Discrete Math

Proofs

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1) Use a **direct proof** to show that the sum of two odd integers is even.

2) Use a **direct proof** to show that the sum of two even integers is even.

Assignment

3) Show that the square of an even number is an even number **using a direct proof**.

5) Prove that if $m + n$ and $n + p$ are even integers, where m, n, and p are integers, then $m + p$ is even. **using a direct proof**.

Mathematical Induction

PRINCIPLE OF MATHEMATICAL INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.

Use mathematical induction

3. Let P (n) be the statement that $1^2 + 2^2 + \cdots + n^2 = n(n + 1)(2n + 1)/6$ for the positive integer n.

a) What is the statement P(1)?

b) Show that P(1) is true, completing the **basis step** of a proof that P(n) is true for all positive integers n.

c) What is the **inductive hypothesis** of a proof that P(n) is true for all positive integers n? "Assume p(n) for n >= 1 is true"

d) What do you need to prove in the **inductive step** of a proof that P(n) is true for all positive integers n? "prove that p(n+1) is true"

e) Complete the inductive step of a proof that P(n) is true for all positive integers n, **identifying where you use the inductive hypothesis**.

A) 1^2 = (1*(1+1)*(2(1)+1))/6

B) 1=1

C) Assume for all positive integers n P(n): 1^2+2^2+....+n^2=(n(n+1)(2n+1))/6 is true

D) P(n+1): 1^2+2^2+....+n^2+(n+1)^2=((n+1)(n+2)(2n+3))/6

E) P(n+1) is true

E)

Consider

P(n):1^2+2^2+....+n^2+(n+1)^2=(n(n+1)(2n+1))/6 + (n+1)^2

Focusing on RHS

=(n(n+1)(2n+1)+6(n+1)^2)/6
=(n+1)(n(2n+1)+6(n+1))/6
=(n+1)(2n^2+n+6n+6)/6
=(n+1)(2n^2+7n+6)/6

Use +4, +3 as factors for 2n^2+7n+6 to get
=((n+1)(n+2)(2n+3))/6

which is equal to P(n+1)

Use mathematical induction

4. Let P (n) be the statement that $1^3 + 2^3 + \cdots + n^3 = (n(n + 1)/2)^2$ for the positive integer n.

a) What is the statement P(1)?

b) Show that P(1) is true, completing the basis step of the proof of P(n) for all positive integers n.

c) What is the inductive hypothesis of a proof that P(n) is true for all positive integers n?

d) What do you need to prove in the inductive step of a proof that P(n) is true for all positive integers n?

e) Complete the inductive step of a proof that P(n) is true for all positive integers n, identifying where you use the inductive hypothesis.

Assignment

5. Prove that $1^2 + 3^2 + 5^2 + \cdots + (2n + 1)^2 = (n + 1)(2n + 1)(2n + 3)/3$ whenever n is a nonnegative integer.

Basis step $n = 0$

$$1^2 + 3^2 + 5^2 + ... + (2n+1)^2 = 1^2 = 1$$
$$\frac{(n+1)(2n+1)(2n+3)}{3} = \frac{(0+1)(2(0)+1)(2(0)+3)}{3} = \frac{(1)(1)(3)}{3} = \frac{3}{3} = 1$$

We then note $P(0)$ is true.

Induction step Let $P(k)$ be true.

$$1^2 + 3^2 + 5^2 + ... + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$$

We need to prove that $P(k+1)$ is also true.

$$\begin{aligned}
&1^2 + 3^2 + 5^2 + ... + (2k+1)^2 + (2(k+1)+1)^2 \\
&= 1^2 + 3^2 + 5^2 + ... + (2k+1)^2 + (2k+3)^2 \\
&= \frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2 \\
&= \left(\frac{(k+1)(2k+1)}{3} + (2k+3)\right)(2k+3) && \text{Factor out } (2k+3) \\
&= \left(\frac{(k+1)(2k+1)}{3} + \frac{3(2k+3)}{3}\right)(2k+3) \\
&= \left(\frac{(k+1)(2k+1) + 3(2k+3)}{3}\right)(2k+3) \\
&= \left(\frac{2k^2 + 2k + k + 1 + 6k + 9}{3}\right)(2k+3) && \text{Use distributive propert} \\
&= \left(\frac{2k^2 + 9k + 10}{3}\right)(2k+3) && \text{Combine like terms} \\
&= \left(\frac{(k+2)(2k+5)}{3}\right)(2k+3) && \text{Factorize numerator} \\
&= \frac{(k+2)(2k+3)(2k+5)}{3} \\
&= \frac{((k+1)+1)(2(k+1)+1)(2(k+1)+3)}{3}
\end{aligned}$$

We then note that $P(k+1)$ is also true.

7. Prove that

$$3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^n=3(5^{n+1}-1)/4$$

whenever n is a nonnegative integer.

Assignment

10. a) Find a formula for

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}$$

by examining the values of this expression for small values of n.

b) Prove the formula you conjectured in part (a).

(a) Given:

$$f(n) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n+1)}$$

We will evaluate the first few values of n and look for a pattern:

$$\begin{array}{ll} n = 1 & \frac{1}{1 \cdot 2} = \frac{1}{2} \\ n = 2 & \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{3}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3} \\ n = 3 & \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{6}{12} + \frac{2}{12} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4} \end{array}$$

We then note a pattern:

$$f(n) = \frac{n}{n+1}$$

(b) To proof: $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for every positive integer n.

PROOF BY INDUCTION

Let $P(n)$ be $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

B, Assignment

Use mathematical induction to prove the inequalities

18. Let P(n) be the statement that *where n is an integer* greater than 1.

$$n! < n^n,$$

a) What is the statement *P(2)?, then show that P(2) is true, completing the **basis step.***

b) What is the **inductive hypothesis** of a proof by mathematical induction that *P(n) is true for all integers n* greater than 1? "let p(k) be true"

c) What do you need to prove in the **inductive step** of a proof by mathematical induction that *P(n) is true for* all integers *n greater than 1?, Then* Complete the inductive step.

"need to prove that p(k+1) is also true", "$(k+1)! < (k+1)^{K+1}$"

$$\begin{aligned}(k+1)! &= (k+1)\cdot k! \\ &< (k+1)\cdot k^k \\ &< (k+1)\cdot (k+1)^k && \text{Since } k < k+1 \\ &= (k+1)^{k+1}\end{aligned}$$

We then note that $P(k+1)$ is also true.