Ring of N Classical or Quantum Harmonic Oscillators

1 The Classsical Oscillator

The equation of motion for the nth oscillator in a ring of N coupled classical harmonic oscillators is given by

$$\ddot{q}_n = \frac{k}{m}(q_{n+1} + q_{n-1} - 2q_n) \tag{1}$$

where $n = n+N$ and a dot denotes differentiation with respect to time. This can be written more compactly as

$$\ddot{\mathbf{q}} = -K\mathbf{q} \tag{2}$$

where $\mathbf{q}$ is a vector of displacements of individual oscillators at time t and K is the matrix with components

$$K_{ij} = \frac{k}{m}(2\delta_{ij} - \delta_{i+1,j} - \delta_{i-1,j}) \tag{3}$$

with δ_{ij} the Kronecker delta. To solve this equation, we can use the method of normal modes, where we assume every oscillator oscillates with the same frequency ω, i.e.

$$\mathbf{q}(t) = \mathbf{A}\cos(\omega t - \delta) \tag{4}$$

where δ is a phase. Hence,

$$\omega^2\mathbf{A} = K\mathbf{A} \quad \Rightarrow \quad \det\left(K - \omega^2\mathbb{1}\right) = 0 \tag{5}$$

where $\mathbb{1}$ is the $N \times N$ identity matrix. So, the normal mode frequencies will be given by the eigenvalues of the matrix K, and $\mathbf{A}$ will be given by the eigenvectors. Since this specifies a basis of solutions, a general solution will be of the form

$$\mathbf{q}(t) = \sum_i a_i\mathbf{A}_i\cos(\omega_i t - \delta) = \sum_i \big(\alpha_i\mathbf{A}_i\cos(\omega_i t) + \beta_i\mathbf{A}_i\sin(\omega_i t)\big) \tag{6}$$

where i runs over the number of normal modes and α_i and β_i are constants. Thus, our differential equation turns into a set of algebraic equations which we can solve given initial conditions.

2 The Quantum Oscillator

2.1 Discrete Case

Start with a ring of N identical, simple (classical) harmonic oscillators with equilibrium separation ϵ. The Hamiltonian is given by

$$H = \sum_{n=0}^{N-1}\left(\frac{p_n^2}{2m} + \frac{\alpha}{2}\big(q_{n+1} - q_n\big)^2\right) \tag{7}$$

The periodicity constraint implies $n = n + N$. To solve, take positions and momenta to Fourier space

$$p_n = \frac{1}{\sqrt{N}}\sum_{j=-N/2}^{N/2} e^{-ik_jx_n}\tilde{p}_j, \qquad q_n = \frac{1}{\sqrt{N}}\sum_{j=-N/2}^{N/2} e^{-ik_jx_n}\tilde{q}_j \tag{8}$$

where

$$k_j \equiv \frac{2\pi j}{N\epsilon}, \quad x_n \equiv n\epsilon \tag{9}$$

First, looking at the kinetic energy term,

$$T = \frac{1}{2mN}\sum_{n=0}^{N-1}\sum_{j=-N/2}^{N/2}\sum_{i=-N/2}^{N/2} e^{-ix_n(k_j+k_i)}\tilde{p}_j\tilde{p}_i = \sum_{-N/2}^{N/2}\frac{|\tilde{p}_j|^2}{2m} \tag{10}$$

where the orthonormality of the basis functions as well as $p_n \in \mathbb{R}$ were used

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{-ix_n(k_j-k_i)} = \delta_{ij}, \qquad \tilde{p}_{-j} = \tilde{p}_j^* \text{ for } p_n \in \mathbb{R} \tag{11}$$

Now, the potential energy term,

$$V = \frac{\alpha}{2N}\sum_{n=0}^{N-1}\sum_{j=-N/2}^{N/2}\sum_{i=-N/2}^{N/2}\Bigg[e^{-ix_{n+1}(k_i+k_j)}\tilde{q}_i\tilde{q}_j + e^{-ix_n(k_i+k_j)}\tilde{q}_i\tilde{q}_j \\ - \left(e^{-i(k_ix_n+k_jx_{n+1})} + e^{-i(k_jx_n+k_ix_{n+1})}\right)\tilde{q}_i\tilde{q}_j\Bigg] \tag{12}$$

Here, the symmetry of the sums over i and j are used to re-index the term in parenthesis. Focusing on this term, we use the periodicity of n to re-index the x_n's using $x_{n\pm1} = (n \pm 1)\epsilon$

$$\sum_{n=0}^{N-1}\left(e^{-i(k_ix_n+k_jx_{n+1})} + e^{-i(k_jx_{n-1}+k_ix_n)}\right) = \sum_{n=0}^{N-1} e^{-ix_n(k_i+k_j)}\left(e^{-ik_j\epsilon} + e^{ik_j\epsilon}\right) = 2N\cos(k_j\epsilon)\delta_{i,-j} \tag{13}$$

which gives

$$V = \frac{\alpha}{2}\sum_{j=-N/2}^{N/2} 2\big(1-\cos(k_j\epsilon)\big)|\tilde{q}_j|^2 \equiv \sum_{j=-N/2}^{N/2}\frac{m\omega_j^2}{2}|\tilde{q}_j|^2 \quad \text{where} \quad \omega_j^2 \equiv \frac{2\alpha}{m}\big(1-\cos(k_j\epsilon)\big) \tag{14}$$

Now, we just quantize by upgrading $\tilde{q}_j \to \tilde{Q}_j$ and $\tilde{p}_j \to \tilde{P}_j$ where $\tilde{Q}_j$ and $\tilde{P}_j$ are Hermitian operators satisfying

$$\big[\tilde{Q}_j, \tilde{P}_{j'}\big] = i\delta_{jj'} \tag{15}$$

We then define creation and annihilation operators

$$\tilde{Q}_j = \frac{1}{\sqrt{2m\omega_j}}\big(\tilde{a}_j^\dagger + \tilde{a}_j\big), \quad \tilde{P}_j = i\sqrt{\frac{m\omega_j}{2}}\big(\tilde{a}_j^\dagger - \tilde{a}_j\big) \tag{16}$$

which satisfy

$$\big[\tilde{a}_j, \tilde{a}_{j'}^\dagger\big] = \delta_{jj'}, \quad \big[\tilde{a}_j, \tilde{a}_{j'}\big] = \big[\tilde{a}_j^\dagger, \tilde{a}_{j'}^\dagger\big] = 0 \tag{17}$$

giving the Hamiltonian of N QHO's

$$H = \sum_{j=-N/2}^{N/2}\omega_j\left(\tilde{a}_j^\dagger\tilde{a}_j + \frac{1}{2}\right) \tag{18}$$

This Hamiltonian acts on the Fock space of states

$$|\psi\rangle = \bigotimes_{j=-N/2}^{N/2}|n_j\rangle \equiv \big|n_{(-N/2)}, n_{(-(N-1)/2)}, \ldots, n_{N/2}\big\rangle \tag{19}$$

where n_j is the number of quanta of energy in mode j. Now, we want to find the creation/annihilation operators in position space. We may guess they take the form

$$a_n = \frac{1}{\sqrt{N}} \sum_{j=-N/2}^{N/2} e^{-ik_j x_n} \tilde{a}_j, \quad a_n^\dagger = \frac{1}{\sqrt{N}} \sum_{j=-N/2}^{N/2} e^{ik_j x_n} \tilde{a}_j^\dagger \tag{20}$$

We check that these do indeed satisfy the correct commutation relations for creation/annihilation operators

$$\left[a_n, a_{n'}^\dagger\right] = \frac{1}{N} \sum_{j=-N/2}^{N/2} \sum_{i=-N/2}^{N/2} e^{-i(k_j x_n - k_i x_{n'})} \left[\tilde{a}_j, \tilde{a}_i^\dagger\right] = \frac{1}{N} \sum_{j=-N/2}^{N/2} e^{-ik_j(x_n - x_{n'})} = \delta_{nn'} \tag{21}$$

with all other commutation relations vanishing as expected. With this, we place a single quanta of energy in the n^{th} position by acting on the vacuum with $a_n^\dagger$

$$|\psi_n\rangle = a_n^\dagger |0\rangle = \frac{1}{\sqrt{N}} \sum_{j=-N/2}^{N/2} e^{ik_j x_n} \tilde{a}_j^\dagger |0\rangle = \frac{1}{\sqrt{N}} \sum_{j=-N/2}^{N/2} e^{ik_j x_n} |j\rangle \tag{22}$$

where $|j\rangle \equiv |0, 0, \ldots, 1, \ldots, 0, 0\rangle$ with a 1 in the j^{th} mode. Now, we want to time-evolve this state with the Hamiltonian and find the amplitude for the particle to be at some different position, m. Neglecting the zero-point energy, the energy of the single-quanta state j is given by

$$H |j\rangle = E_j |j\rangle = \omega_j |j\rangle \tag{23}$$

So, our desired amplitude is given by

$$\langle\psi_m| e^{-iHt} |\psi_n\rangle = \frac{1}{N} \sum_{j=-N/2}^{N/2} e^{ik_j(x_n - x_m)} e^{-i\omega_j t} = \frac{1}{N} \sum_{j=-N/2}^{N/2} e^{i\frac{2\pi j}{N}(n-m)} e^{-i\omega_j t} \tag{24}$$

Note that, as $\epsilon \to 0$ keeping the total length $N\epsilon = L$ of the ring fixed, $k_j\epsilon \ll 1$. Then, our dispersion relation becomes

$$\omega_j^2 = \frac{2\alpha}{m}\big(1 - \cos(k_j\epsilon)\big) = \frac{\alpha}{m}(k_j\epsilon)^2 + O\big((k_j\epsilon)^4\big) \tag{25}$$

in which case, the amplitude becomes

$$\langle\psi_m| e^{-iHt} |\psi_n\rangle = \frac{1}{N} \sum_{j=-N/2}^{N/2} e^{i\frac{2\pi j}{N}\left[(n-m) \pm \sqrt{\alpha/m}\, t\right]} = \delta_{(n-m), \pm\sqrt{\alpha/m}\, t} \tag{26}$$

In other words, the distance the quanta of energy is expected to travel is given exactly by

$$d = n - m = \pm\sqrt{\frac{\alpha}{m}} t \tag{27}$$

which is, of course the equation of a particle travelling at speed $c_s = \sqrt{\alpha/m}$!

2.2 Continuum Case

Let's study this more explicitly in the continuum case. When we take $\epsilon \to 0$, we write our Hamiltonian as

$$H = \sum_{n=0}^{N-1} \epsilon \left(\frac{p_n^2}{2m\epsilon} + \frac{\alpha\epsilon}{2} \left(\frac{q_{n+1} - q_n}{\epsilon} \right)^2 \right) \tag{28}$$

Now, when we take the limit, our position and momentum variables become functions of x and t, the term in parenthesis becomes a derivative, and the sum is upgraded to an integral over position. Defining new fields and constants

$$\pi(x,t) = \epsilon p(x,t), \quad \phi(x,t) = q(x,t), \quad \rho = \frac{m}{\epsilon}, \quad \beta = \epsilon\alpha \tag{29}$$

giving the Hamiltonian

$$H = \int_0^L dx\left(\frac{\pi(x)^2}{2\rho} + \frac{\beta}{2}\left(\frac{\partial\phi}{\partial x}\right)^2\right) \tag{30}$$

Again, transforming to Fourier space (keeping in mind that we must use a discrete Fourier transform due to the periodic boundary conditions)

$$\pi(x,t) = \frac{1}{\sqrt{L}}\sum_{k=-\infty}^{\infty} e^{-ikx}\tilde{\pi}(k,t), \quad \phi(x,t) = \frac{1}{\sqrt{L}}\sum_{k=-\infty}^{\infty} e^{-ikx}\tilde{\phi}(k,t) \tag{31}$$

Plugging these into H,

$$\begin{aligned} H =& \frac{1}{L}\int_0^L dx \sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty} e^{-ix(k+k')}\left[\frac{\tilde{\pi}(k)\tilde{\pi}(k')}{2\rho} + \frac{\beta}{2}(-ik)(-ik')\tilde{\phi}(k)\tilde{\phi}(k')\right] \\ =& \sum_{k=-\infty}^{\infty}\left[\frac{|\tilde{\pi}(k)|^2}{2\rho} + \frac{\beta k^2}{2}|\tilde{\phi}(k)|^2\right] \equiv \sum_{k=-\infty}^{\infty}\left[\frac{|\tilde{\pi}(k)|^2}{2\rho} + \frac{\rho\omega(k)^2}{2}|\tilde{\phi}(k)|^2\right] \end{aligned} \tag{32}$$

where

$$\omega(k)^2 = \frac{\beta}{\rho}k^2 \equiv c_s^2 k^2 \tag{33}$$

We again play the same game with upgrading position/momentum variables to operators and defining position and momentum space creation/annihilation operators such that

$$a(x) = \frac{1}{\sqrt{L}}\sum_{k=-\infty}^{\infty} e^{-ikx}\tilde{a}(k), \quad a(x)^\dagger = \frac{1}{\sqrt{L}}\sum_{k=-\infty}^{\infty} e^{ikx}\tilde{a}(k)^\dagger \tag{34}$$

It is easy to see that these satisfy the correct commutation relations, as before.

Again, we create a single-quanta state at point x and want to find the amplitude of finding it at point y after time t. This amplitude is now given by

$$\langle\psi(y)|\, e^{-iHt}\,|\psi(x)\rangle = \frac{1}{L}\sum_{k=-\infty}^{\infty} e^{ik(x-y)}e^{-i\omega(k)t} = \frac{1}{L}\sum_{k=-\infty}^{\infty} e^{ik\left[(x-y)\pm c_s t\right]} = \delta\big((x-y)\pm c_s t\big) \equiv \delta(d \pm c_s t) \tag{35}$$

which, of course, describes a point particle travelling around the ring at the speed of sound c_s (a phonon). It seems that a theory of quantum fields describes the movement of particles!