To Continue the discussion on eigenvalues/eigenvectors we need to take a further look on how the eigenvectors of a given matrix are profiled. We can do this through some examples.

Example 1:

$$A = \begin{bmatrix} 3 & 1 & 5 \\ 2 & 0 & -1 \\ 2 & 1 & 1 \end{bmatrix}$$

This matrix is a full rank. So the shortcuts might not be easy to apply. So we resolve to using the characteristic equation. $|A - \lambda I| = 0$. I'll use octave tool

>> eig(A) this gives

$\lambda_1 = 5.69$

$\lambda_2 = -0.84 + 0.93i$

$\lambda_3 = -0.84 - 0.93i$

The matrix is close to being anti-symmetric (skew-symmetric), that is because the 0 in the diagonal and the $a_{23} = -a_{32}$. Also, note that λ_2 and λ_3 are complex conjugate

Example 2:

$$A = \begin{bmatrix} 3 & 1 & 5 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$\lambda_1 = 5.72$, $\lambda_2 = -1.81$, $\lambda_3 = 0.09$

Again, we found these by solving the c/c equation or by calling eig(A) in octave.

Example 3:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Obviously the λ's are all $= 3$. A quick insight to find λ's comes from asking the question: What are the values I need to use to make A singular? (singular means make one of the columns without a pivot.

- Since A has 3 columns then we will have 3 λ's (we know that).
- Now let's find the eigenvectors:

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{(A-\lambda I)} x = 3x$$

$\lambda = 3$

x comes from $N(A)$, $N(A) = R^3$

, x can be any vector in R^3 so the eigenvectors are:

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Note: the eigenvectors are independent (actually they are the basis of R^3)

Example 4:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Obviously, $\lambda_1 = \lambda_2 = 3$, $\lambda_3 = 2$

$|A - \lambda I|$ will become zero because 1^{st} & 2^{nd} Columns will become zero

$|A - \lambda I|$ will become zero because 3^{rd} column will become $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

now, let's find the eigenvectors:

$\underline{\lambda_1 \cdot \lambda_2 = 3}$

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{(A-3I)} x = 3x$$

Comes from N(A) which has dim=2

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}, \text{ t any scalar}$$

pretty much the same thing!

$$, x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda_3 = 2$

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{(A-2I)} x = 2x$$

Comes from N(A) which has dim=1

$$x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So now
$$\underbrace{x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{independent}}$$

Example 5:

$$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}, \lambda_1 = \lambda_2 = 3$$
$$x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_2$$

Here λ's are equal (similar to Example 4 and 3) but here we have shortage of eigenvectors. By this we mean that the eigenvectors don't form an independent set of vectors. So if we construct a matrix $S = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$ then S will be singular.

eigenvectors

Conclusion: Equal λ's might or might not produce independent eigenvectors.

Example 6:

$$A = \begin{bmatrix} 0.87 & 0.07 & -0.33 \\ 0.07 & 0.97 & 0.17 \\ -0.33 & 0.17 & 0.17 \end{bmatrix}$$

This matrix is symmetric. $\text{rank}(A) = 2$

$\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 1$ Then this must be a Projection matrix. The Subspace that this P represents has dim $= 2$ (because $\lambda = 1$ is repeated 2 times).