# Abstract

To be written.

This MMSC thesis will further explore general kernel spectral methods for finding equilibrium measures where initial progress made in Gutleb, Carrillo and S. Olver 2020 and Gutleb, Carrillo and S. Olver 2021.

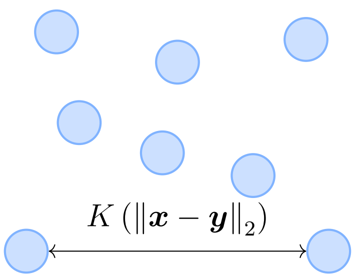
**Keywords:** Equilibrium Measures

**Languages:** C++, Julia, Python

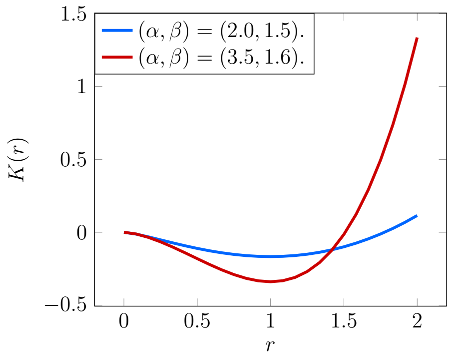
# Contents

# Chapter 1

# Introduction




**(a)** $N = 8$ particles interacting with one another through the potential $K(r)$.




**(b)** Plot of attractive-repulsive potential functions $K(r) = \frac{r^\alpha}{\alpha} - \frac{r^\beta}{\beta}$ for different $\alpha, \beta$.

Cf. Figure 1.1a and Figure 1.1b.

All plots and figures in this thesis were generated using the Makie visualisation tool (Danisch and Krumbiegel 2021), an open-source package available for the Julia computing language (Bezanson et al. 2017).

# Just Notes

This chapter's purpose is the collection of notes, and it will not be included in the final dissertation.

## Special Functions we like

**Pochhammer's falling symbol** $(x)_n := \prod_{k=0}^{n-1}(x-k)\,.$

**Pochhammer's rising symbol** $(x)^n := \prod_{k=0}^{n-1}(x+k)\,.$

**Generalised hypergeometric series**

$$ {}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z) := \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}.\,. $$

**(Gaussian) Hypergeometric function**

$$ {}_2F_1(a,-n;c;z) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{(a)_j}{(c)_j}z^j\,. $$

(A special case of the hypergeometric series with $p=2$, $q=1$).

**Jacobi (=hypergeometric) polynomials**

$$ P_n^{(\alpha,\beta)}(z) := \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(-n,1+\alpha+\beta+n;\alpha+1;\tfrac{1}{2}(1-z)\right)\,. $$

**Gegenbauer (=ultraspherical) polynomials**

$$ C_n^{(\lambda)}(z) := \frac{(2\lambda)_n}{n!}\,{}_2F_1\left(-n,2\lambda+n;\lambda+\frac{1}{2};\frac{1-z}{2}\right) = \frac{(2\lambda)_n}{(\lambda+\frac{1}{2})_n}P_n^{(\lambda-1/2,\lambda-1/2)}(x)\,. $$

They satisfy a three-term recurrence relation (as all orthogonal polynomials do!)

$$\begin{aligned} C_0^{(\lambda)}(x) &= 1 \\ C_1^{(\lambda)}(x) &= 2\lambda x \\ (n+1)C_{n+1}^{(\lambda)}(x) &= 2(n+\lambda)xC_n^{(\lambda)}(x) - (n+2\lambda-1)C_{n-1}^{(\lambda)}(x). \end{aligned} .$$

From Wikipedia: In spectral methods for solving differential equations, if a function is expanded in the basis of Chebyshev polynomials and its derivative is represented in a Gegenbauer/ultraspherical basis, then the derivative operator becomes a diagonal matrix, leading to fast banded matrix methods for large problems (S. Olver and Townsend 2013).

**Three-term recurrence relationship** F. Olver et al. 2018, p. 18.9.1:

$$xC_n^{(\lambda)}(x) = \tfrac{(n+2\lambda-1)}{2(n+\lambda)}C_{n-1}^{(\lambda)}(x) + \tfrac{n+1}{2(n+\lambda)}C_{n+1}^{(\lambda)}(x). \tag{1.1}$$

**1.0.1 Theorem: Two term recurrence of $Q^\alpha$**

The integral operator

$$Q^\alpha\left[u\right](x) = \int_{-1}^{1} |x-y|^\alpha u(y)\,\mathrm{d}y$$

satisfies a two-term recurrence relationship when acting on the ultraspherical polynomials $C_n^{(\lambda)}(y)$ with weight $w(y) = (1-y^2)^{\lambda-\frac{1}{2}}$ such that

$$xQ^\alpha\left[wC_n^{(\lambda)}\right](x) = \kappa_1 Q^\alpha\left[wC_{n-1}^{(\lambda)}\right](x) + \kappa_2 Q^\alpha\left[wC_{n+1}^{(\lambda)}\right](x),$$

where $n \geq 2$ and with the constants

$$\begin{aligned} \kappa_1 &= \frac{(n-\alpha-1)(2\lambda+n-1)}{2n(\lambda+n)}, \\ \kappa_2 &= \frac{(n+1)(2\lambda+n+\alpha+1)}{2(\lambda+n)(2\lambda+n)}. \end{aligned}$$

# Chapter 2

# Conclusion

In the present thesis, we explored the interesting realm of particle-particle interactions. Next to the written part, the reader will find an implementation of the particle simulator, including a Graphical User Interface (GUI), as well as the numerical solver.

# Acronyms, Definitions and Theorems



## Definitions

## Theorems



## Lemmata

## Remarks

# Bibliography

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# Appendix A – Supplemental Proofs