II. One-period Binomial Model

Consider a market with two assets, (B, S), where B stands for a risk-free bank account (or a bond) and S represents a risky asset, e.g., stock. There is a one-period investment horizon with time indices $t = 0, 1$. The deterministic interest rate applied for this period is r, namely, if we deposit \$1 at time $t = 0$ we will receive $\$(1 + r)$ at $t = 1$. We assume that the borrowing and lending rates are the same and shorting on S is allowed without cost.

Let us denote the time-t value of two assets by B_t and S_t, respectively. Suppose the initial prices B_0 (usually set to be \$1) and S_0 are known. The state of market at $t = 1$ is random and supposed to be binomial: "up" with some probability $p \in (0, 1)$ or "down" with probability $1 - p$. The realization of the market determines the value of risky asset at $t = 1$. Formally, the sample space of this one-period binomial model (OPBM) is given by $\Omega = \{\omega_U, \omega_D\}$, and we assume $S_1(\omega_U) = uS_0$ and $S_1(\omega_D) = dS_0$ with multipliers[8] $u > d > 0$. The descriptions of this market are summarized in two ways in Table 3 and Figure 1, respectively.

Table 3: The market of one-period binomial model

$\omega \in \Omega$	ω_U	ω_D
$\mathbb{P}(\omega)$	p	$1 - p$
$S_1(\omega)$	uS_0	dS_0
B_1	$B_0(1 + r)$	$B_0(1 + r)$

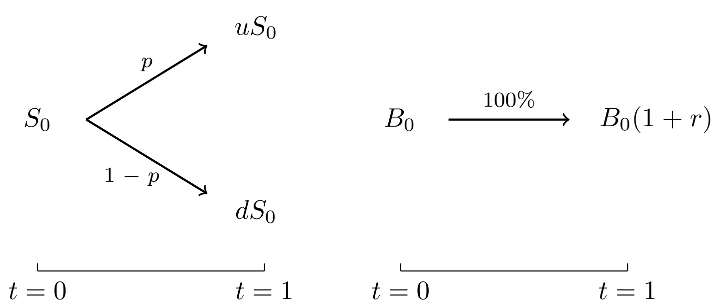

Figure 1: The tree form of assets' dynamics in OPBM

◊ 真实世界的运行显然不会如OPBM这般简单. 但经典的OPBM依然能够为我们提供很多insights.

The no arbitrage is the most fundamental requirement in asset pricing in financial mathematics. In the following, we discuss three equivalent conditions that leads to an arbitrage-free market (B, S) of OPBM.

[8] Also known as total return.

5 No arbitrage: $u > (1+r) > d$

We claim that the free of arbitrage in the market (B, S) requires $u > (1+r) > d > 0$. To see this, let us try to find any arbitrage opportunity if the above condition is violated.

Suppose, for instance, that $u > d > (1+r)$. It indicates that S dominates B: No matter what happens one period later, the total return of risky asset always outperforms that of riskless asset. This motivates us to construct a zero-cost strategy at $t = 0$ where we buy risky asset through borrowing money from the bank. The actions are summarized in Table 4. According to the Definition 3.1, we indeed find an arbitrage when $u > d > (1+r)$, and the arbitrage portfolio is formed by longing one share of S and shorting $\$S_0$ amount of money on B. Similar arguments can be also applied to another situation when $(1+r) > u > d$, but we omit the details here. Therefore, we have demonstrated our claim that the no-arbitrage condition for the OPBM is $u > (1+r) > d > 0$.

Table 4: Arbitrage opportunity when $u > d > (1+r)$

Actions	t=0	t=1	
		ω_H	ω_D
A1	buy one share of stock at price S_0	sell stock at price uS_0	sell stock at price dS_0
A2	borrow $\$S_0$ amount of money from bank at rate r	pay back $S_0(1+r)$ to bank	
cash flow (CF)	0	$uS_0 - S_0(1+r) > 0$	$dS_0 - S_0(1+r) > 0$

6 No arbitrage: Existence of positive state price

Since there are two possible states in OPBM, we now consider two more assets, called Arrow-Debreu (AD) securities, which are not really traded in the market but seem more "fundamental" than B and S. More precisely, let us denote these two assets by AD^U and AD^D, respectively, and their payoffs are illustrated in Figure 2. We can see that they are designed to be exclusively associated with a certain state. Therefore, the price of AD^i at $t = 0$, denoted by AD_0^i, is also known as the state price for each state $i \in \{U, D\}$. Our goal is to find AD_0^i.

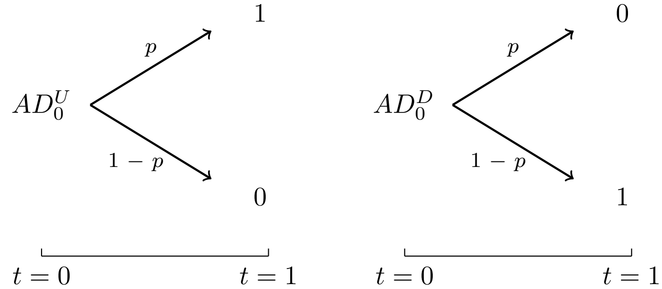

Figure 2: The tree form of AD securities in OPBM

No arbitrage principle guarantees that if two assets/portfolios have the same future cash flows (payoffs), they must have the same current prices (this is also known as law of one price), otherwise there exist arbitrage. Therefore, to price each AD security, we could utilize the existing assets B and S to replicate the payoff of AD^i in both circumstances. Suppose our replicating strategy is to hold Δ shares of S and B amount of deposit at $t = 0$, then we set at $t = 1$

$$\begin{cases} \Delta S_1(\omega_U) + B_1 = AD_1^U(\omega_U), \text{ i.e., } \Delta u S_0 + B(1+r) = 1 \\ \Delta S_1(\omega_D) + B_1 = AD_1^U(\omega_D), \text{ i.e., } \Delta d S_0 + B(1+r) = 0. \end{cases} \tag{14}$$

Solving the above system of equations for two unknowns Δ and B we obtain that

$$\Delta = \frac{1}{S_0(u-d)} > 0 \tag{15}$$

$$B = \frac{d}{(d-u)(1+r)} < 0, \tag{16}$$

where the negative sign of B means borrowing money from bank. Due to the law of one price we must have at $t = 0$

$$AD_0^U = \Delta S_0 + B = \frac{1}{1+r}\frac{(1+r)-d}{u-d}. \tag{17}$$

Applying the similar procedure of replicating portfolio to AD^D, we can obtain

$$AD_0^D = \frac{1}{1+r}\frac{u-(1+r)}{u-d}. \tag{18}$$

Finally, in order to be the real prices of securities with nonnegative payoff and at least one positive realization (e.g., $AD_1^U \geq 0$ and $AD_1^U(\omega_U) = 1 > 0$), AD_0^U and AD_0^D must satisfy

$$AD_0^U > 0 \text{ and } AD_0^D > 0. \tag{19}$$

In summary, no arbitrage of OPBM requires law of one price for replicating portfolios, therefore, when we consider special assets AD securities in OPBM, each for one state, we derive that no arbitrage requests the existence of positive state price.

7 No arbitrage: Existence of risk-neutral probabilities

If we take a further look on the prices of AD securities in the preceding section, we find that $AD_0^U + AD_0^D = 1/(1+r)$. It really makes sense because holding one share of AD^U and one share of AD^D at $t = 0$ locks one dollar for sure at $t = 0$, as shown in Figure 3. There is no doubt that the price of this risk-free portfolio must be the discount factor $1/(1+r)$.

The structures of state prices' formulas (17) and (18) motivate us to define a key quantity

$$q := \frac{(1+r)-d}{u-d}. \tag{20}$$

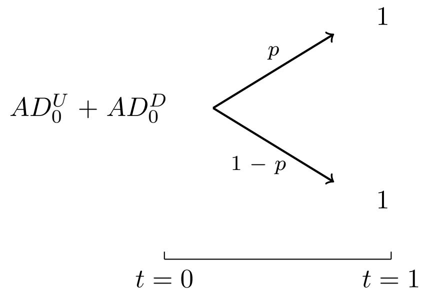

Figure 3: The portfolio $AD_0^U + AD_0^D$ locks \$1 for sure at $t = 1$.

By using q and the special payoffs of AD securities, we can expressed their prices as

$$AD_0^U = \frac{1}{1+r}(q \times 1 + (1-q) \times 0) \tag{21}$$

$$AD_0^D = \frac{1}{1+r}(q \times 0 + (1-q) \times 1), \tag{22}$$

From last section we know that no arbitrage requires the existence of positive state prices $AD_0^U > 0$ and $AD_0^D > 0$. By applying these requirements to (21) and (22), we derive a equivalent no-arbitrage condition: $0 < q < 1$. It is natural to treat $q \in (0,1)$ as a "probability" for the state "up" (and hence $1-q$ for "down"), and these probabilities are irrelevant to the physical probability p in the real world. We also find that under the new probabilities $q_U = q$ and $q_D = 1-q$, pricing AD^i becomes easy: we just use risk-free rate to discount future payoffs of risky asset! Therefore, we have a special name for q_U and q_D: risk-neutral probabilities. We can easily verify this pricing method on the risky asset S by checking

$$S_0 = \frac{1}{1+r}(q_U u S_0 + q_d d S_0). \tag{23}$$

In summary, the no-arbitrage condition for OPBM can be summarized as the existence of risk-neutral probabilities $\mathbb{Q} = \{q_U, q_D\}$, where

$$q_U = \frac{(1+r)-d}{u-d} > 0, \; q_D = 1 - q_U > 0. \tag{24}$$

And the risk-neutral pricing formula is just given by

$$A_0 = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[A_1] = \frac{1}{1+r}[q_U A_1(\omega_U) + q_D A_1(\omega_D)] \tag{25}$$

for any asset A.

Note that the condition $q_U > 0$ and $q_D > 0$ induces $u > (1+r) > d$ which is just the first no-arbitrage condition shown in the previous section.

8 Pricing option in OPBM

Consider, in the no-arbitrage market (B, S) of OPBM, a general European option C whose underlying asset is S. It means that, if you buy this option at $t = 0$, what you can receive at $t = 1$ depends on the performance of the asset S. More specifically, let us denote by C_t the time-t value of this option, and assume that

$$C_1 = f(S_1) \tag{26}$$

for any function f. In the following, we use two equivalent methods to compute the fair price of C at $t = 0$, that is, C_0.

We first replicate the payoff C_1 by using the existing assets S and B in the market. This is a similar process as adopted in pricing AD securities in Section 6. Specifically, let us take Δ shares of S and B amount of deposit at $t = 0$ in order to duplicate the payoff of C at $t = 1$ under any circumstance, namely,

$$\begin{cases} \Delta S_1(\omega_U) + B(1+r) = C_1(\omega_U) \\ \Delta S_1(\omega_D) + B(1+r) = C_1(\omega_D), \end{cases} \tag{27}$$

where $C_1(\omega) = f(S_1(\omega))$. Solving the above system of equations for two unknowns Δ and B we obtain that

$$\Delta = \frac{C_1(\omega_U) - C_1(\omega_D)}{S_1(\omega_U) - S_1(\omega_D)} = \frac{C_1(\omega_U) - C_1(\omega_D)}{S_0(u-d)} \tag{28}$$

$$B = \frac{1}{1+r}\frac{uC_1(\omega_D) - dC_1(\omega_U)}{u-d}. \tag{29}$$

According to the law of one price in the no-arbitrage market, we have

$$\begin{aligned} C_0 = \Delta S_0 + B &= \frac{C_1(\omega_U) - C_1(\omega_D)}{u-d} + \frac{1}{1+r}\frac{uC_1(\omega_D) - dC_1(\omega_U)}{u-d} \\ &= \frac{1}{1+r}\left[\frac{(1+r)-d}{u-d}C_1(\omega_U) + \frac{u-(1+r)}{u-d}C_1(\omega_D)\right]. \end{aligned} \tag{30}$$

We can easily observe from (30) that the risk-neutral pricing formula (25) could lead to the same result of C_0, i.e.,

$$\begin{aligned} C_0 &= \frac{1}{1+r}\left[q_U C_1(\omega_U) + q_D C_1(\omega_D)\right] \\ &= \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[C_1], \end{aligned} \tag{31}$$

where the risk-neutral probabilities are given in (24).

Note that the standard European call option is just a special case of (26), where its payoff function at $t = 1$ is given by

$$f(s) = (s-K)^+ = \max\{S_1 - K, 0\} = \begin{cases} s - K, & s > K \\ 0, & s \le K \end{cases} \tag{32}$$

for some preselected constant $K > 0$ called the strike price. We succeed in pricing the option in the no-arbitrage market (B, S) of OPBM by replicating portfolio as in (30) and adopting risk-neutral pricing formula directly as in (31).