Lecture 1

Definition 0.1. A metric space is a set X equipped with a function $d : X \times X \to \mathbb{R}$, which satisfies the following axioms:

1. For any $x, y \in X$, $d(x, y) = d(y, x)$

2. For any $x, y, z \in X$, we have $d(x, y) \leq d(x, z) + d(z, y)$. This is called the "triangle inequality"

3. For any $x, y \in X$, $d(x, y) = 0$ exactly when $x = y$

Example 0.1. For $x, y \in \mathbb{R}^n$,

$$d(x, y) \stackrel{\text{def}}{=} \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

This is called the Euclidean distance. 2 can be replaced with any real $r \geq 1$, and it will still be a metric.

Example 0.2. In this example, $C[0, 1]$ is the set of all continuous functions $f : [0, 1] \to \mathbb{R}$. Here,

$$d(f, g) \stackrel{\text{def}}{=} \sup_{x \in [0,1]} |f(x) - g(x)|$$

Example 0.3. Let $X = \mathbb{N}$, the natural numbers, including 0. Let p be a fixed prime number. The p-adic metric on $\mathbb{N}$ is defined by

$$d_p(a, b) \stackrel{\text{def}}{=} \frac{1}{p^\alpha}$$

Where p^α is the largest power of p which divides $|a - b|$. So two naturals are "close" if their difference is divisible by a high power of p.

Claim. *This is a metric*

Proof. The 1st and 3rd axioms are clear. So we must prove the triangle inequality. We will consider the three quantities $d_p(a, b), d_p(a, t)$, and $d_p(b, t)$, where $a, b, t \in \mathbb{N}$. Suppose p^β divides both $a - t$ and $t - b$. Then p^β divides $(a - t) + (t - b) = a - b$. Therefore,

$$d_p(a, b) \leq \frac{1}{p^\beta} \leq \max(d_p(a, t), d_p(t, b)) \leq d_p(a, t) + d_p(t, b)$$

■

Definition 0.2. Let $(X, d_X), (Y, d_Y)$ be two metric spaces. For a function $f : X \to Y$, we say that f is continuous at $x_0 \in X$ if, for all $\epsilon > 0$, there is a $\delta > 0$ such that

$$0 < |d_X(x_0, x)| < \delta \implies |d_Y(f(x_0), f(x))| < \epsilon$$

A function $f : X \to Y$ is said to be continuous if it is continuous at x for all $x \in X$.

Example 0.4. Consider a map $(\mathbb{N}, d_5) \to (\mathbb{N}, d_5)$ defined by

$$x \mapsto x^2$$

Is this continuous?
At 0, to be continuous, then for any x, if we want to get within a small distance of 0, then x has to be divisible by large powers of 5.
What about at 11?
This is continuous.

Example 0.5. What about $(\mathbb{N}, d_5) \to (\mathbb{N}, d_{17})$.

Lecture 2

Theorem 0.1. *If $f : (X, d_X) \to (Y, d_Y)$ and $g : (Y, d_Y) \to (Z, d_Z)$ are both continuous, then $g \circ f : (X, d_X) \to (Z, d_Z)$*

Proof. Fix $x \in X$ and $\varepsilon > 0$. Choose $\delta_1 > 0$ so that if $d_Y(f(x), y) < \delta_1$, then $d_Z(gf(x), g(y)) < \varepsilon$.
By continuity of f, we may then choose a $\delta_0 > 0$ such that if $d_X(x, x') < \delta_2$, then $d_Y(f(x), f(x')) < \delta_1$.

■

Definition 0.3. For a metric space (X, d_X), and a real $r > 0$, the open r-ball around a point x is defined as

$$B_r(x) = \{x' \in X \mid d(x, x') < r\}$$

Exercise: State and prove some theorem about the existence of a function from $X \times X' \to Y \times Y'$, given a function $f : X \to Y$ and $g : X' \to Y'$.

Example 0.6. Balls

1. In $\mathbb{R}^2$, consider

$$d_r\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = \left(\sum_{i=1}^{2} (x_i - y_i)^r\right)^{\frac{1}{r}}$$

For $r = 2$, this is the usual euclidean distance. For $r = 1$, the balls look like diamonds. In the limit, as $r \to \infty$, the metric will approach what is known as the "box metric," in which the distance between any point and 0 is it's largest coordinate.

2. On $C[0,1]$, the set of continuous functions from $[0,1]$ to $\mathbb{R}$, we have the sup metric:

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$$

3. We also have

$$d_1(f,g) = \int_0^1 d|f(x) - g(x)|dx$$

Definition 0.4. For a metric space (X, d_X), suppose that $U \subseteq X$ is said to be "open" if, for any $x \in U$, there is a $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

Lemma 1. *$B_\varepsilon(x)$ is always open.*

Proof. Let $y \in B_\varepsilon(x)$. Let $t = d(x,y)$. By construction, $t < \varepsilon$. Let $\delta = \varepsilon - t$. Consider $B_\delta(y)$. For any $z \in B_\delta(y)$, we have by the triangle inequality

$$d(x,z) \leq d(x,y) + d(y,z) < \varepsilon - t + t = \varepsilon$$

and so $z \in B_\varepsilon(x)$. z was arbitrary, so we are done.

■

Lecture 3

Definition 0.5. Let (X,d) be a metric space. A set $U \subseteq X$ is said to be <u>open</u> if for all $x \in U$, there is an $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq U$.

Theorem 0.2. *Let $\{U_\alpha\}_{\alpha\in A}$ be a collection of open sets. Then*

1. *$\cup_{\alpha\in A} U_\alpha$ is open.*

2. *Let $U_1, \ldots, U_n$ be a finite subcollection. Then $\cap_{i=1}^{\infty} U_i$ is open.*

Proof. **Lecture 4**

Definition 0.6. Two metrics d_1, d_2 are said to be <u>equivalent</u> on a space X if any set which is open under the d_1-induced topology is also open under the d_2-induced topology, and vice versa.

Example 0.7. In $\mathbb{R}^2$,

$$\begin{aligned} d_2(0,(x,y)) &= (x^2+y^2)^{\frac{1}{2}} \\ d_1(0,(x,y)) &= |x|+|y| \\ d_\infty(0,(x,y)) &= \max\{|x|,|y|\} \end{aligned}$$

How do these metric's unit balls compare? In fact, d_1's is within d_2's, which is within d_∞'s.
But all of these balls contain a ball of radius $\frac{1}{2}$ in any of the three metric. Thus, these are equivalent.

Definition 0.7. Two metrics d_1, d_2 are called Lipschitz equivalent if there exists some $k \in \mathbb{R}$ such that, for all $x, y \in X$, we have

$$\frac{1}{k}d_2(x,y) < d_1(x,y) < kd_2(x,y)$$

Example 0.8. This is a non-example. The 5-adic and the 17-adics are not equivalent.

Example 0.9. Let

$$\begin{aligned} d_1(f,g) &= \int_0^1 |f(x)-g(x)|\,dx \\ d_\infty(f,g) &= \sup_{x\in[0,1]} |f(x)-g(x)| \end{aligned}$$

One controls for area, one controls for the maximum value of f. We have

$$B_\varepsilon^{d_\infty}(0) \subset B_\varepsilon^{d_1}(0)$$

This is because if we control for the maximum size of f, we can surely control for the area under it. However, no matter how much we limit the area under f, there is some f which has that much area which has a sup greater than some ε which is fixed.

Theorem 0.3. *Let d_1, d_2 be equivalent metrics on X. Then the following are equivalent*

1. *$f : X \to Y$ is d_1, d_Y continuous if and only if it is d_2, d_Y continuous.*

2. *$g : Z \to X$ is d_Z, d_1 continuous if and only if if is d_Z, d_2 continuous.*

Proof. **1.** Let $U \subset Y$ be open. We know that the preimage $f^{-1}(U)$ is open under the d_1 metric. But by the equivalence of d_1, d_2, $f^{-1}(U)$ must also be open under the d_2 metric. The reverse argument also holds.

2. Let $U \subset X$ be an open set under the d_1 metric. By continuity of g, we know $g^{-1}(U)$ is open. However, U must also be open under the d_2 metric, meaning that g must be continuous with respect to both metrics.

■

Recall: If $\{U_\alpha\}_{\alpha \in A}$ is a collection of open sets, then $\cup_{\alpha \in A} U_\alpha$ is open.
We can associate to any set $R \subset X$ an open set, called the interior of R.

Definition 0.8. For any $\mathbb{R} \subset X$, we define it's interior by

$$\text{int}(R) = \bigcup_{U \text{ open}, U \subseteq R} U$$

We can say several things about $\text{int}(R)$.

1. $\text{int}(R)$ is open.

2. If U is open, then $\text{int}(U) = U$, and vice versa.

3. Suppose $A \subseteq B$. Then $\text{int}(A) \subseteq \text{int}(B)$.

Recall: If $\{C_\alpha\}_{\alpha \in A}$ are all closed, then $\cap_{\alpha \in A} C_\alpha$ is closed.

Definition 0.9. If $R \subseteq X$, then the closure of R, denoted by $\overline{R}$, or sometimes $\text{cl}(R)$, is defined as

$$\overline{R} \stackrel{\text{def}}{=} \bigcap_{C \text{ closed}, C \supseteq R} C$$

Analagously,

1. $\overline{R}$ is closed for any R.

2. R is closed if and only if $R = \overline{R}$.

3. If $A \subset B$, then $\text{cl}(A) \subset \text{cl}(B)$.

Proposition 1. *Let $x \in \text{cl}(R)$. Then, for all $\varepsilon > 0$, $B_\varepsilon(x) \cap R \neq \varnothing$, and vice-versa. We will prove this next lecture.*

Lecture 4

Proof. First, suppose that $x \notin \overline{A}$. The complement of $\overline{A}$ is open, so there exists a $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq (\overline{A})^c$, and so $B_\varepsilon(x) \cap \overline{A} = \varnothing$.
Now, suppose that there exists some $\varepsilon > 0$ such that $B_\varepsilon(x) \cap A = \varnothing$. Then $(B_\varepsilon(x))^c$ is a closed set containing A which does not contain x, thus $\overline{A}$ cannot contain x.

■

Now, it is time for the main event.

Topological Spaces

Let X be a set. Let $\mathscr{P}(X)$ denote the power set of X, which is the set of all subsets of X.

Definition 0.10. Let $\mathscr{T} \subset \mathscr{P}(X)$. $\mathscr{T}$ is a topology on X if it has the following properties:

1. $\varnothing, X \in \mathscr{T}$.

2. If $\{U_\alpha\}_{\alpha \in A}$ with each $U_\alpha \in \mathscr{T}$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathscr{T}$.

3. If $A, B \in \mathscr{T}$, then $A \cap B \in \mathscr{T}$. Of course, we can "strengthen" this to the equivalent statement: if $U_1, \ldots, U_k \in \mathscr{T}$, then $\cap_{i=1}^k U_i \in \mathscr{T}$

Elements of $\mathscr{T}$ are called "open sets."

Example 0.10. Here are some simple examples.

1. Open sets in (X, d) form a topology (hence the definition)

2. $\mathscr{T} = \mathscr{P}(X)$ forms the discrete topology.

3. $\mathscr{T} = \{\varnothing, X\}$ forms the indescrete topolgy.

4. If $X = \{0, 1\}$, then $\mathscr{T} = \{\varnothing, X, \{0\}\}$ forms a topology.

5. We can form the Zariski topology on $\mathbb{R}$ by specifying the closed sets, which satisfy a similar but slightly different set of axioms. We define the closed sets to be $\varnothing, \mathbb{R}$, and any finite collection of points.

More generally, the Zariski topology on some ring R is specified by its closed sets, which are the solution locii of some set of polynomials in R.

6. Let $X = \mathbb{R}$, $\mathscr{T} = \{(r, \infty) \mid r \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$.

We have now defined a set of objects (topological spaces). Now, we want to define morphisms, the maps between spaces.

Definition 0.11. Let $(X, \mathscr{T}_X), (Y, \mathscr{T}_Y)$ be topological spaces. We say a function $f : X \to Y$ is continuous if, for all $V \in \mathscr{T}_Y$, $f^{-1}(V) \in \mathscr{T}_X$. In other words, the preimage of any open subset of Y is an open subset of X.

Example 0.11. Some baby examples

1. Any function $f : (X, \text{discrete}) \to (Y, \mathscr{T}_Y)$ is continuous as long as X has the discrete metric, as the preimage of any open set will be a subset of X, all of which are open under the discrete topology.

2. Any function $f : (X, \mathscr{T}_X) \to (Y, \text{ discrete})$ is continous for a similar reason.

3. $\mathrm{Id}_X : (X, \mathscr{T}_X) \to (X, \mathscr{T}_X)$ is continuous.

Theorem 0.4. *Let $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ and $g : (Y, \mathscr{T}_Y) \to (Z, \mathscr{T}_Z)$ be continuous functions. Then $g \circ f : (X, \mathscr{T}_X) \to (Z, \mathscr{T}_Z)$ is continuous.*
In other words, the composition of continuous functions is continuous.

Proof. Pick $W \in \mathscr{T}_Z$. Then $g^{-1}(W) \in \mathscr{T}_Y$ since g is continuous. So, because f is continuous,

$$(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) \in \mathscr{T}_X$$

Hence, $g \circ f$ is continuous.

Definition 0.12. For $\mathscr{T}$ a topology on X, a basis for $\mathscr{T}$ is a subset $\mathscr{B} \subseteq \mathscr{T}$ such that every set in $\mathscr{T}$ is the union of sets in $\mathscr{B}$.

Lecture 5

Proposition 2. *Suppose $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is a function. Then f is continuous if and only if $f^{-1}(B) \in \mathscr{T}_X$ for all $B \in \mathscr{B}$, with $\mathscr{B}$ a basis for $\mathscr{T}_Y$.*

Proof. The forward direction is trivial, as a basis consists of sets which are all open. Now, suppose that $f^{-1}(B) \in \mathscr{T}_X$ for all $B \in \mathscr{B}$. Let $V \in \mathscr{T}_Y$. Write it as $\cup_{\alpha \in A} B_\alpha = V$ for $B_\alpha \in \mathscr{B}$.
Then $f^{-1}(V) = f^{-1}(\cup_{\alpha \in A} B_\alpha) = \cup_{\alpha \in A} f^{-1}(B_\alpha)$. This is the union of open subsets of X, so $f^{-1}(V)$ is open.

■

Example 0.12. Consider $B_{\frac{p}{q}}(\frac{r}{s})$, with $p, q, r, s \in \mathbb{Z}$, $q, s \neq 0$. In other words, the balls of rational radius and rational center. This collection of sets forms a basis for $\mathbb{R}$ using the standard (or, as Darren calls it, the "Mother's Knee") topology.
Exercise: Write down the definition of interiors and closures, and check the following lemma is true:

Lemma 2. *$x \in \mathrm{cl}(A)$ if and only if, for all open $U \ni x$, $U \cap A \neq \varnothing$.*

Example 0.13. Consider ($\mathbb{N}$, Zariski). What is the closure of the collection of prime numbers under this topology?
Recall that the Zariski topology on $\mathbb{N}$ defines closed sets to be either empty, $\mathbb{N}$, or finite. So, for example, the closure of the integers from 1 to 10 is itself.
But the only closed set that can contain an infinite set is $\mathbb{N}$, so the closure of the primes is $\mathbb{N}$. More generally, for any infinite subset of $\mathbb{N}$, the closure is $\mathbb{N}$.

Subobjects and product objects

Subobjects

We have now defined our objects and morphisms, so let's talk about subobjects and product objects.
Let $(X, \mathscr{T}_X)$ be a topological space, and $A \subseteq X$ a nonempty subset.
Denote the inclusion map $\iota : A \to X$. We want to topologize A so that ι is continuous, and is as small as possible, in the sense that any topology for which ι is continuous includes this topology on A.
For any open $U \subseteq X$, we want $\iota^{-1}(U)$ to be open. But $\iota^{-1}(U) = U \cap A$.

Definition 0.13. For $(X, \mathscr{T}_X)$ a topological space and $A \subset X$ a subset, then the subspace topology on A, $\mathscr{T}_A$, consists of

$$\mathscr{T}_A \stackrel{\text{def}}{=} \{U \cap A \mid U \in \mathscr{T}_X\}$$

Proposition 3. *Suppose we have a commutative diagram of the form*

$$\begin{tikzcd}
(Z, \mathscr{T}_Z) \arrow[r, "g"] \arrow[dr, "\iota\circ g"'] & A \arrow[d, "\iota"] \\
& X
\end{tikzcd}$$

Then ι is continuous if and only if $\iota \circ g$ is continuous.

Proof. One direction is trivial, as we know the composition of continuous functions is continuous. Now, suppose that $\iota \circ g$ is continuous.
Let $W \in \mathscr{T}_A$. We know $W = W^* \cap A$ for some $W^* \in \mathscr{T}_X$. $\iota \circ g$ is continuous, so $(\iota \circ g)^{-1}W^* \in \mathscr{T}_Z$.
So, $\mathscr{T}_Z \ni (\iota \circ g)^{-1} = g^{-1}(\iota^{-1}(W^*)) = g^{-1}(W)$.

■

So we can see that if we want the above to hold, we are forced into our definition of $\mathscr{T}_A$.

Proposition 4. *$\mathscr{T}_A$ is the only topology so that the previous proposition is true for all spaces $(Z, \mathscr{T}_Z)$ and functions g.*

Proof. Suppose that $\mathscr{T}_\mu$ (μ for "mystery" topology) such that the previous proposition holds for all choices of $(Z, \mathscr{T}_Z)$ and $g : (Z, \mathscr{T}_Z) \to A$.

Consider

$$\begin{tikzcd}
(A, \mathscr{T}_A) \arrow[r, "\mathrm{Id}"] \arrow[dr, "\iota\circ\mathrm{Id}"'] & (A, \mathscr{T}_\mu) \arrow[d, "\iota"] \\
& (X, \mathscr{T}_X)
\end{tikzcd}$$

Lecture 6

We are trying to prove that $\mathscr{T}_A$ is the unique topology on A such that

$$\begin{tikzcd}
(A, \mathscr{T}_A) \arrow[r, "\mathrm{Id}"] \arrow[dr, "\iota\circ\mathrm{Id}"'] & (A, \mathscr{T}_\mu) \arrow[d, "\iota"] \\
& (X, \mathscr{T}_X)
\end{tikzcd}$$

Proof. Suppose $\mathscr{T}_\mu$ is such a topology on A. Then

$$\begin{tikzcd}
(A, \mathscr{T}_A) \arrow[r, "\mathrm{Id}"] \arrow[dr, "\iota\circ\mathrm{Id}"'] & (A, \mathscr{T}_\mu) \arrow[d, "\iota"] \\
& (X, \mathscr{T}_X)
\end{tikzcd}$$

We have $\iota \circ \mathrm{Id}$ is continuous, so $(A, \mathscr{T}_A) \xrightarrow{\mathrm{Id}} (A, \mathscr{T}_\mu)$ is continuous. So if $W \in \mathscr{T}_\mu$, then $W \in \mathscr{T}_A$. Therefore $\mathscr{T}_\mu \subseteq \mathscr{T}_A$.
We know that $\iota \circ \mathrm{Id}$ is continuous. Let $W \in \mathscr{T}_X$. By continuity, $(\iota \circ \mathrm{Id})^{-1}W \in \mathscr{T}_\mu \implies (\mathrm{Id})^{-1} \circ \iota^{-1}W \in \mathscr{T}_\mu$.
So then $(\mathrm{Id})^{-1}(W \cap A) \in \mathscr{T}_\mu$, and $W \cap A \in \mathscr{T}_\mu$. Therefore for any $W \in \mathscr{T}_\mu$, $W \cap A \in \mathscr{T}_\mu$, so $\mathscr{T}_A \subset \mathscr{T}_\mu$.

Product Objects

Let $(X, \mathscr{T}_X), (Y, \mathscr{T}_Y)$ be topological spaces. We want to topologize $X \times Y$.
Well, if that holds a topology, then the projections damn well better be continuous.

In other words, we want to arrange such that

$$\begin{tikzcd} X \times Y \arrow[r, "\rho_X"] \arrow[d, "\rho_Y"'] & X \\ Y & \end{tikzcd}$$

both ρ_X, ρ_Y are continuous.
Let $U \in \mathscr{T}_X$. Then $\rho_X^{-1} U = U \times Y$. Similarly, for $V \in \mathscr{T}_Y$, $\rho_Y^{-1} V = X \times V$.
Thus, for the projections to be continuous, we need that the intersection of $U \times Y$ and $X \times V$ are in $\mathscr{T}_{X\times Y}$ for all $U \in \mathscr{T}_X, V \in \mathscr{T}_Y$. The sets of this form, $X \times V \cap U \times Y$, form a basis for a topology.
In other words, the product topology has basis $\{U \times V \mid U \in \mathscr{T}_X, V \in \mathscr{T}_Y\}$.

Theorem 0.5. *Consider the commutative diagram*

$$\begin{tikzcd} & Z \arrow[dl, "\rho_X \circ g"'] \arrow[d, "g"] \arrow[dr, "\rho_Y \circ g"] & \\ X & X \times Y \arrow[l, "\pi_j"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

Then g is continuous if and only if $\rho_X \circ g$ and $\rho_Y \circ g$ are continuous.

Proof. Next time

Lecture 7

Relevant digression

Let $(X_i, \mathscr{T}_i)_{i\geq 1}$ be a family of topological spaces. A basis for a topology on $\prod_{i\geq 1} X_i$ is $\{\prod_{i\geq 1} u_i \mid u_i \in \mathscr{T}_i\}$
We can ask that the projection $\prod_{i\geq 1} X_i \to X_j$ is continuous for each j, but then

$$p_j^{-1}(V_j) = X_1 \times \cdots \times X_{j-1} \times V_j \times X_{j+1} \times \cdots$$

Definition 0.14. Let $(X_i, \mathscr{T}_i)_{i\geq 1}$ be a family of topological spaces. Then the Tychonoff's product topology has a basis consisting of sets of the form $\prod_{i\geq 1} V_i$, where $V_j \subseteq X_j$, and $V_j = X_j$ for all but finitely many j.

Example 0.14. Let p be prime. Then $\mathbb{Z}/p^k$ will denote $\mathbb{Z}/p^k\mathbb{Z}$. We may equip it with the discrete topology. Consider $\prod_{k\geq 1}\mathbb{Z}/p^k$. Let's put a metric on this. We will define

$$d(x,y)=\sum_{k\geq 1}\frac{1}{2^k}d_k(x_k,y_k)$$

It is easy to convince yourself this is a metric.
Note that there exists a map $\mathbb{Z}\to\prod_{k\geq 1}\mathbb{Z}/p^k$, given by $x\mapsto(x \pmod p, x \pmod{p^2},\dots)$
An integer m is close to zero in this metric if m is divisible by large powers of p.
This is (basically) the p-adic metric.
Exercise: What is the closure of $\mathbb{Z}$ in this metric space?

Hausdorff Spaces

Definition 0.15. Here is what Darren calls a "reasonable definition" of convergence of a sequence in a general topological space.
Let x_n be a seequence in $(X,\mathscr{T})$. We say that x_n converges to x if, for any $x\in U\in\mathscr{T}$, there exists an N such that $x_k\in U$ for all $k\geq N$.

Example 0.15. Let $\mathcal{P}$ be the set of prime numbers, and let $\mathbb{N}$ have the Zariski topology. What does $\mathcal{P}$ converge to?
Consider 193. An open neighborhood U of this would be a set which contains 193, and all but finitely many primes. So, for any such neighborhood, there will be an N such that $p_k\in U$ for all $k\geq N$. So, the sequence $x_k=p_k$ converges to 193. But 193 was arbitrary, so the sequence converges to any natural number.
The problem here is that the open sets are too big. The Hausdorff condition will get us around this.

Definition 0.16. A topological space $(X,\mathscr{T})$, is said to be Hausdorff if, for any $x,y\in X$, when $x\neq y$, there exists $U_x,U_y\in\mathscr{T}$, such that $x\in U_x$, $y\in U_y$, and $U_x\cap U_y=\varnothing$.

Lemma 3. *(a) Metric spaces are Hausdorff*

(b) If (x_n) has a limit in a Hausdorff space $(X,\mathscr{T})$, then it's unique

Proof. (a) Pick $x\neq y$ in (X,d) with $d(x,y)=\varepsilon>0$. Consider $B_{\frac{\varepsilon}{3}}(x)$ and $B_{\frac{\varepsilon}{3}}(y)$. These obviously are disjoint by the triangle inequality.

(b) Let x_n converge to x, and let $y\neq x$ be some other point besides x. Then, there is some neighborhood of x which is disjoint from a neighborhood of y. Eventually, every x_k is in this neighborhood, meaning none are in the neighborhood of y. Thus, x_n cannot converge to y.

Theorem 0.6. *(a) Suppose X,Y are Hausdorff spaces. Then $X\times Y$ is also Hausdorff.*

(b) If $A \subseteq X$, and X is Hausdorff, then A is also Hausdorff under the subspace topology.

(c) If $(X, \mathscr{T}_X)$ is Hausdorff, and X is homeomorphic to $(Y, \mathscr{T}_Y)$, then Y is also Hausdorff.

Definition 0.17. Let $(X, \mathscr{T}_X), (Y, \mathscr{T}_Y)$ be topological spaces. X and Y are said to be homeomorphic if there exist continuous maps $f : X \to Y$, $g : Y \to X$, such that $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$.