Faculty of Computer Science and Information Technology

Discrete Mathematics

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Lecture 4

Sequences and Summations

Sequences

- **Definition:** A sequence is a set of things (usually numbers) that are in order.

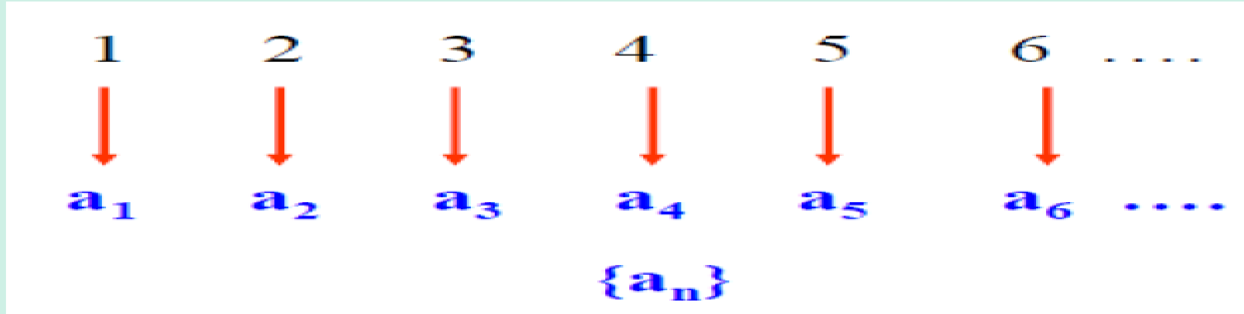
➢ For example, 1 , 2, 3, 5, 8 is a sequence with five terms and 1, 3, 9, 27, 81, . . . , 30, . . . is an infinite sequence.

- We use the notation $\boldsymbol{a_n}$ to denote the image of the integer n. We call $\boldsymbol{a_n}$ a term of the sequence.

- We use the notation $\{a_n\}$ to describe the sequence.

$$\{a_n\} = \{a_1, a_2, a_3, \dots\}$$

Sequences



- **Example :**
- Consider the sequence $\{a_n\}$, where

$$a_n = \frac{\mathbf{1}}{\boldsymbol{n}}$$

The list of the terms of this sequence, beginning with a_1, namely,

$$\boldsymbol{a_1, a_2, a_3, a_4, \ldots,}$$

Starts with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

Sequences

- (1) $a_n = n^2$, where $n = 1,2,3...$
  - What are the elements of the sequence?

    1, 4, 9, 16, 25, ...
- (2) $a_n = (-1)^n$, where $n=0,1,2,3,...$
  - Elements of the sequence?

    1, -1, 1, -1, 1, ...
- 3) $a_n = 2^n$, where $n=0,1,2,3,...$
  - Elements of the sequence?

    1, 2, 4, 8, 16, 32, ...

Geometric

- **Definition:** A geometric progression is a sequence of the form:

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the initial term a and the common ratio r are real numbers.

2, 10, 50, 250, …

Geometric

- **Geometric – Example1**

$$1, -1, 1, -1, 1, \ldots;$$

ar^n

$a, ar, ar^2, \ldots, ar^n$

$$\{ar^n\}, \qquad n = 0,1,2, \ldots$$

$$a = 1$$

$$r = -1$$

$\{ar^n\}_{\boxed{}}$

$\{(1)(-1)^n\}_{n=0}^{\boxed{}}$

$\{\cancel{a}(-1)^n\}_{n=0}^{\boxed{}}$ $\quad \{(-1)^n\}_{n=0}$

Geometric

- **Example2**

$$2, 10, 50, 250, 1250, \ldots;$$

$$\{ar^n\}, \qquad n = 0,1,2, \ldots$$

$$a = 2$$

$$r = 5$$

$$\{ar^n\}_{n=0}$$

$$\{2(5)^n\}_{n=0}^{\infty}$$

$$\overset{\times 5}{} \quad \overset{\times 5}{} \quad \overset{\times 5}{} \quad \overset{\times 5}{}$$

$$2, 10, 50, 250, 1250, \ldots; \qquad a = 2$$

$$5 \quad 5 \quad 5 \quad 5 \qquad r = 5$$

Geometric

- **Example 3**

$$6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots$$

$$6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots$$

$\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$

$a=6$

$r=\frac{1}{3}$

$$\{ar^n\}, \qquad n = 0,1,2, \ldots$$

$$a = 6$$

$$r = 1/3$$

$$\{ar^n\} = \{6(\tfrac{1}{3})^n\}_{n=0}^{\infty}$$

Geometric

- **Example 4**

Find a, r? $\{3 * 4^n\}$, $n = 0, 1, 2, \ldots$

$$\{ar^n\}, \qquad n = 0,1,2, \ldots$$

$$a = 3$$

$$r = 4$$

$3, 12, \ldots$

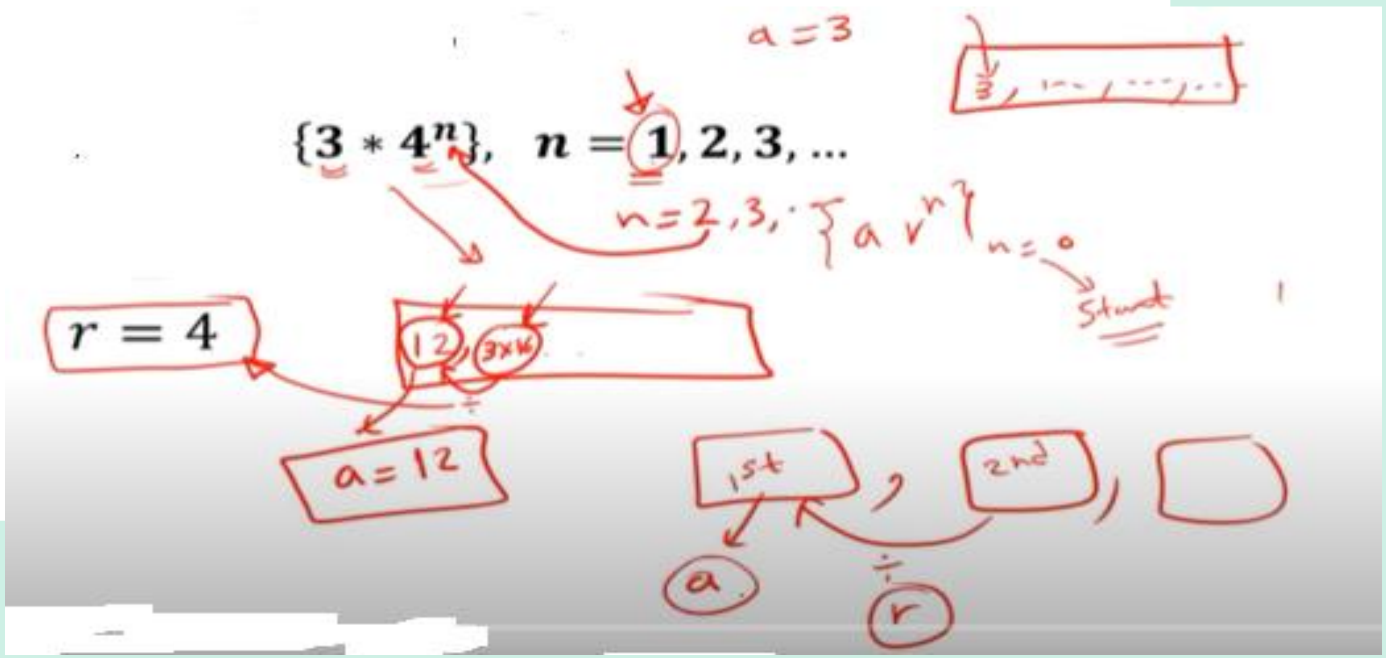
$a = 3$
$r = 4$

Geometric

- **Example 5**

Find a, r? $\{3 * 4^n\}, \ n = 1, 2, 3, \ldots$

$$a = 12$$
$$r = 4$$



Geometric

Examples:

1. Let $a = 1$ and $r = -1$. Then:

$$\{b_n\} = \{b_0, b_1, b_2, b_3, b_4, \ldots\} = \{1, -1, 1, -1, 1, \ldots\}$$

2. Let $a = 2$ and $r = 5$. Then:

$$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}$$

3. Let $a = 6$ and $r = 1/3$. Then:

$$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \ldots\} = \{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\}$$

Arithmetic

Definition:

An arithmetic progression is a sequence of the form:

$$a, a + d, a + 2d, \ldots, a + nd, \ldots$$

where the **initial term a** and the common difference d are real numbers.

Arithmetic

Arithmetic – Example1

$$-1, 3, 7, 11, \ldots,$$

$$\{a + nd\}, \qquad n = 0,1,2, \ldots$$

$$a = -1$$
$$d = 4$$

$a = -1$
$d = 4$

$\{a + nd\}_{n=0}^{\infty}$

$\{-1 + 4n\}_{n=0}$

$-1, 3, 7, 11, \ldots,$

4 4 4

-1

$-1 + 4 = 3$
$3 + 4 = 7$
$7 + 4 = 11$

Arithmetic

Arithmetic – Example 2

$$7, 4, 1, -2, \ldots$$

$$\{a + nd\}, \qquad n = 0,1,2, \ldots$$

$$a = 7$$

$$d = -3$$

$$7, \underbrace{}_{-3} 4, \underbrace{}_{-3} 1, \underbrace{}_{-3} -2, \ldots$$

$a = 7$

$d = -3$

$\{a + nd\}_{n=0}$

$\rightarrow \{7 - 3n\}_{n=0,1,2,\ldots}$

$7, 4, 1, \ldots$

Arithmetic

Examples

1. Let $a = -1$ and $d = 4$:

$$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{-1, 3, 7, 11, 15, \ldots\}$$

2. Let $a = 7$ and $d = -3$:

$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}$$

3. Let $a = 1$ and d = 2:

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Recurrence Relations

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- A sequence is called **a solution** of a recurrence relation if its terms satisfy the recurrence relation.

Recurrence Relations

Examples

Example 1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation

$$a_n = a_{n-1} + 3 \text{ for } n = 1,2,3,4,\ldots.$$

$$a_0 = 2.$$

What are a_1 , a_2 and a_3?

[Here $a_0 = 2$ is the initial condition.]

Solution: We see from the recurrence relation that

$$a_1 = a_0 + 3 = 2 + 3 = 5$$

$$a_2 = 5 + 3 = 8$$

$$a_3 = 8 + 3 = 11$$

Recurrence Relations

Examples

Example 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation

$a_n = a_{n-1} - a_{n-2}$ for $n = 2,3,4,\ldots.$

$a_0 = 3$ and $a_1 = 5$.

What are a_2 and a_3?

[Here the initial conditions are $a_0 = 3$ and $a_1 = 5$.]

Solution: We see from the recurrence relation that

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$

$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

Fibonacci Sequence

The *Fibonacci sequence*, $f_0, f_1, f_2, \ldots,$ is defined by the initial conditions $f_0 = 0, f_1 = 1$, and the recurrence relation

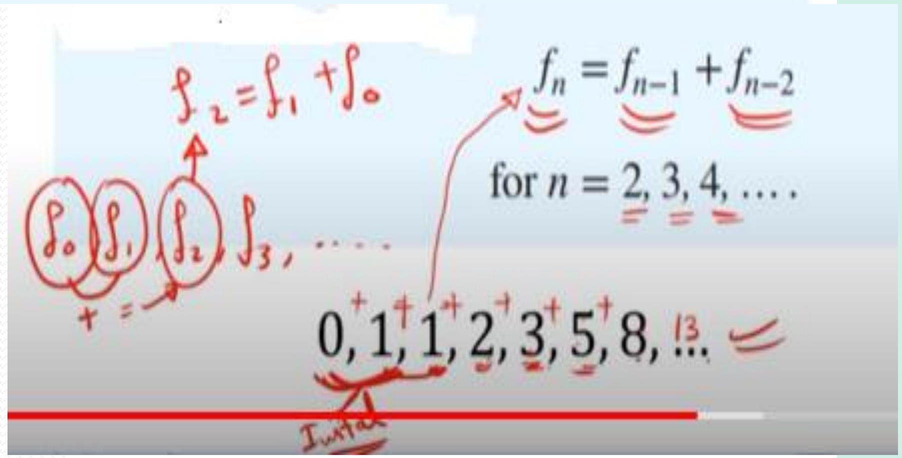
$$f_n = f_{n-1} + f_{n-2}$$

for $n = 2, 3, 4, \ldots$.

Example: Find f_2, f_3, f_4, f_5 and f_6

Answer:

$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8.$$



0, 1, 1, 2, 3, 5, 8, ...

Iterative Solution Example

Method 1: Working upward, forward substitution

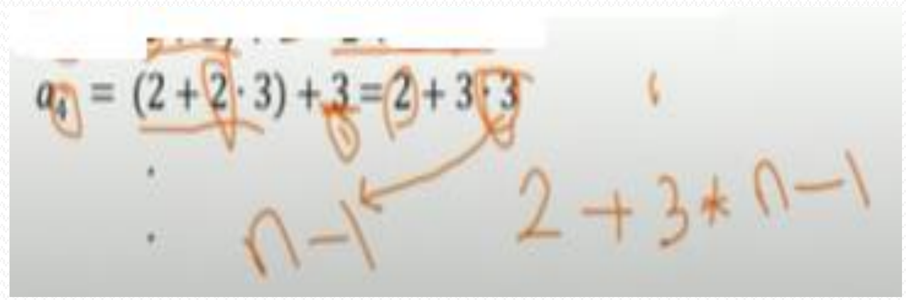
Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

$$a_2 = 2 + 3$$

$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$

$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$$

$$\vdots$$



$$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$$

Iterative Solution Example

Method 2: Working downward, backward substitution

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

$$
\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\
&= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\
&\;\;. \\
&\;\;. \\
&= a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1)
\end{aligned}
$$

Useful Sequences

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, …
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, …
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, …
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, …
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, …
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, …
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …

Summations

Next, we introduce **summation notation.**
We begin by describing the notation used to express the sum of the terms

$$a_m, a_{m+1}, \ldots, a_n$$

from the sequence $\{a_n\}$. We use the notation

$$\sum_{j=m}^{n} a_j, \qquad \textstyle\sum_{j=m}^{n} a_j, \qquad \text{or} \qquad \textstyle\sum_{m \le j \le n} a_j$$

(read as the sum from $j = m$ to $j = n$ of a_j)

to represent

Here, the variable j is called the **index of summation**

$$a_m + a_{m+1} + \cdots + a_n.$$

Summations

$$\sum_{i=1}^{5} a_i$$

$a_1 + a_2 + a_3 + a_4 + a_5$

index

$\{a_n\}_{n=0}^{5}$ ← end ← start

$= a_0, a_1, a_2, a_3, a_4, a_5$

$+ \quad + \quad + \quad + \quad +$

end

$$\sum_{j=m}^{n} a_j, \equiv \sum\nolimits_{j=m}^{n} a_j, \quad \text{or} \quad \equiv \sum\nolimits_{m \le j \le n} a_j$$

start

Summations

$$\sum_{j=m}^{n} a_j = \sum_{i=m}^{n} a_i = \sum_{k=m}^{n} a_k$$

- Here, the index of summation runs through all integers starting with its **lower limit** m and ending with its **upper limit** n. A large uppercase Greek letter sigma, **Σ**, is used to denote summation.

Summations

Example 1

Express the sum of the first 100 terms of the sequence $\{a_n\}$, where $a_n = 1/n$ for $n = 1, 2, 3, \ldots$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{100}$

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{100}$

Answer

$$\sum_{n=1}^{100} 1/n$$

$$\sum_{n=1}^{100} 1/n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{100}$$

The lower limit index of summation is 1, and the upper is 100 .

Summations

Example 2

What is the value of $\sum_{j=1}^{5} j^2$?

Answer

$$\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

$$= 1 + 4 + 9 + 16 + 25$$

$$= 55.$$

Summations

Example 3

What is the value of $\sum_{s \in \{0,2,4\}} s$?

$$\sum_{s \in \{0,2,4\}} s = 0 + 2 + 4 = 6.$$

$\sum_{n=1}^{100}$

$n = 1, 2, 3, \ldots, 100$

$1, 5, 20, 100$

$\sum_{s \in \{0,2,4\}} s$?

$s = 0$

$s = 2$

$s = 4$

s

$n \in \{1, 5, 20, 100\}$

Summations

Example 5

What is the value of $\sum_{k=4}^{8}(-1)^k$?

Solution: We have

$$\begin{aligned}\sum_{k=4}^{8}(-1)^k &= (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8 \\ &= 1 + (-1) + 1 + (-1) + 1 \\ &= 1.\end{aligned}$$

Questions?