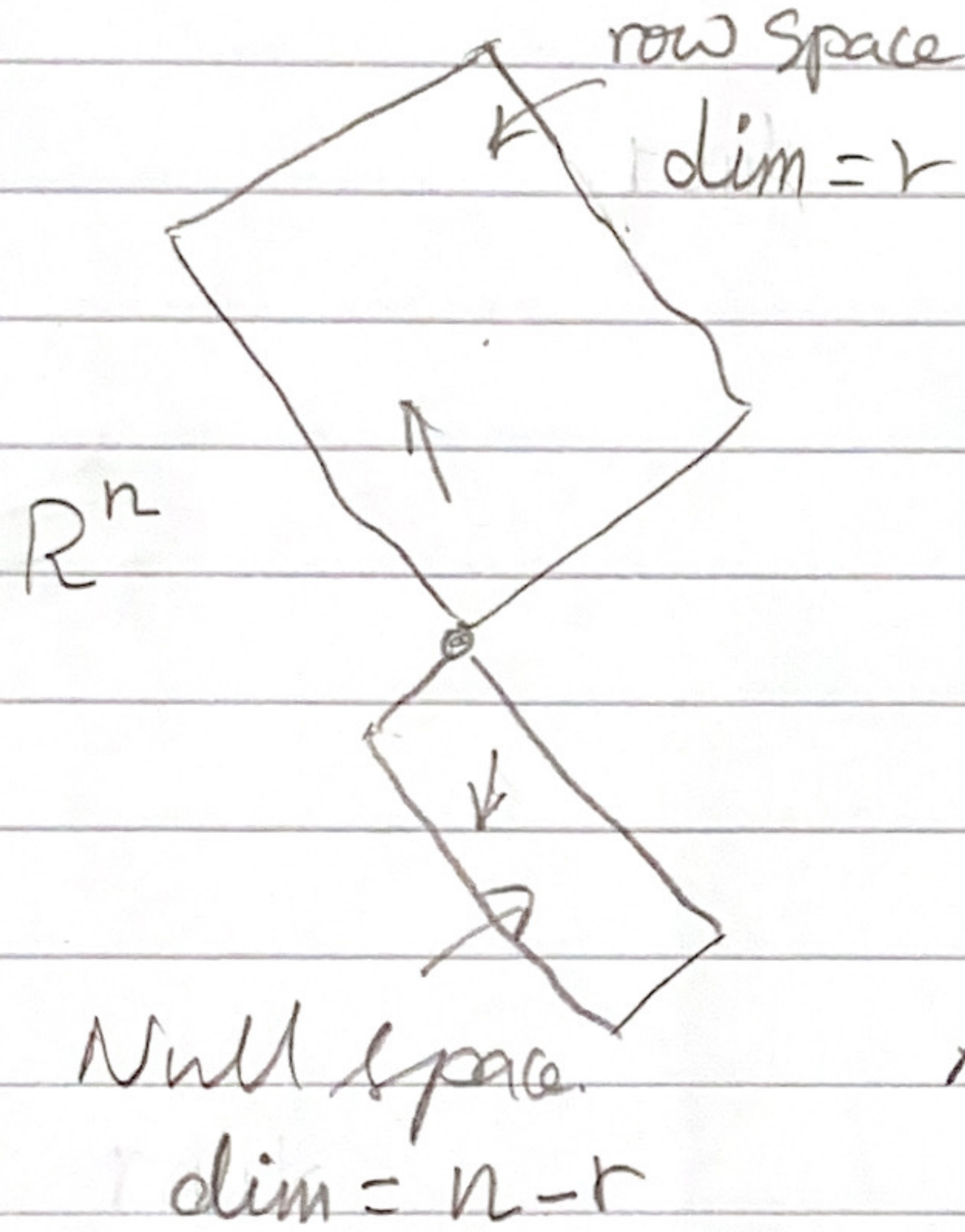
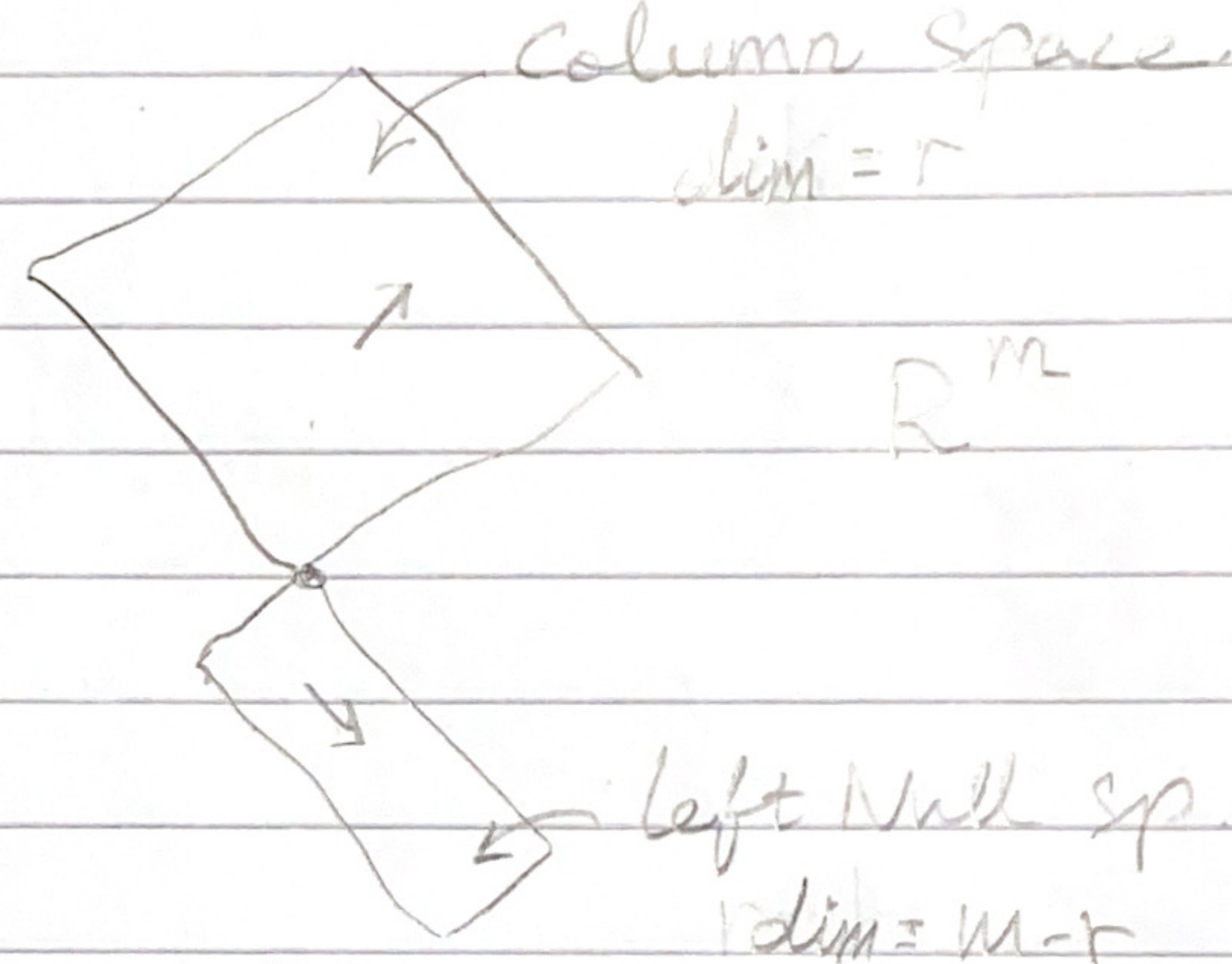
Orthogonal Subspaces:

Any matrix defines 4 subspaces.

row Space } orthogonal — Null Space

Column Space } orthogonal — Left Null





$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix}$$

Orthogonal vectors:

Their dot product is = 0 $\quad (x \cdot y = 0)$

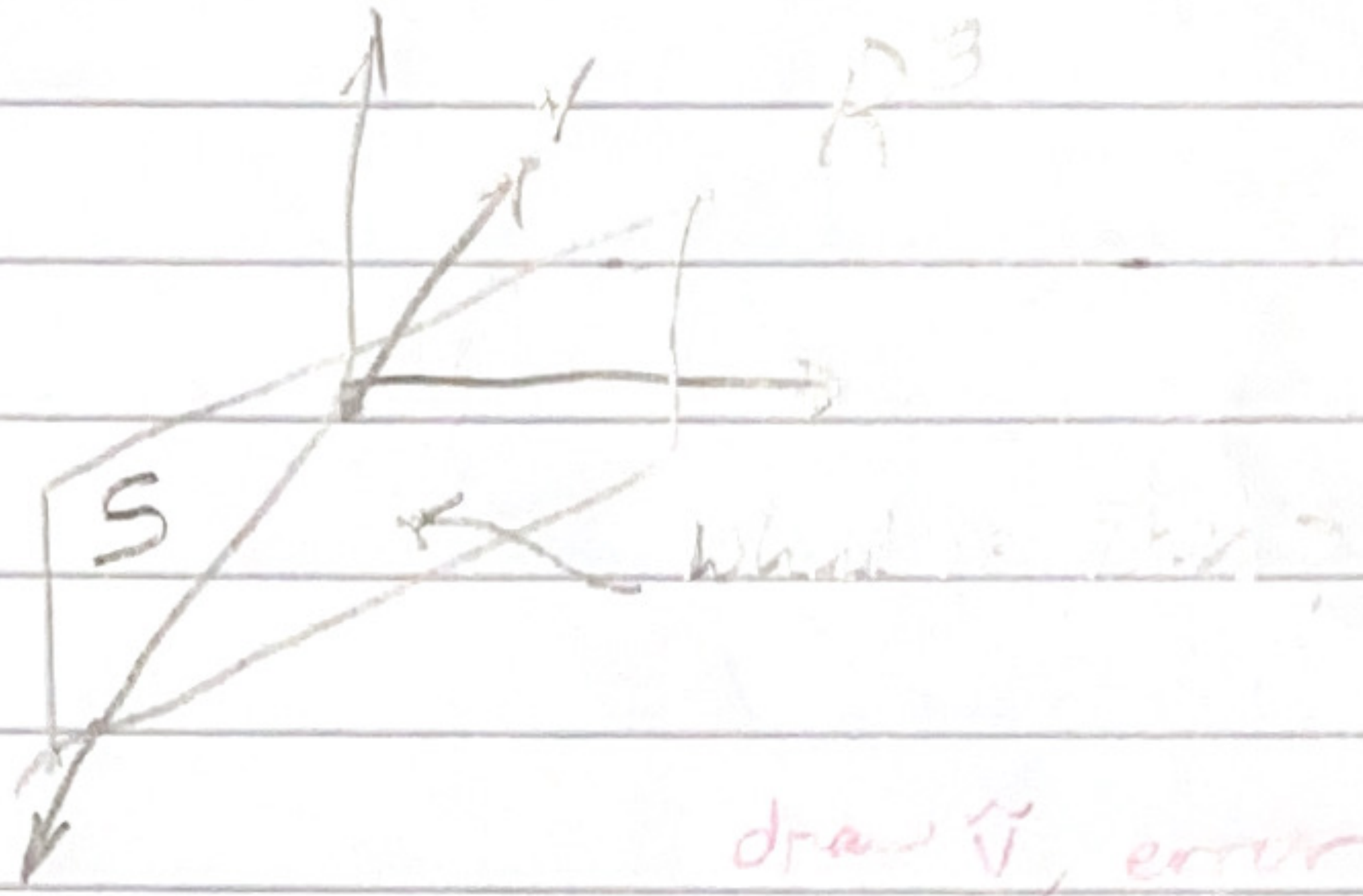
$(x^T y = 0)$

$$\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2.5 \\ -1 \end{bmatrix} = 0$$

R2

Projection:

- A subspace $\underline{S}$ in R^n has a dimension r less than n. (Only the entire R^n has a dim = n).
- Basis for S has r independent vectors ($r<n$)
- This means that there are vectors in R^n that don't belong to S.
- The closest image of the vector v on S is the projection of v on S.



$$\tilde{v} = P_S v$$

draw $\tilde{v}$, error

Where P_S is the projection matrix that can be composed using a basis set of S.

Ex: $b_1 = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ form a basis set of the subspace S. Let's construct P_S

$P_S = B(B^T B)^{-1} B^T$, where $B = \begin{bmatrix} 2 & 1 \\ 4 & 0 \\ 4 & 1 \end{bmatrix}$

$$P_S = \begin{bmatrix} 0.56 & -0.22 & 0.44 \\ -0.22 & 0.89 & 0.22 \\ 0.44 & 0.22 & 0.56 \end{bmatrix}$$

dim P_S =

rank(P_S) = so?!

Try to use a different basis set: $\begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 2 & 5 \end{bmatrix}$ is it in S?

$P_S = ?$

What do you infer about P_S?

Let's take this example:

$B = \begin{bmatrix} 2 & 1 \\ 4 & 0 \\ 4 & 1 \end{bmatrix}$ as basis for Subspace S.

Now assume we have an orthonormal basis for S, call it $B_\perp = \begin{bmatrix} 0.33 & 0.67 \\ 0.67 & -0.67 \\ 0.67 & 0.33 \end{bmatrix}$

How did we get it?

Find P_S using $B_\perp$ =

Do you see the advantage of using $B_\perp$?

How did you find $B_\perp$ from B?

Orthonormal vectors

Orthogonal basis $q_1, \dots q_n$

Orthogonal matrix Q

Gram-Schmidt $A \to Q$ (non-orthonormal $\xrightarrow{A}$ orthonormal $\xrightarrow{Q}$)

Orthonormal vectors: $q_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

basis

⊥ to each other

normalized length.

Why important & preferred to work with orthonormal vector? They never get out of hand, never overflow / underflow.

$$Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \qquad Q^T = \begin{bmatrix} — q_1^T — \\ \vdots \\ — q_n^T — \end{bmatrix}$$

doesn't have to be square.

$$Q^TQ = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix}$$

~~But if Q is □~~

~~then~~ $Q^TQ = I$

if Q is □ $Q^T = Q^{-1}$

Ex: perm $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $Q^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$QQ^T = I$

Ex: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ check ⊥, check normality?

Ex $Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ check ⊥, normality?

is it orthogonal matrix? $q_1 \cdot q_2 = 0$

$\|q_1\|^2 = 2 = \|q_2\|^2 = 2$

then $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ makes it orthogonal + normal

Ex: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

Ex: $Q = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}$ these are orthonormal basis for a 2D Column space. (of course they are indep)

$Q = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$

What is the advtg of Q?
Q has orthonormal Col's., we would like to project onto its Col. Sp. $C(Q)$:

$$P = Q(Q^TQ)^{-1}Q^T = QQ^T = \begin{cases} I & \text{if } Q \text{ is square} \\ \neq I & \text{if } Q \text{ is not square} \end{cases}$$

square, symmetric

$P^2 = P$ projects onto the whole space if $P = I$ (b is in its space!).

$A^TA\hat{x} = A^Tb$ NOW A is Q

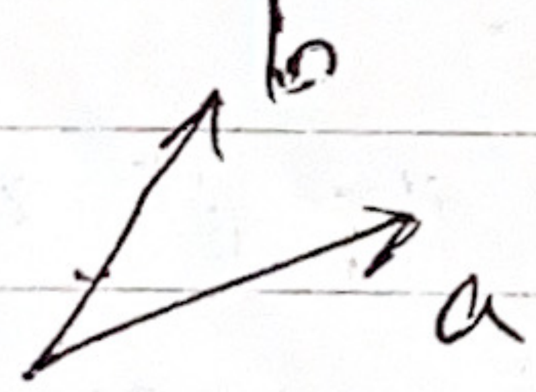
$\underbrace{Q^TQ}_{I}\hat{x} = Q^Tb$

$\hat{x} = Q^Tb$ (no inversion involved)

$\hat{x}_i = q_i^Tb$

the component on the ith basis vector.

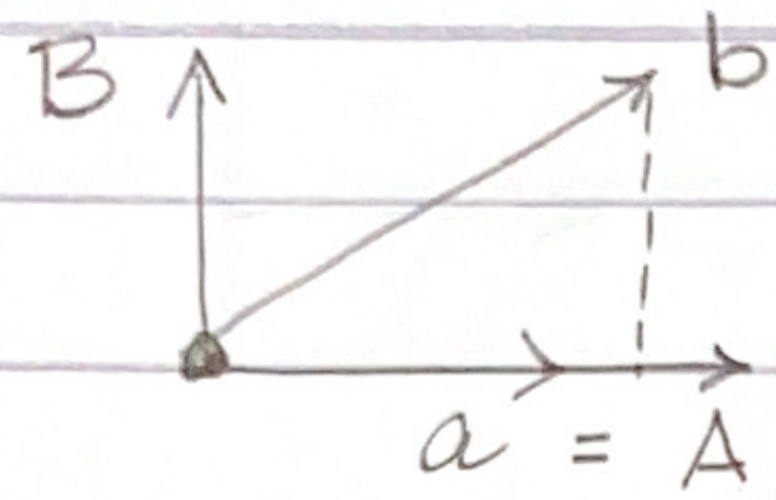
It would be nice to make A to be Q
this way we won't have to do inverse
Gram-Schmidt: it is not elimination
(cos elimination targets making the matrix diag.
Here, we target making the matrix orthon.).



in any $\mathbb{R}^m$ (a, b are indep.).

we want to produce $\underbrace{q_1, q_2}_{\text{orthonormal}}$ from a, b.

How to perform G.S.



$a, b \rightarrow A, B \rightarrow q_1, q_2$

non $\perp$ $\quad \perp \quad \frac{A}{\|A\|} \quad \frac{B}{\|B\|}$

orthonormal

Let $a = A$

Now find B that is $\perp$ to A

$$B = b - \text{its projection on } A$$

$$= b - \frac{A^T b}{A^T A} A$$

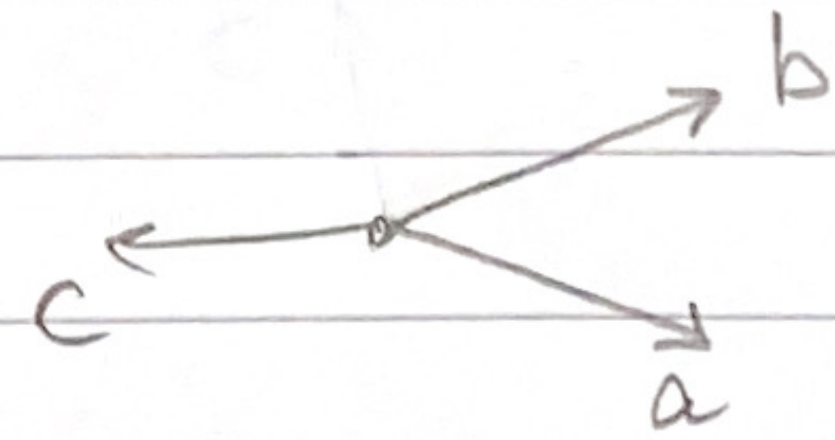
Check $\quad A^T B = A^T \left(b - \frac{A^T b}{A^T A} A \right)$ check for orthogonality

$$A^T B = A^T b - A^T b \frac{A^T A}{A^T A} = 0 \quad \checkmark$$

Now say you added a third vector c. Then you have to make $C \perp A$, $C \perp B \rightarrow$ then $q_3 = \frac{C}{\|C\|}$

$C = c - \{$ its component in A direction and in B direction $\}$

$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B$$



$C \perp A, \quad C \perp B$

Ex: $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $\qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$ This is the old basis set

$$B = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}_{b} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}_{B}$$

$$Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$$ $\leftarrow$ this is the new basis set, it's orthonormal

Q is a better basis set.

note: $C(Q) = C(A)$

Because $C(Q)$ are Lin. Combination of $C(A)$.

Write the Connection between $A \to Q$

indep. Columns $\longrightarrow A = QR$ "Gram-Schmidt Expression"

$\uparrow$ indep. orthonormal Columns.

It means that the columns of A can be generated by manipulating (linearly combine) the columns of Q using weights from the matrix R.

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} \text{includes those numbers as } \frac{1}{\sqrt{3}}, \ldots \end{bmatrix}$$

$$\underset{}{A} = \underset{\text{orthonormal}}{Q} \quad R$$

R is an upper triangular matrix (because q_2 is $\perp$ to a_1, so the weight from R that is multiplied by q_2 to produce a_1 must be 0).

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}} = \underset{Q}{\begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}} \underset{R}{\begin{bmatrix} 3/\sqrt{3} & 3/\sqrt{3} \\ 0 & 2/\sqrt{2} \end{bmatrix}}$$

$Q = AR^{-1}$ R is invertible.

Ex: $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}$, Find R, Q.

$a \quad b$

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{6}{14} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 2 \\ 8 \\ -4 \end{bmatrix}$$

$$q_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad q_2 = \frac{B}{\|B\|} = \frac{1}{\sqrt{84}} \begin{bmatrix} 2 \\ 8 \\ -4 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2/\sqrt{14} & 2/\sqrt{84} \\ 1/\sqrt{14} & 8/\sqrt{84} \\ 3/\sqrt{14} & -4/\sqrt{84} \end{bmatrix} \qquad R = \begin{bmatrix} 14/\sqrt{14} & 6/\sqrt{14} \\ 0 & 4/\sqrt{84} \end{bmatrix}$$

Note: in the previous example $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}$ has $C(A)$ as a linear combination of the column space of the previous $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$. This means that the column space of both matrices is the same. However, we get different set of Q & R.

Example: $A = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 2 & 4 \end{bmatrix}$, Find Q, R.

You will find that Q is the same as the original. $R = 2 R_{original}$.