Math 140 — Tutorial 10

March 17 - 20

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1. Prove the following identities

   (a)
   $$\frac{1+\tanh x}{1-\tanh x} = e^{2x}$$

   (b)
   $$\cosh 2x = \cosh^2 x + \sinh^2 x$$

   (c)
   $$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$$

   For any real number n.

Solution

(a) Remember that

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

We can plug that into the left hand side to get

$$\frac{1 + \frac{e^x - e^{-x}}{e^x + e^{-x}}}{1 - \frac{e^x - e^{-x}}{e^x + e^{-x}}} = \frac{e^x + e^{-x} + e^x - e^{-x}}{e^x + e^{-x} - e^x + e^{-x}} = \frac{2e^x}{2e^{-x}} = e^{2x}$$

(b) We can plug in the definitions of cosh and sinh in the right hand side to get

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{1}{4}\left[e^{2x} + 2 + e^{-2x} + e^{2x} - 2 + e^{-2x}\right] = \frac{e^{2x} + e^{-2x}}{2}$$

and by definition

$$\cosh 2x = \frac{e^{2x} + e^{-2x}}{2}$$

(c) For this one note that

$$\cosh x + \sinh x = e^x$$

Therefore

$$(\cosh x + \sinh x)^n = e^{nx}$$

Similarly

$$\cosh nx + \sinh nx = e^{nx}$$

2. Find the derivative of the function.

   (a) $y = \tan^{-1}(\sinh x)$

   (b) $y = \tanh^{-1}\left(x^3\right)$

   (c) $y = \operatorname{sech}(\tanh x)$

Solution

(a) We will apply the chain rule. Recall that

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\sinh x = \cosh x$$

Therefore

$$\frac{d}{dx}\tanh^{-1}(\sinh x) = \frac{\cosh x}{1+\sinh^2 x} = \frac{\cosh x}{\cosh^2 x} = \frac{1}{\cosh x}$$

(b) We will apply the chain rule. Recall that

$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}$$

Therefore

$$\frac{d}{dx}\tanh^{-1}(x^3) = \frac{3x^2}{1-x^6}$$

(c) We have $\text{sech}x = \frac{1}{\cosh x}$ therefore

$$\frac{d}{dx}\text{sech}x = -\frac{\sinh x}{\cosh^2 x} = -\tanh x \ \text{sech}\ x$$

Therefore

$$\frac{d}{dx}\text{sech}(\tanh x) = -\tanh(\tanh x)\text{sech}(\tanh x)\cdot\frac{1}{1-x^2}$$

3. Verify that $f(x) = \sqrt{x} - \frac{1}{3}x$ satisfies the hypotheses of Rolle's Theorem on the interval $[0, 9]$. Find all numbers c satisfying the conclusion of the theorem.

Solution

First we need to check that f satisfies the hypothesis of Rolle's Theorem

- f is composed of elementary functions whose domain contains the interval $[0, 9]$ therefore, f is continuous on the interval.
- f is differentiable on the interval $(0, 9)$ with derivative

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3}$$

  Note that f is not differentiable at 0, but that is outside of the interval $(0, 9)$.
- $f(0) = f(9) = 0$.

So f satisfies the hypothesis of Rolle's Theorem. Therefore we can conclude that there exists some number c between 0 and 9 such that $f'(c) = 0$. This means

$$\frac{1}{2\sqrt{c}} - \frac{1}{3} = 0$$

We can solve this equation to find that

$$c = \frac{9}{4}.$$

4. Explain why, if the graph of a polynomial function has three x-intercepts, then it must have at least two points at which its tangent line is horizontal. Is this true for any function having three x-intercepts?

Solution

Every polynomial is an elementary function and is continuous and differentiable, and its derivative is a polynomial of smaller degree. If f is a polynomial that has three x-intercepts, this means there exists three numbers, x_1, x_2, x_3 such that

$$f(x_1) = f(x_2) = f(x_3) = 0.$$

We can now apply Rolle's theorem to the interval $[x_1, x_2]$, which shows that there is a number $c_1 \in (x_1, x_2)$, such that

$$f'(c_1) = 0.$$

We can also do the same for the interval $[x_2, x_3]$ to get another number $c_2 \in (x_2, x_3)$ such that

$$f'(c_2) = 0.$$

This means that the tangent line of the graph of f is horizontal at c_1 and c_2.

This will be also true for any other function, which is not necessarily a polynomial, as long as it satisfies the hypothesis to Rolle's theorem.

5. Show that the equation $x^4 + 4x + c = 0$ has at most two solutions which are real numbers.

Solution

We will solve this problem in two steps. First, for the function $f(x) = x^4 + 4x + c$, we will show that $f'(x)$ is equal to 0 exactly once. Then, we will use the previous problem show that if $f(x) = 0$ has more than two solutions, then $f'(x) = 0$ must have at least 2 solutions. This means that $f(x) = 0$ cannot have more than two solutions.

For the first step, differentiate

$$f'(x) = 4x^3 + 4$$

So $f'(x) = 0$, means $x^3 + 1 = 0$. This equation has only one solution, $x = -1$.

We now know that $f(x)$ has only one point with horizontal tangent line (at $x = -1$). If $f(x)$ had three x-intercepts, then this means it has at least two points with horizontal tangent lines (which we know is false). Therefore, f can't have three x-intercepts. So it has at most two x-intercepts.

6. Verify that the function satisfies the hypotheses of the MVT on the given interval. Then, find all values of c that satisfy the conclusion of the MVT.

   (a) $f(x) = \dfrac{x}{x+2}$ on $[1, 4]$

   (b) $f(x) = e^{-2x}$ on $[0, 3]$

Solution

(a) $f(x)$ is composed of elementary functions and is defined everywhere except for $x = -2$, therefore it is continuous on the interval $[1, 4]$. We can also differentiate

$$f'(x) = \frac{2}{(x+2)^2}$$

which is again defined everywhere except for $x = -2$, so f is differentiable on $[1, 4]$. We have shown that f satisfies the hypothesis of the MVT. Therefore we can conclude that there exists a number $c \in (1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1} = \frac{1}{9}$$

This means that

$$\frac{2}{(c+2)^2} = \frac{1}{9}$$

Therefore

$$c = 3\sqrt{2} - 2$$

(b) Same as before $f(x)$ is composed of elementary functions and its domain is all of the real numbers, so it is certainly continuous on the interval $[0, 3]$ and its derivative is

$$f'(x) = -2e^{-2x}$$

which is defined on the interval $[0, 3]$. So we can conclude that f satisfies the hypothesis of the MVT and there exists some $c \in (0, 3)$ such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1} = \frac{e^{-6} - e^{-2}}{2}$$

Therefore,

$$c = -2 \ln\left(\frac{e^{-2} - e^{-6}}{4}\right)$$

7. A number a is called a **fixed point** of a function f if $f(a) = a$. Prove that if f satisfies the hypothesis of the MVT and $f'(x) \neq 1$ for all x, then f has at most one fixed point. *Hint: Suppose there are at least two fixed points, call them a and b, $a \neq b$. Then, use the MVT.*

Solution

If f has two fixed points a and b (So $f(a) = a$ and $f(b) = b$), and f also satisfies the hypothesis of the MVT, then we know there exists some number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{b - a}{b - a} = 1$$

but we have assumed that $f'(x) \neq 1$. So such a number c cannot exist. Therefore, we can't have had two fixed points to start with, otherwise we will have a contradiction.