Faculty of Computers and Artificial intelligence

Discrete Mathematics

Lecture 6

Proof Techniques & Mathematical Induction

Proof Techniques

- **Definition 1:**

A **theorem** is a statement that can be shown to be true. We demonstrate that a theorem is true with a proof.

Definition 2:

A **proof** is a valid argument that establishes the truth of a theorem.

Proof Techniques

❑ Even and Odd Integers

Even Integer:

$$2 * (Any\ Integer) = even$$

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if a is an even number, so you can write it as follows:

2n 2m 2K

$$a = 2n, \qquad \text{where } n \text{ is integer}$$

$$\text{Even} + 1 = \text{Odd}$$
$$\text{Odd} + 1 = \text{Even}$$

Proof Techniques

Odd Integer:

$5 = 2 \cdot 2 + 1$ $\quad$ $2k$ $\quad$ $8 = 2 \cdot 4$

if a is an odd number, so you can write it as follows:

$$a = 2m + 1, \qquad \text{where } m \text{ is integer}$$

$$\neg\text{Even} = \text{Odd}$$
$$\neg\text{Odd} = \text{Even}$$

Prefect Square

if a is a prefect square, so you can write it as follows:

$$a = (n)^2, \qquad \text{where } n \text{ is integer}$$

Proof Techniques

Rational Number

The real number r is rational if there exist integers p and q where q≠0 such that r = p/q

$r = \frac{p}{q} \neq 0$

- **Direct Proof**

– An implication p→q can be proved by showing that if p is true, then q is also true.

$$p \to q$$

$T \to T$ True

1. We assume that p is true
2. We try to prove that q is also true
3. Then $p \to q$ is true.

Proof Techniques

❑ Example1

Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

$1^2 = 1$ $3^2 = 9$ $5^2 = 25$

Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

$$\underbrace{\text{"If } n \text{ is an odd integer,}}_{p} \text{ then } \underbrace{n^2 \text{ is odd."}}_{q}$$

$p \to q$ True

1. We assume that p is true
2. We try to prove that q is also true
3. Then $p \to q$ is true.

$p \to q$

① Assume
② prove
③ ∴ $p \to q$ True

Proof Techniques

❑ Example 1

$n = 2m+1$
$n^2 = (2m+1)^2$

Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

$2 * (Any\ Integer) = even$

1. We assume that p is true

$$n = 2m + 1, \qquad \text{where } m \text{ is integer.}$$

$n = 2m + 1,$
even
odd

2. We try to prove that q is also true

$$n^2 = (2m+1)^2$$
$$= 4m^2 + 4m + 1$$
$$= 2(2m^2 + 2m) + 1$$
$$= even \qquad + 1 = odd$$

$4(m^2+m)$

$2 * (Any\ Integer) = even$

$2(2m^2+2m)$

3. $\therefore p \rightarrow q$ is true.

q is also true

Proof Techniques

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- **Example 2**

➢ Prove that the sum of two rational numbers is rational.

Solution: Assume r and s are two rational numbers.

Then there must be integers p, q and also t, u such that

$$r = p/q, \quad s = t/u, \quad u \neq 0, \; q \neq 0$$

$$r + s = \frac{p}{q} + \frac{t}{u} = \frac{pu+qt}{qu} = \frac{v}{w} \qquad \text{where } v = pu + qt$$

$$w = qu \neq 0$$

Thus the sum is rational.

$$r + s = \frac{p}{q} + \frac{t}{u} = \frac{up + tq}{qu}$$

Proof Techniques

Example 3

Show that the sum of two odd integers is even.

$2(k)$

– Let n = 2k+1, m=2j+1 be odd integers

$2 * (Any\ Integer) = even$

– n+m = 2k+1 + 2j+1 = 2k + 2j + 2 = 2(k+j+1) is even.

Example 4

- If m and n are both perfect squares, then nm is also a perfect squares.
  - Assume m and n are perfect squares, then:
    - $m = s^2, \ n = t^2, \quad s, t \in Z$
    - $mn = s^2t^2 = (ss)(tt) = (st)(st) = (st)^2$
  - → mn is also a perfect square.

$9 = 3^2$

$4 = 2^2$

$25 = 5^2$

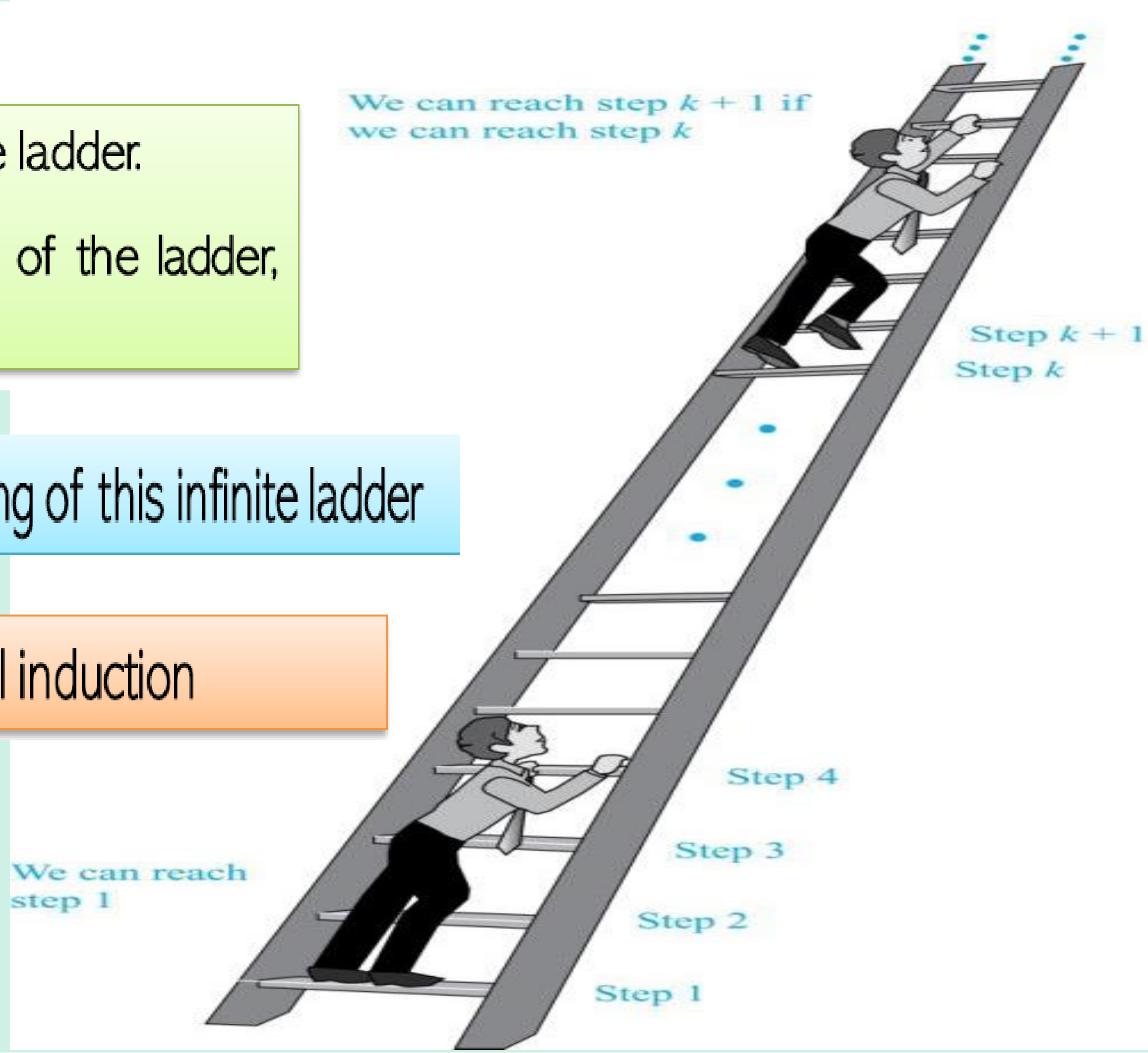
Mathematical Induction

Infinite ladder

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

Therefore, we are able to reach every rung of this infinite ladder

Using proof technique called mathematical induction



Mathematical Induction

Note:

Mathematical induction is <u>not</u> a tool for discovering formulae or theorems.

Mathematical Induction definition:

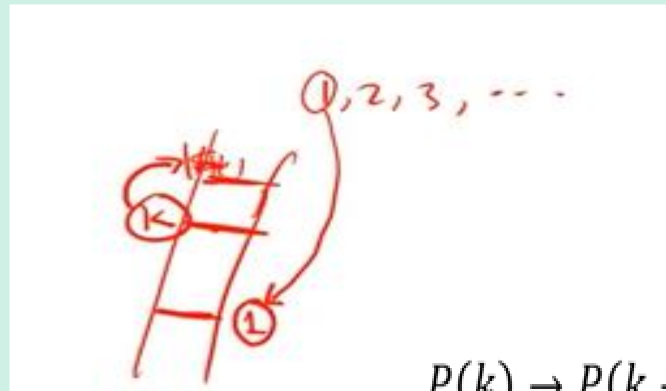
Mathemaical induction can be used to prove statments that assert that P(n) is true for all positive integers n, where P(n) is a propositional function.

Mathematical Induction

Principle of Mathematical Induction

To prove that P(n) is true for all positive integers n, where P(n) is a propositional function,

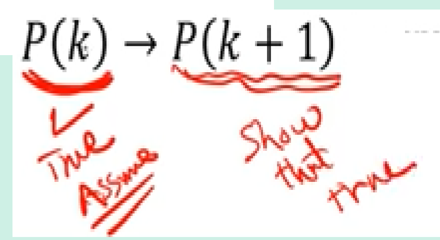
we complete two steps:



Basis Step

We verify that P(1) is true.

Inductive Step



We show that the conditional statement

P(k) $\to$ P(k+1) is true for all positive integers k.

Mathematical Induction

Principle of Mathematical Induction

To complete the inductive step of a proof using the principle of mathematical induction, we assume that $P(k)$ is true for an arbitrary positive integer k and show that under this assumption, $P(k+1)$ must also be true. The assumption that $P(k)$ is true is called the ***inductive hypothesis*** (**IH**).

$$\boxed{\forall k(P(k) \rightarrow P(k+1))}$$

1, 2, 3, ...

P(n) true $n \geq 1$

Remark: In a proof by mathematical induction, it is **not** assumed that $P(k)$ is true for all positive integers! It is only shown that if it is assumed that $P(k)$ is true, then $P(k+1)$ is also true.

Mathematical Induction

Principle of Mathematical Induction

Expressed as a rule of inference, this proof technique can be stated as:

$$[P(1) \wedge \forall k(P(k) \rightarrow P(k+1))] \rightarrow \forall n P(n)$$

when the domain is the set of positive integers.

① B
$P(1)$ = true ✓

non-negative integers
0, 1, 2, ...

$n = 1, 2, 3, \ldots$

$n \geq 4$ 4, 5, 6, ...

Remark: In a proof by mathematical induction, for basis step, we **not always** start at the integer 1. In such a case, the basis step begins at a starting point b where b is an integer.

Mathematical Induction

Notes for Proofs by Mathematical Induction

- Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
  - ✓ for all positive integers n, let $b = 1$, and $P(1)$
  - ✓ for all nonnegative integers n, let $b = 0$, and so on … $P(0)$

- Write out the words "Basis Step." Then show that $P(b)$ is true.

- Write out the words "Inductive Step" and state, and clearly identify, the inductive hypothesis, in the form "Assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."

Mathematical Induction

Notes for Proofs by Mathematical Induction

- State what needs to be proved under the assumption that the inductive hypothesis (IH) is true.
  - ✓ That is, write out what $P(k+1)$ says.
- Show that $P(k+1)$ is true under the assumption that $P(k)$ is true.
  - ✓ The most difficult part of a mathematical induction proof.
  - ✓ This completes the inductive step.
- After completing the basis step and the inductive step, state the conclusion, namely, "By mathematical induction, $P(n)$ is true for all integers n with $n \geq b$".

Mathematical Induction

Example 1:

Use mathematical induction to prove that

$$\sum_{i=1}^{n} i = 1 + 2 \ + 3 \cdots +n = \frac{n(n+1)}{2}$$

For all positive integers n . $(n \geq 1)$

Mathematical Induction

Example 1 – Answer:

Let P(n) be the proposition that

$$1 + 2 \;\; + 3 \;\cdots\; + n \;= \frac{n(n+1)}{2}$$

$n \geq 1$

$\{1, 2, 3, ...\}$

$P(1)$

1) Basis Step:

If $\boldsymbol{n} = \mathbf{1}$. $P(1)$ is **true**, because $1 = \frac{(1)(2)}{2}$

This completes the basis step.

2) Inductive Step:

We first **Assume** that (Inductive Hypothesis (IH)) $P(k)$ is true for the positive integer k, i.e.: $P(k)$

$k \in \{1, 2, 3, ...\}$

$$"1 + 2 \;\; + 3 \;\cdots\; + k = \frac{k(k+1)}{2}".$$

Mathematical Induction

Example 1 – Answer:

$P(k)$ $\boxed{\text{"}1+2\ +3\ \cdots\ +k=\dfrac{k(k+1)}{2}\text{"}.}$

$\dfrac{(k+1)(k+2)}{2}$

We **need to show** that if $P(k)$ is true, then $P(k+1)$ is true.

i. e. , we need to show that $P(k+1)$ is also true.

$$1+2\ +3\ \cdots\ +k+(k+1)=\frac{(k+1)[(k+1)+1]}{2}=\frac{(k+1)(k+2)}{2}$$

Mathematical Induction

Example 1 – Answer:

$P(k)$ $\boxed{\text{"}1+2\ +3\ \cdots\ +k=\dfrac{k(k+1)}{2}\text{".}}$

We **add** $(\boldsymbol{k}+\mathbf{1})$ to both sides of the equation in $P(k)$, we obtain

$$1+2\ \ +3\ \cdots\ +k+\boxed{(k+1)}\overset{\text{IH}}{=}\frac{k(k+1)}{2}+\boxed{(k+1)}$$

$\dfrac{2(k+1)}{2}$

$$=\frac{k(k+1)+2(k+1)}{2}$$

$$=\frac{(k+1)(k+2)}{2}$$

$\dfrac{(k+1)(k+2)}{2}$

- This equation show that $P(k+1)$ is true under the assumption that $P(k)$ is true.
- This completes the inductive step.

Mathematical Induction

Example 1 – Answer:

So, by mathematical induction we know that $P(n)$ is true for all positive integers n.

That is, we proven that

$$1 + 2 \ + 3 \ \cdots \ + n \ = \frac{n(n+1)}{2}$$

for all positive integers n.

Mathematical Induction

Example 2 :

Use mathematical induction to prove that

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 \; + 3^2 \cdots + n^2 \; = \frac{n(n+1)(2n+1)}{6}$$

For all positive integers n . (i.e., $n \geq 1$)

Mathematical Induction

Example 2 – Answer :

Let $P(n)$ be the proposition that

$n \geq 1$

$$1^2 + 2^2 + 3^2 \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

1) Basis Step:

If $\boldsymbol{n} = \mathbf{1}$. $P(1)$ is **true**, because $1^2 = 1 = \frac{(1)(2)(3)}{6}$

$\frac{1(2)(3)}{6} = \frac{6}{6} = 1$

This completes the basis step.

2) Inductive Step:

We first **Assume** that (Inductive Hypothesis (IH)) $P(k)$ is true for the positive integer k, i.e.: $P(k)$

$k \in \{n \geq 1\}$

$$"1^2 + 2^2 + 3^2 \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}".$$

Mathematical Induction

Example 2 – Answer :

$P(k)$

$$"1^2+2^2\ +3^2\ \cdots+k^2=\frac{k(k+1)(2k+1)}{6}".$$

$$\boxed{\frac{(k+1)(k+2)(2k+3)}{6}}$$

We **need to show** that if $P(k)$ is true, then $P(k+1)$ is true.

i. e. : we need to show that $P(k+1)$ is also true.

$$1^2+2^2\ +3^2\ \cdots+k^2+(k+1)^2=\frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

$$1^2+2^2\ +3^2\ \cdots+k^2+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}$$

Mathematical Induction

Example 2 – Answer :

We **add** $(\boldsymbol{k}+\mathbf{1})^{\mathbf{2}}$ to both sides of the equation in $P(k)$, we obtain

$$1^2 + 2^2 + 3^2 \cdots + k^2 + (k+1)^2 \overset{\text{IH}}{=} \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)(k(2k+1) + 6(k+1))}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

2k + 3
k + 2

Mathematical Induction

Example 2 – Answer :

- This equation show that $P(k+1)$ is true under the assumption that $P(k)$ is true.
- This completes the inductive step.

So, by mathematical induction we know that $P(n)$ is true for all positive integers n.

That is, we proven that

$$1^2 + 2^2 \; + 3^2 \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all positive integers n.

Mathematical Induction

Example 3:

Use mathematical induction to prove that

$$1+3+5+7+....+(2n-1) = n^2$$

For all positive integers n. (i.e., $n \geq 1$)

Mathematical Induction

Example 3 – Answer :

Let $P(n)$ be the proposition that

$$1+3+5+7+....+(2n-1) = n^2$$

<u>**1) Basis Step:**</u>

If n = 1 P(1) is true , because $1 = 1^2$

<u>**2) Inductive Step:**</u>

We first **Assume** that (Inductive Hypothesis (IH)) $P(k)$ is true for the positive integer k, i.e.: $P(k)$

$$1+3+5+7+....+(2k-1) = k^2$$

Mathematical Induction

Example 3 – Answer :

We **need to show** that if $P(k)$ is true, then $P(k+1)$ is true.

i. e., we need to show that $P(k+1)$ is also true.

$$1+3+5+7+....+(2k-1)+(2k+1) = (k+1)^2$$

We **add** $(\boldsymbol{k}+\mathbf{1})$ to both sides of the equation in $P(k)$, we obtain

$$\mathbf{1+3+5+7+....+(2k-1) = } k^2$$

$$1+3+5+7+....+(2k-1)+(2k+1) = (k)^2+(2k+1)$$

$$= (k+1)^2$$

Mathematical Induction

Example 3 – Answer :

- This equation show that $P(k+1)$ is true under the assumption that $P(k)$ is true.
- This completes the inductive step.

So, by mathematical induction we know that $P(n)$ is true for all positive integers n.

That is, we proven that

$$1+3+5+7+....+(2n-1) = n^2$$

for all positive integers n.

Mathematical Induction

Example 4 :

Use mathematical induction to prove that

$$n < 2^n$$

For all positive integers n. (i.e., $n \geq 1$)

Mathematical Induction

Example 4 – Answer :

Let $P(n)$ be the proposition that

$$n < 2^n$$

$$1 < 2^1$$

1) Basis Step:

If $\boldsymbol{n = 1}$. $P(1)$ is **true**, because $1 < 2^1$

This completes the basis step.

2) Inductive Step:

We first **Assume** that (Inductive Hypothesis (IH)) $P(k)$ is true for the positive integer k, i.e.: $P(k)$

$$k < 2^k$$

Mathematical Induction

Example 4 – Answer :

$P(k)$ $\boxed{k < 2^k}$

We **need to show** that if $P(k)$ is true, then $P(k+1)$ is true.

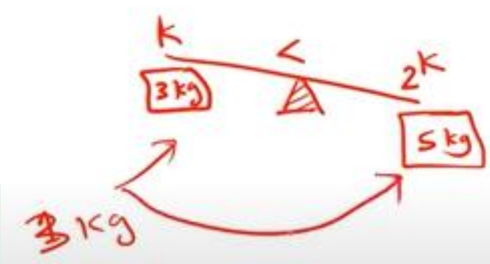
i. e., we need to show that $P(k+1)$ is also true.

$\boxed{k+1 < 2^{k+1}}$

$$(k+1) < 2^{k+1}$$

We **add** $(\mathbf{1})$ to both sides of the equation in $P(k)$, we obtain

$$(k+\boxed{1}) \overset{\text{IH}}{<} 2^k + \boxed{1}$$



Mathematical Induction

Example 4 – Answer :

$P(k)$ $\boxed{k < 2^k}$

IH

$(k+1) < 2^k + \boxed{1}$

$2^k > 1$

$2^1 \cdot 2^k$

$2^k + 2^k$

2^{k+1}

Because the integer $k \geq 1$. Therefore, $2^k > 1$

$(k+1) < 2^k + \boxed{2^k}$

$2^k > 1$

$2^1 > 1$

$n \geq 1$

$\{1, 2, 3, \ldots\}$

$2 + 2 = 2 \times 2$

$(k+1) < 2 \cdot 2^k$

$(k+1) < 2^{k+1}$

$2^m + 2^m$

$2 * 2^m$

$5 + 5 = 5 \times 2$

- This equation show that $P(k+1)$ is true under the assumption that $P(k)$ is true.
- This completes the inductive step.

Mathematical Induction

Example 4 – Answer :

So, by mathematical induction we know that $P(n)$ is true for all positive integers n.

That is, we proven that

$$n < 2^n$$

for all positive integers n.

Questions?