

UNIVERSITÄT STUTTGART

OPTIMAL CONTROL

Solution of Homework Exercise 2

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Problem 2

- a) To compute the equilibrium of the unforced system we consider the equation

$$\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = A \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$

with A as defined on the exercise sheet. The matrix equation is equivalent to the following system of equations:

$$\begin{aligned} x^1 + 3 \cdot x^2 &= x^1 \\ -0.5 \cdot x^1 + x^2 &= x^2. \end{aligned}$$

We receive immediately the unique solution $x^1 = x^2 = 0$, so the euibrilbrium of the unforced system lies in the origin. The stability of the equilibrium will be investigated by computing the eigenvalues of A :

$$\det(\lambda I - A) = \det \left(\begin{bmatrix} \lambda - 1 & -3 \\ 0.5 & \lambda - 1 \end{bmatrix} \right) = (\lambda - 1)^2 + 1.5 = \lambda^2 - 2\lambda + 2.5 \stackrel{!}{=} 0.$$

The solution of the eigenvalue equation is $\lambda_{1/2} = 1 \pm j\frac{\sqrt{6}}{2}$, so the unforced system is not stable.

- b) The given discrete-time system can be formulated in the MPC scheme as follows:

$$\begin{aligned} u_{MPC}(\cdot, x(t_j)) = \underset{u}{\operatorname{argmin}} \quad & \sum_{k=0}^2 x_j^\top x_j + u_j^2 + x_3^\top P x_3 \\ \text{s.t.} \quad & x_{j+1} = A x_j + B u_j \\ & |u_j| \leq 1, x_j \in \mathbb{R}^2, x_3 \in \mathcal{X}_f \end{aligned} \tag{1}$$

with P , A , B and \mathcal{X}_f as defined on the exercise sheet.

- c) Now, we show that the MPC scheme in b) gives a stabilizing controller that is defined by $u_j = K x_j$.

1. The eigenvalue of P is used as a lower limit in the following equation to show the feasibility of the controller:

$$\lambda_{\min}(P)|x|^2 \leq x^\top Px.$$

For all $x \in \mathcal{X}_f$ the relation

$$x^\top Px \leq c = \frac{\lambda_{\min}(P)}{|K|^2}$$

holds by definition. After deviding by $\lambda_{\min}(P)$ we obtain the input constraint

$$|u|^2 = |K|^2|x|^2 \leq 1.$$

That means that $|u| \leq 1$, so the controller is feasible.

2. By inserting the given definitions we gain

$$\begin{aligned} \phi(x_{j+1}) - \phi(x_j) &= x_{j+1}^\top Px_{j+1} - x_j^\top Px_j \\ &= ((A - BK)x_j)^\top P(A - BK)x_j - x_j^\top Px_j \\ &= x_j^\top \underbrace{[(A - BK)^\top P(A - BK) - P]}_L x_j \\ &\leq \lambda_{\max}(L)|x|^2. \end{aligned}$$

The eigenvalues of L were computed via MATLAB. Both of them are smaller than -7. Besides, we use that

$$f_0(x_j, u_j) = |x_j|^2 + |K|^2|x_j|^2 = (1 + |K|^2)|x_j|^2 = 3.05|x_j|^2.$$

It follows that

$$\begin{aligned} \phi(x_{j+1}) - \phi(x_j) &\leq \lambda_{\max}(L)|x|^2 \\ &< -7|x|^2 \\ &< -3.05|x|^2 \\ &= -f_0(x_j, u_j) \end{aligned}$$

which was to be proven.

3. For showing that the terminal region \mathcal{X}_f is invariant, we need to demonstrate that $x_{k+1} \in \mathcal{X}_f$ if $x_k \in \mathcal{X}_f$. From the previous task we know that

$$\phi(x_{j+1}) - \phi(x_j) = x_{j+1}^\top P x_{j+1} - x_j^\top P x_j \leq -f_0(x_j, u_j).$$

It follows from the definition of the terminal region ($x_j^\top P x_j \leq c$) that

$$x_{j+1}^\top P x_{j+1} - c \leq -(|x_j|^2 + |u_j|^2).$$

This is equivalent to

$$x_{j+1}^\top P x_{j+1} \leq c - \underbrace{(|x_j|^2 + |u_j|^2)}_{\geq 0} \leq c.$$

Consequently, $x_{k+1} \in \mathcal{X}$ and the terminal region \mathcal{X}_f is invariant.

- d) The optimal control problem from b) can be written as a quadratically constrained quadratic program with the following variables

$$z = \begin{bmatrix} x_0^\top & x_1^\top & x_2^\top & x_3^\top & u_0 & u_1 & u_2 \end{bmatrix}^\top \in \mathbb{R}^{11}$$

$$H = \begin{bmatrix} \mathbf{I}_2 & 0 & & \cdots & & & 0 \\ 0 & \mathbf{I}_2 & & & & & \\ & & \mathbf{I}_2 & \ddots & & & \\ \vdots & & \ddots & P & & & \vdots \\ & & & & 1 & & \\ & & & & & 1 & 0 \\ 0 & & & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{11 \times 11}$$

$$A_{eq} = \begin{bmatrix} \mathbf{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -A & \mathbf{I}_2 & 0 & 0 & -B & 0 & 0 \\ 0 & -A & \mathbf{I}_2 & 0 & 0 & -B & 0 \\ 0 & 0 & -A & \mathbf{I}_2 & 0 & 0 & -B \end{bmatrix} \in \mathbb{R}^{8 \times 11}$$

$$\begin{aligned}
b_{eq} &= \begin{bmatrix} \bar{x}^\top & 0 & \dots & 0 \end{bmatrix}^\top \in \mathbb{R}^{11} \\
A_{ineq} &= \begin{bmatrix} 0 & \dots & 0 & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \\ \vdots & & \vdots & 0 & 0 & 1 \\ & & & -1 & 0 & 0 \\ & & & 0 & -1 & 0 \\ 0 & \dots & 0 & 0 & 0 & -1 \end{bmatrix} \in \mathbb{R}^{6 \times 11} \\
b_{ineq} &= \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}^\top \in \mathbb{R}^6 \\
T &= \begin{bmatrix} 0 & \dots & 0 \\ & 0 & & \\ & & 0 & \\ \vdots & & P & \vdots \\ & & & 0 \\ & & & & 0 \\ 0 & \dots & & & 0 \end{bmatrix} \in \mathbb{R}^{11 \times 11}.
\end{aligned}$$

The problem is convex, because

- the matrixes H and T are positive definite (P has only positive eigenvalues) and
 - linear equations are always convex.
- e) Solving the optimization problem from d) in MATLAB, we obtain the results shown in Fig. ???. One can see that the calculated input signal is large at first to move the state quickly to the equilibrium. The input constraint prevents higher magnitudes of the input than one. After approximately 10 timesteps both components of the state are moved to the origin and the input signal also equals zero.
- f) If we insert the other initial condition, we observe that that qualitative evolution of the state and the determined input (Fig. ???) is similar compared

to the previous plot. That makes sense because both initial condition do not differ much from each other. The higher absolute value of the new initial condition would request a higher input signal in the unconstrained, optimal case, but we consider a limited input signal.

- g) In contrast to the MPC algorithm, the MATLAB solver for linear-quadratic problems does not include an input constraint. That is why the absolute value of the input signal is not restricted in contrast to the input in the previous tasks. In the LQ solution, we observe that the input is higher than one at first (Fig. ???), because that leads to a smaller value of the cost functional. Another difference is that the MPC scheme only minimizes the cost functional till the prediction horizon which is $N = 3$ in our case. However, the prediction horizon does not have a huge impact on the trajectories and the qualitative evolution of the results stays the same.