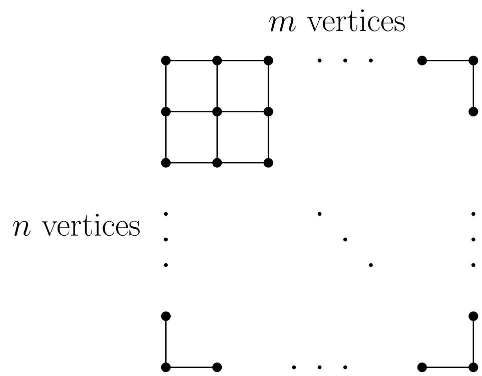
Part A (Eulerian Cycle & Hamiltonian Cycle).

Problem 1.
For which values of m and n does the following graph contain an Eulerian Cycle?



Problem 2.
n-cube is a graph with 2^n vertices, each corresponding to a n-bit string. Two vertices has an edge if the corresponding two n-bit strings differ in exactly one bit. For which values of n does the n-cube contain an Eulerian Cycle?

Problem 3. (Gray Code)
A sequence

$$s_1, s_2, ..., s_{2^n} \tag{1}$$

is called a *Gray code* if each s_i is a string of n bits and satisfying

- Every n-bit string appears somewhere in the sequence.
- s_i and s_{i+1} differ in exactly one bit, $i = 1, ..., 2^n - 1$.
- s_{2^n} and s_1 differ in exactly one bit.

Let G_1 denote the sequence 0, 1. We define G_n in terms of G_{n-1} by the following rules:

(a) Let G_{n-1}^R denote the sequence G_{n-1} written in reverse.

(b) Let G'_{n-1} denote the sequence obtained by prefixing each member of G_{n-1} with 0.

(c) Let G''_{n-1} denote the sequence obtained by prefixing each member of G_{n-1}^R with 1.

(d) Let G_n be the sequence consisting of G'_{n-1} followed by G''_{n-1}.

Complete the following two problems.

(I) Show that G_n is a Gray code for every positive integer n.

(II) Show that n-cube has a Hamiltonian cycle for every positive integer $n \geq 2$.

Part B (Stable Matching & Bipartite Matching).

Problem 4.
For stable marriage problem, show that the Gale-Shapley algorithm is men-optimal, and also women-pessimal.

Problem 5.
For stable marriage problem, show an example that women can lie to get a better partner.

Problem 6.
Show that a bipartite graph has a matching of size $n - k$ if and only if there is no subset S of one side with $|N(S)| + k < |S|$, where n is the number of nodes on one side and so the graph has $2n$ nodes.

Remark: This is a generalization of Hall's theorem.

Problem 7.
Suppose we have

- n jobs (j_i) each taking one unit of time.
- m machines, machine m_i can run c_i jobs.
- And define happily $\sum_{i=1}^{m} c_i =$ n

1. Is it possible to assign jobs to machines so that you only need to wait one unit of time and finish them all?
2. Model this as a bipartite matching problem. Show the correspondence (how you can construct an instance of bipartite matching problem given the sepc of jobs and machines).

Part C (Graph Coloring & Planar Graph).

Problem 8.
What is the chromatic number of the following graph?

- K_n, complete graph with n vertices.
- P_n, a path with n vertices.
- C_n, a circle with n vertices.

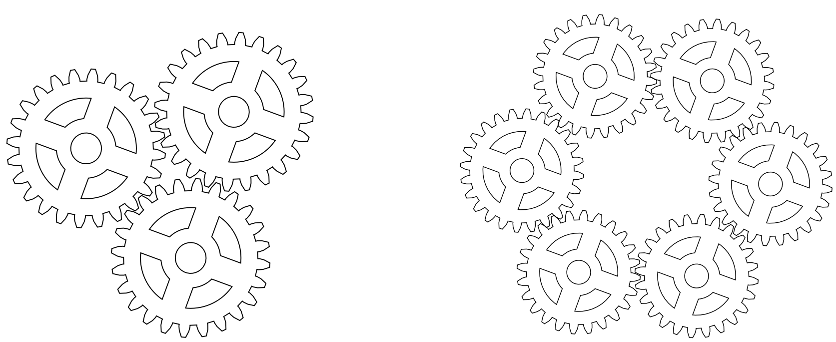
Problem 9.
A company manufactures n chemicals $C_1, C_2, ..., C_n$. Certain pairs of these chemicals are incompatible and would cause explosions if brought into contact with each other. As a precautionary

measure the company wishes to partition its warehouse into compartments, and store incompatible chemicals in different compartments. What is the least number of compartments into which the warehouse should be partitioned? Model this problem as a graph coloring problem and explain your answer.

Problem 10.
Suppose you run a day care for an office building and there are seven children A, B, C, D, E, F, G. You need to assign a locker where each child's parent can put the child's food. The children come and leave so they are not all there at the same time. You have 1 hour time slots starting 7:00 am to 12:00 nn. A star in the following table means a child is present at that time. What is the minimum number of lockers you need to prepare? What's your plan to assign the lockers? Show your steps.

	A	B	C	D	E	F	G
7:00	*			*	*		
8:00	*	*	*				
9:00	*		*	*		*	
10:00	*		*			*	*
11:00	*					*	*
12:00	*				*		

Problem 11.
Recall that in homework 4 you have a "gears" problem (Problem 2). The relation among the gears can be modeled as a graph: the vertex set is the set of gears, and there is an edge between a pair of gears if they touch each other (note that we assume all gears have identical radii and number of teeth). Now, your task is to color the "gear" graph with minimum number of colors. How many colors do you need? Explain your answer.



(a) (b)

Figure 1: Meshing gears

Problem 12.
Show that K_5 (complete graph with 5 vertices) and $K_{3,3}$ (complete bipartite graph with 3 vertices in one side and 3 vertices in the other side) are not planar.

Problem 13 (Optional).
Show that every planar graph is 5-colorable.