Universität Stuttgart

Optimal Control

Solution of Homework Exercise 2

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Problem 2

a) To compute the equilibrium of the unforced system we consider the equation

$$\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = A \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$

with A as defined on the exercise sheet. The matrix equation is equivalent to the following system of equations:

$$\begin{aligned} x^1 + 3 \cdot x^2 &= x^1 \\ -0.5 \cdot x^1 + x^2 &= x^2. \end{aligned}$$

We receive immediately the unique solution $x^1 = x^2 = 0$, so the euibrilibrium of the unforced system lies in the origin. The stability of the equilibrium will be investigated by computing the eigenvalues of A:

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda - 1 & -3 \\ 0.5 & \lambda - 1 \end{bmatrix}\right) = (\lambda - 1)^2 + 1.5 = \lambda^2 - 2\lambda + 2.5 \overset{!}{=} 0.$$

The solution of the eigenvalue equation is $\lambda_{1/2} = 1 \pm \mathrm{j}\frac{\sqrt{6}}{2}$, so the unforced system is not stable.

b) The given discrete-time system can be formulated in the MPC scheme as follows:

$$\begin{aligned} u_{MPC}(\cdot, x(t_j)) = \quad & \underset{u}{\operatorname{argmin}} \quad \sum_{k=0}^{2} x_j^T x_j + u_j^2 + x_3^T P x_3 \\ & \text{s.t.} \quad x_{j+1} = Ax_j + Bu_j \\ & \qquad |u_j| \leq 1, x_j \in \mathbb{R}^2, x_3 \in \mathcal{X}_f \end{aligned} \tag{1}$$

with P, A, B and $\mathcal{X}_f$ as defined on the exercise sheet.

c) Now, we show that the MPC scheme in b) gives a stabilizing controller that is defined by $u_j = Kx_j$.

1. The eigenvalue of P is used as a lower limit in the following equation to show the feasibility of the controller:

$$\lambda_{min}(P)|x|^2 \le x^T P x.$$

For all $x \in \mathcal{X}_f$ the relation

$$x^T P x \le c = \frac{\lambda_{min}(P)}{|K|^2}$$

holds by definition. After deviding by $\lambda_{min}(P)$ we obtain the input constraint

$$|u|^2 = |K|^2|x|^2 \le 1.$$

That means that $|u| \le 1$, so the controller is feasible.

2. By inserting the given definitions we gain

$$\begin{aligned}
\phi(x_{j+1}) - \phi(x_j) &= x_{j+1}^T P x_{j+1} - x_j^T P x_j \\
&= ((A - BK)x_j)^T P (A - BK)x_j - x_j^T P x_j \\
&= x_j^T \underbrace{\left[(A - BK)^T P (A - BK) - P\right]}_{L} x_j \\
&\le \lambda_{max}(L)|x|^2.
\end{aligned}$$

The eigenvalues of L were computed via MATLAB. Both of them are smaller than -7. Besides, we use that

$$f_0(x_j, u_j) = |x_j|^2 + |K|^2|x_j|^2 = (1 + |K|^2)|x_j|^2 = 3.05|x_j|^2.$$

It follows that

$$\begin{aligned}
\phi(x_{j+1}) - \phi(x_j) &\le \lambda_{max}(L)|x|^2 \\
&< -7|x|^2 \\
&< -3.05|x^2| \\
&= -f_0(x_j, u_j)
\end{aligned}$$

which was to be proven.

3. For showing that the terminal region $\mathcal{X}_f$ is invariant, we need to demonstrate that $x_{k+1} \in \mathcal{X}_f$ if $x_k \in \mathcal{X}_f$. From the previous task we know that

$$\phi(x_{j+1}) - \phi(x_j) = x_{j+1}^T P x_{j+1} - x_j^T P x_j \leq -f_0(x_j, u_j).$$

It follows from the definition of the terminal region ($x_j^T P x_j \leq c$) that

$$x_{j+1}^T P x_{j+1} - c \leq -(|x_j|^2 + |u_j|^2).$$

This is equivalent to

$$x_{j+1}^T P x_{j+1} \leq c - \underbrace{(|x_j|^2 + |u_j|^2)}_{\geq 0} \leq c.$$

Consequently, $x_{k+1} \in \mathcal{X}$ and the terminal region $\mathcal{X}_f$ is invariant.

d) TODO

$$\begin{aligned}
y &= \begin{bmatrix} x_1^\top & \cdots & x_N^\top & u_0 & \cdots & u_{N-1} \end{bmatrix}^\top \in \mathbb{R}^{3N} \\
H &= 2h \begin{bmatrix} \mathbf{I}_{2N} & 0 \\ 0 & \alpha \mathbf{I}_N \end{bmatrix} \in \mathbb{R}^{3N \times 3N} \\
A_{ineq} &= 0 \\
b_{ineq} &= 0 \\
A_{eq} &= \begin{bmatrix} \mathbf{I}_2 & 0 & \cdots & 0 & -B_D & 0 & \cdots & 0 \\ -A_D & \mathbf{I}_2 & 0 & \cdots & 0 & -B_D & \ddots & \vdots \\ & \ddots & \ddots & 0 & \cdots & 0 & \ddots & 0 \\ 0 & & -A_D & \mathbf{I}_2 & 0 & \cdots & 0 & -B_D \end{bmatrix} \in \mathbb{R}^{2N \times 3N} \\
b_{eq} &= \begin{bmatrix} (A_D x_0)^\top & 0 & \cdots & 0 \end{bmatrix}^\top \in \mathbb{R}^{2N}
\end{aligned}$$

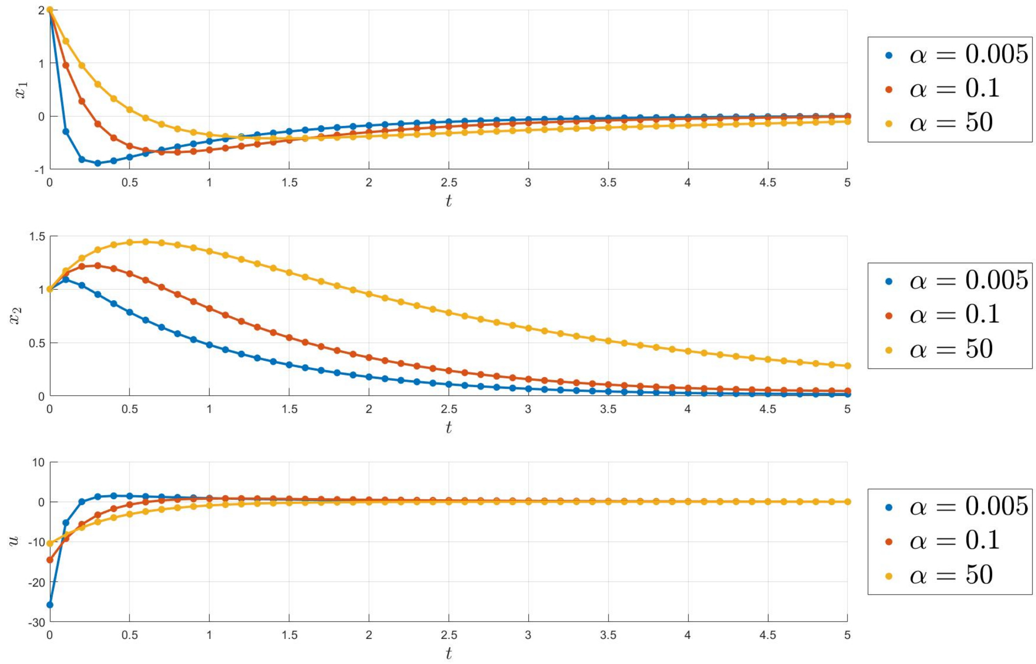

Fig. 1: Trajectories of the solutions for various α

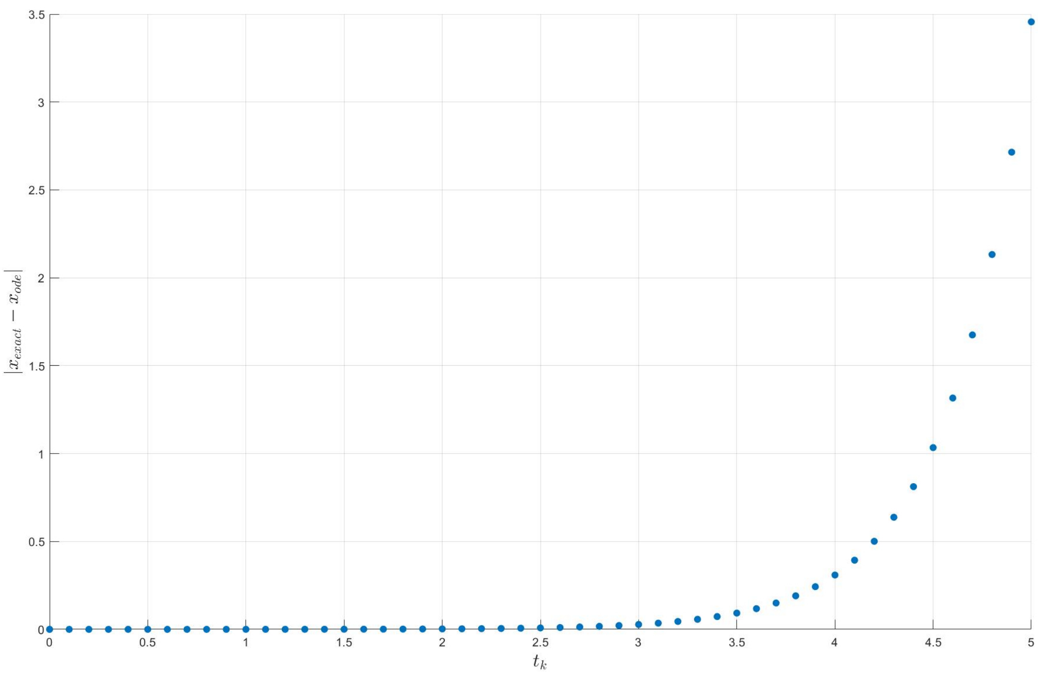

Fig. 2: Difference between the two solutions