

Problem 1.

Is the graph depicted in Fig 2 Eulerian? Either give an Eulerian cycle or justify why it doesn't exist.

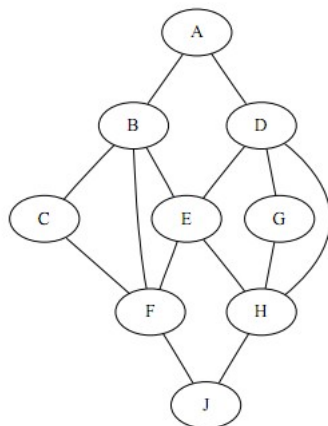


Figure 1: Can you draw this in one stroke?

Solution: Yes. B C F J H G D A B F E D H E B.

Problem 2.

A **leaf** is a vertex with exactly one neighbor. Every tree G with more than one vertex has at least two leaves.

Proof. Let G be an arbitrary connected acyclic graph with more than one vertex.

Because G is connected and has more than one vertex, every vertex has degree at least 1.

Let v_0, v_1, \dots, v_n be a maximal path in G , that is, a path that cannot be made longer by adding a vertex to either end. Because the path is maximal, it must visit every neighbor of v_n . If v_n is adjacent to v_i for any $i < n - 1$, then $v_i, v_{i+1}, \dots, v_n, v_i$ is a cycle in G . Because G is acyclic, no such cycle exists.

Thus, v_n is adjacent to v_{n-1} and nothing else; in other words, v_n is a leaf. By a similar argument, v_0 is a leaf. □

Other facts about trees that you should prove

- Every connected graph has at least $|V| - 1$ edges.
- Every acyclic graph has at most $|V| - 1$ edges.
- Any connected graph with (at most) $|V| - 1$ edges is a tree.
- Any acyclic graph with (at least) $|V| - 1$ edges is a tree.

- Any minimally connected graph is a tree (We've proved it in later exercise).
- Any maximally acyclic graph is a tree. (Maximally acyclic means adding any edge creates a cycle.)
- A graph is a tree if and only if there is a unique path from any vertex to any other vertex.
- Every tree containing a vertex of degree Δ has at least Δ leaves.
- In any tree, a strict majority of the vertices have degree at most 2.

Problem 3 (Tree Characterization).

Let T be a connected simple graph of order n . Then T is a tree iff the size of T is $n - 1$.

Proof. By definition, the order of a tree is how many nodes it has, and its size is how many edges it has.

Necessary Condition

Suppose T is a tree with n nodes. We need to show that T has $n - 1$ edges.

Proof by induction:

Let T_n be a tree with n nodes. For all $n \in \mathcal{N}^*$, let $P(n)$ be the proposition that a tree with n nodes has $n - 1$ edges.

Basis for the Induction:

$P(1)$ says that a tree with 1 vertex has no edges. It is clear that T_1 is N_1 , the edgeless graph, which has 1 node and no edges. So $P(1)$ is (trivially) true. This is our basis for the induction.

Induction Hypothesis:

Now we need to show that, if $P(k)$ is true, where $k \geq 1$, then it logically follows that $P(k + 1)$ is true. So this is our induction hypothesis: Any tree with k nodes has $k - 1$ edges. Then we need to show: Any tree with $k + 1$ nodes has k edges.

Induction Step:

Let T_{k+1} be any tree with $k + 1$ nodes. Take any node v of T_{k+1} of degree 1. Such a node exists from Tree has Degree One Nodes.

Let us consider T_k , the subgraph of T_{k+1} created by removing v and the edge connecting it to the rest of the graph. By Subgraph of Tree, T_k is itself a tree. The order of T_k is k , and it has one less edge than T_{k+1} by definition. By the induction hypothesis, T_k has $k - 1$ edges. So T_{k+1} must have k edges. So $P(k) \implies P(k + 1)$ and the result follows by the Principle of Mathematical Induction.

Alternative Induction Step

Let T_{k+1} be any tree with $k + 1$ nodes. Remove any edge e of T . As T_{k+1} has no circuits, e must be a bridge, from Condition for an Edge to be a Bridge. So removing e disconnects T_{k+1} into two trees T_1 and T_2 , with k_1 and k_2 nodes, where $k_1 + k_2 = k + 1$. By the induction hypothesis, T_1 and T_2 have $k_1 - 1$ and $k_2 - 1$ edges. Putting the edge e back again, we see that T_{k+1} has

$(k_1 - 1) + (k_2 - 1) + 1 = k$ edges. So $P(k) \implies P(k + 1)$ and the result follows by the Principle of Strong Induction. Therefore a tree with n nodes has $n - 1$ edges.

Sufficient Condition

Suppose T is a connected simple graph of order n with $n - 1$ edges. We need to show that T is a tree.

Suppose that T is not a tree. Then it contains a circuit. It follows from Condition for an Edge to be a Bridge that there is at least one edge in T which is not a bridge. So we can remove this edge and obtain a graph T' which is connected and has n nodes and $n - 2$ edges.

Let us try and construct a connected graph with n nodes and $n - 2$ edges. We start with the edgeless graph N_n , and add edges till the graph is connected. We pick any two vertices of N_n , label them u_1 and u_2 for convenience, and use one edge to connect them, labelling that edge e_1 . We pick any other vertex, label it u_3 , and use one edge to connect it to either u_1 or u_2 , labelling that edge e_2 . We pick any other vertex, label it u_4 , and use one edge to connect it to either u_1, u_2 or u_3 , labelling that edge e_3 . We continue in this way, until we pick a vertex, label it u_{n-1} , and use one edge to connect it to either u_1, u_2, \dots, u_{n-2} , labelling that edge e_{n-2} . That's the last of our edges, and we still haven't connected the last vertex. Therefore a graph with n vertices and $n - 2$ edges that such a graph can "not" be connected.

Therefore we can not remove any edge from T without leaving it disconnected. Therefore all the edges in T are bridges. Hence T can contain no circuits and so must be a tree. \square

Problem 4 (k stroke graphs).

For a connected graph, if there are $2k$ odd degree vertices, then the graph can be drawn in k strokes.

Proof. If $k > 1$, connects two odd degree vertices u and v , to make a new problem:

For a connected graph, if there are $2(k - 1)$ odd degree vertices, then the graph can be drawn in $k - 1$ strokes.

The new problem implies the old one because passing through the newly edge (u, v) can be regarded as making a new stroke starting at v (while the previous one ending at u). Hence our problem can reduce to

For a connected graph, if there are 2 odd degree vertices, then the graph can be drawn in 1 strokes.

This is the Euler Path setting, the problem is certainly solvable, so does our original one. \square

Problem 5 (Tree).

Prove a graph is a tree if and only if it is "minimally" connected, i.e. removing any edge would disconnect the graph.

Proof. Tree \rightarrow 'minimally' connected:

By definition, a graph is a tree if and only if there is a unique path between any pair of vertices. If we remove (u, v) , then u and v will be disconnected because (u, v) is the unique path. Thus a tree is ‘minimally’ connected.

‘minimally’ connected \rightarrow tree:

To prove that minimally connected implies a tree, we can prove that for a connected graph having a cycle is not minimally connected (since delete arbitrary one edge on cycle the graph remains connected). The **contrapositive** is that for a connected graph, minimally connected implies no cycles. Then we are done because one definition for tree is that: tree is a connected graph with no cycle.

An alternative proof that minimally connected implies a tree, pick any edge e and delete it. Each connected component is minimally connected otherwise, reduce edges to make every component minimally connected, and then add e back to make the graph connected again, result into a smaller but connected graph, contradiction. Then by induction each connected component is a tree(easy check), and thus the original graph is a tree itself.

Thus finish our proof. □

Problem 6.

Mr. and Mrs. Smith held a party at home and n couples came. A few handshakes took place. Mr Smith observed that:

- No couples shook hands;
 - Nor did anyone shake hands with himself or herself;
 - Nobody shook hands with the same person more than once;
 - Number of handshakes of others (the n couples and Mrs Smith) are distinct.
1. For $n = 3$ and $n = 4$, calculate the number of handshakes Mrs. Smith and Mr. Smith had.
 2. What is the number of handshakes Mr. and Mrs. Smith had for the general case? Prove your answers.

Proof. We prove the general case by induction on n .

Base case: when $n = 1$, there are two guests and it’s easy to verify the only solution is one guest having no handshakes and the other guest shook hands with both the Smiths. And the Smiths both had $n = 1$ handshake(s). Suppose when $n - 1$ couples come and Mr. Smith finds everybody else has different number of handshakes, then the Smiths both had $n - 1$ handshakes, for some $n > 1$.

For notational convenience we denote by P_i the guest (or Mrs. Smith) with i handshakes. P_{2n} shook hands with everybody except his spouse, hence P_0 has to be his spouse. Now we show this couple cannot be the Smiths: We observe that Mr. Smith cannot have 0 handshakes, or P_{2n} can at most have $2n - 1$ handshakes. Neither can he have $2n$ handshakes, or P_1 will have at least 2 handshakes. Now if we take this couple out of the picture, everybody else’s handshake count drops by one. The number of handshakes among the Smiths and the $n - 1$ couples are

$0, 1, 2, \dots, 2n - 2$ and that of Mr. Smith's. By induction hypothesis, the Smiths both had $n - 1$ handshakes with those $n - 1$ couples. Adding back the 1 handshake they each had with the P_{2n} 's, the Smiths both had n handshakes. Therefore for any $n \geq 1$, the Smiths both have n handshakes. \square

Problem 7.

Determine if each pair of graphs that follows are isomorphic.

1. See Fig 2

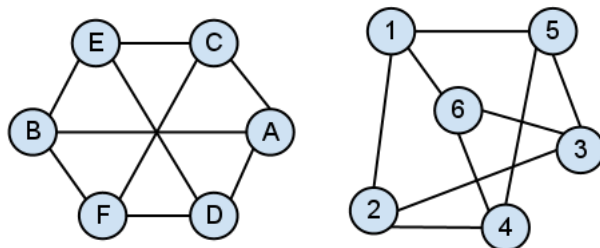


Figure 2: isomorphic graphs determine

2. See Fig 3

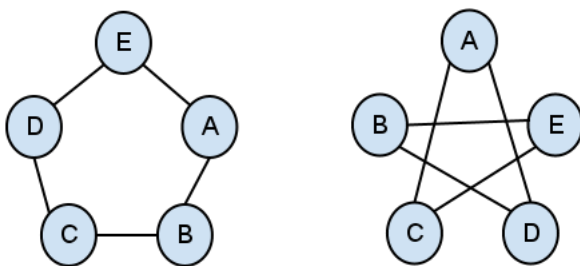


Figure 3: isomorphic graphs determine

Solution:

1. YES. The mapping is follow: $A(2), B(3), C(4), D(1), E(5), F(6)$
2. YES. The mapping is follow: $A(A), B(C), C(E), D(B), E(D)$

Problem 8.

Are the following sequences valid degree sequences of simple graphs? For each sequence, do one of the following:

- Construct a graph with the given degree sequence, or
- Show that the sequence cannot be a degree sequence of any simple graph

1. $(3,3,3,1)$
2. $(4,4,4,2,2)$
3. $(4,3,2,2,1)$
4. $(3,3,3,3,3,3)$

Solution:

1. $(3,3,3,1)$

This sequence is not graphical. Each of the three vertices of degree 3 can at most have two edges from other degree-3 vertices, but then the degree 1 vertex can offer only one more edge.

2. $(4,4,4,2,2)$

Similarly, this sequence is not graphical.

3. $(4,3,2,2,1)$ This sequence corresponds uniquely to a simple graph as shown in Figure 4

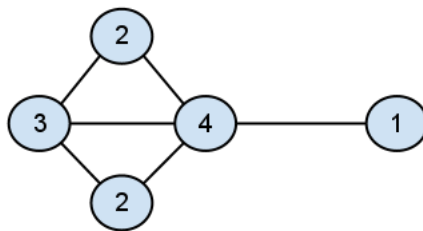


Figure 4: Graph with degree sequence $(4,3,2,2,1)$

4. $(3,3,3,3,3,3)$

This sequence is graphical. The graph in Figure 2 has such a degree sequence.