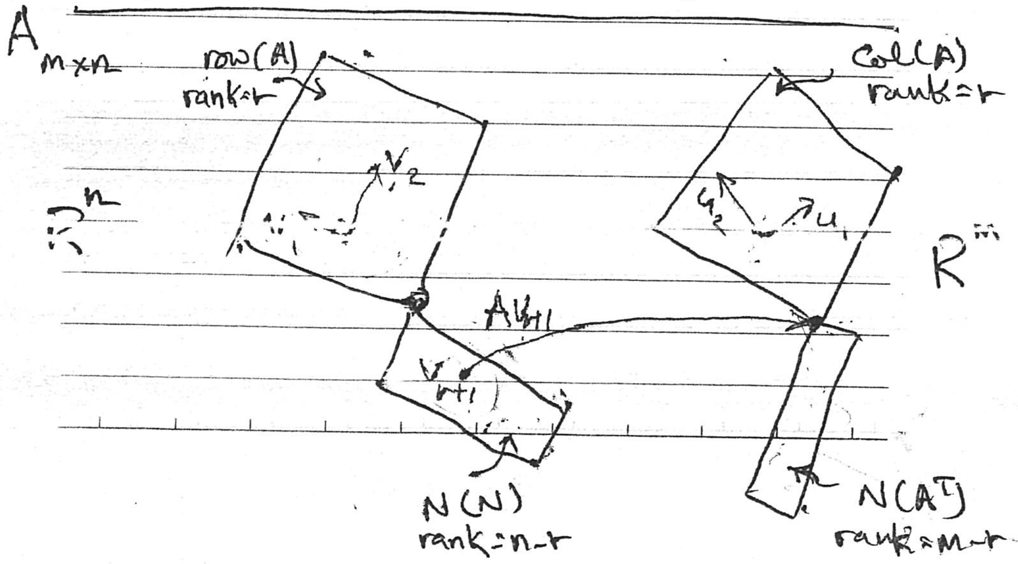
Diagonalization $A = S \Lambda S^{-1}$

eigenvectors (S), e.values (Λ)

1. Limitations:
   A is not square
2. Not enough indep. evectors
   (S is singular, no S^{-1})

Note: if A is symmetric $A = Q \Lambda Q^T$

evectors form an orthogonal matrix (Q)



$$Av_1 = u_1$$

$$\frac{1}{v_{1n}} Av_1 = \frac{1}{v_{1n}} u_1$$

$$Av_1 = \frac{1}{v_{1n}} u_1 \qquad (v_1 \text{ normalized})$$

$$Av_1 = \frac{u_{1n}}{v_{1n}} \frac{u_1}{u_{1n}}$$

$$Av_1 = \underbrace{\frac{u_{1n}}{v_{1n}}}_{\sigma_1} u_1 \qquad (v_1, u_1 \text{ normalized})$$

$$Av_1 = \sigma_1 u_1$$

$$Av_2 = \sigma_2 u_2$$

$$\vdots$$

$$Av_r = \sigma_r u_r$$

All v's are orthogonal

$$A \begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \\ & & & \sigma_r \end{bmatrix}$$

V orthonormal

U orthonormal ? ? ?

Σ diagonal

$$AV = U\Sigma$$

$$A\underbrace{VV^T}_{I} = U\Sigma V^T$$

$$A = U\Sigma V^T$$

$$\begin{aligned} A^TA &= A^T U\Sigma V^T \\ &= (U\Sigma V^T)^T U\Sigma V^T \\ &= V\Sigma^T \underbrace{U^TU}_{I} \Sigma V^T \end{aligned}$$

$$\underbrace{A^TA}_{\text{symmetric}} = V\Sigma^2 V^T$$ (diagonalization in form of $S\Lambda S^{-1}$)

Σ^2 diagonal

Σ^2 has evalues of A^TA

$$A \quad x \quad = \quad \lambda \quad x$$

Matrix Vector = Constant Same Vector

Eigen Equation

$$(A^TA)\,V_1 = \sigma_1^2\,V_1$$

$$(A^TA)\,V_2 = \sigma_2^2\,V_2$$

$$(A^TA)\,V_r = \sigma_r^2\,V_r$$

V's are evectors of (A^TA)

$u_1 \cdot u_2 \overset{?}{=} 0$, $u_1^T u_2 \overset{?}{=} 0$ ✓

$A v_1 = \sigma_1 u_1$, $u_1 = \dfrac{A v_1}{\sigma_1}$

$A v_2 = \sigma_2 u_2$, $u_2 = \dfrac{A v_2}{\sigma_2}$

$$u_1^T u_2 = \left(\frac{Av_1}{\sigma_1}\right)^T \left(\frac{Av_2}{\sigma_2}\right)$$

$$= \frac{1}{\sigma_1\sigma_2}\, V_1^T(A^TA)V_2$$

$$= \frac{\sigma_2^2}{\sigma_1\sigma_2}\, \underbrace{V_1^T V_2}_{=0} \qquad \begin{array}{c}(V_1 \perp V_2)\\ V_1^T V_2 = 0\end{array}$$

$$= 0$$

So U is orthonomal

$A = U \Sigma V^T$ is correct

left singular vectors

singular values

right singular vectors

$\boxed{V \text{ has eigenvectors of } (A^T A)}$

$$AA^T = U\Sigma V^T (U\Sigma V^T)^T$$

$$= U\Sigma \underbrace{V^T V}_{I} \Sigma^T U^T$$

$$AA^T = U\Sigma^2 U^T$$

$\boxed{U \text{ has eigenvectors of } (AA^T)}$

$A_{m \times n}$

$$A[\underbrace{v_1\ v_2 \cdots v_r}\ \underbrace{v_{r+1} \cdots v_n}] = U\ \Sigma$$

$$\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}$$

Examples

$$\underset{2\times 3}{\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}} = \underset{2\times 2}{U}\ \underset{2\times 3}{\Sigma}\ \underset{3\times 3}{V^T}$$

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix}$$

$$\underset{3\times 2}{\begin{bmatrix} 1 & 4 \\ 3 & 0 \\ 2 & 1 \end{bmatrix}} = \underset{3\times 3}{U} \; \underset{3\times 2}{\Sigma} \; \underset{2\times 2}{V^T}$$

$$\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}$$

$$\underset{3\times 3}{\begin{bmatrix} 1 & 4 & 3 \\ 0 & 0 & -1 \\ 2 & 8 & 1 \end{bmatrix}} = \underset{3\times 3}{U} \; \underset{3\times 3}{\Sigma} \; \underset{3\times 3}{V^T}$$

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underset{4\times 4}{\begin{bmatrix} 1 & 2 & 4 & 0 \\ 3 & 1 & 0 & 2 \\ -1 & 4 & 5 & 2 \\ 7 & 3 & -1 & 0 \end{bmatrix}} = \underset{4\times 4}{U} \; \underset{4\times 4}{\Sigma} \; \underset{4\times 4}{V^T}$$

$$\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \sigma_3 & \\ & & & \sigma_4 \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}, \quad \lambda_1 = 18, \quad \lambda_2 = 32$$

$$V = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$A^TA = V(\Sigma^2)V^T$$

$$= \begin{array}{c} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \\ \quad v_1 \qquad\quad v_2 \end{array} \begin{bmatrix} 18 & 0 \\ 0 & 32 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$Av_1 = \sigma_1 u_1$$

$$\overset{A}{\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}} \overset{v_1}{\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}} = \overset{\sigma_1}{\sqrt{18}} \overset{u_1}{\begin{bmatrix} a \\ b \end{bmatrix}}$$

$$\begin{bmatrix} 0 \\ 6/\sqrt{2} \end{bmatrix} = \sqrt{18} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

* Alternative method to find U is by finding evectors of (AA^T)

Keep in mind that (A^TA) and (AA^T) have the same non zero eigenvalues.

Example

$$\overset{A}{\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}} = \underset{2\times 2}{U} \; \underset{2\times 3}{\Sigma} \; \underset{3\times 3}{V^T}$$

$\text{eig}(A^TA) = 0, 0.265, 90.74$

$$\Sigma^2 = \begin{bmatrix} 90.74 & 0 & 0 \\ 0 & 0.265 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 0.22985 & 0.88346 & -0.40825 \\ 0.52474 & 0.24078 & 0.81650 \\ 0.81964 & -0.40190 & -0.40825 \end{bmatrix}$$

v_1

$A v_1 = \sigma_1 u_1$

$$\begin{bmatrix} 5.9023 \\ 7.4765 \end{bmatrix} = \sqrt{90.74}\begin{bmatrix} u_1 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 0.61961 \\ 0.78487 \end{bmatrix}$$

$$A V_2 = \sigma_2 u_2$$

$$\begin{bmatrix} -0.40370 \\ 0.31864 \end{bmatrix} = \sqrt{0.265}\begin{bmatrix} u_2 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} -0.78422 \\ 0.61898 \end{bmatrix}$$

$$A = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix} \begin{bmatrix} \sqrt{90.74} & 0 & 0 \\ 0 & \sqrt{0.265} & 0 \end{bmatrix} \begin{bmatrix} \text{—} & V_1^T & \text{—} \\ \text{—} & V_2^T & \text{—} \\ \text{—} & V_3^T & \text{—} \end{bmatrix}$$