Math 140 **Tutorial 7**

February 17 - 20

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1. Find $\frac{dy}{dx}$ by using implicit differentiation

   (a) $y^2 + x^2 = 1$

   (b) $\sin x + \cos y = x^3 - 3y^2$

   (c) $2(x^2 + y^2)^2 = 9(x^2 - y^2)$

   (d) $\sin x \cos y = \sin^2(x + y)$. (Check out the graph of this curve)

Solution

(a) We start by differentiating term by term with respect to x, to get

$$2y\frac{dy}{dx} + 2x = 0$$

We can move things around and divide by $2y$ to get

$$\frac{dy}{dx} = -\frac{x}{y}$$

(b) Same as before we differentiate term by term to get

$$\cos x - \sin y\frac{dy}{dx} = 3x^2 - 6y\frac{dy}{dx}$$

Now, move all the term that involve $\frac{dy}{dx}$ to one side, and all the other terms to the other side and divide:

$$\frac{dy}{dx} = \frac{\cos x - 3x^2}{\sin y - 6y}$$

(c) Again we want to differentiate term by term. To differentiate $(x^2 + y^2)^2$ we need to use the chain rule with $f(x) = x^2$ and $g(x) = x^2 + y^2$, we know that $f'(x) = 2x$ and $g'(x) = 2x + 2y\frac{dy}{dx}$ so

$$\frac{d}{dx}(x^2 + y^2)^2 = 2(x^2 + y^2)(2x + 2y\frac{dy}{dx})$$

So by differentiating term by term we have

$$8(x^2 + y^2)\left(x + y\frac{dy}{dx}\right) = 18x - 18y\frac{dy}{dx}$$

So by moving things around we can re-write as

$$\frac{dy}{dx} = -\frac{8x(x^2+y^2) - 18x}{8y(x^2+y^2) + 18y}$$

(d) To differentiate $\sin x \cos y$ we need to apply the product rule:

$$\frac{d}{dx} \sin x \cos y = \cos x \cos y - \sin x \sin y \frac{dy}{dx}$$

To differentiate $\sin^2(x+y)$ we can apply the chain rule with $u(x) = \sin^2(x)$ and $v(x) = x + y$. We can apply the chain rule once more to find that $u'(x) = 2 \sin x \cos x$ and $v'(x) = 1 + \frac{dy}{dx}$. Therefore

$$\frac{d}{dx} \sin^2(x+y) = 2\sin(x+y)\cos(x+y) \cdot \left(1 + \frac{dy}{dx}\right)$$

So by implicit differentiation we have

$$\cos x \cos y - \sin x \sin y \frac{dy}{dx} = 2\sin(x+y)\cos(x+y)\left(1 + \frac{dy}{dx}\right)$$

So by moving things around in this equation we have

$$\frac{dy}{dx} = \frac{\cos x \cos y - 2\sin(x+y)\cos(x+y)}{\sin x \sin y + 2\sin(x+y)\cos(x+y)}$$

2. Use implicit differentiation to find an equation for the tangent line to the curve at the given point.

(a) $x^2 + 2xy + 4y^2 = 12$ at $(2, 1)$

(b) $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$ at $(0, \frac{1}{2})$

(c) $y^2(y^2 - 4) = x^2(x^2 - 5)$ at $(0, -2)$

Solution

(a) By implicit differentiation we can find that

$$\frac{dy}{dx} = -\frac{x + 5y}{x + 4y}$$

So we can plug in $x = 2, y = 1$ to get the slope of the tangent line is $\frac{7}{6}$. It also goes through $(2, 1)$ So it has equation

$$y = \frac{7}{6}x - \frac{4}{3}$$

(b) Again by implicit differentiation we find that

$$2x + 2y\frac{dy}{dx} = 2(2x^2 + 2y^2 - x)(4x + 4y\frac{dy}{dx} - 1)$$

So if we plug in $x = 0, y = \frac{1}{2}$ we find that the slope is 1. The tangent line goes through $(0, \frac{1}{2})$ so it has equation

$$y = x + \frac{1}{2}$$

(c) By implicit differentiation we find

$$2y(y^2 - 4)\frac{dy}{dx} + y^2 \cdot (2y)\frac{dy}{dx} = 2x(x^2 - 5)x^2(2x)$$

So by plugging in $x = 0, y = -2$ we find that the slope is 0. Therefore the equation for the tangent line is $y = -2$.

3. Use implicit differentiation to prove that

$$\frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{x\sqrt{x^2 - 1}}$$

Hint: let $y(x) = \sec^{-1}(x)$*, so that* $x = \sec(y(x))$*. Then, draw your triangle and use implicit differentiation.*

Solution

We start by putting $y(x) = \sec^{-1} x$, therfore

$$x = \sec y$$

Now applying implicit differentiation we see that

$$1 = \sec y \tan y \frac{dy}{dx}$$

So

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

We need to write this in terms of x. We know that $y = \sec^{-1} x$, so $\sec y = x$. Also $\tan y = \sqrt{\sec^2 y - 1} = \sqrt{x^2 - 1}$. We can conclude that

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}}$$

4. In this problem, we'll evaluate the derivative of $f(x) = (\sin x)^{\ln x}$ in two 'different' ways.

   (a) Use the fact that $\ln x$ is the inverse of e^x to write $f(x) = \exp(\ln(g(x)))$ and then use the Chain Rule to evaluate $f'(x)$.

   (b) Take logarithms on both sides of $f(x) = (\sin x)^{\ln x}$ and then use implicit differentiation to evaluate $f'(x)$. Note: this technique is known as *logarithmic differentiation.*

Solution

(a) We can write

$$f(x) = e^{\ln\left((\sin x)^{\ln x}\right)} = e^{(\ln x)\ln(\sin x)}$$

So we can apply the chain rule with $u(x) = e^x$ and $v(x) = \ln x \ln(\sin x)$. We know that $u'(x) = e^x$. For $v'(x)$ we can apply the product rule to see that

$$v'(x) = \frac{\ln(\sin x)}{x} + \ln x \frac{d}{dx}\ln(\sin x) = \frac{\ln(\sin x)}{x} + \ln x \frac{\cos x}{\sin x}$$

So the final answer is

$$f'(x) = e^{\ln x \ln(\sin x)}\left(\frac{\ln(\sin x)}{x} + \ln x \frac{\cos x}{\sin x}\right) = (\sin x)^{\ln x}\left(\frac{\ln(\sin x)}{x} + \ln x \frac{\cos x}{\sin x}\right)$$

(b) By taking ln of both sides we can write

$$\ln(f(x)) = \ln x \ln(\sin x)$$

So by implicit differentiation we can write

$$\frac{f'(x)}{f(x)} = \frac{\ln(\sin x)}{x} + \ln x \frac{\cos x}{\sin x}$$

So multiplying both sides by $f(x)$ we find

$$f'(x) = (\sin x)^{\ln x}\left(\frac{\ln(\sin x)}{x} + \ln x \frac{\cos x}{\sin x}\right)$$

5. For the following functions $y(x)$. Write down the domain and range of the function y and find the derivative of the function.

   (a) $y = \sin^{-1}\left(\sqrt{\sin x}\right)$

   (b) $y = \sqrt{\tan^{-1}(x)}$

   (c) $y = \ln(\sec x + \tan x)$

   (d) $y = \ln(xe^{x^2})$

Solution

(a) Recall that

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

This can be found using implicit differentiation. Now we can find $\frac{dy}{dx}$ using the chain rule with $f(x) = \sin^{-1} x$ and $g(x) = \sqrt{\sin x}$ and we get that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin x}} \cdot g'(x)$$

So we need to find $g'(x)$, which we can do by using the chain rule again

$$g'(x) = \frac{\cos x}{2\sqrt{\sin x}}$$

Therefore

$$\frac{dy}{dx} = \frac{\cos x}{2\sqrt{\sin x - \sin^2 x}}$$

(b) Again we will apply the chain rule. Recall that

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

So if we let $f(x) = \sqrt{x}$, $g(x) = \tan^{-1} x$ and apply the chain rule, then

$$\frac{dy}{dx} = \frac{1}{2\tan^{-1} x} \cdot \frac{1}{1+x^2}$$

(c) We will apply the chain rule with $f(x) = \ln x$ so $f'(x) = \frac{1}{x}$ and $g(x) = \sec x + \tan x$, so $g'(x) = \sec x \tan x + \sec^2 x$. Therefore

$$\frac{dy}{dx} = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x$$

(d) We will apply the chain rule to $f(x) = \ln x$, so $f'(x) = \frac{1}{x}$ and $g(x) = xe^{x^2}$. We know that

$$\frac{dy}{dx} = \frac{1}{xe^{x^2}} \cdot g'(x)$$

To find $g'(x)$ we can apply the product rule to get

$$g'(x) = e^{x^2} + x\frac{d}{dx}\left(e^{x^2}\right)$$

To find $\frac{d}{dx}e^{x^2}$ we can apply the chain rule to find

$$\frac{d}{dx}\left(e^{x^2}\right) = 2xe^{x^2}$$

So finally we can write

$$\frac{dy}{dx} = \frac{e^{x^2} + 2x^2 e^{x^2}}{xe^{x^2}} = \frac{1 + 2x^2}{x} = \frac{1}{x} + 2x$$

Alternatively, we could've seen that

$$\ln(xe^{x^2}) = \ln x + \ln e^{x^2} = \ln x + x^2$$