Lecture hours 13-15

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Definitions and Theorems

Definition (Kernel and Image of a linear transformation)**.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation

- The kernel ker(T) is the set of vectors $\vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = 0$.
- The image of T is the set of all vectors $\vec{y} \in \mathbb{R}^m$ such that $T(\vec{x}) = \vec{y}$ for some $\vec{x} \in \mathbb{R}^n$.

Definition (Rank and Nullity of a linear transformation)**.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation

- The rank of T is the dimension of the image of T, rank T = dim(im T).
- The nullity of T is the dimension of the kernel of T , nullity T = dim(ker T).

Theorem (Rank Nullity Theorem)**.**

- In terms of linear transformations:

  Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation

  $$\text{rank T} + \text{nullity T} = n.$$

- In terms of matrices:

  Let A be an $m \times n$ matrix

  $$dim(imA) + dim(kerA) = \text{number of columns of A} = n.$$

Problem 29 (Rank and Nullity)**.** Let $\vec{v} \neq \vec{0}$ be the vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Define a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(\vec{x}) = \vec{v} \times \vec{x}.$$

a) What is the nullity of T?

b) What is the rank of T? Why?

Solution 29 (Rank and Nullity)

a) From Problem 27 in Tutorial 6 we know that if $\vec{x}$ is in the kernel of T then $x = c\vec{v}$ for some $c \in \mathbb{R}$.

   From the definition of cross product, it is straightforward to show that any scalar multiple of vector $\vec{v}$ is in the kernel of T.

   Therefore, Ker T = span $(\vec{v})$ and nullity T = 1.

b) The rank of T is 2. The easiest way to find this is to use that the kernel of T is 1-dimensional and apply the rank-nullity theorem.

Problem 30 (Rank Nullity Theorem)**.** True or false? Justify your answer.

a) If A is a 2×4 matrix with kernel of dimension 2, then the equation $A\vec{x} = \vec{e}_2$ is consistent.

b) There is a 5×5 matrix A such that dim(imA) = dim(kerA) .

Solution 30 (Rank Nullity Theorem)

a) True, by the rank-nullity theorem A has rank 2, so the image of A is $\mathbb{R}^2$.

b) False. Suppose dim(im A) = dim(ker A) for some 5×5 matrix A. By the rank-nullity theorem we have

$$5 = dim(imA) + dim(kerA) = 2dim(imA).$$

   which is impossible because $2dim(imA)$ is always an even integer. It cannot be equal to 5.

Problem 31 (Rank Nullity Theorem)**.** Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a-b \\ c-d \end{bmatrix}.$$

Find the kernel, nullity, image and rank of T.

Solution 31 (Rank Nullity Theorem)

1. If $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is in the kernel of T then $a = b$ and $c = d$. It follows that the kernel of T is given by the span of $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\}$ and nullity T = 2.

2. From the Rank Nullity Theorem we have

$$\text{rank T} + \text{nullity T} = \text{rank T} + 2 = 4.$$

Therefore rank T = 2. In other words, the image of T is a 2-dimensional subspace of $\mathbb{R}^2$. It follows that imT = $\mathbb{R}^2$.

Problem 32 (Rank Nullity Theorem)**.** Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Find the kernel, nullity, image and rank of T.

Solution 32 (Rank Nullity Theorem)

1. If $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in the kernel of T then $a = 0$ and $c = 0$. It follows that the kernel of T is given by the span of $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and nullity T = 1.
2. From the Rank Nullity Theorem we can conclude that the image of T is a 2-dimensional subspace of $\mathbb{R}^2$. It follows that imT = $\mathbb{R}^2$.