Math 141 Tutorial 6 Solutions

Main problems

1. Compute the following using trigonometric substitution.

(a) $\displaystyle\int t^3(3t^2-4)^{5/2}\mathrm{d}t$

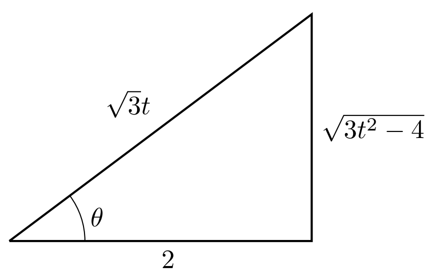
(b) $\displaystyle\int \frac{\sqrt{x^2+16}}{x^4}\mathrm{d}x$

(c) $\displaystyle\int \frac{1}{\sqrt{9x^2-36x+37}}\mathrm{d}x$

(d) $\displaystyle\int \frac{(x+3)^5}{(40-6x-x^2)^{3/2}}\mathrm{d}x$

Solution:

(a) The trigonometric substitution we will use is:



$$2\sec\theta = \sqrt{3}t$$
$$2\sec\theta\tan\theta\,\mathrm{d}\theta = \sqrt{4}t\mathrm{d}t$$
$$\tan\theta = \frac{\sqrt{3t^2-4}}{2}$$

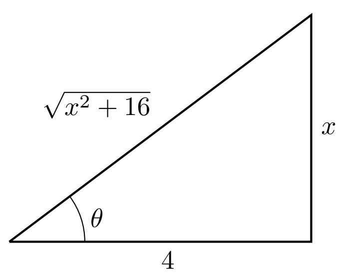
With this, we have

$$\begin{aligned}
\int t^3(3t^2-4)^{5/2}\mathrm{d}t &= \int \frac{2^3\sec^3\theta}{3^{3/2}}(4\sec^2\theta-4)^{5/2}\frac{2\sec\theta\tan\theta}{\sqrt{3}}\mathrm{d}\theta,\\
&= \frac{512}{9}\int \sec^4\theta\tan^6\theta\mathrm{d}\theta,\\
&= \frac{512}{9}\int \sec^2\theta(1+\tan^2\theta)\tan^6\theta\mathrm{d}\theta.
\end{aligned}$$

Using the substitution $u = \tan\theta$, we get $\mathrm{d}u = \sec^2\theta\mathrm{d}\theta$ and hence

$$\begin{aligned}
\int t^3(3t^2-4)^{5/2}\mathrm{d}t &= \frac{512}{9}\int \sec^2\theta(1+\tan^2\theta)\tan^6\theta\mathrm{d}\theta, \\
&= \frac{512}{9}\int (1+u^2)u^6\mathrm{d}u, \\
&= \frac{512}{9}\left(\frac{u^7}{7}+\frac{u^9}{9}\right)+C, \\
&= \frac{512}{9}\left(\frac{\tan^7\theta}{7}+\frac{\tan^9\theta}{9}\right)+C, \\
&= \frac{512}{9}\left(\frac{1}{7}\left(\frac{\sqrt{3t^2-4}}{2}\right)^7+\frac{1}{9}\left(\frac{\sqrt{2t^2-4}}{2}\right)^9\right)+C, \\
&= \frac{4(3t^2-4)^{7/2}}{63}+\frac{(3t^2-4)^{9/2}}{81}+C
\end{aligned}$$

(b) The trigonometric substitution we will use is:



$$\begin{aligned}
4\tan\theta &= x \\
4\sec^2\theta\,\theta &= \mathrm{d}x \\
\sin\theta &= \frac{x}{\sqrt{x^2+16}}
\end{aligned}$$

With this, we have

$$\begin{aligned}
\int \frac{\sqrt{x^2+16}}{x^4}\mathrm{d}x &= \int \frac{\sqrt{16\tan^2\theta+16}}{64\tan^4\theta}4\sec^2\theta\mathrm{d}\theta, \\
&= \frac{1}{16}\int \frac{\sec^3\theta}{\tan^4\theta}\mathrm{d}\theta, \\
&= \frac{1}{16}\int \frac{1}{\cos^3\theta}\frac{\cos^4\theta}{\sin^4\theta}\mathrm{d}\theta, \\
&= \frac{1}{16}\int \frac{\cos\theta}{\sin^4\theta}\mathrm{d}\theta,
\end{aligned}$$

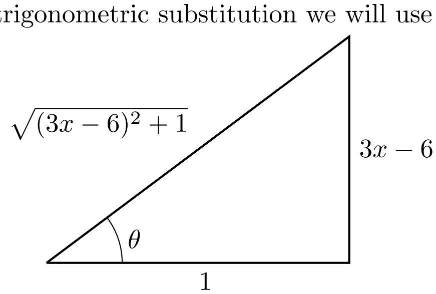
Using the substitution $u = \sin\theta$, we get $\mathrm{d}u = \cos\theta \mathrm{d}\theta$ and hence

$$\begin{aligned}
\int \frac{\sqrt{x^2+16}}{x^4}\mathrm{d}x &= \frac{1}{16}\int \frac{\cos\theta}{\sin^4\theta}\mathrm{d}\theta, \\
&= \frac{1}{16}\int u^{-4}\mathrm{d}u, \\
&= \frac{-1}{48u^3} + C, \\
&= \frac{-1}{48\sin^3\theta} + C, \\
&= \frac{-(x^2+16)^{3/2}}{48x^3}.
\end{aligned}$$

(c) Completing the square in the square root gives

$$9x^2 - 36x + 37 = (3x-6)^2 - 1.$$

Thus the trigonometric substitution we will use is:



$$\tan\theta = 3x - 6$$
$$\sec^2\theta\, \mathrm{d}\theta = 3\mathrm{d}x$$
$$\sec\theta = \sqrt{(3x-6)^2+1}$$

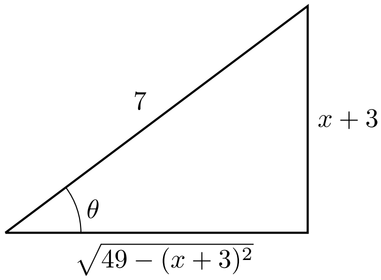
With this, we have

$$\begin{aligned}
\int \frac{1}{\sqrt{(3x-6)^2+1}}\mathrm{d}x &= \int \frac{\sec^2\theta \mathrm{d}\theta}{3\sqrt{\tan^2\theta+1}}, \\
&= \frac{1}{3}\int \frac{\sec^2\theta}{\sec\theta}\mathrm{d}\theta, \\
&= \frac{1}{3}\int \sec\theta \mathrm{d}\theta, \\
&= \frac{1}{3}\ln|\sqrt{(3x^2-6)^2+1} + 3x - 6| + C.
\end{aligned}$$

(d) Completing the square in the square root gives

$$40 - 6x - x^2 = 49 - (x+3)^2.$$

Thus the trigonometric substitution we will use is:



$$\sin\theta = x+3$$
$$7\cos\theta\,\mathrm{d}\theta = \mathrm{d}x$$
$$\cos\theta = \frac{\sqrt{49-(x+3)^2}}{7}$$

With this, we have

$$\begin{aligned}
\int \frac{(x+3)^5}{(49-(x+3)^2)^{3/2}}\mathrm{d}x &= \int \frac{7^5\sin^5\theta}{(49-49\sin^2\theta)^{3/2}}7\cos\theta\mathrm{d}\theta,\\
&= 343\int\frac{\sin^5\theta}{\cos^2\theta}\mathrm{d}\theta,\\
&= 343\int\frac{(1-\cos^2\theta)^2}{\cos^2\theta}\sin\theta\mathrm{d}\theta.
\end{aligned}$$

Using the substitution $u=\cos\theta$, we get $\mathrm{d}u = -\sin\theta\mathrm{d}\theta$ and hence

$$\begin{aligned}
\int \frac{(x+3)^5}{(49-(x+3)^2)^{3/2}}\mathrm{d}x &= 343\int\frac{(1-\cos^2\theta)^2}{\cos^2\theta}\sin\theta\mathrm{d}\theta,\\
&= -343\int\frac{(1-u^2)^2}{u^2}\mathrm{d}\theta,\\
&= -343\int\frac{1-u^2+u^4}{u^2}\mathrm{d}u,\\
&= -343\int u^{-2}-2+u^2\mathrm{d}u,\\
&= -343\left(\frac{-1}{u}-2u+\frac{u^3}{3}\right)+C,\\
&= -343\left(\frac{-1}{\cos\theta}-\cos\theta+\frac{\cos^3\theta}{3}\right)+C,\\
&= 343\left(\frac{7}{\sqrt{49-(x+3)^2}}+2\frac{\sqrt{49-(x+3)^2}}{7}-\frac{\left(49-(x+3)^2\right)^{3/2}}{1029}\right)+C
\end{aligned}$$

2. Use long division to express each of the following functions $f(x)$ as a proper fraction. That is, find polynomials $S(x), R(x), Q(x)$ such that

$$f(x) = S(x) + \frac{R(x)}{Q(x)}$$

and $\deg R < \deg Q$.

(a) $f(x) = \dfrac{x^2+1}{x+1}$

(b) $f(x) = \dfrac{2x^3-x}{x+3}$

(c) $f(x) = \dfrac{x^3+x^2-4x+6}{x^2-2x+2}$

(d) $f(x) = \dfrac{x^4+x+1}{(x^2+1)(x-1)}$

Solutions:

(a)

$$\begin{array}{r|l} & \quad x - 1 \\ \hline x+1 & x^2 \quad + 1 \\ & -x^2 - x \\ \hline & \quad -x + 1 \\ & \quad\ \ x + 1 \\ \hline & \qquad\ 2 \end{array}$$

Hence, $f(x) = x - 1 + \dfrac{2}{x+1}$

(b)

$$\begin{array}{r|l} & \quad 2x^2 - 6x + 17 \\ \hline x+3 & 2x^3 \qquad - x \\ & -2x^3 - 6x^2 \\ \hline & \quad -6x^2 - x \\ & \quad\ \ 6x^2 + 18x \\ \hline & \qquad 17x \\ & \quad -17x - 51 \\ \hline & \qquad\quad -51 \end{array}$$

Hence, $f(x) = 2x^2 - 6x + 17 - \dfrac{51}{x+3}$

(c)

$$\begin{array}{r|l} & \qquad\qquad x + 3 \\ \hline x^2-2x+2 & x^3 + x^2 - 4x + 6 \\ & -x^3 + 2x^2 - 2x \\ \hline & \quad 3x^2 - 6x + 6 \\ & -3x^2 + 6x - 6 \\ \hline & \qquad\qquad 0 \end{array}$$

Hence, $f(x) = x + 3$

(d)

$$
\begin{array}{r}
x+1 \\
x^3-x^2+x-1 \overline{\smash{\big)}\ x^4 \qquad\qquad +x+1} \\
\underline{-x^4+x^3-x^2+x} \\
x^3-x^2+2x+1 \\
\underline{-x^3+x^2-x+1} \\
x+2
\end{array}
$$

Hence, $f(x) = x + 1 + \dfrac{x+2}{(x^2+1)(x-1)}$

3. Write out the partial fraction decomposition for each of the following rational functions.

(a) $f(x) = \dfrac{1}{(x+a)(x+b)}$ when $a \neq b$

(b) $f(z) = \dfrac{3z^2 - z + 8}{z^3 + 4z}$

(c) $f(x) = \dfrac{x^2 + x + 1}{(x+1)^2(x+2)}$

(d) $f(t) = \dfrac{t^2 + t + 1}{t^4 + 2t^2 + 1}$

Solutions:

(a) We have $f(x) = \dfrac{1}{(x+a)(x+b)} = \dfrac{A}{x+a} + \dfrac{B}{x+b}$. Multiplying by the denominator on either side, we obtain

$$1 = A(x+b) + B(x+a) = (A+B)x + (Ab + Ba).$$

Hence $A + B = 0$ or $B = -A$ and $Ab + Ba = 1$. It follows that $a = Ab + Ba = Ab - Aa = A(b-a)$ or $A = \frac{1}{b-a}$. This forces $B = -\frac{1}{b-a} = \frac{1}{a-b}$. In conclusion,

$$f(x) = \frac{A}{x+a} + \frac{B}{x+b} = \frac{1}{(b-a)(x+a)} + \frac{1}{(a-b)(x+b)}.$$

(b) We begin by factorizing the denominator: $z^3 + 4z = \left(z^2 + 4\right) z$. Then, we write

$$f(z) = \frac{3z^2 - z + 8}{z^3 + 4z} = \frac{A}{z} + \frac{Bz + C}{z^2 + 4}.$$

Multiplying by the denominator on either side we see that

$$3z^2 - z + 8 = A(z^2 + 4) + (Bz + C)z = (A+B)z^2 + Cz + 4A.$$

Hence, $A + B = 3, C = -1$ and $4A = 8$. This forces $A = 2, B = 1$ and $C = -1$. Therefore,

$$f(z) = \frac{2}{z} + \frac{z-1}{z^2+4}.$$

(c) We have

$$f(x) = \frac{x^2 + x + 1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}.$$

Multiplying by the denominator on either side we find

$$\begin{aligned} x^2 + x + 1 &= A(x+1)(x+2) + B(x+2) + C(x+1)^2 \\ &= (A+C)x^2 + (3A + B + 2C)x + (2A + 2B + C) \end{aligned}$$

Hence

$$\begin{cases} A \phantom{{}+2B} + C = 1 \\ 3A + B + 2C = 1 \\ 2A + 2B + C = 1 \end{cases}$$

We thus find $C = 1 - A$ and reduce the system to

$$\begin{cases} A + B = -1 \\ A + 2B = 0 \end{cases}$$

This yields $A = -2B$ so $-1 = A + B = -2B + B = -B$. Therefore, $B = 1$. Working backwards, we find $A = -2$ and $C = 1 - A = 3$. Ergo,

$$f(x) = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2} = \frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}.$$

(d) We first factor the denominator: $t^4 + 2t^2 + 1 = (t^2+1)^2$. Hence,

$$f(t) = \frac{t^2+t+1}{t^4+2t^2+1} = \frac{At+B}{t^2+1} + \frac{Ct+D}{(t^2+1)^2}.$$

Multiplying by the denominator on either side we see that

$$t^2 + t + 1 = (At+B)(t^2+1) + Ct + D = At^3 + Bt^2 + (A+C)t + (B+D).$$

Hence $A = 0, B = 1, A + C = 1$ and $B + D = 1$. This yields $C = 1$ and $D = 0$. Therefore

$$f(t) = \frac{At+B}{t^2+1} + \frac{Ct+D}{(t^2+1)^2} = \frac{1}{t^2+1} + \frac{t}{(t^2+1)^2}.$$

4. Integrate the following rational functions.

(a) $\displaystyle\int_0^{1/2} \frac{1}{1-x^2}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{1}{x^2(x+1)^2}\,\mathrm{d}x$

(c) $\displaystyle\int \frac{1}{x^3+x^2-x-1}\,\mathrm{d}x$

(d) $\displaystyle\int \frac{x^4+x+1}{(x^2+1)(x-1)}\,\mathrm{d}x$

(e) $\displaystyle\int_0^4 \frac{y-1}{y^2+4y+3}\,\mathrm{d}y$

(f) $\displaystyle\int_2^4 \frac{t+1}{t^3-t^2}\,\mathrm{d}t$

Solutions:

(a) We use the method of partial fractions:

$$\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}.$$

Multiplying by the denominator on either side by $1-x^2$ we see that

$$1 = A(1+x) + B(1-x) = (A-B)x + (A+B)$$

Hence

$$\begin{cases} A - B = 0 \\ A + B = 1 \end{cases}.$$

Solving, we find $A = B = 1/2$. Thus, we obtain the partial fraction

$$\frac{1}{1-x^2} = \frac{1/2}{1-x} + \frac{1/2}{1+x}.$$

We can now evaluate the integral

$$\begin{aligned}
\int_0^{1/2} \frac{1}{1-x^2}\,\mathrm{d}x &= \int_0^{1/2} \frac{1/2}{1-x} + \frac{1/2}{1+x}\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^{1/2} \frac{1}{1-x}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\int_0^{1/2} \frac{1}{1+x}\frac{\mathrm{d}}{\mathrm{d}x} \\
{\scriptstyle (y=1-x,\ z=1+x)} \qquad &= -\frac{1}{2}\int_1^{1/2} \frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}y} + \frac{1}{2}\int_1^{3/2} \frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z} \\
&= \frac{1}{2}\left(-\ln(|y|)\Big|_{y=1}^{1/2} + \ln(|z|)\Big|_{z=1}^{3/2}\right) \\
&= \frac{1}{2}\left[-\ln(1/2) + \ln(3/2)\right] = \frac{1}{2}\ln(3)
\end{aligned}$$

(b) We write

$$\frac{1}{x^2(x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}.$$

Multiplying by the denominator on either side we find

$$\begin{aligned} 1 &= Ax(x+1)^2 + B(x+1)^2 + Cx^2(x+1) + Dx^2 \\ &= (A+C)x^3 + (2A+B+C+D)x^2 + (A+2B)x + B. \end{aligned}$$

Hence,

$$\begin{cases} A \quad\quad + C \quad\quad = 0 \\ 2A + \; B + C + D = 0 \\ A + 2B \quad\quad\quad\; = 0 \\ \quad\quad B \quad\quad\quad\; = 1 \end{cases}$$

From $B = 1$ and the third equation we find $A = -2$. Then $C = 2$ and, finally, we find $D = 1$. We therefore have

$$\frac{1}{x^2(x+1)^2} = \frac{-2}{x} + \frac{1}{x^2} + \frac{2}{x+1} + \frac{1}{(x+1)^2}.$$

We can now integrate

$$\begin{aligned} \int \frac{1}{x^2(x+1)^2}\,\mathrm{d}x &= \int \left(\frac{-2}{x} + \frac{1}{x^2} + \frac{2}{x+1} + \frac{1}{(x+1)^2} \right) \frac{\mathrm{d}}{\mathrm{d}x} \\ &= -2\ln(|x|) - \frac{1}{x} + 2\ln(|x+1|) - \frac{1}{x+1} + \widetilde{C}. \end{aligned}$$

(c) In order to find our partial fractions, we begin by factorizing the denominator: $x^3 + x^2 - x - 1 = \left(x-1\right)\left(x+1\right)^2$.
We can now write

$$\frac{1}{x^3 + x^2 - x - 1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Multiplying by the denominator on either side, we find

$$1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1) = (A+B)x^2 + (2A+C)x + (A-B-C).$$

Solving

$$\begin{cases} A + B \quad\quad = 0 \\ 2A \quad\quad + C = 0 \\ A - B - C = 1 \end{cases}$$

we find $A = 1/4, B = -1/4$ and $C = -1/2$. Hence,

$$\frac{1}{x^3 + x^2 - x - 1} = \frac{1/4}{x-1} - \frac{1/4}{x+1} - \frac{1/2}{(x+1)^2}.$$

We can now evaluate the integral:

$$\begin{aligned} \int \frac{1}{x^3 + x^2 - x - 1}\frac{\mathrm{d}}{\mathrm{d}x} &= \frac{1}{4}\int \frac{1}{x-1}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{4}\int \frac{1}{x+1}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{2}\int \frac{1}{(x+1)^2}\frac{\mathrm{d}}{\mathrm{d}x} \\ &= \frac{1}{4}\ln(|x-1|) - \frac{1}{4}\ln(|x+1|) + \frac{1}{2(x+1)} \end{aligned}$$

(d) Notice that the numerator in $\dfrac{x^4+x+1}{(x^2+1)(x-1)}$ is of higher degree than the denominator. Hence, we begin with long division:

$$
\begin{array}{r l}
 & \ x+1 \\ \cline{2-2}
x^3-x^2+x-1 \big) & x^4 \qquad\qquad\quad +x+1 \\
 & -x^4+x^3-x^2\ \ +x \\ \cline{2-2}
 & \qquad\ x^3-x^2+2x+1 \\
 & \qquad -x^3+x^2\ \ -x+1 \\ \cline{2-2}
 & \qquad\qquad\qquad\quad x+2
\end{array}
$$

Hence,

$$\frac{x^4+x+1}{(x^2+1)(x-1)} = x+1+\frac{x+2}{(x^2+1)(x-1)}.$$

Now, we write

$$\frac{x+2}{(x^2+1)(x-1)} = \frac{A}{x-1}+\frac{Bx+C}{x^2+1}.$$

Multiplying by the denominator on either side we see that

$$x+2 = A(x^2+1)+(Bx+C)(x-1) = (A+B)x^2+(-B+C)x+(A-C).$$

Therefore, we solve

$$\begin{cases} A+B \qquad\ \ = 0 \\ \quad -B+C = 1 \\ A \qquad -C = 2 \end{cases}$$

to find $A = 3/2, B = -3/2$ and $C = -1/2$. Combining our results, we have

$$\frac{x^4+x+1}{(x^2+1)(x-1)} = x+1+\frac{3}{2(x-1)}-\frac{3x+1}{2(x^2+1)}.$$

We now integrate

$$\begin{aligned} \int \frac{x^4+x+1}{(x^2+1)(x-1)}\,\mathrm{d}x &= \int (x+1)\frac{\mathrm{d}}{\mathrm{d}x}+\frac{3}{2}\int\frac{1}{x-1}\frac{\mathrm{d}}{\mathrm{d}x}-\frac{1}{2}\int\frac{3x+1}{x^2+1}\frac{\mathrm{d}}{\mathrm{d}x} \\ &= \frac{x^2}{2}+x+\frac{3}{2}\ln(|x-1|)-\frac{1}{2}\int\frac{3x+1}{x^2+1}\frac{\mathrm{d}}{\mathrm{d}x} \end{aligned}$$

To evaluate the last integral, we observe that

$$\begin{aligned} \frac{1}{2}\int\frac{3x+1}{x^2+1}\frac{\mathrm{d}}{\mathrm{d}x} &= \frac{3}{2}\int\frac{x}{x^2+1}\frac{\mathrm{d}}{\mathrm{d}x}+\frac{1}{2}\int\frac{1}{x^2+1}\frac{\mathrm{d}}{\mathrm{d}x} \\ {\scriptstyle (u=x^2+1)} \qquad &= \frac{3}{4}\int\frac{1}{u}\frac{\mathrm{d}}{\mathrm{d}u}+\frac{1}{2}\arctan(x) \\ &= \frac{3}{4}\ln(|u|)+\frac{1}{2}\arctan(x)+\widetilde{C} \\ &= \frac{3}{4}\ln(x^2+1)+\frac{1}{2}\arctan(x)+\widetilde{C} \end{aligned}$$

Combining our results, we see that

$$\int \frac{x^4+x+1}{(x^2+1)(x-1)}\,\mathrm{d}x = \frac{x^2}{2} + x + \frac{3}{2}\ln(|x-1|) - \frac{3}{4}\ln(x^2+1) - \frac{1}{2}\arctan(x) + \widetilde{C}$$

(e) We factorize the denominator: $y^2+4y+3 = \big(y+1\big)\,\big(y+3\big)$. We may therefore write

$$\frac{y-1}{y^2+4y+3} = \frac{A}{y+1} + \frac{B}{y+3}.$$

Multiplying by the denominator on either side we obtain

$$y-1 = A(y+3) + B(y+1) = (A+B)y + (3A+B)$$

Hence, $A+B=1$ and $3A+B=-1$ so $A=-1$ and $B=2$. Therefore,

$$\frac{y-1}{y^2+4y+3} = \frac{-1}{y+1} + \frac{2}{y+3}.$$

It follows that

$$\begin{aligned}
\int_0^4 \frac{y-1}{y^2+4y+3}\,\mathrm{d}y &= -\int_0^4 \frac{1}{y+1}\frac{\mathrm{d}}{\mathrm{d}y} + 2\int \frac{1}{y+3}\frac{\mathrm{d}}{\mathrm{d}y} \\
&= -\ln(y+1) + 2\ln(y+3)\Big|_0^4 \\
&= -\ln(5) + 2\ln(7) + \ln(1) - 2\ln(3) = \ln(49/45)
\end{aligned}$$

(f) We begin by writing

$$\frac{t+1}{t^3-t^2} = \frac{t+1}{t^2(t-1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t-1}.$$

Multiplying by the denominator on either side, we see that

$$t+1 = At(t-1) + B(t-1) + Ct^2 = (A+C)t^2 + (-A+B)t - B.$$

Hence, $A+C=0, -A+B=1$ and $-B=1$ which gives $A=-2, B=-1$ and $C=2$. Therefore

$$\frac{t+1}{t^3-t^2} = \frac{-2}{t} + \frac{-1}{t^2} + \frac{2}{t-1}.$$

$$\begin{aligned}
\int_2^4 \frac{t+1}{t^3-t^2}\,\mathrm{d}t &= -2\int_2^4 \frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t} - \int_2^4 \frac{1}{t^2}\frac{\mathrm{d}}{\mathrm{d}t} + 2\int_2^4 \frac{1}{t-1}\frac{\mathrm{d}}{\mathrm{d}t} \\
&= -2\ln(t) + \frac{1}{t} + 2\ln(t-1)\Big|_2^4 \\
&= -2\ln(4) + \frac{1}{4} + 2\ln(3) + 2\ln(2) - \frac{1}{2} + 2\ln(1) \\
&= 2\ln(3/2) - \frac{1}{4}
\end{aligned}$$

5. Integrate the following rational functions by using partial fractions. Use the rational root theorem (stated below) and long division.

(a) $\displaystyle\int_0^1 \frac{x}{x^4+2x^3+2x^2+2x+1}\,\mathrm{d}x$ (b) $\displaystyle\int \frac{x-1}{2x^4+x^3-6x^2+x+2}\,\mathrm{d}x$

Theorem (Rational Root Theorem)**.** *Consider a polynomial*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

where the coefficients $a_0, a_1, \cdots a_n$ are integers and $a_0 \neq 0$. If r is a rational root of $f(x)$, i.e. if $r \in \mathbb{Q}$ and $f(r) = 0$, then writing r in it's lowest terms

$$r = \pm p/q$$

we have that p is a factor of a_0 and q is a factor of a_n.

Solutions:

(a) In order to evaluate $\displaystyle\int_0^1 \frac{x}{x^4+2x^3+2x^2+2x+1}\,\mathrm{d}x$, we wish to use partial fractions. To this end, we must first factorize the denominator $x^4+2x^3+2x^2+2x+1$. Since the only factor of 1 is 1, the rational root theorem implies that the only possible rational roots are

$$\left\{\pm\frac{1}{1}\right\} = \{-1, 1\}\,.$$

We can manually verify that 1 is not a root of $x^4+2x^3+2x^2+2x+1$ while -1 is indeed a root of this polynomial since $(-1)^4+2(-1)^3+2(-1)^2+2(-1)+1 = 0$. Therefore $x-(-1) = x+1$ factors $x^4+2x^3+2x^2+2x+1$. Performing long division, we obtain

$$\begin{array}{r|l}
 & x^3 + x^2 + x + 1 \\ \hline
x+1 & x^4 + 2x^3 + 2x^2 + 2x + 1 \\
 & -x^4 - x^3 \\ \hline
 & x^3 + 2x^2 \\
 & -x^3 - x^2 \\ \hline
 & x^2 + 2x \\
 & -x^2 - x \\ \hline
 & x + 1 \\
 & -x - 1 \\ \hline
 & 0
\end{array}$$

Hence,

$$x^4+2x^3+2x^2+2x+1 = (x^3+x^2+x+1)(x+1).$$

Since, -1 is the only possible rational root of $x^4+2x^3+2x^2+2x+1$, it is also the only possible rational root of x^3+x^2+x+1. Hence, we can check if -1 is a root of x^3+x^2+x+1. Indeed, $(-1)^3+(-1)^2+(-1)+1 = 0$. Hence, we once again perform long division:

$$
\begin{array}{r}
x^2 + 1 \\
x+1 \overline{\big) \; x^3 + x^2 + x + 1} \\
\underline{-x^3 - x^2 } \\
x + 1 \\
\underline{-x - 1} \\
0
\end{array}
$$

We now see that

$$x^4 + 2x^3 + 2x^2 + 2x + 1 = (x^2 + 1)(x + 1)^2.$$

Since $x^2 + 1$ is an irreducible quadratic, we have completely factored our polynomial. We can find the partial fraction decomposition

$$\frac{1}{x^4 + 2x^3 + 2x^2 + 2x + 1} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx + D}{x^2 + 1}.$$

Multiplying by the denominator on either side we obtain

$$\begin{aligned} 1 &= A(x+1)(x^2+1) + B(x^2+1) + (Cx + D)(x+1)^2 \\ &= (A + C)x^3 + (A + B + 2C + D)x^2 + (A + C + 2D)x + (A + B + D). \end{aligned}$$

Hence, we solve the system

$$\begin{cases} A + C = 0 \\ A + B + 2C + D = 0 \\ A + C + 2D = 0 \\ A + B + D = 1 \end{cases}.$$

Since $A + C = 0$ and $A + C + 2D = 0$ we see that $D = 0$. Then, we are left to solve

$$\begin{cases} A + C = 0 \\ A + B + 2C = 0 \\ A + B = 1 \end{cases}.$$

Since $A + C = 0$ and $A + B = 1$, we have $C = -A$ and $B = 1 - A$. Then, we can substitute B and C in $A + B + 2C = 0$ to find that $A + (1 - A) - 2A = 0$ or $A = 1/2$. This forces $B = 1/2$ and $C = -1/2$. Our partial fraction decomposition is therefore

$$\frac{1}{x^4 + 2x^3 + 2x^2 + 2x + 1} = \frac{1}{2(x+1)} + \frac{1}{2(x+1)^2} - \frac{x}{2(x^2+1)}.$$

We can now evaluate the integral:

$$
\begin{aligned}
\int_0^1 \frac{1}{x^4+2x^3+2x^2+2x+1}\frac{\mathrm{d}}{\mathrm{d}x} &= \frac{1}{2}\int_0^1 \frac{1}{x+1}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\int_0^1 \frac{1}{(x+1)^2}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{2}\int_0^1 \frac{x}{x^2+1}\frac{\mathrm{d}}{\mathrm{d}x} \\
{\scriptstyle (u=x^2+1)} \qquad &= \frac{1}{2}\ln(|x+1|) - \frac{1}{2(x+1)}\bigg|_0^1 - \frac{1}{4}\int_1^2 \frac{1}{u}\frac{\mathrm{d}}{\mathrm{d}u} \\
&= \frac{1}{2}\ln(2) - \frac{1}{4} + \frac{1}{2} - \frac{1}{4}\ln(u)\bigg|_1^2 \\
&= \frac{1}{2}\ln(2) + \frac{1}{4} - \frac{1}{4}\ln(2) \\
&= \frac{1}{4}\left(1+\ln(2)\right).
\end{aligned}
$$

(b) We begin by factorizing $2x^4+x^3-6x^2+x+2$. Since the only factors of 2 are 2 and 1, the rational root theorem tells us that the only possible rational roots are

$$\left\{\pm\frac{2}{1}, \pm\frac{2}{2}, \pm\frac{1}{2}, \pm\frac{1}{1}\right\} = \left\{-2, -1, -\frac{1}{2}, \frac{1}{2}, 1, 2\right\}.$$

By plugging in each of these values, we see that the only possible rational roots are $-2, -1/2$ and 1. It follows that $x+2, 2x+1$ and $x-1$ are all factors of $2x^4+x^3-6x^2+x+2$. In particular, their product $(x+2)(2x+1)(x-1) = 2x^3+3x^2-3x+2$ is also a factor of $2x^4+x^3-6x^2+x+2$. To find the last factor, we perform long division:

$$
\begin{array}{r|l}
 & \qquad\qquad\quad\;\; x-1 \\ \hline
2x^3+3x^2-3x-2 & \;\;2x^4 \;+x^3-6x^2\;\;+x+2 \\
 & -2x^4-3x^3+3x^2+2x \\ \cline{2-2}
 & \qquad\;\, -2x^3-3x^2+3x+2 \\
 & \qquad\quad\;\; 2x^3+3x^2-3x-2 \\ \cline{2-2}
 & \qquad\qquad\qquad\qquad\quad\; 0
\end{array}
$$

We conclude that

$$2x^4+x^3-6x^2+x+2 = (x+2)(2x+1)(x-1)^2.$$

Hence,

$$\frac{x-1}{2x^4+x^3-6x^2+x+2} = \frac{1}{(x+2)(2x+1)(x-1)} = \frac{A}{x+2} + \frac{B}{2x+1} + \frac{C}{x-1}.$$

To find A, B and C, we multiply by the denominator on either side of the last equality to deduce that

$$
\begin{aligned}
1 &= A(2x+1)(x-1) + B(x+2)(x-1) + C(x+2)(2x+1) \\
&= (2A+B+2C)x^2 + (-A+B+5C)x + (-A-2B+2C).
\end{aligned}
$$

Solving

$$
\begin{cases}
2A + \;\; B + 2C = 0 \\
-A + \;\; B + 5C = 0 \\
-A - 2B + 2C = 1
\end{cases},
$$

we obtain $A = 1/9, B = -4/9$ and $C = 1/9$. Hence,

$$\begin{aligned}
\frac{x-1}{2x^4+x^3-6x^2+x+2} &= \frac{1}{9(x+2)} - \frac{4}{9(2x+1)} + \frac{1}{9(x-1)} \\
&= \frac{1}{9(x+2)} - \frac{2}{9(x+1/2)} + \frac{1}{9(x-1)}.
\end{aligned}$$

We can now evaluate the integral:

$$\begin{aligned}
\int \frac{x-1}{2x^4+x^3-6x^2+x+2}\,\mathrm{d}x &= \frac{1}{9}\int \frac{1}{x+2}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{2}{9}\int \frac{1}{x+1/2}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{9}\int \frac{1}{x-1}\frac{\mathrm{d}}{\mathrm{d}x} \\
&= \frac{1}{9}\ln(|x+2|) - \frac{2}{9}\ln(x+1/2) + \frac{1}{9}\ln(|x-1|) + C_1 \\
&= \frac{1}{9}\left(\ln(|x+2|) - 2\ln(|2x+1|) + \ln(|x-1|)\right) + C_2.
\end{aligned}$$

Practice Problems

6. Compute the following using any method available to you

(a) $\displaystyle\int \frac{1}{(x^2-1)^2}\,\mathrm{d}x$

(b) $\displaystyle\int_1^2 \frac{3x^2+6x+2}{x^2+3x+2}\,\mathrm{d}x$

(c) $\displaystyle\int \arcsin(x)\,\mathrm{d}x$

(d) $\displaystyle\int \sqrt{x^2+1}\,\mathrm{d}x$

(e) $\displaystyle\int_{-10^{10}}^{10^{10}} x^{100}\sin(x^5)\,\mathrm{d}x$

(f) $\displaystyle\int \frac{e^{1/x}}{x^2}\,\mathrm{d}x$

(g) $\displaystyle\int_1^e \frac{\ln(x)}{x^2}\,\mathrm{d}x$

(h) $\displaystyle\int_1^3 \frac{1}{\sqrt{x}+x\sqrt{x}}\,\mathrm{d}x$

(i) $\displaystyle\int \frac{x^2+x}{x^2+2x}\,\mathrm{d}x$

(j) $\displaystyle\int_{-3}^{-1} (x+2)^{99}\,\mathrm{d}x$

(k) $\displaystyle\int \frac{x}{\sqrt{x^2+2}}\,\mathrm{d}x$

(l) $\displaystyle\int \frac{x^3e^{x^2}}{(x^2+1)^2}\,\mathrm{d}x$

(m) $\displaystyle\int \tan^7(x)\sec^4(x)\,\mathrm{d}x$

(n) $\displaystyle\int \frac{\sqrt{25-x^2}}{x^2}\,\mathrm{d}x$

Hint: for problem (e), use symmetry (i.e. is the function even or odd?).

Challenge Problems

7. Solve the following integrals
   (a) $\int (1+\ln(x))\ln(\ln(x))\mathrm{d}x$
   (b) $\int \sqrt{1-\sqrt{x}}\mathrm{d}x$
   (c) $\int \frac{1}{1+\cos^2(x)}\mathrm{d}x$