

Abstract

To be written.

This MMSC thesis will further explore general kernel spectral methods for finding equilibrium measures where initial progress made in [Gutleb, Carrillo and S. Olver 2020](#) and [Gutleb, Carrillo and S. Olver 2021](#).

Keywords: Equilibrium Measures

Languages: C++, Julia, Python

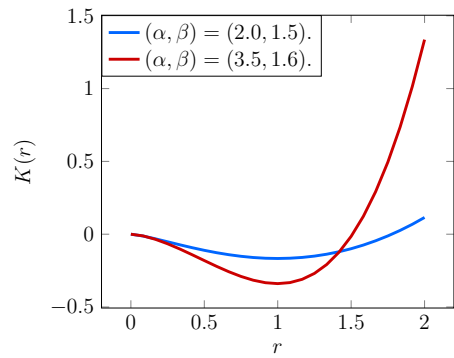
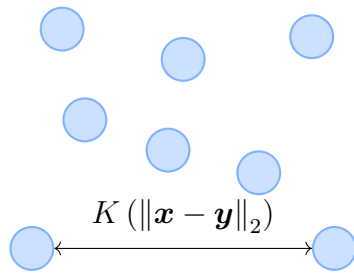
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Chapter 1

Introduction



(a) $N = 8$ particles interacting with one another **(b)** Plot of attractive-repulsive potential functions through the potential $K(r)$.
 $K(r) = \frac{r^\alpha}{\alpha} - \frac{r^\beta}{\beta}$ for different α, β .

Cf. Figure 1.1a and Figure 1.1b.

All plots and figures in this thesis were generated using the Makie visualisation tool ([Danisch and Krumbiegel 2021](#)), an open-source package available for the Julia computing language ([Bezanson et al. 2017](#)).

Just Notes

This chapter's purpose is the collection of notes, and it will not be included in the final dissertation.

Special Functions we like

Pochhammer's falling symbol $(x)_n := \prod_{k=0}^{n-1} (x - k)$.

Pochhammer's rising symbol $(x)^n := \prod_{k=0}^{n-1} (x + k)$.

Generalised hypergeometric series

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}.$$

(Gaussian) Hypergeometric function

$${}_2F_1(a, -n; c; z) = \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{(a)_j}{(c)_j} z^j.$$

(A special case of the hypergeometric series with $p = 2$, $q = 1$).

Jacobi (=hypergeometric) polynomials

$$P_n^{(\alpha, \beta)}(z) := \frac{(\alpha + 1)_n}{n!} {}_2F_1\left(-n, 1 + \alpha + \beta + n; \alpha + 1; \frac{1}{2}(1 - z)\right).$$

Gegenbauer (=ultraspherical) polynomials

$$C_n^{(\lambda)}(z) := \frac{(2\lambda)_n}{n!} {}_2F_1\left(-n, 2\lambda + n; \lambda + \frac{1}{2}; \frac{1 - z}{2}\right) = \frac{(2\lambda)_n}{(\lambda + \frac{1}{2})_n} P_n^{(\lambda-1/2, \lambda-1/2)}(x).$$

They satisfy a three-term recurrence relation (as all orthogonal polynomials do!)

$$\begin{aligned} C_0^{(\lambda)}(x) &= 1 \\ C_1^{(\lambda)}(x) &= 2\lambda x \\ (n+1)C_{n+1}^{(\lambda)}(x) &= 2(n+\lambda)x C_n^{(\lambda)}(x) - (n+2\lambda-1)C_{n-1}^{(\lambda)}(x). \end{aligned}$$

From Wikipedia: In spectral methods for solving differential equations, if a function is expanded in the basis of Chebyshev polynomials and its derivative is represented in a Gegenbauer/ultraspherical basis, then the derivative operator becomes a diagonal matrix, leading to fast banded matrix methods for large problems (S. Olver and Townsend 2013).

Three-term recurrence relationship F. Olver et al. 2018, p. 18.9.1:

$$xC_n^{(\lambda)}(x) = \frac{(n+2\lambda-1)}{2(n+\lambda)}C_{n-1}^{(\lambda)}(x) + \frac{n+1}{2(n+\lambda)}C_{n+1}^{(\lambda)}(x). \quad (1.1)$$

1.0.1 Theorem: Two term recurrence of Q^α

The integral operator

$$Q^\alpha[u](x) = \int_{-1}^1 |x-y|^\alpha u(y) dy$$

satisfies a two-term recurrence relationship when acting on the ultraspherical polynomials $C_n^{(\lambda)}(y)$ with weight $w(y) = (1-y^2)^{\lambda-\frac{1}{2}}$ such that

$$xQ^\alpha[wC_n^{(\lambda)}](x) = \kappa_1 Q^\alpha[wC_{n-1}^{(\lambda)}](x) + \kappa_2 Q^\alpha[wC_{n+1}^{(\lambda)}](x),$$

where $n \geq 2$ and with the constants

$$\begin{aligned} \kappa_1 &= \frac{(n-\alpha-1)(2\lambda+n-1)}{2n(\lambda+n)}, \\ \kappa_2 &= \frac{(n+1)(2\lambda+n+\alpha+1)}{2(\lambda+n)(2\lambda+n)}. \end{aligned}$$

Chapter 2

Particle Interaction Theory

2.0.1 aliases: Molecular Dynamics

Some input from the Wolfson Particle Physicist: Lennard-Jones is an **intermolecular** potential. So length-scale is between-molecules. Therefore, the only relevant interaction is the electromagnetic one. The strong force keeps protons in the nucleus together (a force much stronger than the electromagnetic one).

2.0.2 Structure

- Definition: N-Body System (set of particles with position and velocity)
- Inertia / kinetic energy
- [[Potential]]s motivating a force $F = -\nabla U$
- Write differential equation of movement $\frac{dx_i}{dt}$
- Link to [[Particle Simulator]], give a Screenshot
- Introduce [[Continuous Limit]], write about particle density $\rho(x)$
- [[Friction Term]] -> Energy Dissipation -> Different Plot

Chapter 3

Particle Simulator

3.0.1 aliases: N-Body Simulator, Molecular Dynamics Simulator

is there to solve problems in [[Particle Interaction Theory]].

3.0.2 Structure

- Talk about different integration methods
- Leap-Frog Integration
- Screenshot of GUI

3.0.3 Available Methods:

- [[Integration Routine]]
 - Simple Forward Integration
 - Improvements: Multistep methods
 - [[Leapfrog Integration]]
- [[Fast Multipole Method]]
- [[Multigrid Methods]]

3.0.4 Available Solvers:

- LAMMPS ancient
- [Gromacs](#) has nice homepage
- [OpenMM](#) also has nice homepage
- [OpenFPM](#)
- `[[General Kernel Spectral Method]]` for `[[Equilibrium Measures]]`

3.0.5 Implementations in `[[My Dissertation]]`:

- `[[C++ Particle Integrator with GUI]]`

Chapter 4

Spectral Method

4.1 Definitions

solves an [[Integral Equation]] or [[Differential Equation]] by assuming a solution of the form

$$\rho(x) = \sum_{k=1}^N \rho_k b_k(x)$$

where $\{b_k\}$ is a basis of functions.

4.1.1 Structure

- Introduce [[Chebyshev Polynomials]], [[Gegenbauer Polynomials|Ultraspherical Polynomials]], [[Jacobi Polynomials]], etc.
- Describe the method
- Talk about the resulting [[Operator]].
 - [[Derivation of In-Operator Recurrence]]
- Numerical Analysis ([[Bound on the Error]])
- Show results here? Or in extra results chapter?

4.1.1 Definition: Ansatz

$$\rho(x) = (1 - \|y\|^2)^{m - \frac{\alpha+d}{2}} \sum_{k=1}^N P_k^{(a,b)}(2\|y\|^2 - 1)$$

Todo: - [] is it alpha or beta in the exponent of $(1-y^2)$?

4.1.2 Definition: Bound on the Error

- [] How does one look at this topic? We should have [[Spectral Convergence]], hopefully.

4.1.3 Definition: Chebyshev Polynomials

Of the first kind:

$$T_k(x)$$

Of the second kind:

$$U_k(x)$$

Also have a [[Three-Term Recurrence Relationship]].

4.1.4 Definition: Clarabel

ist ein Tool zur konvexen Optimierung.

Hier ein [[Link den es noch nicht gibt]].

4.1.5 Definition: Derivation of In-Operator Recurrence

Done on paper, 14th of July. Based on the [[Three-Term Recurrence Relationship]].

One can even determine an explicit relationship between the coefficients in the Jacobi expansion by considering the [[Jacobi Matrix]].

4.1.6 Definition: Equilibrium Measures

Are a [[Measure]]

$$\rho : \mathbb{R} \mapsto \mathbb{R}, \rho(x)$$

- [] Need to fix this definition Can be computed using [EquilibriumMeasures.jl](#)

4.1.7 Definition: Function Space

To be defined, but the space our coefficients are in. Could be

$$L := \{f : \mathbb{R} \mapsto \mathbb{R} | f \text{ square integrable?}\}$$

4.1.8 Definition: Gaussian Hypergeometric Function

Written as

$${}_2F_1(a, b; c; z)$$

4.1.9 Definition: Gegenbauer Polynomials**4.1.2 alias: Ultraspherical Polynomials**

Are a special case of the [[Jacobi Polynomials]] and form an [[Orthonormal Basis]] under the weight given by

$$w(x) = (1 + x)^\alpha$$

4.1.10 Definition: Generalised Hypergeometric Series

Is given by

$${}_pF_q$$

Special Case: [[Gaussian Hypergeometric Function]]. The definition involves the [[Rising Factorial]] (Pochhammer Symbol).

4.1.11 Definition: Implementation and Results**4.1.3 Structure**

- Talk about Julia, C++ and the [[C++ Particle Integrator with GUI]]
- Numerical Results
 - Operator plots
 - Plots of Particle Densities
 - Difference between [[Spectral Method]] and [[Particle Simulator]] results

4.1.12 Definition: Integration Routine

Could be done using [Cubature](#). Otherwise, just Forward Euler.

4.1.13 Definition: Jacobi Matrix**4.1.4 aliases: Jacobi Operator**

The [Jacobi operator](#) is the matrix $X \in \mathbb{R}^{N \times N}$ satisfying

$$x \cdot P(x) = P(x) \cdot X^T$$

4.1.14 Definition: Jacobi Polynomials

Are given by

$$J_n^{(a,b)}(x) = \text{prefactor} \cdot {}_2F_1(\dots)$$

So are defined using the [\[\[Gaussian Hypergeometric Function\]\]](#).

4.1.5 Nice Spectral Properties

- Differentiation
- Three-Term Recurrence
- [\[\]](#) why are they better than just Chebyshev?

[\[\[Gegenbauer Polynomials\]\]](#) are a special case. And [\[\[Chebyshev Polynomials\]\]](#) are a special case of them.

4.1.15 Definition: Leapfrog Integration

Nice introduction [here](#). Maybe compare with [Advanced HMC](#)?

4.1.16 Definition: Operator

Either the attractive or the repulsive operator can be sparse.

Obtained using [\[\[Theorem 2.16\]\]](#). Derivation of the exact row/column form on paper ([#include](#) in [\[\[My Dissertation\]\]](#))

- [\[\]](#) What does the solver look like for other kernels?

4.1.17 Definition: Orthogonal Polynomials

Are univariate polynomials

$$p : \mathbb{R} \mapsto \mathbb{R}, p(x) = \sum_{k=1}^N c_k x^k.$$

that form an [[Orthonormal Basis]] under some inner product.

4.1.18 Definition: Outer Optimisation Routine

Perhaps use [[Clarabel]] if we have a convex optimisation problem?

4.1.19 Definition: Rising Factorial**4.1.6 alias: Pochhammer Symbol**

Given by

$$(x)_n = \prod_{k=0}^{n-1} (x + k).$$

4.1.20 Definition: Rust Conference

held on 14th of July, had many cool talks. Link: [here](#).

4.1.7 Notable Stuff for [[My Dissertation]]:

- [[Clarabel]] (Rust + Julia), a convex optimisation package, developed by Paul Goullart (Engineering Dept.).
 - Can be integrated with cvxpy
- Makie Julia Visualisations / Plots
- bempp-rs A collection of crates for the solution of boundary integral equations

4.1.21 Definition: Spectral Convergence

Definition 3.6 (Convergence at spectral speed) An N -point approximation φ_N of a function f converges to f at spectral speed if $|\varphi_N - f|$ decays pointwise in $[-1, 1]$ faster than $O(N^{-p})$ for any $p = 1, 2, \dots$ so $p \in \mathbb{N}$.

Source: https://www.damtp.cam.ac.uk/user/cbs31/Teaching_files/c11.pdf.

4.1.22 Definition: Theorem 2.16

Gives an explicit expression for the main integral $Q^\beta : L \mapsto L$, an operator from the [[Function Space]] L to the function space L , we are interested in:

$$\hat{Q}^\beta[\rho](x) = \int_{B_1} |x - y|^\beta (1 - |y|^2)^{m - \frac{\alpha+d}{2}} P_n^{(m - \frac{\alpha+d}{2}, \frac{d-2}{2})}(2|y|^2 - 1) dy$$

which is used to construct the [[Spectral Method]] [[Operator]] Q^β , acting on the coefficients ρ .

4.1.23 Definition: Three-Term Recurrence Relationship

All [[Orthogonal Polynomials]] have (at least) a three-term recurrence relationship.

- [] how could I prove that?

4.1.1 Theorem: Integration Theorem that needs a name

On the d -dimensional unit ball B_1 the power law potential, with power $\alpha \in (-d, 2 + 2m - d)$, $m \in \mathbb{N}_0$ and $\beta > -d$, of the n -th weighted radial Jacobi polynomial

$$(1 - |y|^2)^{m - \frac{\alpha+d}{2}} P_n^{(m - \frac{\alpha+d}{2}, \frac{d-2}{2})}(2|y|^2 - 1)$$

reduces to a Gaussian hypergeometric function as follows:

$$\begin{aligned} & \int_{B_1} |x - y|^\beta (1 - |y|^2)^{m - \frac{\alpha+d}{2}} P_n^{(m - \frac{\alpha+d}{2}, \frac{d-2}{2})}(2|y|^2 - 1) dy \\ &= \frac{\pi^{d/2} \Gamma(1 + \frac{\beta}{2}) \Gamma(\frac{\beta+d}{2}) \Gamma(m + n - \frac{\alpha+d}{2} + 1)}{\Gamma(\frac{d}{2}) \Gamma(n+1) \Gamma(\frac{\beta}{2} - n + 1) \Gamma(\frac{\beta-\alpha}{2} + m + n + 1)} {}_2F_1 \left(n - \frac{\beta}{2}, -m - n + \frac{\alpha-\beta}{2}; \frac{d}{2}; |x|^2 \right). \end{aligned}$$

4.2 Derivation of Operator

Considering the operator $\hat{Q}^\beta[\rho]$ as in Theorem 4.1.1, from the [[Ansatz]] $\rho(x)$ we have

$$\hat{Q}^\beta(x) = \sum_{k=1}^N \rho_k \int \|x - y\|^\beta (1 - \|y\|^2)^a P_k^{(a,b)}(2\|y\|^2 - 1) dy$$

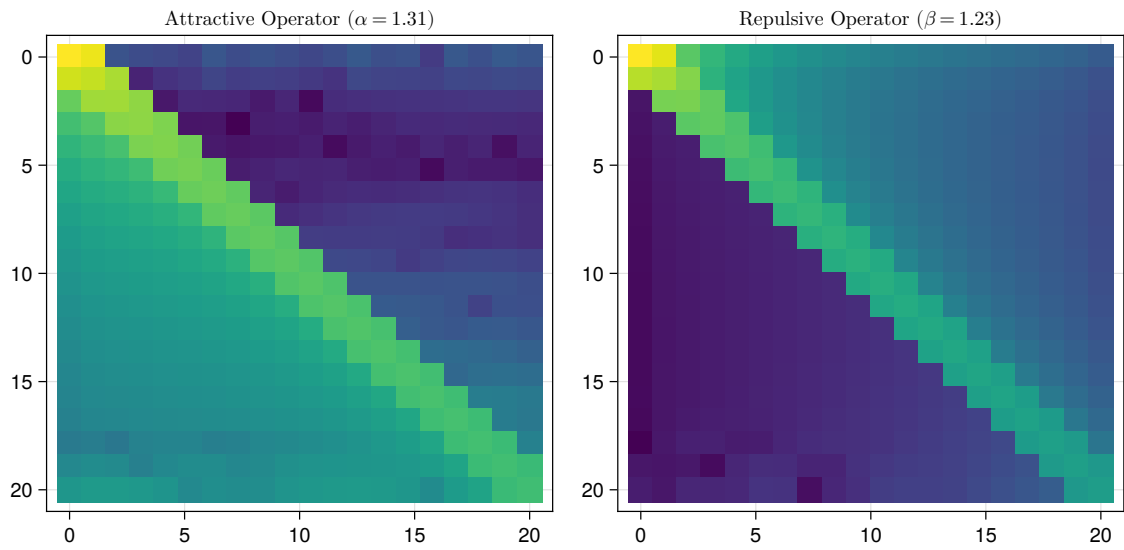


Figure 4.1: The attractive and repulsive operators

Chapter 5

General Kernel Spectral Method

is a [[Spectral Method]] involving an [[Integral Equation]].

5.0.1 Structure

- Was ist ein General Kernel?
- How can we expand?
- Mehr Results als im vorigen Chapter [[Spectral Method]]

Chapter 6

Conclusion

In the present thesis, we explored the interesting realm of particle-particle interactions. Next to the written part, the reader will find an implementation of the particle simulator, including a Graphical User Interface (GUI), as well as the numerical solver.

Acronyms, Definitions and Theorems

GUI Graphical User Interface

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Definitions

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Remarks

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Appendix A – Supplemental Proofs