Math 141 Tutorial 2 Solutions

Main problems

1. Using the summation formulas seen in class, provide a closed form for the following summations in terms of n.

(a) $\displaystyle\sum_{i=0}^{n}(i+1)$

(b) $\displaystyle\sum_{i=2}^{n}(2i+n)$

(c) $\displaystyle\sum_{i=-1}^{n}(i+2)^2$

(d) $\displaystyle\sum_{i=0}^{n}(i-2)^2$

(e) $\displaystyle\sum_{i=-n}^{0}-i^3$

(f) $\displaystyle\sum_{i=-n}^{n}\left(i^3+3i^2n+3in^2+n^3\right)$

Solutions:

(a) By linearity, we have

$$\begin{aligned}\sum_{i=0}^{n}(i+1)=\sum_{i=0}^{n}i+\sum_{i=0}^{n}1=\sum_{i=1}^{n}i+(n+1)&\\ &=\frac{n(n+1)}{2}+(n+1)\\ &=\frac{n(n+1)+2(n+1)}{2}\\ &=\frac{n^2+3n+2}{2}.\end{aligned}$$

(b) We can proceed similarly here, but we must be a little careful when applying the identity $\sum_{i=1}^{n}i=\frac{n(n+1)}{2}$ because our sum begins at $i=2$. To circumvent this issue, we proceed as follows (adding and subtracting 1 in order to introduce the "missing term")

$$\sum_{i=2}^{n}(2i+n)=\sum_{i=2}^{n}2i+\sum_{i=2}^{n}n=2\sum_{i=2}^{n}i+n\underbrace{\sum_{i=2}^{n}1}_{\substack{(n-1)\\ \text{terms}}}$$

$$
\begin{aligned}
&= 2\left[\underbrace{-1+1}_{=0}+\sum_{i=2}^{n} i\right] + n(n-1) \\
&= 2\left[-1+\sum_{i=1}^{n} i\right] + n(n-1) \\
&= 2\left[-1+\frac{n(n+1)}{2}\right] + n(n-1) \\
&= 2\left[\frac{n^2+n-2}{2}\right] + n(n-1) \\
&= n^2+n-2+n^2-n \\
&= 2n^2-2.
\end{aligned}
$$

(c) We have

$$
\begin{aligned}
\sum_{i=-1}^{n}(i+2)^2 &= \left[(-1+2)^2+(0+2)^2+\cdots+(n-1+2)^2+(n+2)^2\right] \\
&= \left[(1)^2+(2)^2+\cdots+(n+1)^2+(n+2)^2\right] \\
&= \sum_{j=1}^{n+2} j^2 \\
&= \frac{(n+2)((n+2)+1)(2(n+2)+1)}{6} \\
&= \frac{(n+2)(n+3)(2n+5)}{6}
\end{aligned}
$$

(d) We write

$$
\begin{aligned}
\sum_{i=0}^{n}(i-2)^2 &= (-2)^2+(-1)^2+0^2+1^2+2^2+\cdots+(n-1-2)^2+(n-2)^2 \\
&= 5+1^2+2^2+\cdots+(n-1-2)^2+(n-2)^2 \\
&= 5+\sum_{j=1}^{n-2} j^2 \\
&= 5+\frac{(2(n-2)+1)(n-2)(n-1)}{6} \\
&= \frac{30+2n^3-9n^2+13n-6}{6} \\
&= \frac{2n^3-9n^2+13n+24}{6}.
\end{aligned}
$$

(e) Simply note that

$$
\sum_{i=-n}^{1} -i^3 = -\sum_{i=-n}^{1} i^3 = -\left((-n)^3+(-(n-1))^3+\cdots+(-2)^3+(-1)^3\right)
$$

$$
\begin{aligned}
&= -\left(-n^3 - (n-1)^3 - \cdots - 2^3 - 1^3\right) \\
&= \left(n^3 + (n-1)^3 + \cdots + 2^3 + 1^3\right) \\
&= \left(1^3 + 2^3 + \cdots + n^3\right) \\
&= \sum_{j=1}^{n} j^3 \\
&= \frac{n^2(n+1)^2}{4}.
\end{aligned}
$$

(f) Notice that $i^3 + 3i^2n + 3in^2 + n^3 = (i+n)^3$, as can be seen by expanding the latter expression. Hence, we see that

$$
\begin{aligned}
&\sum_{i=-n}^{n} \left(i^3 + 3i^2n + 3in^2 + n^3\right) \\
&\quad = \sum_{i=-n}^{n} (n+i)^3 \\
&\quad = (n-n)^3 + (n+(-n+1))^3 + \cdots + (n+0)^3 + (n+1)^3 + \cdots + (n+(n-1))^3 + (n+n)^3 \\
&\quad = 0 + 1 + \cdots + n^3 + (n+1)^3 + \cdots + (2n-1)^3 + (2n)^3 \\
&\quad = \sum_{j=1}^{2n} j^3 \\
&\quad = \frac{(2n)^2(2n+1)^2}{4} \\
&\quad = (2n+1)^2 n^2
\end{aligned}
$$

2. Evaluate the definite integrals by either taking the limit of (left or right) Riemann sums or interpreting the integral as an area.

(a) $\displaystyle\int_0^2 2x\,\mathrm{d}x$

(b) $\displaystyle\int_1^4 (3-x)\,\mathrm{d}x$

(c) $\displaystyle\int_0^2 (2x^2+1)\,\mathrm{d}x$

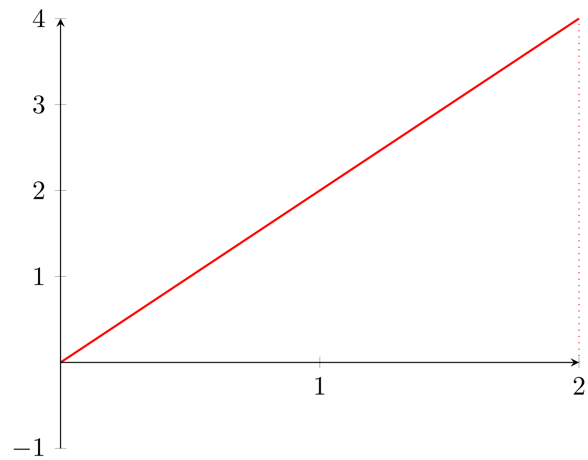
(d) $\displaystyle\int_1^5 (x^2+2x)\,\mathrm{d}x$

(e) $\displaystyle\int_0^3 x^3\,\mathrm{d}x$

(f) $\displaystyle\int_{-2}^2 |2x|\,\mathrm{d}x$

Solutions:

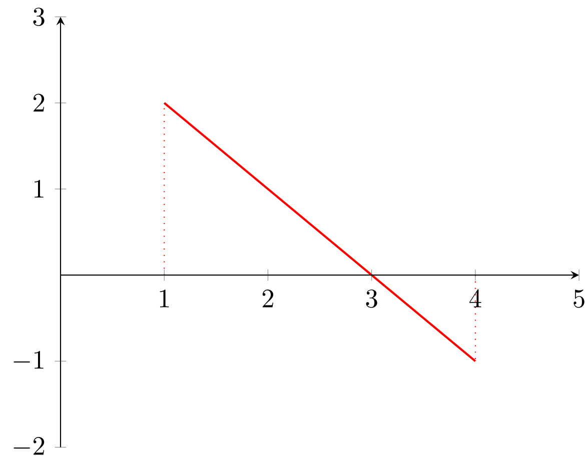
(a) We can interpret this integral as the area of the following triangle:



Geometrically, we see that the triangle has base length 2 and height 4, thus we have

$$\int_0^2 2x\,\mathrm{d}x = \frac{2\cdot 4}{2} = 4.$$

(b) Similarly, we'll calculate the value of the integral as an area:



Notice that we have two triangles - the first has base length 2 and height 2 while the second has base length 1 and height 1. Since the second triangle is below the x-axis, we'll consider this area to be "negative" (we can pretend that the triangle has "height -1"), so we get

$$\int_1^4 (3-x)\,\mathrm{d}x = \frac{3\cdot 2}{2} + \frac{1\cdot(-1)}{2} = \frac{5}{2}.$$

(c) We begin by writing out the right Riemann sum with n-subintervals. The interval $[0,2]$ has length 2, so the width of the rectangles in the Riemann sum will be $\Delta x = 2/n$. Then the nodes/tag points are given by

$$x_i^* = 0 + i\Delta x = \frac{2i}{n}, \quad (i = 1,\dots,n).$$

Then the right Riemann sum for $2x^2+1$ on $[0,2]$ with n-subintervals is

$$\begin{aligned}
R_n = \sum_{i=1}^{n} (2(x_i^*)^2+1)\Delta x &= \sum_{i=1}^{n}\left[2\left(\frac{2i}{n}\right)^2+1\right]\frac{2}{n},\\
&= \sum_{i=1}^{n}\left[\frac{8i^2}{n^2}+1\right]\frac{2}{n},\\
&= \frac{2}{n}\left[\frac{8}{n^2}\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} 1\right]\\
&= \frac{2}{n}\left[\frac{8}{n^2}\frac{n(n+1)(2n+1)}{6}+n\right],\\
&= \frac{8}{3}\frac{(n+1)(2n+1)}{n^2}+2.
\end{aligned}$$

Since the given function is continuous on $[0,2]$, it is integrable there. Consequently, the Riemann sums R_n converge to the value $\int_0^2 (2x^2+1)\,\mathrm{d}x$ as $n\to\infty$. That is, we have

$$\int_0^2 (2x^2+1)\,\mathrm{d}x = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\frac{8}{3}\frac{(n+1)(2n+1)}{n^2}+2,$$

$$= \frac{16}{3} + 2,$$
$$= \frac{22}{3}$$

(d) Since our interval $[1,5]$ has length 4, the width of the rectangles will be $\Delta x = 4/n$. Then, the nodes (or tag points) are given by

$$x_i^* = 1 + i\Delta x = 1 + \frac{4i}{n}, \quad (i = 1, \ldots, n).$$

Thus, the right Riemann sum for $x^2 + 2x$ on $[1,5]$ with n-subintervals is

$$\begin{aligned}
R_n = \sum_{i=1}^{n}((x_i^*)^2 + 2x_i^*)\Delta x &= \sum_{i=1}^{n}\left[\left(1 + \frac{4i}{n}\right)^2 + 2\left(1 + \frac{4i}{n}\right)\right]\frac{4}{n} \\
&= \sum_{i=1}^{n}\left[1 + \frac{8i}{n} + \frac{16i^2}{n^2} + 2 + \frac{8i}{n}\right]\frac{4}{n} \\
&= \frac{4}{n}\sum_{i=1}^{n}\left[3 + \frac{16i}{n} + \frac{16i^2}{n^2}\right] \\
&= \frac{4}{n}\left[\sum_{i=1}^{n}3 + \frac{16}{n}\sum_{i=1}^{n}i + \frac{16}{n^2}\sum_{i=1}^{n}i^2\right] \\
&= \frac{4}{n}\left[3n + \frac{16}{n}\cdot\frac{n(n+1)}{2} + \frac{16}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right] \\
&= \frac{4}{n}\left[3n + 8(n+1) + \frac{8(n+1)(2n+1)}{3n}\right] \\
&= 4\left(11 + \frac{1}{n} + \frac{8(n+1)(2n+1)}{3n^2}\right) \\
&= 4\left(11 + \frac{1}{n} + \frac{16n^2 + 24n + 8}{3n^2}\right).
\end{aligned}$$

Taking the limit, it follows that

$$\begin{aligned}
\int_1^5 (x^2 + 2x)\,\mathrm{d}x = \lim_{n\to\infty} R_n &= \lim_{n\to\infty} 4\left(11 + \frac{1}{n} + \frac{16n^2 + 24n + 8}{3n^2}\right) \\
&= 4\left(11 + \frac{16}{3}\right) \\
&= \frac{196}{3}.
\end{aligned}$$

(e) We proceed as in the previous problem with right Riemann sums. Since $f(x) = x^3$ is continuous on $[0,3]$, it is Riemann integrable there and, furthermore, the right Riemann sums must converge to $\int_0^3 x^3\,\mathrm{d}x$ as $n \to \infty$. Computing the right Riemann sum for $f(x) = x^3$ on $[0,3]$ gives $\Delta x = 3/n$ and

$$R_n(f) = \sum_{i=1}^{n}(x_i^*)^3\Delta x = \sum_{i=1}^{n}\left(0 + i\frac{3}{n}\right)^3\frac{3}{n}$$

$$= \sum_{i=1}^{n} \frac{3^4 i^3}{n^4}$$
$$= \frac{81}{n^4} \sum_{i=1}^{n} i^3$$
$$= \frac{81}{n^4} \cdot \frac{n^2(n+1)^2}{4}$$
$$= \frac{81(n^4 + 2n^3 + 2n)}{4}.$$

Taking the limit as $n \to \infty$, we infer that

$$\int_0^3 x^3 \, \mathrm{d}x = \lim_{n\to\infty} R_n(f) = \lim_{n\to\infty} \frac{81(n^4 + 2n^3 + 2n)}{4} = \frac{81}{4}.$$

(f) We now examine $\int_{-2}^{2} |2x| \, \mathrm{d}x$. To deal with the absolute value that appears, we will break the integral into a sum of two integrals, on which $|2x|$ is either positive or negative. This is done in the following way:

$$\int_{-2}^{2} |2x| \, \mathrm{d}x = \int_{-2}^{0} -(2x) \, \mathrm{d}x + \int_{0}^{2} (2x) \, \mathrm{d}x = \int_0^2 2x \, \mathrm{d}x - \int_{-2}^{0} 2x \, \mathrm{d}x.$$

Clearly, these two integrals should each be easier to evaluate than the original integral we started with. Since $2x$ is Riemann integrable by continuity, the integral $\int_0^2 2x \, \mathrm{d}x$ is the limit of the right Riemann sums as $n \to \infty$, i.e.

$$\int_0^2 2x \, \mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} 2\left(\frac{2i}{n}\right)\frac{2}{n}$$
$$= \lim_{n\to\infty} \frac{8}{n^2} \sum_{i=1}^{n} i$$
$$= \lim_{n\to\infty} \frac{8}{n^2} \cdot \frac{n(n+1)}{2}$$
$$= \lim_{n\to\infty} \frac{4n^2 + 4n}{n^2}$$
$$= 4.$$

In order to compute $\int_{-2}^{0} 2x \, \mathrm{d}x$, we will also make use of right Riemann sums. Here, we also have $\Delta x = \frac{0-(-2)}{n} = \frac{2}{n}$. However, the nodes are instead given by

$$x_i^* = -2 + \frac{2i}{n}, \quad (i = 1, \ldots, n).$$

Thus,

$$\int_{-2}^{0} 2x \, \mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} 2\left(-2 + \frac{2i}{n}\right)\frac{2}{n}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{8}{n} \sum_{i=1}^{n} \left(-1 + \frac{i}{n}\right) \\
&= \lim_{n\to\infty} \left(\frac{8}{n} \sum_{i=1}^{n}(-1) + \frac{8}{n^2} \sum_{i=1}^{n} i\right) \\
&= \lim_{n\to\infty} \left(\frac{8}{n} \cdot (-n) + \frac{8}{n^2} \cdot \frac{n(n+1)}{2}\right) \\
&= \lim_{n\to\infty} \left(-8 + \frac{4n^2 + 4n}{n^2}\right) \\
&= -4.
\end{aligned}
$$

Now, putting together all our work gives:

$$
\int_{-2}^{2} |2x| \, \mathrm{d}x = \int_{0}^{2} 2x \, \mathrm{d}x - \int_{-2}^{0} 2x \, \mathrm{d}x = 4 - (-4) = 8.
$$

3. Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous functions and let $k \in \mathbb{R}$ be a constant. By using the corresponding properties for summations, prove the following

   (a) $\displaystyle\int_a^b (kf(x))\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x$

   (b) $\displaystyle\int_a^b (f(x)+g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$

   Solutions:

   (a) Since f is continuous on the interval $[a, b]$, it must be Riemann integrable on $[a, b]$. Therefore, if we let $R_n(f)$ denote the right Riemann sum of f with n-subintervals, we have

$$\lim_{n\to\infty} R_n(f) = \int_a^b f(x)\,\mathrm{d}x. \tag{1}$$

   That is, the Riemann sums of f converge to $\int_a^b f\,\mathrm{d}x$ as $n \to \infty$. Similarly, because f is continuous on $[a, b]$ and $k \in \mathbb{R}$ is a constant, the function $kf(x)$ is continuous and integrable on $[a, b]$. By the same logic, the right Riemann sum of kf will converge to $\int_a^b kf(x)\,\mathrm{d}x$ as $n \to \infty$, i.e.

$$\lim_{n\to\infty} R_n(kf) = \int_a^b kf(x)\,\mathrm{d}x. \tag{2}$$

   But then, because we can pull constants out of finite sums, we obtain

$$\begin{aligned}\int_a^b kf(x)\,\mathrm{d}x = \lim_{n\to\infty} R_n(kf) &= \lim_{n\to\infty}\sum_{i=1}^n (kf(x_i^*))\Delta x\\ &= \sum_{i=1}^n kf(x_i^*)\Delta x\\ &= k\sum_{i=1}^n f(x_i^*)\Delta x\\ &= kR_n(f).\end{aligned}$$

   Combining this with (1)-(2), this implies that

$$\begin{aligned}\int_a^b kf(x)\,\mathrm{d}x = \lim_{n\to\infty} R_n(kf) &= \lim_{n\to\infty}(kR_n(f))\\ &= k\lim_{n\to\infty} R_n(f)\\ &= k\int_a^b f(x)\,\mathrm{d}x.\end{aligned}$$

   (b) Since f and g are continuous on the interval $[a, b]$, their sum $f + g$ is also continuous on $[a, b]$. Especially, the functions f, g and $f+g$ are Riemann integrable on $[a, b]$. Furthermore, as in the previous exercise, this means that

$$\lim_{n\to\infty} R_n(f) = \int_a^b f(x)\,\mathrm{d}x,$$

$$
\begin{aligned}
\lim_{n\to\infty} R_n(g) &= \int_a^b g(x)\,\mathrm{d}x, \\
\lim_{n\to\infty} R_n(f+g) &= \int_a^b (f(x)+g(x))\,\mathrm{d}x
\end{aligned}
$$

where $R_n(\cdot)$ once again denotes the right Riemann sum of a function. However, the Riemann sums satisfy

$$
\begin{aligned}
R_n(f+g) = \sum_{i=1}^{n} (f+g)(x_i^*)\Delta x &= \sum_{i=1}^{n} (f(x_i^*)+g(x_i^*))\Delta x \\
&= \sum_{i=1}^{n} f(x_i^*)\Delta x + \sum_{i=1}^{n} g(x_i^*)\Delta x \\
&= R_n(f) + R_n(g).
\end{aligned}
$$

Taking the limit as $n \to \infty$ and applying the limit laws from Math 140, we infer that

$$
\begin{aligned}
\int_a^b (f(x)+g(x))\ \mathrm{d}x &= \lim_{n\to\infty} R_n(f+g) \\
&= \lim_{n\to\infty} (R_n(f)+R_n(g)) \\
&= \lim_{n\to\infty} R_n(f) + \lim_{n\to\infty} R_n(g) \\
&= \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x.
\end{aligned}
$$

Challenge problems

4. Using Riemann sums, show that

$$\int_{a(x)}^{b(x)} t \, \mathrm{d}t = \frac{(b(x))^2 - (a(x))^2}{2}$$

5. What do you expect the value of

$$\int_{-\pi/2}^{3\pi/2} \cos x \, \mathrm{d}x$$

to be? *Hint: use symmetry.*

6. Let $f(x) = x^2$.

(a) Using Riemann sums, determine the function

$$F(x) = \int_0^x f(t) \, \mathrm{d}t.$$

(b) Determine the derivative $F'(x)$ of $F(x)$. How does this function relate to f?