1 Lecture hours 1-2

Definitions

Definition (Gaussian elimination)**.** When solving a system, there are certain operations we can apply to our equations, or equivalently to the rows of our augmented matrix, to "simplify" the system. We call this "simplification" process Gaussian elimination, since we are trying to eliminate (i.e. put to 0) as many entries from the augmented matrix as possible. There are 3 types of row operations we can perform:

1. Interchange any two rows.
2. Multiply any row by a (nonzero) constant.
3. Add any multiple of one row to another.

Definition (rref)**.** A matrix is in row reduced echelon form (rref) if :

1. The first non-zero term in each row (the "leading 1") is a 1; there are zeros above and below it.
2. The leading 1 in any row is to the right of the leading 1 in the row above it.
3. Any rows containing all zeros (to the left of the vertical bar) appear at the bottom.

Definition (Leading and Free variables)**.** Recall that in an augmented matrix for a system of equations each row represents the coefficients from one equation and each column of the coefficient matrix represents all the coefficients for a single variable/unknown.

For a matrix in rref, a column with a leading 1 corresponds to a leading variable. Columns that don't have a leading 1, correspond to free variables.

Problem 1 (Linear systems 1)**.** Draw on the xy-plane examples of systems of linear equations for each of the following scenarios:

(a) A consistent system of two equations for two unknowns x and y.

(b) An inconsistent system of two equations for two unknowns x and y.

(c) An inconsistent system of three equations for two unknowns x and y.

(d) A consistent system of three equations for two unknowns x and y.

Solution 1 (Linear systems 1)
(a) A drawing of two lines intersecting at a single point.
(b) A drawing of two parallel lines.
(c) A drawing of three parallel lines, or a drawing of two parallel lines and any third line, or a drawing of three lines that pairwise intersect in three distinct points.
(d) A drawing of three lines intersecting in a single common point.

Problem 2 (Gaussian elimination)**.** For each of the following systems...

- Solve for the unknowns by Gaussian elimination.
- Determine whether the system is consistent or has no solutions (is inconsistent).
- Draw the equations and their solutions on the plane.

(a)

$$\begin{aligned} x + 2y &= -1, \\ 2x - y &= 3, \end{aligned}$$

(b)

$$\begin{aligned} x - 3y &= -2, \\ -2x + 6y &= -4, \end{aligned}$$

(c)

$$\begin{aligned} 2x - 6y &= 4, \\ x - 3y &= 2. \end{aligned}$$

Solution 2 (Gaussian elimination)

(a)

$$\left[\begin{array}{cc|c} 1 & 2 & -1 \\ 2 & -1 & 3 \end{array}\right] \begin{array}{c} \\ -2R_1 \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & -5 & 5 \end{array}\right] \begin{array}{c} \\ \times(-\frac{1}{5}) \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 1 & -1 \end{array}\right] \begin{array}{c} -2R_2 \\ \\ \end{array}$$
$$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right].$$

The system is consistent and there is an unique solution: $x = 1, y = -1$. You can also plug this solution back in the system to make sure it works.

(b)

$$\left[\begin{array}{cc|c} 1 & -3 & -2 \\ -2 & 6 & -4 \end{array}\right] \begin{array}{c} \\ +2R_1 \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & 0 & -8 \end{array}\right].$$

The system is inconsistent by looking at the bottom row.

(c)

$$\left[\begin{array}{cc|c} 2 & -6 & 4 \\ 1 & -3 & 2 \end{array}\right] \begin{array}{c} \\ -\frac{1}{2}R_1 \end{array} \rightarrow \left[\begin{array}{cc|c} 2 & -6 & 4 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{c} \times\frac{1}{2} \\ \\ \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & -3 & 2 \\ 0 & 0 & 0 \end{array}\right].$$

The system is consistent and has an infinite number of solutions: $x = 2 + 3t, y = t$, for any real number t ($t \in \mathbb{R}$).

Problem 3 (Reduced row echelon form)**.** For each of the following matrices...

- Determine whether or not the matrix given is in rref form. If not, explain why not and put them in rref.
- Are the associated systems of equations consistent? If so, give their solution.

(a)

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 5 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b)

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 2 & -7 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & 8 \end{array}\right]$$

(c)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Solution 3 (Reduced row echelon form)
(a) The matrix is not in rref because there is a nonzero entry above the leading 1 in the second row. The rref is found by subtracting row 2 from row 1:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 7 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The solution is $x_1 = 7 - 2t, x_2 = -2, x_3 = 1, x_4 = t$, where $t \in \mathbb{R}$.
(b) The matrix is not in rref. The leading 1 of row 3 is to the left of the leading 1 of row 2. The rref is found by swapping rows 2 and 3:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 2 & -7 \\ 0 & 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 0 & 1 & 0 & 5 \end{array}\right].$$

The system is consistent and has an infinite number of solutions:
$x_1 = -7 - t - 2s, x_2 = t, x_3 = 8, x_4 = 5, x_5 = s$, where $t, s \in \mathbb{R}$.
(c) This matrix is in rref. The system is inconsistent because of the bottom row.

Problem 4 (Linear systems 2). Write down augmented matrices in row reduced echelon form representing systems of linear equations for each of the following scenarios:

(a) A consistent system of two equations for two unknowns x and y.

(b) An inconsistent system of two equations for two unknowns x and y.

(c) An inconsistent system of three equations for two unknowns x and y.

(d) A consistent system of three equations for two unknowns x and y.

Solution 4 (Linear systems 2)
(a) Something like:

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right].$$

(b) Something like:

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 3 \end{array}\right].$$

(c) Something like:

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array}\right].$$

(d) Something like:

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right].$$

Problem 5 (Free and leading variables). For the following system:

$$\begin{aligned} x_1 + x_2 + x_4 &= 1 \\ x_3 + 2x_4 &= 2 \\ x_3 + 2x_4 + x_5 &= 5. \end{aligned}$$

- Write down the coefficient matrix and the augmented matrix.
- Use row operations to put the augmented matrix in rref.
- Which are the free variables and which are the leading variables?
- Give the solution to this system.

Solution 5 (Free and leading variables)
The augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 2 & 1 & 5 \end{array}\right],$$

and the coefficient matrix is

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}.$$

The rref of the augmented matrix is found by subtracting row 2 from row 3:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right].$$

The free variables are x_2 and x_4. The leading variables are x_1, x_3, x_5. The solution is:

$$x_1 = 1 - t - s,\ x_2 = t,\ x_3 = 2 - 2s,\ x_4 = s,\ x_5 = 3,$$

with $t, s \in \mathbb{R}$.