Symmetric matrices and positive definiteness.

Symmetric matrices has two good properties:
1) Their eigenvalues are real (can be proofed)
2) Their eigenvectors are perpendicular:

Proof: $A = S\Lambda S^{-1}$, $A = A^T = (S\Lambda S^{-1})^T$ because A is a symmetric matrix.

$$S\Lambda S^{-1} = (S\Lambda S^{-1})^T$$
$$S\Lambda S^{-1} = S^{-1T} \Lambda^T S^T \quad (\Lambda = \Lambda^T \text{ because } \Lambda \text{ is a diagonal matrix})$$
$$S\Lambda S^{-1} = S^{-1T} \Lambda S^T$$

pre-multiply both sides by S^{-1} and post-multiply both sides by S

$$S^{-1}S\Lambda S^{-1} S = S^{-1}S^{-1T} \Lambda S^TS$$
$$I \ \Lambda \ I = S^{-1}S^{-1T} \Lambda S^TS$$
$$\Lambda = S^{-1}S^{-1T} \Lambda S^TS$$
$$\Lambda = (S^TS)^{-1} \Lambda S^TS$$

This makes $S^TS = I$ which means that $S^T = S^{-1}$, this makes S an orthogonal matrix (i.e. a matrix with perpendicular columns). We usually use the letter Q for such matrices. Therefore,

for a symmetric matrix A, $A = Q \Lambda Q^{-1}$

Definition: the spectrum of a matrix is the set of eigenvalues of and eigenvectors of this matrix.

A symmetric matrix has good properties (real eigenvalues and perpendicular eigenvectors). We can decompose a symmetric matrix:

$$A = Q \Lambda Q^{-1}$$
$$= [q_1 \ q_2 \ ..] \operatorname{diag}(\lambda) \begin{pmatrix} q_1^T \\ q_2^T \\ .. \end{pmatrix}$$
$$= \lambda_1 \underbrace{q_1 q_1^T} + \lambda_2 \underbrace{q_2 q_2^T} + \ldots \qquad \text{(decomposition of symmetric matrix)}$$

projection
matrices

Every symmetric matrix is a combination of projection matrices.

<u>For a given square matrix A</u>
Product of pivots = product of λ's = det(A)
Signs of pivots same as signs of λ's
Number of positive pivots = number of positive λ's
Number of negative pivots = number of negative λ's

<u>Positive definite matrix</u>
A sub class of symmetric matrices that has an excellent properties:

1. All λ's of a positive definite matrix are positive.
2. All pivots of a positive definite matrix are positive.
3. All sub-determinants of a positive definite matrix are positive.

Example: $A = \begin{matrix} 5 & 2 \\ 2 & 3 \end{matrix}$ (A is a positive definite matrix)

Pivots are 5, 11/5
det(A) = 5*11/5 = 11

Find λ's : $\begin{matrix} 5-\lambda & 2 \\ 2 & 3-\lambda \end{matrix}$

$\lambda^2 - 8\lambda + 11 = 0$

$\lambda = 4 \pm \sqrt{5}$

Example: $B = \begin{matrix} -1 & 0 \\ 0 & -3 \end{matrix}$

B is not a positive definite matrix as you can see one of its pivots = -1

<u>Pivots, determinants, eigenvalues, stability condition x^TAx</u>

We have seen how can we tell if a matrix is positive definite. Why are we interested in positive definite matrices?

A 4th property of positive definite matrices is:

4. The quadratic equation $x^TAx > 0$ (this means that if you plug any value for x in the equation x^TAx, you will always get a positive number, except for x =[0]).

Example: $A = \begin{matrix} 2 & 6 \\ 6 & 18 \end{matrix}$

det(A) = 0 this makes A a positive <u>semi-</u>definite. Also, since it is a singular matrix then there must be a λ = 0 (the other λ is 20 in this example).We have one pivot = 2

Now with x^TAx:

$$[x_1 \ \ x_2] \begin{pmatrix} 2 & 6 \\ 6 & 18 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = [x_1 \ \ x_2] \begin{pmatrix} 2x_1 + 6x_2 \\ 6x_1 + 18x_2 \end{pmatrix}$$

$$= \underbrace{2x_1^2 + 12\,x_1x_2 + 18x_2^2}_{f(x_1,x_2)}$$

(this is a quadratic form, all terms are of power 2)

now check for every $[x_1 \ x_2]$ if this quantity > 0 or there could be a non zero $[x_1 \ x_2]$ that produces a quantity < 0. In this example, try (x_1= -3, x_2=1, the equation will =0) other values exist that would make the equation = 0 (for example x_1= 3, x_2= -1). f can never be < 0 in this example, In this case, A is positive semi-definite.

Example: $A = \begin{matrix} 2 & 6 \\ 6 & 7 \end{matrix}$

Here det(A) is negative, pivots are 2, -11 (so it is not positive definite).

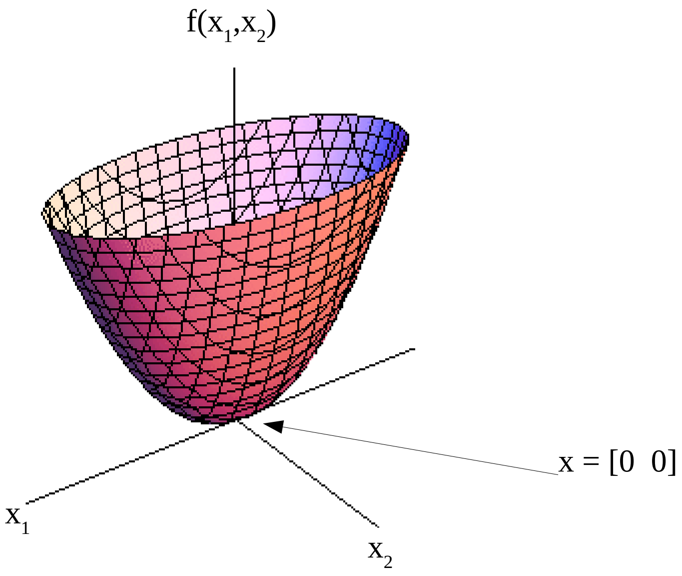
The quadratic equation: $2x_1^2 + 12\,x_1x_2 + 7x_2^2$ it is easy to make it negative (take x_1=1, x_2=-1). In this case, A is not positive definite not positive semi-definite.

Example: $A = \begin{matrix} 2 & 6 \\ 6 & 20 \end{matrix}$

You can easily find that A is positive definite here.

<u>The quadratic equation corresponding to a symmetric matrix A:</u>

In order to build a systematic mechanism to find if f(x1,x2) can be <0 for some values of x (besides x=[0]), let's take a look at the typical shape of a quadratic equation corresponding to a positive definite 2x2 matrix



1. Find the 1st order derivatives of f (i.e. $\partial f/\partial x_1$ and $\partial f/\partial x_2$) and equate them with 0, you will find that at x=[0] there is a minima/maxima. At x=[0] there is a minima/maxima.
2. The second order derivatives make us find out about if we increase/decrease in height as we depart the x=[0] point. Find the 2nd order derivatives and formulate the Hessian matrix:

$$H = \begin{bmatrix} \partial^2 f/\partial x_1^2 & \partial^2 f/\partial x_1 x_2 \\ \partial^2 f/\partial x_2 x_1 & \partial^2 f/\partial x_2^2 \end{bmatrix}$$

This is the Hessian matrix for a 2x2 matrix

3. H must be a positive definite in order for A to be positive semi-definite; in other words det(H) must be > 0 for A to be positive definite.

Note (and recap):
1) We resolve to using H as a check for positive definiteness (instead of the original matrix A) because in some applications H is available while A is not.
2) If A is +def then A^{-1} is also +def. This is because eigenvalues of A^{-1} are $1/\lambda_i$ (i.e. no change in signs of λ) where λ is the eigenvalues of A.
3) If A is a symmetric matrix and det(A) = 0 then A is +sdef. Example, $A = \begin{matrix} 1 & 3 \\ 3 & 9 \end{matrix}$
you still can conduct other checks (on pivot signs, on λ signs, and on the stability of x^TAx) but you don't need to.
4) If A and B are +def then (A+B) is also +def. The reason is:
since A is +def then $x^TAx>0$ and same goes for B; $x^TBx>0$
now we ask the question: is $x^T(A+B)x > 0$?
$x^TAx + x^TBx > 0$? Yes it must be >0 because both terms are > 0

5) if A is mxn matrix (rectangular) then A^TA is square matrix, and is symmetric. But is is +def?
Let's run the test: $x^T(A^TA)x > 0$?
$(x^TA^T)(Ax) > 0$?
$(Ax)^T(Ax) > 0$? this is the length^2 of the vector Ax so it must be ≥ 0
The case when Ax=0 (excluding the option that x is the zero vector) could happen if x comes from the null space of A (i.e. rank(A) < n) and the matrix A^TA is +sdef. Whereas if rank(A) = n (i.e. no null space) then the matrix A^TA is +sdef.

Drill exercise:
Without computing $B = A^T A$, is B +def or +sdef in the following cases:
$A_1 =$

$$\begin{matrix} 2 & 3 \\ 4 & 0 \\ -1 & 5 \end{matrix}$$

$A_2 =$

$$\begin{matrix} 3 & 4 & 0 \\ 1 & 2 & -1 \end{matrix}$$

6) +def matrices are computationally very safe because pivots will not be close to zero (remember we divide by pivots when we do elimination).