Lecture hours 19-20

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Definitions and Theorems

Definition (Coordinates of a vector).

If we have a basis of $\mathbb{R}^n$, $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ and $\vec{x} \in \mathbb{R}^n$, so $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$, for some $c_1, \dots c_n \in \mathbb{R}$. We define the coordinates of $\vec{x}$ in the basis $\mathcal{B}$ as

$$[\vec{x}]_{\mathcal{B}} \stackrel{\text{def}}{=} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Definition (Change of basis matrix).

Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a basis of $\mathbb{R}^n$. We call the matrix $S = [\vec{v}_1, \dots, \vec{v}_n]$ the change of basis matrix of basis $\mathcal{B}$.

For any $\vec{x} \in \mathbb{R}^n$ we have

$$\vec{x} = S\,[\vec{x}]_{\mathcal{B}}\,.$$

So, if we have to find the coordinates $[\vec{x}]_{\mathcal{B}}$ we compute

$$[\vec{x}]_{\mathcal{B}} = S^{-1}\vec{x}.$$

Note that

- If $\mathcal{B} = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ (the standard basis), we have $[\vec{x}]_{\mathcal{B}} = \vec{x}$.

Problem 37 (Coordinates). Let $\mathfrak{B}$ be the basis of $\mathbb{R}^4$ given by

$$\mathfrak{B} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Find the coordinates for vector

$$\vec{v} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

in the basis $\mathfrak{B}$.

Solution 37 (Coordinates) Let $E = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$ be the standard basis of $\mathbb{R}^4$. Define

$$S = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The change of basis matrix from the standard basis E to basis $\mathfrak{B}$ is given by

$$S^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & -2 \end{bmatrix}$$

Thus

$$[\vec{v}]_{\mathfrak{B}} = S^{-1}[\vec{v}]_E = S^{-1}\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

Problem 38 (Coordinates)**.** In this problem we will be working with the following bases of $\mathbb{R}^3$:

$$\mathfrak{B} = \left\{ \begin{bmatrix} -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 9 \end{bmatrix} \right\}, \quad E = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \mathfrak{D} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

.

1. Find the change of basis matrix from basis $\mathfrak{B}$ to the standard basis E.
2. Find the change of basis matrix from the standard basis E to basis $\mathfrak{D}$.
3. Find the change of basis matrix from basis $\mathfrak{B}$ to basis $\mathfrak{D}$.

Solution 38 (Coordinates) Define

$$S = \begin{bmatrix} -3 & 1 & 5 \\ 2 & -1 & 4 \\ -3 & -1 & 9 \end{bmatrix}, \quad \textit{and} \quad R = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 1 \\ 1 & -1 & 3 \end{bmatrix}$$

1. The change of basis matrix from basis $\mathfrak{B}$ to basis E is given by S.
2. The change of basis matrix from basis E to basis $\mathfrak{D}$ is given by

$$R^{-1} = \frac{1}{8} \begin{bmatrix} 7 & 2 & -3 \\ -5 & 2 & 1 \\ -4 & 0 & 4 \end{bmatrix}$$

3. The change of basis matrix from basis $\mathfrak{B}$ to basis $\mathfrak{D}$ is given by

$$R^{-1}S = \frac{1}{8} \begin{bmatrix} 7 & 2 & -3 \\ -5 & 2 & 1 \\ -4 & 0 & 4 \end{bmatrix} \begin{bmatrix} -3 & 1 & 5 \\ 2 & -1 & 4 \\ -3 & -1 & 9 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 2 & -1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

What are the columns of $R^{-1}S$ in terms of coordinates?

$$\begin{tikzcd} & {[\vec{v}]_E} \arrow[dr, "R^{-1}"] & \\ {[\vec{v}]_{\mathfrak{B}}} \arrow[ur, "S"] \arrow[rr, dashed, "SR^{-1}"] & & {[\vec{v}]_{\mathfrak{D}}} \end{tikzcd}$$

Note: Red arrow means

$\vec{v} = [\vec{v}]_E = S[\vec{v}]_{\mathfrak{B}}$

(matrix multiplication).

Problem 39 (Coordinates for linear transformations)**.** In this problem we will be working with the same bases of $\mathbb{R}^3$ as in Problem 38:

$$\mathfrak{B} = \left\{ \begin{bmatrix} -3 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 9 \end{bmatrix} \right\}, \quad E = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \mathfrak{D} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\vec{v} = A\vec{v}$$

where

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Find the matrix $[T]^{\mathfrak{D}}_{\mathfrak{B}}$.

Remember that the matrix $[T]^{\mathfrak{D}}_{\mathfrak{B}}$ satisfies:

$$[T\vec{v}]_{\mathfrak{D}} = [T]^{\mathfrak{D}}_{\mathfrak{B}}[\vec{v}]_{\mathfrak{B}}.$$

for any $\vec{v} \in \mathbb{R}^3$.

Solution 39 (Coordinates for linear transformations)

Define

$$S = \begin{bmatrix} -3 & 1 & 5 \\ 2 & -1 & 4 \\ -3 & -1 & 9 \end{bmatrix}, \quad and \quad R = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 1 \\ 1 & -1 & 3 \end{bmatrix}$$

Take any $\vec{v} \in \mathbb{R}^3$.

To find $[T]^{\mathfrak{D}}_{\mathfrak{B}}$ take the following steps:

1. Since we start with $[\vec{v}]_{\mathfrak{B}}$, we first need to change to standard basis coordinates: $[\vec{v}]_E = S[\vec{v}]_{\mathfrak{B}}$.
2. We know A is the matrix representation of T in the standard basis coordinates. Thus $[T\vec{v}]_E = AS[\vec{v}]_{\mathfrak{B}}$.

3. Finally, we need to transform vector $AS\vec{v}$ from standard coordinates to $\mathfrak{D}$ coordinates. That is $[T\vec{v}]_{\mathfrak{B}} = R^{-1}AS[\vec{v}]_{\mathfrak{B}}$. Therefore $[T]_{\mathfrak{B}}^{\mathfrak{D}} = R^{-1}AS$.

$$\begin{tikzcd}
{[\vec{v}]_E} \arrow[r, "A"] & {[T\vec{v}]_E} \arrow[d, "R^{-1}"] \\
{[\vec{v}]_B} \arrow[u, "S"] \arrow[r, dashed, "R^{-1}AS"] & {[T\vec{v}]_D}
\end{tikzcd}$$

Problem 40 (Elementary matrices)**.** Write the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 2 \\ -1 & 1 & 1 \end{bmatrix}$$

as a product of elementary row matrices. Use your expression to find A^{-1}.

Solution 40 (Elementary matrices) By Gauss-Jordan elimination, we have $E_5E_4E_3E_2E_1A = I$, where

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix},$$

$$E_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}$. We have $A^{-1} = E_5E_4E_3E_2E_1$.