A 3-player game theoretic model of a choice between two queueing systems with strategic managerial decision making

Michalis Panayides

Gregynog 2022

About me

About me

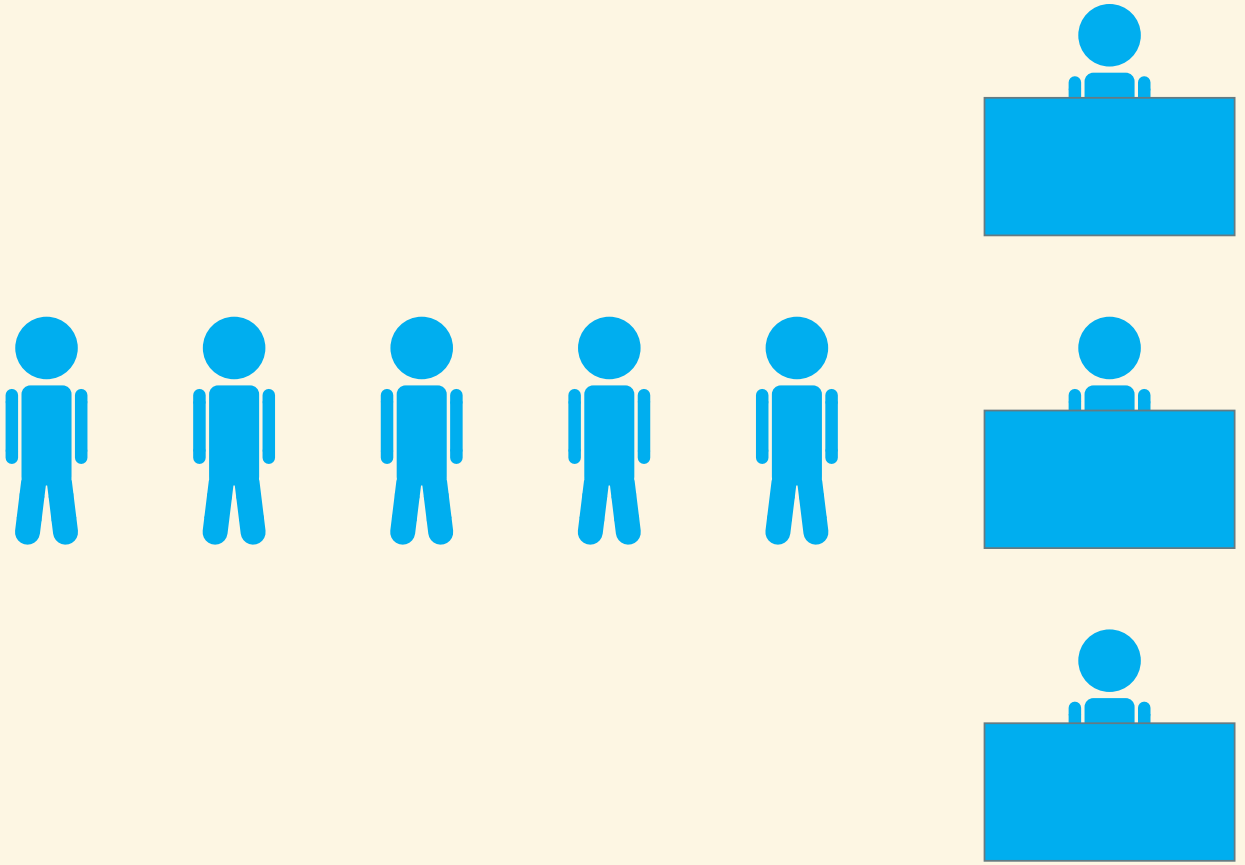
THIS.

Queues - M/M/1

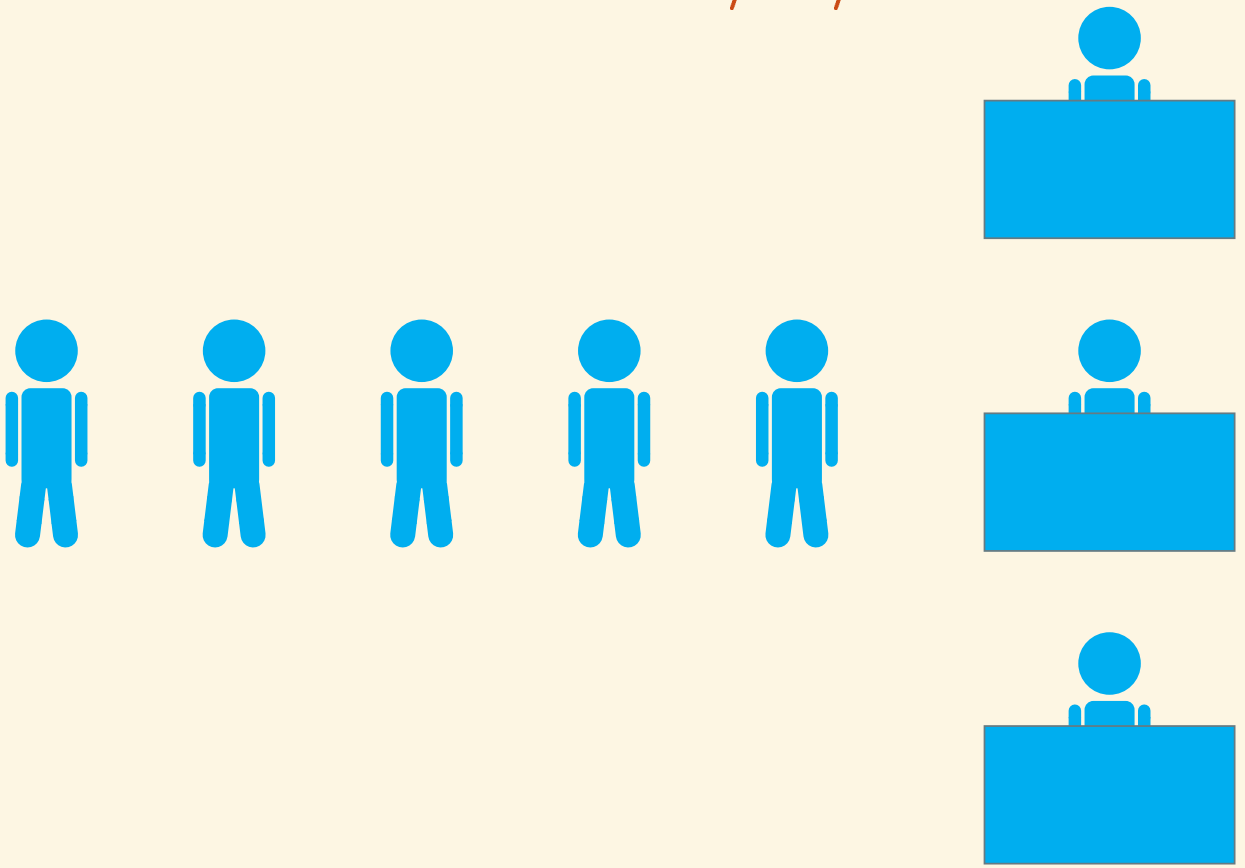
Queues - M/M/1

- Shone R, Knight VA, Williams JE. Comparisons between observable and unobservable M/M/1 queues with respect to optimal customer behaviour
- Kerner Y, Shmuel-Bittner O. Strategic behaviour and optimization in a hybrid M/M/1 queue with retrials.
- Gai Y, Liu H, Krishnamachari B. A packet dropping mechanism for efficient operation of M/M/1 queues with selfish users.

Queues - M/M/3



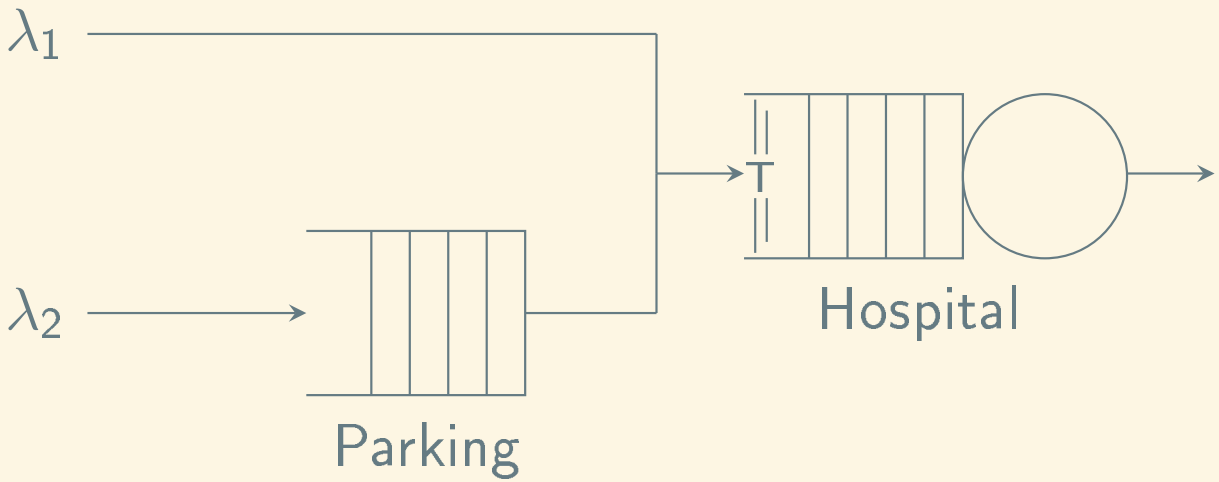
Queues - M/M/3



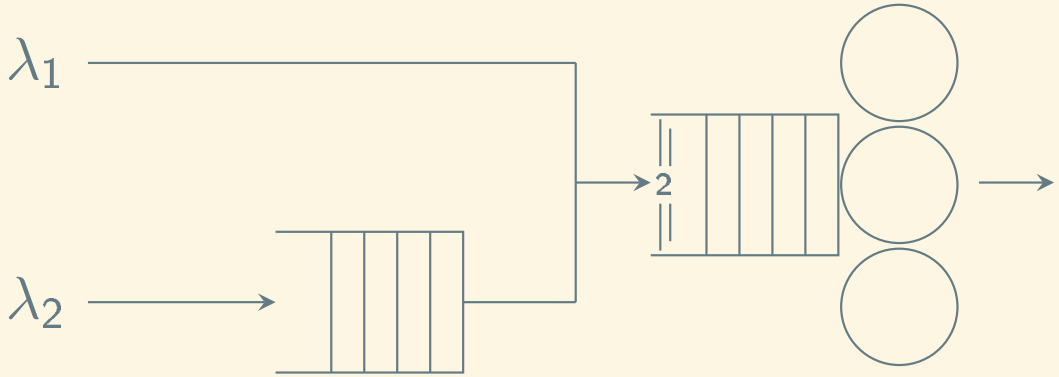
- Knight V, Harper P. The Impact of Choice on Public Services.
- Haviv M, Roughgarden T. The price of anarchy in an exponential multi-server
- Shone R, Knight V, Harper P. A conservative index heuristic for routing problems with multiple heterogeneous service facilities

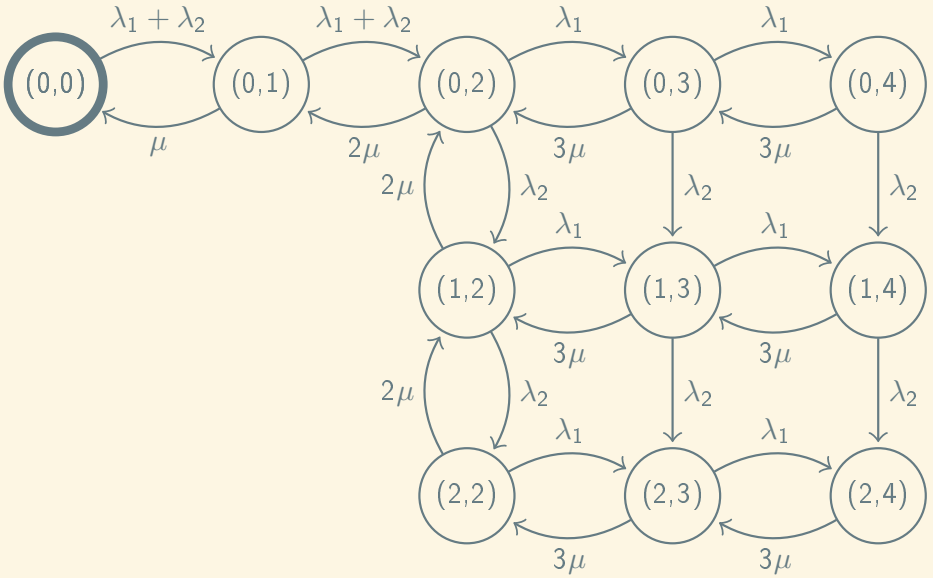
Queues - Custom network of queues

Queueing network structure

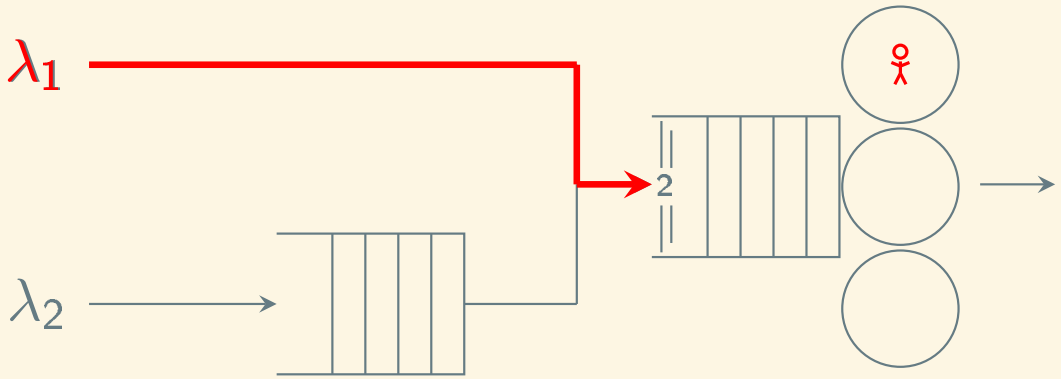


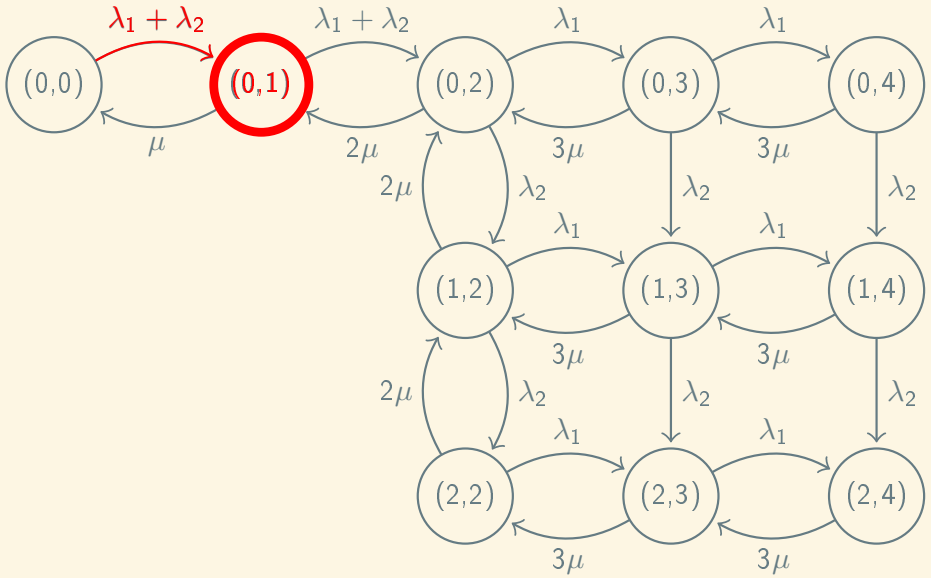
Markov Chain - Custom network



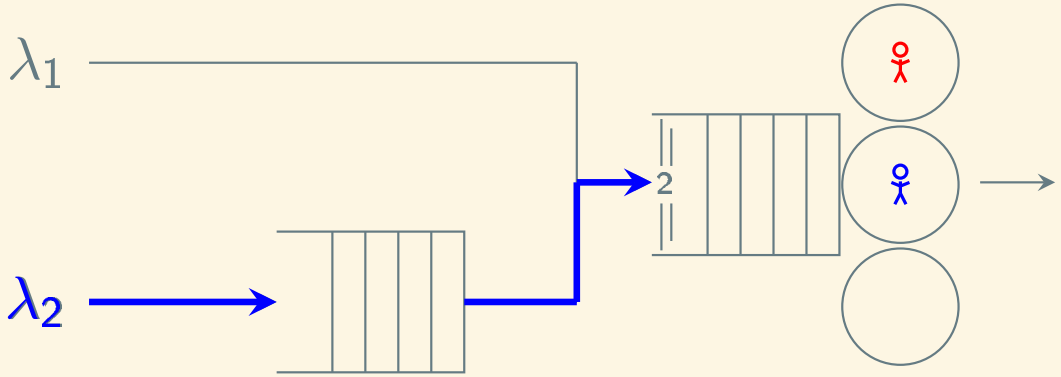


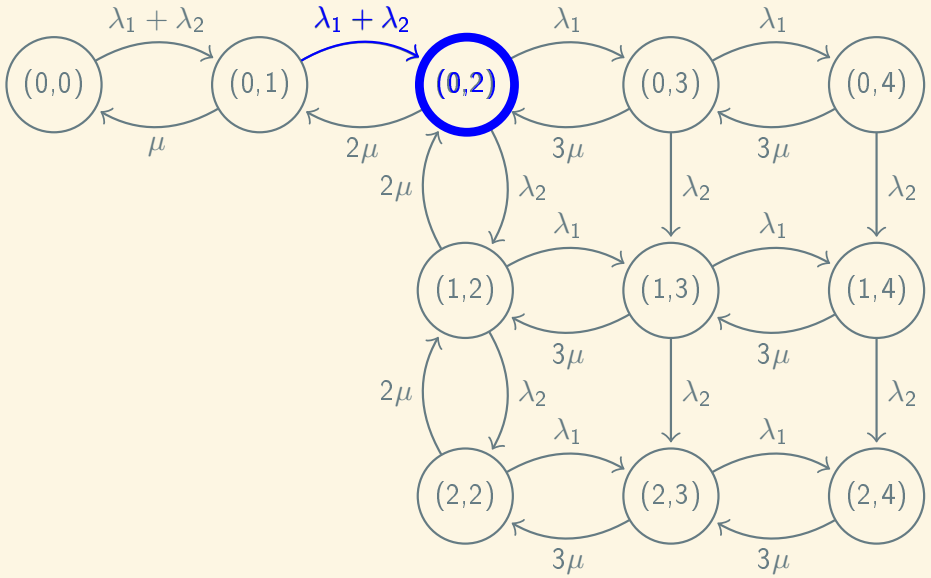
Markov Chain - Custom network



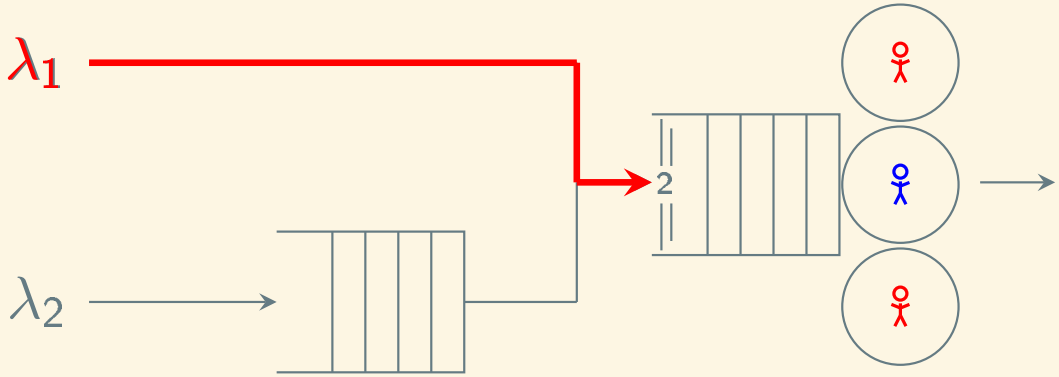


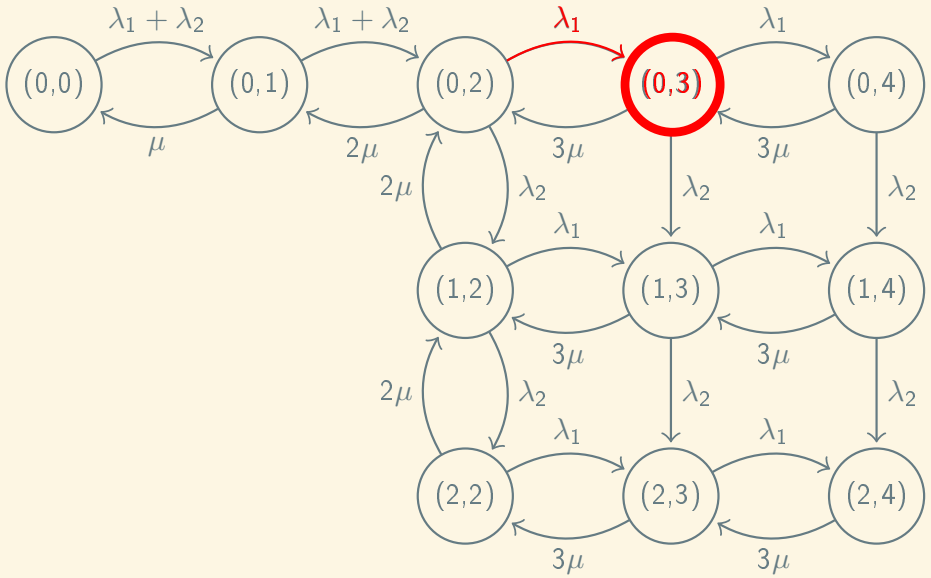
Markov Chain - Custom network



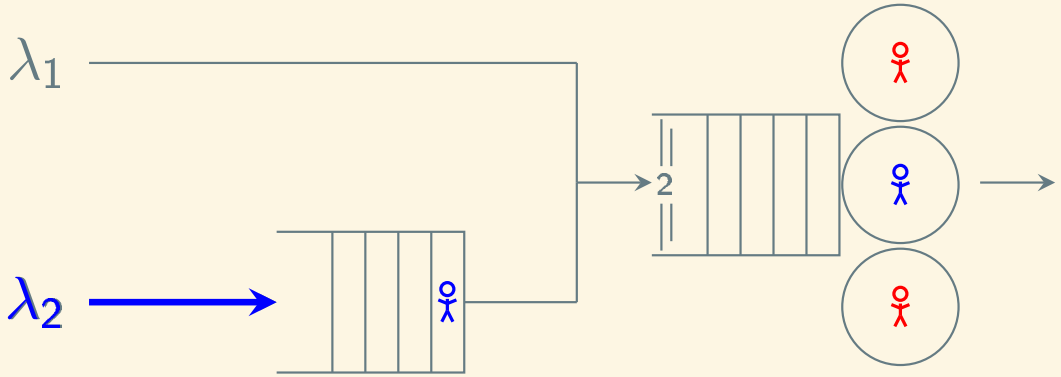


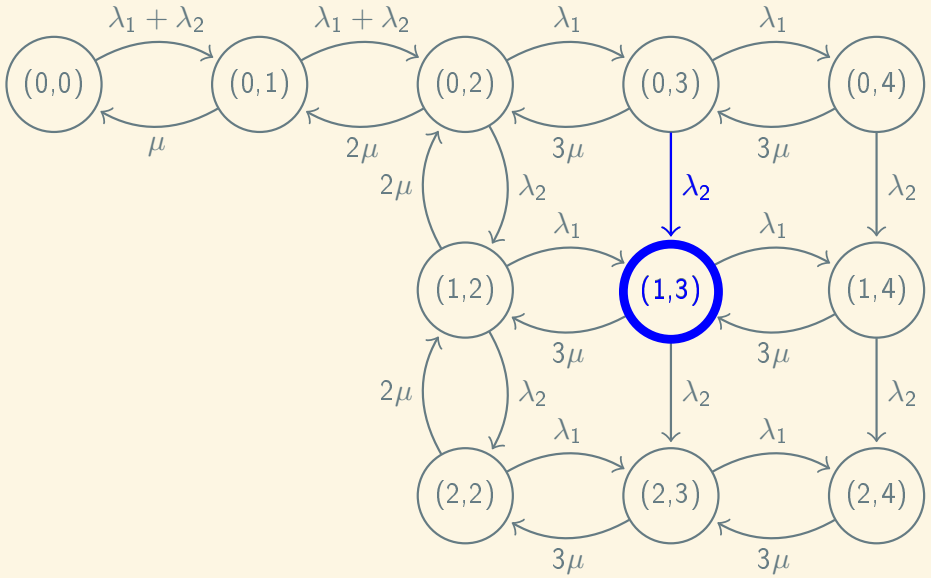
Markov Chain - Custom network



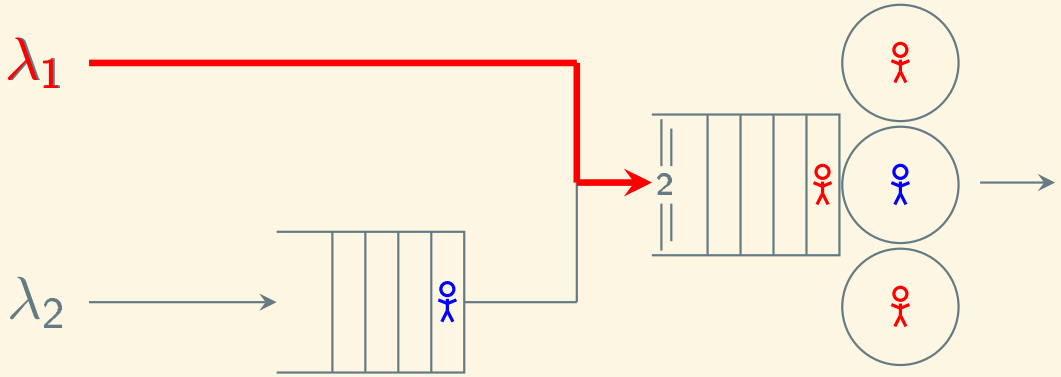


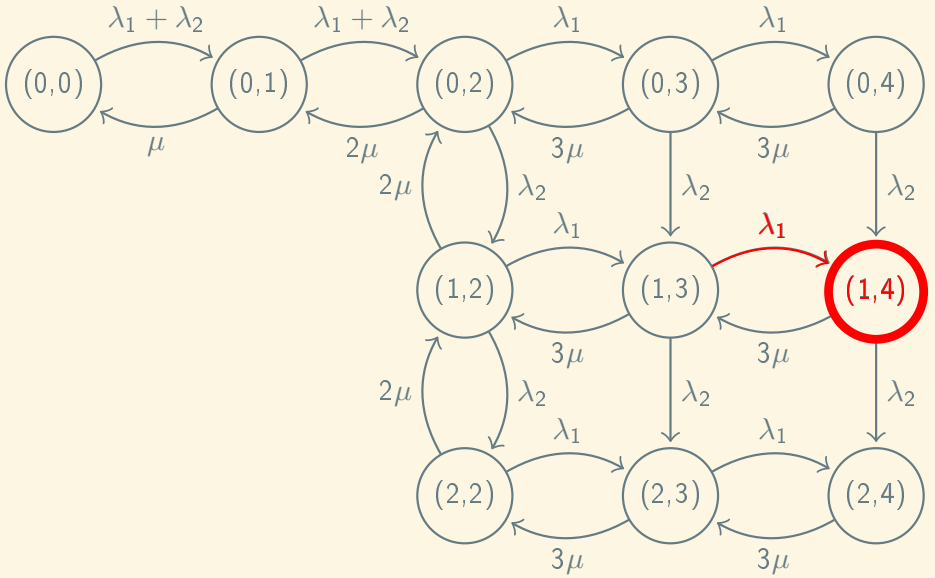
Markov Chain - Custom network



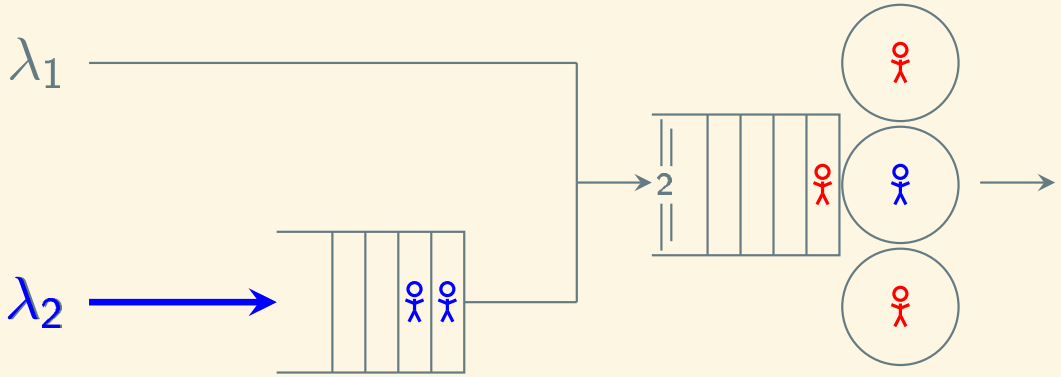


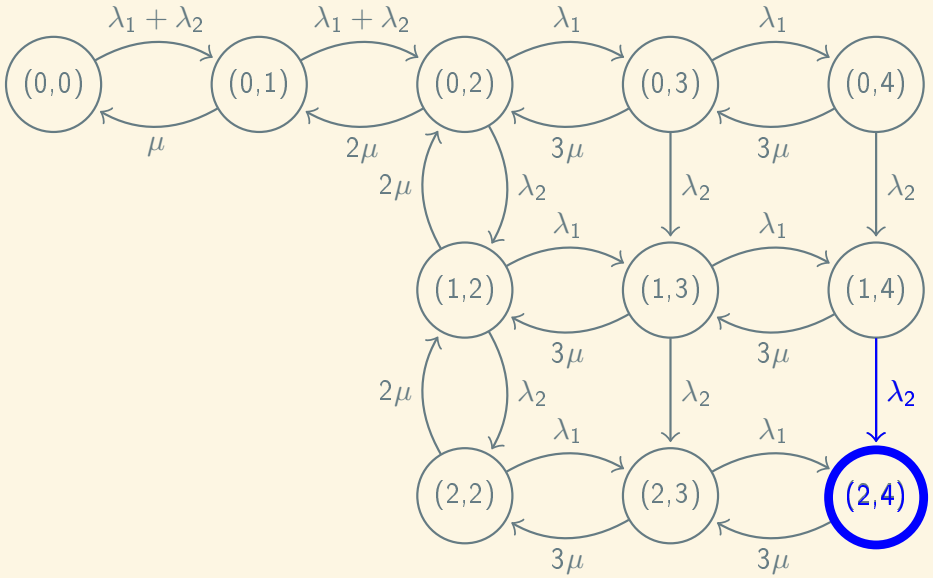
Markov Chain - Custom network



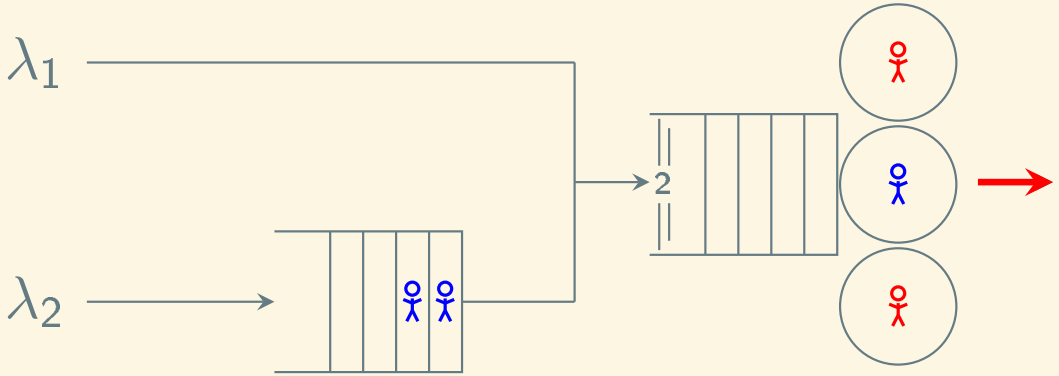


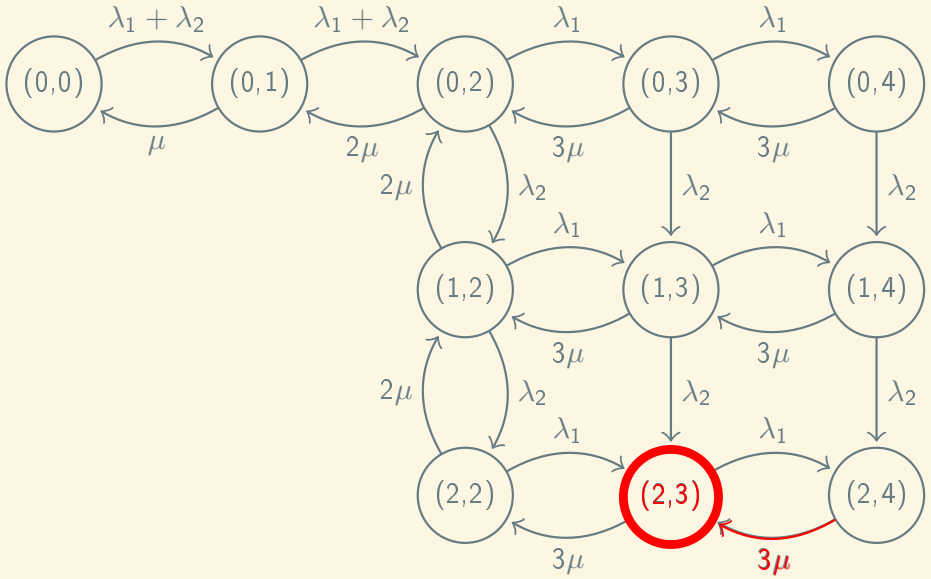
Markov Chain - Custom network



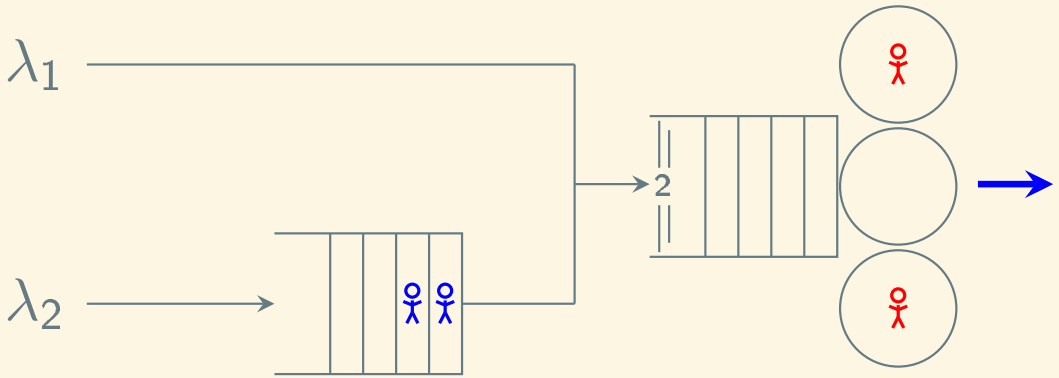


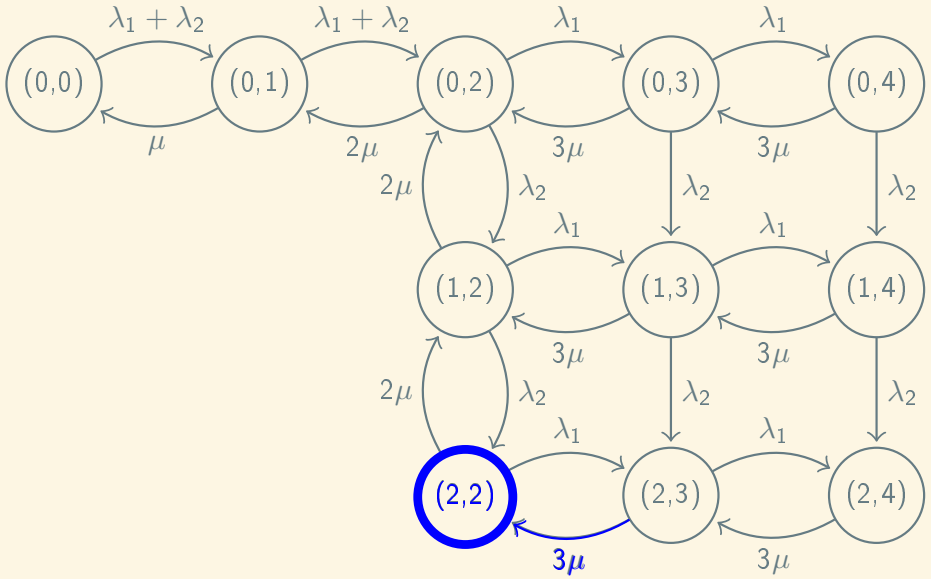
Markov Chain - Custom network



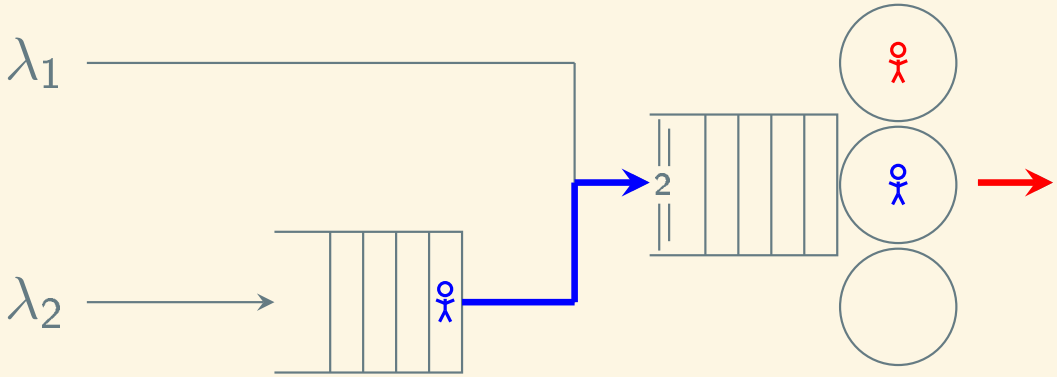


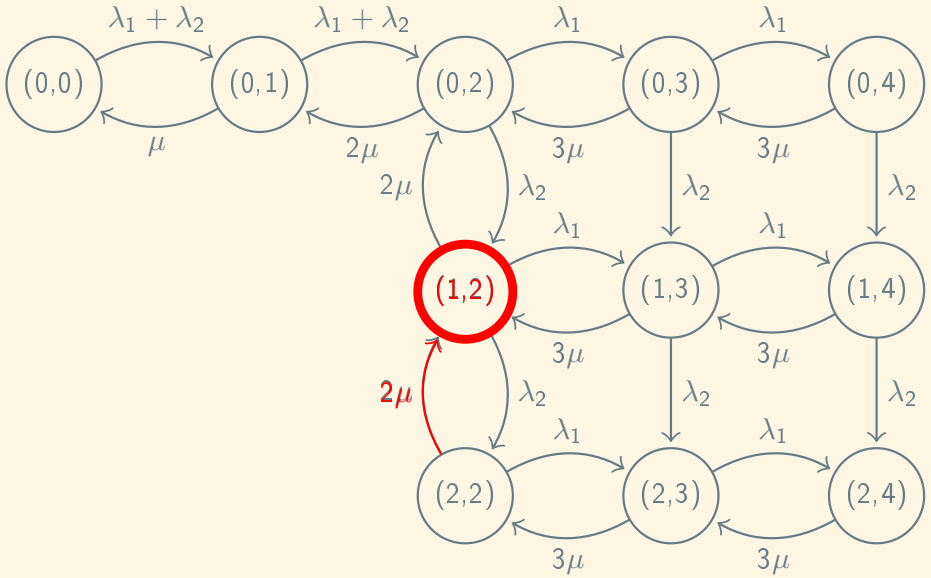
Markov Chain - Custom network





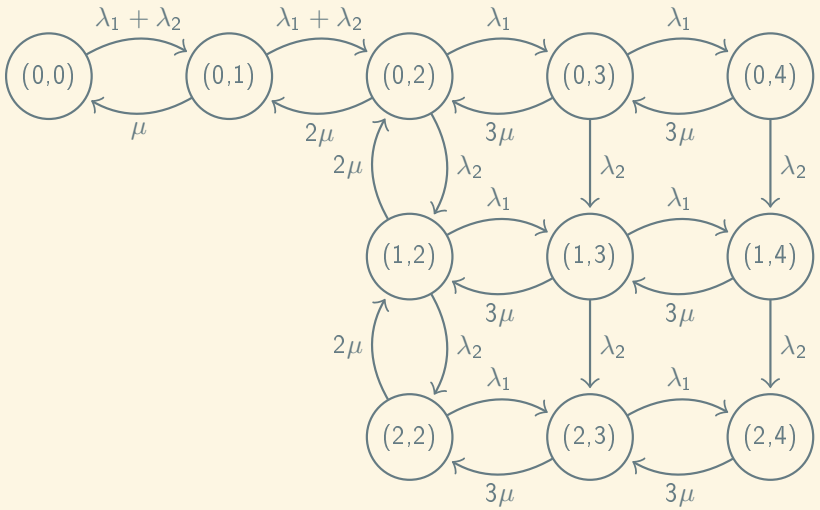
Markov Chain - Custom network





Steady state probabilities - Custom network

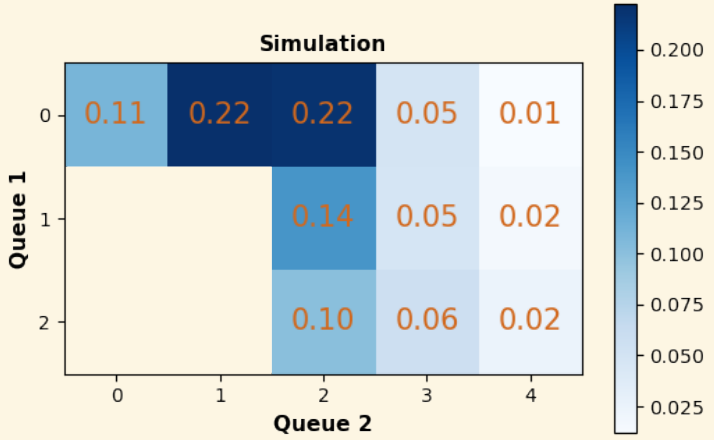
$$
Q = \begin{array}{c} \begin{matrix} (0,0) & (0,1) & (0,2) & \dots & (2,3) & (2,4) \end{matrix} \\ \begin{pmatrix} -\lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 & 0 & \dots & 0 & 0 \\ \mu & -\mu - \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 & \dots & 0 & 0 \\ 0 & 2\mu & -2\mu - \lambda_1 - \lambda_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\lambda_1 - 3\mu & \lambda_1 \\ 0 & 0 & 0 & \dots & 3\mu & -3\mu \end{pmatrix} \end{array} \begin{matrix} (0,0) \\ (0,1) \\ (0,2) \\ \\ \\ (2,3) \\ (2,4) \end{matrix}
$$

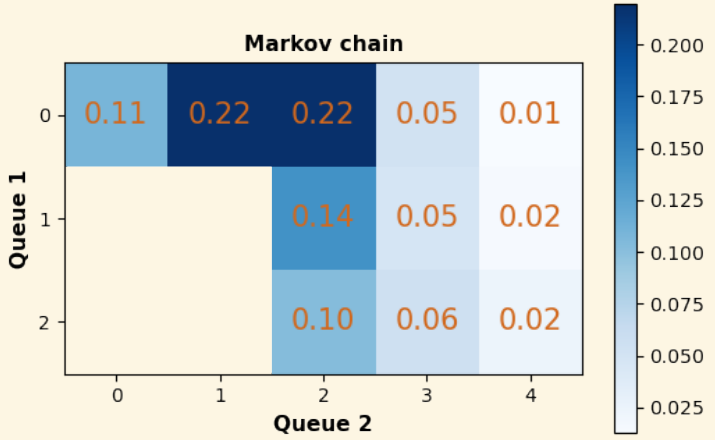


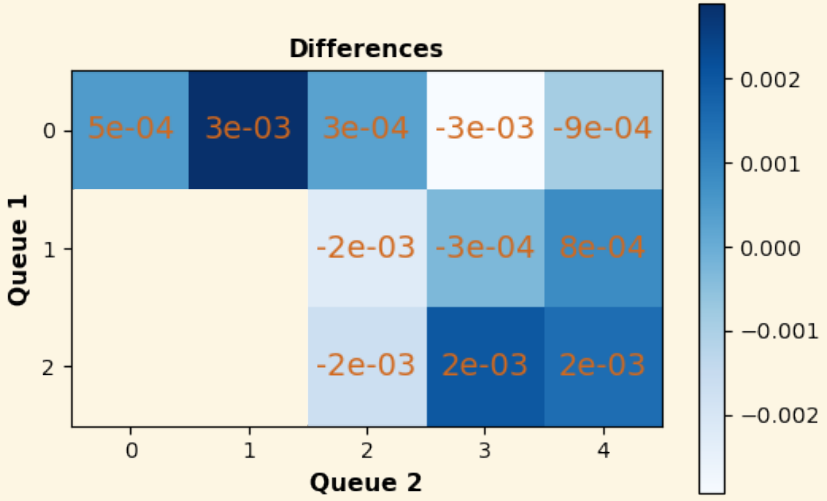
$$
\frac{d\pi}{dt} = \pi Q = 0, \qquad \sum \pi_{(u,v)} = 1
$$

$$
\pi = \begin{bmatrix} \pi_{(0,0)} \\ \pi_{(0,1)} \\ \pi_{(0,2)} \\ \vdots \\ \pi_{(2,3)} \\ \pi_{(2,4)} \end{bmatrix}
$$

Steady state probabilities - Comparison





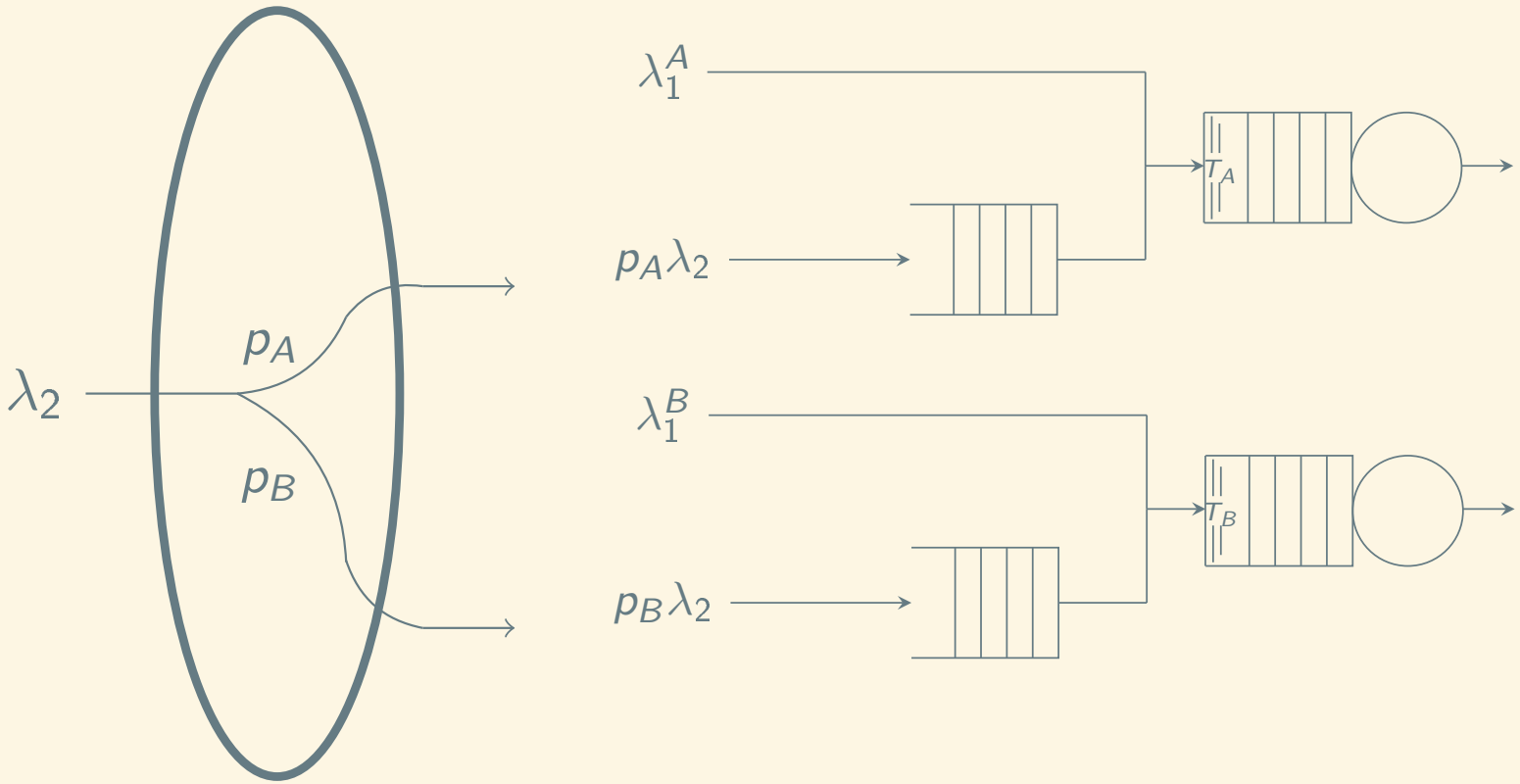


Game - Definition

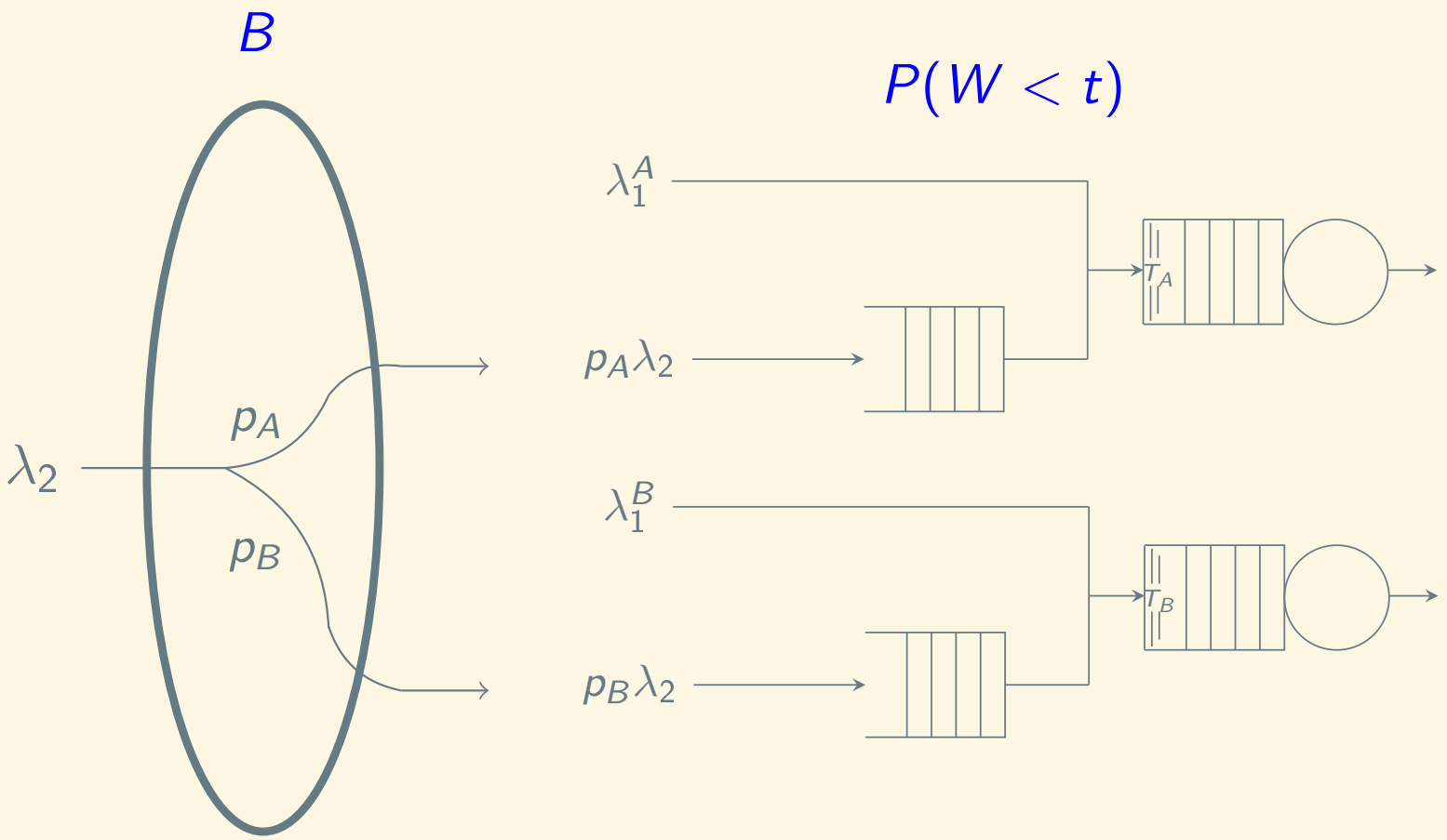
Game - Definition

- Misra S, Sarkar S. Priority-based time-slot allocation in wireless body area networks during medical emergency situations: An evolutionary game-theoretic perspective
- Song J, Wen J. A non-cooperative game with incomplete information to improve patient hospital choice

Game - Players and objectives



Game - Players and objectives



Performance Measures - Blocking time

$$B = \frac{\sum_{(u,v) \in S_A^{(2)}} \pi_{(u,v)} \; b(u,v)}{\sum_{(u,v) \in S_A^{(2)}} \pi_{(u,v)}}$$

Performance Measures - Blocking time

$$B = \frac{\sum_{(u,v) \in S_A^{(2)}} \pi_{(u,v)} \; b(u,v)}{\sum_{(u,v) \in S_A^{(2)}} \pi_{(u,v)}}$$

Performance Measures - Blocking time

$$B = \frac{\sum_{(u,v)\in S_A^{(2)}} \pi_{(u,v)}\ b(u,v)}{\sum_{(u,v)\in S_A^{(2)}} \pi_{(u,v)}}$$

$$b(u,v) = \begin{cases} 0, & \text{if } (u,v) \notin S_b \\ c(u,v) + b(u-1,v), & \text{if } v = N = T \\ c(u,v) + b(u,v-1), & \text{if } v = N \neq T \\ c(u,v) + p_s(u,v)b(u-1,v) + p_a(u,v)b(u,v+1), & \text{if } u > 0 \text{ and } v = T \\ c(u,v) + p_s(u,v)b(u,v-1) + p_a(u,v)b(u,v+1), & \text{otherwise} \end{cases}$$

$$S_b = \{(u,v) \in S \mid u > 0\}$$

$$c(u,v) = \begin{cases} \frac{1}{\min(v,C)\mu}, & \text{if } v = N \\ \frac{1}{\lambda_1 + \min(v,C)\mu}, & \text{otherwise} \end{cases}$$

$$p_s(u,v) = \frac{\min(v,C)\mu}{\lambda_1 + \min(v,C)\mu}, \qquad p_a(u,v) = \frac{\lambda_1}{\lambda_1 + \min(v,C)\mu}$$

Performance Measures - Proportion within time

$$P(W < t) = \frac{\lambda_1 P_{L_1'}}{\lambda_2 P_{L_2'} + \lambda_1 P_{L_1'}} P(W^{(1)} < t) + \frac{\lambda_2 P_{L_2'}}{\lambda_2 P_{L_2'} + \lambda_1 P_{L_1'}} P(W^{(2)} < t)$$

$$P(W^{(1)} < t) = \frac{\sum_{(u,v) \in S_A^{(1)}} P(W_{(u,v)}^{(1)} < t) \pi_{u,v}}{\sum_{(u,v) \in S_A^{(1)}} \pi_{u,v}}$$

$$P(W^{(2)} < t) = \frac{\sum_{(u,v) \in S_A^{(2)}} P(W_{(u,v)}^{(2)} < t) \pi_{u,v}}{\sum_{(u,v) \in S_A^{(2)}} \pi_{u,v}}$$

Performance Measures - Proportion within time

$$P(W^{(i)} < t) = \frac{\sum_{(u,v)\in S_A^{(i)}} P(W_{u,v}^{(i)} < t)\pi_{u,v}}{\sum_{(u,v)\in S_A^{(i)}} \pi_{u,v}}, \quad \textbf{for } i = \{1, 2\}$$

Performance Measures - Proportion within time

$$P(W^{(i)} < t) = \frac{\sum_{(u,v)\in S_A^{(i)}} P(W_{u,v}^{(i)} < t)\pi_{u,v}}{\sum_{(u,v)\in S_A^{(i)}} \pi_{u,v}}, \quad \textbf{for } i = \{1, 2\}$$

$$W_{(u,v)}^{(1)} \sim \begin{cases} \textbf{Erlang}(v, \mu), & \textbf{if } C = 1 \textbf{ and } v > 1 \\ \textbf{Hypo}\left([v - C, 1], [C\mu, \mu]\right), & \textbf{if } C > 1 \textbf{ and } v > C \\ \textbf{Erlang}(1, \mu), & \textbf{if } v \leq C \end{cases}$$

$$W_{(u,v)}^{(2)} \sim \begin{cases} \textbf{Erlang}(\min(v, T), \mu), & \textbf{if } C = 1 \textbf{ and } v, T > 1 \\ \textbf{Hypo}\left([\min(v, T) - C, 1], [C\mu, \mu]\right), & \textbf{if } C > 1 \textbf{ and } v, T > C \\ \textbf{Erlang}(1, \mu), & \textbf{if } v \leq C \textbf{ or } T \leq C \end{cases}$$

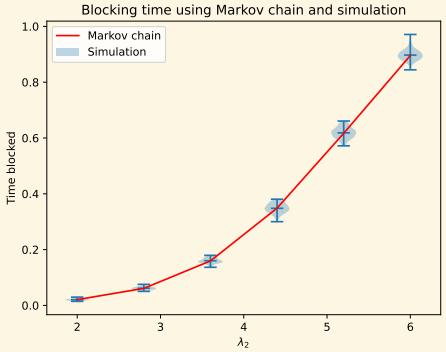
Performance Measures - Proportion within time

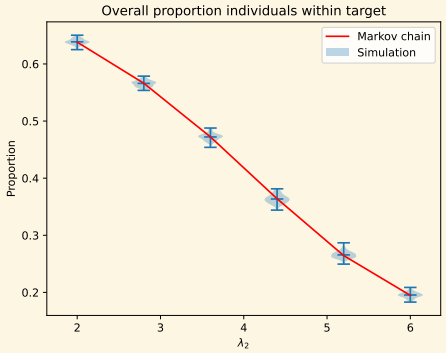
$$P(W^{(i)} < t) = \frac{\sum_{(u,v)\in S_A^{(i)}} P(W_{u,v}^{(i)} < t)\pi_{u,v}}{\sum_{(u,v)\in S_A^{(i)}} \pi_{u,v}}, \quad \textbf{for } i = \{1, 2\}$$

$$P(W_{(u,v)}^{(\mathbf{1})} < t) = \begin{cases} 1 - \sum_{i=\mathbf{0}}^{v-\mathbf{1}} \frac{\mathbf{1}}{i!} e^{-\mu t}(\mu t)^i, & \text{if } C = 1 \text{ and } v > 1 \\ 1 - (\mu C)^{v-C} \mu \sum_{k=\mathbf{1}}^{|\vec{r}|} \sum_{l=\mathbf{1}}^{r_k} \frac{\Psi_{k,l}(-\lambda_k) t^{r_k - l} e^{-\lambda_k t}}{(r_k - l)!(l-\mathbf{1})!}, & \text{if } C > 1 \text{ and } v > C \\ \quad \textbf{where } \vec{r} = (v - C, 1) \textbf{ and } \vec{\lambda} = (C\mu, \mu) & \\ 1 - e^{-\mu t}, & \text{if } v \leq C \end{cases}$$

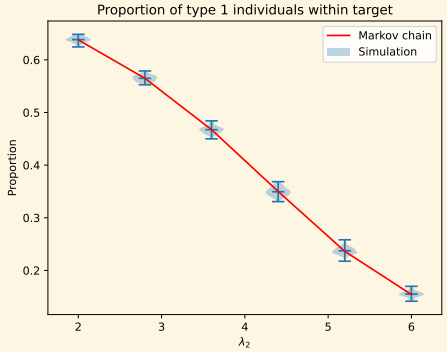
$$P(W_{(u,v)}^{(\mathbf{2})} < t) = \begin{cases} 1 - \sum_{i=\mathbf{0}}^{\min(v,T)-\mathbf{1}} \frac{\mathbf{1}}{i!} e^{-\mu t}(\mu t)^i, & \text{if } C = 1 \text{ and } v, T > 1 \\ 1 - \mu(\mu C)^{\min(v,T)-C} \sum_{k=\mathbf{1}}^{|\vec{r}|} \sum_{l=\mathbf{1}}^{r_k} \frac{\Psi_{k,l}(-\lambda_k) t^{r_k - l} e^{-\lambda_k t}}{(r_k - l)!(l-\mathbf{1})!}, & \text{if } C > 1 \text{ and } v, T > C \\ \quad \textbf{where } \vec{r} = (\min(v, T) - C, 1) \textbf{ and } \vec{\lambda} = (C\mu, \mu) & \\ 1 - e^{-\mu t}, & \text{if } v \leq C \textbf{ or } T \leq C \end{cases}$$

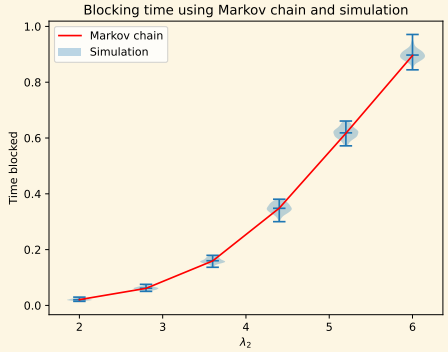
Comparisons

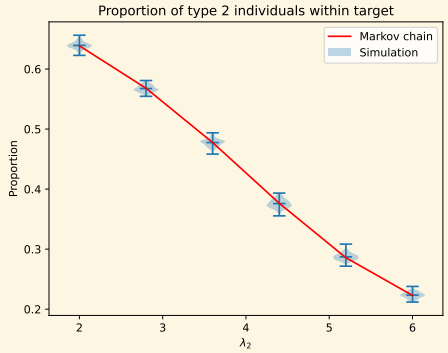




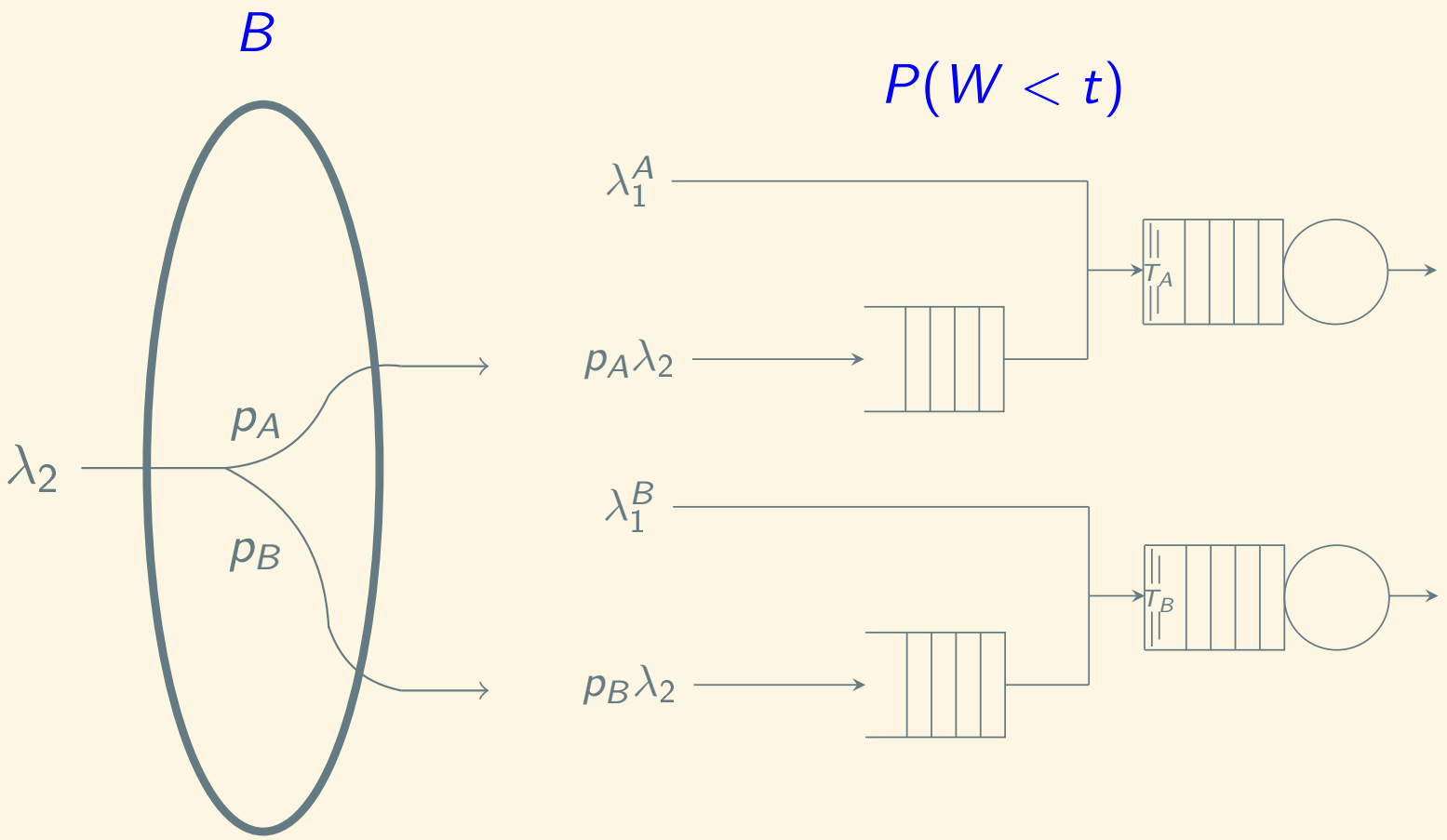
N/A



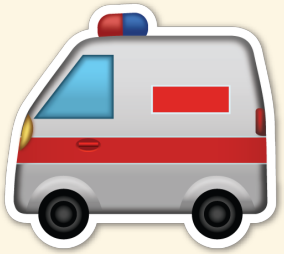


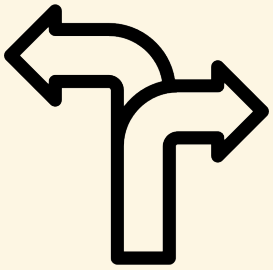


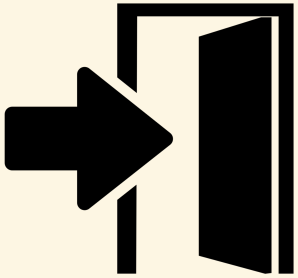
Game - Players and objectives

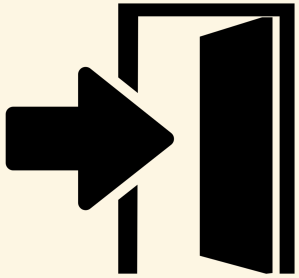


Game - Strategies









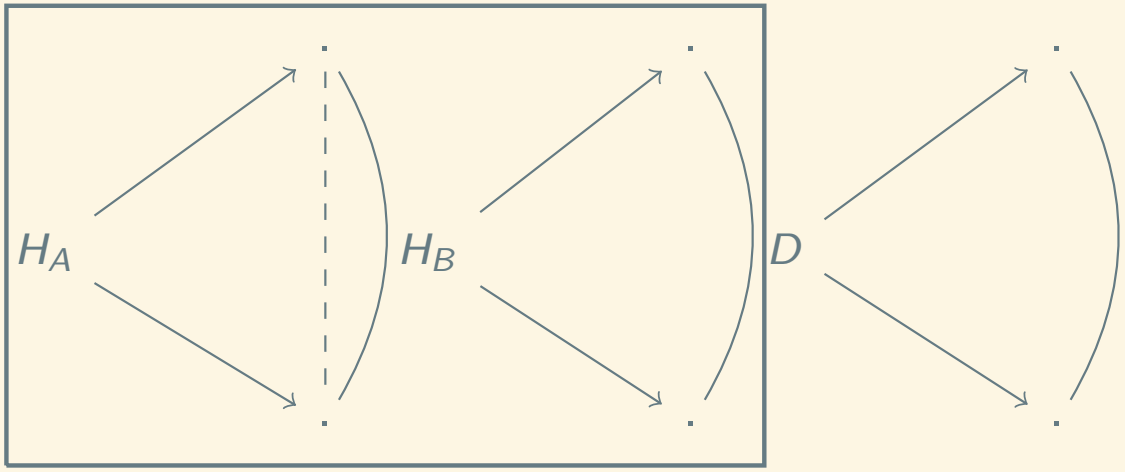
$p_A, p_B \in [0, 1]$

$T_A \in [1, N_A]$

$T_B \in [1, N_B]$

$p_A + p_B = 1$

Game - Formulation

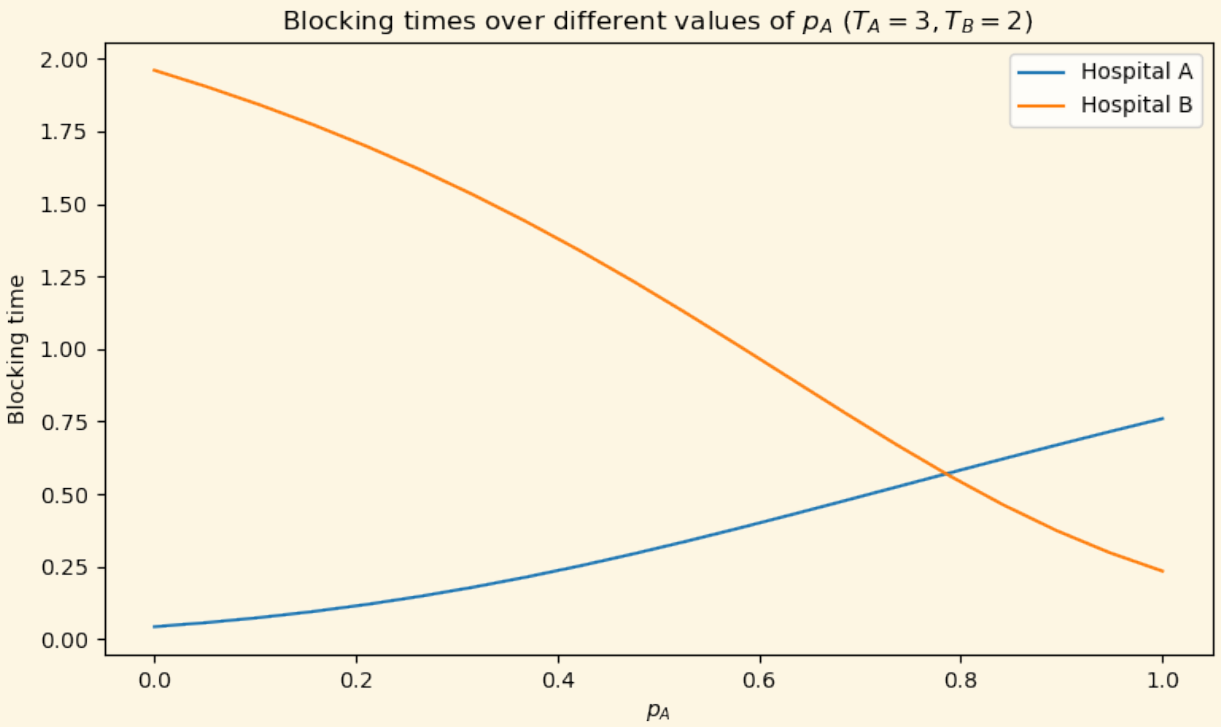


Ambulance's Decision

$$R = \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,N_B} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,N_B} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N_A,1} & p_{N_A,2} & \cdots & p_{N_A,N_B} \end{pmatrix}$$

Ambulance's Decision

$$\begin{matrix} T_A = 3 \\ T_B = 2 \end{matrix} \rightarrow \begin{pmatrix} - & - & - & - \\ - & - & - & - \\ - & x & - & - \\ - & - & - & - \end{pmatrix} \rightarrow B_A(p_A) = B_B(1 - p_A)$$



Hospitals' Decision

$$A = \begin{pmatrix} U^A_{1,1} & U^A_{1,2} & \cdots & U^A_{1,N_B} \\ U^A_{2,1} & U^A_{2,2} & \cdots & U^A_{2,N_B} \\ \vdots & \vdots & \ddots & \vdots \\ U^A_{N_A,1} & U^A_{N_A,2} & \cdots & U^A_{N_A,N_B} \end{pmatrix}, \quad B = \begin{pmatrix} U^B_{1,1} & U^B_{1,2} & \cdots & U^B_{1,N_B} \\ U^B_{2,1} & U^B_{2,2} & \cdots & U^B_{2,N_B} \\ \vdots & \vdots & \ddots & \vdots \\ U^B_{N_A,1} & U^B_{N_A,2} & \cdots & U^B_{N_A,N_B} \end{pmatrix}$$

$$U^{(i)}_{T_A,T_B} = 1 - \left[(P(X^{(i)} < t) - 0.95)^2 \right]$$

Nash Equilibrium

$$R = \begin{pmatrix} 0.5 & 0.1 & 0 & 0 \\ 0.9 & 0.5 & 0.2 & 0 \\ 1 & 0.8 & 0.5 & 0.3 \\ 1 & 1 & 0.7 & 0.5 \end{pmatrix}$$

$$A = \begin{pmatrix} 0.99998394 & 0.99998394 & 0.99998394 & 0.99998394 \\ 0.99998955 & 0.99998848 & 0.99998649 & 0.9999845 \\ 0.99999952 & 0.9999987 & 0.99999596 & 0.99999199 \\ 0.99994372 & 0.99995113 & 0.99998603 & 0.99999911 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.99998394 & 0.99998955 & 0.99999952 & 0.99994372 \\ 0.99998394 & 0.99998848 & 0.9999987 & 0.99995113 \\ 0.99998394 & 0.99998649 & 0.99999596 & 0.99998603 \\ 0.99998394 & 0.9999845 & 0.99999199 & 0.99999911 \end{pmatrix}$$

Nash Equilibria: $\underline{A}$ $(0, 0, 0.4, 0.6)$ $\underline{B}$ $(0, 0, 0.4, 0.6)$

Asymmetric Replicator Dynamics

$$\frac{dx}{dt}_i = x_i((f_x)_i - \phi_x), \quad \text{for all } i$$

$$\frac{dy}{dt}_i = y_i((f_y)_i - \phi_y), \quad \text{for all } i$$

- Fudenberg, Drew, et al. The theory of learning in games. Vol. 2. MIT press, 1998.
- Elvio, Accinelli and Carrera, Edgar. 2011. Evolutionarily Stable Strategies and Replicator Dynamics in Asymmetric Two-Population Games. 10.1007/978-3-642-11456-4_3.

Inefficiency measure

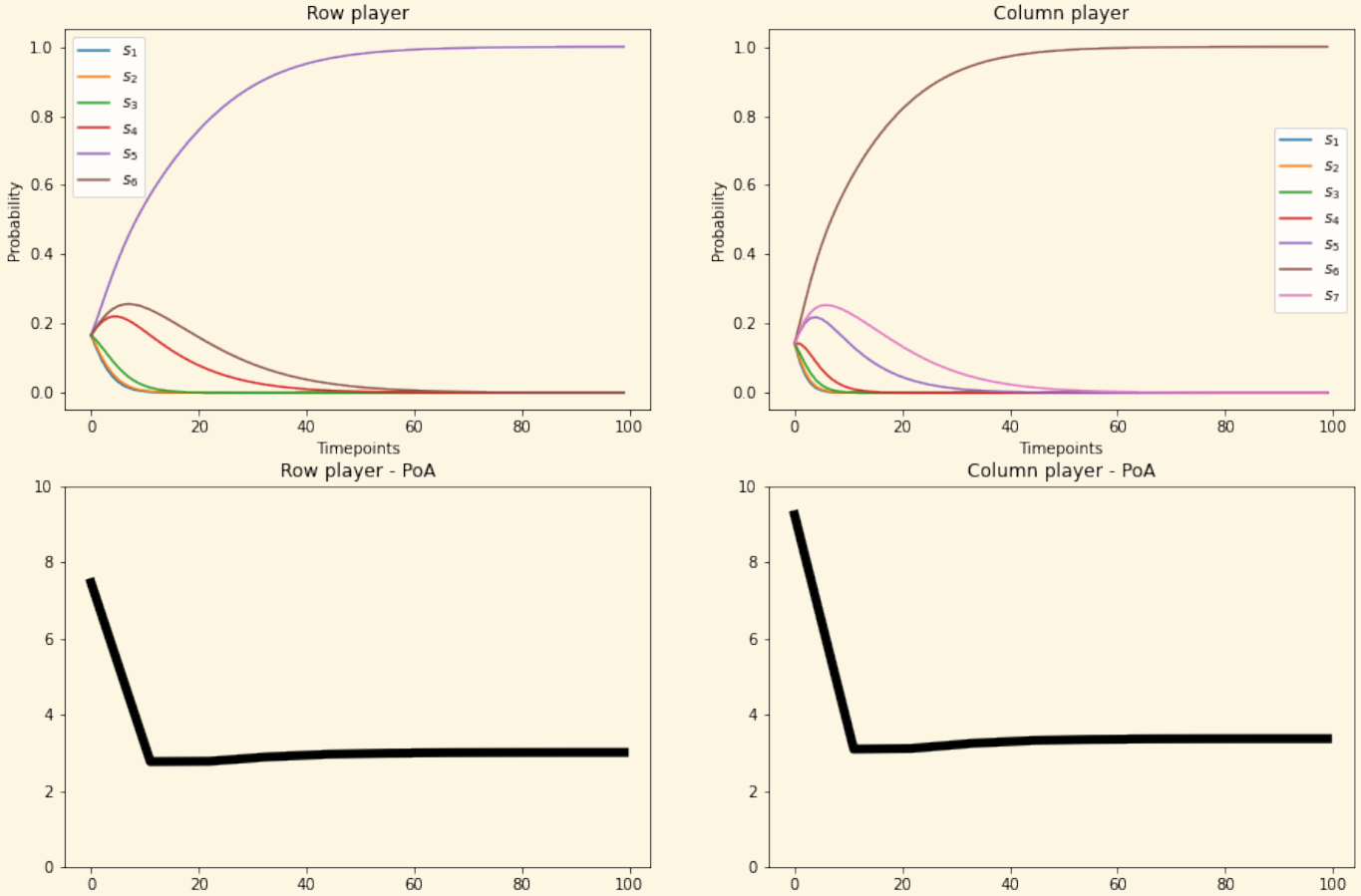
$$PoA = \frac{\max_{s \in E} Cost(s)}{\min_{s \in S} Cost(S)}$$

Inefficiency measure

$$PoA = \frac{\max_{s \in E} Cost(s)}{\min_{s \in S} Cost(S)}$$

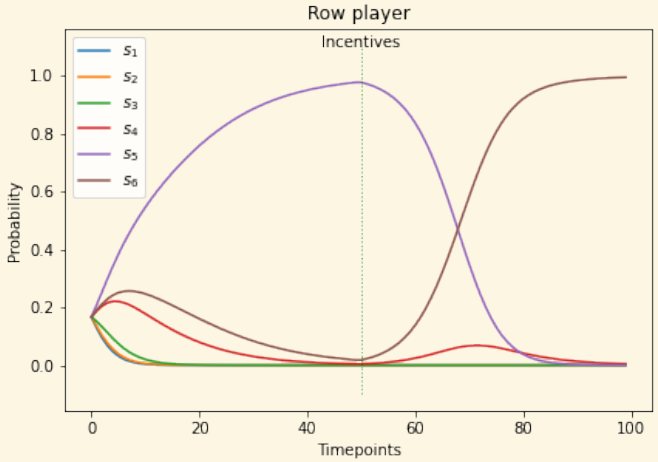
$$PoA_A(s_r) = \frac{Cost(s_r)}{\min_{s \in S} Cost(S)}, \qquad PoA_B(s_c) = \frac{Cost(s_c)}{\min_{s \in S} Cost(S)}$$

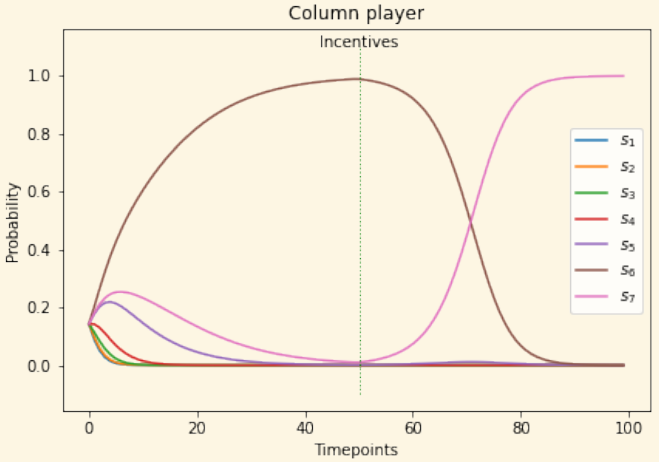
Learning algorithms - Asymmetric replicator dynamics

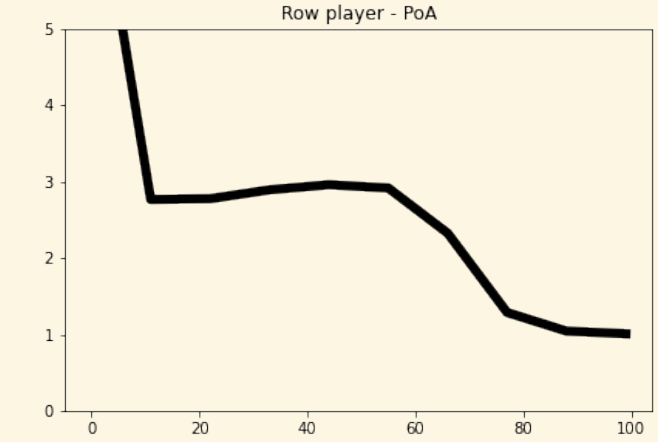


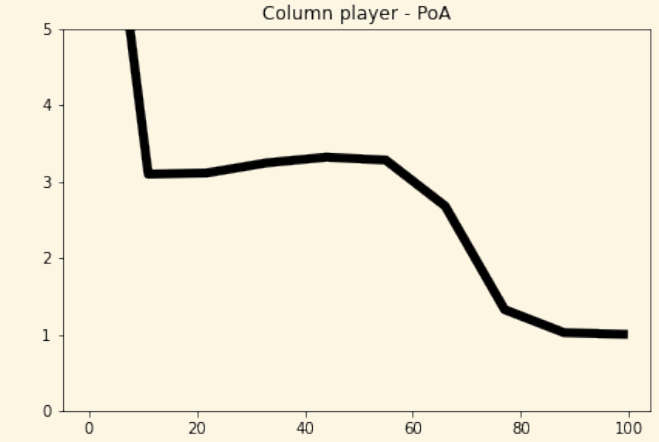
Inefficiencies can be learned and emerge naturally

Learning algorithms - Asymmetric replicator dynamics









Targeted incentivisation of behaviours can help escape learned inefficiencies

Thank you!

"Inefficiencies can be learned and emerge naturally"

"Targeted incentivisation of behaviours can help escape learned inefficiencies"

PanayidesM@cardiff.ac.uk
@Michalis_Pan
@11michalis11
https://github.com/11michalis11/AmbulanceDecisionGame