Lecture hours 16-18

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Definitions and Theorems

Definition (Matrix multiplication)**.** Let A be an $m \times p$ and B an $p \times n$. Matrix AB is an $m \times n$ matrix. Recall, to multiply matrices together: multiply left matrix by each column of the right matrix, those are the columns of the resulting matrix.

Definition (Composition of linear transformations)**.** Let $S : \mathbb{R}^m \to \mathbb{R}^p$ and $T : \mathbb{R}^p \to \mathbb{R}^n$ be linear transformations. The composition $T \circ S : \mathbb{R}^m \to \mathbb{R}^n$ (pronounced "T composed with S") is given by

$$T \circ S(\vec{v}) \stackrel{\text{def}}{=} T(S(\vec{v})),$$

for $\vec{v} \in \mathbb{R}^m$.

Definition (Invertible linear transformation)**.**

In terms of linear transformations:

A linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is invertible if for all $\vec{y} \in \mathbb{R}^n$ (outputs of T) there exists an $\vec{x} \in \mathbb{R}^m$ such that $T(\vec{x}) = \vec{y}$, and this $\vec{x}$ is unique.

In terms of matrices:

A matrix A is invertible if for all $\vec{y} \in \mathbb{R}^n$ there is an unique $\vec{x} \in \mathbb{R}^m$ such that $A\vec{x} = \vec{y}$.

Problem 33 (Matrix algebra)**.** Compute the following matrix products:

a)

$$\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}4 & 5\end{bmatrix}.$$

b)

$$\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}4\\5\\6\end{bmatrix}.$$

Solution 33 (Matrix algebra)

a) This operation is called outer product.

$$\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}4 & 5\end{bmatrix}=\begin{bmatrix}1\cdot 4 & 1\cdot 5\\2\cdot 4 & 2\cdot 5\\3\cdot 4 & 3\cdot 5\end{bmatrix}=\begin{bmatrix}4 & 5\\8 & 10\\12 & 15\end{bmatrix}.$$

b) This operation is called inner product.

$$\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}4\\5\\6\end{bmatrix}=4\cdot 1+2\cdot 5+3\cdot 6=32.$$

Problem 34 (Compositions and inverses)**.** Let $T:\mathbb{R}^3\to\mathbb{R}^3$ be projection from $\mathbb{R}^3$ onto the xy-plane.

a) Is T invertible? Why or why not?

b) Find the matrix A with $T(\vec{x})=A\vec{x}$.

c) Find a linear transformation $S:\mathbb{R}^3\to\mathbb{R}^3$ with $T\circ S=S\circ T$.

d) Find a linear transformation $S:\mathbb{R}^3\to\mathbb{R}^3$ with $T\circ S\neq S\circ T$.

Solution 34 (Compositions and inverses)

a) T is not invertible, because $\ker(T)\neq\{0\}$.

b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

c) The identity transformation.

d) $S =$ rotation of 90 degrees around the x-axis works. The vector $\vec{e}_2$ is in the kernel of $T \circ S$, but $S \circ T(\vec{e}_2) = \vec{e}_3$. There are many other possible answers too.

Problem 35 (Compositions)**.** True or false? If true, explain why; if false, give a counterexample.

a) If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation and $\ker(T) = \{\vec{0}\}$, then T is invertible.

b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with nullity 1 and $S : \mathbb{R}^2 \to \mathbb{R}^2$ is another linear transformation with nullity 1, then $T \circ S$ is the zero transformation.

c) If T and S are linear transformations with the domain of T equal to the codomain of S, then $\operatorname{rank}(T \circ S) \leq \operatorname{rank}(T)$.

d) If T and S are linear transformations with the domain of T equal to the codomain of S, then $\operatorname{rank}(T \circ S) \leq \operatorname{rank}(S)$.

Solution 35 (Compositions)

a) True. If $\ker(T) = \{0\}$, then, by the rank-nullity theorem, the rank of T is 3, so the image of T is $\mathbb{R}^3$. Therefore, the equation $T(\vec{x}) = \vec{y}$ has a solution for any $\vec{y} \in \mathbb{R}^3$. This solution is unique because $\ker(T) = \{0\}$.

b) This is false. For example, both T and S could be projection to the x-axis.

c) This is true, because any vector that is in the image of $T \circ S$ must be in the image of T.

d) This is true. Let A be the matrix for T and B be the matrix for S. Let the columns of B be $\vec{b}_1, \ldots, \vec{b}_n$. We have:

$$\operatorname{image}(S) = \operatorname{span}(\vec{b}_1, \ldots, \vec{b}_n), \quad \operatorname{image}(T \circ S) = \operatorname{span}(A\vec{b}_1, \ldots, A\vec{b}_n)$$

Any linear relation among the $\vec{b}_1, \ldots, \vec{b}_n$ gives a linear relation among the $A\vec{b}_1, \ldots, A\vec{b}_n$ (because A represents a linear transformation). Therefore, the dimension of $\operatorname{span}(A\vec{b}_1, \ldots, A\vec{b}_n)$ cannot be larger than the dimension of $\operatorname{span}(\vec{b}_1, \ldots, \vec{b}_n)$.

Problem 36 (Inverse of a problem)**.** Let $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 3 & 5 \\ 1 & 2 \end{smallmatrix}\right]$.

a) Find A^{-1}.

b) Find B^{-1}.

c) Find all 2×2 matrices X with $AXA^{-1} = B$.

Solution 36 (Inverse of a problem)

a) $A^{-1} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

b) $B^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$.

c) $X = A^{-1}BA = \left[\begin{smallmatrix} 2 & 1 \\ 5 & 3 \end{smallmatrix}\right]$.