Lecture hours 27-29

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Definitions and Theorems

Definition (Determinant of a matrix). Remember that

$$det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

To find the determinant of a 3×3 matrix, use a cofactor expansion (it works any size square matrix).

Example: Cofactor expansion along first row of a 3×3 matrix

$$det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} = a(+1)det \begin{bmatrix} e & f \\ h & j \end{bmatrix} + b(-1)det \begin{bmatrix} d & f \\ g & j \end{bmatrix} + c(+1)det \begin{bmatrix} d & e \\ g & h \end{bmatrix}.$$

The signs in the expansion are given by the "sign matrix"

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix},$$

which has same size and a positive sign in the top left component. Signs are alternate otherwise.

Definition (Adjugate matrix). Let A be a square matrix, the adjugate matrix is given by

$$adj(A) = [\text{matrix of cofactors}]^T.$$

Definition (Cramer's rule). Let A be an invertible $n \times n$ matrix. For vector $\vec{b} \in \mathbb{R}^n$, the solution to $A\vec{x} = \vec{b}$ is given by

$$\vec{x} = \frac{1}{det(A)} adj(A)\vec{b}.$$

Problem 46 (Determinant)**.** Suppose an $n \times n$ matrix A has $detA \neq 0$. Is it true that det rref(A) = det A ?

Problem 47 (Determinant)**.** Use a cofactor expansion to find:

a) $\det \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -1 \\ 0 & 1 & -2 \end{bmatrix}$,

b) $\det \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -2 \end{bmatrix}$,

c) $\det \begin{bmatrix} 1 & -1 & 0 \\ 22 & 33 & -11 \\ 0 & 1 & -2 \end{bmatrix}$.

What do you notice about the results?

Problem 48 (Determinant)**.** Suppose an $n \times n$ matrix A has two identical rows. Explain why the determinant of A is zero.

Problem 49 (Determinant)**.** Assume A is an $n \times n$ matrix. This problem is about the relation between determinants and row operations. In each case, give a justification for your answer by explaining what happens to the cofactor expansion.

a) What happens to the determinant of A if a row is multiplied by a nonzero scalar? (Hint: Try expanding the determinant along the row that is multiplied by the scalar).

b) What happens to the determinant of A if two rows are swapped? (Hint: Try this for a 2×2 matrix first, then use that answer to see what happens for a 3×3... etc.)

c) What happens to the determinant of A if a multiple of one row is added to another row? (Hint: Expand along the new row and split the result into the sum of two different determinants, then use the answer to Problem 2.)

Use your answers to justify the statement that **the determinant of A is nonzero if and only if A is invertible**.

Problem 50 (Determinant and area)**.** Let A be the 2×2 matrix

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$$

a) Let S be the unit square with sides given by $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let $A(\mathsf{S})$ be the parallelogram with sides $A\vec{v}_1$, $A\vec{v}_2$. Verify that

$$\det(A) = \frac{\operatorname{area}(A(\mathsf{S}))}{\operatorname{area}(\mathsf{S})}.$$

b) Let T be the triangle with vertices at the origin, $\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\vec{w}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Le $A(\mathsf{T})$ be the triangle with vertices at the origin, $A\vec{w}_1$, and $A\vec{w}_2$. Compute

$$\frac{\operatorname{area}(A(\mathsf{T}))}{\operatorname{area}(\mathsf{T})}.$$

What do you notice?

c) Repeat parts a and b with your own choice of matrix A. What do you notice about your new answer to part b?

Problem 51 (Cramer's Rule). a) Use a determinant to show that the matrix

$$A = \begin{bmatrix} -1 & 4 & 0 \\ 1 & 3 & -2 \\ -2 & 0 & -1 \end{bmatrix}$$

is invertible.

b) Use Cramer's rule to solve the system

$$A\,\vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$