11 Lecture hours 24-26

Definitions and Theorems

Definition (Transpose of a matrix Matrix).

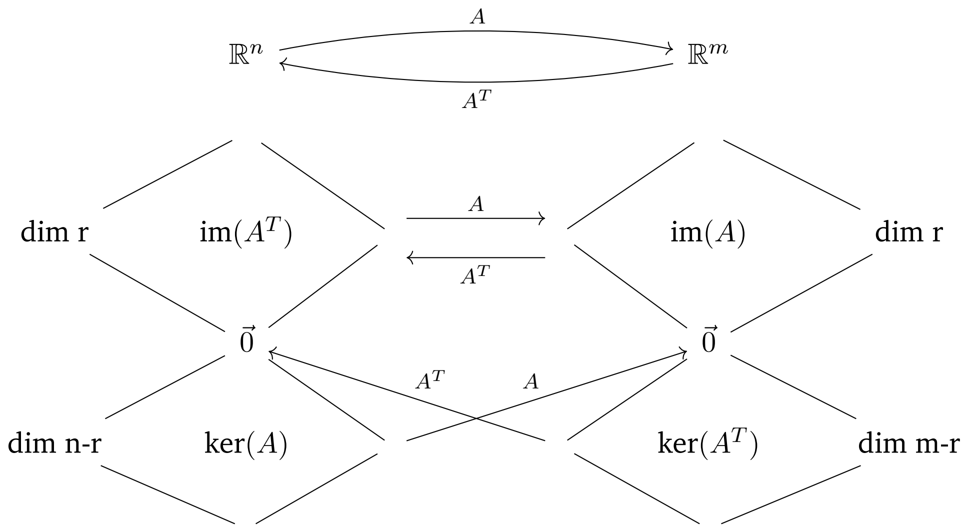
The transpose of a matrix A is A^T, and it has columns the rows of A (same order).

Definition (Perpendicular complement).

Let V be a subspace of $\mathbb{R}^n$, then W is called the "perpendicular complement" of V and denoted $V^\perp$ (pronounced "V perp", symbol $\perp$ is a superscript) if W contains all vector in $\mathbb{R}^n$ that are perpendicular to all vectors in V.

Definition (Fundamental subspaces of linear algebra).

For any m by n matrix A we have



$$(\ker A)^\perp = \mathrm{im}(A^T), \qquad (\mathrm{im} A)^\perp = \ker(A^T).$$

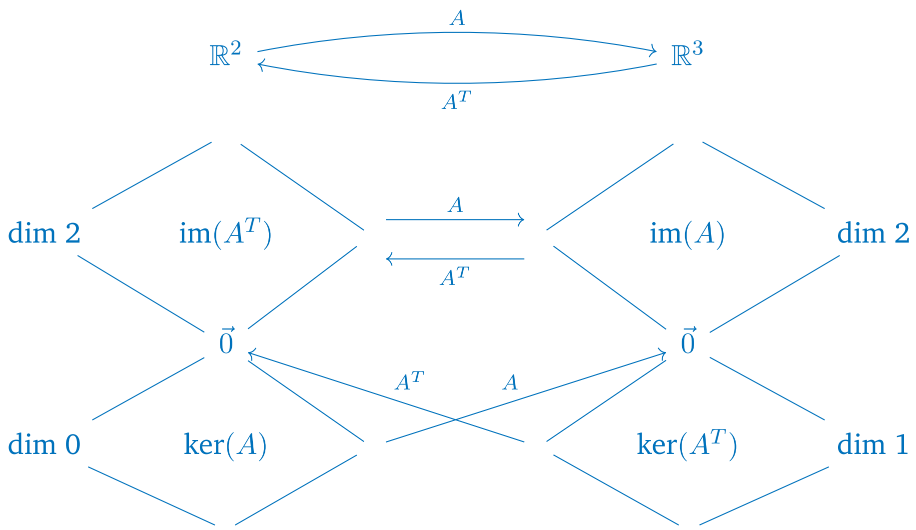
51

Problem 43 (Fundamental subspaces of linear algebra)**.** Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Find $\ker(A)$, $\operatorname{im}(A)$, $\ker(A^T)$, and $\operatorname{im}(A^T)$. For each of these subspaces, determine the value of n for which they are a subspace of $\mathbb{R}^n$.

Solution 43 (Fundamental subspaces of linear algebra) $\ker(A) = \{\vec{0}\} \subset \mathbb{R}^2$, $\operatorname{im}(A) = \operatorname{span}(\vec{e}_1, \vec{e}_2) \subset \mathbb{R}^3$, $\ker(A^T) = \operatorname{span}(\vec{e}_3) \subset \mathbb{R}^3$, and $\operatorname{im}(A^T) = \mathbb{R}^2$.



Problem 44 (Transpose of a matrix)**.** Let A be an invertible $n \times n$ matrix.

a) Explain why A^T is invertible.

b) Explain why $(A^T)^{-1} = (A^{-1})^T$. (Hint: $I^T = I$.)

Solution 44 (Transpose of a matrix)

a) A^T is square, so to show that is invertible it is sufficient to show that $\ker(A^T) = \{\vec{0}\}$. We know that A is invertible, so $\mathrm{im}(A) = \mathbb{R}^n$. We always have $\mathrm{im}(A)^\perp = \ker(A^T)$, and $(\mathbb{R}^n)^\perp = \{\vec{0}\}$, so we do indeed have $\ker(A^T) = \{\vec{0}\}$.

b) We have the formula $A^{-1}A = I$. Transposing both sides gives $A^T(A^{-1})^T = I$. Similarly, transposing both sides of $AA^{-1} = I$ gives $(A^{-1})^T A^T = I$ .Therefore, the inverse matrix of A^T is $(A^{-1})^T$.

Problem 45 (Least squares - Normal Equations)**.** You are given data points $(x, y) = (1, 1), (2, 3), (-1, 3)$. Use a least squares line of best fit to predict the y-value when $x = 7$.

Solution 45 (Least squares - Normal Equations) From fitting a line $y = mx + c$ to the given data points we get equations:

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

We can solve this use the method of normal equations.

Let

$$\vec{x}^* = \begin{bmatrix} m \\ c \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 1 \end{bmatrix} \quad and \quad \vec{b} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}.$$

We know $\vec{x}^*$ is a least squares solution of

$$A\vec{x}^* = \vec{b}$$

if and only if $\vec{b} - A\vec{x}^* \in (ImA)^\perp$ (take a look at figure below).

But we have seen that $(ImA)^\perp = ker(A^T)$, so we $\vec{x}^*$ such that

$$A^T(\vec{b} - A\vec{x}^*) = \vec{0}.$$

Solving for $\vec{x}^*$ we obtain that

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}.$$

$$A^T A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}, \quad (A^T A)^{-1} = \frac{1}{14} \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}, \quad A^T \vec{b} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

Putting this together, our least squares solution has $m = -1/7, c = 17/7$. Therefore, when $x = 7$, we predict $y = 10/7$.