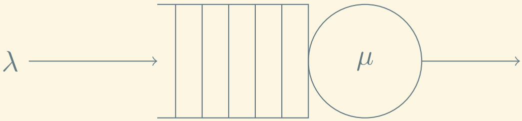
A 3-player game theoretic model of a choice between two queueing systems with strategic managerial decision making

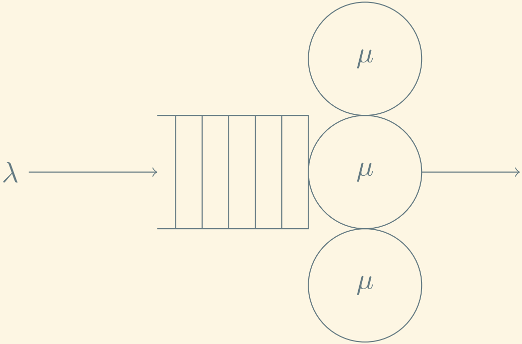
Michalis Panayides

Supervisors:
Dr. Vince Knight,
Prof. Paul Harper

Queues - M/M/1



Queues - M/M/3



Queues - Custom network of queues

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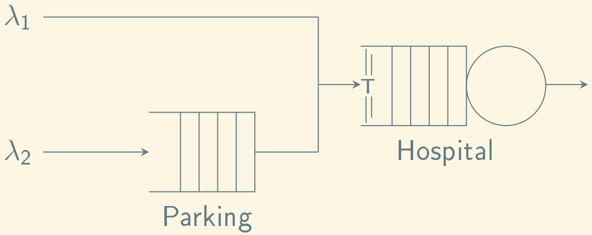
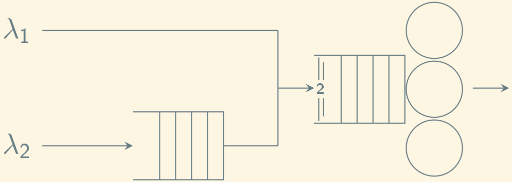
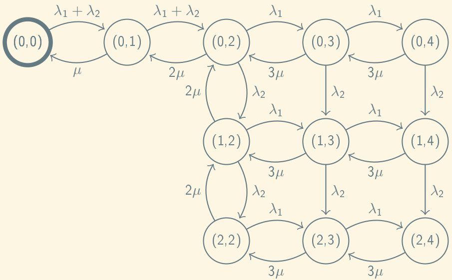


Parameters:

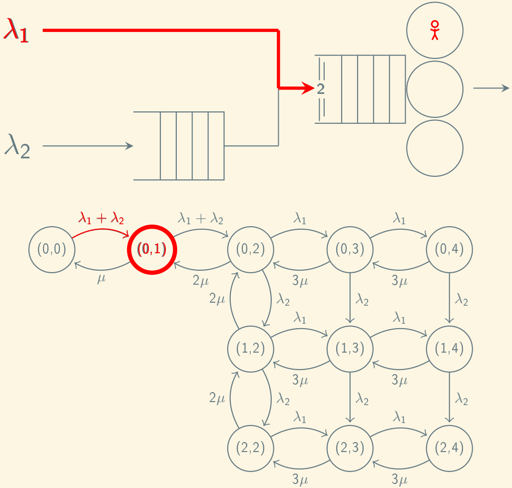
- λ_1: Arrival rate of type 1 individuals
- λ_2: Arrival rate of type 2 individuals
- μ: Service rate
- C: Number of servers
- T: Threshold

Markov Chain - Custom network





Markov Chain - Custom network



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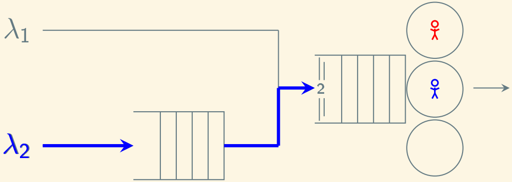

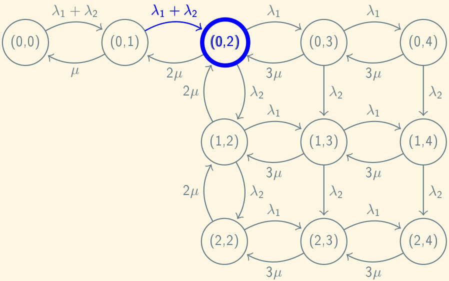

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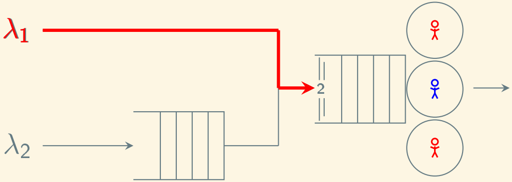

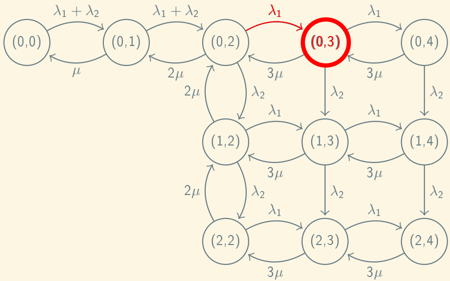

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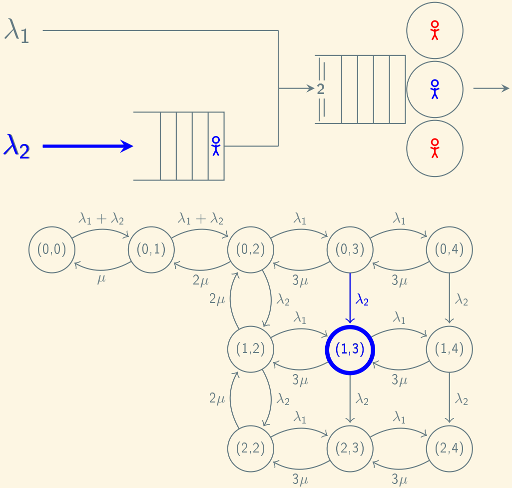

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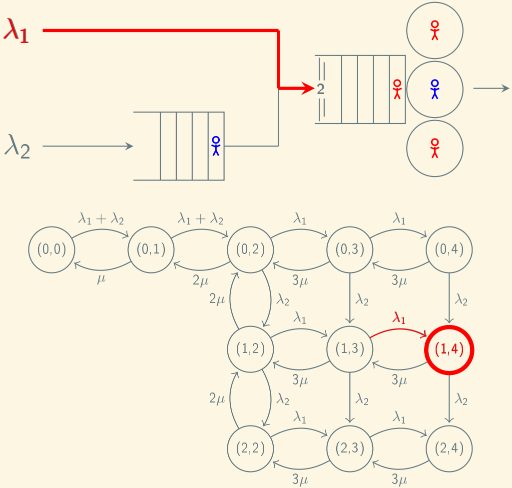
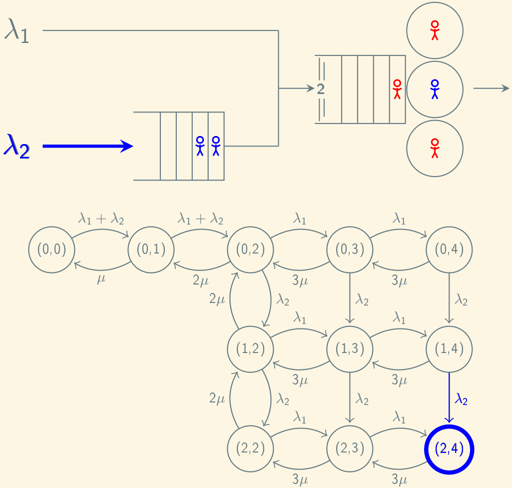
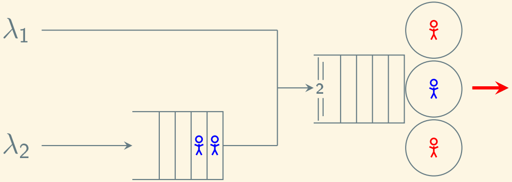
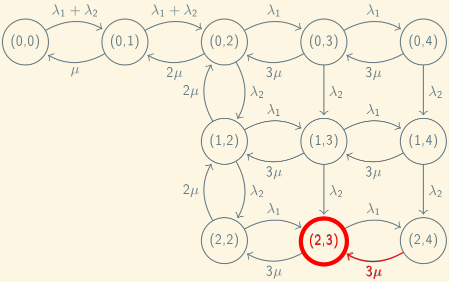


Markov Chain - Custom network

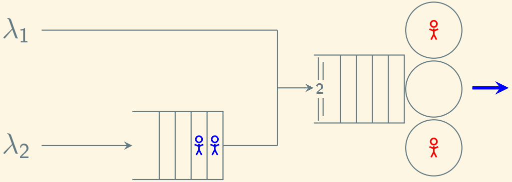


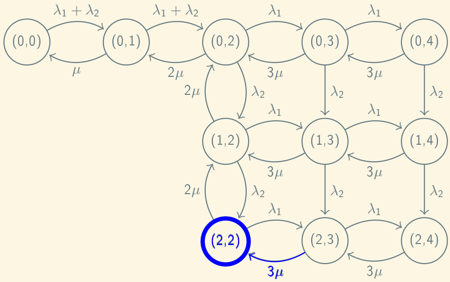
Markov Chain - Custom network



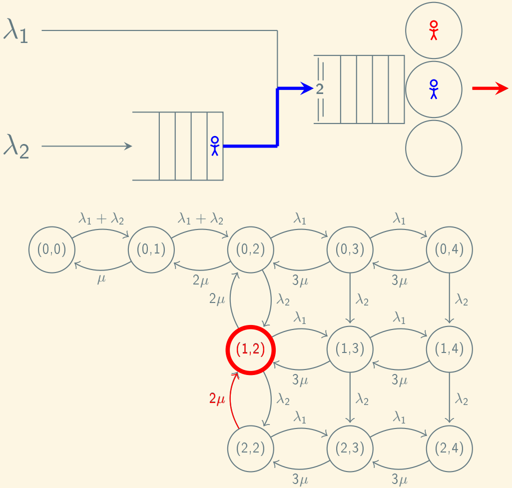


Markov Chain - Custom network



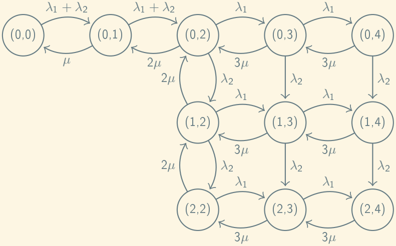


Markov Chain - Custom network



Steady state probabilities - Custom network

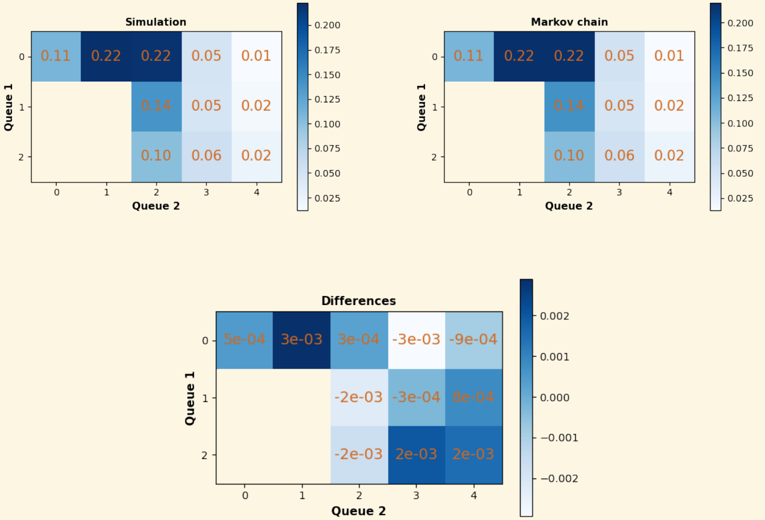
$$Q = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \begin{array}{cccccc} (0,0) & (0,1) & (0,2) & & (2,3) & (2,4) \end{array} & \\ \begin{pmatrix} -\lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 & 0 & \dots & 0 & 0 \\ \mu & -\mu - \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 & \dots & 0 & 0 \\ 0 & 2\mu & -2\mu - \lambda_1 - \lambda_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\lambda_1 - 3\mu & \lambda_1 \\ 0 & 0 & 0 & \dots & 3\mu & -3\mu \end{pmatrix} & \begin{array}{c} (0,0) \\ (0,1) \\ (0,2) \\ \\ (2,3) \\ (2,4) \end{array} \end{array}$$



$$\frac{d\pi}{dt} = \pi Q = 0, \qquad \sum \pi_{(u,v)} = 1$$

$$\pi = \begin{bmatrix} \pi_{(0,0)} \\ \pi_{(0,1)} \\ \pi_{(0,2)} \\ \vdots \\ \pi_{(2,3)} \\ \pi_{(2,4)} \end{bmatrix}$$

Steady state probabilities - Comparison

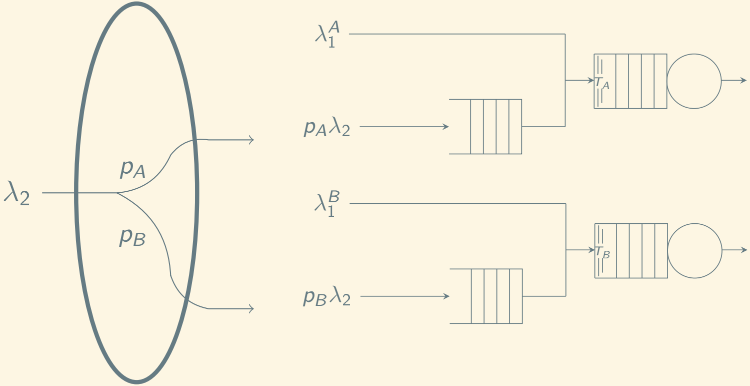


Game - Definition

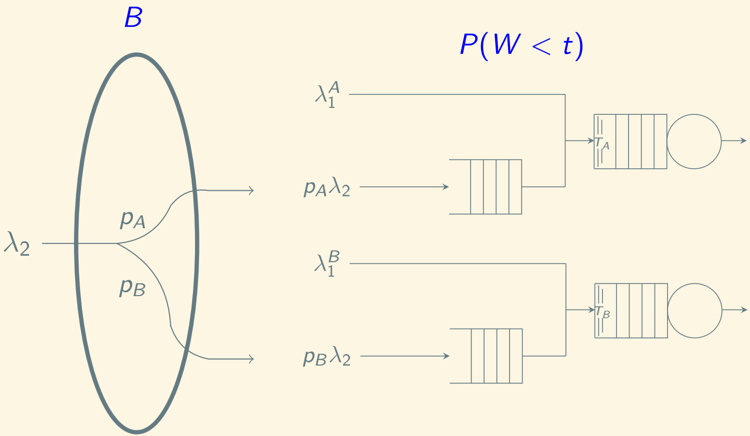
Game - Definition

- Misra S, Sarkar S. Priority-based time-slot allocation in wireless body area networks during medical emergency situations: An evolutionary game-theoretic perspective
- Song J, Wen J. A non-cooperative game with incomplete information to improve patient hospital choice

Game - Players and objectives



Game - Players and objectives



Performance Measures - Blocking time

$$B = \frac{\sum_{(u,v) \in S_A^{(2)}} \pi_{(u,v)} \; b(u,v)}{\sum_{(u,v) \in S_A^{(2)}} \pi_{(u,v)}}$$

Performance Measures - Blocking time

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Performance Measures - Blocking time

$$B = \frac{\sum_{(u,v)\in S_A^{(2)}} \pi_{(u,v)}\ b(u,v)}{\sum_{(u,v)\in S_A^{(2)}} \pi_{(u,v)}}$$

$$b(u,v) = \begin{cases} 0, & \text{if } (u,v) \notin S_b \\ c(u,v) + b(u-1,v), & \text{if } v = N = T \\ c(u,v) + b(u,v-1), & \text{if } v = N \neq T \\ c(u,v) + p_s(u,v)b(u-1,v) + p_a(u,v)b(u,v+1), & \text{if } u > 0 \text{ and} \\ & \quad v = T \\ c(u,v) + p_s(u,v)b(u,v-1) + p_a(u,v)b(u,v+1), & \text{otherwise} \end{cases}$$

$$S_b = \{(u,v) \in S \mid u > 0\}$$

$$c(u,v) = \begin{cases} \frac{1}{\min(v,C)\mu}, & \text{if } v = N \\ \frac{1}{\lambda_1 + \min(v,C)\mu}, & \text{otherwise} \end{cases}$$

$$p_s(u,v) = \frac{\min(v,C)\mu}{\lambda_1 + \min(v,C)\mu}, \qquad p_a(u,v) = \frac{\lambda_1}{\lambda_1 + \min(v,C)\mu}$$

Performance Measures - Proportion within time

$$P(W < t) = \frac{\lambda_1 P_{L'_1}}{\lambda_2 P_{L'_2} + \lambda_1 P_{L'_1}} P(W^{(1)} < t) + \frac{\lambda_2 P_{L'_2}}{\lambda_2 P_{L'_2} + \lambda_1 P_{L'_1}} P(W^{(2)} < t)$$

$$P(W^{(1)} < t) = \frac{\sum_{(u,v) \in S_A^{(1)}} P(W^{(1)}_{(u,v)} < t) \pi_{u,v}}{\sum_{(u,v) \in S_A^{(1)}} \pi_{u,v}}$$

$$P(W^{(2)} < t) = \frac{\sum_{(u,v) \in S_A^{(2)}} P(W^{(2)}_{(u,v)} < t) \pi_{u,v}}{\sum_{(u,v) \in S_A^{(2)}} \pi_{u,v}}$$

Performance Measures - Proportion within time

$$P(W^{(i)} < t) = \frac{\sum_{(u,v) \in S_A^{(i)}} P(W_{u,v}^{(i)} < t) \pi_{u,v}}{\sum_{(u,v) \in S_A^{(i)}} \pi_{u,v}}, \quad \textbf{for } i = \{1, 2\}$$

Performance Measures - Proportion within time

$$P(W^{(i)} < t) = \frac{\sum_{(u,v)\in S_A^{(i)}} P(W_{u,v}^{(i)} < t)\pi_{u,v}}{\sum_{(u,v)\in S_A^{(i)}} \pi_{u,v}}, \quad \textbf{for } i = \{1, 2\}$$

$$W_{(u,v)}^{(1)} \sim \begin{cases} \textbf{Erlang}(v, \mu), & \textbf{if } C = 1 \textbf{ and } v > 1 \\ \textbf{Hypo}\left([v - C, 1], [C\mu, \mu]\right), & \textbf{if } C > 1 \textbf{ and } v > C \\ \textbf{Erlang}(1, \mu), & \textbf{if } v \leq C \end{cases}$$

$$W_{(u,v)}^{(2)} \sim \begin{cases} \textbf{Erlang}(\min(v, T), \mu), & \textbf{if } C = 1 \textbf{ and } v, T > 1 \\ \textbf{Hypo}\left([\min(v, T) - C, 1], [C\mu, \mu]\right), & \textbf{if } C > 1 \textbf{ and } v, T > C \\ \textbf{Erlang}(1, \mu), & \textbf{if } v \leq C \textbf{ or } T \leq C \end{cases}$$

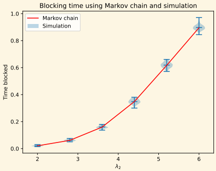
Performance Measures - Proportion within time

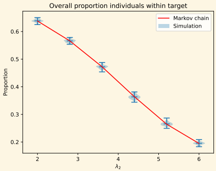
$$P(W^{(i)} < t) = \frac{\sum_{(u,v)\in S_A^{(i)}} P(W_{u,v}^{(i)} < t)\pi_{u,v}}{\sum_{(u,v)\in S_A^{(i)}} \pi_{u,v}}, \quad \textbf{for } i = \{1, 2\}$$

$$P(W_{(u,v)}^{(1)} < t) = \begin{cases} 1 - \sum_{i=0}^{v-1} \frac{1}{i!} e^{-\mu t} (\mu t)^i, & \text{if } C = 1 \text{ and } v > 1 \\ 1 - (\mu C)^{v-C} \mu \sum_{k=1}^{|\vec{r}|} \sum_{l=1}^{r_k} \frac{\Psi_{k,l}(-\lambda_k) t^{r_k - l} e^{-\lambda_k t}}{(r_k - l)!(l-1)!}, & \text{if } C > 1 \text{ and } v > C \\ \quad \text{where } \vec{r} = (v - C, 1) \text{ and } \vec{\lambda} = (C\mu, \mu) & \\ 1 - e^{-\mu t}, & \text{if } v \leq C \end{cases}$$

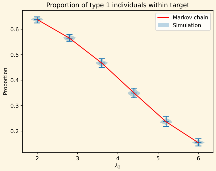
$$P(W_{(u,v)}^{(2)} < t) = \begin{cases} 1 - \sum_{i=0}^{\min(v,T)-1} \frac{1}{i!} e^{-\mu t} (\mu t)^i, & \text{if } C = 1 \text{ and } v, T > 1 \\ 1 - \mu(\mu C)^{\min(v,T)-C} \sum_{k=1}^{|\vec{r}|} \sum_{l=1}^{r_k} \frac{\Psi_{k,l}(-\lambda_k) t^{r_k - l} e^{-\lambda_k t}}{(r_k - l)!(l-1)!}, & \text{if } C > 1 \text{ and } v, T > C \\ \quad \text{where } \vec{r} = (\min(v, T) - C, 1) \text{ and } \vec{\lambda} = (C\mu, \mu) & \\ 1 - e^{-\mu t}, & \text{if } v \leq C \text{ or } T \leq C \end{cases}$$

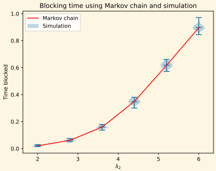
Comparisons

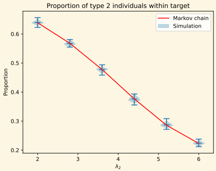




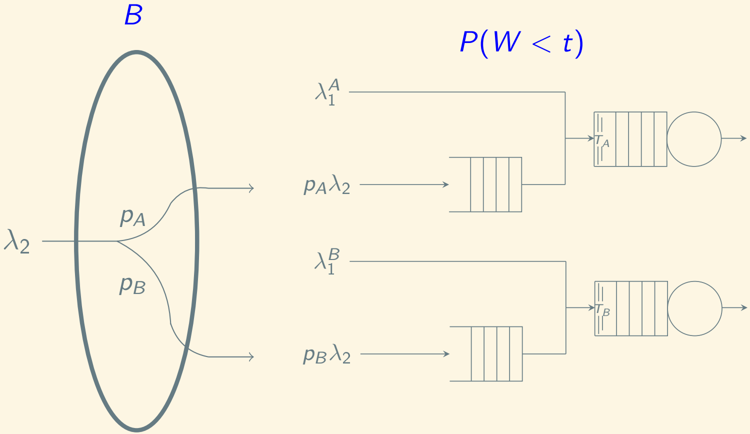
N/A



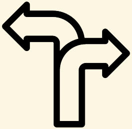




Game - Players and objectives

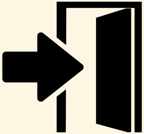


Game - Strategies

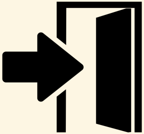


$p_A, p_B \in [0, 1]$

$p_A + p_B = 1$

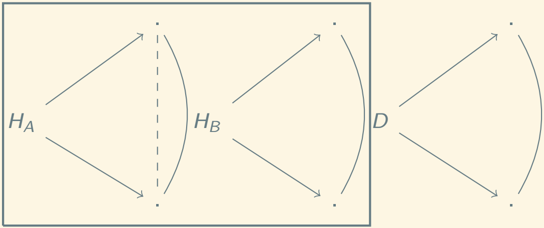


$T_A \in [1, N_A]$



$T_B \in [1, N_B]$

Game - Formulation

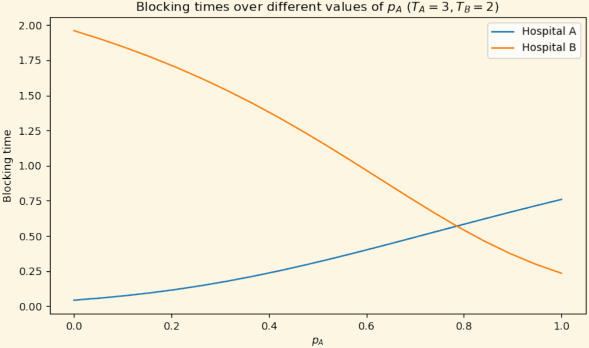


Ambulance's Decision

$$R = \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,N_B} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,N_B} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N_A,1} & p_{N_A,2} & \cdots & p_{N_A,N_B} \end{pmatrix}$$

Ambulance's Decision

$$\begin{array}{l} T_A = 3 \\ T_B = 2 \end{array} \rightarrow \begin{pmatrix} - & - & - & - \\ - & - & - & - \\ - & \text{x} & - & - \\ - & - & - & - \end{pmatrix} \rightarrow B_A(p_A) = B_B(1 - p_A)$$



Hospitals' Decision

$$A = \begin{pmatrix} U^A_{1,1} & U^A_{1,2} & \cdots & U^A_{1,N_B} \\ U^A_{2,1} & U^A_{2,2} & \cdots & U^A_{2,N_B} \\ \vdots & \vdots & \ddots & \vdots \\ U^A_{N_A,1} & U^A_{N_A,2} & \cdots & U^A_{N_A,N_B} \end{pmatrix}, \quad B = \begin{pmatrix} U^B_{1,1} & U^B_{1,2} & \cdots & U^B_{1,N_B} \\ U^B_{2,1} & U^B_{2,2} & \cdots & U^B_{2,N_B} \\ \vdots & \vdots & \ddots & \vdots \\ U^B_{N_A,1} & U^B_{N_A,2} & \cdots & U^B_{N_A,N_B} \end{pmatrix}$$

$$U^{(i)}_{T_A,T_B} = 1 - \left[(P(X^{(i)} < t) - 0.95)^2 \right]$$

Nash Equilibrium

$$R = \begin{pmatrix} 0.5 & 0.1 & 0 & 0 \\ 0.9 & 0.5 & 0.2 & 0 \\ 1 & 0.8 & 0.5 & 0.3 \\ 1 & 1 & 0.7 & 0.5 \end{pmatrix}$$

$$A = \begin{pmatrix} 0.99998394 & 0.99998394 & 0.99998394 & 0.99998394 \\ 0.99998955 & 0.99998848 & 0.99998649 & 0.9999845 \\ 0.99999952 & 0.9999987 & 0.99999596 & 0.99999199 \\ 0.99994372 & 0.99995113 & 0.99998603 & 0.99999911 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.99998394 & 0.99998955 & 0.99999952 & 0.99994372 \\ 0.99998394 & 0.99998848 & 0.9999987 & 0.99995113 \\ 0.99998394 & 0.99998649 & 0.99999596 & 0.99998603 \\ 0.99998394 & 0.9999845 & 0.99999199 & 0.99999911 \end{pmatrix}$$

Nash Equilibria:	A	B
	$(0, 0, 0.4, 0.6)$	$(0, 0, 0.4, 0.6)$

Asymmetric Replicator Dynamics

$$\frac{dx}{dt}_i = x_i((f_x)_i - \phi_x), \quad \text{for all } i$$

$$\frac{dy}{dt}_i = y_i((f_y)_i - \phi_y), \quad \text{for all } i$$

- Fudenberg, Drew, et al. The theory of learning in games. Vol. 2. MIT press, 1998.
- Elvio, Accinelli and Carrera, Edgar. 2011. Evolutionarily Stable Strategies and Replicator Dynamics in Asymmetric Two-Population Games. 10.1007/978-3-642-11456-4_3.

Inefficiency measure

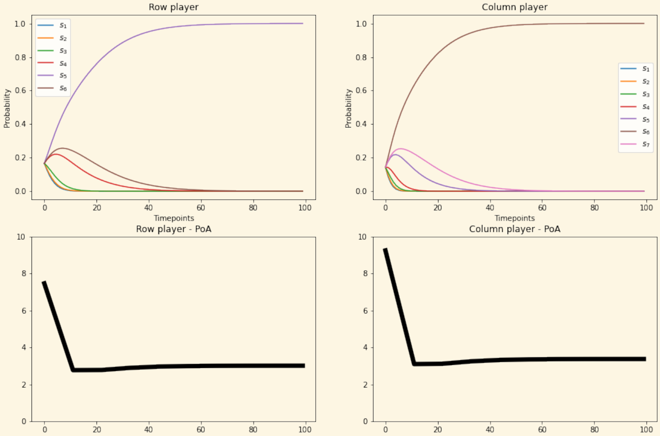
$$PoA = \frac{\max_{s \in E} Cost(s)}{\min_{s \in S} Cost(S)}$$

Inefficiency measure

$$PoA = \frac{\max_{s \in E} Cost(s)}{\min_{s \in S} Cost(S)}$$

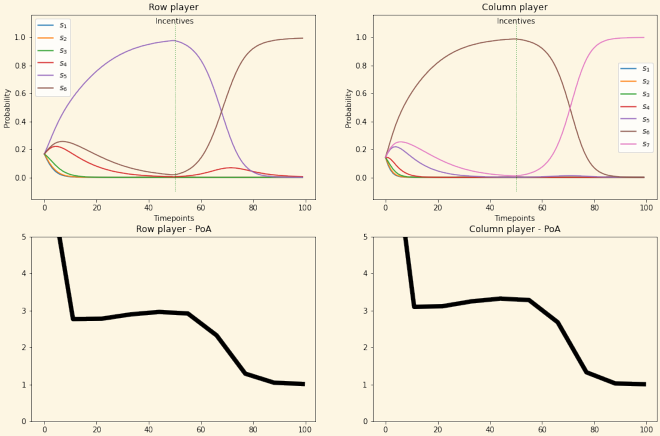
$$PoA_A(s_r) = \frac{Cost(s_r)}{\min_{s \in S} Cost(S)}, \qquad PoA_B(s_c) = \frac{Cost(s_c)}{\min_{s \in S} Cost(S)}$$

Learning algorithms - Asymmetric replicator dynamics



Inefficiencies can be learned and emerge naturally

Learning algorithms - Asymmetric replicator dynamics



Targeted incentivisation of behaviours can help escape learned inefficiencies

Thank you!

"Inefficiencies can be learned and emerge naturally"

"Targeted incentivisation of behaviours can help escape learned inefficiencies"

PanayidesM@cardiff.ac.uk
@Michalis_Pan
@11michalis11
https://github.com/11michalis11/AmbulanceDecisionGame