

# Abstract

To be written.

This MMSC thesis will further explore general kernel spectral methods for finding equilibrium measures where initial progress made in [Gutleb, Carrillo and Olver 2020](#) and [Gutleb, Carrillo and Olver 2021](#).

**Keywords:** Equilibrium Measures

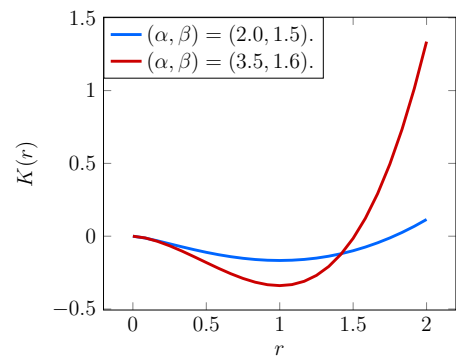
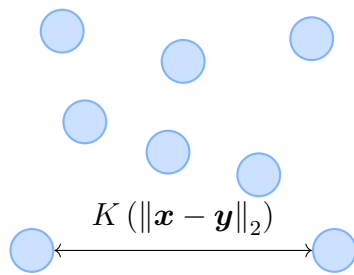
**Languages:** C++, Julia, Python

# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
<b>2</b>	<b>Conclusion</b>	<b>6</b>
	<b>Acronyms, Definitions and Theorems</b>	<b>7</b>
	<b>Bibliography</b>	<b>8</b>
	<b>List of Figures and Tables</b>	<b>9</b>
<b>A</b>	<b>Supplemental Proofs</b>	<b>10</b>

# Chapter 1

## Introduction



(a)  $N = 8$  particles interacting with one another through the potential  $K(r)$ . (b) Plot of attractive-repulsive potential functions  $K(r) = \frac{r^\alpha}{\alpha} - \frac{r^\beta}{\beta}$  for different  $\alpha, \beta$ .

Cf. Figure 1.1a and Figure 1.1b.

# Just Notes

This chapter's purpose is for the collection of notes, and it will not be included in the final dissertation.

## Special Functions we like

**Pochhammer's falling symbol**  $(x)_n := \prod_{k=0}^{n-1} (x - k)$ .

**Pochhammer's rising symbol**  $(x)^n := \prod_{k=0}^{n-1} (x + k)$ .

**Generalised hypergeometric series**

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}.$$

**(Gaussian) Hypergeometric function**

$${}_2F_1(a, -n; c; z) = \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{(a)_j}{(c)_j} z^j.$$

(A special case of the hypergeometric series with  $p = 2$ ,  $q = 1$ ).

**Jacobi (=hypergeometric) polynomials**

$$P_n^{(\alpha, \beta)}(z) := \frac{(\alpha + 1)_n}{n!} {}_2F_1\left(-n, 1 + \alpha + \beta + n; \alpha + 1; \frac{1}{2}(1 - z)\right).$$

**Gegenbauer (=ultraspherical) polynomials**

$$C_n^{(\lambda)}(z) := \frac{(2\lambda)_n}{n!} {}_2F_1\left(-n, 2\lambda + n; \lambda + \frac{1}{2}; \frac{1 - z}{2}\right) = \frac{(2\lambda)_n}{(\lambda + \frac{1}{2})_n} P_n^{(\lambda-1/2, \lambda-1/2)}(x).$$

They satisfy a three-term recurrence relation (as all orthogonal polynomials do!)

$$\begin{aligned}C_0^{(\lambda)}(x) &= 1 \\C_1^{(\lambda)}(x) &= 2\lambda x \\(n+1)C_{n+1}^{(\lambda)}(x) &= 2(n+\lambda)x C_n^{(\lambda)}(x) - (n+2\lambda-1)C_{n-1}^{(\lambda)}(x).\end{aligned}$$

From Wikipedia: In spectral methods for solving differential equations, if a function is expanded in the basis of Chebyshev polynomials and its derivative is represented in a Gegenbauer/ultraspherical basis, then the derivative operator becomes a diagonal matrix, leading to fast banded matrix methods for large problems ([Olver and Townsend 2013](#)).

## Chapter 2

## Conclusion

In the present thesis, we explored the interesting realm of particle-particle interactions. Next to the written part, the reader will find an implementation of the particle simulator, including a Graphical User Interface (GUI), as well as the numerical solver.

# Acronyms, Definitions and Theorems

GUI   Graphical User Interface

6

**Definitions**

**Theorems**

**Lemmata**

**Remarks**

# Bibliography

- Gutleb, Timon S., José A. Carrillo and Sheehan Olver (Oct. 2020). ‘Computing Equilibrium Measures with Power Law Kernels’. In: *arXiv*. DOI: [10.1090/mcom/3740](https://doi.org/10.1090/mcom/3740). eprint: [2011.00045](https://arxiv.org/abs/2011.00045).
- (Sept. 2021). ‘Computation of Power Law Equilibrium Measures on Balls of Arbitrary Dimension’. In: *arXiv*. DOI: [10.1007/s00365-022-09606-0](https://doi.org/10.1007/s00365-022-09606-0). eprint: [2109.00843](https://arxiv.org/abs/2109.00843).
- Olver, Sheehan and Alex Townsend (Aug. 2013). ‘A Fast and Well-Conditioned Spectral Method’. In: *SIAM Rev.* URL: <https://epubs.siam.org/doi/10.1137/120865458>.



# List of Figures and Tables

List of Figures

List of Tables

## Appendix A – Supplemental Proofs