Math 140 **Tutorial 13**
April 12 - 13

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1. For the following exercises you are given $\frac{df}{dx}$. Can you come up with some function $f(x)$ such that its derivative is the given $\frac{df}{dx}$

   (a) $\dfrac{df}{dx} = x^3 + x + 1$

   (b) $\dfrac{df}{dx} = \sin x$

   (c) $\dfrac{df}{dx} = e^{x+2} + \dfrac{x}{2}$

Solution

(a)
$$f(x) = \frac{x^4}{4} + \frac{x^2}{2} + x$$

(b)
$$f(x) = \cos x$$

(c)
$$f(x) = e^{x+2} + x^2$$

2. Find the most general antiderivative.

   (a) $f(x) = 0$

   (b) $f(x) = 3x^3 + 2x^2 + x + 1$

   (c) $h(y) = 17e^{-2y} + 123\sec^2 x$

   (d) $y(\theta) = \cos(\theta) - \sin(\theta)$

   (e) $f(x) = 5e^x - 3\cosh x$

   (f) $g(t) = \sin t + 2\sinh t$

Solution

(a)
$$g(x) = c$$

where c is any constant real number

(b)

$$g(x) = \frac{3}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + x + c$$

where c is any constant real number

(c)

$$g(x) = 123\tan x - \frac{17}{2}e^{-2y} + c$$

where c is any constant real number

(d)

$$g(\theta) = \sin\theta + \cos\theta + c$$

where c is any constant real number

(e)

$$g(x) = 5e^x - 3\sinh x + c$$

where c is any constant real number

(f)

$$h(t) = -\cos t + 2\cosh t + c$$

where c is any constant real number

3. Find a function f which satisfies the given conditions.

   (a) $f''(x) = 6x + 12x^2$

   (b) $f''(x) = 2e^t + 3\sin t$ with $f(0) = f(\pi) = 0$

   Parts (c) and (d) are more difficult that usual, and are certainly more difficult than questions to come on the final exam.

   (c) $f'(x) = f(x)$ with $f(0) = 1$

   $\Big[$*hint: Try to re-write this equation in terms of the function* $g(x) = e^{-x}f(x)\Big]$

   (d) $f''(x) = f(x)$ with $f(0) = 2$ and $f'(0) = 0$

   $\Big[$*Try writing* $g(x) = e^x f(x)$ *and show that* g *satisfies the equation*

   $$g''(x) = 2g'(x)$$

   Then write an equation in terms of the function $h(x) = e^{-2x}g'(x)\Big]$

Solution

(a) By taking the anti-derivative we know that

$$f'(x) = 3x^2 + 4x^3 + c_1$$

for some constant c_1, and taking the anti-derivative again, we find that

$$f(x) = x^3 + x^4 + c_1x + c_2$$

for some constant c_2. In particular, one function that satisfies this condition is

$$x^3 + x^4$$

(b) Similar to before we can take the anti-derivative to find that

$$f'(t) = 2e^t - 3\cos t + c_1$$

for some constant c_1, and then

$$f(t) = 2e^t - 3\sin t + c_1t + c_2$$

for some constant c_2. Now plugging in $f(0)$ and $f(\pi)$ we get the equations

$$2 + c_2 = 0$$

and

$$2e^\pi + c_1\pi + c_2 = 0$$

Therefore, $c_2 = -2$ and $c_1 = \frac{2-2e^\pi}{\pi}$. So the function is

$$f(t) = 2e^t - 3\sin t + \frac{2-2e^\pi}{\pi} - 2$$

(c) This one is slightly more tricky. Let's follow the hint. We calculate that

$$g'(x) = e^{-x}(f'(x) - f(x)) = 0$$

This means that $g(x) = c$ for some constant c. Therefore

$$f(x) = ce^x$$

Plugging in $f(0) = 1$ gives us that

$$f(x) = e^x$$

(d) Again we follow the hint and calculate

$$g'(x) = e^x(f'(x) + f(x))$$

and

$$g''(x) = e^x(f''(x) + 2f'(x) + f(x))$$

so using the relation $f''(x) = f(x)$ we get

$$g''(x) = e^x(2f'(x) + 2f(x)) = 2g'(x)$$

Moving forward with the hint, we calculate that

$$h'(x) = e^{-2x}(g'(x) - 2g(x)) = 0$$

Therefore

$$h(x) = c_1$$

for some constant c_1, so

$$g'(x) = c_1 e^{2x}$$

so by finding the anti-derivative we find that

$$g(x) = c_2 + \frac{c_1}{2}e^{2x}$$

for some constant c_2 and finally we can conclude that

$$f(x) = c_2 e^{-x} + \frac{c_1}{2}e^x$$

plugging in $f(0) = 2$ and $f'(0) = 0$ gives us that

$$f(x) = e^x + e^{-x}$$