Sets

1. **T/F** $\{1, 2, 3\} = \{3, 2, 3, 1\}$.
2. **SQ** What are the following sets?
   (a) $\{x \in Z | x = y^2 \text{ for some integer } y \leq 3\}$
   (b) $\{x \in Z | x^2 = y \text{ for some integer } y \leq 3\}$
   (c) $\{x \in R | x^2 = y \text{ for some integer } y \leq 3\}$
3. **T/F** Which of the followings is/are true?
   (a) $a \in \{a\}$.
   (b) $a \in \{\{a\}\}$.
   (c) $\{a\} \in \{a\}$.
   (d) $\{a\} \in \{\{a\}\}$.
4. **T/F** Which of the followings is/are true? Give counter example(s) for the false one(s).
   (a) $\emptyset \in S$ for any set S.
   (b) $\{\emptyset\} \in S$ for any set S.
5. **T/F** Which of the followings is/are true? Give counter example(s) for the false one(s).
   (a) If $\{A, B\} = \{C, D\}$, then $A = C$ and $B = D$.
   (b) If $\{A, \{B\}\} = \{C, \{D\}\}$, then $A = C$ and $B = D$.
   (c) If $\{A, B\} = \{C\}$, then $A = B = C$.
6. **T/F** Which of the followings is/are true?
   (a) $a \subseteq \{a\}$.
   (b) $a \subseteq \{\{a\}\}$.
   (c) $\{a\} \subseteq \{a\}$.
   (d) $\{a\} \subseteq \{\{a\}\}$.
7. **T/F** Which of the followings is/are true? Give counter example(s) for the false one(s).
   (a) $\emptyset \subseteq S$ for any set S.
   (b) $\emptyset \subset S$ for any set S.
   (c) $S \nsubseteq \emptyset$ for any set S.
   (d) $S \not\subset \emptyset$ for any set S.

8. **T/F** Which of the followings is/are true? Give counter example(s) for the false one(s).
   (a) If $A \subseteq B$, then $A \subset B$.
   (b) If $A \subset B$, then $A \subseteq B$.

9. **T/F** Which of the followings is/are true?
   (a) $\emptyset \in \{0\}$
   (b) $\emptyset \subset \{0\}$
   (c) $\{0\} \subset \{0\}$
   (d) $\{\emptyset\} \subseteq \{\emptyset\}$

10. **SQ** Suppose the universal set is $\{-1, 0, 1, 2\}$, $A = \{0, 1, 2\}$ and $B = \{-1, 2\}$. What are the following sets?
   (a) $A \cap B$
   (b) $A \cup B$
   (c) A^c
   (d) $A - B$
   (e) $A \times B$
   (f) $P(B)$

11. **T/F** $A \times B = B \times A$.

12. **T/F** Which of the followings is/are true? Give counter example(s) for the false one(s).
   (a) $\emptyset \in P(S)$ for any set S.
   (b) $\emptyset \subseteq P(S)$ for any set S.
   (c) $\{\emptyset\} \in P(S)$ for any set S.
   (d) $\{\emptyset\} \subseteq P(S)$ for any set S.

13. **T/F** Which of the followings is/are true? Give counter example(s) for the false one(s).
   (a) $S \in P(S)$ for any set S.
   (b) $S \subseteq P(S)$ for any set S.
   (c) $\{S\} \in P(S)$ for any set S.
   (d) $\{S\} \subseteq P(S)$ for any set S.

14. **T/F** Which of the followings is/are true? Give counter example(s) for the false one(s).
   (a) $\{a, b, c\} - \{b\} = \{a, c\}$ for any integers a, b, c.
   (b) $\{a, b, c\} - \{b\} \subseteq \{a, c\}$ for any integers a, b, c.

15. **T/F** Which of the followings is/are true? Give counter example(s) for the false one(s).

(a) $A \cap B^c = (A^c \cup B)^c$.

(b) $(A \cup B) \cap B = A$.

(c) $(A \cup B) \cup B^c = U$, where U is the universal set.

(d) $(A \cup B) \cap B^c = A - B$.

16. **T/F** Which of the followings is/are true? Give counter example(s) for the false one(s).

    (a) $(A - B) \cup B = A \cup B$.

    (b) $(A \cup B) - B = A - B$.

    (c) $(A - B) \cup (B - A) = A \cup B$.

    (d) $(A - B) \cup (A - C) = A - (B \cup C)$.

17. **T/F** $\emptyset \times S = \emptyset$ for any set S.

18. **T/F** Which of the followings is/are true? Give counter example(s) for the false one(s).

    (a) If $A - B = \emptyset$, then $A = B$.

    (b) If $A \cap B = \emptyset$, then $A \subseteq B^c$.

Basic Counting

1. **SQ** Count the number of passwords with following constraints. Assuming digits $= \{0, 1, \ldots, 9\}$, letters $= \{a, b, \ldots, y, z\}$

    (a) 5 characters which are digits or letters.

    (b) 4 characters which are digits or letters, with at least 1 digit.

    (c) 4 characters which are digits. Adjacent characters are not the same.

    (d) 6 characters which are letters. The first half must not equal to the last half.

    (e) 5 characters which are digits. The sum of digits is a multiple of 10.

    (f) 6 characters which are digits. The i-th digit must not equal to the $7-i$-th digit for $1 \leq i \leq 6$.

    (g) 7 characters which are digits. The i-th digit must not equal to the $8-i$-th digit for $1 \leq i \leq 7$.

2. **SQ** Count the number of 4 digit numbers by using digits $\{3, 4, 5, 7\}$ when

    (a) The first digit must be 3 and repetition of digits is allowed?

    (b) The first digit must be 3 and repetition of digits is not allowed?

    (c) The 4 digit number must be divisible by 2 and repetion is allowed?

    (d) The 4 digit number must be divisible by 2 and repetion is not allowed?

3. **SQ** How many tickets in Mark Six (49 numbers in total) are there that wins

    (a) first prize? (All 6 drawn numbers)

    (b) second prize? (5 drawn numbers plus the extra number)

(c) third prize? (5 drawn numbers)

(d) forth prize? (4 drawn numbers plus the extra number)

4. **SQ** Count the number of rectangles with following constraints.

   (a) The width and height are both positive integers between 3 and 10.

   (b) The width and height are both positive integers. The area is no more than 6.

   (c) The width and height are both positive integers. The perimeter is no more than 20.

   (d) The width and height are both positive integers between 3 and 10. Swapping the width and height is considered to be the same rectangle.

   (e) The width and height are both square numbers. The width is between 3 and 10. The height is between 11 and 30.

   (f) The width and height are both positive integers between 3 and 10. The width is not equals to the height.

5. **SQ** How many rectangles are there in a 5×6 grid?

6. **SQ** I want to visit 10 cities, each exactly once. Count the number of possible routes (order of visiting). How about with the following constraints?

   (a) I must start at city 1.

   (b) I must start at city 1 and end at city 10.

   (c) There is no flight from city 1 to city 2.

   (d) There is no flights between city 1 and city 2.

7. **SQ** How many different 5-card poker hands are there is

   (a) Royal flush? 10, J, Q, K, A of the same suit.

   (b) Straight flush? Five adjacent values of the same suit, but not a royal flush. Note that A, 2, 3, 4, 5 is valid but K, A, 2, 3, 4 is not.

   (c) Four of a kind? Four cards of one value.

   (d) Full horse? Three cards of one value, and the other two of another value.

   (e) Flush? Five cards of the same suit, but not a straight flush or royal flush.

   (f) Straight? Five adjacent values, but not a straight flush or royal flush.

   (g) Three of a kind? Three cards of one value, and the other two of different values.

   (h) Two pairs? Two cards of one value, two cards of second value, and the fifth cards of the third.

   (i) One pair? Two cards of one value and the other three of different values.

   (j) No pair? Five cards of different values but not a straight, flush, straight flush or royal flush.

8. **SQ** Count the number of 01-strings with following constraints.

   (a) The length is 8. Number of 1s is 2 more than number of 0s.

(b) The length is 8. Number of 1s is 3 more than number of 0s.

(c) The length is 9. Number of 1s is 3 more than number of 0s.

(d) The length is $2n$. Number of 1s is $2k$ more than number of 0s.

(e) The length is 8. 00 or 11 must appear somewhere in the string.

(f) The length is 8. 000000 must not appear in the string.

9. **SQ** Suppose the group of twelve consists of five men and seven women.

   (a) How many five-person teams can be chosen that consist of three men and two women?

   (b) How many five-person teams contain at least one man?

   (c) How many five-person teams contain at most one man?

10. **SQ** A student council consists of 15 students.

    (a) In how many ways can a committee of six be selected from the membership of the council?

    (b) Two council members have the same major and are not permitted to serve together on a committee. How many ways can a committee of six be selected from the membership of the council?

    (c) Two council members always insist on serving on committees together. If they can't serve together, they won't serve at all. How many ways can a committee of six be selected from the council membership?

    (d) Suppose the council contains eight men and seven women. How many committees of six contain at least one woman?

    (e) Suppose the council consists of three freshmen, four sophomores, three juniors, and five seniors. How many committees of eight contain two representatives from each class?

11. **SQ** Consider a digital circuit that reads an nbit input string and writes a 1bit output. Two circuits are considered different if they have a different output for some input string, otherwise they are considered the same (i.e. two circuits are considered the same if they have the same output bit for every possible input string). Count the number of different circuits.

Binomial Coefficients

1. **SQ** Find the required coefficients.

   (a) Coefficient of x^3y^2 in $(x+y)^5$.

   (b) Coefficient of x^2y^3 in $(x-y)^5$.

   (c) Coefficient of x^7y^{13} in $(x+y)^{19}$.

Combinatorial Proof, Inclusion-Exclusion Principle

1. **LQ** Proof the following equality (by combinatorial argument). Assume $k \le r \le n$.

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{n-r}$$

2. **LQ** Proof the following equality (by combinatorial argument). Assume $k \leq n$.

$$\binom{2n}{k} = \binom{n}{0}\binom{n}{k} + \binom{n}{1}\binom{n}{k-1} + \cdots + \binom{n}{k}\binom{n}{0}$$

3. **SQ** Count the number of pairs (x, y) with following constraints.

   (a) x and y are between 1 and 100. 2 does not divide both of them.

   (b) x and y are between 1 and 100. 3 does not divide both of them.

   (c) x and y are between 1 and 100. Both 2 and 3 do not divide both of them.

   (d) x and y are between 1 and 100. 2, 3 and 5 do not divide both of them.

Functions

1. **T/F** Which of the followings is/are true? Give counter example(s) to the false one(s).

   (a) If $f : A \to B$ is injective, then $|A| \leq |B|$.

   (b) If $|A| < |B|$, then any fucntion $f : A \to B$ is injective.

   (c) If $f : A \to B$ is surjective, then $|A| > |B|$.

   (d) If $|A| > |B|$, then any function $f : A \to B$ is surjective.

   (e) If $f : A \to B$ is bijective, then $|A| = |B|$.

   (f) If $|A| = |B|$, then any function $f : A \to B$ is bijective.

2. **SQ** Specify a codomain such that the following functions are surjective.

   (a) $f(x) = x$, domain is $[0, 1]$.

   (b) $f(x) = x + 1$, domain is $[3, 5]$.

   (c) $f(x) = x^2$, domain is $\mathbb{R}$.

   (d) $f(x) = \sqrt{x}$, domain is $[1, 4]$.

   (e) $S = \{0, 1, 2, 3\}$, $f(x) = |x|$, domain is $P(S)$.

3. **T/F** Which of the followings is/are true?

   (a) $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$ is surjective.

   (b) $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^3$ is surjective.

   (c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is surjective.

   (d) **Optional** $f : \mathbb{Q} \to \mathbb{Q}$, $f(x) = x^3$ is surjective. ($\mathbb{Q}$ is the set of rational numbers)

4. **T/F** Which of the followings is/are true?

   (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 1$ is bijective.

   (b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 1$ is bijective.

(c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is bijective.

(d) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = (x^2 + 1)/(x^2 + 2)$ is bijective.

5. **T/F** Which of the followings is/are true?

   (a) The function $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$ has inverse function.

   (b) The function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$, $f(x) = x^2$ has inverse function.

   (c) The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ has inverse function.

   (d) The function $f : \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = x^2$ has inverse function.

Counting Functions

1. **SQ** How many functions are there from $\{a, b, c, d, e\}$ to $\{1, 2, 3, 4\}$?

2. **SQ** How many functions are there from $\{1, 2, 3, 4\}$ to $\{a, b, c, d, e\}$?

3. **SQ** How many injective functions are there from $\{a, b, c\}$ to $\{1, 2, 3, 4, 5, 6\}$?

4. **SQ** How many injective functions are there from $\{a, b, c, d, e\}$ to $\{1, 2, 3, 4\}$?

5. **SQ** How many bijective functions are there from $\{a, b, c, d, e\}$ to $\{1, 2, 3, 4, 5\}$?

6. **SQ** How many functions are there from $\{a, b, c, d, e, f\}$ to $\{True, False\}$ with half-input-true and half-input-false?

Pigeonhole Principle

1. **T/F** Which of the followings is/are true? Give counter example(s) to the false one(s).

   (a) If there are 101 pigeons and 20 holes, then some hole has at least 6 pigeons.

   (b) If there are 100 pigeons and 20 holes, then some hole has at least 5 pigeons.

   (c) If there are 100 pigeons and 20 holes, then every hole has at most 6 pigeons.

   (d) If there are 101 pigeons and 20 holes, then at least 2 holes has at least 3 pigeons.

   (e) If there are 100 pigeons and 3 holes, then every hole has at least 33 pigeons.

   (f) If there are 100 pigeons and 3 holes, then every hole has at most 34 pigeons.

2. **SQ** There are many red, blue and green balls in a box. How many balls do you need to draw in order to guaranttee that

   (a) you have drawn two balls of the same color?

   (b) you have drawn five balls of the same color?

3. **SQ** How many integerss from 100 through 999 must you pick in order to be sure that at least two of them have a digit in common?(For example, 256 and 530 have the common digit 5)

4. **LQ** Show that if you choose $n + 1$ different numbers from $\{1, 2, 3, \ldots, 2n\}$, then one of your chosen number is a multiple of another chosen number.

5. **LQ** Show that every positive integer has a multiple which consists of 0 and 7 only. For example, 70 is a multiple of 2, 777 is a multiple of 3.

6. **LQ** Suppose there are 10 dots in a square with side length 1. Show that there are two dots whose distance is less than 0.5.

Counting by Mapping

1. **SQ** You are standing at 0. Each step you move one step to right $(+1)$ or left (-1). Count the number of paths with following constraints.
   (a) Total number of steps is 6.
   (b) Total number of steps is 8. After the 8 steps you go back to 0.
   (c) Total number of steps is 8. After the 8 steps you stand at -2.
   (d) Total number of steps is 8. After the 8 steps you stand at 1.
   (e) Total number of steps is 7. After the 7 steps you stand at 3.

2. **SQ** You are standing at $(0,0)$. Each step you move one step right $((+1,0))$ or one step up $((0,+1))$. Count the number of paths with following constraints.
   (a) Stand at $(3,5)$ after 8 steps.
   (b) Stand at $(3,5)$ or $(4,4)$ after 8 steps.
   (c) Stand at $(6,6)$ after 12 steps. Also you cannot pass through $(3,3)$.
   (d) Stand at $(10,11)$ after 21 steps. Also you cannot pass through $(3,5)$ or $(4,4)$.
   (e) Stand at $(10,11)$ after 21 steps. Also you cannot pass through $(3,5)$ and $(4,9)$ simultaneously.
   (f) Stand at $(10,11)$ after 21 steps. Also you cannot pass through $(3,5)$ or $(4,9)$.

3. **SQ** Count the number of integer solutions with following constraints.
   (a) $x_1 + x_2 + x_3 + x_4 = 20$, $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$.
   (b) $x_1 + x_2 + x_3 + x_4 = 20$, $5 \geq x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$.
   (c) $x_1 + x_2 + x_3 = 10$, $x_1 \geq -1$, $x_2 \geq -2$, $x_3 \geq 1$.
   (d) $x_1 + x_2 + x_3 \leq 10$, $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$.
   (e) $x_1 + x_2 \geq 10$, $x_1 \leq 8$, $x_2 \leq 9$.
   (f) $x_1 + x_2 = 10$, $0 \leq x_1 \leq 6$, $0 \leq x_2 \leq 7$.
   (g) $x_1 + x_2 + x_3 = 10$, $0 \leq x_1 \leq 6$, $0 \leq x_2 \leq 7$, $0 \leq x_3 \leq 3$.

4. **SQ** What is the number of 'Hello's printed by the pseudocode below? (`for i from lo to hi` exhaust `i` between `lo` and `hi` inclusive, and is a empty loop when `lo` is greater than `hi`)
   (a)
   ```
   for i from 1 to n
     for j from 1 to i - 1
       for k from 1 to j - 1
         print 'Hello'
   ```

(b)
```
for i from 1 to 10
  for j from i to 10
    for k from i to j
      print 'Hello'
```
(c)
```
for i from 1 to 10
  for j from i + 1 to 10
    for k from i + 2 to j - 2
      print 'Hello'
```
(d)
```
for i from 1 to 10
  for j from i to 10
    for k from j to 10
      for l from i to j
        print 'Hello'
```

5. **SQ** 9 red balloons and 6 blue balloons are distributed to 4 children, how many distributions are possible under the following situations?
   (a) No constraints?
   (b) Every child must receive a balloon of red color?
   (c) Every child must receive a balloon of each color?
6. **SQ** Count the number of n-bit strings of length $n \geq 4$ that contain exactly two occurrences of 10.

More on Counting

1. **SQ** Consider the word `MILLIMICRON` in this question.
   (a) How many distinguishable ways can the letters of the word be arranged in order?
   (b) How many distinguishable orderings of the letters of the word begin with `M` and end with `N`?
   (c) How many distinguishable orderings of the letters of `MILLIMICRON` contain the letters `CR` next to each other in order and also the letters `ON` next to each other in order?
2. **SQ** Consider the expansion of $(1 + x - y)^{37}$
   (a) What is the coefficient of $x^{15}y^{13}$?
   (b) What is the coefficient of $x^{28}y^4$?
   (c) What is the coefficient of x^4y^{24}?
   (d) What is the coefficient of $x^{20}y^{19}$?
3. **SQ** Consider the expansion of $(x + y + z)^{33}$
   (a) What is the coefficient of $x^{11}y^{11}z^{11}$?
   (b) What is the coefficient of $x^7y^8z^{18}$?
   (c) What is the coefficient of $x^{13}y^{15}z^5$?

4. **SQ** n boys and m girls line up in a row. Count the number of ways for lining up with following constraints. Different orders among boys/girls are considered different ways.
   (a) $n = 10, m = 15$.
   (b) $n = 5, m = 20$. There are no adjacent boys.
   (c) $n = 5, m = 20$. There are at least 2 girls between different boys.
5. **SQ** Count the number of functions $f : A \to B$ with the following constraints:
   (a) $A = \{1, 2, ..., 10\}, B = \{1, 2, 3\}$.
   (b) $A = \{1, 2, ..., 10\}, B = \{1, 2\}$, there are exactly 4 elements in A.
   (c) $A = \{1, 2, ..., 10\}, B = \{1, 2, 3\}$, exactly 5 elements in A are mapped to 1, and exactly 4 elements in A are mapped to 2.
6. **SQ (Optional)** Count the number of ways to partition $\{1, 2, ..., 100\}$ into 5 (possibly empty) disjoint sets with the following constraints:
   (a) The sizes of the sets are $0, 10, 20, 30, 40$ respectively.
   (b) The sizes of the sets are $10, 10, 10, 30, 40$ respectively.
   (c) The sizes of the sets are all 20.
7. **LQ (Optional)** Find a bijective mapping between the following two sets:
   (a) The set of length n strings with characters x or y, such that in any prefix of the string, the number of x's is no less than the number of y's. For example, when $n = 4$, the set is $\{xxxx, xxxy, xxyx, xxyy, xyxx, xyxy\}$.
   (b) The set of length n monotone paths starting from $(0, 0)$. Recall a monotone path only moves right $(+1, 0)$ or up $(0, +1)$, and it does not exceed the diagonal line $x = y$. An example of a length 4 monotone path is $(0, 0) \to (1, 0) \to (1, 1) \to (2, 1) \to (3, 1)$.
8. **SQ (Optional)** A circular necklace consists of 5 beads, each of the beads is either red, blue, or green. What is the number of different necklaces if rotations of a necklace is considered the same? For example, the necklace `RBBGR` is the same as the necklaces `BBGRR` and `BGRRB`, but not the same as `RGBBR`. How about with the following changes?
   (a) There are 7 beads instead of 5.
   (b) There are 7 beads instead of 5, and 4 colors instead of 3.

Number Sequences

1. **SQ** Find a general form of the following sequences:
   (a) $a_1 = -2, a_2 = 1, a_3 = 4, a_4 = 7, a_5 = 10, \dots$
   (b) $a_1 = \frac{1}{2}, a_2 = \frac{3}{4}, a_3 = \frac{5}{8}, a_4 = \frac{7}{16}, a_5 = \frac{9}{32}, \dots$
   (c) $a_1 = -\frac{3}{10}, a_2 = \frac{9}{13}, a_3 = -\frac{27}{16}, a_4 = \frac{81}{19}, a_5 = -\frac{243}{22}, \dots$

(d) $a_1 = 1, a_{i+1} = a_i \times (i+1)$ for every $i \geq 1$.

2. **SQ** Evaluate the following:

   (a) $\displaystyle\sum_{i=0}^{n} \frac{1}{i^2 + 3i + 2}$.

   (b) $\displaystyle\sum_{i=0}^{n} (i-1)^3$.

   (c) $\displaystyle\prod_{i=0}^{n} \frac{2i+3}{2i+1}$.

   (d) $\displaystyle\sum_{j=0}^{2n} \prod_{i=0}^{j} (-\frac{1}{2})^i$.

3. **SQ** Show that $\displaystyle\sum_{i=1}^{n} \frac{1}{i^2} < 2$ for any $n \geq 2$.
   (Hint: You may use the fact that $i^2 > i(i-1)$ for any $i \geq 2$, then use the idea of telescoping sum.)

4. **SQ** Show that $\displaystyle\sum_{i=1}^{n} \frac{1}{i!} < 2$ for any $n \geq 2$.
   (Hint: Try to upper-bound this sum by a geometric series.)

5. **SQ** You have borrowed \$8000 from the bank. Suppose you want to repay a fixed amount of money for each of the following n years (except possibly the last year), and the annual interest rate r does not change in these n years. For example, if $r = 10\%$ and you repay \$4000 each year, then you will own the bank $\$(8000+800-4000) = \$4800$ next year, $\$(4800+480-4000) = \$1280$ two years after, and at the end of the third year you only need to repay $\$(1280 + 128) = \$1408$.

   (a) If $r = 10\%$ and you want to repay all the money in 10 years, how much do you need to pay each year?

   (b) If $r = 20\%$ and you want to repay all the money in 10 years, how much do you need to pay each year?

   (c) If $r = 20\%$ and you want to repay \$2500 each year (except possibly the last year). How many years do you need to repay all the money?

   (d) If $r = 24\%$ and you want to repay \$2000 each year (except possibly the last year). How many years do you need to repay all the money

6. **SQ** You have won a price! Now you have 2 options. The first is to get $\$1,000,000$ in each of the following 100 days. The second is to get \$1 in the first day, and 20% more than the previous day in each of the following 99 days, that means you will get \$1.2 next day, and \$1.44 two days after, and so on. Which one would you prefer?

Recursion

1. **SQ** What are a_1, a_2, a_3?
   (a) $a_0 = 4, a_k = 3a_{k-1} + 2k$, for all integers $k > 1$.
   (b) $a_0 = 3, a_k = (k+2)^{a_{k-1}-2}$, for all integers $k > 1$.
   (c) $a_0 = 0, b_0 = 1, a_k = a_{k-1} + b_k, b_k = a_{k-1} - b_{k-1}$, for all integers $k > 1$.
   (d) Let c be a constant. $a_0 = c, a_1 = 3, a_k = ca_{k-1}a_{k-2}$, for all integers $k > 1$.

2. **SQ** We have a single pair of rabbits (male and female) initially. Assume that:
   - The rabbit pairs are not fertile during their first two months of life, but thereafter give birth to three new male/female pairs at the end of every month;
   - The rabbits will never die.

   Let r_n be the number of pairs of rabbits alive at the end of month n. Find a recurrence relation for $r_0, r_1, r_2 \ldots$

3. **SQ** We have a single pair of rabbits (male and female) initially. Assume that:
   - The rabbit pairs are not fertile during their first three months of life, but thereafter give birth to four new male/female pairs at the end of every month;
   - The rabbits will never die.

   Let r_n be the number of pairs of rabbits alive at the end of month n. Find a recurrence relation for $r_0, r_1, r_2 \ldots$

4. **SQ** A palindrome is a word that can be read the same way in either direction, for example "abbcbba", "bcddcb", "a", etc. Let P_n be the number of palindromes made of "a" to "z".
   (a) What are P_1, P_2, P_3, P_4?
   (b) Give a recurrence equation for P_n.
   (c) Find the closed form for P_n.
      i. When n is even.
      ii. When n is odd.

5. **SQ** A straight line split a plane into 2 regions, let R_n be the maximum number of region split by n straight lines.
   (a) What are R_1, R_2, R_3, R_4?
   (b) Give a recurrence equation for R_n.
   (c) Find the closed form for R_n.

6. **SQ** Suppose that the Tower of Hanoi problem has four poles in a row instead of three. Disks can be transferred one by one from one pole to any other pole, but at no time may a larger disk be placed on top of a smaller disk. Let s_n be the minimun number of moves needed to transfer the entire tower of n disks from the left-most to the right-most pole.

(a) Find s_1, s_2, s_3, s_4.

(b) Show that $s_k \leq 2s_{k-2} + 3$ for all integers $k \geq 3$.

7. **SQ** Consider the set of all strings of 0's, 1's, and 2's.

   (a) For each integer $n \geq 0$, let s_n be the number of strings of 0's, 1's, and 2's of length n that do not contain the pattern 00. Find s_0, s_1, s_2, s_3.

   (b) Find a recurrence relation for $s_0, s_1, s_2, \ldots$

   (c) Use the results from previous parts to find the number of strings of 0's, 1's, and 2's of length four that do not contain the pattern 00.

8. **SQ** There are n identical balls and m different bins. Let $B_{n,m}$ be the total number of way to put n balls in m bins.

   (a) What are $B_{1,1}$, $B_{2,1}$, $B_{1,2}$, $B_{2,2}$, $B_{3,2}$?

   (b) Express $B_{n,m}$ in terms of $B_{k,m-1}$, for $k = 0, \ldots, n$.

9. **SQ (Optional)** A permuation w of the set $\{1, 2, \ldots, n\}$ is called *stack-sortable if* $S(w) = (1, 2, \ldots, n)$ (i.e. the identity permutation), where $S(w)$ is defined recursively as follows: write $w = unv$ where n is the largest element in w and u, v are shorter sequences, and set $S(w) = S(u)S(v)n$, with S being the identity for one-element sequences. Show that the number of stack-sortable permutations of $\{1, 2, \ldots, n\}$ is exactly the Catalan number C_n.

10. **SQ** Show by iteration that the closed form of recurrence $C_0 = 1$, $C_n = \frac{4n-2}{n+1} \cdot C_{n-1}$ is the Catalan number.

11. **SQ** Let P_n be the number of partitions of a set with n elements. Show that:

$$P_n = \binom{n-1}{0} P_{n-1} + \binom{n-1}{1} P_{n-2} + \cdots + \binom{n-1}{n-1} P_0$$

for any integers $n \geq 1$.