Lecture hours 5-7

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Definitions

Definition (Linear relations)**.** Consider the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ in $\mathbb{R}^n$. An equation of the form $c_1\vec{v}_1 + \cdots + c_r\vec{v}_r = \vec{0}$ is called a linear relation among the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$. If at least one of the c_i is nonzero, then we call this a nontrivial linear relation among $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$.

Definition (Linear Independent vectors)**.** We say vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ in $\mathbb{R}^n$ are linearly independent if and only if the only linear relation between them is the trivial one. In other words, $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ in $\mathbb{R}^n$ are linearly independent if and only if the only way that $c_1\vec{v}_1 + \cdots + c_r\vec{v}_r = \vec{0}$ is if all the c_i are 0.

Definition (Subspace - Span version)**.** A subspace of $\mathbb{R}^n$ is a set of vectors in $\mathbb{R}^n$ that can be described as a span of vectors.

Definition (Subspace - Standard version)**.** A subspace of $\mathbb{R}^n$ subset V of $\mathbb{R}^n$ with the following properties:

(i) V is a non-empty set.

(ii) If $\vec{u}$ is in V , $\vec{u}$ is also in V for any scalar $k \in \mathbb{R}$ (We say V is closed under scalar multiplication.)

(iii) If $\vec{u}$ and $\vec{w}$ are in V, their sum $\vec{u} + \vec{w}$ is also in V. (We say V is closed under addition.)

Definition (Basis)**.** The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are a basis of a subspace V if they span V and are linearly independent. In other words, a basis of a subspace V is the minimal set of vectors needed to span all of V.

Definition (Dimension of a subspace)**.** The dimension of the subspace V is the number of vectors in a basis of V.

Problem 12 (Linear dependence)**.** True or false? If false, give a counter-example. If true, explain why.

a) If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent vectors in $\mathbb{R}^2$, then $\vec{v}_3$ is in the span of $\vec{v}_1$ and $\vec{v}_2$.

b) Any collection of 4 vectors in $\mathbb{R}^3$ is linearly dependent.

Solution 12 (Linear dependence)

a) This is false, consider the $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. A nontrivial linear relation is $2\vec{v}_1 - \vec{v}_2 = 0$, but $\vec{v}_3$ is not in the span of $\vec{v}_1$ and $\vec{v}_2$.

b) This is true. Pick three of the vectors. Either they span $\mathbb{R}^3$ or they do not. If they do not span $\mathbb{R}^3$, then they must be linearly dependent, and hence all four vectors are linearly dependent. If they do span $\mathbb{R}^3$, then they are a basis of $\mathbb{R}^3$ and the fourth vector can be expressed in terms of them, which gives a linear relation.

Problem 13 (Definition of subspace)**.** Give examples of:

a) A subset V of $\mathbb{R}^2$ that is closed under scalar multiplication, but not closed under addition.

b) A subset V of $\mathbb{R}^2$ that is closed under addition, but not closed under scalar multiplication.

Solution 13 (Definition of subspace)

a) The set consisting of both the x-axis and the y-axis.

Here if you take an element in this set, it will be a vector on the x-axis or the y-axis. If you multiply this vector by a scalar, it is still on the x-axis or the y-axis, thus closed under scale multiplication.

However, if you take two vectors in the set (e.g. (1 0) on x axis and (0 1) on y axis) and try to add them, you may not get a vector that is still on the x-axis or the y-axis (e.g. (1 1) in this case), therefore no longer in the set. So it is not closed under addition.

b) The set of vectors $\begin{bmatrix} x \\ 0 \end{bmatrix}$ with just $x \geq 0$.

Take any 2 vectors that are in that set, so say $\begin{bmatrix} x \\ 0 \end{bmatrix}$ and $\begin{bmatrix} y \\ 0 \end{bmatrix}$. Then $x \geq 0$ and $y \geq 0$. Add them up, and the result is still in the set since $\begin{bmatrix} x+y \\ 0 \end{bmatrix}$ still has $(x+y) \geq 0$. So the set is closed under vector addition.

However, take any vector in the set, say $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and multiply it by any negative scalar, say (-1). Then the new vector is $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$, which is not in the set. So the set is not closed under scalar multiplication.

Problem 14 (Subspaces of $\mathbb{R}^n$). Give an example of:

a) A subspace of $\mathbb{R}$.

b) A subset of $\mathbb{R}^2$ that is not a subspace of $\mathbb{R}^2$. Explain why it is not a subspace.

c) A subspace of $\mathbb{R}^3$ of dimension 2. Explain why it has dimension 2.

d) A subset of $\mathbb{R}^3$ that contains infinitely many vectors, but is not a subspace of $\mathbb{R}^3$. Explain why it is not a subspace.

Solution 14 (Subspaces of $\mathbb{R}^n$)

a) The subsets $\mathbb{R}$ and $\left\{\vec{0}\right\}$ are subspaces of $\mathbb{R}$.

b) The set of all vectors $\begin{bmatrix} a \\ 0 \end{bmatrix} \in \mathbb{R}^2$ such that $a \leq 0$. It is not closed under scalar multiplication, so it cannot be a subspace of R^2.

c) The subspace consisting of vectors of the form $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$. It has dimension 2 because it has a basis consisting of two vectors: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

d) The set of vectors (v_1, v_2, v_3) in $\mathbb{R}^3$ such that $v_1 + v_2 + v_3 = 1$ (the unit sphere).

It does not contain the zero vector, so it cannot be a subspace of R^3.

Problem 15 (Basis and dimension)**.** Find all linear relations between the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$$

. What is the dimension of $\mathrm{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$? Give a basis for this subspace.

Solution 15 (Basis and dimension)

1. Use Gaussian elimination to show that the linear combination

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4$$

   equals the zero vector if c_1, c_2, c_3, and c_4 take the form

$$c_1 = -2s - t, \;\; c_2 = s - t, \;\; c_3 = s, \;\; c_4 = t,$$

   where s and t are any real numbers.

2. Now, take any vector $\vec{v}$ in $\mathrm{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$. By definition of span we know there are $a, b, c, d \in \mathbb{R}$ such that

$$\vec{v} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 + d\vec{v}_4.$$

   Using 1 we have:

$$(-2c - d)\vec{v}_1 + (c - d)\vec{v}_2 + c\vec{v}_3 + d\vec{v}_4 = \vec{0}.$$

   From the two expressions above follows that

$$\vec{v} = a\vec{v}_1 + b\vec{v}_2 - (-2c - d)\vec{v}_1 - (c - d)\vec{v}_2,$$

   in other words

$$\vec{v} = (a + 2c + d)\vec{v}_1 + (b - c + d)\vec{v}_2.$$

   This means that any vector in $\mathrm{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ can be written as a linear combination of the vectors v_1 and v_2.

3. Since $\vec{v}_1$ is not a scalar multiple of $\vec{v}_2$, these two vectors are linear independent.

By 2 and 3 we can conclude that $\{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathrm{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ and that the dimension for this subspace is 2.

Observe that $\{\vec{v}_3, \vec{v}_4\}$ is also a basis for $\mathrm{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$.

Here are other 2 possible solutions:

- Pick $s = 1$ and $t = 0$ to show v_3 can be written as a linear combination of v_1 and v_2; then pick $s = 0$ and $t = 1$ to show v_4 can be written as a linear combination of v_1 and v_2; see that v_1 and v_2 are not multiples of each other; conclude that v_1 and v_2 are a possible basis.
- Pick $s = 0$ and $t = 1$ to show v_4 can be written as a linear combination of v_1 and v_2 so remove it from the list; now check if v_1, v_2, v_3 are linear independent by putting as columns in a matrix and row-reducing - you get the same first 3 columns as the previous rref so you know they are dependent too so remove v_3 from the list (since you can write it in terms of v_1 and v_2); but v_1 and v_2 are independent so a basis is v_1 and v_2.