Faculty of Computer Science and Information Technology

Discrete Mathematics

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Lecture 2

Logic and Proofs

Part 2

Compound Propositions

➢ **Implication**($p \rightarrow q$)

- If p and q are propositions, then $p \rightarrow q$ is a conditional statement or implication which is read as "**if p, then q** ".

"P implies q "

The Truth Table for the Conditional Statement $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Compound Propositions

- **Implication**($p \rightarrow q$)

Example 1

Let p be the statement "Maria learns discrete mathematics" and q the statement "Maria will find a good job." Express the statement $p \rightarrow q$ as a statement in English.

"If Maria learns discrete mathematics, then she will find a good job."

"Maria will find a good job when she learns discrete mathematics."

Example 2

"If today is Friday, then $2 + 3 = 6$."

is true every day except Friday, even though $2 + 3 = 6$ is false.

Compound Propositions

- **Implication**($p \to q$)

The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise.

- In $p \to q$, p is the hypothesis and q is the conclusion.

"if p, then q"
"if p, q"
"p is sufficient for q"
"q if p"
"q when p"
"a necessary condition for p is q"
"q unless $\neg p$"

"p implies q"
"p only if q"
"a sufficient condition for q is p"
"q whenever p"
"q is necessary for p"
"q follows from p"

Compound Propositions

➢ **Biconditional**($p \leftrightarrow q$)

- If p and q are propositions, then we can form the biconditional proposition $p \leftrightarrow q$, read as "**p if and only if q .**"

"p iff q"

The Truth Table for the Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Compound Propositions

- **Biconditional($p \leftrightarrow q$)**

The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called *bi-implications*.

- **Expressing the Biconditional:**

• $p \leftrightarrow q$ biconditional "if p then q , and if q then p"

• $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

Example:

" you can take the flight if and only if you buy a ticket."

Compound Propositions

➢ **Truth Tables of Compound Propositions**

Example 1

- Construct the truth table of the Compound Proposition $(p \vee \neg q) \rightarrow (p \wedge q)$

The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Compound Propositions

- **Precedence of Logical Operators**

$p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$

If the intended meaning is $p \vee (q \rightarrow \neg r)$

then parentheses must be used.

Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

Compound Propositions

➢ Truth Tables of Compound Propositions

Example 2

- Construct the truth table for $p \vee q \rightarrow \neg r$

p	q	r	$\neg r$	$p \vee q$	$p \vee q \rightarrow \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

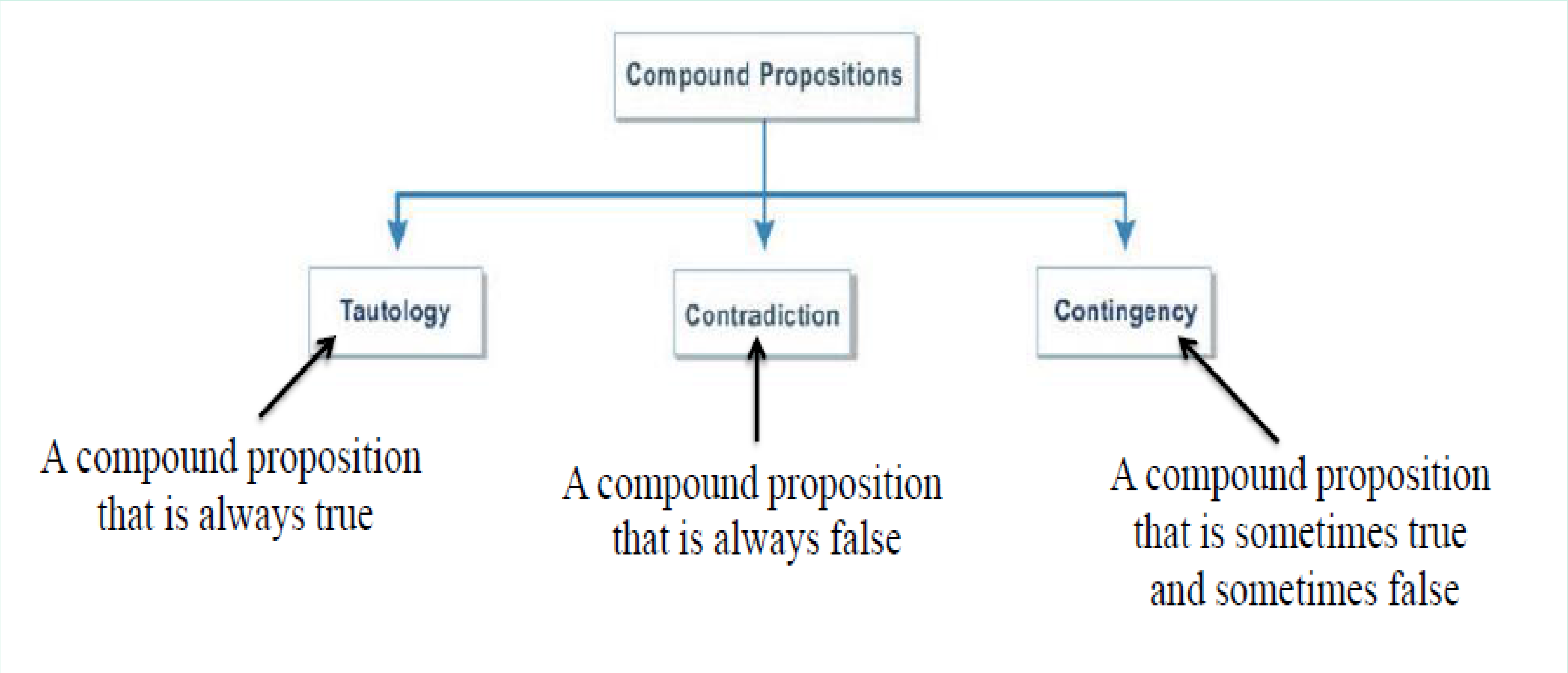
Compound Propositions

Example 3

• Construct the truth table of the Compound Proposition $(p \wedge \neg q) \rightarrow r$

p	q	r	$\neg q$	$p \wedge \neg q$	$(p \wedge \neg q) \rightarrow r$
T	T	T	F	F	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	F	T
F	F	F	T	F	T

Compound Propositions Classification



Compound Propositions Classification

Example 4:

- Show that following conditional statement is a tautology by using truth table

$(p \land q) \rightarrow p$

p	q	$p \land q$	$(p \land q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Compound Propositions Classification

- **A tautology** is a proposition which is always true.

 Example: $p \vee \neg p$

- **A contradiction** is a proposition which is always false. Example: $p \wedge \neg p$

- **A contingency** is a proposition that is sometimes true and sometimes false, such as p

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logical Equivalences

➢ Logically equivalent:

- We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.
- Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree.

Logical Equivalences

- **Example 5:**

Show that $\neg p \vee q$ and $p \to q$ are Logically equivalent .

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Logical Equivalences

- **Example 6:**
- Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are Logically equivalent .

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Logical Equivalences

- **De Morgan's Laws**

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

This truth table shows that De Morgan's **first** Law holds.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg p$	$\neg p \vee \neg p$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Logical Equivalences

- ## De Morgan’s Laws

This truth table shows that De Morgan’s Second Law holds.

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Logical Equivalences

➢ **Using a Truth Table to Show Nonequivalence**

Show that $p \rightarrow q$ and $\neg p \rightarrow \neg q$ are not equivalent .

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \rightarrow \neg q$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

Applications of Propositional Logic

Translating English Sentences

- ➢ Steps to convert an English sentence to a statement in propositional logic

- Identify atomic propositions and represent using propositional variables.

- Determine appropriate logical connectives

Example 7

You can access the Internet from campus only if you are a computer science major or you are not a student.

Applications of Propositional Logic

Translating English Sentences

Example 7

(You can access the Internet from campus) only if (you are a computer science major or you are not a student).

$p \to q$
"p only if q"

Solution:

Let p, q and r be the propositions:

p: You can access the Internet from campus.

q: You are a computer science major.

r: You are a student.

<u>The sentence can be represented by logic as</u>

$$\boxed{p \to (q \lor \neg r)}$$

Applications of Propositional Logic

Translating English Sentences

Example 8

(The automated reply cannot be sent) when (the file system is full.)

$p \to q$

"q when p"

Solution:

Let p and q be the propositions:

p: The automated reply can be sent .

q: The file system is full.

The sentence can be represented by logic as

$$\boxed{q \to \neg p}$$

Applications of Propositional Logic

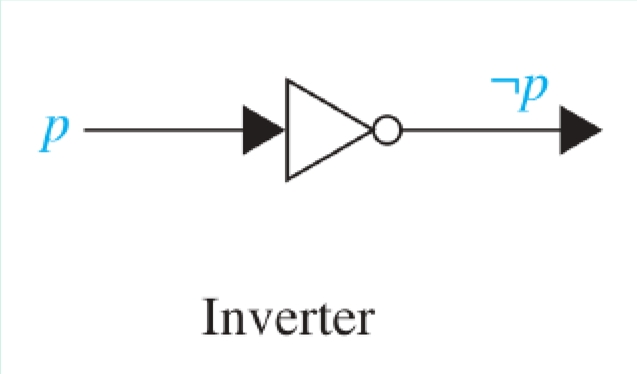
Logic Circuits

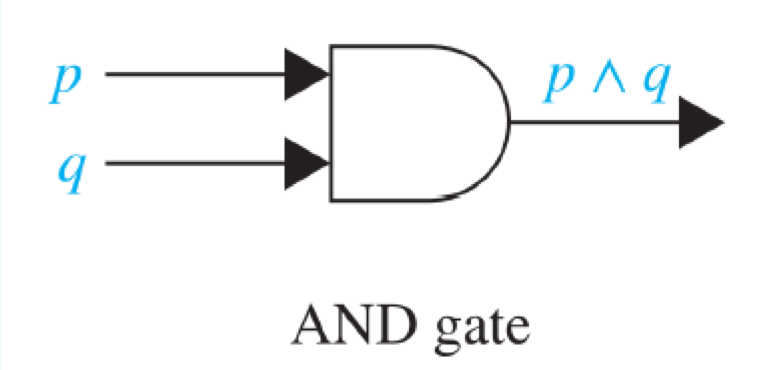
- A logic circuit (or digital circuit) receives input signals $p1, p2, \ldots , pn$, each a bit [either 0 (off) or 1 (on)], and produces output signals $s1, s2, \ldots , sn$, each a bit.

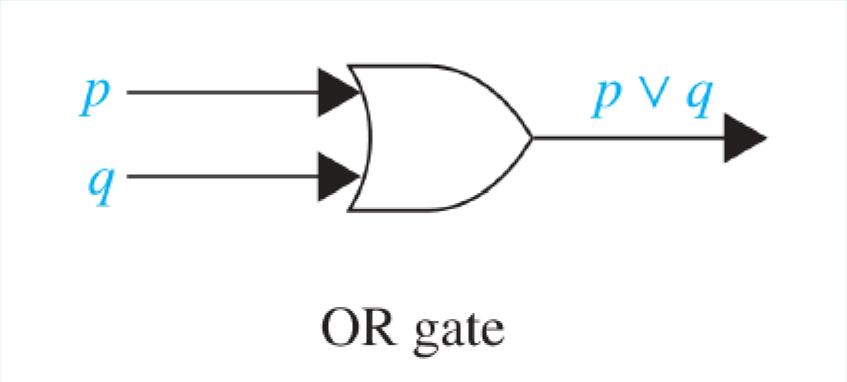
Applications of Propositional Logic

Logic Circuits

The three basic circuits, called gates.





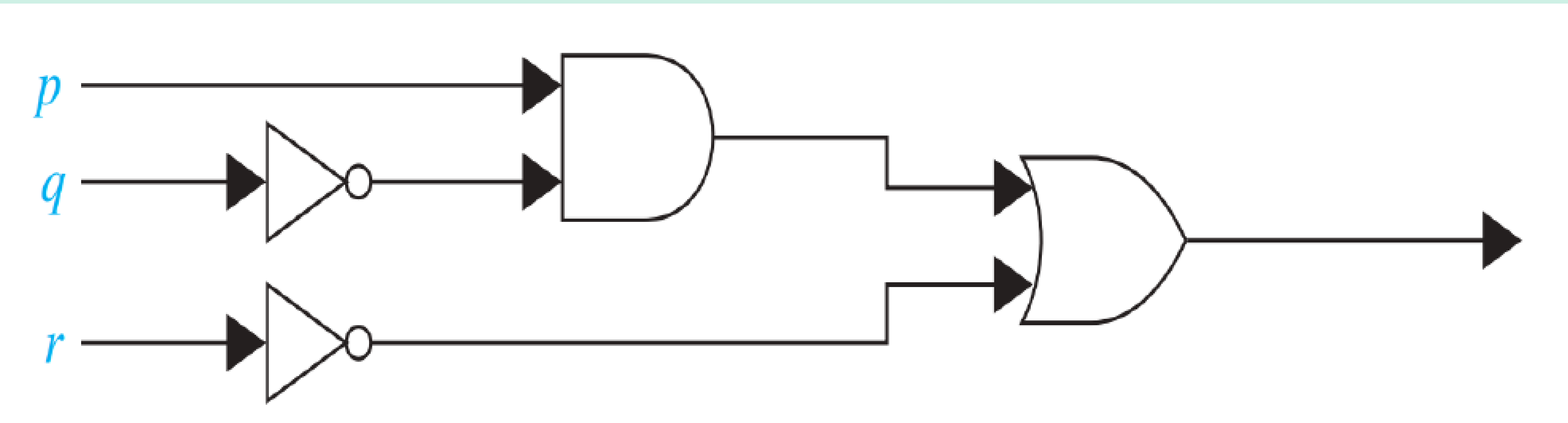


Applications of Propositional Logic

Logic Circuits

Example 9

- Determine the output for the combinatorial circuit in the following figure.

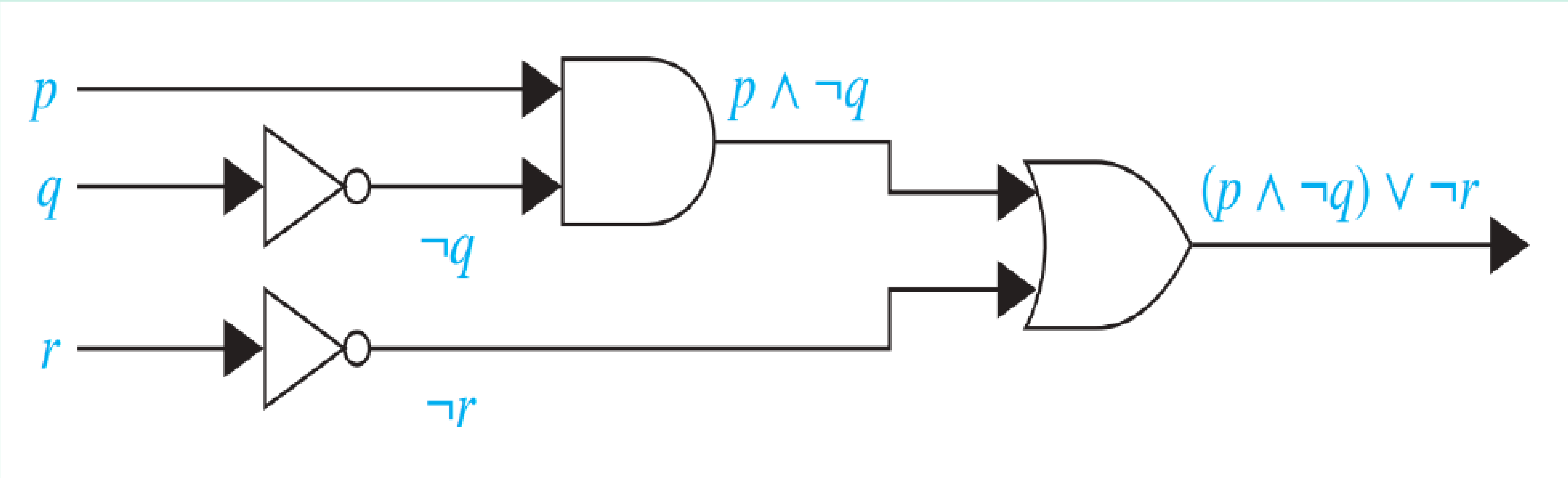


Applications of Propositional Logic

Logic Circuits

Example 9

Solution:

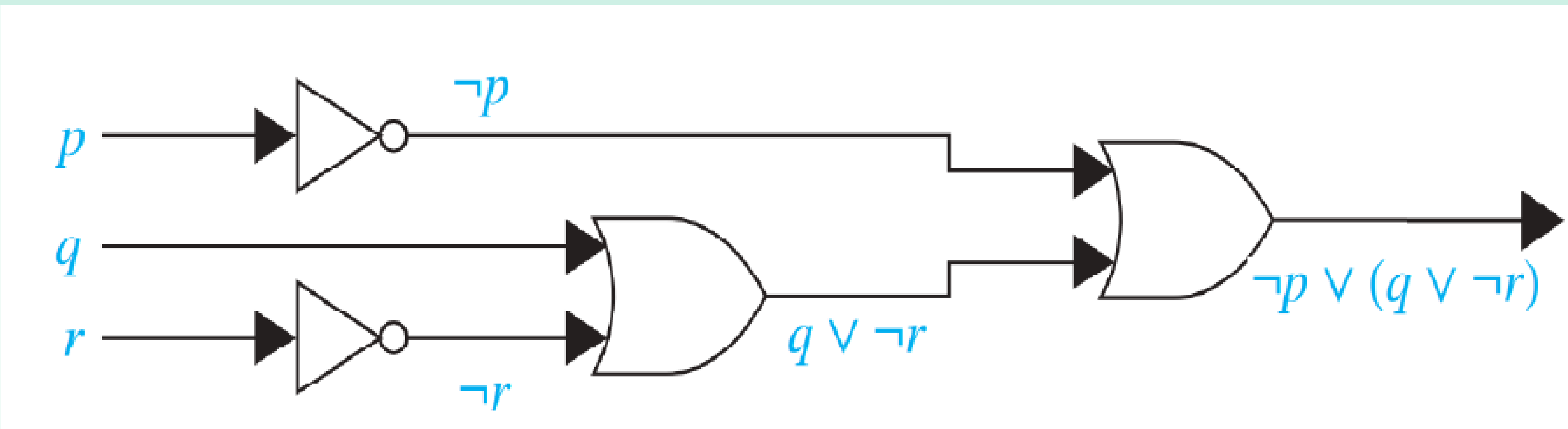


Applications of Propositional Logic

Logic Circuits

Example 10

- **Determine the output for the combinatorial circuit in the following figure.**

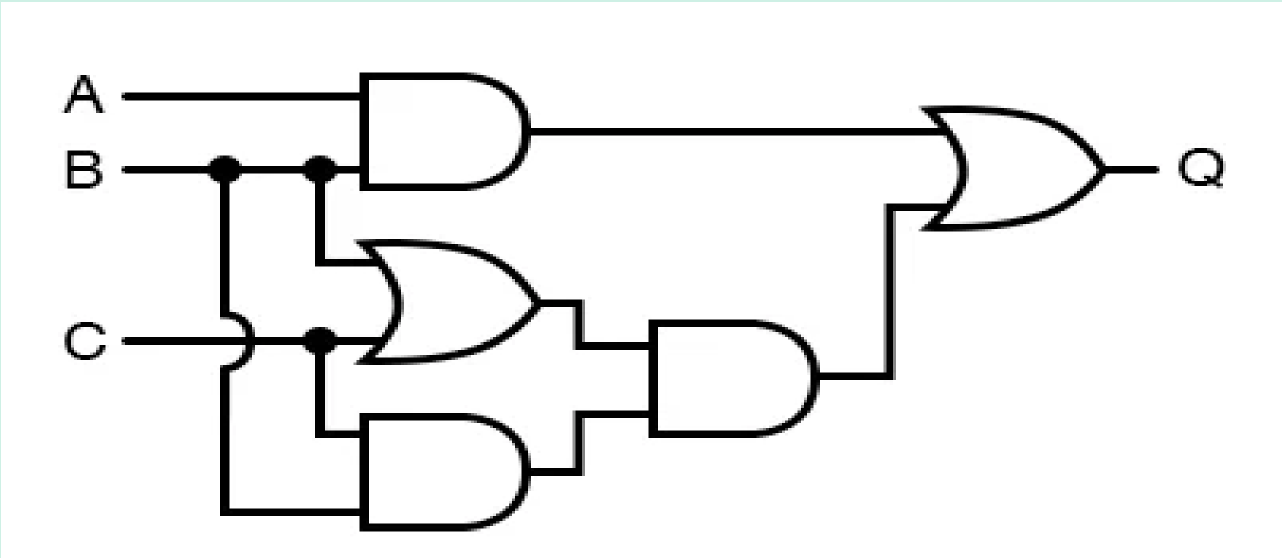


Applications of Propositional Logic

Logic Circuits

Example 11

- **Determine the output for the combinatorial circuit in the following figure.**



$[A \wedge B] \vee [\,(B \vee C) \wedge (B \wedge C)\,]$

Questions?