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# PERFORMANCE BOUNDS FOR LEVEL-ORIENTED TWO-DIMENSIONAL PACKING ALGORITHMS*

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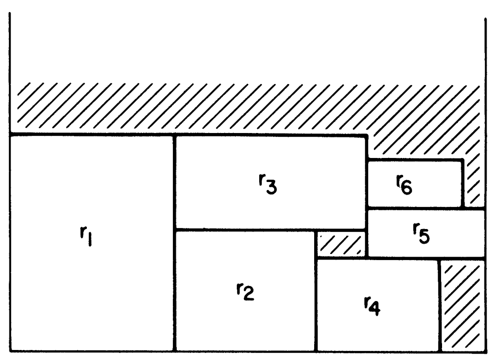
**Abstract.** We analyze several "level-oriented" algorithms for packing rectangles into a unit-width, infinite-height bin so as to minimize the total height of the packing. For the three algorithms we discuss, we show that the ratio of the height obtained by the algorithm to the optimal height is asymptotically bounded, respectively, by 2, 1.7, and 1.5. The latter two improve substantially over the performance bounds for previously proposed algorithms. In addition, we give more refined bounds for special cases in which the widths of the given rectangles are restricted and in which only squares are to be packed.

**Key words.** level-oriented packing algorithm, bin-packing, two-dimensional packing

**1. Introduction.** We consider the following two-dimensional packing problem, first proposed in [1]: Given a collection of rectangles, and a bin with fixed width and unbounded height, pack the rectangles into the bin so that no two rectangles overlap and so that the height to which the bin is filled is as small as possible. We shall assume that the given rectangles are oriented, each having a specified side that must be parallel to the bottom of the bin. We also assume, with no loss of generality, that the bin width has been normalized to 1. Fig. 1 provides an illustration, where the first dimension




*Dimensions for* $r_i$, $1 \leqq i \leqq 6$: $\frac{7}{20} \times \frac{9}{20}$, $\frac{3}{10} \times \frac{1}{4}$, $\frac{2}{5} \times \frac{1}{5}$, $\frac{1}{4} \times \frac{1}{5}$, $\frac{1}{4} \times \frac{1}{10}$, $\frac{1}{5} \times \frac{1}{10}$

FIG. 1. *A packing of oriented rectangles in a unit-width bin.*

specified for each rectangle corresponds to the side that must be parallel to the bottom of the bin (we use this convention throughout the paper).

This problem is a natural generalization of the one-dimensional bin-packing problem studied in [9]. Indeed, if all rectangles are required to have the same height, then the two problems coincide. On the other hand, the case in which all rectangles have the same *width* corresponds to the well-known makespan minimization problem of combinatorial scheduling theory [3]. Both these restricted problems are known to be NP-complete [3], [7], from which it follows trivially that the two-dimensional packing

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problem is also NP-complete. For this reason we shall focus on fast heuristic algorithms for solving this problem, seeking to prove close bounds on the extent to which they can deviate from optimality. Those readers unfamiliar with this "approximation algorithms" approach may wish to consult one or more of [4], [6], [7], for general background and for examples of other problems to which it has been applied.

For $L$ an arbitrary list of rectangles, all assumed to have width no more than 1, let $\mathrm{OPT}(L)$ denote the minimum possible bin height within which the rectangles in $L$ can be packed, and let $A(L)$ denote the height actually used by a particular algorithm when applied to $L$. The results in [1] are concerned primarily with demonstrating *absolute performance bounds* for various algorithms $A$, i.e., bounds of the form

$$A(L) \leqq \beta \cdot \mathrm{OPT}(L)$$

for all lists $L$. In contrast, we will be interested in proving *asymptotic performance bounds* of the form

$$A(L) \leqq \beta \cdot \mathrm{OPT}(L) + \gamma$$

for all lists $L$. This is of interest because in many cases the worst absolute performance can be achieved only by highly specialized, "small" examples. The constant $\beta$ in such an asymptotic bound is intended to characterize the behavior of the algorithm as the ratio between $\mathrm{OPT}(L)$ and the maximum height rectangle in $L$ goes to infinity. For this purpose, we may, without loss of generality, normalize the height of the tallest rectangle in $L$ to 1. (Of course, any height other than 1 would serve just as well for this normalization; a different choice would affect only the additive constant $\gamma$ in our bounds, leaving the multiplicative constant $\beta$ unchanged.)

The difference between these two types of bounds is illustrated by the Next-Fit Decreasing Height (NFDH) algorithm, to be defined in § 2, where it is known [2], [8] that

$$\mathrm{NFDH}(L) \leqq 3 \cdot \mathrm{OPT}(L)$$

for all lists $L$, and that there exist lists $L$ for which $\mathrm{NFDH}(L)$ is arbitrarily close to $3 \cdot \mathrm{OPT}(L)$. We shall show, however, that if the height as well as the width of each rectangle is no more than 1, then

$$\mathrm{NFDH}(L) \leqq 2 \cdot \mathrm{OPT}(L) + 1,$$

for all $L$, and the multiplicative constant 2 cannot be improved upon. In the case of NFDH, as with the other algorithms we shall consider, asymptotic performance bounds seem to provide more accurate and useful information, properly relegating "transient" effects to the additive constant. (However, as we note in the conclusion, many of our asymptotic results also provide good absolute bounds.)

In addition to proving asymptotic bounds that hold for all lists $L$, we will also be interested in proving such bounds for special cases in which the rectangles in $L$ satisfy additional width restrictions or all are required to be squares. Section 2 examines several approximation algorithms that are natural analogues of the one-dimensional packing algorithms considered in [9], and provides best possible performance bounds for them. Perhaps surprisingly, these bounds turn out to be essentially the same as those for the one-dimensional case. In § 3 we propose a new algorithm for the two-dimensional packing problem, and prove tight bounds on its performance. These bounds demonstrate that the new algorithm is a substantial improvement over the algorithms in § 2.

Finally, in § 4, we compare the performance bounds for our algorithms with those for the non-level-oriented algorithms of [1] and discuss several variants of our algorithms that might perform somewhat better in practice.

**2. The NFDH and FFDH algorithms.** The packing algorithms that we analyze in this section both assume that the rectangles in the list $L$ are ordered by decreasing (actually, nonincreasing) height, and they pack the rectangles in the order given by $L$ so as to form a sequence of *levels*. All rectangles will be placed with their bottoms resting on one of these levels. The first level is simply the bottom of the bin. Each subsequent level is defined by a horizontal line drawn through the top of the first (and hence maximum height) rectangle placed on the previous level. Notice how this corresponds with one-dimensional bin-packing; the horizontal slice determined by two adjacent levels can be regarded as a bin (lying on its side) whose width is determined by the maximum height rectangle placed in that bin. The following two level algorithms are suggested by analogous algorithms studied for one-dimensional bin-packing:

(1) *Next-Fit Decreasing-Height* (NFDH). With this algorithm, rectangles are packed left-justified on a level until there is insufficient space at the right to accommodate the next rectangle. At that point, the next level is defined, packing on the current level is discontinued, and packing proceeds on the new level.

(2) *First-Fit Decreasing-Height* (FFDH). At any point in the packing sequence, the next rectangle to be packed is placed left-justified on the first (i.e., lowest) level on which it will fit. If none of the current levels will accommodate this rectangle, a new level is started as in the NFDH algorithm.

Fig. 2 shows the results of applying the two packing rules to the same list. The essential difference between them is that whereas FFDH can always return to a previous level for packing a new rectangle, NFDH always places subsequent rectangles at or above the currrent level.

Some notation will be useful for conducting our analysis of these two algorithms. Let the list $L$ be given as $r_1, r_2, \cdots, r_n$, and let $w(r)$ and $h(r)$ denote the width and height of rectangle $r$. By our previous assumptions we have $0 \leqq w(r) \leqq 1$ and $0 \leqq h(r) \leqq 1$, and we have $h(r_1) \geqq h(r_2) \geqq \cdots \geqq h(r_n)$. The space between two consecutive levels will be called a *block*. Packings will be regarded as a sequence of blocks $B_1, B_2, \cdots, B_k$, where the index increases from the bottom to the top of the packing. Let $A_i$ denote the total area of the rectangles in block $B_i$, and let $H_i$ denote the height of block $B_i$. Note that, by the manner in which these algorithms define levels, we have $H_1 \geqq H_2 \geqq \cdots \geqq H_k$.

Consider a particular rectangle $r$ packed in block $B_i$. If block $B_{i+1}$ was nonempty at the time $r$ was packed, then we say that $r$ is a *fallback item*. If, on the other hand, block $B_i$ was the highest nonempty block at the time $r$ was packed, then $r$ is called a *regular item*. Note that all rectangles in an NFDH packing are regular items, and in any block in an FFDH packing all regular items are taller than and appear to the left of any fallback items in that block.

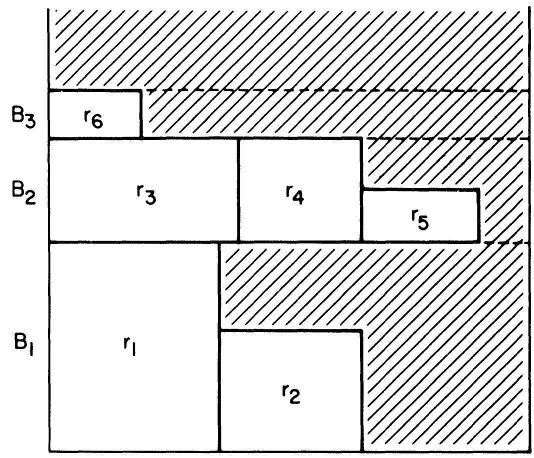
Our first result provides a tight bound on the performance of the NFDH algorithm.

THEOREM 1. *For any list $L$ ordered by nonincreasing height,*

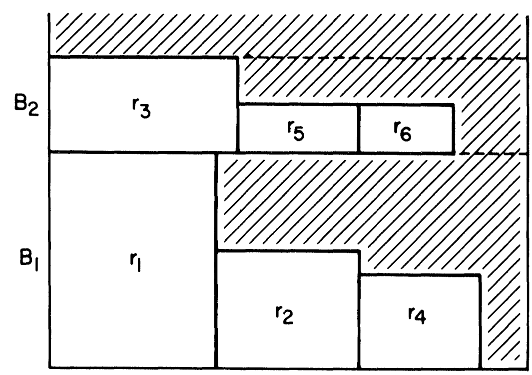
$$\mathrm{NFDH}(L) \leqq 2 \cdot \mathrm{OPT}(L) + 1.$$

*Moreover, the multiplicative constant* 2 *is the smallest possible.*

*Proof.* Consider the NFDH packing of such a list $L$, with blocks $B_1, B_2, \cdots, B_t$. For each $i$, let $x_i$ be the width of the first rectangle in $B_i$, and $y_i$ be the total width of rectangles in $B_i$. For each $i < t$, the first rectangle in $B_{i+1}$ does not fit in $B_i$. Therefore $y_i + x_{i+1} > 1$, $1 \leq i < t$. Since each rectangle in $B_i$ has height at least $H_{i+1}$, and the first




NFDH$(L)=\frac{3}{4}$




FFDH$(L)=\frac{13}{20}$

FIG. 2. NFDH *and* FFDH *algorithms applied to the list* $L=(r_1, r_2, r_3, r_4, r_5, r_6)$, *with the dimensions of* $r_i$ *as given in Fig.* 1.

rectangle in $B_{i+1}$ has height $H_{i+1}$, $A_i+A_{i+1}\geqq H_{i+1}(y_i+x_{i+1})>H_{i+1}$. Therefore, if $A$ denotes the total area of all the rectangles,

$$\begin{aligned}\text{NFDH}(L)&=\sum_{i=1}^{t} H_i\leqq H_1+\sum_{i=1}^{t-1} A_i+\sum_{i=2}^{t} A_i\\&\leqq H_1+2A\\&\leqq 1+2\ \text{OPT}(L),\end{aligned}$$

which is the desired bound.

Examples showing that the coefficient of 2 is smallest possible are derived trivially from the corresponding examples for the Next-Fit algorithm of one-dimensional bin-packing. The list $L$ has $n$ rectangles, where $n$ is a multiple of 4. All the rectangles have height 1, the odd numbered ones have width $\frac{1}{2}$, and the even numbered ones have width $\varepsilon$, for a suitably small $\varepsilon>0$. The optimum and NFDH packings of $L$ are shown in Fig. 3. In this case we have NFDH$(L)=n/2$ and OPT$(L)=n/4+1$, so the ratio NFDH$(L)/$OPT$(L)$ can be made arbitrarily close to 2 by choosing $n$ suitably large and $\varepsilon$ suitably small. □

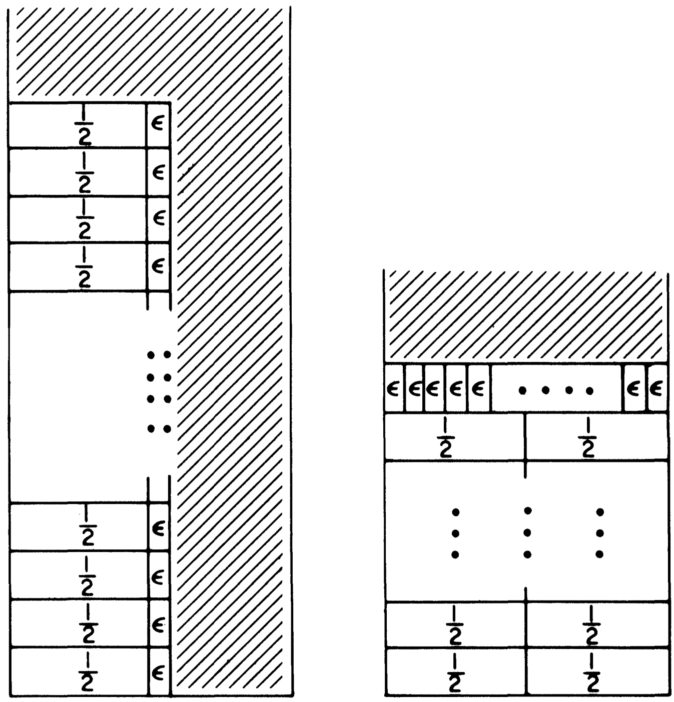



FIG. 3. *Worst-case examples for Theorem* 1.

It is perhaps surprising that the multiplicative constant of 2 in Theorem 1 is the same as the multiplicative constant for Next-Fit in the one-dimensional case. One might expect the two-dimensional algorithm to perform more poorly, since only the first item in a block need have height equal to the block's height, and there may be wasted space above the subsequent items that does not correspond to any waste in the one-dimensional case.

This wasted space, however, can only affect the additive constant in our result. Every item in $B_i$, $1 \leqq i \leqq t$, must have height at least $H_{i+1}$ (set $H_{t+1} = 0$ by convention). Thus the total wasted space in $B_i$ *above* items in $B_i$ is at most $H_i - H_{i+1}$. The cumulative waste is consequently bounded by

$$\sum_{i=1}^{t} (H_i - H_{i+1}) = H_1 - H_{t+1} = H_1 = 1$$

(by our normalization assumption on heights). This same "collapsing sum" principle is at work in all results of this paper, and helps explain why in general the multiplicative constants do not change as we go from the one- to the two-dimensional case (although the proof techniques certainly do).

We turn now to the FFDH algorithm. It is routine to prove that FFDH($L$) $\leqq$ NFDH($L$) for all lists $L$. In the following we show that the worst-case bounds are significantly lower for the FFDH algorithm. We first prove the bound for the general case, and then we consider the special cases in which all rectangles have width no more than some fixed $\alpha < 1$ and in which all the rectangles are squares. The multiplicative constants we obtain in all three situations are best possible, and equal (when relevant) the corresponding constants in the one-dimensional case [9].

THEOREM 2. *For any list L ordered by nonincreasing height,*

$$\mathrm{FFDH}(L) \leqq 1.7 \cdot \mathrm{OPT}(L) + 1.$$

*Furthermore, the multiplicative constant* 1.7 *is the smallest possible.*

*Proof.* The proof is based on the analogous proof for the First-Fit algorithm of one-dimensional bin-packing [5], [9]. We begin by defining the following weighting function.

$$W(x) = \begin{cases} \frac{6}{5}x & \text{if } 0 \leqq x \leqq \frac{1}{6}, \\ \frac{9}{5}x - \frac{1}{10} & \text{if } \frac{1}{6} < x \leqq \frac{1}{3}, \\ \frac{6}{5}x + \frac{1}{10} & \text{if } \frac{1}{3} < x \leqq \frac{1}{2}, \\ \frac{6}{5}x + \frac{4}{10} & \text{if } \frac{1}{2} < x \leqq 1. \end{cases}$$

We extend this function to rectangles $r$ by writing $W(r) = W(w(r))$, and set

$$A = \sum_{r \in L} h(r) \cdot W(r).$$

It is proved in [5] that no collection of numbers $x$ summing to 1 or less can have $W(x)$ summing to more than 1.7. We can apply this result to our case by cutting the optimal packing into horizontal slices, formed by drawing a line through the top and bottom of each rectangle. Summing over all the slices, we have $A \leqq 1.7\ \mathrm{OPT}(L)$.

Thus, all that remains is to show that $A \geqq \mathrm{FFDH}(L) - 1$. Let $T_1$ be the set of blocks in the FFDH packing whose first rectangle has width at most $\frac{1}{2}$, and $T_2$ the set of blocks whose first rectangle has width greater than $\frac{1}{2}$. For $i = 1, 2$, let $H(T_i)$ be the total height of blocks in $T_i$. Since the first item in block $B_i$ has height $H_i$, and since $W(x) > 1$ for $x > \frac{1}{2}$, we have

$$\sum_{B \in T_2} \sum_{r \in B} h(r) \cdot W(r) \geqq H(T_2).$$

We will show that

$$\sum_{B \in T_1} \sum_{r \in B} h(r) \cdot W(r) \geqq H(T_1) - 1.$$

Consequently $A \geqq H(T_1) + H(T_2) - 1 = \mathrm{FFDH}(L) - 1$, as desired.

Let $L_1$ be the sublist of $L$ consisting of the rectangles in blocks of $T_1$. Note that the FFDH packing of $L_1$ would yield a set of blocks identical to $T_1$. Let $B_1, B_2, \cdots, B_t$ denote these blocks, with index increasing from the bottom to the top of the packing, and let $H_i$ be the height of $B_i$, $1 \leqq i \leqq t$, with $H_{t+1} = 0$ by convention. Classify items as regular items or fallback items according to their roles in the FFDH packing of $L_1$. Let $f_i$ be the first regular item in $B_i$, and let $R_i$ be the set of all regular items in $B_i$. For $1 < i \leqq t$ and $1 \leqq j < i$, define $F_{ij}$ to be the set of fallback items packed in $B_j$ after the last regular item was packed in $B_{i-1}$ but before the first regular item was packed in $B_i$. For $1 < i \leqq t$ and $1 \leqq j < i$, define $S_{ij}$ to be the set of all fallback items packed in $B_j$ after the first regular item was packed in $B_i$ but before the last regular item was packed in $B_i$. See Fig. 4. Note that the sets $F_{ij}$ and $S_{ij}$ are all disjoint, and that $F_{i,i-1} = \varnothing$, $1 < i \leqq t$. Moreover, we have

$$L_1 = \bigcup_{i=1}^{t} \left[ R_i \cup \bigcup_{j<i} F_{ij} \cup \bigcup_{j<i} S_{ij} \right].$$

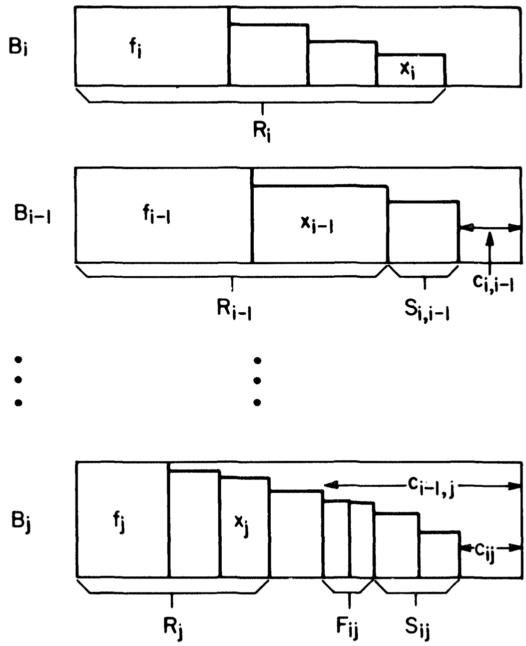



FIG. 4. *Illustration for proof of Theorem 2 (blocks have been separated for display purposes). Note that all items in $F_{ij}$ come between $x_{i-1}$ and $f_i$, while all items in $S_{ij}$ and $S_{i,i-1}$ come between $f_i$ and $x_i$ in the list.*

For each $i$, $1<i\leqq t$, and each $j$, $1\leqq j<i$, define the *coarseness* $c_{ij}$ to be the width of the empty space at the bottom right end of $B_j$ when the last regular item is packed in $B_i$. Again see Fig. 4. Note that the following relations hold:

$$w(f_i)\geqq c_{ij}+\sum_{r\in S_{ij}} w(r), \qquad 1<i\leqq t, \quad 1\leqq j<i, \tag{2.1}$$

$$c_{ij}=c_{i-1,j}-\sum_{r\in S_{ij}\cup F_{ij}} w(r), \qquad 2<i\leqq t, \quad 1\leqq j<i. \tag{2.2}$$

Let us define $c_i=\max_{j<i} c_{ij}$ for $1<i\leqq t$, with $c_1=0$. We claim that for every $i$, $1<i<t$, there exists an $i'<i$ such that

$$\sum_{r\in R_{i-1}} W(r)+\tfrac{6}{5}\sum_{r\in F_{ii'}\cup S_{ii'}} w(r)\geqq 1+\tfrac{6}{5}(c_{i-1}-c_{ii'}). \tag{2.3}$$

Using this claim (to be proved later), we can complete the proof as follows: We first observe that

$$\sum_{r\in L_1} h(r)\cdot W(r)\geqq \sum_{i=1}^{t} H_{i+1}\sum_{r\in R_i} W(r)+\sum_{i=1}^{t} W(f_i)(H_i-H_{i+1})$$
$$+\sum_{i=2}^{t} H_i\sum_{r\in F_{ii'}} W(r)+\sum_{i=2}^{t} H_{i+1}\sum_{r\in S_{ii'}} W(r).$$

This is because every regular item in $B_i$ has height at least $H_{i+1}$, $1\leqq i\leqq t$ (recall that by convention $H_{t+1}=0$), while the first item in $B_i$ has height $H_i$, and all elements of $F_{ii'}$ and $S_{ii'}$ have heights at least $H_i$ and $H_{i+1}$, respectively, $1<i\leqq t$.

Now we use (2.1) and the fact that for all rectangles $r$, $W(r) \geqq \frac{6}{5}w(r)$, to conclude that the above sum is at least as large as

$$\sum_{i=2}^{t}\left[H_i \sum_{r\in R_{i-1}} W(r) + \tfrac{6}{5}(H_i - H_{i+1})\Big(c_{ii'} + \sum_{r\in S_{ii'}} w(r)\Big) + \tfrac{6}{5}H_i \sum_{r\in F_{ii'}} w(r) + \tfrac{6}{5}H_{i+1} \sum_{r\in S_{ii'}} w(r)\right]$$

$$\geqq \sum_{i=2}^{t}\left[H_i \sum_{r\in R_{i-1}} W(r) + \tfrac{6}{5}(H_i - H_{i+1})c_{ii'} + \tfrac{6}{5}H_i \sum_{r\in F_{ii'}\cup S_{ii'}}\right]$$

$$\geqq \sum_{i=2}^{t} [H_i + \tfrac{6}{5}H_i(c_{i-1} - c_{ii'}) + \tfrac{6}{5}(H_i - H_{i+1})c_{ii'}],$$

by (2.3). We thus conclude that

$$\sum_{r\in L_1} h(r)\cdot W(r) \geqq \sum_{i=2}^{t} [H_i - \tfrac{6}{5}H_{i+1}c_{ii'} + \tfrac{6}{5}H_i c_{i-1}]$$

$$\geqq H(T_1) - H_1 - \tfrac{6}{5}\sum_{i=2}^{t} H_{i+1}c_{ii'} + \tfrac{6}{5}\sum_{i=1}^{t-1} H_{i+1}c_i$$

$$\geqq H(T_1) - H_1 + \tfrac{6}{5}\sum_{i=2}^{t-1} H_{i+1}(c_i - c_{ii'}) \geqq H(T_1) - 1,$$

as desired, since $c_i \geqq c_{ii'}$ and $c_1 = H_{t+1} = 0$ by definition, and since $H_1 = 1$ by our normalization assumption (the tallest item has height exactly 1).

Thus it only remains to prove the claim, i.e., to show that for any $i$, $1 < i \leqq t$, there exists an $i' < i$ such that (2.3) holds. First, suppose $\sum_{r\in R_{i-1}} W(r) \geqq 1$. If $c_i \geqq c_{i-1}$, then for some $i' < i$, $c_i = c_{ii'} \geqq c_{i-1}$, yielding

$$\sum_{r\in R_{i-1}} W(r) + \tfrac{6}{5}\sum_{r\in F_{ii'}\cup S_{ii'}} w(r) \geqq 1 \geqq 1 + \tfrac{6}{5}(c_{i-1} - c_{ii'}),$$

as desired. If $c_i < c_{i-1}$, then $c_{i-1} > 0$ and hence $i-1 \geqq 2$, so for some $i' < i-1 < i$, $c_{i-1} = c_{i-1,i'}$. Then by (2.2) $F_{ii'} \cup S_{ii'}$ contains items of total width at least $c_{i-1,i'} - c_{ii'}$, and once again (2.3) follows.

Now suppose $\sum_{r\in R_{i-1}} W(r) < 1$. We apply Lemma 4 of [5], which says that if $B$ is a set of one or more numbers $x$ satisfying $c < x \leqq 1$ and $\sum_{x\in B} W(x) < 1$, then either $|B| = 1$ and the single element $x \in B$ satisfies $x \leqq \frac{1}{2}$, or else

$$\sum_{x\in B} x \leqq 1 - c - \tfrac{5}{6}\Big(1 - \sum_{x\in B} W(x)\Big).$$

Consider the set $R_{i-1}$. By the definition of the packing $T_1$, the first item in $R_{i-1}$ must have width $\frac{1}{2}$ or less, which implies by (2.1) that $c_{i-1} \leqq \frac{1}{2}$. Since every item in $R_{i-1}$ must have width exceeding $c_{i-1}$, the hypotheses of the lemma apply to the set of widths of items in $R_{i-1}$, with $c = c_{i-1}$, and so one of the two possibilities must apply. The first possibility cannot occur since it would imply that $w(f_i) > \frac{1}{2}$, contrary to the definition of $T_1$. Thus we conclude,

$$\sum_{r\in R_{i-1}} w(r) \leqq 1 - c_{i-1} - \tfrac{5}{6}\Big(1 - \sum_{r\in R_{i-1}} W(r)\Big).$$

Letting $i' = i-1$, we have

$$\begin{aligned} c_{ii'} &= 1 - \sum_{r \in R_{i-1}} w(r) - \sum_{r \in F_{ii'} \cup S_{ii'}} w(r) \\ &\geqq 1 - \left(1 - c_{i-1} - \tfrac{5}{6}\left(1 - \sum_{r \in R_{i-1}} W(r)\right)\right) - \sum_{r \in F_{ii'} \cup S_{ii'}} w(r) \\ &\geqq c_{i-1} + \tfrac{5}{6}\left(1 - \sum_{r \in R_{i-1}} W(r)\right) - \sum_{r \in F_{ii'} \cup S_{ii'}} w(r). \end{aligned}$$

which once again implies (2.3), as desired. This completes the proof of the claim and hence of the performance bound for FFDH.

Examples showing that the multiplicative constant 1.7 is the smallest possible follow immediately from the corresponding examples in [9], simply by taking the sizes given there to be widths and taking all heights to be 1. □

THEOREM 3. *Let $L$ be any list of rectangles ordered by nonincreasing height such that no rectangle in $L$ has width exceeding $1/m$ for some fixed $m \geqq 2$. Then*

$$\mathrm{FFDH}(L) \leqq \left(1 + \frac{1}{m}\right) \mathrm{OPT}(L) + 1.$$

*Furthermore, the multiplicative constant $(1 + 1/m)$ is the smallest possible.*

*Proof.* We would like to argue that for each block $B_i$ of the FFDH packing, the total area $A_i$ of rectangles in $B_i$ is at least $(m/(m+1))H_{i+1}$. This would imply that

$$\mathrm{OPT}(L) \geqq \sum_{i=1}^{t-1} A_i \geqq \frac{m}{m+1} \sum_{i=1}^{t-1} H_{i+1} = \frac{m}{m+1}(\mathrm{FFDH}(L) - 1),$$

and the performance bound would follow. Unfortunately, it may be the case that for some $i < t$, $A_i < (m/(m+1))H_{i+1}$. It is the task of our proof to show that such shortfalls cannot hurt us.

Define $A_{ij}$ and $w_{ij}$, $1 < i < t$ and $i \leqq j < i$, to be the total area and width, respectively, of items packed in $B_j$ when the last regular item is packed in $B_i$ (note that $A_{ii}$ and $w_{ii}$ are the total area and width, respectively, of regular items in $B_i$). Define $\Delta_i = \max(0, m/(m+1) - w_{ij})$. We shall prove that for all $i$, $1 \leqq i \leqq t$,

$$\sum_{j=1}^{i} A_{ij} \geqq \frac{m}{m+1} \sum_{j=1}^{i} H_{j+1} - \Delta_i H_{i+1}, \tag{3.1}$$

where, as usual, $H_{t+1} = 0$ by convention.

The proof is by induction. Inequality (3.1) clearly holds for $i = 1$, in which case it reduces to

$$A_{11} \geqq H_2\left(\frac{m}{m+1} - \max\left(0, \frac{m}{m+1} - w_{11}\right)\right),$$

which is true since $A_{11} \geqq H_2 \cdot w_{11}$. Consider any $i$ satisfying $1 < i \leqq t$, and suppose (3.1) holds for $i-1$. If $\Delta_{i-1} = 0$, then we have

$$\sum_{j=1}^{i-1} A_{ij} \geqq \sum_{j=1}^{i-1} A_{i-1,j} \geqq \frac{m}{m+1} \sum_{j=1}^{i-1} H_{j+1},$$

and (3.1) will follow for arbitrary $i$ in the same way it did for $i = 1$. So suppose $\Delta_{i-1} > 0$. There remain two cases to consider.

(i) $B_i$ contains no regular item with width less than $1/(m+1)$. Then since the width of the first item in $B_i$ is at least $1/(m+1)+\Delta_{i-1}$, and there must be at least $m$ regular items, we have $w_{ii} \geqq m/(m+1)+\Delta_{i-1}$. We thus can conclude,

$$A_{ii} \geqq \left(\frac{m}{m+1}+\Delta_{i-1}\right) H_{i+1}+\left(\frac{1}{m+1}+\Delta_{i-1}\right)(H_i-H_{i+1})$$

$$\geqq \frac{m}{m+1} H_{i+1}+\Delta_{i-1} H_i.$$

Combining this with (3.1) for $i-1$, we obtain

$$\sum_{j=1}^{i} A_{ij} \geqq A_{ii}+\sum_{j=1}^{i-1} A_{i-1,j} \geqq \frac{m}{m+1} \sum_{j=1}^{i} H_{j+1}+\Delta_{i-1} H_i-\Delta_{i-1} H_i,$$

so (3.1) continues to hold for $i$ in case (i).

(ii) $B_i$ contains a regular item of width less than $1/(m+1)$. This can only happen if $B_{i-1}$ received fallback items of total width exceeding $\Delta_{i-1}$ before the last regular item in $B_i$ was packed. We thus have

$$A_{i,i-1}-A_{i-1,i-1} \geqq \Delta_{i-1} H_{i+1}.$$

Combining this inequality with (3.1) for $i-1$, and the fact that the first item in $B_i$ has width at least $1/(m+1)+\Delta_{i-1}$, we obtain

$$\begin{aligned}\sum_{j=1}^{i} A_{ij} &\geqq A_{ii}+\sum_{j=1}^{i-1} A_{i-1,j}+(A_{i,i-1}-A_{i-1,i-1})\\ &\geqq \left(\frac{m}{m+1}-\Delta_i\right) H_{i+1}+\left(\frac{1}{m+1}+\Delta_{i-1}\right)(H_i-H_{i+1})+\frac{m}{m+1} \sum_{j=1}^{i-1} H_{j+1}-\Delta_{i-1} H_i\\ &\qquad +\Delta_{i-1} H_{i+1}\\ &\geqq \frac{m}{m+1} \sum_{j=1}^{i} H_{j+1}-\Delta_i H_{i+1}.\end{aligned}$$

So (3.1) continues to hold for $i$ in this case also.

By induction we can thus conclude that

$$\begin{aligned}A \geqq \sum_{j=1}^{t} A_{t,j} &\geqq \frac{m}{m+1} \sum_{j=1}^{t} H_{j+1}-\Delta_t H_{t+1}\\ &=\frac{m}{m+1} \sum_{j=1}^{t} H_{j+1}=\frac{m}{m+1}(\mathrm{FFDH}(L)-1).\end{aligned}$$

The desired performance bound follows.

Examples showing that the multiplicative constant of $(1+1/m)$ is the smallest possible once again follow in a straightforward way from the corresponding one-dimensional examples in [9]. □

Our next result concerns the interesting special case in which only *squares* are being packed. Note that this case does not have a nontrivial one-dimensional counterpart—if all items have the same height they must also all have the same width, and the problem becomes trivial. Note also that for this special case we can no longer make our normalizing assumption that the tallest item has height 1, and must settle for assuming that no square has size (size = width = height) exceeding 1. Our result for squares is as follows.

THEOREM 4. *For all lists of squares ordered by nonincreasing size,*

$$\mathrm{FFDH}(L) \leqq \tfrac{3}{2}\mathrm{OPT}(L) + 1.$$

*Furthermore, the multiplicative constant* $\frac{3}{2}$ *is the smallest possible.*

*Proof.* We divide the blocks of the FFDH packing into three groups. $G_1$ contains all blocks up to the highest block that contains a square of size exceeding $\frac{1}{2}$ but no square in the range $(\frac{1}{3}, \frac{1}{2}]$. (If no such block exists, $G_1$ is empty.) $G_2$ consists of all blocks above $G_1$ that contain at least one square in the range $(\frac{1}{3}, \frac{1}{2}]$. Note that each block of $G_2$ will thus contain exactly two items of size exceeding $\frac{1}{3}$, except possibly the last block, which may contain only one. $G_3$ contains all remaining blocks. Note that these blocks are above all of those of $G_1$ and $G_2$. For $i = 1, 2, 3$, let $g_i$ denote the total height of blocks in $G_i$. We consider two cases, depending on the relative sizes of $g_1$ and $g_3$.

First, suppose $g_3 \leqq g_1/2$. Let us say that two items "overlap" in a packing if a horizontal line can be drawn which passes through the interiors of both items. Consider the first items in all the blocks of $G_1$. Note that none of them can overlap each other in an optimal packing, since all have size exceeding $\frac{1}{2}$. Furthermore, none of the items in all the blocks of $G_2$ with size exceeding $\frac{1}{3}$ can overlap any of these first squares from $G_1$ in an optimal packing. For if any such item from $G_2$ did, it would surely fit with the first square from the top block of $G_1$, and by definition of $G_1$ no such item did fit. Thus at least a total of $g_1$ of the height of the optimal packing is not overlapped by any item of size exceeding $\frac{1}{3}$ from $G_2$. If $g'$ is the total height of such items, we therefore must have $\mathrm{OPT}(L) \geqq g_1 + g'/2$. Since every block of $G_2$ has its height determined by the size of its first square, which is less than $\frac{2}{3}$, and since every block except possibly the last has a second square of size exceeding $\frac{1}{3}$, we have $g' \geqq \frac{3}{2}g_2 - \frac{1}{3}$. We thus conclude that $\mathrm{OPT}(L) \geqq g_1 + \frac{3}{4}g_2 - \frac{1}{6}$.

If $g_3 \leqq g_1/2$, then $\mathrm{FFDH}(L) = g_1 + g_2 + g_3 \leqq \frac{3}{2}g_1 + g_2 \leqq \frac{3}{2}(g_1 + \frac{3}{4}g_2 - \frac{1}{6}) + \frac{1}{4} \leqq \frac{3}{2}\mathrm{OPT}(L) + \frac{1}{4}$, and we are done. So suppose $g_3 > g_1/2$.

For $i = 1, 2, 3$, let $A_i$ denote the total area of squares in $G_i$, and let $A = A_1 + A_2 + A_3$. We will show that $\mathrm{FFDH}(L) \leqq \frac{3}{2}A + 1 \leqq \frac{3}{2}\mathrm{OPT}(L) + 1$, and thus complete the proof of the performance bound.

We first consider $A_3$. Let $p$ be the size of the first (hence largest) square packed in $G_3$, and let $m$ be such that $1/(m+1) < p \leqq 1/m$. Note that $m \geqq 3$. Since the widest square in $G_3$ has width (and height) no greater than $1/m$, the proof of Theorem 3 yields

$$A_3 \geqq \frac{m}{m+1}\left(g_3 - \frac{1}{m}\right) = \frac{m}{m+1}g_3 - \frac{1}{m+1}. \tag{4.2}$$

Now consider $A_2$. Any block of $G_2$ with two squares of width $x$ and $y$, $x \geqq y \geqq \frac{1}{3}$, is at least $(x^2 + y^2)/x$ full. Since this expression is minimized at $x = y = \frac{1}{3}$, the block is at least $\frac{2}{3}$ full. Only the top block of $G_2$ may fail to meet these conditions. If so, that block has height at most $\frac{1}{2}$ (or else it would be in $G_1$). Thus

$$A_2 \geqq \tfrac{2}{3}g_2 - \tfrac{1}{3}(\tfrac{1}{2}) = \tfrac{2}{3}g_2 - \tfrac{1}{6}. \tag{4.3}$$

Finally, consider $A_1$. We claim that each block of $G_1$ is at least $(m+2)/(2m+2)$ full, and hence

$$A_1 \geqq \frac{m+2}{2m+2}g_1. \tag{4.4}$$

Let $B$ be a block of $G_1$, and let $x > \frac{1}{2}$ be the size of the first square in $B$. If $m = 3$, either $x > \frac{2}{3}$ or $B$ contains a second square of size at least $p > 1/(m+1)$ and hence is at least

$(x^2+(1/(m+1))^2)/x$ full. In both cases it is easy to verify that $B$ is at least $(m+2)/(2m+2)=\frac{5}{8}$ full. For $m \geqq 4$, note that $B$ is filled to width at least $1-p$ by squares of size at least $p$, and is therefore at least $(x^2+(1-x-1/m)/(m+1))/x$ full. Since this expression is minimized at $x=\frac{1}{2}$, we see that the block is at least

$$\frac{1}{2}+\frac{2(\frac{1}{2}-1/m)}{m+1}=\frac{m+1+2-4/m}{2m+2}\geqq\frac{m+2}{2m+2}$$

full. Thus (4.4) holds. Combining it with the formulas for $A_3$ and $A_2$ and using the fact that $g_3 \geqq g_1/2$, we obtain

$$\begin{aligned}
A_1+A_2+A_3 &\geqq \frac{m+2}{2m+2}g_1+\tfrac{2}{3}g_2-\tfrac{1}{6}+\frac{m}{m+1}g_3-\tfrac{1}{4}\\
&\geqq \frac{m+2}{2m+2}g_1+\tfrac{2}{3}g_2+\tfrac{2}{3}g_3+\left(\frac{m}{m+1}-\tfrac{2}{3}\right)g_3-\tfrac{2}{3}\\
&\geqq \left(\left(\frac{m+2}{2m+2}+\tfrac{1}{2}\left(\frac{m}{m+1}-\tfrac{2}{3}\right)\right)\right)g_1+\tfrac{2}{3}(g_2+g_3-1)\\
&\geqq \left(\frac{2m+2}{2m+2}-\tfrac{1}{3}\right)g_1+\tfrac{2}{3}(g_2+g_3-1)=\tfrac{2}{3}(g_1+g_2+g_3-1)\\
&\geqq \tfrac{2}{3}(\mathrm{FFDH}(L)-1).
\end{aligned}$$

The performance bound follows.

The construction of examples showing that the multiplicative constant $\frac{3}{2}$ is the best possible is quite simple (see Fig. 5). For suitably small $\varepsilon>0$, the list $L$ consists of $n$

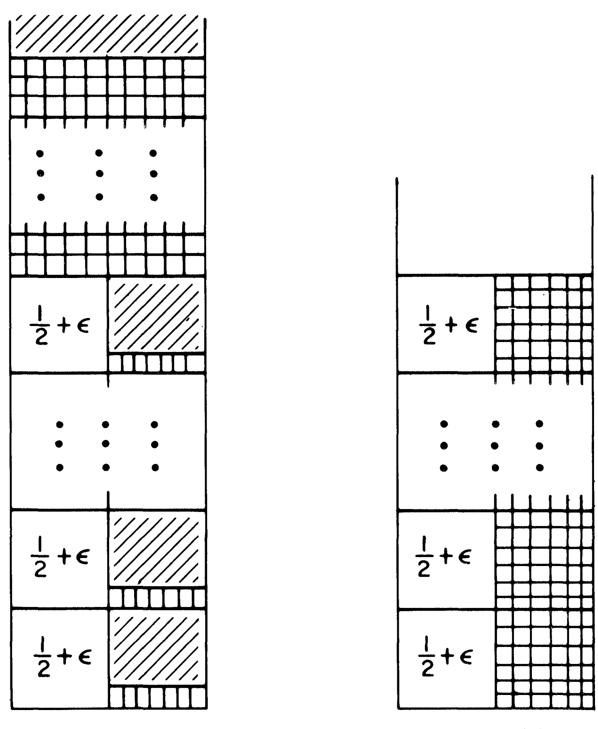



FIG. 5. *Worst-case examples for Theorem* 4.

squares of size $\frac{1}{2}+\varepsilon$ followed by $\lfloor(\frac{1}{2}-\varepsilon)/\varepsilon\rfloor \cdot \lfloor n(\frac{1}{2}+\varepsilon)/\varepsilon\rfloor$ squares of size $\varepsilon$. Then $\mathrm{OPT}(L) \leqq (n/2)+n\varepsilon$ and

$$\mathrm{FFDH}(L) \geqq ((n/2)+n\varepsilon)+\left\lceil\frac{\frac{1}{2}-\varepsilon}{\varepsilon}\right\rceil\left\lceil\frac{n/2}{\varepsilon}\right\rceil\varepsilon^2$$

$$\geqq (3n/4)-\varepsilon/2,$$

so $\mathrm{FFDH}(L)/\mathrm{OPT}(L)$ approaches $\frac{3}{2}$ as $\varepsilon$ approaches 0. □

**3. The Split-Fit algorithm.** In this section we describe an algorithm which is slightly more complicated than those analyzed in the previous section but which, as we shall see, performs significantly better. We call this algorithm Split-Fit (abbreviated SF), and it operates as follows.

Let $m \geqq 1$ be the largest integer such that all the given rectangles have width $1/m$ or less. Divide the given list $L$ of rectangles into two lists $L_1$ and $L_2$, both ordered by nonincreasing height, such that $L_1$ contains all the rectangles from $L$ that have width greater than $1/(m+1)$ and $L_2$ contains all the rectangles from $L$ that have width $1/(m+1)$ or less. Then pack the rectangles in $L_1$ using the FFDH algorithm (note that there will be *no* fallback items). Rearrange the blocks of this packing so that all blocks having total width more than $(m+1)/(m+2)$ are below all those blocks that have total width less than or equal to $(m+1)/(m+2)$. This leaves sufficient room that we can create a rectangle $R$ of width $1/(m+2)$ to the right of the latter group of blocks, as shown in Fig. 6.

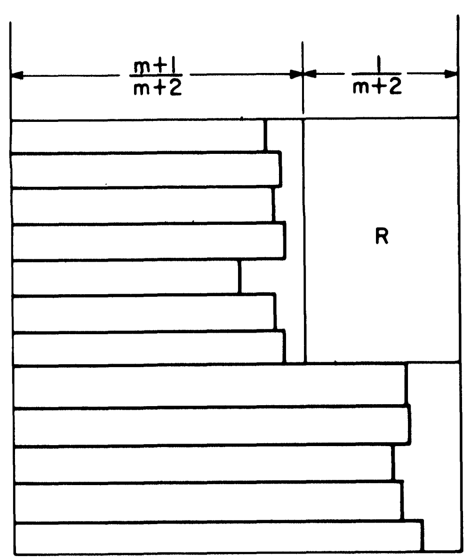



FIG. 6. *The rectangle $R$ created in the space left by the rearranged* FFDH *packing of $L_1$.*

Finally, pack the rectangles in $L_2$ using the FFDH algorithm, allowing new blocks to be established within $R$ (independently of the blocks in the packing of $L_1$) but not allowing any rectangle to overlap the top of $R$. For purposes of applying the FFDH algorithm, blocks within $R$ are regarded as being below those established above the packing of $L_1$. Thus, in general, some of the rectangles from $L_2$ will be packed in $R$, and the remainder (those that fail to fit in $R$) will be packed above the packing for $L_1$.

The following theorem characterizes the worst-case performance of Split-Fit.

THEOREM 5. *For any list L of rectangles, if all rectangles in L have width less than or equal to* $1/m$ (*m an integer*), *then*

$$\mathrm{SF}(L) \leqq \frac{m+2}{m+1}\,\mathrm{OPT}(L)+2.$$

*Furthermore, the multiplicative constant* $(m+2)/(m+1)$ *is the smallest possible.*

*Proof.* Let $T$ denote the region of the SF packing that contains rectangles from $L_1$ (not including $R$), and let $S$ denote the region above the packing for $L_1$, thus dividing the SF packing into three disjoint regions, $R$, $S$ and $T$. Let $H(S)$, $H(R)$ and $H(T)$ denote the heights of these regions. Index the blocks within each of these regions from bottom to top as $B_1(S), B_2(S), \cdots, B_{k(S)}(S)$; $B_1(T), B_2(T), \cdots, B_{k(T)}(T)$; and $B_1(R), B_2(R), \cdots, B_{k(R)}(R)$. For $X \in \{R, S, T\}$ and $1 \leqq i \leqq k(X)$, let $H_i(X)$ denote the height of block $B_i(X)$. We divide the proof into two cases, depending on whether $S$ contains any rectangles of width $1/(m+2)$ or less.

*Case* 1. All rectangles in $S$ have width larger than $1/(m+2)$.

In this case, since all rectangles in $S$ also have width at most $1/(m+1)$, each block in $S$ except possibly the last one contains exactly $m+1$ regular items (and no fallback items). The total height of all these rectangles is thus at least

$$\begin{aligned}(m+1)\sum_{i=1}^{k(S)-1} H_{i+1}(S) &\geqq (m+1)\sum_{i=2}^{k(S)} H_i(S)\\ &\geqq (m+1)(H(S)-1).\end{aligned}$$

Similarly, all rectangles in $T$ have width greater than $1/(m+1)$ and no more than $1/m$, so each block in $T$ except possibly the last contains exactly $m$ regular items (and no fallback items). The total height of all these rectangles is at least

$$\begin{aligned}m\sum_{i=1}^{k(T)-1} H_{i+1}(T) &\geqq m\sum_{i=2}^{k(T)} H_i(T)\\ &\geqq m(H(T)-1).\end{aligned}$$

Now consider any optimal packing of $L$. We can divide this packing into "slices" by drawing a horizontal line from one side of the bin to the other through the top and bottom of each rectangle. For $0 \leqq i \leqq m$, let $Z_i$ denote the cumulative height of all such slices that contain exactly $i$ rectangles (subrectangles of the original rectangles) of width exceeding $1/(m+1)$. Notice that

$$\mathrm{OPT}(L) = \sum_{i=0}^{m} Z_i.$$

The total height of all rectangles from $L$ having width greater than $1/(m+1)$ is then $\sum_{i=0}^{m} iZ_i$, so we must have

$$m(H(T)-1) \leqq \sum_{i=0}^{m} iZ_i.$$

Rewriting, we obtain

$$H(T) \leqq \frac{1}{m}\sum_{i=0}^{m} iZ_i + 1.$$

The total height of all rectangles from $L$ having width greater than $1/(m+2)$ but no more than $1/(m+1)$ is at most $\sum_{i=0}^{m}(m+1-i)Z_i$, so we must have

$$(m+1)(H(S)-1)\leqq \sum_{i=0}^{m}(m+1-i)Z_i,$$

or rewriting,

$$H(S)\leqq \frac{1}{m+1}\sum_{i=0}^{m}(m+1-i)Z_i+1.$$

Combining these, we obtain

$$\begin{aligned}\mathrm{SF}(L)&=H(S)+H(T)\\&\leqq \frac{1}{m}\sum_{i=0}^{m} iZ_1+\frac{1}{m+1}\sum_{i=0}^{m}(m+1-i)Z_i+2\\&=\sum_{i=0}^{m}\left(\frac{i}{m}+1-\frac{i}{m+1}\right)Z_i+2\\&=\sum_{i=0}^{m}\left(1+\frac{i}{m(m+1)}\right)Z_i+2\\&\leqq \sum_{i=0}^{m}\left(1+\frac{1}{m+1}\right)Z_i+2=\frac{m+2}{m+1}\mathrm{OPT}(L)+2,\end{aligned}$$

and hence the claimed bound holds in Case 1.

*Case* 2. Some rectangle in $S$ has width $1/(m+2)$ or less.

We first claim the following:

$$A(T)\geqq \frac{m+1}{m+2}H(T)-\frac{1}{(m+1)(m+2)}H(R)-\frac{m}{m+2}. \tag{5.1}$$

Consider the FFDH packing of the rectangles of $L_1$ (which we later rearrange to form $T$ in the process of performing Split-Fit). Index the bins from bottom to top as $B_1, B_2, \cdots, B_{k(T)}$, and denote the height of $B_i$ by $H_i$, $1\leqq i\leqq k(T)$. Let $T'$ be the subset consisting of those blocks which have width less than $(m+1)/(m+2)$. Since all items packed are regular items of width at least $1/(m+1)$, and all blocks except possibly the last have width at least $m/(m+1)$, we have the following (assuming by convention that $H_{k(T)+1}=0$):

$$\begin{aligned}A(T)&\geqq \frac{m+1}{m+2}\sum_{i=1}^{k(T)-1}H_{i+1}+\frac{1}{m+1}\sum_{i=1}^{k(T)}(H_i-H_{i+1})-\frac{1}{(m+1)(m+2)}\sum_{B_i\in T'}H_{i+1}\\&\geqq \frac{m+1}{m+2}(H(T)-H_1)+\frac{1}{m+1}H_1-\frac{1}{(m+1)(m+2)}\sum_{B_i\in T'}H_i\\&\geqq \frac{m+1}{m+2}H(T)-\frac{m}{m+2}-\frac{1}{(m+1)(m+2)}H(R),\end{aligned}$$

as desired, since $H_1\leqq 1$ and $H(R)=\sum_{B_i\in T'}H_i$.

Our second claim concerns region $R$:

$$A(R)\geqq \frac{1}{2(m+2)}(H(R)-2). \tag{5.2}$$

Since $S$ by assumption contains items of width less than or equal to $1/(m+2)$ and rectangle $R$ has width $1/(m+2)$, we must have $\sum_{i=1}^{k(R)} H_i(R) \geqq H(R)-1$. The packing of $R$ must contain at least as much area as would its packing under NFDH, so by the proof of Theorem 1 (normalized to a bin width of $1/(m+2)$) we obtain

$$A(R) \geqq \tfrac{1}{2}\left(\sum_{i=1}^{k(R)} H_i(R) - H_1(R)\right) \cdot \frac{1}{m+2}$$

$$\geqq \tfrac{1}{2}(H(R)-1-H_1(R)) \cdot \frac{1}{m+2} \geqq \frac{1}{2(m+2)}(H(R)-2),$$

as desired.

Finally, let us turn to region $S$. This is packed by FFDH using items all of width $1/(m+1)$ or less. Thus by the proof of Theorem 3

$$A(S) \geqq \frac{m+1}{m+2}(H(S)-1).$$

Putting together (5.1), (5.2) and (5.3) we obtain

$$\begin{aligned}\mathrm{OPT}(L) &\geqq A(T)+A(R)+A(S)\\ &\geqq \frac{m+1}{m+2}H(T) - \frac{1}{(m+1)(m+2)}H(R) - \frac{m}{m+2} + \frac{1}{2(m+2)}H(R)\\ &\quad - \frac{1}{m+2} + \frac{m+1}{m+2}H(S) - \frac{m+1}{m+2}\\ &\geqq \frac{m+1}{m+2}(H(T)+H(S)-2) = \frac{m+1}{m+2}(\mathrm{SF}(L)-2),\end{aligned}$$

and the performance bound follows.

The fact that the multiplicative constant $(m+2)/(m+1)$ cannot be improved for Split-Fit follows immediately from the examples used to show that the bound of Theorem 3 is best possible. For a fixed value of $m$, we simply take the examples having rectangles of width no more than $1/(m+1)$ for which First-Fit achieves $((m+2)/(m+1)) \cdot \mathrm{OPT}(L)$ and add to the beginning of $L$, $m$ rectangles of height 1 and width $1/m$. The performance of Split-Fit in region $S$ is the dominant factor in such cases, so it performs essentially as First-Fit and achieves the same bound. □

The overall bound for Split-Fit in the unrestricted case is thus

$$\mathrm{SF}(L) \leqq \tfrac{3}{2}\,\mathrm{OPT}(L)+2,$$

which compares quite favorably with the corresponding bound for FFDH. Further improvements may well be obtainable if the idea of "splitting" implicit in Split-Fit is used in more elaborate ways [2], [8].

**4. Discussion.** The level-oriented packing methods analyzed here would seem to be wasteful of space, since they never consider packing rectangles in the space between the tops of the shorter items in a block and the top half of the block itself. Thus one might expect that the non-level-oriented Bottom-Left methods of [1], which pack by always placing the next rectangle from $L$ as low as possible and then as far to the left as possible, would perform significantly better. However, straightforward extensions of the "checkerboard construction" from [1] can be used to show that these algorithms cannot satisfy an asymptotic performance bound better than $2 \cdot \mathrm{OPT}(L)$, regardless of whether $L$ is ordered by decreasing height, increasing height, decreasing width, or increasing width. Thus, in contrast to expectations, the Bottom-Left methods turn out to be substantially *worse* than the level methods.

In a sense, the wasted space referred to above for level-oriented packing methods has an effect that is more apparent than real. In each of our proofs, we essentially showed that it was bounded by a constant, independent of OPT($L$), and that is why the results for the one-dimensional case carried over so nicely. The wasted space does become significant, however, if we consider absolute bounds, bounds of the form $A(L) \leqq \alpha \cdot \mathrm{OPT}(L)$, where no additive constant is allowed. Such bounds may well be of interest in situations where optimal packings are not very tall, so we have proved our asymptotic results in such a way that absolute bounds can be derived as corollaries.

Note that in each of our proofs except the one for squares, we have assumed the height of the tallest rectangle to be exactly 1. This means that $\mathrm{OPT}(L) \geqq 1$, and hence a performance bound of the form

$$A(L) \leqq \beta \cdot OPT(L) + \gamma$$

implies the absolute bound

$$A(L) \leqq (\beta + \gamma) \cdot \mathrm{OPT}(L).$$

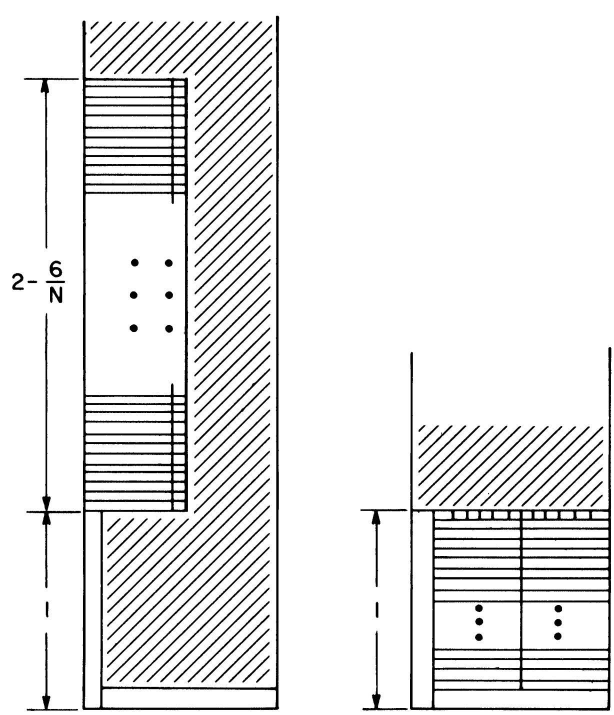



FIG. 7. *Example of a list L with* $\mathrm{NFDH}(L) = (3-6\varepsilon)\,\mathrm{OPT}(L)$ *for any* $\varepsilon = 1/N$. *The list consists of one* $2\varepsilon \times 1$ *rectangle, one* $(1-2\varepsilon) \times 2\varepsilon$ *rectangle, and* $2N-6$ *pairs of rectangles with dimensions* $(\frac{1}{2}-\varepsilon) \times \varepsilon$ *and* $3\varepsilon \times \varepsilon$.

Hence we obtain from Theorem 1 the previously known [2], [8] absolute bound $\mathrm{NFDH}(L) \leqq 3 \cdot \mathrm{OPT}(L)$ for all lists $L$. Fig. 7 provides an example to show that this is the best possible absolute bound (all the other absolute lower bounds we cite can be proved using the same basic ideas in conjunction with lower bound examples for the corresponding one-dimensional algorithms). Theorem 2 yields the absolute bound $\mathrm{FFDH}(L) \leqq 2.7 \cdot \mathrm{OPT}(L)$. This bound also can be shown to be tight. Theorem 3 yields the absolute bound $\mathrm{FFDH}(L) \leqq (2+1/m) \cdot \mathrm{OPT}(L)$ for all lists $L$ in which no rectangle has width exceeding $1/m$, and this bound is also tight. Theorem 5 yields the corresponding absolute bound $\mathrm{SF}(L) \leqq (3+1/(m+1)) \cdot \mathrm{OPT}(L)$, and although this bound is not known to be tight for any fixed value of $m$, both lower and upper bounds approach 3 as $m$ goes to infinity. Thus for all sufficiently large $m$, Split-Fit is worse than First-Fit in absolute ratio even though it is better asymptotically. The case for $m=1$ is less clear. Our proof of Theorem 5 can be tightened to yield a bound of $\mathrm{SF}(L) \leqq \frac{3}{2}\mathrm{OPT}(L) + \frac{3}{2}$ in this case, by observing that in region $T$ each block $B_i$ contains a single regular item of height $H_i$, and so there is no wasted space above regular items in this region. This yields an absolute bound of $\mathrm{SF}(L) \leqq 3 \cdot \mathrm{OPT}(L)$, but the best lower bound we can prove is $2.7 \cdot \mathrm{OPT}(L)$ and we suspect that this, the same bound as for FFDH, is the correct answer. Sleator [10] has found a method similar to Split-Fit with an absolute bound of $2.5 \cdot \mathrm{OPT}(L)$.

The question arises: Are there any techniques which might be used to help reduce the wasted space in practice? Certain techniques are worth noting.

The first of these is that, instead of packing every block in a left-to-right manner, one might alternately pack blocks from left-to-right and then from right-to-left. Then the tallest rectangle in the block will be above the shortest, rather than the tallest, rectangle in the preceding block, and some reduction in total height might be achieved by dropping each rectangle until it touches some rectangle below it. Another approach would be to allow more general packings within blocks, say by allowing new, shortened blocks to be created in the space above the regular items in a block, which otherwise would remain empty.

However, the essentially one-dimensional nature of the worst case examples mentioned in the paper show that the asymptotic performance bounds cannot be improved by these modifications, and so one can only hope that they will do better in practice.

One possible modification of our basic model deserves mention. Suppose that, in packing, we are allowed to rotate rectangles by 90° if we so wish. We have not yet analyzed this variant in detail, but we note that Theorems 1 and 3, depending as they do solely on area-arguments for their proof, continue to hold even if $\mathrm{OPT}(L)$ is interpreted to be the minimum height packing with such rotations allowed. If we are to make use of the possibility of rotations in our algorithmic packings, one appealing heuristic would be to rotate all rectangles so that their width is no larger than their height, and then apply one of our standard algorithms which, according to Theorems 3 and 5, yield better bounds for smaller widths. Our result for squares (Theorem 4) indicates the limits of this approach for algorithm FFDH, but there are no doubt many interesting questions left to be answered.

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