# Linear Algebra II

TUTORIAL 6

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# Tests for Positive Definiteness:

Problem set 6.2: 1, 2, 4, 7, 8, 19, 23, 30

# Tutorial (6)
## Positive Definite Matrices

- All tests are done on ((Symmetric Matrices))
  $\hookrightarrow A = A^T$

* Tests for Positive Definiteness:

1- All EigenValues ($\lambda$) are Positive

2- All Pivots are Positive

3- All Sub-determinants are Positive

4- The Quadratic equation $x^T A x > 0$

\# (if you Put any Value for $x$ in the equation $x^T A x$, you'll always get a Positive number except for $x = 0$)

**1.** For what range of numbers $a$ and $b$ are the matrices $A$ and $B$ positive definite?

$$A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & b & 8 \\ 4 & 8 & 7 \end{bmatrix}.$$

1) $A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$

$\rightarrow$ A is Positive definite for $a > 2$

$B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & b & 8 \\ 4 & 8 & 7 \end{bmatrix}$

$\begin{vmatrix} 1 & 2 \\ 2 & b \end{vmatrix} > 0 \Rightarrow b - 4 > 0$

$b > 4$

$$\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 1 & 2 & 4 \\ 2 & b & 8 \\ 4 & 8 & 7 \end{vmatrix} \end{array} = \begin{vmatrix} b & 8 \\ 8 & 7 \end{vmatrix} - 2 \begin{vmatrix} 2 & 8 \\ 4 & 7 \end{vmatrix} + 4 \begin{vmatrix} 2 & b \\ 4 & 8 \end{vmatrix}$$

$= 7b - 64 - 28 + 64 + 64 - 16b$

$= 36 - 9b > 0$

$9b > 36$ $\quad$ $b > 4$

$\Rightarrow b > 4$

**2.** Decide for or against the positive definiteness of

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}^2 .$$

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

2- Pivots test : $R_2 + \frac{1}{2}R_1 \rightarrow R_2$

$R_3 + \frac{1}{2}R_1 \rightarrow R_3$

$$\Rightarrow \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3 + R_2 \rightarrow R_3$

$\rightarrow$ 3rd Pivot $= 0$ (not Positive)

3- Determinant test

$\det(A_1) = |2| = 2$

$\det(A_2) = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3$

$$\det(A_3) = \begin{vmatrix} \overset{+}{2} & \overset{-}{-1} & \overset{+}{-1} \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ -1 & 2 \end{vmatrix} - \begin{vmatrix} -1 & 2 \\ -1 & -1 \end{vmatrix}$$

$= 2(3) - 3 - 3 = 0 \rightarrow$ not Positive

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

4- Quadratic form ($X^TAX = 0$) test

$$X^TAX = [x_1 \;\; x_2 \;\; x_3] \begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & 2 & -1 & -1 \\ x_2 & -1 & 2 & -1 \\ x_3 & -1 & -1 & 2 \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$= 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3$

Put $X = (1, 1, 1)$

$X^TAX = 2 + 2 + 2 - 2 - 2 - 2 = 0$

not Positive

$\rightarrow$ This matrix is not Positive Definite matrix

# you can try only one test

**4.** Show from the eigenvalues that if $A$ is positive definite, so is $A^2$ and so is $A^{-1}$.

(4) if $A$ is Positive definite, it means that it's eigenValues ($\lambda$s) are Positive

$\because$ eigenValues of $A^2 = \lambda^2$

if $\lambda$ is +Ve $\rightarrow \therefore \lambda^2$ is +Ve

$\therefore A^2$ is Positive definite

$\because$ eigenValues of $A^{-1} = 1/\lambda$

$\therefore$ if $\lambda$ is +Ve $\rightarrow \therefore 1/\lambda$ is +Ve

$\therefore A^{-1}$ is Positive definite

**19.** Which 3 by 3 symmetric matrices $A$ produce these functions $f = x^{\mathrm{T}}Ax$? Why is the first matrix positive definite but not the second one?

(a) $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$.

(b) $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3)$.

19) $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$

A that Produce $f = x^T A x$

$\Rightarrow 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$\Rightarrow$ This matrix is Positive definite because it has Pivots : $2, 3/2, 4/3 > 0$

$f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3)$

$\Rightarrow 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3$

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$\Rightarrow$ This matrix is not Positive definite because if we Put $x = (1, 1, 1)$

$f = 2 + 2 + 2 - 2 - 2 - 2 = 0$ not Positive

**23.** Give a quick reason why each of these statements is true:

(a) Every positive definite matrix is invertible.

(b) The only positive definite projection matrix is $P = I$.

(c) A diagonal matrix with positive diagonal entries is positive definite.

(d) A symmetric matrix with a positive determinant might not be positive definite!

(23) a) Every Positive definite matrix is invertible

⇒ True, since the determinant of any Positive definite matrix is always greater than Zero, then it's not Singular
∴ invertible

b) The only Positive definite projection matrix is $P = I$

⇒ True, All Projection matrices except $I$ are singular, since the Projection matrix has eigen values: 0, 1 and $I$ has the eigenvalues 1 only.

c) A diagonal matrix with Positive diagonal entries is Positive definite

⇒ True, The diagonal entries of a diagonal matrix are its eigenvalues

d) A symmetric matrix with a +ve det might not be +ve def!
⇒ True, if the det is +ve it doesn't mean that the subdet of submatrices are +ve.

# THANK YOU