Lecture hours 16-18

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Definitions and Theorems

Definition (Matrix multiplication)**.** Let A be an $m \times p$ and B an $p \times n$. Matrix AB is an $m \times n$ matrix. Recall, to multiply matrices together: multiply left matrix by each column of the right matrix, those are the columns of the resulting matrix.

Definition (Composition of linear transformations)**.** Let $S : \mathbb{R}^m \to \mathbb{R}^p$ and $T : \mathbb{R}^p \to \mathbb{R}^n$ be linear transformations. The composition $T \circ S : \mathbb{R}^m \to \mathbb{R}^n$ (pronounced "T composed with S") is given by

$$T \circ S(\vec{v}) \stackrel{\text{def}}{=} T(S(\vec{v})),$$

for $\vec{v} \in \mathbb{R}^m$.

Definition (Invertible linear transformation)**.**

In terms of linear transformations:

A linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is invertible if for all $\vec{y} \in \mathbb{R}^n$ (outputs of T) there exists an $\vec{x} \in \mathbb{R}^m$ such that $T(\vec{x}) = \vec{y}$, and this $\vec{x}$ is unique.

In terms of matrices:

A matrix A is invertible if for all $\vec{y} \in \mathbb{R}^n$ there is an unique $\vec{x} \in \mathbb{R}^m$ such that $A\vec{x} = \vec{y}$.

Problem 33 (Matrix algebra). Compute the following matrix products:

a)

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 \end{bmatrix}.$$

b)

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

Problem 34 (Compositions and inverses). Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be projection from $\mathbb{R}^3$ onto the xy-plane.

a) Is T invertible? Why or why not?

b) Find the matrix A with $T(\vec{x}) = A\vec{x}$.

c) Find a linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ with $T \circ S = S \circ T$.

d) Find a linear transformation $S : \mathbb{R}^3 \to \mathbb{R}^3$ with $T \circ S \neq S \circ T$.

Problem 35 (Compositions). True or false? If true, explain why; if false, give a counterexample.

a) If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation and $\ker(T) = \{\vec{0}\}$, then T is invertible.

b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with nullity 1 and $S : \mathbb{R}^2 \to \mathbb{R}^2$ is another linear transformation with nullity 1, then $T \circ S$ is the zero transformation.

c) If T and S are linear transformations with the domain of T equal to the codomain of S, then $\operatorname{rank}(T \circ S) \leq \operatorname{rank}(T)$.

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Problem 36 (Inverse of a problem)**.** Let $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 3 & 5 \\ 1 & 2 \end{smallmatrix}\right]$.

a) Find A^{-1}.

b) Find B^{-1}.

c) Find all 2×2 matrices X with $AXA^{-1} = B$.