FinMath Topic 3

The Capital Asset Pricing Model

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The Markowitz mean-variance model exhibits the relationship between the expected return and the volatility of an *efficient* portfolio. After more than a decade, William Sharpe (who was awarded the Nobel Prize in Economics in 1990 together with Markowitz and Miller), John Lintner, and Jan Mossin[1] independently discovered the Capital Asset Pricing Model (CAPM), which reveals the mean-risk relationship but for an individual asset or a portfolio that may be inefficient in an equilibrium market setting. In this topic, we first fully review how the CAMP is derived from the mean-variance model through rigorous proof and then see its usage on making financial decisions.

I. Market Equilibrium and CAPM Derivation

1 Ideal market and its equilibrium

Recall that in Topic2 we have introduced the Markowitz mean-variance model based on two types of assumptions, and they are

- market conditions: both shorting (on all risky assets) and borrowing (at the same risk-free rate r_F) are allowed; and
- technical conditions: the covariance matrix $\Sigma \in \mathbb{R}^{n\times n}$ is invertible and the excess return of all risky assets $\bar{\mu} = \mu - r_F \mathbb{1}_n \neq 0$.

We talked about the concepts like the minimum-variance set (which is the parabolic boundary of the risky feasible region), the minimum-variance point (MVP) and the efficient frontier (upper branch of the boundary) of a general market with n risky assets. We also involved a risk-free asset and established the one-fund theorem saying that the allocation between a single risky fund (called tangent portfolio) and the risk-free asset is enough to construct any efficient portfolio for this new market, and we eventually derived the efficient frontier which is simply a half line on the σ-μ graph that originates at $(0, r_F)$ and goes through the tangent portfolio.[2]

Note that the above investigation is from a single mean-variance investor's perspective. Let us now consider an idealized market such that

i) every investor applies the Markowitz mean-variance framework; and

[1] See Further Readings at the end of this note for more details on their publications.

[2] We suppose here and hereafter that $r_F < \mu_{\text{MVP}}$. Thus, the efficient frontier is tangent at the tangent portfolio on the *upper* portion of risky boundary. More discussions on this issue can be found in the remark section at the end of Topic2.

ii) everyone has the same assessment of mean vector and covariance matrix for all risky assets and faces the same risk-free asset; and

iii) no transaction costs or tax.

Therefore, there should be a *common* tangent portfolio for *all* investors, and the actions that everyone buys this common risky fund to build up their own efficient portfolios (according to their own risk attitudes or target expected returns) leads to an *equilibrium market* such that this public fund is nothing but just the *market portfolio* with the *capitalization weight*[3] for each single risky asset currently in the market.

⚠ 本专题所讲的equilibrium状态下的market portfolio, 其weights是各个资产的capitalization weight, 可通过观测当前市场各资产价格和发行数量计算到; 对比Topic1,2中我们把tangent portfolio也叫做market portfolio, 其weights $w_M = \Sigma^{-1}\bar{\mu}/(\mathbb{1}'\Sigma^{-1}\bar{\mu})$是通过对各资产收益率的估计而后计算得到; 两者出发点完全不同, 严格意义上讲, 后者属于个别投资者的personal market portfolio. 通常情况下, 即便所有投资者都拥有相同的估计, w_M和当前capitalization weight也不会一样, 原因在于现实世界其实并不符合上述equilibrium所需要的全部假设.

According to the one-fund theorem, all the investors share the same efficient frontier as a half line that connects the riskless asset F with risk-free rate r_F and the market portfolio M with the market expected return μ_M and the volatility σ_M. This line is also called the *capital market line* (CML) for this equilibrium market, and its expression is given by

$$\mu_0 = r_F + \frac{\mu_M - r_F}{\sigma_M}\sigma_0, \ \sigma_0 \geq 0, \tag{1}$$

where μ_0 and σ_0 denote, respectively, the expected return and the volatility of any *efficient* portfolio on CML, and the slope $(\mu_M - r_F)/\sigma_M$ is known as the Sharpe ratio of the market or the market risk premium. In other words, CML indicates the relationship between the expected return of any efficient portfolio and its volatility in terms of the market risk premium in this equilibrium market. Then how about the mean-risk relationship for an *individual* asset or *inefficient* portfolio? This is what the CAPM tells us in the next section.

⚠ 严格意义上讲, 此处的market portfolio应该包含当前世界全部的risky assets available, 如equity, corporate bond, derivatives, gold and other metals, and even cryptcurrency. 但实操中, 我们一般只用一个equity market的index来代表这里的market portfolio, 直接估计其μ_M和σ_M (注意, 不是用$w_M'\mu$和$w_M'\Sigma w_M$计算!), 并且stock-market index往往也是用capitalization weight编制的.

2 Derivation of CAPM

The Capital Asset Pricing Model (CAPM) links the expected return of any individual asset (or portfolio) to its individual risk (as comparison to the market risk). We provide the formal statement and give the proof as well.

[3]The capitalization weight of an asset is defined as the proportion of this asset's total capital value to the total market capital value (summation of capital values of all assets in the market). The capital value of a stock, for instance, equals the product of its price and shares.

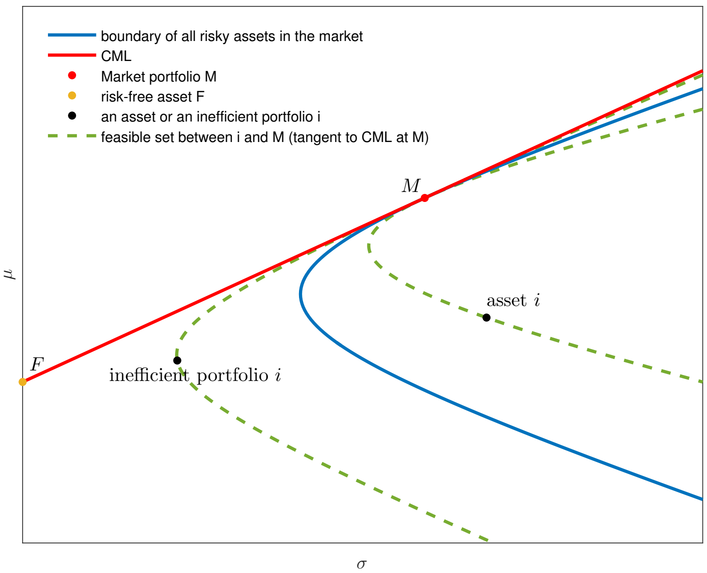


Figure 1: Derivation of CAPM: The feasible set between the market portfolio M and an individual asset (or an inefficient portfolio) i is tangent to CML at M.

CAPM. If the market portfolio (with capitalization weights) M is efficient[4], then the expected return μ_i of any individual asset (or portfolio) i satisfies

$$\mu_i = r_F + \beta_i(\mu_M - r_F), \tag{2}$$

where $\beta_i = \sigma_{iM}/\sigma_M^2$ is known as the beta of this asset and σ_{iM} is the covariance between the returns of asset i and market portfolio.

Proof. Consider an asset (or a portfolio) i in the market whose expected return and volatility are denoted by μ_i and σ_i, respectively. Suppose we form a portfolio by investing the proportion $\alpha \in \mathbb{R}$ on the asset i and the rest $1 - \alpha$ on the market portfolio M, then the expected return and volatility of this new portfolio are easily obtained, namely,

$$\begin{aligned}\mu_\alpha &= \alpha\mu_i + (1-\alpha)\mu_M,\\ \sigma_\alpha &= (\alpha^2\sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha)^2\sigma_M^2)^{\frac{1}{2}}.\end{aligned}$$

From the previous topics we know that $\{(\sigma_\alpha, \mu_\alpha)|\alpha \in \mathbb{R}\}$ represents the feasible set formed by these two assets, which should be a parabolic curve on the σ-μ diagram passing through these two points. Moreover, this curve *cannot* cross the CML, otherwise it contradicts with the fact that CML is the efficient frontier for this equilibrium market. Therefore, the curve must be tangent to CML exactly at the market point! (See Figure 1 for two concrete examples.) In other words, the first-order derivative of this curve when $\alpha = 0$ should equal the slope of

[4] This condition is equivalent to saying that all the assumptions required for the equilibrium market are satisfied so that the market portfolio is the single common risky fund for all investors.

CML, namely,

$$
\begin{aligned}
\frac{\mu_M - r_F}{\sigma_M} &= \left.\frac{d\mu_\alpha}{d\sigma_\alpha}\right|_{\alpha=0} = \left.\frac{d\mu_\alpha}{d\alpha}\frac{d\alpha}{d\sigma_\alpha}\right|_{\alpha=0} \\
&= \left.\frac{\mu_i - \mu_M}{\frac{1}{2\sigma_\alpha}[2\alpha\sigma_i^2 + 2\sigma_{iM} - 4\alpha\sigma_{iM} - 2(1-\alpha)\sigma_M^2]}\right|_{\alpha=0} \\
&= \frac{\mu_i - \mu_M}{(\sigma_{iM} - \sigma_M^2)/\sigma_M},
\end{aligned} \tag{3}
$$

and hence we simplify the above result and get

$$
\mu_i = r_F + \beta_i(\mu_M - r_F), \text{ where } \beta_i = \frac{\sigma_{iM}}{\sigma_M^2}, \tag{4}
$$

as desired. □

CAPM is an equilibrium pricing model deduced right from the Markowitz mean-variance theory under the preceding assumptions i) to iii) required by an equilibrium market. It predicts a “fair” expected return of an individual asset (or a portfolio) under the mean-variance framework. Therefore, it provides a benchmark for evaluating an asset or making investment decisions. We will see some of its applications in the next section.

II. Applications of CAPM

Throughout the discussions below, suppose the assumptions of CAPM are always fulfilled and $\mu_M > r_F$.

3 Special cases in CAPM

In general, the CAPM holds for any individual asset (or portfolio). Therefore, it definitely covers those special assets or portfolios we are going to verify in the following.

The market portfolio M. Set the asset i in (2) to be M and we should have

$$\mu_M = r_F + \beta_M(\mu_M - r_F),$$

which indicates that $\beta_M = 1$. We could also get this result directly from the definition of beta, that is, $\beta_M = \mathrm{Cov}(R_M, R_M)/\sigma_M^2 = 1$.

The risk-free asset F. Set the asset i in (2) to be F and we should have

$$r_F = r_F + \beta_F(\mu_M - r_F),$$

which indicates that $\beta_F = 0$. Similarly, we could also get this result from the definition of beta, that is, $\beta_F = \mathrm{Cov}(r_F, R_M)/\sigma_M^2 = 0$.

An efficient portfolio E_α. Suppose there is an efficient portfolio E_α with $\alpha \le 1$ portion invested on F and the rest $1-\alpha \ge 0$ on M. We know that its random return is given by $R_E = \alpha r_F + (1-\alpha)R_M$, thus its volatility is $\sigma_E = (1-\alpha)\sigma_M$ and the correlation between E_α and M is always 1. To see this,

$$\begin{aligned}\rho_{EM} &= \frac{\mathrm{Cov}(R_E, R_M)}{\sigma_E\sigma_M} \\ &= \frac{\mathrm{Cov}(\alpha r_F + (1-\alpha)R_M, R_M)}{(1-\alpha)\sigma_M^2} \\ &= \frac{(1-\alpha)\sigma_M^2}{(1-\alpha)\sigma_M^2} \\ &= 1. \end{aligned} \tag{5}$$

Therefore, the beta of the efficient portfolio E_α should satisfy

$$\beta_E = \frac{\mathrm{Cov}(R_E, R_M)}{\sigma_M^2} = \frac{\rho_{EM}\sigma_E\sigma_M}{\sigma_M^2} = \frac{\sigma_E}{\sigma_M} \ge 0. \tag{6}$$

Thus, we have $\sigma_E = \beta_E\sigma_M$, which means, within CAPM, the risk (volatility) of an efficient portfolio is fully captured by its beta (and also the market volatility). Moreover, the expression of CML coincides with CAPM when the "asset" considered is an efficient portfolio. To see this, we know that E lies on CML and together with (6) we have

$$
\begin{aligned}
\mu_E &= r_F + \frac{\mu_M - r_F}{\sigma_M}\sigma_E \\
&= r_F + \frac{\sigma_E}{\sigma_M}(\mu_M - r_F) \\
&= r_F + \beta_E(\mu_M - r_F).
\end{aligned}
$$

4 CAMP v.s. Sharpe's single-index model

The Sharpe's single-index model says the asset's (random) return is driven by a single market index return and white noise. More precisely, it assumes at any time t,

$$
R_t^i - r_F = \alpha_i + \beta_i(R_t^M - r_F) + \epsilon_t^i, \tag{7}
$$

where R_t^i and R_t^M denote, respectively, the (random) returns of asset i and market index at time t, and the random variable ϵ_t^i is a Gaussian white noise, namely, $\epsilon_t^i \sim \mathcal{N}(0, \sigma_{\epsilon_i}^2)$ and is independent of R_t^M. Moreover, α_i and β_i are the intercept and the coefficient of this linear model, respectively, which are a.k.a. the *alpha* and *beta* of this asset. The CAPM coincides with this simple model if we further assume $\alpha_i = 0$. To see this, let us set $\alpha_i = 0$ and take expectation on both sides of (7) and denote $\mu_i = \mathbb{E}[R_t^i]$, $\mu_M = \mathbb{E}[R_t^M]$, $\sigma_i^2 = \mathrm{Var}(R_t^i)$ and $\sigma_M^2 = \mathrm{Var}(R_t^M)$, we then have

$$
\begin{aligned}
\mathbb{E}[R_t^i] - r_F &= \beta_i(\mathbb{E}[R_t^M] - r_F) + \mathbb{E}[\epsilon_t^i] \\
\Rightarrow \mu_i &= r_F + \beta_i(\mu_M - r_F),
\end{aligned}
$$

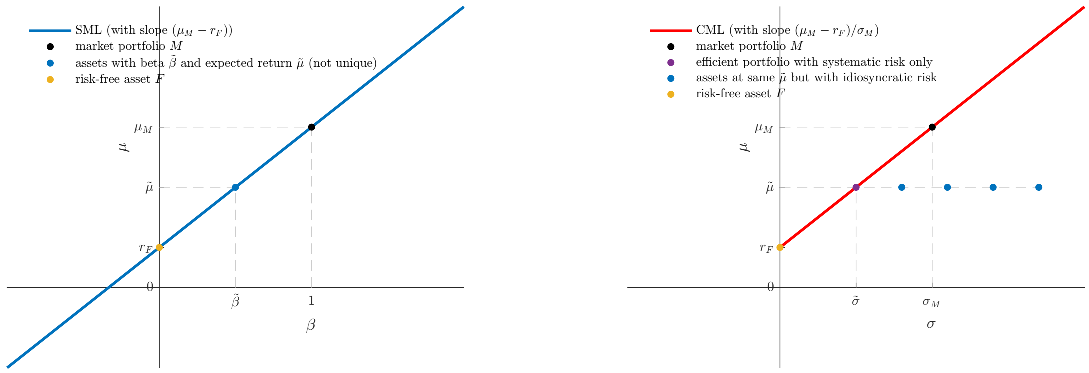
which leads to CAPM. In this case, the beta in zero-intercept Sharpe model serves as the beta in CAPM. On the other hand, the variance of return under the Sharpe's single-index model is given by

$$
\begin{aligned}
\mathrm{Var}(R_t^i) &= \beta_i^2 \mathrm{Var}(R_t^M) + \mathrm{Var}(\epsilon_t^i) \\
\Rightarrow \sigma_i^2 &= \beta_i^2\sigma_M^2 + \sigma_{\epsilon_i}^2, \qquad (8)
\end{aligned}
$$

namely, it can be decomposed into two parts: $\beta_i^2\sigma_M^2$ is known as the *systematic risk* which is represented by a certain leverage (determined by beta) of market risk, while $\sigma_{\epsilon_i}^2$ is called the *idiosyncratic risk* (or individual risk). We have seen from (6) that, if we treat CAPM as a special Sharpe model, the efficient portfolio has only the systematic risk remaining while no idiosyncratic risk since as an *efficient* portfolio it has been fully diversified.

5 The security market line

We have already learnt that all the *efficient* portfolios must lie on the CML, a half-line on the σ-μ plane given by (1), and this describes a relationship between the volatility and the expected return of any efficient portfolio. On the other hand, any inefficient portfolio or individual asset must lie to the right of CML on the σ-μ graph, which indicates higher risk



(a) SML (b) CML

Figure 2: Comparison between SML and CML: any asset should locate on SML while only efficient portfolio lie on CML.

(in terms of volatility) compared to the efficient portfolio with the same level of the expected return.

In contrast, CAPM tells us that any individual asset or portfolio should satisfy (2). If we deem the CAPM equation as a linear function between the asset beta and its expected return, we could obtain another straight line but on the β-μ graph, and this is called the *security market line* (SML). Under the equilibrium conditions required by CAPM, all the assets (including individuals or portfolios, efficient or not) must fall on SML.

Figure 2 compares SML and CML. We can see that assets with the same beta $\tilde{\beta}$ must have the same expected return $\tilde{\mu}$ according to CAPM thus they overlap on SML at the same point $(\tilde{\beta}, \tilde{\mu})$ (see blue point in Figure 2(a)). However, when we plot them on the σ-μ diagram, we find that only the efficient portfolio lies on CML which means only systematic risk remains in the efficient portfolio, while other assets scatter at the same level of expected return but right to CML which indicates they possess idiosyncratic risk (see Figure 2(b)). Note that the beta could be negative while the volatility cannot.

More generally, we know that the correlation between any asset (or a portfolio, efficient or not) i and the market portfolio M, denoted by ρ_{iM}, satisfies $|\rho_{iM}| \leq 1$. Therefore,

$$|\beta_i| = \left|\frac{\mathrm{Cov}(R_i, R_M)}{\sigma_M^2}\right| = \frac{|\rho_{iM}|\sigma_i\sigma_M}{\sigma_M^2} \leq \frac{\sigma_i}{\sigma_M}.$$

Together with CAPM formula, we have

$$\begin{aligned}
&\mu_i = r_F + \beta_i(\mu_M - r_F),\\
&\sigma_i \geq |\beta_i|\sigma_M, \text{ (the equality holds for the efficient portfolios).}
\end{aligned}$$

The above results tell us three important properties:

- the volatility of a general asset has two parts: the first one is known as the systematic risk which is determined by its beta and market risk ($\beta_i\sigma_M$) and the rest ($\sigma_i - \beta_i\sigma_M$) usually represents its idiosyncratic risk; for efficient portfolios the rest part is zero, meaning that they are actually sufficiently diversified; and

- the size and direction (positive, zero or negative) of beta indicates how large fluctuation of the asset is relevant (with same direction or opposite) to the market movement; and
- the expected return is fully determined by the beta; in other words, within CAPM, only the market-related risk (measured by β_i) is concerned in pricing.

Further readings

Recommended for business school students:
Chapter 9 of *Investments*, Bodie, B. Z., Kane, A., and Marcus, A., McGraw-Hill, 2020.
Recommended for math or engineering students:
Chapter 7 of *Investment Science*, Luenberger, D. G., Oxford University Press, 2013.
Chapter 5 of *Portfolio Theory and Risk Management*, Capiński, M. J., and Kopp E., Cambridge University Press, 2014.