Math 141 Tutorial 10 Solutions

Main problems

1. Determine whether the following sequences $\{a_n\}_{n=1}^{\infty}$ converge/diverge and justify your answer.

   (a) $a_n = \frac{\ln n}{n}$

   (b) $a_n = \frac{n^2}{\sqrt{n^3+4n}}$

   (c) $a_n = \sin(n\pi)$

   (d) $a_n = 2^{-n}\cos(n\pi)$

   (e) $a_n = \left(1+\frac{1}{n}\right)^n$

   (f) $a_n = \sqrt{n} - \sqrt{n+1}\sqrt{n+3}$

   Solutions:

   (a) Instead of using a_n, we replace a_n with $f(x) = \frac{\ln(x)}{x}$:

   $$\begin{aligned}\lim_{n\to\infty} \frac{\ln n}{n} &= \lim_{n\to\infty} \frac{\ln x}{x}, \\ &= \lim_{n\to\infty} \frac{\frac{1}{x}}{1}, \quad \text{(L'Hôpital's Rule)} \\ &= \lim_{n\to\infty} \frac{1}{x}, \\ &= 0,\end{aligned}$$

   so the sequence converges.

   (b) We'll compute the limit by factoring out power of n in the numerator and denominator:

   $$\begin{aligned}\lim_{n\to\infty} \frac{n^2}{\sqrt{n^3+4n}} &= \lim_{n\to\infty} \frac{n^2}{n^{\frac{3}{2}}\sqrt{1+\frac{4}{n^2}}}, \\ &= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{1+\frac{4}{n^2}}}, \\ &= \infty,\end{aligned}$$

   so the sequence diverges.

   (c) Notice that for every $n \geq 1$, we actually have $\sin(n\pi) = 1$ - so the sequence $\{a_n\}_{n=1}^{\infty}$ is constantly zero. Hence

   $$\lim_{n\to\infty} \sin(n\pi) = \lim_{n\to\infty} 0 = 0,$$

   so the sequence converges.

(d) Notice that we have

$$\cos(n\pi) = \begin{cases} 1 & \text{if } n \text{ even,} \\ -1 & \text{if } n \text{ odd,} \end{cases} = (-1)^n.$$

So, from what we've seen in class we get

$$\lim_{n\to\infty} 2^{-n}\cos(n\pi) = \lim_{n\to\infty} 2^{-n}(-1)^N = \lim_{n\to\infty}\left(\frac{-1}{2}\right)^n = 0,$$

so the sequence converges.

(e) Let

$$C = \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n,$$

then we see that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^C.$$

It remains to solve for C:

$$\begin{aligned}
C &= \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n, \\
&= \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right), \\
&= \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}, \\
&= \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n}}\frac{-1}{n^2}}{\frac{-1}{n^2}}, \quad \text{(L'Hôpital's Rule)} \\
&= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}, \\
&= 1.
\end{aligned}$$

So it follows that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e,$$

so the sequence converges.

(f) We compute the limit by rationalizing the expression,

$$
\begin{aligned}
\lim_{n\to\infty} \sqrt{n} - \sqrt{n+1}\sqrt{n+3} &= \lim_{n\to\infty} (\sqrt{n} - \sqrt{n+1}\sqrt{n+3})\frac{\sqrt{n}+\sqrt{n+1}\sqrt{n+3}}{\sqrt{n}+\sqrt{n+1}\sqrt{n+3}}, \\
&= \lim_{n\to\infty} \frac{n-(n^2+4n+1)}{\sqrt{n}+\sqrt{n+1}\sqrt{n+3}}, \\
&= \lim_{n\to\infty} \frac{-n^2-3n-1}{\sqrt{n}+\sqrt{n+1}\sqrt{n+3}}, \\
&= -\lim_{n\to\infty} \frac{n^2(1+\frac{3}{n}+\frac{1}{n^2})}{n\left(\frac{1}{\sqrt{n}}+\sqrt{1+\frac{1}{n}}\sqrt{1+\frac{3}{n}}\right)}, \\
&= -\lim_{n\to\infty} \frac{n(1+\frac{3}{n}+\frac{1}{n^2})}{\frac{1}{\sqrt{n}}+\sqrt{1+\frac{1}{n}}\sqrt{1+\frac{3}{n}}}, \\
&= -\infty.
\end{aligned}
$$

So the sequence diverges.

2. Compute the limit of the following convergent, recursively defined sequences

(a) $a_{n+1} = \frac{1}{2}(a_n + 6)$, $a_0 = 2$ (b) $a_{n+1} = \frac{a_n}{1+a_n}$, $a_0 = 1$ (c) $a_{n+1} = \sqrt{2a_n - 1}$, $a_0 = 2$

Solutions:

(a) Set

$$L = \lim_{n\to\infty} a_n,$$

then using the recursive expression for a_{n+1}, we get

$$L = \lim_{n\to\infty} \frac{1}{2}(a_n + 6) = \frac{1}{2}(L + 6),$$

and solving for L we find that $L = 6$.

(b) Set

$$L = \lim_{n\to\infty} a_n,$$

then using the recursive expression for a_{n+1}, we get

$$L = \lim_{n\to\infty} \frac{a_n}{1 + a_n} = \frac{L}{1 + L},$$

and solving for L, we get

$$\begin{aligned} L = \frac{L}{1+L} &\iff L^2 + L = L, \\ &\iff L^2 = 0, \\ &\iff L = 0. \end{aligned}$$

So the limit of the sequence is 0.

(c) Set

$$L = \lim_{n\to\infty} a_n,$$

then using the recursive expression for a_{n+1}, we get

$$L = \lim_{n\to\infty} \sqrt{2a_n - 1} = \sqrt{2L - 1},$$

and solving for L, we get

$$\begin{aligned} L = \sqrt{2L - 1} &\iff L^2 = 2L - 1, \\ &\iff L^2 - 2L + 1 = 0, \\ &\iff (L - 1)^2 = 0, \\ &\iff L = 1. \end{aligned}$$

So the limit of the sequence is 1.

3. Determine whether the following sequence converges:

$$a_n = \frac{1}{n^3} \cos(2n).$$

Solution:

Using the fact that $-1 \leq \cos(2n) \leq 1$, we get that

$$\lim_{n\to\infty} \frac{-1}{n^3} \leq \lim_{n\to\infty} \frac{\cos(2n)}{n^3} \leq \lim_{n\to\infty} \frac{1}{n^3}.$$

But the limits on both the left and right sides converge:

$$\lim_{n\to\infty} \frac{-1}{n^3} = 0 = \lim_{n\to\infty} \frac{1}{n^3},$$

so by the squeeze theorem it follows that

$$\lim_{n\to\infty} \frac{\cos(2n)}{n^3} = 0.$$

4. Compute the value of the following series:

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$ (c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ (e) $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

(b) $\sum_{n=1}^{\infty} 2^n e^{-n}$ (d) $\sum_{n=1}^{\infty} \frac{3^{n+1}}{2^{2n+2}}$ (f) $\sum_{n=1}^{\infty} \frac{1+3^n}{4^n}$

Solution:

(a) We can factor the terms in the sum as

$$\frac{1}{n^2+3n+2} = \frac{1}{(n+1)(n+2)}.$$

By doing a partial fraction decomposition

$$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2},$$

we see that $A(n+2) + B(n+1) = 1$, and so we get the system of equations

$$\begin{aligned} A + B &= 0 \\ 2A + B &= 1, \end{aligned}$$

from which we find that $A = 1$ and $B = -1$, i.e.,

$$\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}.$$

Looking at the sequence of partial sums S_k, we find that

$$S_k = \sum_{n=1}^{k} \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{2} - \frac{1}{k+2}.$$

Finally, letting $k \to \infty$ we compute

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2} = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{1}{2} - \frac{1}{k+2} = \frac{1}{2}.$$

(b) Since $2 < e$, the sum is a convergent geometric series, and so

$$\sum_{n=1}^{\infty} 2^n e^{-n} = \sum_{n=1}^{\infty} \left(\frac{2}{e}\right)^n = \frac{2}{e} \cdot \frac{1}{1-\frac{2}{e}} = \frac{2}{e}\frac{e}{2-e} = \frac{2}{2-e}.$$

(c) We rationalize the terms in the series to find

$$\frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{\sqrt{n+1}-\sqrt{n}}{n+1-n} = \sqrt{n+1}-\sqrt{n}.$$

Looking at the sequence of partial sums S_k, we find that

$$S_k = \sum_{n=1}^{k} \sqrt{n+1}-\sqrt{n} = \sqrt{k+1}-1,$$

and by taking the limit we see that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \sqrt{k+1}-1 = \infty.$$

(d) We rewrite the sum as

$$\sum_{n=1}^{\infty} \frac{3^{n+1}}{2^{2n+2}} = \frac{3}{4}\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n,$$

which is a convergent geometric series. Hence we find that

$$\sum_{n=1}^{\infty} \frac{3^{n+1}}{2^{2n+2}} = \frac{3}{4}\cdot\frac{3}{4}\frac{1}{1-\frac{3}{4}} = \frac{9}{4}.$$

(e) Using log properties, we can write

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} \ln(n+1)-\ln(n).$$

Looking at the sequence of partial sums S_k, we find that

$$S_k = \sum_{n=1}^{k} \ln(n+1)-\ln(n) = \ln(k+1),$$

and so

$$\sum_{n=1} \ln\left(\frac{n+1}{n}\right) = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \ln(k+1) = \infty.$$

(f) Using properties of sums, we can rewrite the series as

$$\sum_{n=1}^{\infty} \frac{1+3^n}{4^n} = \sum_{n=1}^{\infty}\frac{1}{4^n} + \sum_{n=1}^{\infty}\frac{3^n}{4^n}.$$

Both of these are convergent geometric series, so we can compute their values individually:

$$\sum_{n=1}^{\infty}\frac{1}{4^n} = \frac{1}{4}\frac{1}{1-\frac{1}{4}} = \frac{1}{3},$$
$$\sum_{n=1}^{\infty}\frac{3^n}{4^n} = \frac{3}{4}\frac{1}{1-\frac{3}{4}} = 3.$$

Putting them together, we get

$$\sum_{n=1}^{\infty} \frac{1+3^n}{4^n} = \sum_{n=1}^{\infty} \frac{1}{4^n} + \sum_{n=1}^{\infty} \frac{3^n}{4^n} = \frac{1}{3} + 3 = \frac{10}{3}.$$