A complete design matrix or arbitrary matrix

B partial design matrix or arbitrary matrix

M arbitrary matrix

Def. MPI (A arbitrary matrix):

$$
\begin{aligned}
AA^+A &= A && (1)\\
A^+AA^+ &= A^+ && (2)\\
(A^+A)^t &= A^tA && (3)\\
(AA^t) &= AA^t && (4)\\
& && (5)
\end{aligned}
$$

gilt:

$$
\begin{aligned}
(A^t)^+ &= (A^+)^t && (6)\\
(A^+)^+ &= A && (7)\\
(AA^t)^+ &= (A^t)^+A^+ && (8)
\end{aligned}
$$

The last equality can be found in [1, p6,Prop. 3.2].
if appearing inverses exist:

$$
\begin{aligned}
A^+ &= A^t(AA^t)^{-1}\\
A^+ &= (A^tA)^{-1}A^t
\end{aligned}
$$

B partial design matrix

$$
\begin{aligned}
BB^+ &= I_m\\
B^+B &\neq I_n
\end{aligned}
$$

B partial design matrix $D = \mathrm{diag}(v)$, $E = \mathrm{diag}(\sqrt{v})$, X, Y any matrices:

$$
\begin{aligned}
(BDB')^+ &= (B(EE)B')^+ && // \; D = EE\\
&= (B(EE')B')^+ && // \; E = E'\\
&= ((BE)(E'B'))^+ && // \text{ associativity}\\
&= ((BE)(BE)')^+ && // \; (XY)' = Y'X', \checkmark\\
&= ((BE)')^+(BE)^+ && // \text{ Eqn. 8, where } A := (BE)\\
&= ((BE)^+)'(BE)^+ && // \text{ Eqn. 6}
\end{aligned}
$$

References

[1] J. Barata and M. Hussein. The moore-penrose pseudoinverse: A tutorial review of the theory. *Braz. J. Phys.*, 42:146–165, 2012.