(x) = 0$. This is

$$1 - \frac{1}{x^2} = 0$$

, which has two solutions ± 1, only one of which is on our interval. Since f is continuous, the absolute maximum and minumum will either be critical points or will be on the boundary. So we have to check all three points.

$$f(0.2) = 5.2,\ f(1) = 2,\ f(4) = 4.25$$

Therefore the absolute maximum is 5.2 and the absolute minimum is 2.

(c) We solve the equation $f'(x) = 0$. This is

$$ax^{a-1}(1-x)^b - bx^a(1-x)^{b-1} = 0$$

We can take $x^{a-1}(1-x)^{b-1}$ as a common factor and divide by it to get (note that $x^{a-1}(1-x)^{b-1}$ is equal to 0 only at $x = 0, 1$ which are our boundary points.)

$$a(1-x) - bx = 0$$

The solution to this equation is $x = \frac{a}{a+b}$. We can conclude that the absolute maximum is

$$\frac{a^a b^b}{(a+b)^{a+b}}$$

and the absolute minimum is 0.

(d) As before, we will solve $f'(t) = 0$. This is

$$-2\sin t + 2\cos 2t = 0$$

Now we can use the trig identity

$$\cos 2t = 1 - 2\sin^2 t$$

. We get the equation

$$2\sin^2 t + \sin t - 1 = 0$$

This is a quadratic equation in $\sin t$, which we can solve. It has solutions $\sin t = \frac{1}{2}, -1$. Therefore, in the interval $[0, \pi/2]$ the only critical point is at $t = \pi/6$. To find the absolute maximum and minimum we just need to check the three points

$$f(0) = 2,\ f(\pi/6) = 2.598,\ f(\pi/2) = 0$$

Therefore, the absolute max is 2.598 and the absolute min is 0.

3. Prove that the function

$$f(x) = x^{101} + x^{51} + x + 1$$

has neither a local maximum or a local minimum.

Solution

The function is differentiable, therefore the critical points will correspond to points where $f'(x) = 0$. We know that

$$f'(x) = 101x^{100} + 51x^{50} + 1$$

This is never 0, because $x^100 = (x^{50})^2$ which is always greater than or equal to 0, and similarly $x^{50} = (x^{25})^2$.

4. Find two numbers whose difference is 10 and whose product is minimal.

Solution

We will label the two numbers a, b, such that $a - b = 10$. We want to minimize ab. Note that we can write $a = 10 + b$. So $ab = (10 + b)b$. Therefore, we are trying to minimize the function $f(b) = (10 + b)b$. First, we check the critical points $f'(b) = b + (b + 10) = 2b + 10$. So the only critical point is at $b = -5$. This is a local minimum. This can also be seen by completing the square

$$(10 + b)b = (b + 5)^2 - 25$$

which has a minimum at $b = -5$.

5. Which point on the line

$$y = 2x + 1$$

is closest to the origin?

Solution

For a given point (x, y) on the line, its distance to the origin is $\sqrt{x^2 + y^2}$. Therefore, we are trying to minimize the value of $x^2 + y^2$. By plugging in the equation for the line, we find that we are trying to minimize the value of

$$f(x) = x^2 + (2x + 1)^2 = 5x^2 + 4x + 1$$

We can again find the critical point by differentiation

$$f'(x) = 10x + 4$$

, therefore the minimum will be at $x = \frac{-2}{5}$, $y = \frac{1}{5}$. Once again, this can be seen by completing the square

$$5x^2 + 4x + 1 = 5(x + \frac{2}{5})^2 + \frac{1}{5}$$

6. Find the points on the ellipse $4x^2 + y^2 = 4$ which are farthest from the point $(0, 1)$.

Solution

Given coordinates (x, y), the square of the distance to the point $(0, 1)$ is

$$x^2 + (y - 1)^2$$

This is the function we are trying to maximize. We can plug in the constraint of the ellipse, which is

$$x^2 = 1 - \frac{y^2}{4}$$

So, we are trying to maximize

$$1 - \frac{y^2}{4} + y^2 - 2y + 1$$

which is proportional to

$$3y^2 - 8y + 5$$

We are trying to maximize this function on the interval

$$-2 \leq y \leq y$$

because we need $4x^2 + y^2 = 4$. To find the critical points we need to differentiate, and solve the equation

$$6y - 8 = 0$$

which has solution

$$y = \frac{4}{3}$$

Now we want to figure out the x-coordinate. They satisfy $4x^2 + y^2 = 4$, plugging in $y = \frac{4}{3}$ gives

$$x^2 = \frac{8}{9}$$

so

$$x = \pm\frac{2\sqrt{2}}{3}$$

The two critical points

$$(\pm\frac{2\sqrt{2}}{3}, \frac{4}{3})$$

Now we need to compare the critical points to the boundary points, which are $(0, \pm 2)$ by comparing these points we see that the one that is furthest is $(0, -2)$.

7. A cylindrical can is made from material such that the top and bottom cost twice as much as the sides. What dimensions will minimize the cost of a 4L can?

Solution

A cylinder has height h, and radius r. Its volume is

$$\pi r^2 h = 4$$

The cost of the material is

$$4\pi r^2 + 2\pi r h$$

So we are trying to minimize

$$2r^2 + rh$$

we can also plug in the constraint $h = \frac{4}{\pi r^2}$. So we are trying to minimize

$$2r^2 + \frac{4}{\pi r}$$

where r lies in the interval

$$0 < r$$

We differentiate and set to 0, to get the equation $4r - \frac{4}{\pi r^2} = 0$ which is equivalent to $r^3 = \frac{1}{\pi}$. So $r = \pi^{-1/3}$, therefore $h = 4\pi^{-1/3}$.

8. A person is standing at coordinates $(1, 4)$. They are making their way back to their house which has coordinates $(6, 8)$, but first they need to pass by a river, which goes down the x-axis. What is the fastest path for this trip? At what x-coordinate should this person reach the river?

Solution

There are several methods to solve this problem. I will present two, the first one being a longer tedious solution and the second is slightly simpler.

Solution 1

Clearly the fastest path would be a straight line to some point $(x, 0)$ on the x-axis and then a straight line back to the house at $(6, 8)$. The total distance is then

$$\sqrt{(x-1)^2+4^2}+\sqrt{(x-6)^2+8^2}$$

which is the function we are trying to minimize. To do so, we differentiate and equate it to 0. This gives

$$\frac{x-1}{\sqrt{(x-1)^2+4^2}}+\frac{x-6}{\sqrt{(x-6)^2+8^2}}=0$$

which is the same as

$$(x-1)\sqrt{(x-6)^2+8^2}+(x-6)\sqrt{(x-1)^2+4^2}=0$$

by moving things around and squaring we get

$$\frac{(x-1)^2}{(x-6)^2}=\frac{(x-1)^2+4^2}{(x-6)^2+8^2}$$

This is equivalent to

$$4(x-1)^2=(x-6)^2$$

which has positive solution $x=\frac{8}{3}$, which is the critical point. Therefore, the fastest path is to go straight to the point $(\frac{8}{3}, 0)$, and then straight to the house.

Solution 2

As noted before, the fastest path must be a straight line to the river, and then a straight line to the house. Notice that if the house had coordinates $(6, -8)$, then the fastest path to the house is exactly the line from $(1, 4)$ to $(6, -8)$ and this line passes through the x-axis at $x=\frac{8}{3}$. Notice that the fastest path then to the house would be to go to the river at $(\frac{8}{3}, 0)$ and then reflect the line which goes to $(6, -8)$.