13 Lecture hours 30-32

Definitions and Theorems

Definition (Eigenvectors - Eigenvalues). A vector $\vec{v} \in \mathbb{R}^n$ is an eigenvector of $n \times n$ matrix A if there exists a scalar λ ("lambda") such that $A\vec{v} = \lambda\vec{v}$, and $\vec{v} \neq \vec{0}$. This λ is called the corresponding eigenvalue.

Definition (Eigenspace). If A is an $n \times n$ matrix and λ is a scalar, the λ-eigenspace of A (denoted E_λ) is the set of all vector $\vec{v} \in \mathbb{R}^n$ such that $A\vec{v} = \lambda\vec{v}$. So, the nonzero vector in E_λ are exactly the eigenvector of A with eigenvalues λ. This set is a subspace.

Definition (Geometric Multiplicity). The geometric multiplicity of an eigenvalue λ is the dimension of its eigenspace (dim (ker(λ I - A))).

Definition (Algebraic Multiplicity). The algebraic multiplicity of an eigenvalue λ_1 is the number of times the factor $(\lambda - \lambda_1)$ appears in the characteristic polynomial $c_A(\lambda) \stackrel{\text{def}}{=} \det(\lambda I - A)$.

Definition (Diagonalization). We say a $n \times n$ matrix A is diagonalizable if there exists an invertible matrix S and a diagonal matrix B such that $A = SBS^{-1}$.

Problem 52. For what values of λ is the following matrix invertible?

$$\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & -2 & 7 \\ 0 & -1 & 2-\lambda & 3 \\ 0 & 0 & 0 & 4-\lambda \end{bmatrix}.$$

Solution 52 The determinant of this matrix may be computed to be

$$-\lambda^2 (\lambda - 3) (\lambda - 4).$$

A matrix is invertible if and only if its determinant is nonzero. Therefore, this matrix is invertible if and only if λ is not equal to 0, 3, or 4.

Problem 53. For which values of k does the matrix

$$\begin{bmatrix} k+1 & k \\ -k & 1-k \end{bmatrix}$$

have an eigenbasis?

Solution 53 The characteristic polynomial of this matrix is $(\lambda - 1)^2$, so for all values of k it has an eigenvalue $\lambda = 1$ with algebraic multiplicity 2. On the other hand, the kernel of the matrix

$$I - A = \begin{bmatrix} -k & -k \\ k & k \end{bmatrix}$$

is 2-dimensional only when $k = 0$. Therefore, this matrix has an eigenbasis only when $k = 0$.

Problem 54 (Eigenvalues)**.** Two $n \times n$ matrices A and B are called *similar* if there is an invertible matrix S with $A = SBS^{-1}$.

a) Show that if X and Y are similar, then $\det X = \det Y$.

b) Show that if A is similar to B, then the matrix $\lambda I - A$ is similar to $\lambda I - B$.

c) Use the previous two parts to show that similar matrices have the same eigenvalues. (Hint: use the characteristic polynomial).

Solution 54 (Eigenvalues and similar matrices)

a) If X and Y are similar, then $X = SYS^{-1}$. Then

$$\begin{aligned} \det X &= \det(SYS^{-1}) \\ &= \det S \det Y \det(S^{-1}) \\ &= \frac{\det S}{\det S} \det Y \\ &= \det Y. \end{aligned}$$

b) If A is similar to B, then $A = SBS^{-1}$. Now,

$$\begin{aligned} S\,(\lambda I - A)\,S^{-1} &= S\lambda IS^{-1} - SAS^{-1} \\ &= \lambda SS^{-1} - B \\ &= \lambda I - B, \end{aligned}$$

so $\lambda I - A$ is similar to $\lambda I - B$.

c) By combining parts a and b, if A and B are similar, then

$$\det(\lambda I - A) = \det(\lambda I - B).$$

Therefore the characteristic polynomials of A and B are the same, so A and B must have the same eigenvalues (with the same algebraic multiplicities).

Problem 55 (Diagonalization and Dynamical Systems)**.** The matrix A has the following eigenvectors and eigenvalues: $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ with eigenvalue $\lambda_1 = 1$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ with eigenvalue $\lambda_2 = -1$, and $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ with eigenvalue $\lambda_3 = 2$.

a) Diagonalise A, i.e., find a matrix S such that $S^{-1}AS$ is a diagonal matrix.

b) Find a closed form for A^{2021}.

c) Consider the discrete dynamical system given by $\vec{x}(t+1) = A\vec{x}(t)$ with initial condition $\vec{x}(0) = \begin{bmatrix} 1000 \\ 2000 \\ 3000 \end{bmatrix}$. Find $x(2021)$.

Solution 55 (Diagonalization and Dynamical Systems)

a) Let S be the matrix whose columns are the eigenvectors of A:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}.$$

By expressing A in terms of the eigenbasis given by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, we have that $A = SDS^{-1}$, where D is the diagonal 3×3 matrix with entries $1, -1, 2$.

b) From the previous part, we know that $A = SDS^{-1}$. It follows that $A^{2021} = SD^{2021}S^{-1}$, so

$$\begin{aligned} A^{2021} &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (-1)^{2021} & 0 \\ 0 & 0 & 2^{2021} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{2021} \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ 2 & 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2(2^{2021}-1) & 2^{2021}-1 & 1-2^{2021} \\ 2(2^{2021}-1) & 2(2^{2021}-1) & 2-2^{2021} \end{bmatrix} \end{aligned}$$

c) We know that $\vec{x}(2021)$ is given by $A^{2021}\vec{x}(0)$.