Eigenvalues and Eigenvectors

5.1 Introduction

Eigenvalue equation $\quad Ax = \lambda x.$

Now we have the fundamental equation of this chapter. It involves two unknowns λ and x. It is an algebra problem, and differential equations can be forgotten! The number λ (lambda) is an ***eigenvalue*** of the matrix A, and the vector x is the associated ***eigenvector***. Our goal is to find the eigenvalues and eigenvectors, λ's and x's, and to use them.

Notice that $Ax = \lambda x$ is a nonlinear equation; λ multiplies x. If we could discover λ, then the equation for x would be linear. In fact we could write λIx in place of λx, and bring this term over to the left side:

$$(A - \lambda I)x = 0. \tag{9}$$

The identity matrix keeps matrices and vectors straight; the equation $(A - \lambda)x = 0$ is shorter, but mixed up. This is the key to the problem:

The vector x is in the nullspace of $A - \lambda I$.
The number λ is chosen so that $A - \lambda I$ has a nullspace.

Of course every matrix has a nullspace. We want a *nonzero* eigenvector x, the nullspace of $A-\lambda I$ must contain vectors other than zero. In short, $A-\lambda I$ ***must be singular***.

5A The number λ is an eigenvalue of A if and only if $A-\lambda I$ is singular:

$$\det(A-\lambda I)=0. \tag{10}$$

This is the characteristic equation. Each λ is associated with eigenvectors x:

$$(A-\lambda I)x=0 \qquad \text{or} \qquad Ax=\lambda x. \tag{11}$$

In our example, we shift A by λI to make it singular:

$$\textbf{Subtract } \lambda I \qquad A-\lambda I=\begin{bmatrix}4-\lambda & -5\\ 2 & -3-\lambda\end{bmatrix}.$$

Note that λ is subtracted only from the main diagonal (because it multiplies I).

$$\textbf{Determinant} \qquad |A-\lambda I|=(4-\lambda)(-3-\lambda)+10 \qquad \text{or} \qquad \lambda^2-\lambda-2.$$

factoring into $\lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2)$. That is zero if $\lambda = -1$ or $\lambda = 2$, as the general formula confirms:

$$\textbf{Eigenvalues} \qquad \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{9}}{2} = -1 \text{ and } 2.$$

There are two eigenvalues, because a quadratic has two roots. Every 2 by 2 matrix $A - \lambda I$ has λ^2 (and no higher power of λ) in its determinant.

The values $\lambda = -1$ and $\lambda = 2$ lead to a solution of $Ax = \lambda x$ or $(A - \lambda I)x = 0$. A matrix with zero determinant is singular, so there must be nonzero vectors x in its nullspace. In fact the nullspace contains a whole *line* of eigenvectors; it is a subspace!

$$\lambda_1 = -1: \qquad (A - \lambda_1 I)x = \begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solution (the first eigenvector) is any nonzero multiple of x_1:

$$\textbf{Eigenvector for } \boldsymbol{\lambda_1} \qquad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

You might notice that the columns of $A-\lambda_1 I$ give x_2, and the columns of $A-\lambda_2 I$ are multiples of x_1. This is special (and useful) for 2 by 2 matrices.

In the 3 by 3 case, I often set a component of x equal to 1 and solve $(A-\lambda I)x=0$ for the other components. Of course if x is an eigenvector then so is $7x$ and so is $-x$. All vectors in the nullspace of $A-\lambda I$ (which we call the *eigenspace*) will satisfy $Ax=\lambda x$. In our example the eigenspaces are the lines through $x_1=(1,1)$ and $x_2=(5,2)$.

Before going back to the application (the differential equation), we emphasize the steps in solving $Ax=\lambda x$:

1. ***Compute the determinant of*** $A-\lambda I$. With λ subtracted along the diagonal, this determinant is a polynomial of degree n. It starts with $(-\lambda)^n$.

2. ***Find the roots of this polynomial***. The n roots are the eigenvalues of A.

3. ***For each eigenvalue solve the equation*** $(A-\lambda I)x=0$. Since the determinant is zero, there are solutions other than $x=0$. Those are the eigenvectors.

The key equation was $Ax = \lambda x$. Most vectors x will not satisfy such an equation. They change direction when multiplied by A, so that Ax is not a multiple of x. This means that ***only certain special numbers are eigenvalues, and only certain special vectors x are eigenvectors***. We can watch the behavior of each eigenvector, and then combine these "normal modes" to find the solution. To say the same thing in another way, *the underlying matrix can be diagonalized.*

Example 1. Everything is clear when A is a ***diagonal matrix***:

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{has} \quad \lambda_1 = 3 \quad \text{with} \quad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 2 \quad \text{with} \quad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

On each eigenvector A acts like a multiple of the identity: $Ax_1 = 3x_1$ and $Ax_2 = 2x_2$.

Other vectors like $x = (1,5)$ are mixtures $x_1 + 5x_2$ of the two eigenvectors, and when A multiplies x_1 and x_2 it produces the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 2$:

$$A \quad \text{times} \quad x_1 + 5x_2 \quad \text{is} \quad 3x_1 + 10x_2 = \begin{bmatrix} 3 \\ 10 \end{bmatrix}.$$

This is Ax for a typical vector x—not an eigenvector. But the action of A is determined by its eigenvectors and eigenvalues.

Example 2. The eigenvalues of a ***projection matrix*** are 1 or 0!

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{has} \quad \lambda_1 = 1 \quad \text{with} \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = 0 \quad \text{with} \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We have $\lambda = 1$ when x projects to itself, and $\lambda = 0$ when x projects to the zero vector.

The column space of P is filled with eigenvectors, and so is the nullspace. If those spaces have dimension r and $n-r$, then $\lambda = 1$ is repeated r times and $\lambda = 0$ is repeated $n-r$ times (*always* n λ's):

$$\textbf{Four eigenvalues allowing repeats} \qquad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{has} \quad \lambda = 1, 1, 0, 0.$$

There is nothing exceptional about $\lambda = 0$. Like every other number, zero might be an eigenvalue and it might not. If it is, then its eigenvectors satisfy $Ax = 0x$. Thus x is in the nullspace of A. A zero eigenvalue signals that A is singular (not invertible); its determinant is zero. Invertible matrices have all $\lambda \neq 0$.

Example 3. The eigenvalues are on the main diagonal when A is ***triangular***:

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 4 & 5 \\ 0 & \frac{3}{4}-\lambda & 6 \\ 0 & 0 & \frac{1}{2}-\lambda \end{vmatrix} = (1-\lambda)(\tfrac{3}{4}-\lambda)(\tfrac{1}{2}-\lambda).$$

The determinant is just the product of the diagonal entries. It is zero if $\lambda = 1$, $\lambda = \frac{3}{4}$, or $\lambda = \frac{1}{2}$; the eigenvalues were already sitting along the main diagonal.

5B The ***sum*** of the n eigenvalues equals the sum of the n diagonal entries:

$$\textbf{Trace of}\quad A = \lambda_1 + \cdots + \lambda_n = a_{11} + \cdots + a_{nn}. \tag{15}$$

Furthermore, the ***product*** of the n eigenvalues equals the ***determinant*** of A.

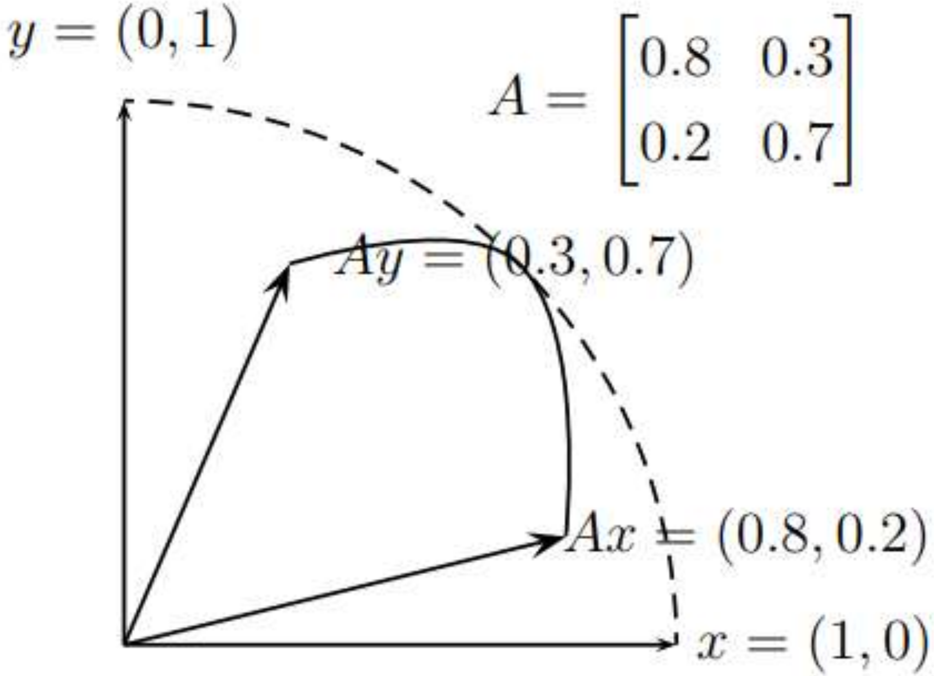
The projection matrix P had diagonal entries $\frac{1}{2}$, $\frac{1}{2}$ and eigenvalues 1, 0. Then $\frac{1}{2}+\frac{1}{2}$ agrees with $1+0$ as it should. So does the determinant, which is $0 \cdot 1 = 0$. A singular matrix, with zero determinant, has one or more of its eigenvalues equal to zero.

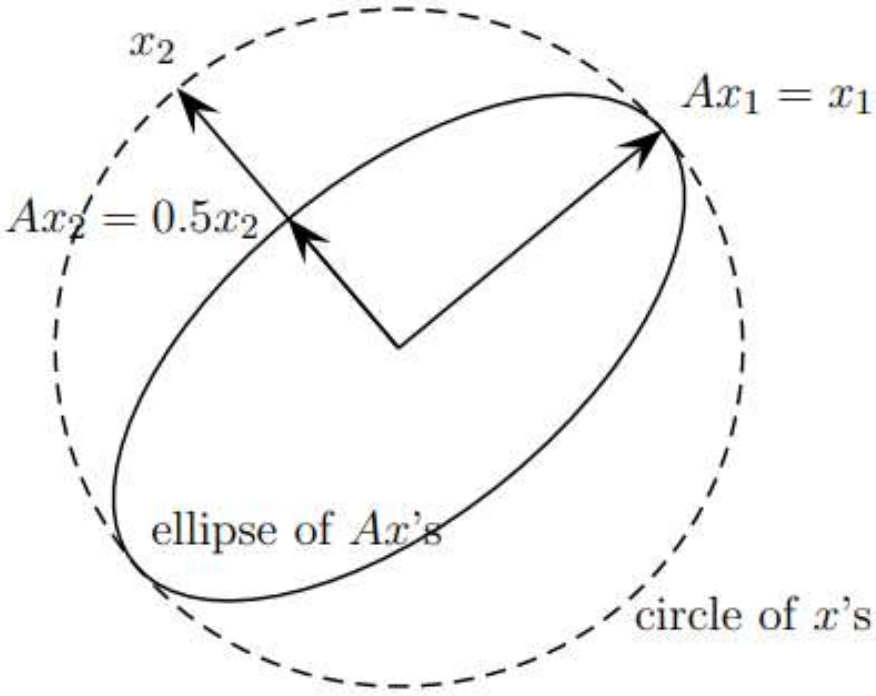
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has trace } a+d\text{, and determinant } ad-bc$$

$$\det(A-\lambda I) = \det\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (\text{trace})\lambda + \text{determinant}$$

Eigshow

There is a MATLAB demo (just type eigshow), displaying the eigenvalue problem for a 2 by 2 matrix. It starts with the unit vector $x = (1,0)$. *The mouse makes this vector move around the unit circle*. At the same time the screen shows Ax, in color and also moving. Possibly Ax is ahead of x. Possibly Ax is behind x. *Sometimes Ax is parallel to x*. At that parallel moment, $Ax = \lambda x$ (twice in the second figure).





The eigenvalue λ is the length of Ax, when the unit eigenvector x is parallel. The built-in choices for A illustrate three possibilities: 0, 1, or 2 real eigenvectors.

1. There are *no real eigenvectors. Ax stays behind or ahead of x.* This means the eigenvalues and eigenvectors are complex, as they are for the rotation Q.

2. There is only *one* line of eigenvectors (unusual). The moving directions Ax and x meet but don't cross. This happens for the last 2 by 2 matrix below.

3. There are eigenvectors in *two* independent directions. This is typical! Ax crosses x at the first eigenvector x_1, and it crosses back at the second eigenvector x_2.

Suppose A is singular (rank 1). Its column space is a line. The vector Ax has to stay on that line while x circles around. One eigenvector x is along the line. Another eigenvector appears when $Ax_2 = 0$. Zero is an eigenvalue of a singular matrix.

You can mentally follow x and Ax for these six matrices. How many eigenvectors and where? When does Ax go clockwise, instead of counterclockwise with x?

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Problem Set 5.1

1. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. Verify that the trace equals the sum of the eigenvalues, and the determinant equals their product.

3. If we shift to $A - 7I$, what are the eigenvalues and eigenvectors and how are they related to those of A?

$$B = A - 7I = \begin{bmatrix} -6 & -1 \\ 2 & -3 \end{bmatrix}.$$

7. Suppose that λ is an eigenvalue of A, and x is its eigenvector: $Ax = \lambda x$.

(a) Show that this same x is an eigenvector of $B = A - 7I$, and find the eigenvalue. This should confirm Exercise 3.

(b) Assuming $\lambda \neq 0$, show that x is also an eigenvector of A^{-1}—and find the eigenvalue.

11. ***The eigenvalues of*** A ***equal the eigenvalues of*** A^{T}. This is because $\det(A - \lambda I)$ equals $\det(A^{\mathrm{T}} - \lambda I)$. That is true because ____. Show by an example that the eigenvectors of A and A^{T} are *not* the same.

12. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

22. Compute the eigenvalues and eigenvectors of A and A^2:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

A^2 has the same ____ as A. When A has eigenvalues λ_1 and λ_2, A^2 has eigenvalues ____.

25. From the unit vector $u = \left(\frac{1}{6}, \frac{1}{6}, \frac{3}{6}, \frac{5}{6}\right)$, construct the rank-1 projection matrix $P = uu^{\mathrm{T}}$.

(a) Show that $Pu = u$. Then u is an eigenvector with $\lambda = 1$.

(b) If v is perpendicular to u show that $Pv =$ zero vector. Then $\lambda = 0$.

(c) Find three independent eigenvectors of P all with eigenvalue $\lambda = 0$.

34. This matrix is singular with rank 1. Find three λ's and three eigenvectors:

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}.$$