3 Lecture hours 5-7

Definitions

Definition (Linear relations)**.** Consider the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ in $\mathbb{R}^n$. An equation of the form $c_1\vec{v}_1 + \cdots + c_r\vec{v}_r = \vec{0}$ is called a linear relation among the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$. If at least one of the c_i is nonzero, then we call this a nontrivial linear relation among $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$.

Definition (Linear Independent vectors)**.** We say vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ in $\mathbb{R}^n$ are linearly independent if and only if the only linear relation between them is the trivial one. In other words, $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ in $\mathbb{R}^n$ are linearly independent if and only if the only way that $c_1\vec{v}_1 + \cdots + c_r\vec{v}_r = \vec{0}$ is if all the c_i are 0.

Definition (Subspace)**.** A subspace of $\mathbb{R}^n$ is a non-empty set of vectors in $\mathbb{R}^n$ that can be described as a span of vectors.

Here is an equivalent definition of subspace:

Definition (Subspace)**.** A subspace of $\mathbb{R}^n$ is a non-empty subset V of $\mathbb{R}^n$ with the following properties:

(i) If $\vec{u}$ is in V , $\vec{u}$ is also in V for any scalar $k \in \mathbb{R}$ (We say V is closed under scalar multiplication.)

(ii) If $\vec{u}$ and $\vec{w}$ are in V, their sum $\vec{u} + \vec{w}$ is also in V. (We say V is closed under addition.)

Definition (Basis)**.** The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are a basis of a subspace V if they span V and are linearly independent. In other words, a basis of a subspace V is the minimal set of vectors needed to span all of V.

Definition (Dimension of a subspace)**.** The dimension of the subspace V is the number of vectors in a basis of V.

Problem 12 (Linear dependence)**.** True or false? If false, give a counter-example. If true, explain why.

a) If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent vectors in $\mathbb{R}^2$, then $\vec{v}_3$ is in the span of $\vec{v}_1$ and $\vec{v}_2$.

b) Any collection of 4 vectors in $\mathbb{R}^3$ is linearly dependent.

Problem 13 (Subspaces)**.** Give examples for:

a) A subset V of $\mathbb{R}^2$ that is closed under scalar multiplication, but not closed under addition.

b) A subset V of $\mathbb{R}^2$ that is closed under addition, but not closed under scalar multiplication.

Problem 14 (Subspaces of $\mathbb{R}^n$). Give an example of:

a) A subspace of $\mathbb{R}$.

b) A subset of $\mathbb{R}^2$ that is not a subspace of $\mathbb{R}^2$. Explain why it is not a subspace.

c) A subspace of $\mathbb{R}^3$ of dimension 2. Explain why it has dimension 2.

d) A subset of $\mathbb{R}^3$ that contains infinitely many vectors, but is not a subspace of $\mathbb{R}^3$. Explain why it is not a subspace.

Problem 15 (Basis and dimension)**.** Find a basis for $\mathrm{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}.$$