Universität Stuttgart

Optimal Control

Solution of Homework Exercise 2

Winter Term 17/18

Authors:
Silvia Gramling
[2867885]
Markus Schmidgall
[2880655]

January 30, 2018

Problem 1

a) The exercise yields a discrete-time infinte-horizon optimal control problem

$$\begin{aligned}
\min_{u_k} \; & \sum_{k=0}^{\infty} 0.9^k\, f_0(x_k, u_k) \\
\text{s.t.} \quad & x_{k+1} = f(x_k, u_k), \quad k = 0, ..., \infty \\
& x_k \in \mathcal{X} \\
& u_k \in \mathcal{U}
\end{aligned}$$

The function f is the transition function which donates the state x_{k+1} given x_k and u_k. It can be read from the graph e.g. $f(\xi_1, u_0) = \xi_2$ (soll ich des hier alles einfügen?).
The Value function iteration

$$V^{new}(\xi_k) = (TV^{old})(\xi_k), \quad \xi_k \in \mathcal{X}$$

is defined by the DP operator

$$(TV)(\cdot) = \min_{u \in \mathcal{U}} \{ f_0(\cdot, u) + 0.9\, V(f(\cdot, u)) \}.$$

b) Trusting the MATLAB program the Value function reads

$$V^{\star} = \begin{bmatrix} 4.16 & 4.9 & 2.4 & 1 & 3.9 & 2.35 & 1.5 & 0 \end{bmatrix}$$

where every entry represents the optimal cost when starting at the corresponding state. The optimal control feedback is given by

$$\begin{aligned}
v_k(\xi_1) &= 2 \\
v_k(\xi_2) &= 2 \\
v_k(\xi_3) &= 0 \\
v_k(\xi_4) &= 1 \\
v_k(\xi_5) &= 0
\end{aligned}$$

$$v_k(\xi_6) = 1$$
$$v_k(\xi_7) = 0 \text{ (or 1 or 2)}$$
$$v_k(\xi_8) = 0 \text{ (or 1 or 2)}$$

This leads to the optimal path (starting at ξ_1)

$$\begin{bmatrix} \xi_1 & \xi_3 & \xi_4 & \xi_8 & \xi_8 & \cdots \end{bmatrix}$$

with the optimal input sequence

$$\begin{bmatrix} u_2 & u_0 & u_1 & u_0 & u_0 & \cdots \end{bmatrix}.$$

c) This proof will be split into two parts. First of all we show that $V^k \to V^\star$, afterwards $V^0(\xi_i) \leq V^\star(\xi_i)$ will be proven.
By defining
$$V^1(\xi_i) := TV^0(\xi_i)$$
we get a series $\{V^j\}_{j\in\mathbb{N}_0}$ with

$$V^{j+1}(\xi_i) := TV^j(\xi_i).$$

This leads to the explicit form

$$V^j(\xi_i) := T^j V^0(\xi_i).$$

To show the first statement we claim that T is a contraction on $\mathbb{R}^{\mathcal{X}}$ the function space of all functions mapping an point in $\mathcal{X}$ into $\mathbb{R}$, i.e.

$$||TV^1 - TV^2||_\infty \leq \beta\,||V^1 - V^2||_\infty, \quad 0 < \beta < 1.$$

This holds true since

$$\begin{aligned} ||TV^1 - TV^2||_\infty = \max_{x\in\mathcal{X}} \Big\{ & \min_{u\in\mathcal{U}}\{f_0(x,u) + \alpha\,V^1(f(x,u))\} \\ & - \min_{u\in\mathcal{U}}\{f_0(x,u) + \alpha\,V^2(f(x,u))\} \Big\} \end{aligned}$$

$$
\begin{aligned}
&\leq \max_{x\in\mathcal{X}}\left\{\min_{u\in\mathcal{U}}\{\alpha\, V^1(f(x,u)) - \alpha\, V^2(f(x,u))\}\right\} \\
&\leq \alpha \max_{x\in\mathcal{X}}\left\{\max_{u\in\mathcal{U}}\{\, V^1(f(x,u)) - \, V^2(f(x,u))\}\right\} \\
&\leq \alpha ||V^1 - V^2||_\infty
\end{aligned}
$$

Due to the fact that $\alpha = 0.9 < 1$, the requirements are satisfied. We now use the Banach Fix Point-Theorem which now yields that there is exactly one $V^\star \in \mathbb{R}^{\mathcal{X}}$ s.t. $TV^\star = V^\star$ and every series $\{V^j\}_{j\in\mathbb{N}_0}$ with $V^0 \in \mathbb{R}^{\mathcal{X}}$ and $V^{j+1} = TV^j$ converges to $V^\star$ which we wanted to show.
For the second statement we use the given inequality

$$V^0(\xi_i) \leq TV^0(\xi_i) =: T^1(\xi_i) \quad \forall \xi_i \in \mathcal{X}$$

By using the monotonicity property of T we derive that

$$V^0(\xi_i) \leq V^1(\xi_i) \Rightarrow V^1(\xi_i) = TV^0(\xi_i) \leq TV^1(\xi_i) = V^2(\xi_i).$$

From $V^1(\xi_i) \leq V^2(\xi_i)$ we arrive at $V^2(\xi_i) \leq V^3(\xi_i)$ applying the same property. We can now go on and make use of it consecutively. This results in a series of inequations

$$V^0(\xi_i) \leq V^1(\xi_i) \leq V^2(\xi_i) \leq \ldots \leq V^\infty(\xi_i)$$

We already know that $V^\infty(\xi_i) = V^\star(\xi_i)$ and so we have $V^0(\xi_i) \leq V^\star(\xi_i)$.

$\square$

d) We consider the feasible set $F = \{V : V(\xi_i) \leq TV(\xi_i),\ 1 \leq i \leq n\}$ of the given optimization problem. Furthermore consider $\tilde{V} \in F$ with $\tilde{V} \neq V^\star$ (obviously $V^\star \in F$). From c) we immediately have $\tilde{V}(\xi_i) \leq V^\star(\xi_i),\ i = 1,\ldots,n$ (with at least one $j \in \{1,2,\ldots,n\}$ such that $\tilde{V}(\xi_j) < V^\star(\xi_j)$). This results in

$$\sum_{i=1}^{n} \tilde{V}(\xi_i) < \sum_{i=1}^{n} V^\star(\xi_i)$$

and hence $\tilde{V}$ cannot be the optimal solution. Since this argument holds for all $\tilde{V} \in F/\{V^\star\}$ the only possible solution left is $V^\star$.

$\square$

e) As the constraints in d) we got

$$V(\xi_i) \leq TV(\xi_i) \quad i = 1, \dots, n$$

with

$$TV(x) = \min_{u \in \mathcal{U}(\xi_i)} \{f_0(x, u) + \alpha\, V(f(x, u))\}.$$

This holds true if

$$V(\xi_i) \leq f_0(\xi_i, u) + \alpha\, V(f(\xi_i, u)) \quad \forall u \in \mathcal{U}(\xi_i)$$

In other words we can transform the n constraints into $n \cdot \sum_i |\mathcal{U}(\xi_i)|$ linear constraints.
To obtain the form

$$\begin{aligned} &\min_V c^\top V \\ &\text{s.t. } AV \leq b \end{aligned}$$

we need to define the matricies c, A and b. You'll find them on the solution sheet.

f) In order to examine the MATLAB program take a look into *run1.m*, *valFunIteration.m*, *valFunItAsLinProg.m* and *onehot.m*.
valFunItAsLinProg already computes and returns the the optimal control feedback. To compute them we use the fact that $V^\star$ is a fix point of T, i.e. there is at least one u for every $i \in \{1, 2, \dots, 8\}$ such that

$$V^\star(\xi_i) = f_0(\xi_i, u) + \alpha\, V(f(\xi_i, u)).$$

In the LQ approach this translates to: For every $i \in \{1, 2, \dots, 8\}$ there has to be at least one $k \in \{i, i+8, i+16\}$, s.t.

$$a_k V^\star(\xi_i) = b_k$$

where a_k is the k-th row of A. Due to numerical inaccuracies it is very likely that $a_k\bar{V}(\xi_i) \approx b_k$ (with $\bar{V}$ the numerical solution) so we look for $\operatorname{argmin}_{k\in\{i,i+8,i+16\}} |a_k\bar{V}(\xi_i) - b_k|$ from which we can derive the corresponding $u \in \{u_0, u_1, u_2\}$.

Problem 2

a) To compute the equilibrium of the unforced system we consider the equation

$$\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = A \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$

with A as defined on the exercise sheet. The matrix equation is equivalent to the following system of equations:

$$\begin{aligned} x^1 + 3 \cdot x^2 &= x^1 \\ -0.5 \cdot x^1 + x^2 &= x^2. \end{aligned}$$

We receive immediately the unique solution $x^1 = x^2 = 0$, so the euibrilibrium of the unforced system lies in the origin. The stability of the equilibrium will be investigated by computing the eigenvalues of A:

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -3 \\ 0.5 & \lambda - 1 \end{bmatrix} = (\lambda - 1)^2 + 1.5 = \lambda^2 - 2\lambda + 2.5 \overset{!}{=} 0.$$

The solution of the eigenvalue equation is $\lambda_{1/2} = 1 \pm \mathrm{j}\frac{\sqrt{6}}{2}$, so the unforced system is not stable.

b) The given discrete-time system can be formulated in the MPC scheme as follows:

$$\begin{aligned} u_{MPC}(\cdot, x(t_j)) = \quad \underset{u}{\operatorname{argmin}} \quad & \sum_{j=k}^{k+2} x_j^\top x_j + u_j^2 + x_{k+3}^\top P x_{k+3} \\ \text{s.t.} \quad & x_{j+1} = A x_j + B u_j \\ & |u_j| \leq 1,\ x_j \in \mathbb{R}^2,\ x_{k+3} \in \mathcal{X}_f \\ & x_k = \bar{x} \end{aligned} \tag{1}$$

with P, A, B and $\mathcal{X}_f$ as defined on the exercise sheet.

c) Now, we show that the MPC scheme in b) gives a stabilizing controller that is defined by $u_j = Kx_j$.

1. The eigenvalue of P is used as a lower limit in the following equation to show the feasibility of the controller:

$$\lambda_{min}(P)|x|^2 \leq x^\top Px.$$

For all $x \in \mathcal{X}_f$ the relation

$$x^\top Px \leq c = \frac{\lambda_{min}(P)}{|K|^2}$$

holds by definition. After deviding by $\lambda_{min}(P)$ we obtain the input constraint

$$|u|^2 = |K|^2|x|^2 \leq 1.$$

That means that $|u| \leq 1$, so the controller is feasible.

2. By inserting the given definitions we gain

$$\begin{aligned}\phi(x_{j+1}) - \phi(x_j) &= x_{j+1}^\top Px_{j+1} - x_j^\top Px_j \\ &= ((A - BK)x_j)^\top P(A - BK)x_j - x_j^\top Px_j \\ &= x_j^\top \underbrace{[(A - BK)^\top P(A - BK) - P]}_{L} x_j \\ &\leq \lambda_{max}(L)|x|^2.\end{aligned}$$

The eigenvalues of L were computed via MATLAB. Both of them are smaller than -7. Besides, we use that

$$f_0(x_j, u_j) = |x_j|^2 + |K|^2|x_j|^2 = (1 + |K|^2)|x_j|^2 = 3.05|x_j|^2.$$

It follows that

$$\begin{aligned}\phi(x_{j+1}) - \phi(x_j) &\leq \lambda_{max}(L)|x|^2 \\ &< -7|x|^2 \\ &< -3.05|x^2|\end{aligned}$$

$$= -f_0(x_j, u_j)$$

which was to be proven.

3. For showing that the terminal region $\mathcal{X}_f$ is invariant, we need to demonstrate that $x_{k+1} \in \mathcal{X}_f$ if $x_k \in \mathcal{X}_f$. From the previous task we know that

$$\phi(x_{j+1}) - \phi(x_j) = x_{j+1}^\top P x_{j+1} - x_j^\top P x_j \leq -f_0(x_j, u_j).$$

It follows from the definition of the terminal region ($x_j^T P x_j \leq c$) that

$$x_{j+1}^\top P x_{j+1} - c \leq -(|x_j|^2 + |u_j|^2).$$

This is equivalent to

$$x_{j+1}^\top P x_{j+1} \leq c - \underbrace{(|x_j|^2 + |u_j|^2)}_{\geq 0} \leq c.$$

Consequently, $x_{k+1} \in \mathcal{X}$ and the terminal region $\mathcal{X}_f$ is invariant.

$\square$

d) The optimal control problem from b) can be written as a quadratically constrained quadratic program with the following variables

$$z = \begin{bmatrix} x_0^\top & x_1^\top & x_2^\top & x_3^\top & u_0 & u_1 & u_2 \end{bmatrix}^\top \in \mathbb{R}^{11}$$

$$H = \begin{bmatrix} \mathbf{I}_2 & 0 & & \cdots & & & 0 \\ 0 & \mathbf{I}_2 & & & & & \\ & & \mathbf{I}_2 & \ddots & & & \\ \vdots & & \ddots & P & & & \vdots \\ & & & & 1 & & \\ & & & & & 1 & 0 \\ 0 & & & \cdots & & 0 & 1 \end{bmatrix} \in \mathbb{R}^{11\times 11}$$

$$
A_{eq} = \begin{bmatrix} \mathbf{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -A & \mathbf{I}_2 & 0 & 0 & -B & 0 & 0 \\ 0 & -A & \mathbf{I}_2 & 0 & 0 & -B & 0 \\ 0 & 0 & -A & \mathbf{I}_2 & 0 & 0 & -B \end{bmatrix} \in \mathbb{R}^{8\times 11}
$$

$$
b_{eq} = \begin{bmatrix} \bar{x}^\top & 0 & \cdots & 0 \end{bmatrix}^\top \in \mathbb{R}^{11}
$$

$$
A_{ineq} = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \\ \vdots & & \vdots & 0 & 0 & 1 \\ & & & -1 & 0 & 0 \\ & & & 0 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & -1 \end{bmatrix} \in \mathbb{R}^{6\times 11}
$$

$$
b_{ineq} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^\top \in \mathbb{R}^{6}
$$

$$
T = \begin{bmatrix} 0 & & & \cdots & & & 0 \\ & 0 & & & & & \\ & & 0 & & & & \\ \vdots & & & P & & & \vdots \\ & & & & 0 & & \\ & & & & & 0 & \\ 0 & & & \cdots & & & 0 \end{bmatrix} \in \mathbb{R}^{11\times 11}.
$$

The problem is convex, because

- the matrizes H and T are positive definite (P has only positive eigenvalues) and
- linear equations are always convex.

e) Solving the optimization problem from d) in MATLAB, we obtain the results shown in Fig. ???. One can see that the calculated input signal is large at first to move the state quickly to the equilibrium. The input constraint prevents higher magnitudes of the input than one. After approximately 10 timesteps both components of the state are moved to the origin and the input signal also equals zero.

f) If we insert the other initial condition, we observe that that qualitative evolution of the state and the determined input (Fig. ???) is similar compared to the previous plot. That makes sense because both initial condition do not differ much from each other. The higher absolute value of the new initial condition would request a higher input signal in the unconstrained, optimal case, but we consider a limited input signal.

g) In contrast to the MPC algorithm, the MATLAB solver for linear-quadratic problems does not include an input constraint. That is why the absolute value of the input signal is not restricted in contrast to the input in the previous tasks. In the LQ solution, we observe that the input is higher than one at first (Fig. ???), because that leads to a smaller value of the cost functional. Another difference is that the MPC scheme only minimizes the cost functional till the prediction horizon which is $N = 3$ in our case. However, the prediction horizon does not have a huge impact on the trajectories and the qualitative evolution of the results stays the same.