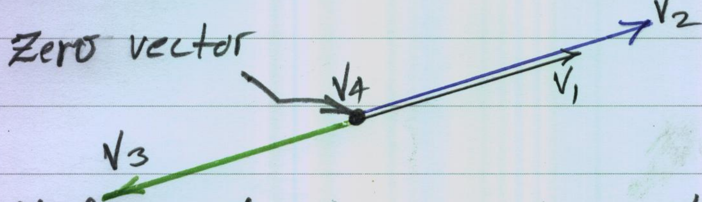
Eigenvectors & Eigenvalues..

1. Parallel Vectors.



Here, V_1, V_2, V_3, V_4 are all parallel

parallel vectors are vectors that are scalar multiplication of one another

$$V_2 = \begin{pmatrix}\text{positive}\\ \text{scalar}\end{pmatrix} V_1$$

$$V_3 = \begin{pmatrix}\text{negative}\\ \text{scalar}\end{pmatrix} V_1$$

$$V_4 = (\text{Zero})\, V_1$$

the zero vector

- this means that the zero vector is parallel to any vector because we can produce it by multiplying any vector with zero.

Transformation of vectors.

When we multiply vectors by matrices, we infact transform the vector to another vector. For example:

$$\underset{A}{\begin{bmatrix} 2 & 1 \\ 3 & -1 \\ 0 & 2 \end{bmatrix}} \underset{x}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix}}$$

Here, $\begin{bmatrix} x \end{bmatrix}$ which is a vector in R^2 was transformed to a vector in R^3 (i.e. b)

- The relation between x and b is very scope limited as they are in different R's.
- b is always a linear combination of the Columns of A.

- Let's take another example where the relation between x and b is more close.

$$\begin{bmatrix} 3 & 1 & 5 \\ 0 & 2 & 3 \\ 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \\ 11 \end{bmatrix}$$

$$A \quad x \quad = \quad b$$

Here, x was transformed to b (via A). But also x and b are in R^3.

You can already see that the relation between x and b is closer now as they belong to the same R.
b is still a linear combination of the columns of A.

- Let's take another example where the connection between x and b is even more tighter.

$$\begin{bmatrix} 3 & 1 & 5 \\ 0 & 2 & 3 \\ 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$A \quad x \quad = \quad b$$

Here, no change from the previous example except that: $b = 3x$

This means that b and x are parallel vectors. And we are particularly interested in this case (b and x are parallel vectors, and this requires that A is nxn). We call this an eigen equation case: $Ax = \lambda x$. x that satisfies that is called an eigenvector. The scalar λ is called an eigenvalue corresponding to the eigenvector. Specifically in this example, $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\lambda = 3$

eigenvector

eigenvalue corresponding to the eigenvector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

We are still at the same example.
Note that rank(A) = 3 (full rank).
Which means that N(A) is trivial
(i.e. has only the zero vector).

$$A \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We can't get $b=0$ unless we are using $x=0$. Which means that this could happen for any λ. We can't say that $\lambda=0$ is an eigenvalue for $x=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$A \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, $\lambda=0$ is not an eigenvalue for a full rank matrix.

- Now let's take the case when rank(A) isn't full.

Example:

$$\underbrace{\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{b}$$

$A \rightarrow \text{rank}(A)=2$ $\quad x \rightarrow \in N(A)$

<u>remember</u>: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is parallel to $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Which means that we can write $Ax=b$ as an eigen equation

$$Ax = \lambda x$$

and the only way for this case to happen is when $\lambda=0$. Therefore, for a rank deficient (singular) matrix an eigenvalue $\lambda=0$ must exist.

Now let's make two important notes before we proceed to other key cases.

1st note :

$$\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Any $x \in N(A)$ will fit in this eigen equation. (ex: $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}, \dots$)

This means that eigen-vectors come from a whole space (called the eigenspace) where any vector in that space will solve the eigenequation using the same λ. Let's take a look at the example previous to the last one.

$$\begin{bmatrix} 3 & 1 & 5 \\ 0 & 2 & 3 \\ 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ 0 \\ 0 \end{bmatrix}$$

This vector is parallel to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (which we used in the original example). Still you can see that $\begin{bmatrix} 15 \\ 0 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$

same eigenvalue corresponding to any eigenvector of the form $\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$

2nd note:

How many eigenvalue (and eigenvectors) does an $n \times n$ matrix have?
In order to answer this question let's take a closer look at the eigen equation.

$$Ax = \lambda x$$

$$Ax - \lambda x = 0$$

$$(A - \lambda I)x = 0$$

this I is used for mathematical necessity.

This is the null equation for the matrix $(A-\lambda I)$. Solving this equation boils down to finding the values that make the matrix $(A-\lambda I)$ singular (i.e. rank deficient).
[Because if it was a full rank then only $x=0$ can solve it). So, again, we need $(A-\lambda I)$ to be singular. In other words, $|A-\lambda I|$ must $=0$. This is called the characteristic equation $|A-\lambda I|=0$.

If you write the expression of the determinant of $|A-\lambda I|$, it's a polynomial of the n^{th} order where A is $n \times n$ matrix. This means that for an $n \times n$ matrix A, there are n eigenvectors and typically each eigenvector has an eigenvalue.

Let's look at a simple example before we conclude this note.

Example: $A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

$$(A - \lambda I) = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 2 & -1 \\ 0 & 1-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

Now we write the expression of $|A-\lambda I|$ using first column:

$$(3-\lambda)(1-\lambda)(1-\lambda) = 0$$

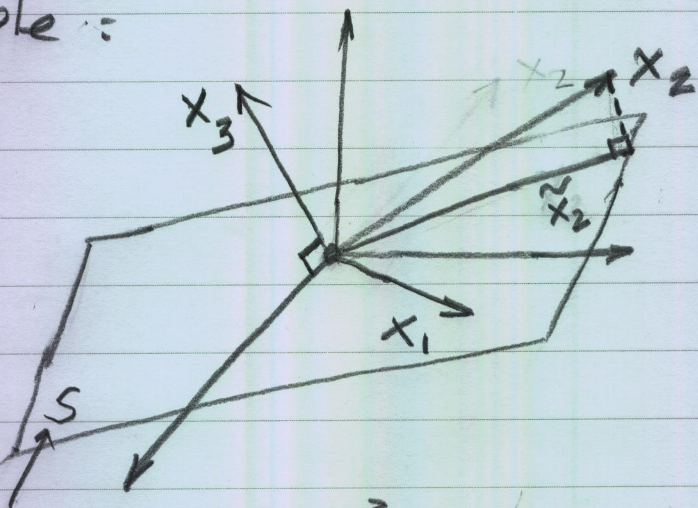
polynomial equation of order 3. There are 3 values (roots) for this equation to make it $=0$

Finding these 3 values of λ is basically finding the eigenvalues. We will look at that again later. But now we know:

A 3×3 matrix has 3 eigenvalues/vectors
2×2 has 2
4×4 has 4
.. and so on.

Eigenvalues/vectors of a projection matrix:

Example:



S is a plane in R^3, consequently:

✓ S has basis set(s) of two vectors (dim = 2)

✓ S has a projection matrix $P_s \neq I$

✓ rank$(P_s) = 2$, P_s is singular, 3×3, symmetric

Let's find its eigenvalues: $P_s x = \lambda x$

So basically we are asking the question: what, possibly, the vectors x which when multiplied by P_s we get a parallel vector to x and what is λ in each possible case?

we have three possibilities for x:

x_1 (any vector in S) we know that

$P_s x_1 = x_1$

which means that $\lambda = 1$ must be an eigenvalue of any Projection matrix. x_1 is an eigen vector

x_2 (any vector $\notin$ S. $\not\perp$ S)

$P_s x_2 = \tilde{x}_2$

$\tilde{x}_2$ is not parallel to x_2 (that is for sure). So this is not an eigen equ.

x_3 (any vector $\notin$ S, $\perp$ S)

$P_s x_3 = 0$

because x_3 is $\perp$ to all vectors in S including the columns of P_s. $\lambda = 0$ must be an eigenvalue. x_3 is an eigenvector.

Note: $\lambda = 0$ must be an eigenvalue for a projection matrix P shall not be surprising to you; because P is a singular matrix.

So, now we know; a projection matrix has an eigenvalue $= 0$ and another eigenvalue $= 1$

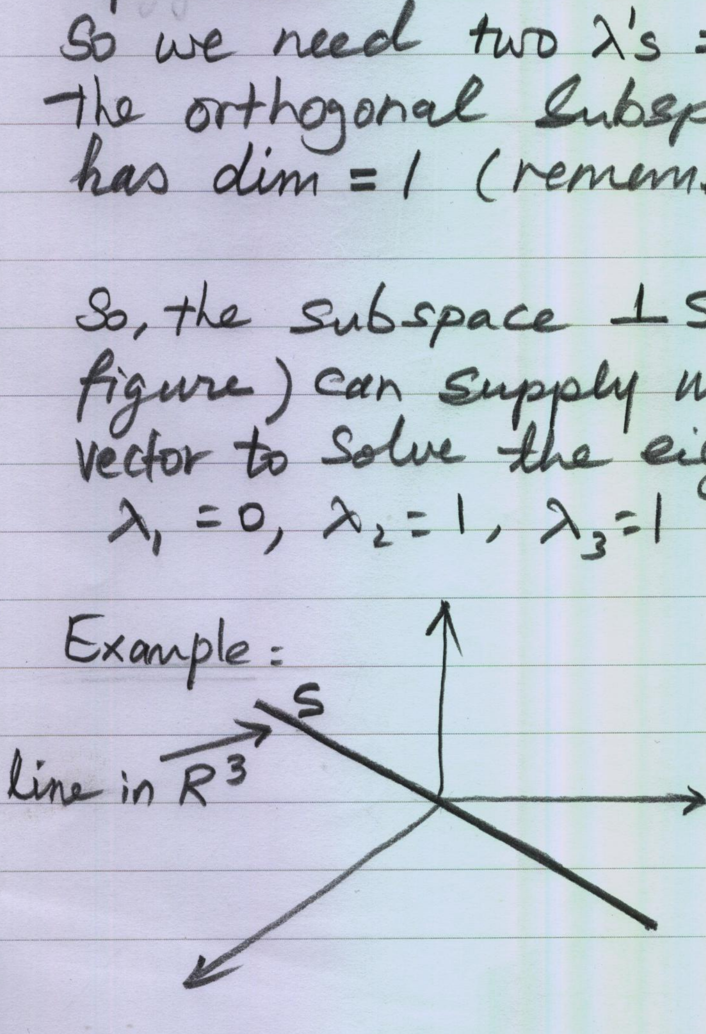
What if P was a 3×3 projection matrix (i.e. there must be three λ's)? We know that $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = ?$ (has to be $= 1$ or $= 0$) There is no other possibility (see figure in page 7). In our current example, $\dim(S) = 2$ this means that S can supply us with 2 independent vectors to solve the eigen-equation (similar to x_1 in figure in page 7). So we need two λ's $= 1$. On the other hand, the orthogonal subspace, that contains x_3, has $\dim = 1$ (remember $\underset{2}{\dim(S)} + \underset{1}{\dim(\perp S)} = \underset{3}{\dim(R^3)}$)

$$2 + 1 = 3$$

So, the subspace $\perp S$ (containing x_3 in the figure) can supply us with 1 independent vector to solve the eigen-equation. In conclusion:

$\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 1$

Example:



$\dim(S) = 1$ $\quad \dim(\perp S) = 2$

$\text{rank}(P_s) = 1$,

$\underbrace{\lambda_1 = 0,\ \lambda_2 = 0}_{\text{eigenvectors corresponding to } \lambda = 0 \text{ are supplied from } (\perp S)}$, $\underbrace{\lambda_3 = 1}_{\text{same words} \dots \text{from } (S)}$

Example : a projection matrix in R^2 has $\lambda_1 = 0$, $\lambda_2 = 1$ (direct and easy!)

Example : a projection matrix in R^4 has

$$\underbrace{\lambda_1 = ? \ , \ \lambda_2 = ? \ , \ \lambda_3 = ? \ , \ \lambda_4 = ?}$$

A property of projection matrices tells that they can be 0 or 1. But how can we split them?

- If the projection matrix is for a subspace in R^4 that is a line, then

$$\lambda = \begin{cases} \lambda_1 = \lambda_2 = \lambda_3 = 0 \\ \lambda_4 = 1 \end{cases}$$

- If the projection matrix is for a subspace, S in R^4 that has $\dim(S) = 2$, then

$$\lambda = \begin{cases} \lambda_1 = \lambda_2 = 0 \\ \lambda_3 = \lambda_4 = 1 \end{cases}$$

- If the projection matrix is for a subspace, S in R^4 that has $\dim(S) = 3$, then

$$\lambda = \begin{cases} \lambda_1 = 0 \\ \lambda_2 = \lambda_3 = \lambda_4 = 1 \end{cases}$$

Let's take another example; permutation matrix

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ← this matrix changes the row order of the vector that gets multiplied to it

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ ← (this is not an eigen equation).

So, in eigen sense: we need a vector x that is a permuted version of itself:

$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ← (this is a valid eigen eqn. with $\lambda = 1$ and $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$).

$A\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ $\lambda = 1$ $x = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

$A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\lambda = -1$ $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

<u>Fact</u>: For any $n \times n$ matrix, sum of λ's equals sum of diagonal elements.

Now let's work more on finding λ's and x's, you should look at $|A - \lambda I| = 0$ as follows: What are the values of λ that take A to singularity (sort of shifts the matrix A to rank deficiency). This explains that if A is singular at the first place (without any shifting) then there must be a $\lambda = 0$ (in other words, we don't need to shift A to singularity always. If it's singular in a given dimension then no need to shift and $\lambda = 0$ corresponding to that dimension (eigenvector)).

Example: $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Find the eigenvalues and the eigenvectors for A.

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = 0$$

$$\lambda^2 - 6\lambda + 8 = 0$$

this is the trace of A ↑ (6)   this is $|A|$ ↑ (8)

$tr(A) = \sum$ diagonal elements

$= \sum$ eigenvalues

now we solve $\lambda^2 - 6\lambda + 8 = 0$

$$(\lambda - 4)(\lambda - 2) = 0$$

$$\therefore \lambda_1 = 4 \text{ or } \lambda_2 = 2$$

✓ Now we can find the eigenvectors. Eigenvectors are the vectors in the null space, N(A), such that when we shift A by λ we make the matrix singular

for $\lambda_1 = 4$: $|A - 4I| = 0$ (must be singular)

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ find x_1 corresponding to λ_1

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} x_1 = 0$$

a vector in the Null space of $(A - \lambda I)$. $\rightarrow x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

We can choose any scalar version of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

✓ Do the same for $\lambda_2 = 2$: $(A - 2I) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Another basis for a whole subspace.

Example: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, find eigenvalues/vectors.

$$(A - \lambda I) = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$$

$$|A - \lambda I| = (-\lambda)^2 - 1 = 0$$
$$\lambda^2 = 1$$
$$\lambda = \pm 1, \quad \lambda_1 = 1, \quad \lambda_2 = -1$$

Now find x's (eigenvectors)

use $\lambda_1 = 1$: $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} x_1 = 0 \rightarrow x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

use $\lambda_2 = -1$: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x_2 = 0 \rightarrow x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

From the last two examples we can see that adding a constant to A doesn't change the eigenvectors but only adds the same constant to the eigenvalues. Proof:

$$Ax = \lambda x$$
$$(A + kI)x = Ax + kx$$
$$= \lambda x + kx$$
$$= (\lambda + k)x \quad \leftarrow \text{same } x\text{, add } k \text{ to } \lambda\text{'s.}$$

Eigenvalues/vectors of a rotation matrix:

A rotation matrix is a matrix when post multiplied by a vector, it produces a rotated version of that vector. Ex: $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

90° rotation matrix

Finding the eigenvectors means that the vector, when operated on by the matrix, gives itself (or a multiple of it). It made sense when we were studying the projection matrix case. But in the rotation matrix case it's hard to find. (A vector when rotated produces itself!).

Let's solve that for $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

$(Q - \lambda I) = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$, $|Q - \lambda I| = \lambda^2 + 1 = 0$

$\lambda^2 = -1$

✓ Complex eigenvalues $\lambda_1 = i$, $\lambda_2 = -i$

✓ they come in conjugate pairs

Note:

Anti-Symmetric matrix $(Q = -Q^T)$ — Complex (imaginary) λ's $\longleftrightarrow$ real λ's — Symmetric matrix $(A = A^T)$

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$. $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

closer to Symmetric

Example: $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$, find λ's and x's.

$(3-\lambda)^2 = 0$, $\lambda_1 = 3$, $\lambda_2 = 3$

$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

there is no second independent eigenvector. This means that there's a shortage in indep. eigenvectors.

- Note: we had repeated λ's in the case of projection matrix but we didn't have shortage in eigenvectors. Why?