Math 141 Tutorial 7 Solutions

Main problems

1. For the following problem compute the area enclosed by the given functions in the specified region:
   (a) between $y = x - 1$ and $y = x^2 - x - 2$
   (b) between $y = \cos(2x)$ and $y = \sin(x)$ for $\frac{-\pi}{2} \le x \le \frac{3\pi}{2}$
   (c) between $x = y^2 - 4y$ and $x = 2y - y^2$
   (d) between $y = e^x$ and $y = e^{5x}$ for $-2 \le x \le 1$

   Solutions:

   (a) In order to find the enclosed area, we first determine where the curves $y = x - 1$ and $y = x^2 - x - 2$ intersect. To this end, we set

   $$x - 1 = x^2 - x - 2 \iff x^2 - 2x - 1 = 0.$$

   The solutions to this problem are $\frac{1}{2}\left(2 \pm \sqrt{8}\right) = 1 \pm \sqrt{2}$.
   Hence, the region in question is contained between $x = 1 - \sqrt{2}$ and $x = 1 + \sqrt{2}$. In this interval, the function $x - 1$ will be above $x^2 - 2x - 2$ (recall from differential calculus that this can be seen by simply verifying the value of each function at any value between $1 - \sqrt{2}$ and $1 + \sqrt{2}$). The area between the curves is therefore

   $$\begin{aligned}\int_{1-\sqrt{2}}^{1+\sqrt{2}} \big((x-1) - (x^2 - 2x - 1)\big)\,\mathrm{d}x &= \int_{1-\sqrt{2}}^{1+\sqrt{2}} \left(-x^2 + 3x\right)\mathrm{d}x \\ &= -\frac{x^3}{3} + \frac{3x^2}{2}\bigg|_{1-\sqrt{2}}^{1+\sqrt{2}} = \frac{8\sqrt{2}}{3}\end{aligned}$$

   (b) Once, again, we first find the intersection of the curves. Since $\cos(2x) = 1 - 2\sin^2(x)$, we see that

   $$\cos(2x) = \sin(x) \iff 1 - 2\sin^2(x) = \sin(x) \iff 2\sin^2(x) + \sin(x) - 1 = 0.$$

   Factorizing, we obtain

   $$(2\sin(x) - 1)(\sin(x) + 1) = 0.$$

   Hence, either $\sin(x) = 1/2$ or $\sin(x) = -1$. Since we are in the interval $[-\pi/2, 3\pi/2]$, the solutions of $\sin(x) = 1/2$ are $\pi/6$ and $5\pi/6$ while the solutions of $\sin(x) = -1$ are $-\pi/2$ and $3\pi/2$.
   By plugging in an appropriate value, we can verify that

- $\cos(2x) \geq \sin(x)$ on $[-\pi/2, \pi/6]$
- $\cos(2x) \leq \sin(x)$ on $[\pi/6, 5\pi/6]$
- $\cos(2x) \geq \sin(x)$ on $[5\pi/6, 3\pi/2]$

The area between the curves is therefore

$$\int_{-\pi/2}^{\pi/6} (\cos(2x) - \sin(x))\,\mathrm{d}x + \int_{\pi/6}^{5\pi/6} (\sin(x) - \cos(2x))\,\mathrm{d}x + \int_{5\pi/6}^{3\pi/2} (\cos(2x) - \sin(x))\,\mathrm{d}x$$
$$= \left[-\frac{1}{2}\sin(2x) + \cos(x)\right]_{-\pi/2}^{\pi/6} + \left[-\cos(x) - \frac{1}{2}\sin(2x)\right]_{\pi/6}^{5\pi/6} + \left[-\frac{1}{2}\sin(2x) + \cos(x)\right]_{5\pi/6}^{3\pi/2}$$
$$= \frac{3\sqrt{3}}{4} + \frac{3\sqrt{3}}{2} + \frac{3\sqrt{3}}{4} = 3\sqrt{3}$$

(c) between $x = y^2 - 4y$ and $x = 2y - y^2$

We find the value of y such that $y^2 - 4y = 2y - y^2$. Rearranging, we have instead the equation $2y^2 - 6y = 2y(y-3) = 0$. The solutions are $y = 0$ or $y = 3$. In this interval, we notice that $y^2 - 4y \leq 2y - y^2$. Hence, the area between the curves is

$$\int_0^3 \left((2y - y^2) - (y^2 - 4y)\right)\mathrm{d}y = \int_0^3 \left(6y - 2y^2\right)\mathrm{d}y = 3y^2 - \frac{2y^3}{3}\Bigg|_0^3 = 9.$$

(d) Notice that the curves $y = e^x$ and $y = e^{5x}$ intersect when $x = 5x$. Thus, the only intersection is at $x = 0$. For $x > 0$, $e^{5x} > e^x$ and for $x < 0$ we have $e^{5x} < e^x$. Hence, for $-2 \leq x \leq 1$ the are between the curves is

$$\int_{-2}^{0} \left(e^x - e^{5x}\right)\mathrm{d}x + \int_0^1 \left(e^{5x} - e^x\right)\mathrm{d}x = \left[e^x - \frac{1}{5}e^{5x}\right]_{-2}^{0} + \left[\frac{1}{5}e^{5x} - e^x\right]_0^1$$
$$= \frac{8 + e^5 - 5e - 5e^{-2} + e^{-10}}{5} \approx 28.4$$

2. Find the area enclosed by the curves $y = \ln x$, $y = \ln 2 + \ln(x-1)$ and $y = 2$. Which approach is easier, integrating with respect to x or y?

Solutions:

We first find where the given curves intersect. Setting

$$\ln x = \ln 2 + \ln x - 1 = \ln 2(x-1)$$

and exponentiating both sides, we obtain the relation

$$x = 2(x-1) \iff x = 2.$$

Thus, the curves $y = \ln x$ and $y = \ln 2 + \ln(x-1)$ intersect at the point $(2, \ln 2)$. Note that the curve $y = \ln x$ intersects the line $y = 2$ at $(e^2, 2)$. Similarly, the curve $y = \ln 2 + \ln(x-1) = \ln(2x-2)$ meets the line $y = 2$ when $x = \dfrac{e^2}{2} + 1$.

With respect to x. In order to integrate with respect to x, we first place our points of intersection in increasing x-order:

$$(2, \ln(2)), \quad \left(\frac{e^2}{2} + 1, 2\right), \quad \left(e^2, 2\right).$$

From $x = 2$ to $x = \frac{e^2}{2} + 1$, observe that

$$\ln(x) \leq \ln(2) + \ln(x-1) \leq 2.$$

From $x = \frac{e^2}{2} + 1$ to $x = e^2$ however, we have

$$\ln(x) \leq 2 \leq \ln(2) + \ln(x-1).$$

Hence, the area between the curves is

$$\int_2^{\frac{e^2}{2}+1} ((\ln(2) + \ln(x-1)) - \ln(x))\, \mathrm{d}x + \int_{\frac{e^2}{2}+1}^{e^2} (2 - \ln(x))\, \mathrm{d}x.$$

Evaluating these integrals (one may use a substitution and integration by parts), we obtain

$$\Big[(\ln(2)x + (x-1)\ln(x-1) - x\ln(x)\Big]_2^{\frac{e^2}{2}+1} + \Big[2x - x\ln(x) + x\Big]_{\frac{e^2}{2}+1}^{e^2}$$
$$= \frac{1}{2}e^2 - 3 + \ln(2)$$

With respect to y. Notice that the curve $y = \ln(x)$ is precisely $x = e^y$ and the curve $y = \ln(2) + \ln(x-1) = \ln(2x-2)$ is $x = \frac{1}{2}e^y + 1$. Inspecting the points of intersection, we see that the desired area is

$$\int_{\ln(2)}^{2} \left(e^y - \left(\frac{1}{2}e^y + 1\right)\right) \mathrm{d}y.$$

Evaluating the integral, we obtain

$$\int_{\ln(2)}^{2} \left(\frac{1}{2}e^y - 1\right) \mathrm{d}y = \frac{1}{2}e^y - y\Big|_{\ln(2)}^{2} = \frac{1}{2}e^2 - 3 + \ln(2)$$

3. Consider an object S whose base is a circle of radius r. Suppose that the cross-sections along one of the diameters of this circle are isosceles right triangles such that the hypotenuse does *not* rest on the base.

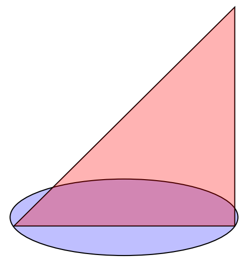


Figure 1: A cross section of S depicted in red

Show that the volume of S is

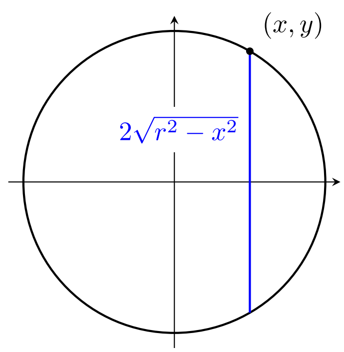
$$V = \frac{8}{3}r^3.$$

Solution:

In order to find the volume, we centre the circle of radius r on the xy-plane such that the cross-sections along the x-axis are isosceles right triangles. The volume is then

$$V = \int_{-r}^{r} A(x)\mathrm{d}x$$

where $A(x)$ is the area of the cross section at a fixed value of x. Observe that at x, the corresponding positive y value is $\sqrt{r^2 - x^2}$. Hence, the length of on side of the triangle is $2\sqrt{r^2 - x^2}$.



The height of the triangle is therefore also $2\sqrt{r^2 - x^2}$ since we have an isosceles right triangles. Therefore,

$$A(x) = \frac{1}{2}\left(2\sqrt{r^2 - x^2}\right)^2 = r^2 - x^2.$$

We conclude that the volume is

$$V = \int_{-r}^{r} A(x) \mathrm{d}x = 2 \int_{-r}^{r} \left(r^2 - x^2\right) \, \mathrm{d}x = 2 \left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} = \frac{8}{3} r^3$$

4. Consider the following problems on volumes of revolution using disks/washers.

   (a) Find the volume of the object created by rotating the region trapped between $f(x) = x - x^3$ and $y = 0$ about the x-axis.

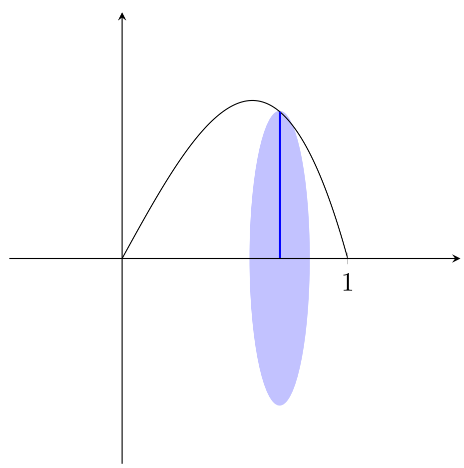
   (b) Find the volume of the object created by rotating the region trapped between $y = x$, $x = 0$ and $y = 3$ about the y-axis.

   (c) Find the volume of the object created by rotating the region trapped between $y = (x - 2)^2$, $y = 0$ and $x = 1$ about the the line $x = 1$.

   (d) Find the volume of the object created by rotating the region trapped between $y = \cos(x)$, $y = -\cos(x)$ *which contains the origin* about the the line $y = -1$.

   Solutions:

   (a) Notice first that $f(x) = x - x^3$ meets the line $y = 0$ at $x = 0$ and at $x = 1$. Thus, in the region trapped between $x - x^3$ and $y = 0$, x ranges from 0 to 1. Now, notice that the area of the region can be evaluated by integrating with respect to x. One can imagine that we are "adding up" infinitely many vertical segments. Rotating one of the vertical segment at x about the x-axis, we obtain a disk. These disks are precisely the cross-sections of the object.



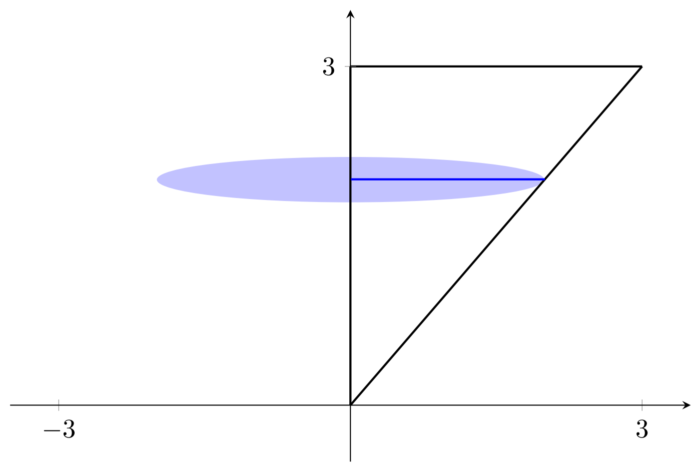
   Now, the radius of each disk is $r(x) = x - x^3$. Thus, the area is

$$A(x) = \pi\,(r(x))^2 = \pi\left(x - x^3\right)^2 = \pi\left(x^2 - 2x^4 + x^6\right).$$

   Ergo, the volume of the region rotated about the x-axis is

$$V = \int_0^1 A(x)\mathrm{d}x = \pi\int_0^1 \left(x^2 - 2x^4 + x^6\right)\mathrm{d}x = \pi\left[\frac{x^3}{3} - \frac{2x^5}{5} + \frac{x^7}{7}\right]_0^1 = \frac{8}{105}\pi$$

   (b) The region trapped between $y = x$, $x = 0$ and $y = 3$ is a triangle. One can evaluate the area by integrating with respect to y from $y = 0$ to $y = 3$. We can imagine that we are "adding up" infinitely many horizontal segments. For a fixed y, rotating one of these segments about the y-axis yields a disk which corresponds to the cross-section of the solid of revolution.



Observe that, with respect to y, the radius of this disk is $r(y) = y$. Hence, the area of the disk is

$$A(x) = \pi \left(r(y)\right)^2 = \pi y^2$$

Ergo, the volume of the region rotated about the y-axis is

$$V = \int_0^3 A(y)\mathrm{d}y = \pi \int_0^3 y^2 \mathrm{d}y = \pi \left[\frac{y^3}{3}\right]_0^3 = 9\pi.$$

(c) Note first that, for $x \leq 2$, the curve $y = (x-2)^2$ corresponds to $x = 2 - \sqrt{y}$. Therefore, the region that we will rotate is enclosed by $x = 2 - \sqrt{y}$, $y = 0$ and $x = 1$. The area of this region can be evaluated by integrating with respect to y from 0 to 1. At a given y, we imagine one of the horizontal segments that contributes to the area. Rotating this segment about $x = 1$ yields a disk which corresponds to the cross-section of the solid of revolution. The radius of this disk is $r(y) = (2 - \sqrt{y}) - 1 = 1 - \sqrt{y}$. Hence, the area of the disk is

$$A(x) = \pi \left(r(y)\right)^2 = \pi \left(1 - \sqrt{y}\right)^2 = \pi \left(1 - 2\sqrt{y} + y\right).$$

Therefore, the volume of our object is

$$V = \int_0^1 A(y)\mathrm{d}y = \pi \int_0^1 \left(1 - 2\sqrt{y} + y\right)\mathrm{d}y = \pi \left[y - \frac{4}{3}y^{3/2} + \frac{1}{2}y^2\right]_0^1 = \frac{\pi}{6}.$$

(d) Notice that $y = \cos(x)$ intersects $y = -\cos(x)$ precisely when $\cos(x) = 0$. In other words, when $x = \frac{\pi}{2} + k\pi$ for some integer k. Thus, in the region enclosed between these curves *which contains the origin*, the x value ranges from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. The area of this region can be obtained by integrating with respect to x. At a given point x, we can imagine one of the vertical segments that "add up" to the area. Rotating one of these segments about $y = -1$ yields a washer (i.e. the difference of two disks). This washer is a cross-section of the solid of revolution. The outer radius is

$$R(x) = \cos(x) - (-1) = \cos(x) + 1.$$

The inner radius is

$$r(x) = -\cos(x) - (-1) = 1 - \cos(x).$$

Hence, the area of the washer is

$$A(x) = \pi\left(R(x)\right)^2 - \pi\left(r(x)\right)^2 = \pi\left((\cos(x)+1)^2 - (1-\cos(x))^2\right) = 4\pi\cos(x).$$

Hence, the volume of the enclosed region rotated about $y = -1$ is

$$V = \int_{-\pi/2}^{\pi/2} A(x)\mathrm{d}x = 4\pi\int_{-\pi/2}^{\pi/2}\cos(x)\mathrm{d}x = 4\pi\sin(x)\Big|_{-\pi/2}^{\pi/2} = 8\pi.$$