

# Abstract

To be written.

This MMSC thesis will further explore general kernel spectral methods for finding equilibrium measures where initial progress made in [Gutleb, Carrillo and S. Olver 2020](#) and [Gutleb, Carrillo and S. Olver 2021](#).

**Keywords:** Equilibrium Measures

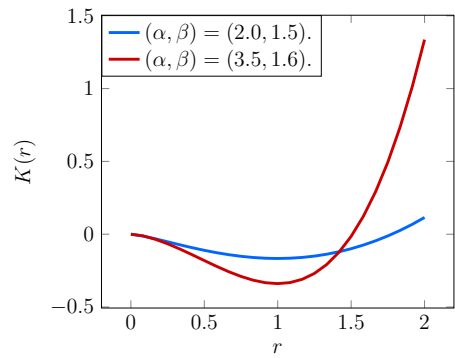
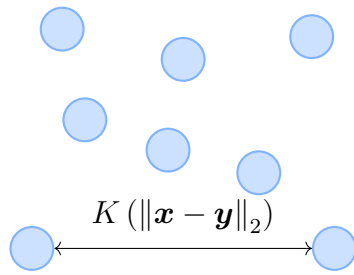
**Languages:** C++, Julia, Python

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# Chapter 1

## Introduction



**(a)**  $N = 8$  particles interacting with one another **(b)** Plot of attractive-repulsive potential functions through the potential  $K(r)$ .  
 $K(r) = \frac{r^\alpha}{\alpha} - \frac{r^\beta}{\beta}$  for different  $\alpha, \beta$ .

Cf. Figure 1.1a and Figure 1.1b.

All plots and figures in this thesis were generated using the Makie visualisation tool ([Danisch and Krumbiegel 2021](#)), an open-source package available for the Julia computing language ([Bezanson et al. 2017](#)).

# Just Notes

This chapter's purpose is the collection of notes, and it will not be included in the final dissertation.

## Special Functions we like

**Pochhammer's falling symbol**  $(x)_n := \prod_{k=0}^{n-1} (x - k)$ .

**Pochhammer's rising symbol**  $(x)^n := \prod_{k=0}^{n-1} (x + k)$ .

**Generalised hypergeometric series**

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}.$$

**(Gaussian) Hypergeometric function**

$${}_2F_1(a, -n; c; z) = \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{(a)_j}{(c)_j} z^j.$$

(A special case of the hypergeometric series with  $p = 2$ ,  $q = 1$ ).

**Jacobi (=hypergeometric) polynomials**

$$P_n^{(\alpha, \beta)}(z) := \frac{(\alpha + 1)_n}{n!} {}_2F_1\left(-n, 1 + \alpha + \beta + n; \alpha + 1; \frac{1}{2}(1 - z)\right).$$

**Gegenbauer (=ultraspherical) polynomials**

$$C_n^{(\lambda)}(z) := \frac{(2\lambda)_n}{n!} {}_2F_1\left(-n, 2\lambda + n; \lambda + \frac{1}{2}; \frac{1 - z}{2}\right) = \frac{(2\lambda)_n}{(\lambda + \frac{1}{2})_n} P_n^{(\lambda-1/2, \lambda-1/2)}(x).$$

They satisfy a three-term recurrence relation (as all orthogonal polynomials do!)

$$\begin{aligned} C_0^{(\lambda)}(x) &= 1 \\ C_1^{(\lambda)}(x) &= 2\lambda x \\ (n+1)C_{n+1}^{(\lambda)}(x) &= 2(n+\lambda)x C_n^{(\lambda)}(x) - (n+2\lambda-1)C_{n-1}^{(\lambda)}(x). \end{aligned}$$

From Wikipedia: In spectral methods for solving differential equations, if a function is expanded in the basis of Chebyshev polynomials and its derivative is represented in a Gegenbauer/ultraspherical basis, then the derivative operator becomes a diagonal matrix, leading to fast banded matrix methods for large problems (S. Olver and Townsend 2013).

**Three-term recurrence relationship** F. Olver et al. 2018, p. 18.9.1:

$$x C_n^{(\lambda)}(x) = \frac{(n+2\lambda-1)}{2(n+\lambda)} C_{n-1}^{(\lambda)}(x) + \frac{n+1}{2(n+\lambda)} C_{n+1}^{(\lambda)}(x). \quad (1.1)$$

### 1.0.1 Theorem: Two term recurrence of $Q^\alpha$

The integral operator

$$Q^\alpha[u](x) = \int_{-1}^1 |x-y|^\alpha u(y) dy$$

satisfies a two-term recurrence relationship when acting on the ultraspherical polynomials  $C_n^{(\lambda)}(y)$  with weight  $w(y) = (1-y^2)^{\lambda-\frac{1}{2}}$  such that

$$x Q^\alpha[w C_n^{(\lambda)}](x) = \kappa_1 Q^\alpha[w C_{n-1}^{(\lambda)}](x) + \kappa_2 Q^\alpha[w C_{n+1}^{(\lambda)}](x),$$

where  $n \geq 2$  and with the constants

$$\begin{aligned} \kappa_1 &= \frac{(n-\alpha-1)(2\lambda+n-1)}{2n(\lambda+n)}, \\ \kappa_2 &= \frac{(n+1)(2\lambda+n+\alpha+1)}{2(\lambda+n)(2\lambda+n)}. \end{aligned}$$

# Chapter 2

## Spectral Method

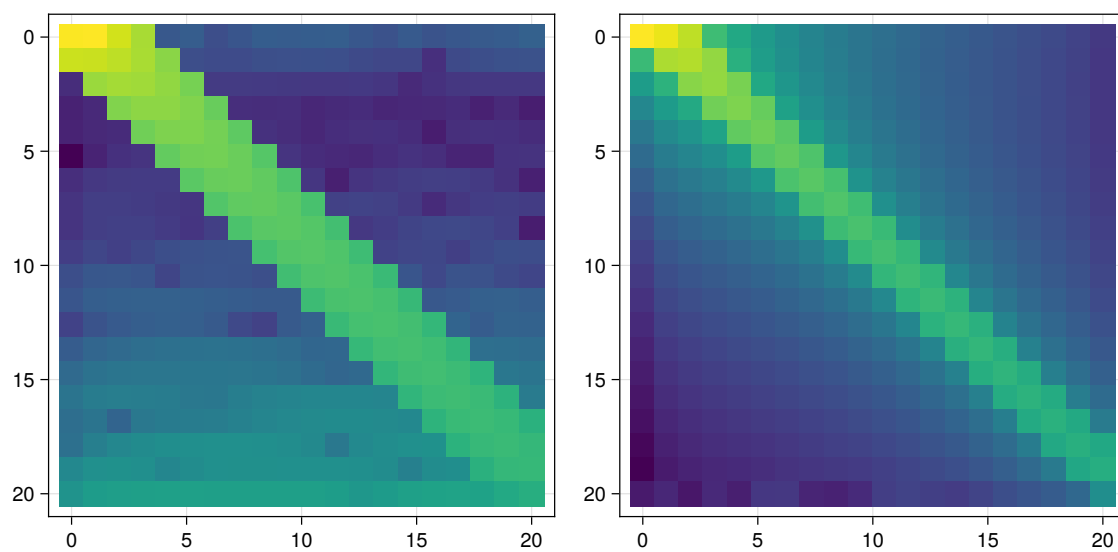
### 2.0.1 Theorem: Integration Theorem that needs a name

On the  $d$ -dimensional unit ball  $B_1$  the power law potential, with power  $\alpha \in (-d, 2 + 2m - d)$ ,  $m \in \mathbb{N}_0$  and  $\beta > -d$ , of the  $n$ -th weighted radial Jacobi polynomial

$$(1 - |y|^2)^{m - \frac{\alpha + d}{2}} P_n^{(m - \frac{\alpha + d}{2}, \frac{d - 2}{2})}(2|y|^2 - 1)$$

reduces to a Gaussian hypergeometric function as follows:

$$\begin{aligned} & \int_{B_1} |x - y|^\beta (1 - |y|^2)^{m - \frac{\alpha + d}{2}} P_n^{(m - \frac{\alpha + d}{2}, \frac{d - 2}{2})}(2|y|^2 - 1) dy \\ &= \frac{\pi^{d/2} \Gamma(1 + \frac{\beta}{2}) \Gamma(\frac{\beta + d}{2}) \Gamma(m + n - \frac{\alpha + d}{2} + 1)}{\Gamma(\frac{d}{2}) \Gamma(n + 1) \Gamma(\frac{\beta}{2} - n + 1) \Gamma(\frac{\beta - \alpha}{2} + m + n + 1)} {}_2F_1 \left( n - \frac{\beta}{2}, \quad -m - n + \frac{\alpha - \beta}{2}; \frac{d}{2}; |x|^2 \right). \end{aligned}$$



**Figure 2.1:** The attractive and repulsive operators

## Chapter 3

## Conclusion

In the present thesis, we explored the interesting realm of particle-particle interactions. Next to the written part, the reader will find an implementation of the particle simulator, including a Graphical User Interface (GUI), as well as the numerical solver.



# Acronyms, Definitions and Theorems

GUI Graphical User Interface

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## Definitions

## Theorems

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# Bibliography

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