FinMath Topic 1

Feasible Set and Efficient Frontier of Two Assets

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In this topic, we discuss how to use two assets to construct a portfolio that is efficient in the mean-variance sense.

I. Preliminaries

1 Statistics on returns

Consider two assets, A and B, whose returns are denoted by the random variables, R_A and R_B, respectively. Besides, the mean, variance and covariance of returns are given as

$$\mathbb{E}[R_A] = \mu_A,\ \mathrm{Var}(R_A) = \sigma_A^2; \tag{1}$$
$$\mathbb{E}[R_B] = \mu_B,\ \mathrm{Var}(R_B) = \sigma_B^2; \tag{2}$$
$$\mathrm{Cov}(R_A, R_B) = \rho\sigma_A\sigma_B,\ \rho \in [-1, 1]. \tag{3}$$

⚠ The above statistics are in terms of returns, not of the asset prices.

We use *variance* σ^2 (or *standard deviation* $\sigma \geq 0$) of return to measure the *risk* of an asset[1]. We say that the asset is *risky* if $\sigma > 0$ and that it is *risk-free* if $\sigma = 0$. Moreover, the larger the σ, the riskier the asset would be.

2 Dominance between two assets

We say the asset A *dominates* asset B (in term of the *mean-variance criterion*) if $\mu_A \geq \mu_B$ and $\sigma_A \leq \sigma_B$. And if it is the case, the mean-variance investor would fully invest on A instead of B.

◊ 这个很好理解: 如果A的期望收益比B大, 同时风险(用variance衡量)来得更小, 那么A当然比B“占优”.

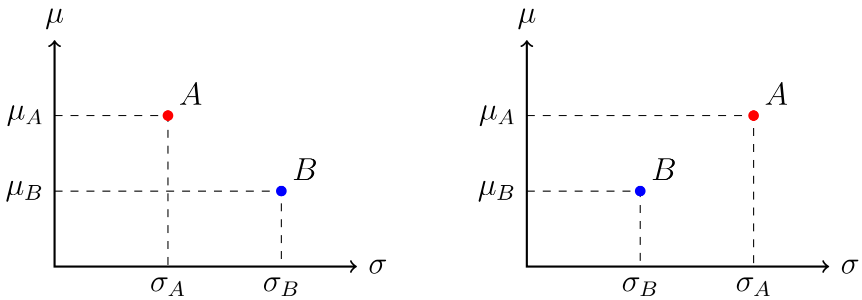
We can draw (σ_A, μ_A) and (σ_B, μ_B) on σ-μ graph in order to easily see their dominance relationship. Figure 1 illustrates two possible situations.

⚠ 我们习惯用σ-μ graph, 也即横轴是σ纵轴是μ, 而不是μ-σ graph.

From now on, in order to construct a portfolio with both A and B, we would only consider the case where A and B cannot dominate each other. And without loss of generality, we assume

$$\sigma_A > \sigma_B, \text{ and hence } \mu_A > \mu_B. \tag{4}$$

[1]In fact, there are other proxies of risk, for example, you may have heard about value-at-risk (VaR). It is however out of scope of current topic.



(a) A dominates B.

(b) A&B cannot dominate each other.

Figure 1: Dominance relationship between A and B in the mean-variance sense.

3 Portfolio and its statistics

A *portfolio* P is formed by allocating w_A proportion of money[2] on A and w_B on B, where

$$w_A + w_B = 1. \tag{5}$$

If no further restrictions are applied on the weights w_A and w_B, it means *shorting* is allowed. For instance, we could borrow B ($w_B < 0$) and sell it in order to buy more on A ($w_A = 1 - w_B > 1$). In contrast, if we further impose the condition $w_A, w_B \in [0, 1]$, it indicates that we are considering the *no-shorting* situation.

Let us denote by R_P the (random) return of the portfolio P, then we have

$$R_P = w_A R_A + w_B R_B. \tag{6}$$

◊ 简单推导: 假设我们有X这笔钱, 其中Xw_A投资在A上回报$Xw_A(1+R_A)$, B类似, 所以总回报$Xw_A(1+R_A)+Xw_B(1+R_B) = X(1+w_AR_A+w_BR_B) = X(1+R_P)$, 则对应投资组合收益率是$R_P = w_AR_A + w_BR_B$.

The expectation and variance of portfolio return can be easily computed:

$$\begin{aligned} \mu_P &= \mathbb{E}[R_P] = \mathbb{E}[w_A R_A + w_B R_B] = w_A\mu_A + w_B\mu_B \\ &= w_A\mu_A + (1 - w_A)\mu_B, \end{aligned} \tag{7}$$

$$\begin{aligned} \sigma_P^2 &= \mathrm{Var}(R_P) = \mathrm{Var}(w_A R_A + w_B R_B) = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2\rho w_A w_B \sigma_A \sigma_B \\ &= w_A^2\sigma_A^2 + (1 - w_A)^2\sigma_B^2 + 2\rho w_A(1 - w_A)\sigma_A\sigma_B \\ &= (\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B)w_A^2 + 2(\rho\sigma_A\sigma_B - \sigma_B^2)w_A + \sigma_B^2. \end{aligned} \tag{8}$$

Here we arrange their expressions as functions of w_A for later use.

[2] If the current price of A is S_A and the total amount of money is X, then it means we buy Xw_A/S_A shares of A; similar statement for B.

II. General Definitions

4 Feasible set

The *feasible set* (a.k.a. *attainable set*) of two assets A and B, with a general group of parameters $\mu_A, \sigma_A, \mu_B, \sigma_B, \rho$ given, is the collection of all portfolios that can be constructed from them (by varying w_A), namely,

$$\{(\mu_P, \sigma_P) : \forall w_A \text{ s.t. } \mu_P \text{ and } \sigma_P \text{ are given in (7) and (8), respectively}\}. \tag{9}$$

Theoretically, there are infinitely many portfolios in the feasible set, since there are infinitely many choices of w_A, no matter $w_A \in [0,1]$ for no-shorting situation or $w_A \in \mathbb{R}$ where shorting is allowed.

If we resolve w_A from (7) and plug the result into (8), we can get an algebraic relation between σ_P and μ_P, that is,

$$\begin{aligned}\sigma_P^2 = &\left(\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B\right)\left(\frac{\mu_P - \mu_B}{\mu_A - \mu_B}\right)^2 \\ &+ 2\left(\rho\sigma_A\sigma_B - \sigma_B^2\right)\frac{\mu_P - \mu_B}{\mu_A - \mu_B} + \sigma_B^2,\ \sigma_P \geq 0.\end{aligned} \tag{10}$$

It seems, however, not that straightforward to plot by hand the above σ_{P}-μ_{P} relation on the σ-μ graph to see its geometric shape. We will examine more on its shape later by assigning some special values on ρ. Figure 2 illustrates a typical curve of a feasible set, which is generated by Matlab.

5 Minimum-variance point

Meanwhile, we are also curious about how lowest the risk we could achieve among all possible portfolios that we could construct from A and B. To answer this question, we investigate directly the formula (8), the relationship between our choice w_A and the portfolio variance σ_P^2.

Due to the assumption in (4) and the correlation $\rho \in [-1, 1]$, we have

$$\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B \geq \sigma_A^2 + \sigma_B^2 - 2\sigma_A\sigma_B = (\sigma_A - \sigma_B)^2 > 0, \tag{11}$$

thus when we treat σ_P^2 as a function of w_A, it has an unique minimum value which achieves at

$$w_A^* = \frac{\sigma_B^2 - \rho\sigma_A\sigma_B}{\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B}. \tag{12}$$

◊ 公式(8)和(11)告诉我们，$\sigma_P^2(w_A)$是一个开口向上的抛物线，有唯一的最小值，而以上的w_A^*正是耳熟能详的"2a分之负b"，下文的${\sigma_P^*}^2$正是源自倒背如流的"4a分之4ac减b平方".

After taking w_A^* back into (8) and making some simplifications, we get the minimum value of risk

$$\sigma_P^{*\,2} = \sigma_B^2 - \frac{(\rho\sigma_A - \sigma_B)^2\sigma_B^2}{\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B}. \tag{13}$$

On the other hand, the expected portfolio return when the portfolio achieves the minimum variance is calculated through

$$\mu_P^* = w_A^*\mu_A + (1 - w_A^*)\mu_B. \tag{14}$$

We call the point (σ_P^*, μ_P^*) on σ-μ graph the *minimum variance point* (MVP) (of a certain feasible set parameterized by $\mu_A, \sigma_A, \mu_B, \sigma_B, \rho$). Graphically speaking, MVP by its very name is the left-most point of the corresponding feasible set.

⚠ 注意应与NBA的MVP区分开来.

A further glance on the expression of minimum variance. Notice that the fractional part of (13) is always larger than or equal to zero, leading to the property

$$\sigma_P^* \leq \sigma_B, \tag{15}$$

where the equality holds if and only if $\rho = \sigma_B/\sigma_A$. This means that through investing on the both assets instead of single, we could *always* form a portfolio[3] which reduces risk, in the sense that the volatility[4] of this portfolio does not exceed[5] the volatility of any of the original two assets. It reveals the beauty of *diversification*[6].

◊ 用数学告诉你听妈妈的话: 鸡蛋不要放在同一个篮子里.

6 Efficient frontier

Leveraged by MVP, the feasible set of (10) can be partitioned at this point into two parts that are usually named upper branch and lower branch. This separation is determined by how much you invest on the asset A compared to w_A^* of MVP. More precisely, the portfolios on the upper branch invest more on the asset A with $w_A > w_A^*$ and thus have the expected returns larger than μ_P^*. To see this, the formula (7) tells us that any portfolio return is an increasing function of w_A,

$$\mu_P(w_A) = (\mu_A - \mu_B)w_A + \mu_B, \tag{16}$$

since we already assume $\mu_A > \mu_B$ in (4). Therefore, $\mu_P(w_A) > \mu_P(w_A^*) = \mu_P^*$ when $w_A > w_A^*$. In contrast, for the portfolios on the lower branch, we have $w_A < w_A^*$ and hence $\mu_P < \mu_P^*$.

[3]For instance, just take the MVP; when σ_P^* is strictly less than σ_B, there will be infinitely many portfolios that satisfy the statement.

[4]We call the standard deviation of asset return the *volatility*. For example, the volatility of the portfolio P is just σ_P, which is the standard deviation of portfolio return R_P.

[5]In fact, it is always lower than, except when $\sigma_P^* = \sigma_B$ (equivalently $\rho = \sigma_B/\sigma_A$).

[6]Diversification helps reduce risk, then how about its influence on the expected return? More discussion will come.

However, no matter which branch the portfolio lies on, its volatility is always larger than σ_P^* of MVP. This results in a phenomenon that we could always find two portfolios, where one of them lies on the upper branch and another one lies on the lower branch with the same level of risk but achieves lower expected return. In other words, the upper branch dominates the lower branch. Therefore, the portfolios on the upper branch (including MVP) form an efficient set for the investor which is called the *efficient frontier* of these two assets. See Figure 2 for an illustrative example.

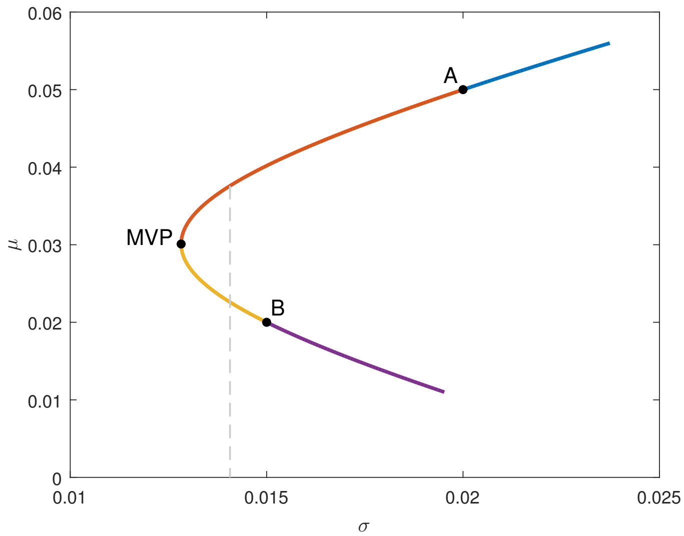


Figure 2: An example of feasible set between two assets A and B with $\mu_A = 0.05, \sigma_A = 0.02, \mu_B = 0.02, \sigma_B = 0.015, \rho = 0.15$. Under no-shorting condition, red+yellow=feasible set, red alone=efficient frontier; if shorting is allowed, all four colors form the feasible set (with blue and purple being infinitely lengthened), red+blue=efficient frontier. The dashed gray line demonstrates that for a certain level of risk, the upper branch dominates the lower branch.

III. From Special ρ to General

We have already defined the feasible set which is represented by the concrete relation between the portfolio's volatility $\sigma_P \geq 0$ and the expected return $\mu_P \in \mathbb{R}$ in (10), but how does it look like on the σ-μ graph? To answer this question, we start from examining some special values of the correlation ρ.

7 Two extreme values of correlation

We first investigate two extreme cases when two assets are fully negatively correlated ($\rho = -1$) and fully positively correlated ($\rho = 1$).

When $\rho = -1$. After we set $\rho = -1$, we could simplify (10) as

$$\begin{aligned}\sigma_P^2 &= (\sigma_A + \sigma_B)^2 \left(\frac{\mu_P - \mu_B}{\mu_A - \mu_B}\right)^2 - 2\,(\sigma_A + \sigma_B)\,\sigma_B \frac{\mu_P - \mu_B}{\mu_A - \mu_B} + \sigma_B^2 \\ &= \left((\sigma_A + \sigma_B)\frac{\mu_P - \mu_B}{\mu_A - \mu_B} - \sigma_B\right)^2, \end{aligned} \tag{17}$$

and we further get

$$\sigma_P = \left|(\sigma_A + \sigma_B)\frac{\mu_P - \mu_B}{\mu_A - \mu_B} - \sigma_B\right|, \tag{18}$$

⚠ 开根号之后记得带绝对值, 因为根据定义$\sigma_P \geq 0$. 下文脱掉绝对值符号记得带±.

or equivalently, if we write μ_P as a function of σ_P we obtain

$$\begin{aligned}\mu_P &= \frac{\mu_A - \mu_B}{\sigma_A + \sigma_B}(\pm\sigma_P + \sigma_B) + \mu_B \\ &= \begin{cases} \frac{\mu_A - \mu_B}{\sigma_A + \sigma_B}(\sigma_P + \sigma_B) + \mu_B \\ -\frac{\mu_A - \mu_B}{\sigma_A + \sigma_B}(\sigma_P - \sigma_B) + \mu_B \end{cases}, \ \sigma_P \geq 0. \end{aligned} \tag{19}$$

⚠ 不要忘记在(19)中保留 $\sigma_P \geq 0$ 这个条件.

The above result shows that in the case of $\rho = -1$, the feasible set on σ-μ graph could be reduced from (10) into a combination of two half-lines given in (19). We can easily draw them on the graph but notice that σ_P should be nonnegative. See Figure 3 blue parts as an illustration. Surprisingly, the lowest volatility in this case could be $\sigma_P^* = 0$ and meanwhile we have $\mu_P^* = \mu_B + \frac{\mu_A - \mu_B}{\sigma_A + \sigma_B}\sigma_B$, which is definitely the MVP of this feasible set, and this MVP can be realized by the weights

$$\begin{aligned} w_A^* &= \frac{\sigma_B}{\sigma_A + \sigma_B}, \\ w_B^* &= 1 - w_A^*. \end{aligned} \tag{20}$$

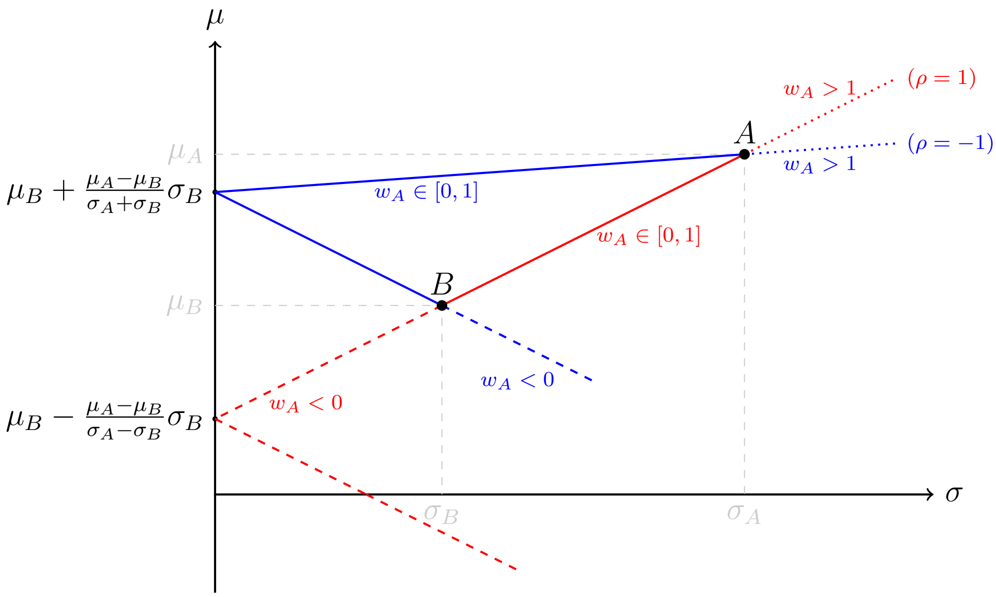


Figure 3: Feasible sets for $\rho = -1$ (blue lines) and $\rho = 1$ (red lines), with solid segments for $w_A \in [0, 1]$, dotted for $w_A > 1$ and dashed for $w_A < 0$.

Note that the above w_A^* can be computed through the general formula (12) under $\rho = -1$ or directly by solving from

$$w_A^*\mu_A + (1 - w_A^*)\mu_B = \mu_P^* = \mu_B + \frac{\mu_A - \mu_B}{\sigma_A + \sigma_B}\sigma_B. \tag{21}$$

Although we could plot the figure immediately by (19) with absence of w_A, never forget that the weight w_A is the common driving force behind μ_P and σ_P as shown in (7) and (8). In other words, any portfolio in the feasible set can be *uniquely* determined by w_A. Conversely, given a portfolio with μ_P and σ_P given, we can find an *unique* w_A[7] that constructs this portfolio. The above fact helps us cut the feasible set into several pieces according to the region that w_A lies in. More specifically, when $w_A \in [0, 1]$, (16) leads to

$$\mu_B = \mu_P(0) \leq \mu_P(w_A) \leq \mu_P(1) = \mu_A, \tag{22}$$

and this corresponds to the no-shorting situation. Similarly, $\mu_P > \mu_A$ when $w_A > 1$ and $\mu_P < \mu_B$ when $w_A < 0$, and these could only be achieved when shorting is allowed. We also point out the status of w_A for each segment of the feasible set in Figure 3.

When $\rho = 1$. We simplify (10) again and get

$$\sigma_P = \left| (\sigma_A - \sigma_B)\frac{\mu_P - \mu_B}{\mu_A - \mu_B} + \sigma_B \right|, \tag{23}$$

[7]Simply through resolving w_A from (7)

or equivalently,

$$\begin{aligned}\mu_P &= \frac{\mu_A-\mu_B}{\sigma_A-\sigma_B}(\pm\sigma_P-\sigma_B)+\mu_B \\ &= \begin{cases} \frac{\mu_A-\mu_B}{\sigma_A-\sigma_B}(\sigma_P-\sigma_B)+\mu_B \\ -\frac{\mu_A-\mu_B}{\sigma_A-\sigma_B}(\sigma_P+\sigma_B)+\mu_B \end{cases}, \ \sigma_P \geq 0. \end{aligned} \tag{24}$$

Similarly, the feasible set of this kind also consists of two half-lines, like a sunlight crossing through two points A and B and reflecting at the point $(0, \mu_B - \frac{\mu_A-\mu_B}{\sigma_A-\sigma_B}\sigma_B)$ on μ-axis, and this point is also the MVP of this case if shorting is allowed, and we have $w_A^* = \frac{\sigma_B}{\sigma_B-\sigma_A} < 0$. See Figure 3 red parts.

Interestingly, if shorting is not allowed, MVP will be confined to the asset B itself, since from the figure we know that at this moment the portfolio that fully invests on B ($w_A^*=0$) is the left-most point on the set, and it achieves the lowest risk (i.e., σ_B) among all possible portfolios that can be constructed from A and B with $\rho=1$. As a comparison, the MVP maintains at the point $(0, \mu_B + \frac{\mu_A-\mu_B}{\sigma_A+\sigma_B}\sigma_B)$ no matter whether shorting is allowed or not when $\rho=-1$.

There is another interesting finding in these two extreme cases of ρ: By selecting proper w_A we could build up a portfolio with *no* risk! For example, when $\rho=-1$, this is attained by forming the MVP, and even if shorting is not allowed, we are still able to do so. As a consequence, although the portfolio risk is reduced to zero ($\sigma_P^*=0$), we could still achieve a not-bad portfolio expected return $\mu_P^* = \mu_B + \frac{\mu_A-\mu_B}{\sigma_A+\sigma_B}\sigma_B$ that is larger than μ_B. It explains, to some extend, why bringing negatively correlated assets into a portfolio is attractive.

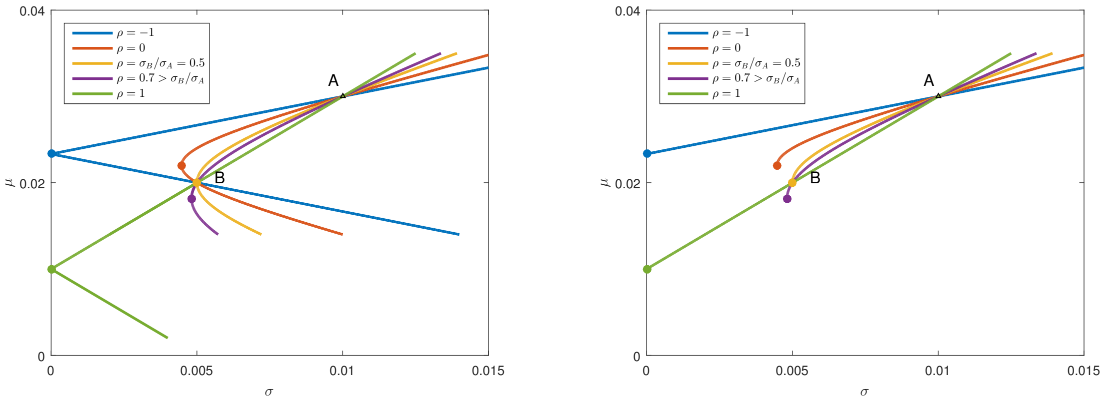
8 Feasible set under a general ρ

Except for the above two extreme values of ρ that make it reduced to straight lines, the feasible set under a general correlation should be a smooth curve as evidenced in (10). By invoking Matlab, we visualize the feasible sets and also efficient frontiers using an artificial example of two assets A and B, with fixed $\mu_A=0.03, \sigma_A=0.01, \mu_B=0.02, \sigma_B=0.005$ but different values of ρ for comparisons. We consider three more meaningful numbers of ρ, and they are $\rho=0$ when the two assets are uncorrelated, $\rho=\sigma_B/\sigma_A$ and a $\rho\in(\sigma_B/\sigma_A,1)$. We also distinguish between shorting-allowed and shorting-forbidden circumstances. See Figures 4 and 5, respectively.

The reason why we study $\rho=\sigma_B/\sigma_A$ is because remember that we have already introduced in (15) a very important property of MVP, that is, $\sigma_P^*(\rho)\leq\sigma_B$ and the equality holds exactly at this crucial value of ρ. This means that the MVP under $\rho=\sigma_B/\sigma_A$ is just the asset B itself hence we have $w_A^*=0$. Moreover, if we treat $w_A^*(\rho)$ in (12) as a function of ρ (given all other parameters fixed), and we take the first-order derivative and get

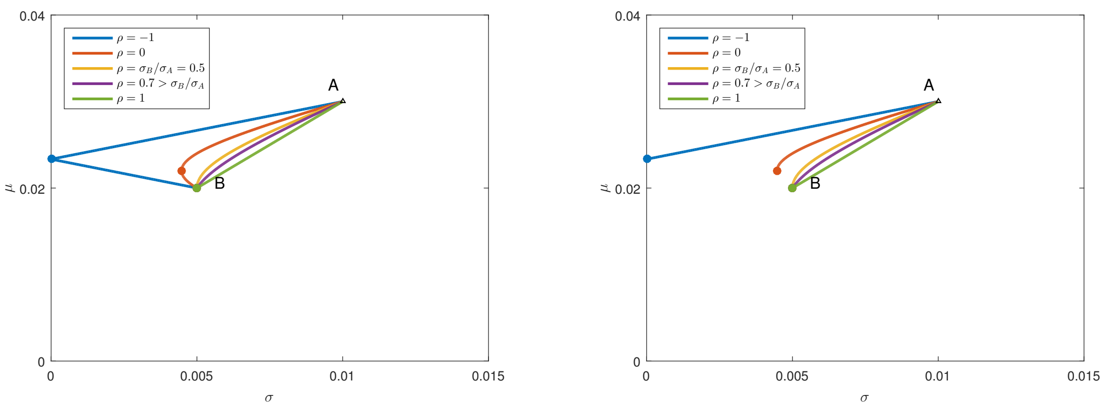
$$\frac{d}{d\rho}w_A^*(\rho) = \frac{(\sigma_B^2-\sigma_A^2)\sigma_A\sigma_B}{(\sigma_A^2+\sigma_B^2-2\rho\sigma_A\sigma_B)^2} < 0. \tag{25}$$

This indicates that w_A^* is decreasing in ρ. According to the above observations, we have the following conclusions: Once we fix $\mu_A, \sigma_A, \mu_B, \sigma_B$ but can vary ρ from -1 to 1, we can find



(a) Feasible set (b) Efficient frontier

Figure 4: Feasible set and efficient frontier of two assets when shorting is allowed. The solid dots on each curve represent the certain MVP.



(a) Feasible set (b) Efficient frontier

Figure 5: Feasible set and efficient frontier of two assets under no shorting. The solid dots on each curve represent the certain MVP.

- the larger the ρ, the smaller the w_A^* of MVP hence the smaller the μ_P^* of MVP[8], and this can be easily observed in Figure 4[9];
- when $\rho \in [-1, \sigma_B/\sigma_A]$, the *global* MVP[10] coincides with the *local* MVP. The volatility of MVP is increasing in ρ during this region and it reaches to σ_B when $\rho = \sigma_B/\sigma_A$, where at that point global MVP=local MVP=asset B; when $\rho \in (\sigma_B/\sigma_A, -1]$ the local MVP has to stay with the asset B while the global MVP starts to move again and this time its volatility becomes decreasing towards zero when $\rho = -1$.

◊ 我会额外补充一个有关feasible set和MVP随ρ变动的动态图来进一步解释上述两点结论.

[8] We have already discussed before that μ_P is increasing in w_A.
[9] There we can see that MVPs are dropping down in terms of μ-axis when ρ is increasing.
[10] It is the MVP when shorting is allowed. In contrast, the local MVP is the MVP when shorting is forbidden.

But pay attention that in real life, once we are given the two assets A and B, their correlation is always determined as well.

To end this section, we summarize in Table 1 the formula to calculate the weight of asset A in MVP under any possible correlation ρ between A and B. As we mentioned before, there is a slight difference on $w_A^*(\rho)$ under no-shorting constraint, compared to that of shorting-allowed situation, if ρ is larger than σ_B/σ_A.

Table 1: The weight on A in MVP for a general ρ

MVP	Shorting Permitted	Shorting Forbidden
$w_A^*(\rho)$	$\dfrac{\sigma_B^2-\rho\sigma_A\sigma_B}{\sigma_A^2+\sigma_B^2-2\rho\sigma_A\sigma_B}$ $\forall\rho\in[-1,1]$	$\max\left\{0,\dfrac{\sigma_B^2-\rho\sigma_A\sigma_B}{\sigma_A^2+\sigma_B^2-2\rho\sigma_A\sigma_B}\right\} = \begin{cases}\dfrac{\sigma_B^2-\rho\sigma_A\sigma_B}{\sigma_A^2+\sigma_B^2-2\rho\sigma_A\sigma_B},\ \rho\in\left[-1,\dfrac{\sigma_B}{\sigma_A}\right] \\ 0,\ \rho\in\left[\dfrac{\sigma_B}{\sigma_A},1\right]\end{cases}$
μ_P^*	$w_A^*\mu_A+(1-w_A^*)\mu_B$	
σ_P^*	$(\sigma_A^2+\sigma_B^2-2\rho\sigma_A\sigma_B){w_A^*}^2+2(\rho\sigma_A\sigma_B-\sigma_B^2)w_A^*+\sigma_B^2$	

IV. Inclusion of A Risk-free Asset

A special case of feasible set between two assets is that one of them is a risk-free asset whose volatility is zero. If it is the case, then what kind of portfolios could they form? Besides, if we want to add a risk-free asset into an existing portfolio that has already been constructed by the two risky assets, then what does the new feasible set look like? These are the questions that we are going to answer in this part.

9 Feasible set between a risky asset and a riskless asset

Consider a risky asset A with the expected return μ_A and volatility $\sigma_A > 0$ and a risk-free asset F with $\mu_F < \mu_A$ and $\sigma_F = 0$. Moreover, a well-known fact is that the correlation between risk-free return (deterministic variable) and risky return (random variable) equals to zero, that is, $\rho = 0$. The first step is to write out the expectation and variance of the portfolio return, and these can be easily obtained from (7) and (8) by setting $\sigma_B = 0$ and $\rho = 0$ there, namely,

$$\mu_P = w_A\mu_A + (1 - w_A)\mu_F, \tag{26}$$
$$\sigma_P^2 = \sigma_A^2 w_A^2. \tag{27}$$

◊ 是不是简洁到难以置信.

Similarly, we resolve w_A from (26) and plug the result into (27) and get

$$\sigma_P = \sigma_A \left| \frac{\mu_P - \mu_F}{\mu_A - \mu_F} \right|, \tag{28}$$

or equivalently,

$$\mu_P = \begin{cases} \frac{\mu_A - \mu_F}{\sigma_A}\sigma_P + \mu_F \\ -\frac{\mu_A - \mu_F}{\sigma_A}\sigma_P + \mu_F \end{cases}, \ \sigma_P \geq 0. \tag{29}$$

⚠ 再次提醒开根号之后要带绝对值而脱掉绝对值符号要带±, 因为σ_P要求非负, 而w_A一般而言(在卖空允许时)可负可正.

Apparently, the feasible set of one risky asset and one risk-free asset, as documented in (29), consists of two straight lines when both short-selling (i.e., $w_A < 0$) and borrowing (i.e., $w_A > 0$ thus $w_F < 0$) are allowed[11]. The upper branch passes the risky asset A at slope $(\mu_A - \mu_F)/\sigma_A$ and intersects with the μ-axis at riskless asset F with $(0, \mu_F)$ (but cannot exceed it since we require $\sigma_P \geq 0$). Another line starting from F at the negative slope constitutes the lower branch of this feasible set. Figure 6 provides a typical example of this type and based on the previous knowledge, we are readily to figure out the MVP (that is, the riskless asset F, which serves as both global and local MVP) and efficient frontier when borrowing is permitted (that is, the solid line segment plus the dotted part) or borrowing is not available (i.e., the solid line segment only).

[11]The word "short-selling" is used for risky assets. When we want to express the shorting on risk-free asset (such as a bank account), we always say "borrowing" (money).

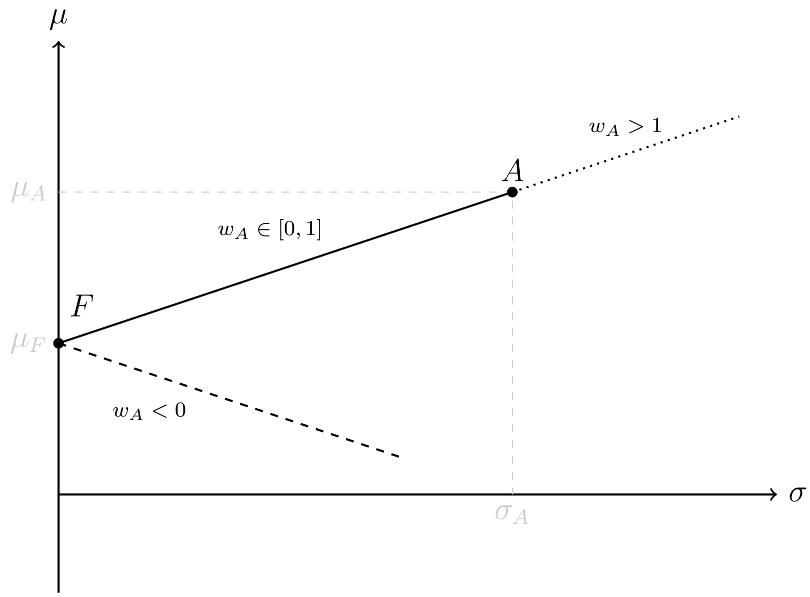


Figure 6: Feasible set between risky asset A and risk-free asset F, with solid segment for $w_A \in [0, 1]$, dotted for $w_A > 1$ (when borrowing is available) and dashed for $w_A < 0$ (when shorting is allowed).

10 Feasible set among two risky and one riskless assets

Suppose now we have two risky assets, A and B with $\mu_A > \mu_B$, $\sigma_A > \sigma_B > 0$ and their correlation ρ_{AB} given, and one more risk-free asset F whose deterministic return is $\mu_F < \mu_A$. And we know that the correlations between F and any risky asset are zeros. If we follow directly the framework used in the previous content, we should embark on formalizing the expectation and variance of return of portfolio that can be constructed from the above *three* assets, namely, $\mu_P = \mathbb{E}[R_P] = \mathbb{E}[w_A R_A + w_B R_B + w_F \mu_F] = w_A \mu_A + w_B \mu_B + w_F \mu_F$, and $\sigma_P^2 = \mathrm{Var}(R_P) = \mathrm{Var}(w_A R_A + w_B R_B + w_F \mu_F) = \mathrm{Var}(w_A R_A + w_B R_B) = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2\rho w_A w_B \sigma_A \sigma_B$ (note that we ignore the last term "$w_F \mu_F$" in the variance operator above because it is deterministic).

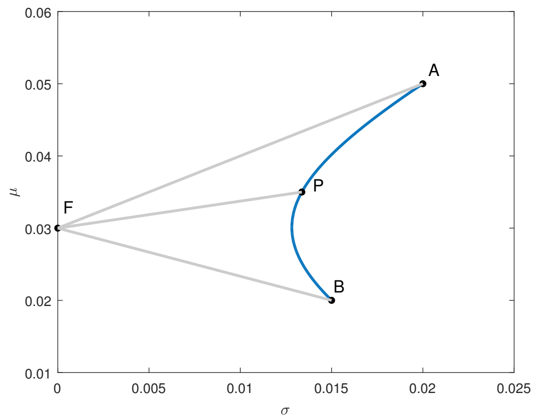
Theoretically, the feasible set we want consists of all the possible portfolios whose expected return μ_P and volatility σ_P satisfy the above formulas, subject to the very important constraint on the weights (when shorting and borrowing are permitted)

$$w_A + w_B + w_F = 1. \tag{30}$$

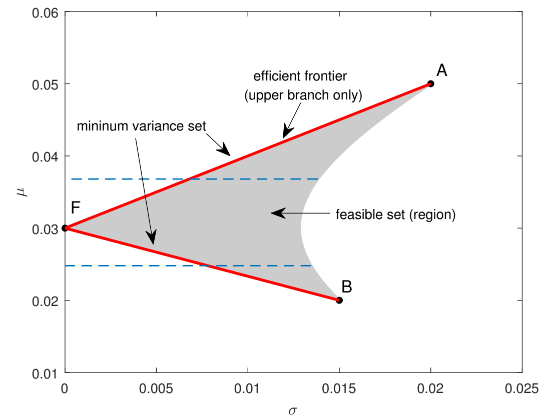
This time, however, the relationship between σ_P and μ_P is driven by multiple weights[12] and hence the feasible set of these three assets becomes a *region* on the σ-μ graph instead of a curve of previous two-asset case. Moreover, there is a tricky way to depict this region. The idea has three steps:

i) We first build up the feasible set between the original two risky assets A and B, as what we did before, and it is a smooth parabolic curve on the σ-μ graph that passes through two points of the risky assets;

[12] In fact two weights in this case, for instance, we take w_A and w_B as our decision variables so that w_F could be further determined by them through $w_F = 1 - w_A - w_B$.



(a) Initial steps to construct feasible set



(b) A typical feasible set

Figure 7: Feasible set, minimum variance set, and efficient frontier among two risky assets and one risk-free asset under no-shorting and no-borrowing conditions. Type I.

ii) we then treat any portfolio P on this curve as a single risky asset, and obtain the feasible set between the original riskless asset F and P, which is a straight line that connects them;

iii) since there are infinitely many such risky portfolios on the curve, the desired feasible set turns out to be a region that sweeps along the curve.

Figure 7 provides a simple example in order to illustrate the above method. To make our life easier, we first confine ourself in the world of no shorting and no borrowing. Therefore, the feasible set between two risky assets is the blue parabolic curve but does not go beyond two points A and B, as shown in Figure 7(a). We then select, for example, a risky portfolio P on the curve and connect it with the risk-free asset F by a line segment (since we assume no borrowing, this line cannot go beyond the point P). As we have infinitely many such risky portfolios on the curve (of course including A and B themselves), we finally obtain the demanded feasible set as a sort-of closed triangle area (including three boundaries) exhibited in Figure 7(b) and shaded in gray. The portfolios *outside* this region *cannot* be constructed by these three assets when shorting and borrowing are not allowed. There emerges a new notion called *minimum variance set* (red lines in the figure). It collects the portfolios with *lowest* risk for *every* expected return level. The blue dashed lines in the figure are used to demonstrate this statement. Obviously, it is the left-most boundary of the region, and the upper branch (separated from the lower branch by the MVP) is the efficient frontier of this case.

V. Tangent Portfolio

Figure 7 in the previous section exhibits a typical, but *not* unique, shape of feasible region among two risky and one riskless assets. In fact, this region highly depends on the assumptions we impose (shorting or borrowing is allowed or not) and the concrete risky assets we face. Among all these circumstances, the key issue is to identify the so-called tangent portfolio.

10.1 Feasible region without shorting and borrowing

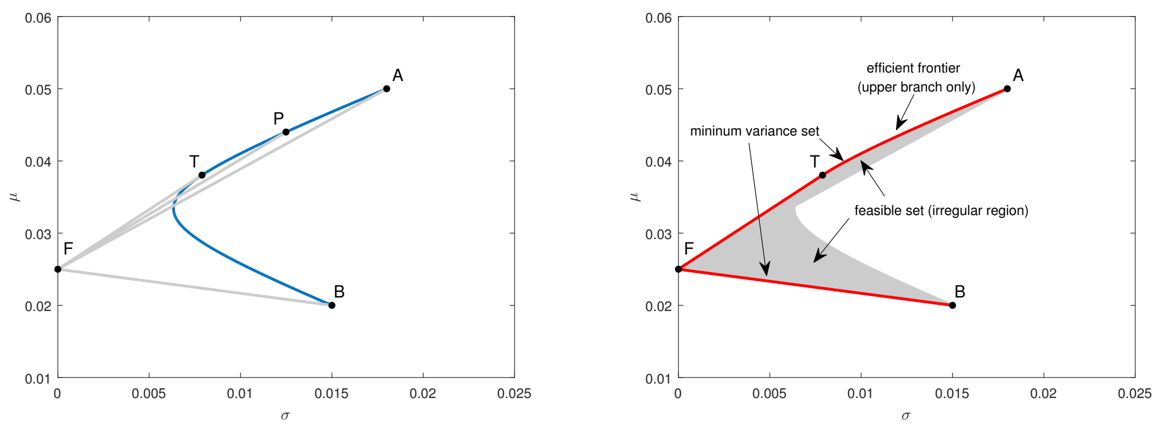
Suppose we are still in the no-shorting and no-borrowing environment. Let us consider another pair of risky assets A and B such that when we connect the risk-free asset F with A, we find that the line segment actually *crosses* the curve founded by A and B, for example, as shown in Figure 8(a). This indicates that if we select a risky portfolio P that is close to A along the curve, we could obtain some more efficient investment opportunities existing on the line segment between P and F than those on the previous line across A and F. This phenomenon continues until we reach a special portfolio such that the line starting from F is *tangent* with the curve at this point. Therefore, this portfolio is called *tangent portfolio*, for instance, the portfolio T in Figure 8. We then follow the similar steps and finally get an entire feasible region of this case which is an irregular shape. This time, the efficient frontier is a combination of the straight line segment from F to T and the curve from T to A as part of the original feasible set established between A and B. See the upper red front of Figure 8(b). We are now clear that Figure 7 demonstrates the case where the tangent portfolio should theoretically exist but would be tangent at the part of curve when $w_A > 1$. Therefore, it cannot be reached if we are not allowed to short sell the asset B.

10.2 Feasible region when borrowing is allowed

Now, let us relax the condition a little bit so that we are allowed to borrow money at the same rate of μ_F (but short-selling on risky assets is still restricted), then what will happen to the shape of feasible region? Recall that when borrowing is available, the original line segment between one risk-free asset and one risky asset would be extended from the risky point to the infinity (see Figure 6 for $w_A \geq 0$). Therefore, we only need to slightly modify the previous method for analyzing the feasible region. As illustrated by the dashed blue part in Figure 9(a), the line segment between the riskless asset F and a risky portfolio P on the curve of the original two risky assets is extended accordingly. We can conclude from Figure 9 that no matter whether there is a tangent portfolio on the original curve or not, the efficient front of the feasible region when borrowing is permitted while shorting is forbidden is always an unlimited straight line.

10.3 Feasible region without any restriction

The last situation would be the unconstrained world, that is, we could either borrow money to invest more on the risky assets or short sell any risky asset if necessary. We could easily imagine that due to the parabolic property of the curve between two risky assets, there is always an available tangent portfolio T in this case, no matter whether $\mu_T \leq \mu_A$ or $\mu_T > \mu_A$.



(a) To find the tangent portfolio T (b) Irregular shape of feasible set

Figure 8: Feasible set, minimum variance set, and efficient frontier among two risky assets and one risk-free asset under no-shorting and no-borrowing conditions. Typer II.

See Figure 10 for an explanation. We know that compared to other portfolios in the feasible region, the portfolios on the efficient frontier are the only efficient ones we should select. But which efficient portfolio should I invest? The answer is that it depends on your own risk attitude: the larger the risk you are willing to take, the larger the *expected* return you could pray for.

◇ 以上结论符合那句老话: 高风险高收益. 但注意两点: 1) 该投资组合模型告诉我们, 应当投资efficient portfolios (同等期望收益下能够构造出的风险最小的投资组合); 2) "高收益"是针对平均意义而言的理论结果, 而非针对单次投资结果的评判.

Note that theoretically, there should be a similar tangent portfolio from below (i.e., on the lower branch of the curve), but since the lower half is always inefficient thus out of interest, we ignore further discussion on it.

11 Calculations on tangent portfolio

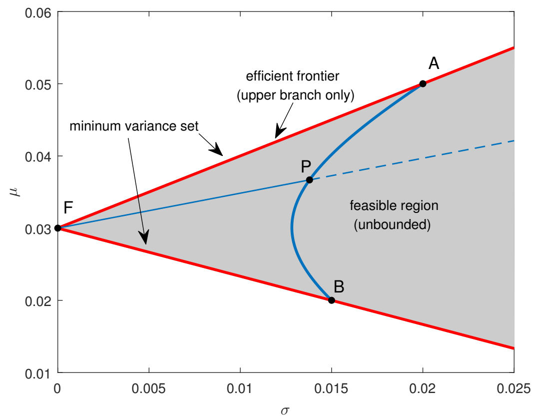
To end this topic, we provide an elementary approach to find the tangent portfolio when shorting is allowed[13]. The basic idea is to maximize the slope ($\tan\theta$) of the straight line crossing through the risk-free asset and any risky portfolio on the curve. Note that in the tangent portfolio there is no weight on the riskless asset. Therefore, our target is to find proper w_A and w_B such that they maximizes

$$\tan\theta = \frac{\mu_P - \mu_F}{\sigma_P} = \frac{w_A\mu_A + w_B\mu_B - \mu_F}{\sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\rho\sigma_A\sigma_B}} \tag{31}$$

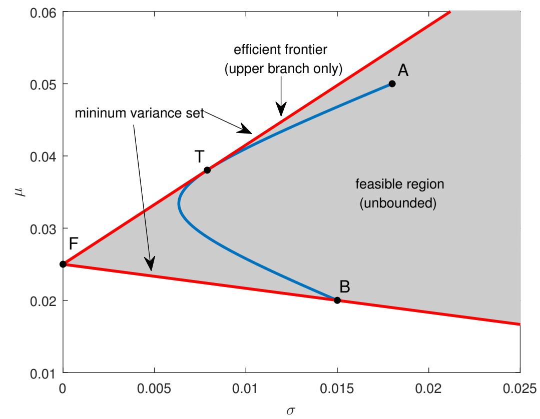
subject to the constraint

$$w_A + w_B = 1. \tag{32}$$

[13]In the next topic, we will provide an advanced method to solve for the tangent portfolio in the general multiple-asset setting.



(a) Tangent portfolio cannot be reached



(b) Tangent portfolio can be formed

Figure 9: Two types of feasible set, minimum variance set, and efficient frontier among two risky assets and one risk-free asset under no-shorting condition only (borrowing is allowed).

Note that the expression $(\mu_P - \mu_F)/\sigma_P$ is also known as the *Sharpe ratio* of a certain asset, and it measures the (excess) expected reward for bearing one unit of risk. Therefore, we sometimes call it the *price of risk* as well.

The basic idea for solving the above problem is to set the first-order derivative w.r.t. w_A and w_B to be zeros, respectively, without considering the constraint (32) first. We all know that the first-order condition is only *necessary* to the optimality. However, if we could only find one unique pair that satisfies both the necessary condition *and* the constraint, then these results would be the optimal weights as desired for constructing the tangent portfolio. To begin with, let us denote the function in (31) by

$$f(w_A, w_B) = [w_A(\mu_A - \mu_F) + w_B(\mu_B - \mu_F)] \times \\ (w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\rho\sigma_A\sigma_B)^{-\frac{1}{2}},$$

then by the chain rule we have

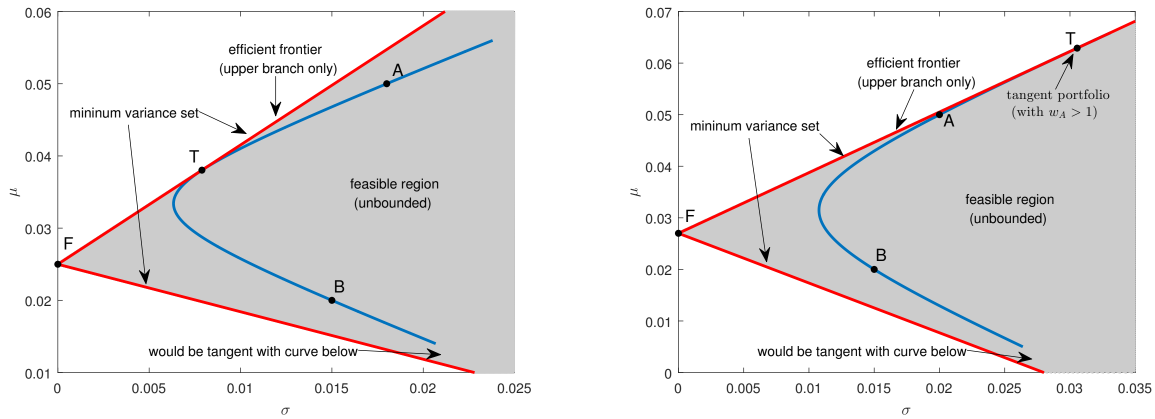
$$\frac{df}{dw_A} = (\mu_A - \mu_F)(w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\rho\sigma_A\sigma_B)^{-\frac{1}{2}} - \\ (w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\rho\sigma_A\sigma_B)^{-\frac{3}{2}}(w_A\sigma_A^2 + w_B\rho\sigma_A\sigma_B),$$

and similar formula for w_B. Then we set both df/dw_A and df/dw_B to be zeros and after some simplifications we get

$$w_A\sigma_A^2 + w_B\rho\sigma_A\sigma_B = (w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\rho\sigma_A\sigma_B)^{-1}(\mu_A - \mu_F), \\ w_B\sigma_B^2 + w_A\rho\sigma_A\sigma_B = (w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\rho\sigma_A\sigma_B)^{-1}(\mu_B - \mu_F),$$

where w_A and w_B are our decision variables while $\mu_A, \sigma_A, \mu_B, \sigma_B, \rho$ are given. If we put the above conditions into a matrix form, we would get a system of linear equations on the weights

$$\begin{pmatrix} \sigma_A^2 & \rho\sigma_A\sigma_B \\ \rho\sigma_A\sigma_B & \sigma_B^2 \end{pmatrix} \begin{pmatrix} w_A \\ w_B \end{pmatrix} = \lambda \begin{pmatrix} \mu_A - \mu_F \\ \mu_B - \mu_F \end{pmatrix}, \tag{33}$$



(a) Tangent portfolio with $w_A \leq 1$

(b) Tangent portfolio with $w_A > 1$

Figure 10: Two types of feasible set, minimum variance set, and efficient frontier among two risky assets and one risk-free asset when shorting and borrowing are both available.

where the trick here is that we denote

$$\lambda = (w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\rho\sigma_A\sigma_B)^{-1},$$

which is unknown at this stage but we know that once w_A and w_B are determined, it is a constant. Therefore, solving (33) we would get (assume invertibility)

$$\begin{aligned} \begin{pmatrix} w_A \\ w_B \end{pmatrix} &= \lambda \begin{pmatrix} \sigma_A^2 & \rho\sigma_A\sigma_B \\ \rho\sigma_A\sigma_B & \sigma_B^2 \end{pmatrix}^{-1} \begin{pmatrix} \mu_A - \mu_F \\ \mu_B - \mu_F \end{pmatrix} \\ &= \frac{\lambda}{\sigma_A^2\sigma_B^2(1-\rho)} \begin{pmatrix} \sigma_B^2 & -\rho\sigma_A\sigma_B \\ -\rho\sigma_A\sigma_B & \sigma_A^2 \end{pmatrix} \begin{pmatrix} \mu_A - \mu_F \\ \mu_B - \mu_F \end{pmatrix}. \end{aligned} \tag{34}$$

All we are left to do is to determine λ. And this can be done through *normalizing* the results in (34) by the extra constraint on the weights of (32), namely,

$$\begin{aligned} 1 = w_A + w_B = \frac{\lambda}{\sigma_A^2\sigma_B^2(1-\rho)}[&\sigma_B^2(\mu_A - \mu_F) - \rho\sigma_A\sigma_B(\mu_B - \mu_F) + \\ &\sigma_A^2(\mu_B - \mu_F) - \rho\sigma_A\sigma_B(\mu_A - \mu_F)], \end{aligned}$$

and we finally get

$$\lambda^* = \frac{\sigma_A^2\sigma_B^2(1-\rho)}{\sigma_B^2(\mu_A - \mu_F) - \rho\sigma_A\sigma_B(\mu_B - \mu_F) + \sigma_A^2(\mu_B - \mu_F) - \rho\sigma_A\sigma_B(\mu_A - \mu_F)}.$$

Therefore, after we plug λ^* back into (34), the weights on A and B in the tangent portfolio, denoted by w_A^* and w_B^*, respectively, are eventually given by

$$\begin{aligned} w_A^* &= \frac{\sigma_B^2(\mu_A - \mu_F) - \rho\sigma_A\sigma_B(\mu_B - \mu_F)}{\sigma_B^2(\mu_A - \mu_F) - \rho\sigma_A\sigma_B(\mu_B - \mu_F) + \sigma_A^2(\mu_B - \mu_F) - \rho\sigma_A\sigma_B(\mu_A - \mu_F)}, \\ w_B^* &= \frac{\sigma_A^2(\mu_B - \mu_F) - \rho\sigma_A\sigma_B(\mu_A - \mu_F)}{\sigma_B^2(\mu_A - \mu_F) - \rho\sigma_A\sigma_B(\mu_B - \mu_F) + \sigma_A^2(\mu_B - \mu_F) - \rho\sigma_A\sigma_B(\mu_A - \mu_F)}. \end{aligned}$$

Once we find the tangent portfolio, any efficient portfolio constructed from the risk-free asset F and two risky assets A and B can be expressed by the weights in this portfolio, namely,

$$\begin{pmatrix} w_F \\ w_A \\ w_B \end{pmatrix} = \begin{pmatrix} \alpha \\ (1-\alpha)\begin{pmatrix} w_A^* \\ w_B^* \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \alpha \\ (1-\alpha)w_A^* \\ (1-\alpha)w_B^* \end{pmatrix}$$

for any $\alpha \in [0,1]$ if borrowing is forbidden, or $\alpha \geq 0$ if borrowing is available. Note that α represents the weight (wealth proportion) investing on the risk-free asset and $(1-\alpha)$ goes to the tangent portfolio which is formed by two risky assets in the proportion of w_A^* and w_B^*, respectively. Note also that $w_F + w_A + w_B = 1$ while $w_A^* + w_B^* = 1$.

Further readings

Part II all chapters of *Investments*, Bodie, B. Z., Kane, A., and Marcus, A., McGraw-Hill, 2020. (Recommended for business school students.)
Chapter 6 of *Investment Science*, Luenberger, D. G., Oxford University Press, 2013. (Recommended for math or engineering students.)

