Discrete Math

Relations

1) List the ordered pairs in the relation R from
$A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if

a) $a = b$.

b) $a + b = 4$.

c) $a > b$.

Answer:

a) $R=\{ (0,0),(1,1),(2,2),(3,3) \}$

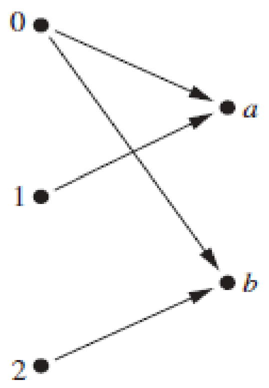
b) $R=\{ (1,3),(2,2),(3,1),(4,0) \}$

c) $R=\{ (1,0),(2,1),(2,0),(3,2),(3,1),(3,0),(4,3),(4,2),(4,1),(4,0) \}$

2) Let A = {0, 1, 2} and B = {a, b}.
Then {(0, a), (0, b), (1, a), (2, b)} is a relation from A to B.

Using arrows to represent ordered pairs and represent this relation is to use a table.

Answer:



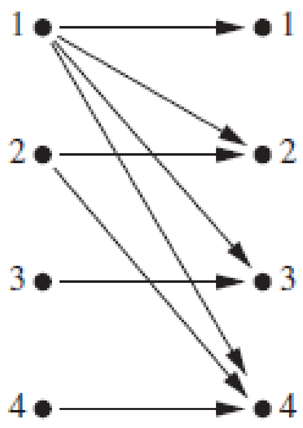
R	a	b
0	×	×
1	×	
2		×

3) Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

Using arrows to represent ordered pairs and represent this relation is to use a table.

Answer:

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}.$$



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

4) Consider the relation R={(a,b)|a divides b} on the set {1,2,3,4,5,6}
   (a) List all the ordered pairs in R.
   (b) Draw the digraph of R

Assignment

Properties of Relations

A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Which of these relations are reflexive?

Solution: The relations R_3 and R_5 are reflexive because they both contain all pairs of the form (a, a), namely, $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$. The other relations are not reflexive because they do not contain all of these ordered pairs. In particular, R_1, R_2, R_4, and R_6 are not reflexive because $(3, 3)$ is not in any of these relations. ◀

Symmetric/Antisymmetric Relations

A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$. A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Which of the relations are symmetric and Which are antisymmetric?

Solution:

The relations R2 and R3 are symmetric, because in each case (b, a) belongs to the relation whenever (a, b) does.
For R2, the only thing to check is that both (2, 1) and (1, 2) are in the relation.
For R3, it is necessary to check that both (1, 2) and (2, 1) belong to the relation, and (1, 4) and (4, 1) belong to the relation.

R4, R5, and R6 are all antisymmetric.
For each of these relations there is no pair of elements a and b with a ≠ b such that both (a, b) and (b, a) belong to the relation.

Transitive Relations

A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Which of the relations in are transitive?

Solution:

R4, R5, and R6 are transitive.

R4 is transitive, because (3, 2) and (2, 1), (4, 2) and (2, 1), (4, 3) and (3, 1), and (4, 3) and (3, 2) are the only such sets of pairs, and (3, 1), (4, 1), and (4, 2) belong to R4.

R1 is not transitive because (3, 4) and (4, 1) belong to R1, but (3, 1) does not.
R2 is not transitive because (2, 1) and (1, 2) belong to R2, but (2, 2) does not.
R3 is not transitive because (4, 1) and (1, 2) belong to R3, but (4, 2) does not.

5) For each of these relations on the set {1, 2, 3, 4}, decide whether it is **reflexive**, whether it is **symmetric**, whether it is **antisymmetric**, and whether it is **transitive**.

a) {(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}
b) {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)} "reflexive, symmetric, transitive"
c) {(2, 4), (4, 2)}
d) {(1, 2), (2, 3), (3, 4)}

Assignment

6) Let R1 = {(1, 2), (2, 3), (3, 4)} and R2 = {(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4)} be relations from {1, 2, 3} to {1, 2, 3, 4}.

Find a) **R1 ∪ R2** *"union"*

b) **R1 ∩ R2** *"intersect"*

c) **R1 – R2** *"subtract"*

d) **R2 – R1** *"subtract"*

Answer:

a) $R_1 \cup R_2 = \{(1,1),(1,2),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(3,4)\}$

b) $R_1 \cap R_2 = \{(1,2),(2,3),(3,4)\}$

c) $R_1 - R_2 = \{(1,2),(2,3),(3,4)\} - \{(1,1),(1,2),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(3,4)\}$

$= \phi$

D) $R_2 - R_1 = \{(1,1),(1,2),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(3,4)\} - \{(1,2),(2,3),(3,4)\}$

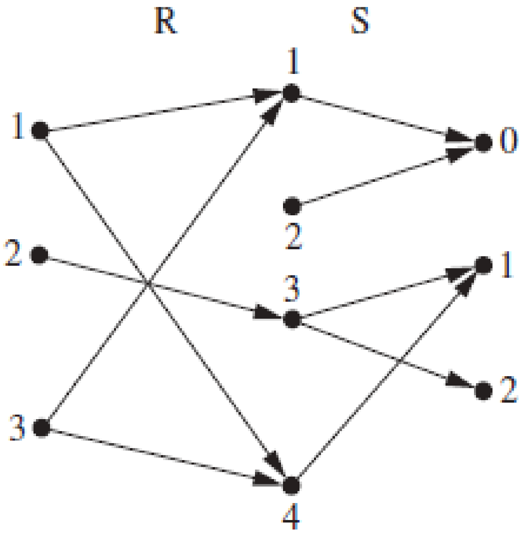
$= \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3)\}$

7) What is the composite of the relations R and S, find **S∘R**

where R is the relation from {1, 2, 3} to {1, 2, 3, 4} with **R = {(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)}**

and S is the relation from {1, 2, 3, 4} to {0, 1, 2} with **S = {(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)}**?

Answer:



$1 \to 1 \to 0$	$(1, 0)$
$1 \to 4 \to 1$	$(1, 1)$
$2 \to 3 \to 1$	$(2, 1)$
$2 \to 3 \to 2$	$(2, 2)$
$3 \to 1 \to 0$	$(3, 0)$
$3 \to 4 \to 1$	$(3, 1)$

$$S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}.$$

8) Let **R={(1,3),(2,2),(3,2)}** and **S={(2,1),(3,2),(2,3)}** be two relations on set A={1,2,3}.
Find RoS is equal

RoS ={(2,3),(3,2),(2,2)}

Assignment

Powers of a Relation

Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \ldots$, are defined recursively by

$$R^1 = R \qquad \text{and} \qquad R^{n+1} = R^n \circ R.$$

The definition shows that $R^2 = R \circ R$, $R^3 = R^2 \circ R = (R \circ R) \circ R$, and so on.

Example: Let $R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$. Find the powers R^n, $n = 2, 3, 4, \ldots$.

Solution:

$R^2 = R \circ R$, $R^2 = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$.

$R^3 = R^2 \circ R$, $R^3 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$.

$R^4 = R^3 \circ R$, $R^4 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$

9) Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

Solution: Because $R = \{(2, 1), (3, 1), (3, 2)\}$, the matrix for R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The 1s in $\mathbf{M}_R$ show that the pairs (2, 1), (3, 1), and (3, 2) belong to R. The 0s show that no other pairs belong to R. ◀

10) Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$$ ◀

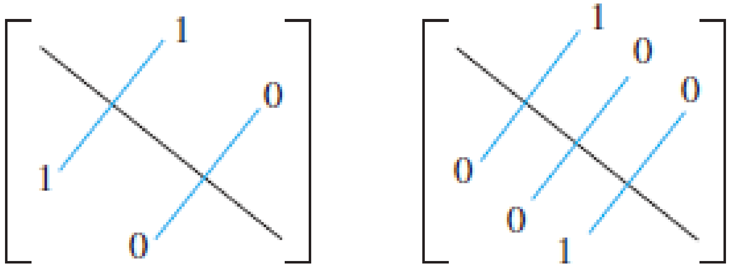
11) Represent each of these relations on {1, 2, 3} with a matrix (with the elements of this set listed in increasing order).

a) {(1, 1), (1, 2), (1, 3)}
b) {(1, 2), (2, 1), (2, 2), (3, 3)}
c) {(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)}

a) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(B,C)Assignment

The zero–one matrices for symmetric and antisymmetric relations



(a) Symmetric (b) Antisymmetric

Let R be a binary relation on a set and let M be its zero-one matrix. R is symmetric if and only if $M = M^t$. In other words, $M_{ij} = M_{ji}$ for all i and j.

Let R be a binary relation on a set and let M be its zero-one matrix. R is antisymmetric if and only if $M_{ij} = 0$ or $M_{ji} = 0$ for all $i \neq j$.

The zero–one matrix for a reflexive relation.

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

Let R be a binary relation on a set and let M be its zero-one matrix. R is reflexive if and only if $M_{ii} = 1$ for all i. In other words, all elements are equal to 1 on the main diagonal.

12) Determine whether the relations represented by the matrices are **reflexive**, **symmetric**, **antisymmetric**.

(a) $M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(a) Reflexive, Symmetric

(b) $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(b) Antisymmetric

(c) $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(c) Symmetric

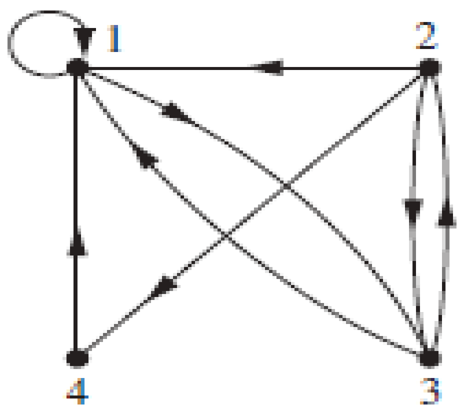
Representing Relations Using Digraphs

A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a, b), and the vertex b is called the *terminal vertex* of this edge.

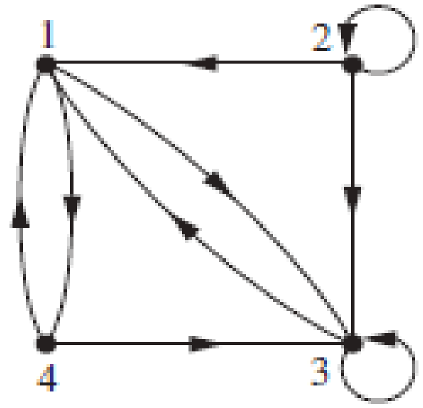
An edge of the form (a, a) is represented using an arc from the vertex a back to itself. Such an edge is called a **loop**.

Example: The directed graph of the relation $R1 = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$ on the set $\{1, 2, 3, 4\}$

Solution:



Example: What are the ordered pairs in the relation R_2 represented by the directed graph?



Solution: R_2 = {(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)}.

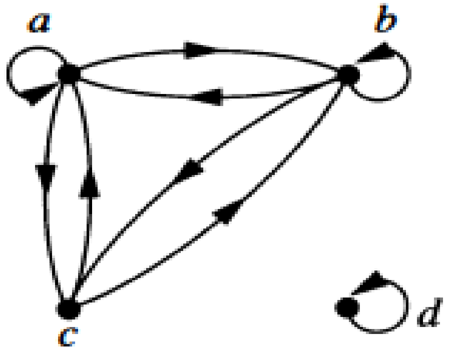
Example: Draw the directed graphs representing each of the relations

a) {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}

b) {(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)}

(B,C)Assignment

Example: List the ordered pairs in the relation R represented by the directed graph?



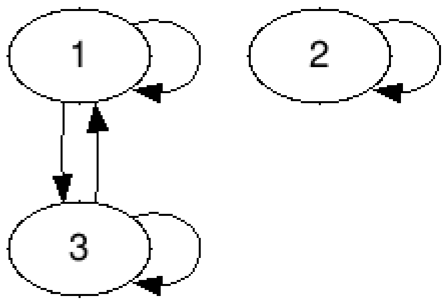
(B,C)Assignment

Example: Draw the directed graph representing each of the relations

(a) $M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Solution(a): *R*= {(1, 1), (1, 3), (2, 2), (3, 1),(3, 3)}

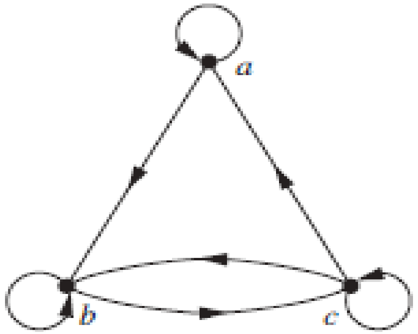
Direct graph:



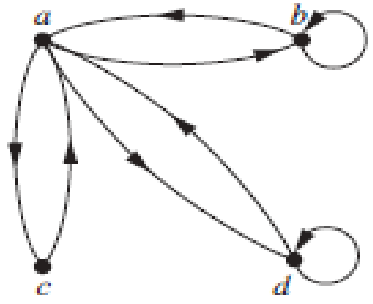
(b) $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(B)Assignment

Example: Determine whether the relations for the directed graphs of S1 and S2 are reflexive, symmetric, antisymmetric, and/or transitive.



(a) Directed graph of S_1



(b) Directed graph of S_2

Solution: **S_1 Relation:**

It is reflexive, Because there are loops at every vertex.

It is neither symmetric nor antisymmetric, Because there is an edge from a to b but not one from b to a, but there are edges in both directions connecting b and c.

It is not transitive, Because there is an edge from a to b and an edge from b to c, but no edge from a to c.

(B) Assignment

Equivalence Relations

A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Two elements a and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

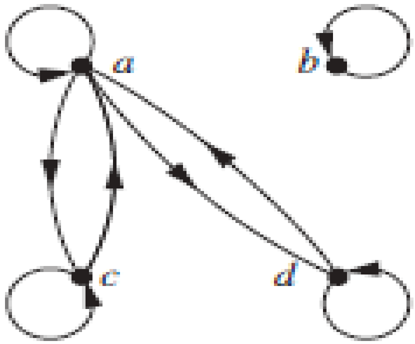
Example: Which of these relations on {0, 1, 2, 3} are equivalence relations? Determine the properties of an equivalence relation that the others lack.
a) {(0, 0), (1, 1), (2, 2), (3, 3)}
b) {(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}
c) {(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}
d) {(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}

Solution:
a) This is an equivalence relation because it is reflexive, symmetric, and transitive.
b) This is not an equivalence relation because it is neither reflexive nor transitive. Missing (1, 1) for reflexive and missing (0, 3) for the path (0, 2),(2, 3) for transitive.

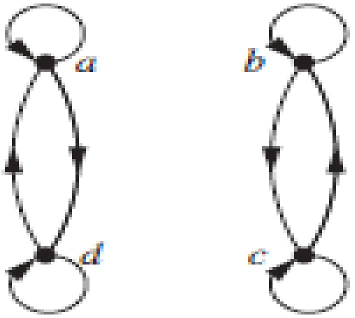
(C,D) Assignment

Example: Determine whether the relation with the directed graph shown is an equivalence relation.

a)



b)



Solution:

a) - As there is a loop at every vertex of the directed graph, so this relation is **<u>reflexive</u>**.

- Also for every edge that appears in the directed graph, there is an edge involving the same two vertices but pointing in the opposite direction. So the relation is **<u>symmetric</u>**.

- But this relation is **<u>not transitive</u>** because there is an edge from vertex c to vertex a, and an edge from vertex a to vertex d in the directed graph, but no edge from vertex c to vertex d.

Hence, this relation is **not** an equivalence relation.

(B) Assignment