Math 140 **Tutorial 9**

March 10 - 13

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1. Find the derivative of the following functions:

   (a) $x^4 + 3x^3 + 3x + 1$

   (b) $(x+1)^4$

   (c) $3e^x - \dfrac{2}{x} + \dfrac{3}{x^2}$

   (d) $\sin x e^x$

   (e) $\dfrac{e^x}{x^3}$

   (f) $\dfrac{\sin x e^x}{x^3}$

   (g) $(2x^2 + 3x + 1)^6$

   (h) $x^2 e^{2x}$

   (i) 3^x

   (j) $e^{\cos x} + \cos(e^x)$

   (k) $\sec\left(1 + x^2\right)$

   (l) $(x^2+1)^{\sin x}$

2. Find the equations of the tangent lines through the given points:

   (a) $x^2 + \cancel{1}4xy + y^2 = 13, \quad (2,1)$

   (b) $y = (2+x)e^{-x}, \quad (0,2)$

3. For which non-zero point P on the curve given by

$$\mathcal{C} : y = (x+1)^3 - 1$$

   does the tangent line to $\mathcal{C}$ at P go through the origin?

4. Consider the curve given by

$$\mathcal{C} : y^2 = x^3 + 17.$$

   The point $P = (-2, 3)$ is on this curve, and the tangent line going through P intersects the curve $\mathcal{C}$ at another point. Find the coordinates of this second point of intersection.

1. (a) $\frac{d}{dx}\left(x^4+3x^3+3x+1\right) = 4x^3 + 9x^2+3$.

(b) Either: $(x+1)^4 = x^4+4x^3+6x^2+4x+1 \Rightarrow \frac{d}{dx}(x+1)^4 = 4x^3+12x^2+12x+4$

or apply the chain rule with $u=x+1$ to get $\frac{d}{dx}\left((x+1)^4\right) = 4(x+1)^3$.

(c) $\frac{d}{dx}\left(3e^x - \frac{2}{x}+\frac{3}{x^2}\right) = 3e^x+\frac{2}{x^2}-\frac{6}{x^3}$

(d) PRODUCT RULE : $\frac{d}{dx}(\sin x\, e^x) = \cos x\, e^x + \sin x\, e^x$.

(e) QUOTIENT RULE : $\frac{d}{dx}\frac{e^x}{x^3} = \frac{e^x x^3 - 3x^2 e^x}{x^6} = e^x\left(\frac{1}{x^3}-\frac{3}{x^4}\right)$

(f) $\frac{d}{dx}\left(\frac{\sin x\, e^x}{x^3}\right) = \frac{(\cos x\, e^x + \sin x\, e^x)x^3 - 3x^2\sin x\, e^x}{x^6}$

(g) CHAIN RULE: $u = 2x^2+3x+1$, $\frac{du}{dx} = 4x+3 \Rightarrow \frac{d}{dx}\left((2x^2+3x+1)^6\right) = 6(4x+3)(2x^2+3x+1)^5$

(h) $\frac{d}{dx}e^{2x} = 2e^{2x} \Rightarrow \frac{d}{dx}(x^2e^{2x}) = 2xe^{2x}+2x^2e^{2x}$

(i) $3^x = e^{\ln(3^x)} = e^{x\ln(3)} \Rightarrow \frac{d}{dx}(3^x) = \frac{d}{dx}\left(e^{x\ln(3)}\right) = \ln(3)\,e^{x\ln(3)} = \ln(3)\,3^x$.

(j) $\frac{d}{dx}\left(e^{\cos x}+\cos(e^x)\right) = -\sin x\, e^{\cos x} - e^x\sin(e^x)$.

(k) $\frac{d}{dx}\sec x = \tan^2 x \Rightarrow \frac{d}{dx}\sec(1+x^2) = 2x\tan(1+x^2)$

(l) $(x^2+1)^{\sin x} = e^{\ln\left((x^2+1)^{\sin x}\right)} = e^{\sin x\ln(x^2+1)} = \left(\cos\ln(x^2+1)+\frac{2x\sin x}{x^2+1}\right)e^{\sin x\ln(x^2+1)}$

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2. (a) $x^2+4xy+y^2=13$, IMPLICIT DIFFERENTIATION:

$2x\,dx+4y\,dx+4x\,dy+2y\,dy=0 \Rightarrow (2x+4y)\,dx = -(4x+2y)\,dy \Rightarrow \frac{dy}{dx} = -\frac{2x+4y}{4x+2y}$ so at $(2,1) = -\frac{8}{10} = -\frac{4}{5}$

So line passing through $(2,1)$ with slope $-\frac{4}{5} \Rightarrow 4x+5y=13$.

(b) $y=(2+x)e^{-x} \Rightarrow \frac{dy}{dx} = 2e^{-x}-(2+x)e^{-x}$, at $(0,2)$ $= 2-2=0$. slope $=0$ through $(0,2)$, the line is $y=2$.

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3. $y = (x+1)^3-1$

$\frac{dy}{dx} = 3(x+1)^2-1$, so if (x_0,y_0) are on the curve, then the tangent line to the curve at (x_0,y_0) has slope $3x_0^2+6x_0+2$, so the line has equation

$$y=(3x_0^2+6x_0+2)x+C,$$

lets plug in (x_0,y_0) to determine the constant! $y_0 = 3x_0^3+6x_0^2+2x_0+C$

Now recall, (x_0,y_0) is on the curve C and so $y_0 = x_0^3+3x_0^2+3x_0$

Combining the two equations $\Rightarrow$ $C = -2x_0^3-3x_0^2+x_0 = -x_0(2x_0^2+3x_0-1)$, so $C=0$ when $x_0=0$ or $x_0 = \frac{3\pm\sqrt{17}}{2}$.

4. $y^2 = x^3+17 \Rightarrow 2y\,dy = 3x^2\,dx \Rightarrow \frac{dy}{dx} = \frac{3x^2}{2y}$, so at $(-2,3)$ the slope is 2. The line with slope 2 going through $(-2,3)$ is $y = 2x+7$. Substituting that for the equation of the curve gives

$$4x^2 + 28x+49 = x^3+17$$

$$\Updownarrow$$

$$x^3-4x^2-28x-32=0$$

The cubic equation would be difficult to solve, but we already know $x=-2$ is a solution so we can divide by

$$\begin{array}{r|l} & x^2-6x-16 \\ \hline x+2 & x^3-4x^2-28x-32 \\ & x^3+2x^2 \\ \hline & -6x^2-28x-32 \\ & -6x^2-12x \\ \hline & -16x-32 \\ & -16x-32 \\ \hline & 0\quad 0 \end{array}$$

, so we have to solve

$$x^2-6x-16=0$$

which has solutions: $\frac{6\pm\sqrt{36+64}}{2} = 3\pm5$ so the solutions are $x=-2, x=8$

Therefore, the other point of intersection is $(8, 23)$.

5. Find the points on the ellipse $x^2 + 2y^2 = 1$ such that the tangent line has slope 1.

6. The volume of a cube is increasing at a rate of $10\text{cm}^3/\text{min}$. How fast is the surface area increasing when the length of an edge is 30 cm?

7. *Recall that the temperature of an object changes at a rate proportional to the temperature difference of the object and its surrounding.* A cup of hot chocolate has temperature 80°C in a room kept at 20°C. After half an hour the hot chocolate cools to 60°C.

   (a) What is the temperature of the chocolate after another half hour?

   (b) When will the chocolate have cooled to 40°C?

5. $x^2+2y^2=1 \Rightarrow 2x\,dx + 4y\,dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{2y}$. we want $\frac{dy}{dx}=1$ so $\frac{-x}{2y}=1$

$\Rightarrow$ $x=-2y$, so we can plug that into $x^2+2y^2=1$ to get $6y^2=1$ i.e $y=\pm\frac{1}{\sqrt{6}}$

$x=-2y \Rightarrow$ the two points are $\left(\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right), \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

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6. Let s be the length of an edge. Then the volume is given by $V=s^3$, and the surface area is $A=6s^2$. These quatities are changing with time. In particular, we know that

$$\frac{dV}{dt} = 10\,cm^3/min$$

To figure out $\frac{ds}{dt}$ we will use differentiate $V=s^3 \Rightarrow \frac{dV}{dt} = 3s^2\frac{ds}{dt}$, so when $s=30cm$

$10 = 2700\frac{ds}{dt} \Rightarrow \frac{ds}{dt} = \frac{1}{270}\,cm/min$. Differentiating $A=6s^2$ gives $\frac{dA}{dt} = 12s$, so

$$\frac{dA}{dt} = \frac{12}{270} = \frac{1}{45}\,cm^2/min$$

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7. Let T denote temperature and t denote time, then we know $\frac{dT}{dt} = k(T-T_0)$ where T_0 is the surrounding temperature. This implies

$T-T_0 = c_0 e^{kt}$ so $T = c_0 e^{kT} - T_0$

we know that $T_0=20$, when $t=0$, $T=80$ and when $T=30$, $t=60$, plugging this in gives

$t=0, T=80 \Rightarrow c_0 = 80-20 = 60 \Rightarrow c_0 = 60$

$t=30, T=60 \Rightarrow 40 = 60e^{30k}$, so $e^{30k} = \frac{2}{3}$ take ln of both sides

$\Rightarrow 30k = \ln(2/3) \Rightarrow k = \frac{1}{30}\ln(2/3)$.

Makes sense that k is a negative number.

(a) when $t=60$, we just plug in

$T = 60e^{2\ln\frac{2}{3}} + 20 = 60\times\frac{4}{9} + 20 = \frac{80}{3} + 20$

(b) Now we know $T=40$, so $20 = 60e^{\frac{1}{30}\ln(\frac{2}{3})t}$ so $\frac{1}{3} = \left(\frac{2}{3}\right)^{t/30} \Rightarrow t = 30\frac{\ln(1/3)}{\ln(2/3)}$.