Determinant

4.2 Properties of the Determinant

4.2 Properties of the Determinant

definition in the 2 by 2 case,

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

$\det A$ and $|A|$. Properties 4–10 will be deduced from the previous ones.

1. *The determinant of the identity matrix is* 1.

$$\det I = 1 \qquad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad \text{and} \qquad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \qquad \text{and} \ldots$$

2. *The determinant changes sign when two rows are exchanged.*

$$\textbf{Row exchange} \qquad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

The determinant of every *permutation matrix* is $\det P = \pm 1$. By row exchanges, we can turn P into the identity matrix. Each row exchange switches the sign of the determinant, until we reach $\det I = 1$. Now come all other matrices!

3. *The determinant depends linearly on the first row.* Suppose A, B, C are the same from the second row down—and row 1 of A is a linear combination of the first rows of B and C. Then the rule says: $\det A$ *is the same combination of* $\det B$ *and* $\det C$.

Linear combinations involve two operations—adding vectors and multiplying by scalars. Therefore this rule can be split into two parts:

Add vectors in row 1 $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}.$

Multiply by t in row 1 $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$

Notice that the first part is *not* the false statement $\det(B+C) = \det B + \det C$. You cannot add all the rows: only one row is allowed to change. Both sides give the answer $ad + a'd - bc - b'c$.

The second part is not the false statement $\det(tA) = t \det A$. The matrix tA has a factor t in *every* row (and the determinant is multiplied by t^n). It is like the volume of a box, when all sides are stretched by 4. In n dimensions the volume and determinant go up by 4^n. If only one side is stretched, the volume and determinant go up by 4; that is rule 3. By rule 2, there is nothing special about the first row.

4. *If two rows of A are equal, then* $\det A = 0$.

$$\textbf{Equal rows} \qquad \begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ba = 0.$$

This follows from rule 2, since if the equal rows are exchanged, the determinant is supposed to change sign. But it also has to stay the same, because the matrix stays the same.

5. *Subtracting a multiple of one row from another row leaves the same determinant.*

$$\textbf{Row operation} \qquad \begin{vmatrix} a - \ell c & b - \ell d \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Rule 3 would say that there is a further term $-\ell \begin{vmatrix} c & d \\ c & d \end{vmatrix}$, but that term is zero by rule 4. The usual elimination steps do not affect the determinant!

7. *If A is triangular then* $\det A$ *is the product* $a_{11}a_{22}\cdots a_{nn}$ *of the diagonal entries. If the triangular A has 1s along the diagonal, then* $\det A = 1$.

$$\textbf{Triangular matrix} \qquad \begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad \qquad \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = ad.$$

Proof. Suppose the diagonal entries are nonzero. Then elimination can remove all the off-diagonal entries, without changing the determinant (by rule 5). If A is lower triangular, the steps are downward as usual. If A is upper triangular, the *last* column is cleared out first—using multiples of a_{nn}. Either way we reach the diagonal matrix D:

$$D = \begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{bmatrix} \quad \text{has} \quad \det D = a_{11}a_{22}\cdots a_{nn} \det I = a_{11}a_{22}\cdots a_{nn}.$$

If a diagonal entry is zero then elimination will produce a zero row. By rule 5 these elimination steps do not change the determinant. By rule 6 the zero row means a zero determinant. This means: When a triangular matrix is *singular* (because of a zero on the main diagonal) its determinant is *zero*.

This is a key property. **All singular matrices have a zero determinant**.

8. *If A is singular, then* $\det A = 0$. *If A is invertible, then* $\det A \neq 0$.

$$\textbf{Singular matrix} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{is not invertible if and only if} \quad ad - bc = 0.$$

If A is singular, elimination leads to a zero row in U. Then $\det A = \det U = 0$. If A is nonsingular, elimination puts the pivots $d_1, \ldots, d_n$ on the main diagonal. We have a "product of pivots" formula for $\det A$! The sign depends on whether the number of row exchanges is even or odd:

$$\textbf{Product of pivots} \qquad \det A = \pm \det U = \pm d_1 d_2 \cdots d_n. \tag{1}$$

9. *The determinant of AB is the product of* $\det A$ *times* $\det B$.

$$\textbf{Product rule } |A||B| = |AB| \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & f \\ g & h \end{vmatrix} = \begin{vmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{vmatrix}.$$

A particular case of this rule gives the determinant of A^{-1}. It must be $1/\det A$:

$$\det A^{-1} = \frac{1}{\det A} \quad \text{because} \quad (\det A)(\det A^{-1}) = \det AA^{-1} = \det I = 1. \tag{2}$$

10. *The transpose of A has the same determinant as A itself:* $\det A^{\mathrm{T}} = \det A$.

$$\textbf{Transpose rule} \qquad \left|A\right| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = \left|A^{\mathrm{T}}\right|.$$

Determinant

4.3 Formulas for the Determinant

4.3 Formulas for the Determinant

For $n = 2$, we will be proving that $ad - bc$ is correct. For $n = 3$, the determinant formula is again pretty well known (it has six terms):

$$\begin{vmatrix} a_{11} & \boldsymbol{a}_{12} & a_{13} \\ a_{21} & a_{22} & \boldsymbol{a}_{23} \\ \boldsymbol{a}_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} +a_{11}a_{22}a_{33} + \boldsymbol{a}_{12}\boldsymbol{a}_{23}\boldsymbol{a}_{31} + a_{13}a_{21}a_{32} \\ -a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \end{matrix} \tag{2}$$

Our goal is to derive these formulas directly from the defining properties 1–3 of $\det A$. If we can handle $n = 2$ and $n = 3$ in an organized way, you will see the pattern.

To start, each row can be broken down into vectors in the coordinate directions:

$$\begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} a & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} c & d \end{bmatrix} = \begin{bmatrix} c & 0 \end{bmatrix} + \begin{bmatrix} 0 & d \end{bmatrix}.$$

Then we apply the property of linearity, first in row 1 and then in row 2:

Separate into $n^n = 2^2$ easy determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$
$$= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}. \tag{3}$$

Every row splits into n coordinate directions, so this expansion has n^n terms. Most of those terms (all but $n! = n$ factorial) will be automatically zero. When two rows are in the same coordinate direction, one will be a multiple of the other, and

$$\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} = 0, \qquad \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} = 0.$$

the numbers $1,2,\ldots,n$. The 3 by 3 case produces $3! = 6$ determinants:

$$\begin{vmatrix} a_{11} & \boldsymbol{a_{12}} & a_{13} \\ a_{21} & a_{22} & \boldsymbol{a_{23}} \\ \boldsymbol{a_{31}} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & \boldsymbol{a_{12}} & \\ & & \boldsymbol{a_{23}} \\ \boldsymbol{a_{31}} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix}$$
$$+ \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}. \tag{4}$$

All but these $n!$ determinants are zero, because a column is repeated. (There are n choices for the first column α, $n-1$ remaining choices for β, and finally only one choice for the last column ν. All but one column will be used by that time, when we "snake" down the rows of the matrix). In other words, ***there are*** $n!$ ***ways to permute the numbers*** $1,2,\ldots,n$. The column numbers give the permutations:

Column numbers $\quad (\alpha,\beta,\nu) = (1,2,3),\ (\mathbf{2},\mathbf{3},\mathbf{1}),\ (3,1,2),\ (1,3,2),\ (2,1,3),\ (3,2,1).$

$$\det A = a_{11}a_{22}a_{33}\begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + \boldsymbol{a_{12}a_{23}a_{31}}\begin{vmatrix} & \mathbf{1} & \\ & & \mathbf{1} \\ \mathbf{1} & & \end{vmatrix} + a_{13}a_{21}a_{32}\begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix}$$
$$+ a_{11}a_{23}a_{32}\begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} + a_{12}a_{21}a_{33}\begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} + a_{13}a_{22}a_{31}\begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}. \tag{5}$$

Every term is a product of $n = 3$ entries a_{ij}, with *each row and column represented once*. If the columns come in the order $(\alpha, \ldots, \nu)$, that term is the product $a_{1\alpha} \cdots a_{n\nu}$ times the determinant of a permutation matrix P. The determinant of the whole matrix is the sum of these $n!$ terms, and ***that sum is the explicit formula we are after***:

$$\textbf{Big Formula} \qquad \det A = \sum_{\text{all } P\text{'s}} (a_{1\alpha}a_{2\beta}\cdots a_{n\nu}) \det P. \tag{6}$$

$\det P = +1$ *or* -1 ***for an even or odd number of row exchanges.***

$$(1,3,2) \quad \text{is odd so} \quad \begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} = -1 \qquad (3,1,2) \quad \text{is even so} \quad \begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix} = 1$$

$(1,3,2)$ requires one exchange and $(3,1,2)$ requires two exchanges to recover $(1,2,3)$. These are two of the six $\pm$ signs. For $n=2$, we only have $(1,2)$ and $(2,1)$:

$$\det A = a_{11}a_{22} \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_{12}a_{21} \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21} \quad (\text{or } ad - bc).$$

$$\boxed{\textbf{Cofactors along row 1} \qquad \det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}. \qquad (8)}$$

This shows that $\det A$ depends linearly on the entries $a_{11}, \ldots, a_{1n}$ of the first row.

Example 2. For a 3 by 3 matrix, this way of collecting terms gives

$$\boxed{\det A = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}). \quad (9)}$$

The ***cofactors*** C_{11}, C_{12}, C_{13} are the 2 by 2 determinants in parentheses.

Cofactor splitting

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}.$$

For a determinant of order n, this splitting gives n smaller determinants (***minors***) of order $n-1$; you see the three 2 by 2 submatrices. The submatrix M_{1j} is formed by *throwing away row* 1 *and column* j. Its determinant is multiplied by a_{1j}—and by a plus or minus sign. These signs alternate as in $\det M_{11}$, $-\det M_{12}$, $\det M_{13}$:

Cofactors of row 1 $\qquad C_{1j} = (-1)^{1+j} \det M_{1j}.$

4B The determinant of A is a combination of any row i times its cofactors:

$$\det A \textbf{ by cofactors} \qquad \det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}. \tag{10}$$

The cofactor C_{1j} is the determinant of M_{ij} with the correct sign:

$$\textbf{delete row } i \textbf{ and column } j \qquad C_{ij} = (-1)^{i+j} \det M_{ij}. \tag{11}$$

There is one more consequence of $\det A = \det A^{\mathrm{T}}$. We can expand in cofactors of a *column* of A, which is a row of A^{T}. Down column j of A,

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}. \tag{12}$$

Example 3. The 4 by 4 second difference matrix A4 has only two nonzeros in row 1:

$$\textbf{Use cofactors} \qquad A4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

C_{11} comes from erasing row 1 and column 1, which leaves the $-1, 2, -1$ pattern:

$$C_{11} = \det A_3 = \det \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

For $a_{12} = -1$ it is column 2 that gets removed, and we need its cofactor C_{12}:

$$C_{12} = (-1)^{1+2} \det \begin{bmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = +\det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \det A_2.$$

This left us with the 2 by 2 determinant. Altogether row 1 has produced $2C_{11} - C_{12}$:

$$\det A_4 = 2(\det A_3) - \det A_2 = 2(4) - 3 = 5$$

The same idea applies to A_5 and A_6, and every A_n:

$$\textbf{Recursion by cofactors} \qquad \det A_n = 2(\det A_{n-1}) - \det A_{n-2}. \tag{13}$$

This gives the determinant of increasingly bigger matrices. At every step the determinant of A_n is $n+1$, from the previous determinants n and $n-1$:

$$\textbf{−1, 2, −1 matrix} \qquad \det A_n = 2(n) - (n-1) = n+1.$$

The answer $n+1$ agrees with the product of pivots at the start of this section.

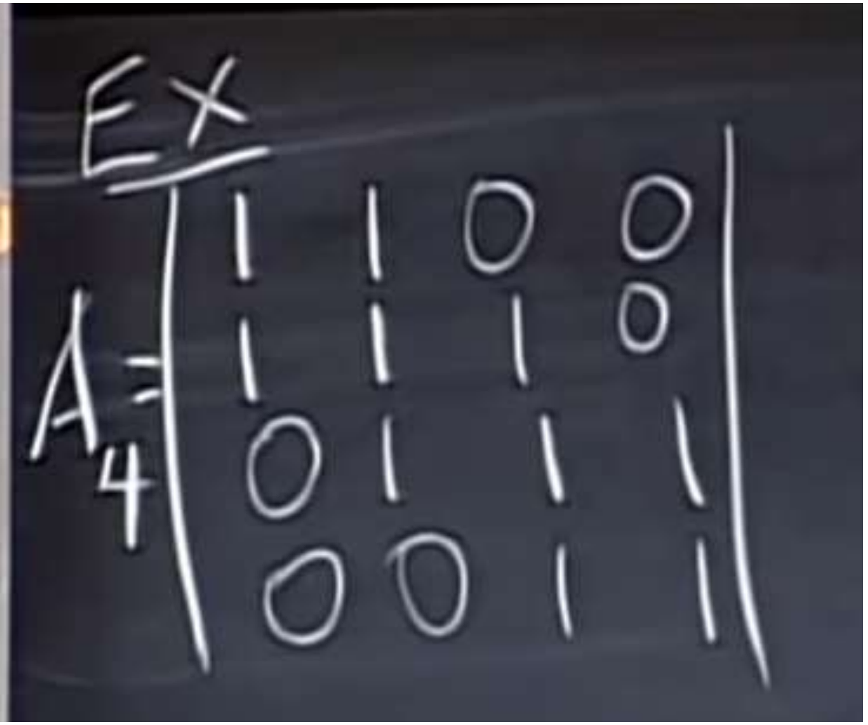
Problem Set 4.3

7. (a) Evaluate this determinant by cofactors of row 1:

$$\begin{vmatrix} 4 & 4 & 4 & 4 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 \end{vmatrix}.$$

(b) Check by subtracting column 1 from the other columns and recomputing.

Compute the determinants of A_2, A_3, A_4. Can you predict A_n?



8. Compute the determinants of A_2, A_3, A_4. Can you predict A_n?

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Use row operations to produce zeros, or use cofactors of row 1.

25. Find the cofactor matrix C and compare AC^{T} with A^{-1}:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$