Discrete Math

Sequences

Tutorial 4

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1. Find these terms of the sequence $\{a_n\}$, where $a_n = 2 \cdot (-3)^n + 5^n$.

a) a_0 **b)** a_1 **c)** a_4 **d)** a_5

Answer:

a) a_0

$$2 \cdot (-3)^0 + 5^0 = 3$$

b) a_1

$$2 \cdot (-3)^1 + 5^1 = -1$$

c) a_4

$$2 \cdot (-3)^4 + 5^4 = 2 \cdot 81 + 625 = 162 + 625 = 787$$

D , Assignment

5. List the first 10 terms of each of these sequences.

a) the sequence that begins with 2 and in which each successive term is 3 more than the preceding term

b) the sequence that lists each positive integer three times, in increasing order

c) the sequence that lists the odd positive integers in increasing order, listing each odd integer twice

d) the sequence whose *nth term is* $n! - 2^n$

e) the sequence that begins with 3, where each succeeding term is twice the preceding term

f) the sequence whose first term is 2, second term is 4, and each succeeding term is the sum of the two preceding terms

(a) 2, 5, 8, 11, 14, 17, 20, 23, 26, 29
(b) 1, 1, 1, 2, 2, 2, 3, 3, 3, 4
(c) 1, 1, 3, 3, 5, 5, 7, 7, 9, 9
(d) $-1, -2, -2, 8, 88, 656, 4912, 40064, 362368, 3627776$
(e) 3, 6, 12, 24, 48, 96, 192, 384, 768, 1536
(f) 2, 4, 6, 10, 16, 26, 42, 68, 110, 178

E and F , Assignment

(a) The sequence starts with 2, thus the first term is 2.

$$a_1 = 2$$

Each successive term is the previous term increased by 3.

$$a_2 = a_1 + 3 = 2 + 3 = 5$$
$$a_3 = a_2 + 3 = 5 + 3 = 8$$
$$a_4 = a_3 + 3 = 8 + 3 = 11$$
$$a_5 = a_4 + 3 = 11 + 3 = 14$$
$$a_6 = a_5 + 3 = 14 + 3 = 17$$
$$a_7 = a_6 + 3 = 17 + 3 = 20$$
$$a_8 = a_7 + 3 = 20 + 3 = 23$$
$$a_9 = a_8 + 3 = 23 + 3 = 26$$
$$a_{10} = a_9 + 3 = 26 + 3 = 29$$

(b) The sequence contains positive integers in increasing order, thus the first term is 1.

$$a_1 = 1$$

Each term occurs three times:

$$a_2 = 1$$
$$a_3 = 1$$

The next positive integer is 2, which also has to occur three times.

$$a_4 = 2$$
$$a_5 = 2$$
$$a_6 = 2$$

The next positive integer is 3, which also has to occur three times.

$$a_7 = 3$$
$$a_8 = 3$$
$$a_9 = 3$$

The next positive integer is 4.

$$a_{10} = 4$$

(c) The sequence contains odd positive integers in increasing order, thus the first term is 1.

$$a_1 = 1$$

Each term occurs twice:

$$a_2 = 1$$

The next odd positive integer is 3, which also has to occur twice.

$$a_3 = 3$$
$$a_4 = 3$$

The next odd positive integer is 5, which also has to occur twice.

$$a_5 = 5$$
$$a_6 = 5$$

The next odd positive integer is 7, which also has to occur twice.

$$a_7 = 7$$
$$a_8 = 7$$

The next odd positive integer is 9, which also has to occur twice.

$$a_9 = 9$$
$$a_{10} = 9$$

(d) Given:

$$a_n = n! - 2^n$$

Evaluate a_n at $n = 1, 2, 3, ..., 9, 10$:

$$a_1 = 1! - 2^1 = 1 - 2 = -1$$
$$a_2 = 2! - 2^2 = 2 - 4 = -2$$
$$a_3 = 3! - 2^3 = 6 - 8 = -2$$
$$a_4 = 4! - 2^4 = 24 - 16 = 8$$
$$a_5 = 5! - 2^5 = 120 - 32 = 88$$
$$a_6 = 6! - 2^6 = 720 - 64 = 656$$
$$a_7 = 7! - 2^7 = 5040 - 128 = 4912$$
$$a_8 = 8! - 2^8 = 40320 - 256 = 40064$$
$$a_9 = 9! - 2^9 = 362880 - 512 = 362368$$
$$a_{10} = 10! - 2^{10} = 3628800 - 1024 = 3627776$$

10. Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.

a) $a_n = -2a_{n-1}$, $a_0 = -1$

b) $a_n = a_{n-1} - a_{n-2}$, $a_0 = 2$, $a_1 = -1$

c) $a_n = 3a_{n-1}^2$, $a_0 = 1$

d) $a_n = na_{n-1} + a_{n-2}^2$, $a_0 = -1$, $a_1 = 0$

C and D , Assignment

(a) Given:

$$a_n = -2a_{n-1}$$
$$a_0 = -1$$

The first term is -1. Each term is the previous term multiplied by -2.

$$a_0 = -1$$
$$a_1 = -2a_0 = -2(-1) = 2$$
$$a_2 = -2a_1 = -2(2) = -4$$
$$a_3 = -2a_2 = -2(-4) = 8$$
$$a_4 = -2a_3 = -2(8) = -16$$
$$a_5 = -2a_4 = -2(-16) = 32$$

(b) Given:

$$a_n = a_{n-1} - a_{n-2}$$
$$a_0 = 2$$
$$a_1 = -1$$

The first term is 2 and the second term is -1. Each term is the difference of the previous two terms.

$$a_0 = 2$$
$$a_1 = -1$$
$$a_2 = a_1 - a_0 = -1 - 2 = -3$$
$$a_3 = a_2 - a_1 = -3 - (-1) = -2$$
$$a_4 = a_3 - a_2 = -2 - (-3) = 1$$
$$a_5 = a_4 - a_3 = 1 - (-2) = 3$$

11. Let $a_n = 2^n + 5 \cdot 3^n$ for $n = 0, 1, 2, \ldots$.

a) Find a_0, a_1, a_2, a_3, and a_4.

b) Show that $a_2 = 5a_1 - 6a_0$, $a_3 = 5a_2 - 6a_1$, and $a_4 = 5a_3 - 6a_2$.

c) Show that $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers n with $n \geq 2$.

B, Assignment

Given:

$$a_n = 2^n + 5 \cdot 3^n$$

$$n = 0, 1, 2, \ldots$$

(a) Replace n in the given expression for a_n with $0, 1, 2, 3, 4$ and then evaluate:

$$a_0 = 2^0 + 5 \cdot 3^0 = 1 + 5 = 6$$

$$a_1 = 2^1 + 5 \cdot 3^1 = 2 + 15 = 17$$

$$a_2 = 2^2 + 5 \cdot 3^2 = 4 + 45 = 49$$

$$a_3 = 2^3 + 5 \cdot 3^3 = 8 + 135 = 143$$

$$a_4 = 2^4 + 5 \cdot 3^4 = 16 + 405 = 421$$

(b) Let us determine $5a_1 - 6a_0$:

$$5a_1 - 6a_0 = 5(17) - 6(6) = 85 - 36 = 49 = a_2$$

Let us determine $5a_2 - 6a_1$:

$$5a_2 - 6a_1 = 5(49) - 6(17) = 245 - 102 = 143 = a_3$$

Let us determine $5a_3 - 6a_2$:

$$5a_3 - 6a_2 = 5(143) - 6(49) = 715 - 294 = 421 = a_4$$

(c) Given: $a_n = 2^n + 5 \cdot 3^n$

To prove: $a_n = 5a_{n-1} - 6a_{n-2}$, $n \geq 2$.

PROOF

Replace n in $a_n = 2^n + 5 \cdot 3^n$ by $n-1$:

$$a_{n-1} = 2^{n-1} + 5 \cdot 3^{n-1}$$

Replace n in $a_n = 2^n + 5 \cdot 3^n$ by $n-2$:

$$a_{n-2} = 2^{n-2} + 5 \cdot 3^{n-2}$$

We will start from the expression $5a_{n-1} - 6a_{n-2}$ and prove that this term has to be equal to a_n (when $n \geq 2$). Let us use the two previous expressions derived for a_{n-1} and a_{n-2}

$$5a_{n-1} - 6a_{n-2} = 5(2^{n-1} + 5 \cdot 3^{n-1}) - 6(2^{n-2} + 5 \cdot 3^{n-2})$$

Use distributive property:

$$= 5 \cdot 2^{n-1} + 5 \cdot 5 \cdot 3^{n-1} - 6 \cdot 2^{n-2} - 6 \cdot 5 \cdot 3^{n-2}$$

Let us group the terms contains powers of 2:

$$= (5 \cdot 2^{n-1} - 6 \cdot 2^{n-2}) + (5 \cdot 5 \cdot 3^{n-1} - 6 \cdot 5 \cdot 3^{n-2})$$
$$= (5 \cdot 2 \cdot 2^{n-2} - 6 \cdot 2^{n-2}) + (5 \cdot 5 \cdot 3 \cdot 3^{n-2} - 6 \cdot 5 \cdot 3^{n-2})$$

Let us factor out 2^{n-2} from the first term and 3^{n-2} from the second term:

$$\begin{aligned} &= 2^{n-2}(5 \cdot 2 - 6) + 3^{n-2}(5 \cdot 5 \cdot 3 - 6 \cdot 5) \\ &= 2^{n-2}(10 - 6) + 3^{n-2}(75 - 30) \\ &= 2^{n-2}(4) + 3^{n-2}(45) \\ &= 2^{n-2}(2^2) + 3^{n-2}(5 \cdot 3^2) \\ &= 2^n + 5 \cdot 3^n \\ &= a_n \end{aligned}$$

□

(a) 6, 17, 49, 143, 421
(b) $a_2 = 5a_1 - 6a_0$, $a_3 = 5a_2 - 6a_1$, $a_4 = 5a_3 - 6a_2$
(c) $a_n = 5a_{n-1} - 6a_{n-2}$, $n \geq 2$.

15. Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ if

a) $a_n = -n + 2$.

b) $a_n = 5(-1)^n - n + 2$.

c) $a_n = 3(-1)^n + 2^n - n + 2$.

B and C ,
Assignment

(a) Given:

$$a_n = -n + 2$$

$$n = 0, 1, 2, ...$$

To proof: $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$, $n \geq 2$.

PROOF

Replace n in $a_n = -n + 2$ by $n - 1$:

$$a_{n-1} = -(n-1) + 2 = -n + 1 + 2 = -n + 3$$

Replace n in $a_n = -n + 2$ by $n - 2$:

$$a_{n-2} = -(n-2) + 2 = -n + 2 + 2 = -n + 4$$

We will start from the expression $a_{n-1} + 2a_{n-2} + 2n - 9$ and prove that this term has to be equal to a_n (when $n \geq 2$). Let us use the two previous expressions derived for a_{n-1} and a_{n-2}

$$a_{n-1} + 2a_{n-2} + 2n - 9 = (-n + 3) + 2(-n + 4) + 2n - 9$$

Use distributive property:

$$= -n + 3 - 2n + 8 + 2n - 9$$

Combine like terms:

$$= -n + 2$$

$$= a_n$$

26. For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.

a) 3*, 6, 11, 18, 27, 38, 51, 66, 83, 102,…*

b) 7*, 11, 15, 19, 23, 27, 31, 35, 39, 43,…*

c) 1*, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011,…*

d) 1*, 2, 2, 2, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, 5,…*

e) 0*, 2, 8, 26, 80, 242, 728, 2186, 6560, 19682,…*

f) 1*, 3, 15, 105, 945, 10395, 135135, 2027025, 34459425,…*

g) 1*, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1,…*

E, f and g , Assignment

(a)

$$3, 6, 11, 18, 27, 38, 51, 66, 83, 102, ...$$

We note 6 is the previous term increased by 3, 11 is the previous term increased by 5, 18 is the previous term increased by 7, and so on. Thus each term is the previous term increased by $2n + 1$.

$$a_n = a_{n-1} + 2n - 1$$
$$a_1 = 3$$

If the pattern continues, then the 11th, 12th and 13th term become

$$a_{11} = a_{10} + 2(11) - 1 = 102 + 22 - 1 = 123$$
$$a_{12} = a_{11} + 2(12) - 1 = 123 + 24 - 1 = 146$$
$$a_{13} = a_{12} + 2(13) - 1 = 146 + 26 - 1 = 171$$

Thus the next three terms in the sequence are then 123, 146 and 171.

(b)

$$7, 11, 15, 19, 23, 27, 31, 35, 39, 43, , ...$$

We note each term is the previous term increased by 4:

$$a_n = a_{n-1} + 4$$

If the pattern continues, then the 11th, 12th and 13th term become

$$a_{11} = a_{10} + 4 = 43 + 4 = 47$$
$$a_{12} = a_{11} + 4 = 47 + 4 = 51$$
$$a_{13} = a_{12} + 4 = 51 + 4 = 55$$

Thus the next three terms in the sequence are then 47, 51 and 55.

(c)

$$1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, ...$$

We note that the given sequence are the positive integers in increasing order in their binary form.

If this pattern continues, the next terms would then be 12, 13 and 14 in binary form (since 1011 corresponds with 11).

$$1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, ...$$

Thus the next three terms in the sequence are then 1100,1101 and 1110.

(d)

$$1, 2, 2, 2, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, 5, ...$$

We note the integers are repeated once, thrice, 5 times, 7 times, etc. Thus the following integer will be repeated 9 times.

We note that each integer is the sum of the previous two integers (not taking into account the repetition of the integers), since $1 + 2 = 3$ and $2 + 3 = 5$. The next integer would then be $3 + 5 = 8$.

$$1, 2, 2, 2, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, 5, 8, 8, 8, 8, 8, 8, 8, 8, 8, ...$$

Thus the next three terms in the sequence are then 8, 8 and 8.