Faculty of Computers and Artificial intelligence

Discrete Mathematics

Lecture 8

Relations

Part 2

Relations and Their Properties

- **Definition – Composite**

Let R be a relation from a set A to a set B and S a relation from B to a set C. The ***composite*** of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote **the composite of R and S** by $S \circ R$.

Relations and Their Properties

- **Example 1:**

$S \circ R$

➢ What is the composite of the relations R **and** S,

where R is the relation from {1, 2, 3} to {1, 2, 3, 4} with

$R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and

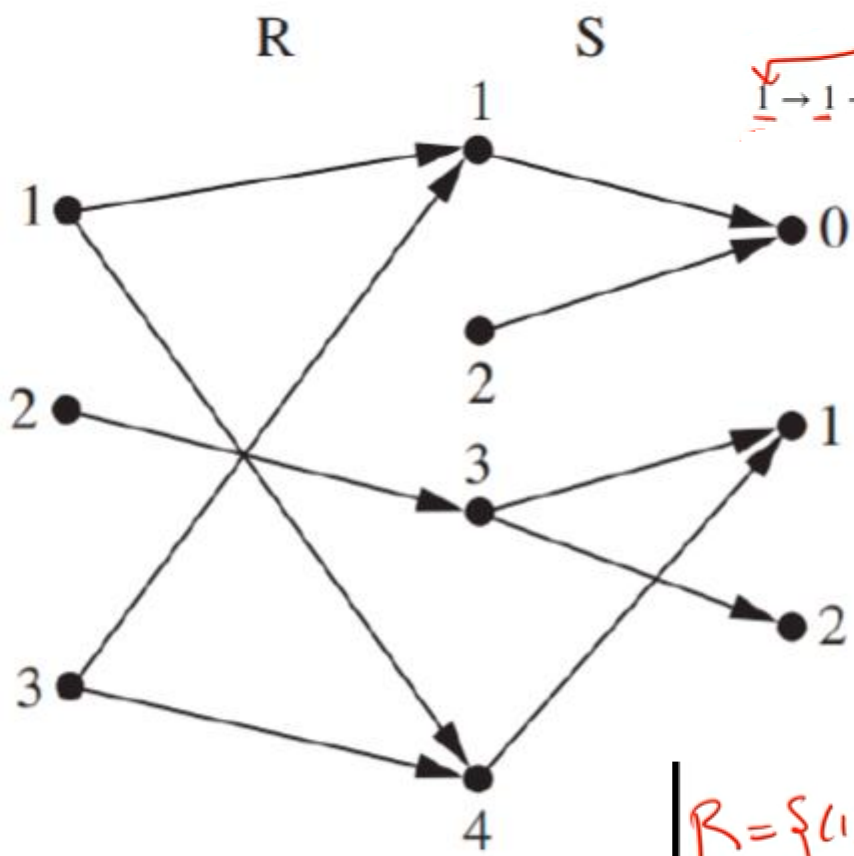
S is the relation from {1, 2, 3, 4} to {0, 1, 2} with

$S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Relations and Their Properties

❑ Definition – Composite

$s = \{(1,0), (2,0), (3,1), (3,2), (4,1)\}$ $\boxed{S \circ R}$



$1 \to 1 \to 0$ $\{(1, 0),$

$S \circ R$

$1 \to 1 \to 0$	$(1, 0)$
$1 \to 4 \to 1$	$(1, 1)$
$2 \to 3 \to 1$	$(2, 1)$
$2 \to 3 \to 2$	$(2, 2)$
$3 \to 1 \to 0$	$(3, 0)$
$3 \to 4 \to 1$	$(3, 1)$

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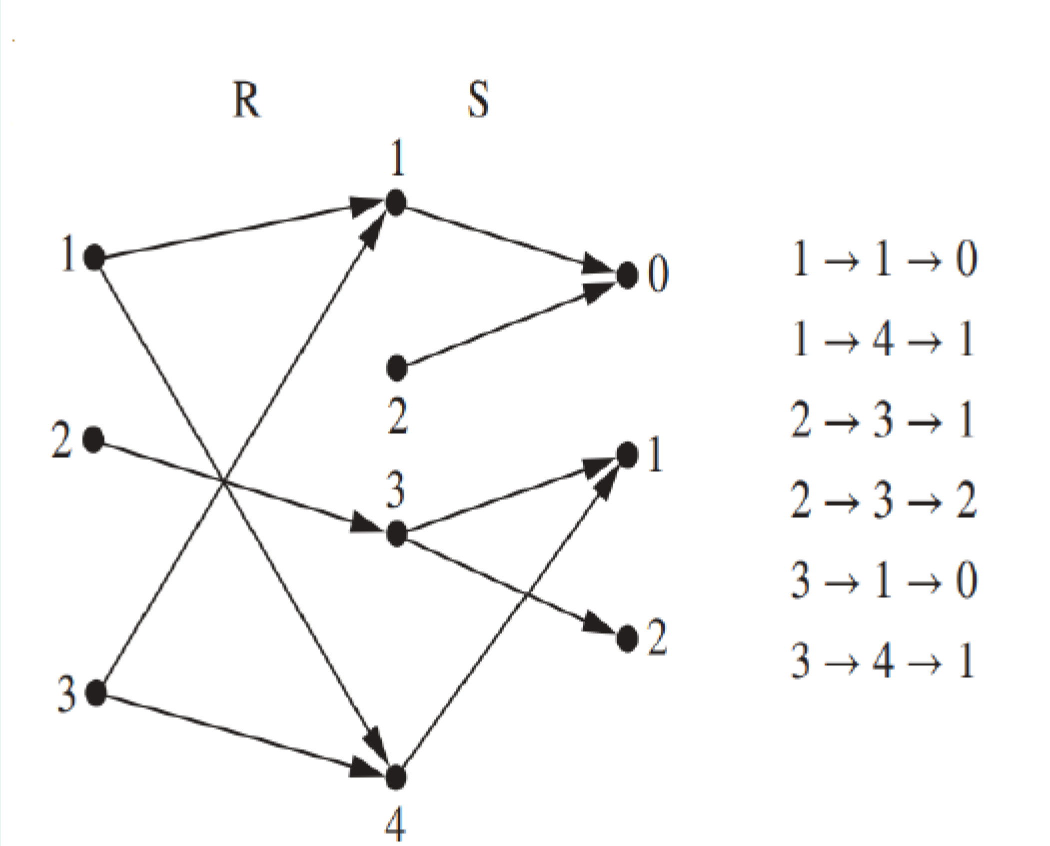
$R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$

Relations and Their Properties

- **Example 1:**

Solution:



$$S \circ R = \{(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\}$$

Relations and Their Properties

❑ Definition – Powers

Let R be a relation on the set A.

The powers R^n , $n = 1, 2, 3, \ldots,$ are defined recursively by

$$R^1 = R \text{ and } R^{n+1} = R^n \circ R$$

The definition shows that $R^2 = R \circ R$, $R^3 = R^2 \circ R$, and so on.

$$R^4 = R^3 \circ R \qquad R^5 = R^4 \circ R$$

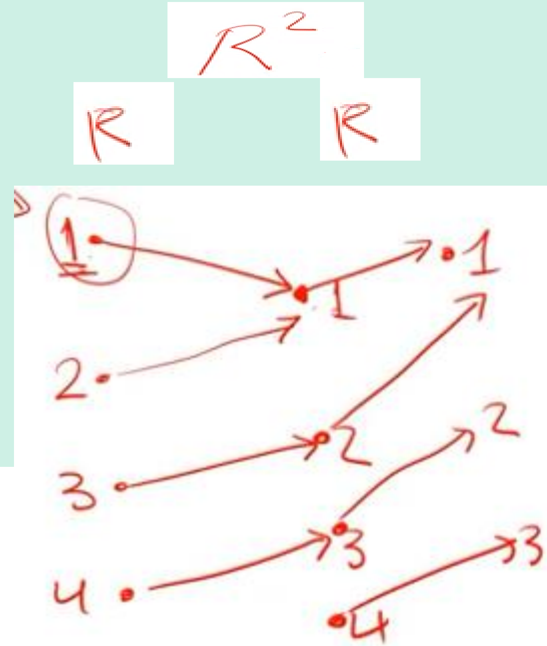
Relations and Their Properties

- **Example 2:**

S o R R → S

Let $R = \{(1,1),(2,1),(3,2),(4,3)\}$.

Find the powers R^n, $n = 2,3,4,\ldots$

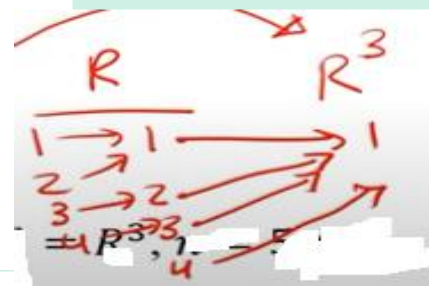


Solution:

$$R^2 = R \circ R = \{(1,1),(2,1),(3,1),(4,2)\}$$

$$R^3 = R^2 \circ R = \{(1,1),(2,1),(3,1),(4,1)\}$$

$$R^4 = R^3 \circ R = \{(1,1),(2,1),(3,1),(4,1)\}$$



R^4 is the same R^3, it is also following that $R^n = R^3$, $n = 5,6,7,\ldots$

Relations and Their Properties

❑ n-ary Relations

Let $A_1, A_2, \dots, A_n$ be sets. An ***n-ary relation*** on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$.

The sets $A_1, A_2, \dots, A_n$ are called the ***domains*** of the relation, and n is called its ***degree***.

❑ Example 1:

Let R be the relation on $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$ consisting of triples

ordered (a, b, c), where a, b, and c are integers with $a < b < c$.

Then $(1, 2, 3) \in R$, but $(2, 4, 3) \notin R$. The degree of this relation is 3. Its domains are all equal to the set of natural numbers $\mathbf{N}$.

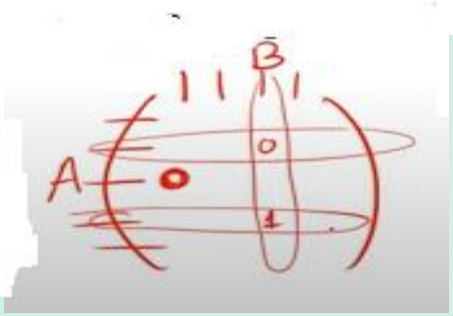
Representing Relations

❑ Representing Relations Using Matrices

A relation between finite sets can be represented using a **zero–one matrix**. Suppose that R is a relation from $A = \{a_1, a_2, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$.

The relation R can be represented by the matrix $\mathbf{M}_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$



Representing Relations

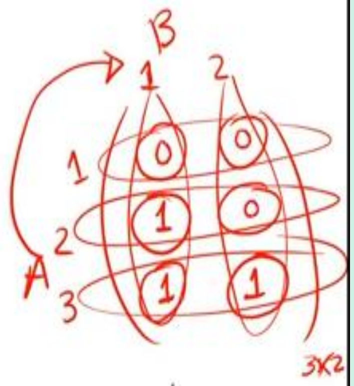
- **Example 1:**

$A \times B = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}$

Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$.

What is the matrix representing R ($\mathbf{M}_R$) if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

M_R



Solution:

What is the matrix representing if

$R = \{(2, 1), (3, 1), (3, 2)\}$

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

$a > b$

Representing Relations

- **Example 2:**

Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d, e\}$. Which ordered pairs are in the relation R represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

$$\mathbf{M}_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}?$$

Solution:

$$R = \{(1, b), (2, a), (2, c), (2, d), (3, a), (3, c), (3, e)\}$$

Representing Relations

- **Example 3:**

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that:

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), \{(a_3, b_3), (a_3, b_5)\}.$$

Representing Relations

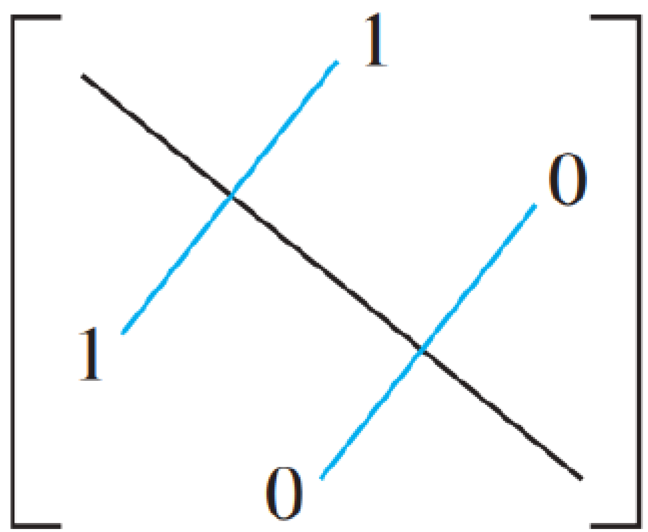
The zero–one Matrix for a Reflexive Relation.

(Off diagonal elements can be 0 or 1):

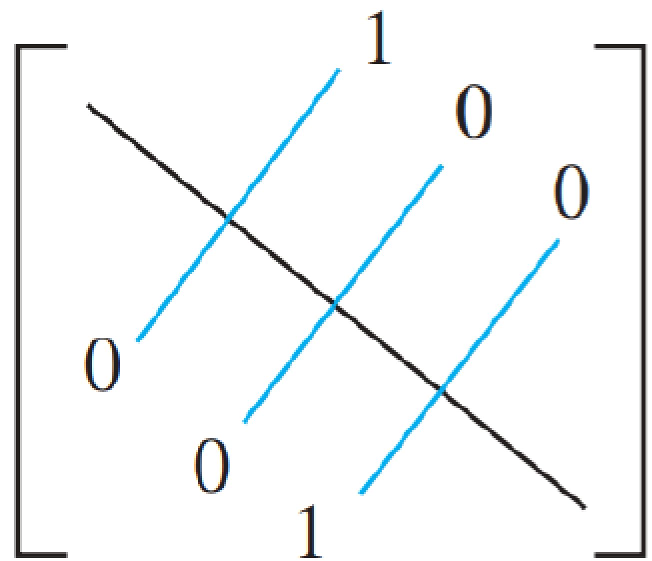
$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

Representing Relations

Matrices for Symmetric and Antisymmetric Relations (Diagonal elements can be 0 or 1):



(a) Symmetric



(b) Antisymmetric

Representing Relations

- **Example 4:**

Suppose that the relation R on a set is represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution:

Because all the diagonal elements of this matrix are equal to **1**, R is **reflexive**. Moreover, $\mathbf{M}_R$ is **symmetric**. It is also easy to see that R is **not antisymmetric**.

Representing Relations

❑ The Boolean Operations

The Boolean operations *join* and *meet* can be used to find the matrices representing the union and the intersection of two relations.

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} \quad \text{and} \quad \mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}.$$

join *meet*

$0 \vee 0 = 0$ $1 \wedge 1 = 1$

Representing Relations

- **Example 4:**

Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

$\vee$ $\wedge$

Representing Relations

- **Example 4:**

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Solution:

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Questions?