Faculty of Computers and

Artificial intelligence

Discrete Mathematics

Lecture 7

Relations

Relations and Their Properties

❑ Introduction

$\overset{2}{A} \times \overset{3}{B} = \{(a,b), (,)$

- Relationships between elements of sets are represented using the structure called a **relation**, which is just a subset of the Cartesian product of the sets.

- The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements. For this reason, sets of ordered pairs are called binary relations.

Relations and Their Properties

❑ Definition 1:

Let A and B be sets. A *binary relation* from A to B is a subset of $A \times B$.

$A \times B = \{(a,b)\}$

A *binary relation* from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.

$a\,R\,b$

We use the notation $a\,R\,b$ to denote that $(a,b) \in R$ and $a \not{R}\, b$ to denote that $(a,b) \notin R$. Moreover, when (a,b) belongs to R, a is said to be related to b by R.

$\in$

$R = \{(a,b)\}$ $\quad$ $a\,R\,b$

Relations and Their Properties

❑ Example 1:

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$.

Then $\{(\mathbf{0}, \boldsymbol{a}), (\mathbf{0}, \boldsymbol{b}), (\mathbf{1}, \boldsymbol{a}), (\mathbf{2}, \boldsymbol{b})\}$ is a relation from A to B.

Roster notation (Roster form of set):

$$R = \{(0, a), (0, b), (1, a), (2, b)\}$$

$A \times B = \{(0,a), (0,b), (1,a), (1,b), (2,a), (2,b)\}$

R_2 R_3 R_1 $R_4 = \phi$

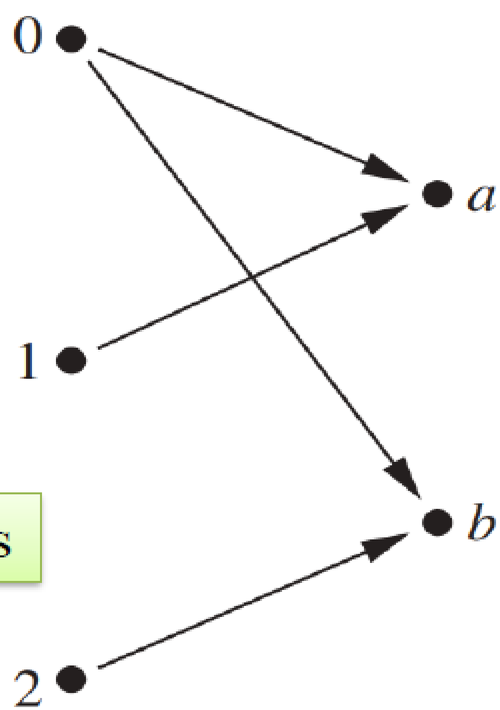
Relations and Their Properties

❑ Example 1:

We can represent relations from a set A to a set B graphically or using a table:

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$.

Then $\{(\mathbf{0}, \boldsymbol{a}), (\mathbf{0}, \boldsymbol{b}), (\mathbf{1}, \boldsymbol{a}), (\mathbf{2}, \boldsymbol{b})\}$ is a relation from A to B.



Using arrows

B

A

R	a	b
0	×	×
1	×	
2		×

Using table

Relations and Their Properties

Example: Let A={a,b,c} and B={1,2,3}.

- Is R={(a,1),(b,2),(c,2)} a relation from A to B? **Yes.**
- Is Q={(1,a),(2,b)} a relation from A to B? **No.**
- Is P={(a,a),(b,c),(b,a)} a relation from A to A? **Yes**

❑ Functions as Relations

Recall that a function f from a set A to a set B assigns exactly one element of B to each element of A. The graph of f is the set of ordered pairs (a, b) such that $b = f(a)$. Because the graph of f is a subset of $A \times B$, it is a relation from A to B.

Relations and Their Properties

A

❑ Relations on a Set

$A \longrightarrow B \quad (a,b)$

$A \rightarrow A$

<u>Definitions:</u>

- A relation on the set A is a relation from A to A. In other words, a relation on a set A is a subset of $A \times A$.
- The identity relation I_A on a set A is the set $\{(a,a) | a \in A\}$
  - Ex. If $A = \{1,2,3\}$, then $I_A = \{(1,1),(2,2),(3,3)\}$

Relations and Their Properties

❑ Example 2:

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) | a \text{ divides } b\}$?

condition $\longrightarrow \frac{b}{a}$ ✓ $r = 0$

Set builder notation:

$R = \{(a, b) | a \text{ divides } b\}$

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) | \underline{a \text{ divides } b}\}$?

$1 | 2$

May change to be:

$a = b$

$a > b$

$a < b$

...

$A = \{1, 2, 3, 4\}$ $A = \{1, 2, 3, 4\}$

$A \times A$

$= \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2),$
$(2,3), (2,4), (3,1), (3,2), (3,3),$
$(3,4), (4,1), (4,2), (4,3), (4,4)\}$

Relations and Their Properties

- **Example 2:**

Solution:

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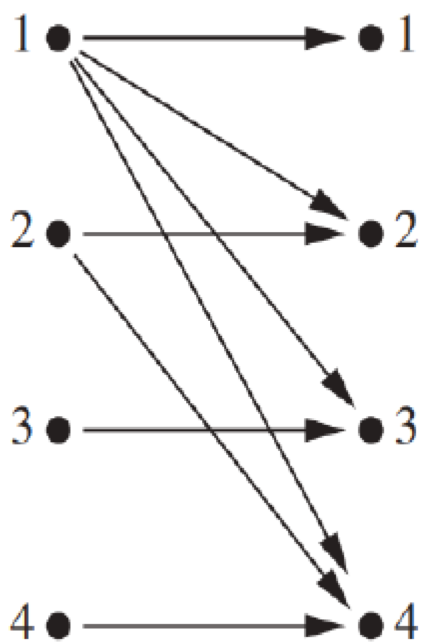
$(1,2) \neq (2,1)$

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$$

A×A

16



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Relations and Their Properties

Example 3:

Let A be the set $\{-1, 0, 1, 2\}$. Which ordered pairs are in the following relations:

$R_1 = \{(a, b) | a < b\}$ $= \{(-1, 0)$

$R_2 = \{(a, b) | a > b\}$

$R_3 = \{(a, b) | a = b\}$

$R_4 = \{(a, b) | a = -b\}$

$R_5 = \{(a, b) | a = b \text{ or } a = -b\}$

$R_6 = \{(a, b) | 0 \le a + b \le 1\}$

$A = \{-1, 0, 1, 2\}$ $A = \{-1, 0, 1, 2\}$

$A \times A = \{(-1,-1), (-1,0), (-1,1), (-1,2), (0,-1), (0,0), (0,1), (0,2), (1,-1), (1,0), (1,1), (1,2), (2,-1), (2,0), (2,1), (2,2)\}$

Relations and Their Properties

- **Example 3:** **Solution:**

Let A be the set $\{-1, 0, 1, 2\}$. Which ordered pairs are in the following relations:

$$R_1 = \{(a\,,b)|a < b\}$$

$$= \{(-1, 0), (-1, 1), (-1, 2), (0, 1), (0, 2), (1, 2)\}$$

$$R_2 = \{(a\,,b)|a > b\}$$

$$= \{(0, -1), (1, 0), (1, -1), (2, 1), (2, 0), (2, -1)\}$$

Relations and Their Properties

- **Example 3: <u>Solution:</u>**

Let A be the set $\{-1, 0, 1, 2\}$. Which ordered pairs are in the following relations:

$$R_3 = \{(a, b) | a = b\} = \{(-1, -1), (0, 0), (1, 1), (2, 2)\}$$

$$R_4 = \{(a, b) | a = -b\} = \{(-1, 1), (0, 0), (1, -1)\}$$

$a = -b$
$-a = b$

$$R_5 = \{(a, b) | a = b \overset{\text{U}}{\text{ or }} a = -b\}$$

$$= \{(-1, -1), (0, 0), (1, 1), (2, 2), (-1, 1), (1, -1)\}$$

Relations and Their Properties

- **Example 3: <u>Solution:</u>**

Let A be the set $\{-1, 0, 1, 2\}$. Which ordered pairs are in the following relations:

$[0, 1]$

$$R_6 = \{(a, b) | 0 \leq a + b \leq 1\}$$

$$= \{(-1, 1), (-1, 2), (0, 0), (0, 1), (1, -1), (1, 0), (2, -1)\}$$

Relations and Their Properties

- **Example 4:**

$|A| = 3$ $A = \{1, 2, 3\}$ R_1 R_2 R_3

How many relations are there on a set with n elements?

It is not hard to determine the number of relations on a finite set, because a relation on a set A is simply a **subset** of $A \times A$.

Note: $|A \times A| = |A|^2 = n^2$

$|A| \times |A|^2$ $|A| = n$

Solution:

$|A| = 3$ $A = \{1, 2, 3\}$ $\boxed{2^9}$

A relation on a set A is a subset of $A \times A$. Because $A \times A$ has n^2 elements when A has n elements, there are 2^{n^2} subsets of $A \times A$.

Relations and Their Properties

- **Properties of Relations**

- There are several properties that are used to classify **relations on a set**. We will introduce the most important of these relations.

- Reflexive
- Irreflexive
- Symmetric
- Antisymmetric
- Transitive

Relations and Their Properties

❑ **Reflexive and Irreflexive**

A relation R on a set A is called ***reflexive*** if $(a, a) \in R$ for every element $a \in A$.

A relation R on a set A is called ***irreflexive*** if $(a, a) \notin R$ for every element $a \in A$.

not reflexive ≠ irreflexive

Relations and Their Properties

- **Example 1:**

Consider the following relations on $\{1,2,3,4\}$ are reflexive or irreflexive or not?

$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$,

$R_2 = \{(1,1),(1,2),(2,1)\}$,

$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$,

$R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$,

$R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$,

$R_6 = \{(3,4)\}$.

Relations and Their Properties

❑ Example 1:

Consider the following relations on $\{1, 2, 3, 4\}$ are reflexive or irreflexive or not?

Solution: R_3 and R_5 are reflexive

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$\boxed{R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},}$

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$\boxed{R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},}$

$R_6 = \{(3, 4)\}$.

Relations and Their Properties

❑ Example 1:

Consider the following relations on $\{1, 2, 3, 4\}$ are reflexive or irreflexive or not?

Solution: R_3 and R_5 are reflexive | R_4 and R_6 are irreflexive

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$\boxed{R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$\boxed{R_6 = \{(3, 4)\}}$.

Relations and Their Properties

❑ Example 1:

Consider the following relations on $\{1, 2, 3, 4\}$ are reflexive or irreflexive or not?

Solution:

R_3 and R_5 are reflexive

R_4 and R_6 are irreflexive

R_1 and R_2 are
Not reflexive
Not irreflexive

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Relations and Their Properties

- **Example 2:**

1, 2, 3, ---

Is the "divides" relation on the set of positive integers reflexive?

$a \mid b$ (a,b) $(a,a) \in R$

Solution:

$1 \mid 1$

Because $a|a$ whenever a is a positive integer, the "divides" relation is **reflexive**.

$a|a$ $2|2$ $3|3$

Example 3:

-2, -1, 0, 1, 2, ---

Is the "divides" relation on the set of integers reflexive?

$a|a$

$0|0$ $\boxed{\frac{0}{0}}$

Solution:

The relation is **not reflexive** because 0 does not divide 0.

Relations and Their Properties

- **Example 4:**

Is the following relations on the integers are reflexive or not?

$$R_1 = \{(a, b) | a \leq b\}$$

$$R_2 = \{(a, b) | a > b\}$$

$$R_3 = \{(a, b) | a = b\}$$

$$R_4 = \{(a, b) | a = b + 1\}$$

$$R_5 = \{(a, b) | a = b \text{ or } a = -b\}$$

$$R_6 = \{(a, b) | a + b \leq 3\}$$

Relations and Their Properties

- **Example 4:**

Is the following relations on the integers are reflexive or not?

Solution: R_1, R_3, and R_5 are reflexive

$\boxed{R_1 = \{(a, b) | a \le b\}}$ ✓

$R_2 = \{(a, b) | a > b\}$

$\boxed{R_3 = \{(a, b) | a = b\}}$ ✓

$R_4 = \{(a, b) | a = b + 1\}$

$\boxed{R_5 = \{(a, b) | a = b \text{ or } a = -b\}}$ ✓

$R_6 = \{(a, b) | a + b \le 3\}$

Relations and Their Properties

- Example 4:

Solution:

R_1, R_3, and R_5 are reflexive

R_2, R_4, and R_6 are not reflexive

$$R_1 = \{(a, b) | a \leq b\}$$

$$\boxed{R_2 = \{(a, b) | a > b\}} \qquad (\text{Counter example, } 2 \ngtr 2)$$

$$R_3 = \{(a, b) | a = b\}$$

$$\boxed{R_4 = \{(a, b) | a = b + 1\}} \qquad (\text{Counter example, } 2 \neq 2 + 1)$$

$$R_5 = \{(a, b) | a = b \text{ or } a = -b\}$$

$$\boxed{R_6 = \{(a, b) | a + b \leq 3\}} \qquad (\text{Counter example, } 2 + 2 \nleq 3)$$

Relations and Their Properties

❑ Symmetric and Antisymmetric

$\{(1,2) \ldots\ldots (2,1)\}$

A relation R on a set A is called ***symmetric*** if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.

-- $\{(5,7) \ldots\ldots (7,5)\}$

A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called ***antisymmetric***.

$(2, 2)$

$\{(1,2) \ldots\ldots \cancel{(2,1)}\}$

$\left\{ \begin{array}{l} (1,2) \longrightarrow (2,1) \\ (3,1) \longrightarrow \times \end{array} \right\}$

$(1,2) \longrightarrow$

$(3,1) \longrightarrow$

Relations and Their Properties

- **Example 4:**

Which of the following relations are symmetric and which are antisymmetric? $A = \{1, 2, 3, 4\}$

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$.

$R_7 = \{(1,1), (2,2)\}$.

Relations and Their Properties

- **Example 4:**

Solution:

$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\},$ not Symmetric
not Antisymmetric

$R_2 = \{(1,1), (1,2), (2,1)\},$ symmetric (not Antisymmetric)

$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\},$ symmetric

$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\},$ antisymmetri not Symmetric

$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\},$

antisymmetric

$R_6 = \{(3,4)\}.$ antisymmetric

$R_7 = \{(1,1), (2,2)\}.$ symmetric and antisymmetric

Relations and Their Properties

- **Example 5:**

1, 2, 3, ---

Is the "divides" relation on the set of positive integers symmetric?

Solution:

This relation is **not symmetric** because $1 \mid 2$, $2 \nmid 1$.

$a \mid b \longrightarrow (a,b) \in R$

$b \mid a \longrightarrow (b,a) \in R$

$1 \mid 2$ $(1,2) \in R$

$2 \nmid 1$

Relations and Their Properties

- **Example 6:**

Is the "divides" relation on the set of positive integers antisymmetric?

$2 | 4$ ✓
$4 | 2$ ✗

$3 | 6$ ✓
$6 | 3$ ✗

Solution:

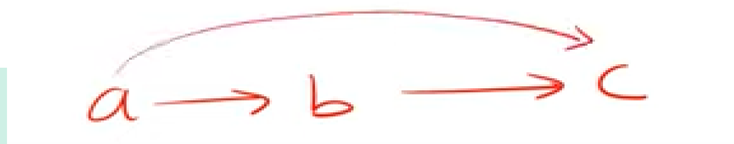
This relation is **antisymmetric**.

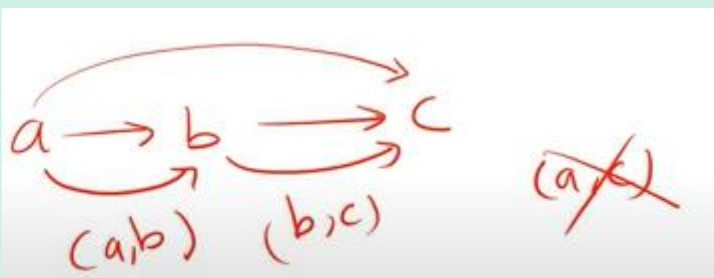
To see this, note that if a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$.

Relations and Their Properties

❑ Transitive

A relation R on a set A is called ***transitive***
If whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$,
for all $a, b, c \in A$





Relations and Their Properties

- **Example 1:**

Which of the following relations are transitive?

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$

$R_2 = \{(1, 1), (1, 2), (2, 1)\},$

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$

$R_6 = \{(3, 4)\}.$

$R_7 = \{(1,1), (2,2)\}.$

Relations and Their Properties

❑ Example 1: <u>Solution:</u>

Which of the following relations are transitive?

$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$, (2,1) — (1,2)

$R_2 = \{(1,1),(1,2),(2,1)\}$, (2,1) (1,2)

$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$,

$R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$, **transitive**

$R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$,

transitive

$R_6 = \{(3,4)\}$. **transitive**

$R_7 = \{(1,1),(2,2)\}$. **transitive**

Relations and Their Properties

❑ **Example 2:**

$1, 2, 3, \ldots \quad Z^+$

Is the "divides" relation on the set of positive integers transitive?

Solution:

This relation is **transitive.**

$a \mid b \quad (a, b) \in R$

$b \mid c \quad (b, c) \in R$

$a \mid c \quad (a, c) \in R$

Relations and Their Properties

❑ Combining Relations

The relations

$R_1 = \{(1,1), (2,2), (3,3)\}$ and

$R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$

can be combined to obtain

Solution:

$$R_1 \cup R_2 = \{(1,1), (2,2), (3,3), (1,2), (1,3), (1,4)\}$$

$$R_1 \cap R_2 = \{(1,1)\}$$

$$R_1 - R_2 = \{(2,2), (3,3)\}$$

$$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$$

$$\begin{aligned} R_1 \oplus R_2 &= R_1 \cup R_2 - R_1 \cap R_2 \\ &= \{(2,2), (3,3), (1,2), (1,3), (1,4)\} \end{aligned}$$

Questions?