10 Lecture hours 22 - 23

This is also a review for the second midterm.

Problem 41. Let T be the linear transformation induced by the matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 2 \end{bmatrix},$$

and S be the linear transformation induced by the matrix

$$\begin{bmatrix} -2 & 0 \\ 0 & 1 \\ 1 & 3 \end{bmatrix}.$$

a) Find the matrix that induces the linear transformation $S \circ T$.

b) Find the rank and nullity of $S \circ T$.

c) Find a basis for the image of $S \circ T$ and a basis for the kernel of $S \circ T$.

d) Is the linear transformation $S \circ T$ invertible? If so, find the inverse. If not, explain why.

e) If $\mathcal{B}$ is the standard basis of $\mathbb{R}^3$ find the matrix for $S \circ T$ in the $\mathcal{B}$ coordinates.

f) Let $\mathcal{D}$ the basis of $\mathbb{R}^3$ given by

$$\mathfrak{D} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}.$$

Find the matrix for $S \circ T$ in the $\mathcal{D}$ coordinates.

Solution 41 a) The matrix is given by

$$\begin{bmatrix} -2 & 0 \\ 0 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -4 \\ 0 & -1 & 2 \\ 1 & -3 & 8 \end{bmatrix}.$$

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b) We have that

$$rref \begin{bmatrix} -2 & 0 & -4 \\ 0 & -1 & 2 \\ 1 & -3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

There is one free variable. Thus, nullity $S \circ T = 1$. By the Rank Nullity theorem, we have that rank $S \circ T = 2$.

c) From the row reduced echelon form in part b) we have that

$$\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix} \right\}$$

is a basis for im $S \circ T$. And

$$\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\}$$

is a basis for the kernel of $S \circ T$.

d) Since nullity $S \circ T$ =1, the linear transformation is not invertible.

e) The matrix for this linear transformation in the $\mathcal{B}$ coordinates is

$$\begin{bmatrix} -2 & 0 & -4 \\ 0 & -1 & 2 \\ 1 & -3 & 8 \end{bmatrix}$$

f) The change of basis matrix from $\mathcal{D}$ to $\mathcal{B}$ is given

$$R \stackrel{\text{def}}{=} \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 1 \\ 1 & -1 & 3 \end{bmatrix}$$

Then, the matrix for this linear transformation in the $\mathcal{D}$ coordinates is

$$R^{-1} \begin{bmatrix} -2 & 0 & -4 \\ 0 & -1 & 2 \\ 1 & -3 & 8 \end{bmatrix} R = \begin{bmatrix} 7/8 & 1/4 & -3/8 \\ -5/8 & 1/4 & 1/8 \\ -1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} -2 & 0 & -4 \\ 0 & -1 & 2 \\ 1 & -3 & 8 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 1 \\ 1 & -1 & 3 \end{bmatrix}.$$

If $A \stackrel{\text{def}}{=} \begin{bmatrix} -2 & 0 & -4 \\ 0 & -1 & 2 \\ 1 & -3 & 8 \end{bmatrix}$,

$$\begin{tikzcd} {[v]_\mathcal{B}} \arrow[r, "A"] & {[S \circ T(v)]_\mathcal{B}} \arrow[d, "R^{-1}"] \\ {[v]_\mathcal{D}} \arrow[u, "R"] \arrow[r, bend right, "R^{-1}AR"] & {[S \circ T(v)]_\mathcal{D}} \end{tikzcd}$$

Problem 42 (Distances)**.** Let P be the plane spanned by the vectors

$$\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{z} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}.$$

1. Assume S is the projection onto P
   a) Find the distance from the vector $\vec{y} \times \vec{z}$ to the vector $S(\vec{y} \times \vec{z})$.
   b) Find the distance from the vector $\vec{z}$ to the vector $S(\vec{z})$.
2. Assume T is the reflection across P.
   c) Find the distance from the vector $\vec{y} \times \vec{z}$ to the vector $T(\vec{y} \times \vec{z})$.
   d) Find the distance from the vector $\vec{y}$ to the vector $T(\vec{y})$.

Solution 42 (Distances) We have that

$$\vec{y} \times \vec{z} = \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}.$$

1. a)
$$\|\vec{y} \times \vec{z} - S(\vec{y} \times \vec{z})\| = \|\vec{y} \times \vec{z}\| = \sqrt{(-2)^2 + 4^2 + 2^2} = \sqrt{24}$$

   b)
$$\|\vec{z} - S(\vec{z})\| = \|\vec{z} - \vec{z}\| = 0$$

2. c)
$$\|\vec{y} \times \vec{z} - T(\vec{y} \times \vec{z})\| = \|\vec{y} \times \vec{z} + \vec{y} \times \vec{z}\| = \sqrt{(-4)^2 + 8^2 + 4^2} = 2\sqrt{24}$$

   d)
$$\|\vec{y} - T(\vec{y})\| = \|\vec{y} - \vec{y}\| = 0$$