<u>Concluding the discussion on independent/dependent eigenvectors</u>

It hope it is clear by now when to expect to have independent eigenvectors and when not.

- If no repeated eigenvalues then eigenvectors are certainly going to be independent.
- If eigenvalues are repeated then we might or might not have independent eigenvectors.
  - *Case of independent eigenvectors:*
    If the null space of the shifted matrix ($A-\lambda I$) has a dimension that is equal to the multiplicity of λ then the eigenvectors corresponding to this value of λ are independent.
    Example: $A = \begin{matrix} 3 & 0 \\ 0 & 3 \end{matrix}$
    here we have $\lambda=3$ (repeated 2 times) and the dimension of the null space of the shifted matrix ($A-3I$) is 2. Therefore, we have independent eigenvectors.

    Example: $A = \begin{matrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{matrix}$
    here we have $\lambda=3$ (repeated 2 times) and the dimension of the null space of the shifted matrix ($A-3I$) is 2. Therefore, we have independent eigenvectors. We also have $\lambda=5$ and this will produce another independent eigenvector.
  - *Case of dependent eigenvectors:*
    If the null space of the shifted matrix ($A- \lambda I$) has a dimension that is less than the multiplicity of λ then we have dependent eigenvectors.
    Example: $A = \begin{matrix} 3 & 1 \\ 0 & 3 \end{matrix}$
    here we have $\lambda=3$ (repeated 2 times) and the dimension of the null space of the shifted matrix ($A-3I$) is 1 (which is less than the multiplicity of $\lambda=3$). Therefore, we have dependent eigenvectors (in other words, there is a shortage in eigenvectors).

<u>Diagonalizing a matrix</u>

We now know that we can write $Ax = \lambda x$. This in face is very useful in putting the matrix in simpler form (factorizing). Particularly, if the matrix A has independent eigenvectors then we can sort of foresee a matrix with numbers only (the eigenvalues, λ's) and another matrix with independent vectors (the eigenvectors, x's).

We know that $Ax_1 = \lambda_1 x_1$, also $Ax_2 = \lambda_2 x_2$, and so on. Now let's write all x's together in one matrix and same thing on the right hand side:

$$A\,[x_1\ x_2\ ..\ x_n] = [\lambda_1 x_1\ \ \lambda_2 x_2\ ..\ \lambda_n x_n]$$
$$= [x_1\ x_2\ ..\ x_n] \begin{pmatrix} \lambda_1 & 0 & .. & 0 \\ 0 & \lambda_2 & 0\ .. & 0 \\ . & . & . & \\ 0 & 0 & .. & \lambda_n \end{pmatrix}$$

diagonal matrix with λ's in the diagonal

$$A\,S = S\,\Lambda$$
$$S^{-1}AS = \Lambda \qquad \text{(diagonalization of A), or}$$
$$A = S\,\Lambda\,S^{-1} \qquad \text{(factorization of A)}$$

Note: this diagonalization/factorization can only be valid if A has independent eigenvectors (due to the inverse operation needed on S).

As a result, consider A^2 , if $Ax = \lambda x$

$A^2x = \lambda Ax = \lambda^2 x$ (eigenvalues of A^2 are λ^2, eigenvectors of A^2 are same) #1

$A = S \Lambda S^{-1}$
$A^2 = (S \Lambda S^{-1})(S \Lambda S^{-1})$
$= S \Lambda^2 S^{-1}$ (so this is similar to #1 but in a matrix form)
Furthermore, $A^k = S \Lambda^k S^{-1}$
We can see that eigenvalues tell a lot about the power of a matrix.

Theorem: $A^k \rightarrow 0$ as $k \rightarrow \infty$ if all $|\lambda_i| < 1$. If all $|\lambda_i| \geq 1$ then $A^k = S \Lambda^k S^{-1}$ is not going to work. (becasue we don't know how far k could go)

Notes:

- A pure eigenvalue/eigenvector approach needs independent eigenvectors. If we don't have independent eigenvectors then A can't be diagonalized using this approach.
- If A is diagonal then $A = \Lambda$

Example: $A = \begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}$

$|A-\lambda I| = (2-\lambda)^2 = 0$, $\lambda_1 = \lambda_2 = 2$, $(A-2I) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

We have one independent eigenvector $x = \begin{matrix} 1 \\ 0 \end{matrix}$

Therefore, we can't diagonalize A using eigenvalue, eigenvector approach.

Example: $A = \begin{matrix} 3 & 2 & 1 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{matrix}$

finding the eigenvalues using the characteristic equation; $\lambda_1 = 5$, $\lambda_2 = \lambda_3 = 1$
Take a quick look at the shifted A matrix using the repeated value, $(A-1I) = \begin{matrix} 2 & 2 & 1 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{matrix}$

you can see that its null space has a dimension of 1 which is less than 2, the multiplicity of $\lambda = 1$.
Therefore, once again we can't diagonalize A using eigenvalue, eigenvector approach.

Example: $A = \begin{matrix} 3 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{matrix}$

use the characteristic equation to find the eigenvalues: $\lambda_1 = 0.238$, $\lambda_2 = 1.636$, $\lambda_3 = 5.124$. since all λ's are different, then we know for sure that A can be written as $S \Lambda S^{-1}$. All we have to do now is to find the eigenvectors.

Look at this snapshot in octave, I am sure you can understand it by now.

```
>> A = [3 2 1; 2 3 0; 1 0 1]
>> [S, Lambda]=eig(A)
S =

  -0.55481   0.42186   0.71709
   0.40181  -0.61887   0.67495
   0.72852   0.66260   0.17385

Lambda =

Diagonal Matrix
```

You can just run eig(A) to return the eigenvalues of A. You can also return the arguments of eig(A) in two matrices S, Lambda

The eigenvectors are here. Corresponding to the λ's in the order they exist in the matrix Lambda. $S = [x_1\ x_2\ x_3]$

```
   0.23844         0         0
         0   1.63667         0
         0         0   5.12489
```

Lambda = λ_1 0 0
0 λ_2 0
0 0 λ_3

To double check:
>> S*Lambda*inv(S)
ans =

```
   3.00000   2.00000   1.00000
   2.00000   3.00000   0.00000
   1.00000  -0.00000   1.00000
```

which is obviously A