TECHNICAL UNIVERSITY OF DENMARK - DTU COMPUTE -

[SUPERSTRINGWITHEXPANSION] Problem

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1 DESCRIPTION OF THE PROBLEM

In this decisional problem we have two different alphabet Σ and Γ, where $\Gamma = \{\gamma_1, ..., \gamma_m\}$ contains m symbol, a string s which is made up of symbols that belongs to the Σ alphabet (formally $s \in \Sigma^*$), k strings $t_1, ..., t_k$, which is made up of symbols from both alphabets, and in the end, we have m subsets $R_1, ..., R_m \subseteq \Sigma^*$. These subsets contain symbols of Σ alphabet, and as we can notice, they are as many as the symbols in the alphabet Γ. That's because for each γ_i there is a subset R_i. We can understand better this relation explaining the core of our problem. The output of the problem will be YES if exits a sequence of words $r_1 \in R_1$, $r_2 \in R_2$,...,$r_m \in R_m$, such that for all $1 \leq i \leq k$ the so-called *expansion* $e(t_i)$ is substring of s; the expansion of a string t_i consists in the replacement of every symbols $\gamma_j \in \Gamma$ that appears in the string t_i with its expansion, where the expansion of a symbol is defined as follow: $e(\gamma_j) := r_j$. That is, we have to choose for every symbol $\gamma_j \in \Gamma$ a symbol $r_j \in R_j$, and we know that $r_j \in \Sigma^*$. We have to replace the symbol γ_j in each string t_i with the symbol r_j that was chosen. We have to do that for each $\gamma \in \Gamma$. In this way, using these replacements, we are transforming our string t_i in its expansion $e(t_i)$.
In the end we will obtain that every expansion-string is made up of symbols of the alphabet Σ as our string s. Hence, if every new constructed string $e(t_i)$ is a substring of s, the answer for our problem will be YES, otherwise NO.
In other words, the answer of our problem will be YES if there is a selection of the $r_i \in R_i$ such that, replacing each symbols $\gamma_i \in \Gamma$ with the respective symbol r_i (we have to remember that for each γ_i we have to chose one symbol from the subset $R_i \subseteq \Sigma^*$) in every string $t \in T$, where $|T| = k$, we will have that every new string that we obtain with this replacement is a substring of the string s. If a selection of $r_i \in R_i$ like this doesn't exist, the answer will be NO.

1.1 Simple examples

With the text of the problem we receive also some problem instances on the alphabets $\Sigma = \{\texttt{a,b,...,z}\}$, $\Gamma = \{\texttt{A,B,...,Z}\}$. The first line of the file contains the number k, which is the number of strigs t we have. The second line contains the string s and the following k lines the strings $t_1, ..., t_k$. Finally, the last lines (at most 26, the number of letters in the alphabet) start with a letter $\gamma_j \in \Gamma$ followed by a colon and the contents of the set R_j belonging to the letter, where the elements of the set are separated by commas. Hence, we can clearly see that for each symbols $\gamma_i \in \Gamma$, in this case letter, there is a subset R_i and in this subset we have to chose one symbol.

For example:

```
4
abdde
ABD
DDE
AAB
ABd
A:a,b,c,d,e,f,dd
B:a,b,c,d,e,f,dd
C:a,b,c,d,e,f,dd
D:a,b,c,d,e,f,dd
E:aa,bd,c,d,e
```

In this example we can observe that there is NO possible selection of symbols r_i such that every string t would be a substring of `s=abdde`. In fact if we have a look to t_2=`DDE` and t_3=`AAB`, we can notice that they have the same structure, and in the same time in our string s the only letter that is repeated twice is the `d`. So we conclude that `A=d` and `D=d`. Moreover from t_4=`ABd` we see that, since t_4 will be a substring, or `B=d` or `B=b`. If we try to choose our selection of r_i following these rules, we can noticed that there is no possible solution, such that at the same time, all $e(t_i)$ are substring of s.

2 SuperStringWithExpansion is in $\mathcal{NP}$

We have to show and prove that our problem is in $\mathcal{NP}$. To do that we use the so called *guess-verify algorithms* proof, so we have to start designing a deterministic algorithm A which takes as input a problem instance X and a random sequence R.

Proof. We have to show that there is a polynomial p and a randomized p-bounded algorithm A which satisfies the condition of the class $\mathcal{NP}$.

1. Let Σ be an alphabet and Γ another alphabet. We know that $|\Gamma| = m$ (cardinality of the alphabet Γ is m). Let s be a string made up of symbols of Σ^* and let $t_1, ..., t_k$ be k strings

made up of symbols of $(\Sigma \cup \Gamma)^*$. Then let $R_1, ..., R_m$ be subsets that contains symbols of Σ^*. So the first part of the input is:

$$X = (\Sigma, \Gamma, s, t_1, ..., t_k, R_1, ..., R_m)$$

We could simplify this symbolism, considering the set $T = \{t_1, ..., t_k\}$ and $R = \{R_1, ..., R_m\}$. In this way we obtain that the first part of the input for our algorithm A now is:

$$X = (\Sigma, \Gamma, s, T, R)$$

a) The random sequence G consists of **integers** in the range $\{1, ..., n\}$, where $n = max(|R_i|)$ for $1 \leq i \leq m$.

b) Now on input $((\Sigma, \Gamma, s, T, R), \mathbf{G})$, algorithm A checks whether G contains at least m integers, and for every integer it has to check whether:

$$int_j \leq |R_j|$$

which means that A has to check if the integer contains in the position j, where $1 \leq j \leq m$, is less or equal than the cardinality of the subset R_j. The reason of that will be clear immediately.
If G is shorter, or the first m integer don't satisfy the condition above, A will return NO. Otherwise A uses the first m integers $g_1, ..., g_m$ of G to select a symbols in every subsets $R_1, ..., R_m$. In this way:

$$r_{g_i} \in R_i$$

That is, the integer g_i, which is in the position i inside the random string G, indicates the position of the symbol that we have to choose inside the subset R_i. That's why it has to satisfy the condition above, in fact if the integer g_i is greater than the cardinality of R_i we are not able to select a symbol.

c) Now we have to choose an element for every subset R_i, where $1 \leq i \leq m$, and then using the expansion for the symbols of the alphabet Γ, we are able to "link" a letter $\gamma_i \in \Gamma$ with a symbol in R_i. (That's because for each letter γ_i we have to choose one and only one symbol in the subset R_i, the i is the same, because the subset is linked to the letter).

$$\gamma_i \in \Gamma \Longrightarrow e(\gamma_i) = r_j \in R_i$$

where $j = g_i$ of the string G

Now we could substitute every letters $\gamma_i \in \Gamma$, $1 \leq i \leq m$, which are in the k strings t with their expansion that we obtained thanks to the formula above. After that, we will have k strings $e(t_i)$ (which are the expansion of our strings) made up of only symbols in Σ^*.
Then we have to check if every string that we obtain $e(t_l)$, $1 \leq l \leq k$, is a substring of our string s. If every string is a substring the algorithm A returns YES, otherwise it returns NO.

2. Now we have to show that the two conditions of the class $\mathcal{NP}$ are met:

   a) Let us first assume that the true answer is YES, so there is a selection of symbols $r_1, ..., r_m$ from the subsets $R_1, ..., R_m$ such that, using the expansion for the symbols $\gamma \in \Gamma$ first, and then for the k strings t, we will obtain that every string $e(t)$ (the expanded one) is a substring of s. So the symbols $r_1, ..., r_m$ is our solution. That means, that our subsets $R_1, ..., R_m$ contains one symbol each to expand $\gamma_1, ..., \gamma_m \in \Gamma$ in the right way, and we have to choose exactly that symbol in every subset R_i.
   Hence, there will be a string $G = g_1 ... g_m$ (which is the random string and contains the position of the element we will select) such that we will select the right symbol inside every subset R, so we are able to obtain that every string $e(t_i)$ is a substring of s. In this case if G is given to A, A will correctly return YES. Hence, there is a string of length at most m which lead to a correct answer YES. To calculate the probability to obtain this string we try to see how to built this string that satisfy our problem: we show this probability in the two cases: the first, where all the subsets have exactly n elements (which is useful to understand how change the probability in regard to the number of elements of the subsets and the number of the subsets) and the second, in which we consider every cardinality of the subset. We have to choose exactly one element in each subset, so our g_i has to be the exactly position of the right symbols. In the first case for each g_i we have to choose 1 position between n, ans so we have:

   $$G = g_1 g_2 ... g_{m-1} g_m$$

   $$\mathbf{P}[\text{G is the right string}] = \underbrace{\left(\frac{1}{n}\right) \times ... \times \left(\frac{1}{n}\right)}_{m \text{ times}} = \left(\tfrac{1}{n}\right)^m$$

   Otherwise, if we want to use subset with no fixed cardinality, we have this probability:

   $$G = g_1 g_2 ... g_{m-1} g_m$$

   $$\mathbf{P}[\text{G is the right string}] = \left(\tfrac{1}{|R_1|}\right) \times ... \times \left(\tfrac{1}{|R_m|}\right)$$

   We could see that this probability is small, especially if n and m are bigger, but itâĂŹs positive. Hence, the first condition is satisfied.

   b) Now, we assume that the true answer is NO, so there is no possible selection of symbols $r_1, ..., r_m$ from $R_1, ..., R_m$ such that every expanded-strings $e(t_i)$ are substring of s. In other words, for each random string G we pass to A, there is no possible sequence of position $g_1 ... g_m$ such that let us select the right symbols to obtain that every expanded-strings $e(t_i)$ are substring of s. Hence, regardless the random string G, the algorithm A will always answer NO in this case, so also the second condition is satisfied.

3. Now we have to show that the running time of the algorithm A that we've built is p-bounded for some polynomial p. We could see that our algorithm have to:
   - check if G contains at least m integer: *O(m)*;
   - for each integer check if the condition is satisfied: so for each integer has to calculate the cardinality of the subset R_i related to the position g_i and compare to the integer value. If we are going to use arrays we have to compare an integer to the `length` of the array. So it will be *O(1)* for each element in G. For m element it will be *O(m)*;
   - select the exact element in R_i using the position g_i for every subset. If we will work with arrays we have to enter one cell, so *O(1)*, but for all the letters of the alphabet Γ is $O(m \times 1) = O(m)$;
   - read all the k strings and substitute all the γ with their expansions. The worst case is when the length of the strings are almost n, so we have to read all the k strings and substitute. If we think that a substitution is *O(1)*, then it will be $O(k \times n)$;
   - in the end check if the all k expanded-strings are substring of s. Using the simplest algorithm, we have that it will be $O(n+q)$, where n is the length of the string s and q the length of strings t. Hence, for all k strings t it will be $O(k(n+q))$

   Therefore, we can see that our algorithm give the result in time:

   $$m + m + m + kn + k(n+q) = 3m + kn + k(n+q)$$

   In the worst case, we could have n symbols in Γ, and n strings t, so the algorithm will be $O(n^2)$, which is polynomial.

□

3 SuperStringWithExpansion is $\mathcal{NP}-$complete

4 Heuristic algorithm

In order to construct a heuristic algorithm for the SuperStringWithExpansion problem we decided to implement an algorithm that works in several steps.

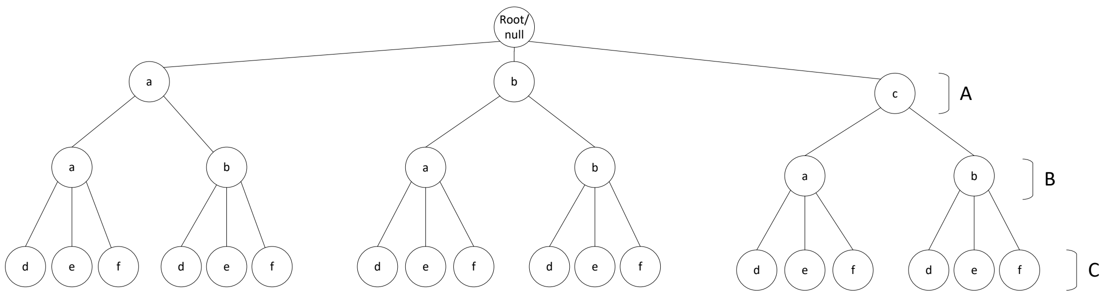
1. Decoding the file and storing the values in appropiate datastructures. This is done in $O(n)$ time complexity.
2. Filtering the input thereby potentially making the problem smaller is done in multiple steps.
   a) The strings $t_1 \dots t_k \in (\Sigma \cup \Gamma)^*$ are filtered by removing duplicates and t strings that are substrings of other t strings for instance if $t_1 = C$, $t_2 = DAC$ and $t_3 = DACD$ then both t_1 and t_2 will be removed as they are substrings of t_3. This is done in $O(n^2)$ time complexity as we need to run through all t strings once and place them

in a HashSet which has to to be run through each time to check whether or not the HashSet contains the value or if the value is a substring of a previously added t string.

b) When filtering t strings each $\gamma \in \Gamma$ is stored in a HashSet in order to keep track of which subsets $R_1 \ldots R_m \subseteq \Sigma^*$ should be used. This is done in $O(n)$ time complexity.

c) We filter the subsets $R_1 \ldots R_m \subseteq \Sigma^*$ by checking if the subset should be used. If the symbol γ is found in the HashSet from the previous step then we move on to filter the actual set. This is done by checking if each element is a substring of the string s. If an element is not a substring of s, the element is removed as it is not an option. Otherwise it is kept in the set. This can be done in $O(mn^2)$ by running through all r elements of of all $R_1 \ldots R_m$ subsets and checking if the string s contains the string as a substring.

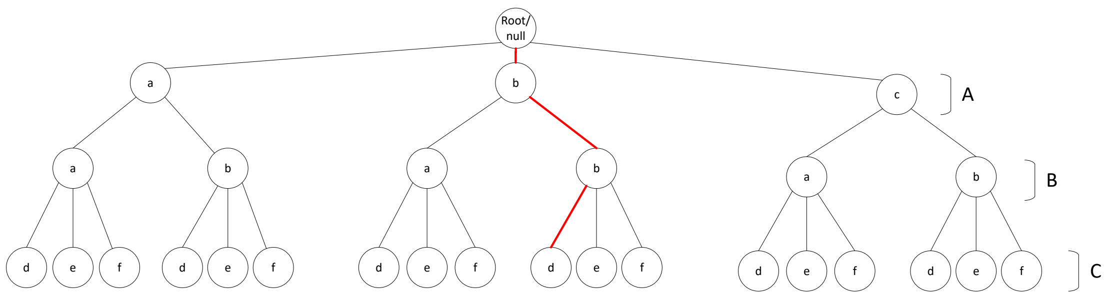
3. In order to be able to search through all the possible combinations we create a tree that follows this pattern:



This is done by recursively taking the $R_1 \ldots R_m$ subsets and then adding each string r in the subset as a child of all nodes in the the previous layer. For instance in the picture the elements of $\gamma = A$ is $\{a, b, c\}$, $\gamma = B$ is $\{a, b\}$ and $\gamma = C$ is $\{d, e, f\}$. With m sets this gives a time complexity of $O(m^n)$

4. Searching the tree is basically a pre-order Depth-First-Search done in a couple of steps.

a) Check if the node is a leaf and a solution is not found already. If the node is a leaf and a solution has not been found then traverse up the tree visiting all parents to get their γ value and r value and store them in a Hashtable as a possible solution.



This represents a possible solution where $A = b, B = b$ and $C = d$

b) Check the solution stored in the Hashtable against all the t strings to see if the solution works for all t strings.

c) If the node is not a leaf and a solution has not been found then call traverseTree function recursively on all children of the node. This means exponential time complexity as each node potentially has n children. Assuming that our alphabet Γ is limited to m this gives $O(m^n)$ time complexity.

d) When/if a solution is found it is stored in a separate Hashtable.

The process of searching the tree can be made iteratively. This can be done such that when creating the tree and adding leafs to the tree, then adding them to a List as well which can be iterated over. This still gives a worst case time complexity of $O(m^n)$ for searching all possible combinations.

5. Add the unused $R_1 \dots R_m \subseteq \Sigma^*$ subsets to the solution. This can be done in $O(n)$ time complexity.

6. Sort the solution by γ and print to a .SOL file. Using the java built in Collections.sort which uses a modified mergesort this gives a time complexity of $O(n \log(n))$

This gives a combined worst-case running time of $O(m^n)$