1. sketch the graph of a function f that satisfies all of the following conditions

   (a) $\lim_{x \to 0} f(x) = +\infty$

   (b) $\lim_{x \to 2^+} f(x) = -\infty$

   (c) $\lim_{x \to 2^-} f(x) = 3$

   (d) $\lim_{x \to 1} f(x) = 0$

   (e) $\lim_{x \to -1} f(x)$ does not exist.

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2. Evaluate the following limits

   (a) $\lim_{x \to -1} x^2 + 1$

   (b) $\lim_{x \to -1} \dfrac{x+1}{x^3-1}$

   (c) $\lim_{x \to -1} \dfrac{x+1}{x^3+1}$

   (d) $\lim_{x \to 0} \sin\left(\dfrac{1}{x}\right)$

   (e) $\lim_{x \to 1^+} \dfrac{x+1}{x^3-1}$

   (f) $\lim_{x \to 1^-} \dfrac{x+1}{x^3-1}$

   (g) $\lim_{x \to 2} \dfrac{2-x}{\sqrt{x+2}-2}$

Solution

(a) x^2+1 is continuous (polynomial) and -1 is in its domain so we can just plug it in to get 2.

(b) Again the function is continuous and -1 is in the domain, so by plugging in we get 0.

(c) Here -1 is not in the domain of the function, but this function can be simplified

$$\frac{x+1}{x^3+1} = \frac{1}{x^2-x+1}$$

Plugging in -1 we get $\frac{1}{3}$.

(d) The limit does not exist.

(e) $+\infty$ Here we get a vertical asymptote.

(f) $-\infty$

(g) By multiplying the fraction on the top and bottom by $\sqrt{x+2}+2$ we can simplify it to get

$$\frac{(2-x)(\sqrt{x+2}+2)}{x-2} = -(\sqrt{x+2}+2)$$

we can now plug in 2 to get the limit is -4

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3. Find the vertical asymptotes of the functions

(a) $f(x) = \dfrac{x^2+1}{3x-2x^2}$

(b) $f(x) = \ln(x^2-1)$

(c) $\ln(x - \dfrac{1}{x})$

Solution

(a) $x = 0,\ x = \frac{2}{3}$, the only asymptotes that can occur are when the denominator is equal to 0.

(b) $x = \pm 1$, we know that $\ln(x)$ has a vertical asymptote when $x = 0$ to $\ln(x^2-1)$ will have vertical asymptotes when $x^2 - 1 = 0$

(c) $x = \pm 1,\ x = 0$ Similar to above, we will get vertical asymptotes when $x - \frac{1}{x} = 0$ and also we see that there is another vertical asymptote because of the $\frac{1}{x}$

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4. Given that

$$\lim_{x\to 2} f(x) = 4,\ \lim_{x\to 2} g(x) = -2,\ \lim_{x\to 2} h(x) = 0$$

find the value of the following limits if they exist, or explain why the limits don't exist.

(a) $\lim_{x\to 2}(f(x) + 5g(x))$

(b) $\lim_{x\to 2} g(x)^3$

(c) $\lim_{x\to 2} \dfrac{f(x)g(x)}{h(x)}$

(d) $\lim_{x\to 2} \cos(h(x))$

Solution

For most of these you can just plug the values of the limits in, because we can add, multiply, subtract, compose and divide (unless we're dividing by 0) limits.

(a) Just plugging in we'll get $4 + 5 \times (-2) = -6$

(b) plugging in: -8

(c) Here the numerator is -8 but the denominator is 0, therefore the limit does not exist.

(d) As x tends to 2, the function $h(x)$ tends to 0, and so $\cos(h(x))$ tends to $\cos(0) = 1$.

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5. In this exercise we will be using the $\epsilon - \delta$ definition of limits to calculate some limits. For

(a) For $f(x) = x + 1$, for each value of ϵ find a value of δ such that

$$\text{if } |x - 1| < \delta \text{ then } |f(x) - f(1)| < \epsilon$$

i. $\epsilon = 0.1$

ii. $\epsilon = 0.01$

iii. $\epsilon = 0.001$

Can you write a formula for δ in terms of ϵ that will work for any value of ϵ? Write the limit statement for $f(x)$ that we are trying to justify.

(b) For $f(x) = \dfrac{x}{5}$, for each value of ϵ find a value of δ such that

$$\text{if } |x - 3| < \delta \text{ then } |f(x) - f(3)| < \epsilon$$

i. $\epsilon = 0.1$

ii. $\epsilon = 0.01$

iii. $\epsilon = 0.001$

Can you write a formula for δ in terms of ϵ that will work for any value of ϵ? Write the limit statement for $f(x)$ that we are trying to justify.

(c) For $f(x) = x^3$, for each value of ϵ find a value of δ such that

$$\text{if } |x - 0| < \delta \text{ then } |f(x) - f(0)| < \epsilon$$

i. $\epsilon = 0.1$

ii. $\epsilon = 0.01$

iii. $\epsilon = 0.001$

Can you write a formula for δ in terms of ϵ that will work for any value of ϵ? Write the limit statement for $f(x)$ that we are trying to justify.

(d) For $f(x) = \dfrac{1}{x^2}$, for each value of M find a value of δ such that

$$\text{if } |x - 0| < \delta \text{ then } f(x) > M$$

i. $M = 10$

ii. $M = 100$

iii. $M = 1000$

Can you write a formula for δ in terms of M that will work for any value of M? Write the limit statement for $f(x)$ that we are trying to justify.

Solution

(a) For this one we can take $\delta = \epsilon$. The limit statement is

$$\lim_{x \to 1} x + 1 = 2.$$

(b) We can choose $\delta = 5\epsilon$. The limit statement is

$$\lim_{x \to 3} \frac{x}{5} = \frac{3}{5}.$$

(c) We can choose $\delta = \epsilon^{1/3}$. The limit statement is

$$\lim_{x \to 0} x^3 = 0.$$

(d) We can choose $\delta = \frac{1}{\sqrt{M}}$. The limit statement is

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$