6. $I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\, dy\, dx$

Podstawienie:

$x = r \cdot \cos\theta$

$y = r \cdot \sin\theta$

$r \in [0, \infty]$

$\theta \in [0, \pi]$

Macierz Jacobiego:

$$\begin{bmatrix} \frac{\partial r\cos\theta}{\partial r} & \frac{\partial r\cos\theta}{\partial \theta} \\ \frac{\partial r\sin\theta}{\partial r} & \frac{\partial r\sin\theta}{\partial \theta} \end{bmatrix}$$

Jakobian:

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r$$

$dy\, dx \to r\, dr\, d\theta$

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2\cos^2\theta + r^2\sin^2\theta}{2}}\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2(\cos^2\theta + \sin^2\theta)}{2}}\, r\, dr\, d\theta =$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-\frac{1}{2}r^2}\, r\, dr\, d\theta = \begin{vmatrix} t = -\frac{1}{2}r^2 \\ dt = -r\, dr \\ dr = -\frac{dt}{r} \end{vmatrix} = \int_0^{2\pi}\int_{-\infty}^{0} e^t\, dt\, d\theta =$$

$$= \int_0^{2\pi} \Big[e^t\Big]_{-\infty}^{0}\, d\theta = \int_0^{2\pi} \underset{1}{\underset{\shortparallel}{e^0}} - \underset{0}{\underset{\shortparallel}{\lim_{t\to-\infty} e^t}}\, d\theta = \int_0^{2\pi} 1\, d\theta = \Big|_0^{2\pi} \theta = 2\pi - 0 = 2\pi$$