Lecture 1

We begin by trying to gain a deeper understanding of the Cauchy-Riemann equations.
Let $f : X \to \mathbb{C}$, where $X \subset \mathbb{C}^n$. For now, let's say $X \subset \mathbb{C}$. In real analysis, we have a notion of differentiability for $f : \mathbb{R}^n \to \mathbb{R}^k$. We can say that f is differentiable at a point $p \in X$ when

$$f(p+h) = f(p) + (df_p)h + \rho(h)$$

Where $(df)_p : \mathbb{R}^n \to \mathbb{R}^k$ is a linear map $\in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^k)$, and $\frac{|\rho(h)|}{|h|} \to 0$ as $h \to 0$.
So we can think of the "real differential" as a linear map in $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C})$.

Definition 0.1. Let $X \subset \mathbb{C}$, and $f : X \to \mathbb{C}$. Differentiability refers to the existence of a $(df)_p \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$.
So, f is complex differentiable at $p \in X$ means that

$$f(p+h) = f(p) + f'(p)h + \rho(h)$$

Where $f'(p)$ is a complex number and $\frac{|\rho(h)|}{|h|} \to 0$.
If $A \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$, $A(z) = \alpha z, \alpha \in \mathbb{C}$.
So $(df)_p \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$.
$\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$ is a $\mathbb{C}$-vector space of dimension 1.
$\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R}^2)$ is a $\mathbb{C}$-vector space of dimension 2.
So where did the extra dimension go? What happened?
Consider an element of $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$ given by $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x - iy$.
We also have $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x + iy$.
From the real analysis point of view, these two functions are equal to their differentials. The first is called $d\overline{z}$, and the second is called dz.
$dz = dx + idy$ and $d\overline{z} = dx - idy$

$$dx \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = h_1$$
$$dy \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = h_2$$

On a complex vector space, suppose $\phi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{C}^n, \mathbb{C})$, we have $(\overline{\phi})(v) = \overline{\phi(v)}$. So $\overline{dz} = d\overline{z}$.
Now, $\mathbb{C}$-valued real differentiable functions are just pairs of $\mathbb{R}$-valued real differentiable functions.

Example 0.1. If $k, m \in \mathbb{N}$, then $z^k\overline{z}^m : \mathbb{C} \to \mathbb{C}$ is a real smooth function (when viewed as an element of $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R}^2)$), with

$$d(z^k\overline{z}^m) = kz^{k-1}\overline{z}^m + m\overline{z}^{m-1}z^k d\overline{z}$$

We will study the differences between $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C})$ versus $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$, with complex dimensions 2 and 1, respectively.

Definition 0.2. Let V be a real vector space.
A complex structure on V is a $J \in \mathrm{End}_{\mathbb{R}}(V)$ which satisfies $J^2 = -\mathrm{Id}_V$

Proposition 1. *Define $V_J = V$ as a set and group, with a $\mathbb{C}$-action $\mathbb{C} \times V_J \to V_J$ defined by $((\alpha + i\beta), x) \mapsto \alpha x + \beta Jx$.*

Proof. Check $z(wx) = (zw)x$ for all $z, w \in \mathbb{C}$ and $x \in V_J$.

Proposition 2. *If a vector space V admits a complex structure J, then $\dim_{\mathbb{R}} V = 2n$. Further, $\dim_{\mathbb{R}} V = 2\dim_{\mathbb{C}} V_J$.*

Proof. First, $\det(J^2) = \det(-\mathrm{Id}_V) = (-1)^{\dim_{\mathbb{R}} V}$, so the dimension must be even. Alternatively, if $e_1, \ldots, e_n$ is a basis of V_J, then check $e_1, \ldots, e_n, Je_1, \ldots, Je_n$ is a basis of V over R.

Example 0.2. For $\mathbb{R}^2$, let $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We see that $J_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$. This is like $i(x + iy) = ix - y$.
So $A : (\mathbb{R}^2)_{J_0} \to \mathbb{C}$ is an isomorphism of $\mathbb{C}$-vector spaces.
Let W be a vector space over $\mathbb{C}$. Consider $W_{\mathbb{R}}$, a real vector space. We see $\dim_{\mathbb{R}} W_{\mathbb{R}} = 2\dim_{\mathbb{C}} W$. Consider $J : W_{\mathbb{R}} \to W\mathbb{R}$ given by $x \mapsto ix$. Then $J^2 = -\mathrm{Id}_{W_{\mathbb{R}}}$.
Let V be a real vector space with complex structure J. Consider $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} V^*$.
$J^t : \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \to \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$, we can express $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \ni \phi = \phi_1 + \phi_2$, and by definition,

$$J^t\phi = \phi \circ J = \phi_1 \circ J + i\phi_2 \circ J$$

So $(J^t)^2 = -1$.
$J^t \in \mathrm{End}_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}))$.
Main observation: $\phi \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ is $\mathbb{C}$-linear in V_j, meaning $\phi(ix) = i\phi(x)$, which is equivalent to $\phi(Jx) = i\phi(x)$.
In other words, such a ϕ is only $\mathbb{C}$-linear if ϕ is an eigenfunction of J^t with eigenvalue i.

Definition 0.3. ϕ is $\mathbb{C}$-antilinear on V_J means

$$\phi((\alpha + i\beta)x) = \overline{(\alpha + i\beta)}\phi(x)$$

for all $x \in V$.

We denote the space of $\mathbb{C}$-antilinear functionals by $\overline{\text{Hom}}_{\mathbb{C}}(V_J, \mathbb{C})$.
In fact, there is an isomorphism between $\text{Hom}_{\mathbb{C}}(V_J, \mathbb{C})$ and $\overline{\text{Hom}}_{\mathbb{C}}(V_J, \mathbb{C})$ as real vector spaces.

Theorem 0.1. *Let V be a real vector space with complex structure J. Then*

1. $\text{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \text{Hom}_{\mathbb{C}}(V_J, \mathbb{C}) \oplus \overline{\text{Hom}}_{\mathbb{C}}(V_J, \mathbb{C})$.

2. *If* $\text{Hom}_{\mathbb{C}}(V_J, \mathbb{C}) := V^{1,0}$, *and* $\overline{\text{Hom}}_{\mathbb{C}}(V_J, \mathbb{C}) := V^{0,1}$, *then* $\text{Hom}_{\mathbb{R}}(V, \mathbb{C}) = V^{1,0} \oplus_{\mathbb{C}} V^{0,1}$.

3. $\dim_{\mathbb{C}} V^{1,0} = \dim_{\mathbb{C}} V^{0,1} = \frac{\dim_{\mathbb{C}}(\text{Hom}_{\mathbb{R}}(V,\mathbb{C}))}{2}$

Proof. Observe that $\phi \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$ can be written as

$$\phi = \frac{\phi - i\phi \circ J}{2} + \frac{\phi + i\phi \circ J}{2} = \frac{\phi(Jx) + i\phi(x)}{2} = i\frac{\phi - i\phi \circ J}{2}(x) = \phi$$

Further, $V^{1,0} \cap V^{0,1} = 0$ by the definitions, so we are done. ■

Thus, any differential can be split into a $\mathbb{C}$-linear and a $\mathbb{C}$-antilinear part.

Definition 0.4. $\pi^{1,0}$ is projection on the first factor, $\pi^{0,1}$ is projection onto the second. We have

$$\phi = \pi^{1,0}\phi + \pi^{0,1}\phi$$

Corollary 0.2. *If $\phi \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$, then ϕ being $\mathbb{C}$-linear (i.e $\phi \in V^{1,0}$) if and only if $\pi^{0,1}\phi = 0$.*

Definition 0.5. Applying to $(df)_p \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$, then

$$(df)_p = \pi^{1,0} df_p + \pi^{0,1} df_p$$

Say $\pi^{1,0} df_p = \underbrace{\partial f_p}_{\text{complex linear}}$ and $\pi^{0,1} df_p = \underbrace{\overline{\partial} df_p}_{\text{complex antilinear}}$

Theorem 0.3. *A function $f : X \to \mathbb{C}$ is $\mathbb{C}$-differentiable at $p \in X$ if and only if f is $\mathbb{R}$-differentiable at p and $df_p = \partial f_p$, which happens if and only if $\overline{\partial} f_p = 0$.*

Proof. We have $\mathbb{C} \cong \mathbb{R}^2_{J_0}$, which has standard basis $\mathbb{R}^2 = \langle e_1, e_2 \rangle_{\mathbb{R}^2}$. This has a dual basis in $\text{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R})$ given by dx and dy. That is, $\text{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{C}) = \langle dx, dy \rangle_{\mathbb{C}}$.
$J_0 e_1 = e_2$ and $J_0 e_2 = -e_1$, so $dx \circ J_0 = -dy$ and $dy \circ J_0 = dx$.

$$\begin{aligned} \pi^{0,1} dx &= \frac{1}{2}(dx - i dx \circ J_0) \\ &= \frac{1}{2}(dx + i dy) && := dz \end{aligned}$$

Further,

$$\pi^{0,1}dx = \frac{1}{2}d\overline{z}$$
$$\pi^{1,0}dy = \frac{1}{2}dz$$

So

$$df = f_x dx + f_y dy = \frac{f_x - if_y}{2}dz + \frac{f_x + if_y}{d}\overline{z} = \partial f + \overline{\partial} f$$

Definition 0.6.

$$\frac{\partial}{\partial z} = \frac{\partial_x - i\partial_y}{2}$$
$$\frac{\partial}{\partial \overline{z}} = \frac{\partial_x + i\partial_y}{2}$$

So

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}$$

So analyticity is equivalent to $df = \partial f$, meaning $\overline{\partial} f = 0$, which is equivalent to $\frac{\partial f}{\partial \overline{z}} = 0$, which means

$$\frac{\partial(u+iv)}{\partial \overline{z}} = 0$$

So $(\partial_x + i\partial_y)(u+iv) = 0$. Multiplying out, we get

$$u_x = v_y$$
$$u_y = -v_x$$

Lecture 2

The focus for the first bit of this course will be the so-called (by Dennis) $\overline{\partial}$-calculus. Suppose $f : X \to \mathbb{C}$ is differentiable for some $X \subseteq \mathbb{R}^{2n}$. It has a differential $df_p \in \operatorname{Hom}_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{C})$.
f is holomorphic if and only if $df_p \in \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$. Last time, we talked about how the second sits in the first, and how they interact.
Question: How to make $\mathbb{C}^n$ out of $\mathbb{R}^{2n}$. The abstract algebra way to do it is with a complex structure J. If V is a vector space over $\mathbb{R}$, then $\dim_{\mathbb{R}} V = 2n$ $J \in \operatorname{End}_{\mathbb{R}}(V)$ with $J^2 = -\operatorname{Id}_V$.

For all $x \in V_J$, we define $ix = Jx$, so V_J is a vector space over $\mathbb{C}$, and $\dim_{\mathbb{C}} V_j = \frac{\dim_{\mathbb{R}} V}{2}$

Example 0.3. Let $V = \mathbb{R}^2$, $J = J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. $\mathbb{R}^2_{J_0} \cong \mathbb{C}$
Last time, we showed that for any $\phi \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = H$, then

$$\phi = \underbrace{\frac{\phi - i\phi \circ J}{2}}_{\in \mathrm{Hom}_{\mathbb{C}}(V_J, \mathbb{C})} + \underbrace{\frac{\phi + i\phi \circ J}{2}}_{\in \overline{\mathrm{Hom}_{\mathbb{C}}}(V_J, \mathbb{C})}$$

So, $H = \mathrm{Hom}_{\mathbb{C}}(V_J, \mathbb{C}) \oplus \overline{\mathrm{Hom}_{\mathbb{C}}(V_J, \mathbb{C})} = V^{1,0} \oplus V^{0,1}$
Where $V^{1,0}$ and $V^{0,1}$ are what we call $\mathrm{Hom}_{\mathbb{C}}(V_J, \mathbb{C})$ and $\overline{\mathrm{Hom}}_{\mathbb{C}}(V_J, \mathbb{C})$ by tradition.
Let $V = \mathbb{R}^2$, $J = J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
Let $\phi = df_p = \frac{\partial f(p)}{\partial x} dx_p + \frac{\partial f(p)}{\partial y} dy_p$
After doing some computations, we get

$$d_p f = \pi^{1,0} df + \pi^{0,1} df = A dz + B d\overline{z}$$

Where $dz = dx + idy$ and $d\overline{z} = dx - idy$. You can check that the former is in $V^{1,0}$ and the latter in $V^{0,1}$.
The coefficient A is denoted by tradition as $\frac{\partial f}{\partial z}(p)$, and B as $\frac{\partial f}{\partial \overline{z}}(p)$.
<u>Here, the presence of ∂ does not imply any limit taking or anything, they are just notation.</u>
Note $A = \frac{1}{2}(\partial_x f - i\partial_y f)|_p$ and $B = \frac{1}{2}(\partial_x f + i\partial_y f)|_p$.

Definition 0.7. We define

$$\partial f_p \overset{\text{def}}{=} f_z(p) dz_p$$
$$\overline{\partial} f_p \overset{\text{def}}{=} f_{\overline{z}}(p) d\overline{z}_p$$

The former is $\mathbb{C}$-linear, and the second $\mathbb{C}$-antilinear.

Claim. *f is $\mathbb{C}$-differentiable at p if and only if f is $\mathbb{R}$-differentiable at p and $\overline{\partial} f_p = 0$*

Proof. If $f = u + iv$, we get

$$\overline{\partial} f_p = \frac{\partial f}{\partial \overline{z}}(p) = 0$$

Which gives you the Cauchy-Riemann equations.
So f is analytic if $d_p f = f_z dz$.

Example 0.4. What are the following?

1. $\frac{\partial |z|}{\partial z}$

2. $\frac{\partial |z|}{\partial \overline{z}}$

How do we manage these problems?

Claim. *If $m, k \in \mathbb{Z}\backslash\{0\}$, we will consider $d(z^m\overline{z}^k)$. We have $f(z+h) = (z+h)^m(\overline{z} + \overline{h})^m = f(z) + (mz^{m-1}\overline{z}^k)h + (k\overline{z}^{k-1}z^m)\overline{h} + \mathcal{O}(h^2)$*
Where $\frac{|LHS-RHS|}{|z-2|} \to 0$ as $z \to 2$
Then $\overline{\partial}(z^m\overline{z}^k) = (k\overline{z}^{k-1}z^m)d\overline{z}$

Proof. Let's do some examples. Consider $\frac{\overline{z}-1}{z+1}$. We have

$$\begin{aligned}
\frac{\overline{z}-1}{z+1} &= \frac{(\overline{z}-2)+2-1}{(z-2)+2+1} \\
&= \frac{(\overline{z}-2)+1}{3}\frac{1}{1+\frac{z-2}{3}} \\
&= \frac{\overline{z}-1}{3}\left(1 - \frac{z-2}{3} + \mathcal{O}(H^2)\right) \\
&= c + Az + B\overline{z} + \rho(z)
\end{aligned}$$

There are two building blocks for doing problems:

1. First, remember you are really doing real analysis.

2. Use the formula $d_p f = \pi^{1,0} \cdots$

Example 0.5. We will calculate $d|z| = d\sqrt{z\overline{z}}$.

$$\begin{aligned}
d|z| &= d\sqrt{z\overline{z}} \\
&= \frac{1}{2\sqrt{z\overline{z}}}d(z\overline{z}) \\
&= \frac{1}{2|z|}d(z\overline{z}) \\
&= \frac{1}{2|z|}zd\overline{z} + \overline{z}dz
\end{aligned}$$

So the answer would be $\frac{z}{2|z|}$.
So, just express your function as a function of $z\overline{z}$ and proceed to do real analysis.
We know $\partial_z f, \partial_{\overline{z}} f$, and want to find $\partial_z(\overline{f}), \partial_{\overline{z}}(\overline{f}) =?$
We have

$$df = \partial f + \overline{\partial} f$$
$$d(\overline{f}) = \overline{df} = \overline{(\partial f)} + \overline{(\overline{\partial} f)}$$

Now, $\overline{\partial}(\overline{f}) = \overline{\partial f}$. The bottom is equal to $\frac{\partial \overline{f}}{\partial \overline{z}} d\overline{z}$
So

$$\frac{\partial \overline{f}}{\partial \overline{z}} = \overline{\frac{\partial f}{\partial z}}$$

The conjugate of something which is complex anti-linear is complex linear.
What is $\frac{\partial \overline{f}}{\partial z}$? This is $\overline{\frac{\partial f}{\partial \overline{z}}}$.
So, the general procedure is to decompose your function as something linear + something antilinear, and use the sentence I just wrote.
How to compute $\frac{\partial f \circ g}{\partial \overline{z}}$?
Well $\underbrace{d(f \circ g) = df \circ df}_{\text{The chain rule expresses the functoriality of the derivative}}$, which is equal to
$(\partial f + \overline{\partial} f) \circ (\partial g + \overline{\partial} g) = \partial f \circ \partial g + \overline{\partial f} \circ dg + \cdots$.
Now, $\overline{\partial} f \circ \partial g = f_{\overline{z}} d\overline{z} \circ (g_z dz) = f_{\overline{z}} \overline{g_z}$.

Definition 0.8. Suppose $\frac{\partial f}{\partial \overline{z}}(p) = 0$. Then $df_p = \frac{\partial f}{\partial z}(p) dz_p$, and we write this as $f'(p)dz$.

Homework problem: We know f holomorphic, compute $\frac{\partial}{\partial}\overline{z} F(|f|)$, where $F : \mathbb{R} \to \mathbb{R}$ is smooth.
So

$$dF(|f|) = F'(|f|)d|f| = F'(|f|)d\sqrt{f\overline{f}}$$

And $d\sqrt{u} = \frac{1}{2u} du$, so the above is equal to

$$F'(|f|)\frac{1}{2|f|} d(f\overline{f})^{-1} = \frac{F'(|f|)}{2|f|}(\overline{f} df + f d\overline{z}) = \frac{F'(|f|)}{2|f|}(\overline{f} f_z dz + f \overline{f_z} d\overline{z})$$

Our answer is thus whatever we get in front of $d\overline{z}$, so in this case the solution is

$$\frac{\partial}{\partial \overline{z}} F(|f|) = \frac{F'(|f|)}{2|f|} f \overline{f'}$$

The $|z| = \sqrt{z\overline{z}}$ is a very useful trick.
Complex analysis is kind of a local study of $f : X \to \mathbb{C}$ for some $X \subseteq \mathbb{C}$, where f is differentiable, $\overline{\partial} f = 0$, or equivalently $\frac{\partial f}{\partial \overline{z}} = 0$ in X. Really, we are studying solutions to a certain PDE (Cauchy-Riemann equation).

Suppose you want $u_{xx} - u_{yy} = 0$. If this is the case (and it turns out exactly when this is the case), we can express $u = \phi(x-y) + \psi(x+y)$ for ϕ, ψ arbitrary of 1 variable. What if we want to study $u_y = u_{xx}$ in, say, $y > -\varepsilon$? We actually have a formula:

$$u(x,y) = \frac{1}{2\pi y}\int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4y}} u(s,0)ds, y > 0$$

Suppose we want to study $u_{yy} + u_{xx} = 0$ in $y > -\varepsilon$, or maybe an open ball around the origin. We have a formula

$$u(x,y) = \int_{-\infty}^{\infty} P_H(x, y-s)u(s,0)ds$$

Where P_H is the Poisson Kernel.

MAIN LOCAL THM

Let $f : B_{R+\varepsilon} \to \mathbb{C}$. Then the following statements are all equivalent.

1. For any $p \in B_R$, f is differentiable and $df_p = \partial f_p$, so $\overline{\partial} f_p = 0$.

2. $f(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{z-w} dw$

3. Like 8 other things

What exactly is complex integration?
Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth map. We integrate functions over maps. If ϕ is a continuous function, the formula is

$$\int_\gamma \phi(z)dz = \int_a^b \phi(\gamma(t))\dot{\gamma}(t)dt$$

where $\dot{\gamma}$ is the time-derivative of γ, so will be a complex number.
Let $t \in [0, 2\pi]$, $C_R(t) = Re^{it}$, a circle going counterclockwise. Let $\gamma(t) = Re^{-it}$. Then $\gamma^{-1} : [a,b] \to \mathbb{C}$ is the same but in the opposite direction. We have

$$\int_{\gamma^{-1}} \phi\, dz = \int_\gamma \phi\, dz$$

Lecture 3

We continue with the Local Theorem for analytic functions. Let $f : B_{R_0+\varepsilon} \to \mathbb{C}$. Then the following statements are all equivalent.

(i) $\forall z \exists$ a finite $\lim_{\Delta z \to \infty} \frac{f(z+\Delta z) - f(z)}{\Delta z} \stackrel{\text{def}}{=} f'(z)$.

(ii) $\forall z$, f is $\mathbb{R}$-differentiable at z and $df_z = \partial f_z$, which is equivalent to $\overline{\partial} f_z \equiv 0$, which gives the Cauchy-Riemann equations.

(iii) f is continuous, and for all $R < R_0$, for all z such that $|z - a| < R$, then

$$f(z) = \frac{1}{2\pi i} \int_{C_R(a)} \frac{f(w)}{w - z} dw$$

(iv) $f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$ in $B_R(a)$ with $a_n = \frac{1}{2\pi i} \int_{C_R(a)} \frac{f(w)}{(w-a)^{n+1}} dw$ for all $R < R_0$.

(v) For some coefficient c_n, $f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$ for all $|z - a| < R_0$ for some c_n.

(vi) For all z, $\exists f'(z), f''(z), \ldots, f^{(n)}(z), \ldots$. Moreover, $f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$ for all $|z - a| < R_0$ with

$$c_n = \frac{f^{(n)}(a)}{n!}$$

and

$$f'(z) = \sum_{n=1}^{\infty} n c_n (z - a)^{n-1}$$

for all $|z - a| < R_0$.

Proof. We already proved (i) and (ii) equivalent. $(ii) \implies (iii)$ will come later, and the steps $(iv) \implies (v) \implies (vi)$ are proven by undergrad power series techniques. Then of course $(vi) \implies (i)$ is a triviality.

Claim. $(iii) \implies (iv)$.

Proof. Recall the setup: (iii) holds in a disk of radius R, which is δ less than R_0. We have $f(z) = \frac{1}{2\pi i} \int_{C_R(a)} \frac{f(w)}{z-w} dw$. Now,

$$\frac{1}{w - z} = \frac{1}{(w - z) - (z - a)} = \frac{1}{w - a} \cdot \frac{1}{1 - \frac{z-a}{w-a}} = \sum_{n=0}^{\infty} \frac{(z - a)^n}{(w - a)^{n+1}}$$

This works because $|z - a| < |w - a|$. So

$$f(z) = \frac{1}{2\pi i} \int_{C_R(a)} \left(\sum_{n=0}^{\infty} (z - a)^n \frac{f(w)}{(w - a)^{n+1}} \right) dw$$

We want to swap the integral and the summation, and then we will be done. We can swap an integral and a sum provided the sum converges uniformly on $[a, b]$. We will show that this sum converges uniformly using the M test.

Let $M_n = \sup_{[a,b]} |f_n|$. Then if $\sum_{n\in\mathbb{N}} M_n$ converges, then $\sum_{n\in\mathbb{N}} f_n$ converges uniformly
.
We have to calculate $\sup_{|w-a|=R} |(z-a)^n \frac{f(w)}{(w-a)^{n+1}}| = M_n$. We find

$$M_n = \frac{(R-\delta)^n}{R^{n+1}} \left(\frac{\sup_{\overline{B_R(a)}} |f|}{R} \right)$$

so $\sum_{n\in\mathbb{N}} M_n < \infty$

■

We make use of the trick where $\frac{1}{A+B} = \frac{1}{A} \cdots \frac{1}{1+\frac{B}{A}}$, and if $\frac{B}{A}$ is small we can do a series expansion.
Now, we will prove the $\triangle$ inequality for complex integrals.
Let $\gamma : [a,b] \to \mathbb{C}$ be C^1 or piecewise smooth. , and let $\phi : \mathbb{C} \to \mathbb{C}$ be continuous. By definition, we have

$$\begin{aligned} \left| \int_\gamma f(z)\, dz \right| &= \int_a^b \phi(\gamma(t))\dot{\gamma}(t)\, dt \\ &\leq \int_a^b |\phi(\gamma(t))||\dot{\gamma}(t)|\, dt \\ &=: \int_\gamma |\phi(z)||dz| \end{aligned}$$

We have used the fact that for continuous real functions $f : \mathbb{R} \to \mathbb{R}$ we have

$$\left| \int_a^b f(t)\, dt \right| \leq \int_a^b |f(t)|\, dt$$

We are moving towards proving $(ii) \implies (iii)$. The next result we need Goursat's Lemma.

Definition 0.9. Let $X \subset \mathbb{C}$ be open. If $f'(z)$ exists for all $z \in X$, we say $f \in A(X)$ or $f \in \mathscr{H}(X)$.

Lemma 1. *(Goursat's Lemma)*
Let a solid $\triangle \subset \Omega \subset \mathbb{C}$, with Ω open ($\triangle$ is a 2-simplex).
Denote the boundary of $\triangle$ by T, with the counter-clockwise orientation. Then for any $f \in \mathscr{H}(\Omega)$, we have

$$\int_T f\, dz = 0$$

Proof. Recall the Cauchy-Riemann system

$$u_x = v_y$$
$$u_y = -v_x$$

We have from Calc 3 the classic Green's theorem:

$$\int_{\partial\Omega} P dx + Q dy = \int \int_{\Omega} (\text{something}) dx dy$$

Using these together will give the proof.

Step 1

First, we will seek a contradiction. Suppose that there is some $\varepsilon_0 > 0$ such that

$$\left| \int_T f\, dz \right| \geq \varepsilon_0$$

We can perform barycentric subdivision to $\triangle$ (pictures forthcoming until I figure out how to insert them in TeXwriter), and we can express this integral as the sum of the integrals of four sub-simplices, $T_1, \dots, T_4$. We must have $|\int_{T_i} f\, dz| \geq \frac{\varepsilon_0}{4}$. Denote this sub-simplex by T_1, and it's interior by $\triangle_1$.
So we have

$$\left| \int_{T_1} f\, dz \right| \geq \frac{\varepsilon_0}{4}$$

Now, $\text{length}(T_1) = \frac{\text{length}(T)}{2} = \frac{c_0}{2}$, and $\text{diam}(\triangle_1) = \frac{\text{diam}(\triangle)^2}{2} = \frac{c_1}{2}$.
We can keep proceeding by doing barycentric subdivision to T_1.
So we have a sequence $\triangle_1 \supset \triangle_2 \supset \dots$, with $\partial \triangle_i = T_i$, with $|\int_{T_i} f\, dz| \geq \frac{\varepsilon_0}{4^i}$. Further, the length of T_j is $\frac{c_0}{2^j}$, and the diameter of $\triangle_j$ is $\frac{c_1}{2^j}$.
We can see that $\cap_{j=1}^{\infty} \triangle_j = \{p\}$, and without loss of generality we can assume $p = 0 \in \Omega$.

Step 2

We have $f(z) = f(0) + f'(0)z + \rho(z)$, where $\frac{|\rho(z)|}{|z|} \to 0$. So

$$\int_{\gamma} 1\, dz = \int_a^b 1 \cdot \dot{\gamma}(t) dt = \gamma(b) - \gamma(a)$$

So

$$\int_\gamma z\,dz = \int_a^b \gamma(t)\dot\gamma(t)\,dt = \frac{1}{2}\int_a^b ((\gamma(\dot{t})^2))\,dt = \frac{\gamma(b)^2 - \gamma(a)^2}{2}$$

So

$$\int_{T_j} f\,dz = \int_{T_j} f(0)\,dz + \int_{T_j} f'(0)\,dz + \int_{T_j} \rho(z)\,dz = 0 + 0 + \left|\int_{T_j}\rho(z)\,dz\right|$$

So

$$\begin{aligned}\left|\int_{T_j} f\,dz\right| &\le \int_{T_j} |\rho(z)||dz| = \int_{T_j} \frac{|\rho(z)|}{|z|}|z||dz| \\ &\le \sup_{\triangle_j} \frac{|\rho(z)|}{|z|}\operatorname{diam}(\triangle_j)\operatorname{length}(T_j)\end{aligned}$$

But $\frac{\varepsilon_0}{c_0c_1 4^i} \le \sup_{bigt_j} \frac{|\rho(z)|}{|z|}\frac{c_0c_1}{4^j}$, which for large d contradicts our assumption that $\frac{|\rho(z)|}{|z|} \to 0$, proving Gousat's Lemma.
Now that we have Goursat's Lemma, we can prove $(ii) \implies (iii)$.

Proposition 3. $(ii) \implies (iii)$.

Proof. **Step 1**

By Goursat, for $F \in \mathscr{H}(B_{R+\varepsilon}(a))$, $\int_{C_R(a)} F\,dw = \int_{C_\varepsilon(z)} F\,dw$ when $z \in B_R(a)$. We will show this in step 2.
Now, consider the map $w \mapsto \frac{f(w)}{w-z}$. We have

$$\int_{C_R(a)} \frac{f(w)}{w-z}\,dz = \int_{C_\varepsilon(z)} \frac{f(w)}{w-z}\,dz = \int_0^{2\pi} \frac{f(z+\varepsilon e^{it})}{\varepsilon e^{it}}\,dz = i\int_0^{2\pi} f(z+\varepsilon e^{it})\,dt = if(z)2\pi i$$

Lecture 4

Suppose we have a function F, which is holomorphic in $B_a(R+\varepsilon)\backslash\{z\}$. $F(w) = \frac{f(w)}{w-z}$, with $f \in \mathscr{H}(B_{R+\varepsilon}(a)$. We have

$$\int_{C_R(a)} F(w)\,dw =_* \int_{C_\varepsilon(z)} F(w)\,dw$$

Once we have established this, we can take the limit as $\varepsilon \to 0$ to get the Cauchy formula $f(z) = \cdots$.
We need some facts to prove $*$

Fact 1

Suppose a circle has N points distributed on it's circumference. Any two adjacent points forms an angle with the center. Let φ_N be the max of these angles. Note $\varphi_N \to 0$ as $N \to \infty$. Let L_n be the polygonal curve formed by the secants between adjacent points. Then

$$\int_{L_n} \phi(w)\, dw \to \int_{C_r(a)} \phi(w)\, dw$$

for ϕ continuous. The proof is easy, just break up the integrals and use uniform convergence.

Fact 2

Let A_1, A_2, A_3, A_4 be points in a a ball, and consider the loop going from $A_1 \to A_2$, then $A_2 \to A_3$, then $A_3 \to A_4$, and finally $A_4 \to A_1$.
Suppose that z is not in the convex hull of $\{A_1, A_2, A_3, A_4\}$.
Then $\int_{A_1 \to \cdots \to A_1} F(w)\, dw = 0$. We calculate this by breaking this loop up into the sum of two simplices, and the integral over the simplices will be zero by Goursat's lemma.
We can then approximate the region in between $B_a(R+\varepsilon)$ and $B_z(\varepsilon)$ by a collection of loops of the above form.
This completes the proof

$\blacksquare$

Theorem 0.4. *(Inverse Function Theorem)*
Let $f \in \mathscr{H}(\Omega)$, with $\Omega \subseteq \mathbb{C}$ open. Let $z_0 \in \Omega$.
Suppose that $f'(z_0) \neq 0$. Then there exists a $\varepsilon > 0$ such that $f|_{B_\varepsilon(z_0)}$ is a biholomorphism onto its image, which is open.
Moreover, there is a $g : f(B_\varepsilon(z_0)) \to B_\varepsilon(z_0))$ which is holomorphic, such that $g \circ f = \mathrm{Id}$.
In other words, g is a "local inverse" to f (think two branches of square root, etc.)

Proof. Let $z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$
Just looking at f purely as a smooth function from $\mathbb{R}^2 \to \mathbb{R}^2$, we have

$$\det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = u_x v_y - u_y v_x$$

By Cauchy-Riemann, we have $f' = u_x + iv_x$, so $u_x^2 + v_x^2 = |f'|^2|_{(x_0,y_0)} \neq 0$. This immediately gives bijectivity of the map f, and the fact that its image is open.
Now $f^{-1} : f(B_\varepsilon(z_0)) \to B_\varepsilon(z_0)$ is a real C^∞ function. We just need to prove it is analytic.

We know $\mathrm{Id}_{B_\varepsilon(z_0)} = g \circ f$, so

$$\begin{aligned}\mathrm{Id} &= dg \circ df \\ &= (\partial g + \overline{\partial} g) \circ (\partial f + \overline{\partial} f) \\ &= \partial g \circ \partial f + \overline{\partial} g \circ \partial f\end{aligned}$$

We must conclude that $\overline{\partial} g \circ \partial f$ is both $\mathbb{C}$-linear and $\mathbb{C}$-antilinear, meaning it is identically zero. Now, ∂f is linear and bijective, thus, $\overline{\partial} g = 0$, so $dg = \partial g$, so g is holomorphic.

■

If $\det df = 0$ for a real smooth function, then we can't really say anything about f. However, suppose $f'(z) = 0$, but $f'(z) \not\equiv 0$. Then we know a lot about the structure of f.
A homework problem asks to show the following: let $K \subseteq \Omega \subseteq \mathbb{C}$, with $\Omega \neq K$, K compact, Ω open.
By compactness, $d(K, \mathbb{C} \backslash \Omega) = \delta > 0$. So

$$f(z) = \frac{1}{2\pi i} \int_{C_{\frac{\delta}{2}}(a)} \frac{f(w)}{w - z}\, dz$$

Then

$$f'(a) = \frac{1}{2\pi i} \int_{C_{\frac{\delta}{2}}(a)} \frac{f(w)}{(w - a)^2}\, dw$$

And now you can do some bounding shenanigans to get $\leq \frac{\sup |f|}{(\frac{\delta}{2})^2}$
By controlling the derivative, we may control the derivative. The reverse is not true: no matter how we restrict the variance of f, the derivative can do strange things.
But for solutions to PDEs, the control goes the other way. That is, if you control $\sup |f|$, this controls the derivative, a fact which is not true in general. And every holomorphic function is a solution to a PDE.
So if a sequence of functions converges uniformly, so does the sequence of their derivatives, and so on.

Theorem 0.5. *(Liouville)*
Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function (meaning holomorphic on all of $\mathbb{C}$), such that $\sup_{\mathbb{C}} |f| < \infty$. Then $f(z)$ is constant.

Remark. This gives the easiest proof of the fundamental theorem of algebra:
Suppose we have a nice non-constant polynomial p without a root. Then $\frac{1}{p}$ is entire and bounded, so $\frac{1}{p}$ must be constant, so p is constant, a contradiction (unless p is constant in the first place).
Thus p must have a root.

Proof. Write

$$f(z) = \frac{1}{2\pi i}\int_{C_R(z_0)} \frac{f(w)}{w - z_0}\,dw$$

Then

$$f' = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{(w - z_0)^2}\,dw$$

The complex triangle inequality tells us

$$|f'| \le c\left|\int_{C_r(z_0)} \frac{f(w)}{(w-z_0)^2}\right||dw| \le c\frac{\sup_{\mathbb{C}}|f|}{R^2}2\pi R = c\frac{\sup_{\mathbb{C}}|f|}{R}$$

Taking a limit as $R \to \infty$, we see that $|f'(z_0)| = 0$. But z_0 was arbitrary, so f is constant.

Theorem 0.6. *(Factorization)*
Let $f : B_\varepsilon(a) \to \mathbb{C}$ be holomorphic, and suppose $f(a) = 0$ and $f' \not\equiv 0$.
Then there exists an $N \in \mathbb{N}$ such that $f(z) = (z-a)^N \phi(z)$, with ϕ holomorphic in $B_\varepsilon(a)$, and $\phi(a) \neq 0$.

Proof. Write
Find the first N such that the Nth derivative of f is nonzero.

$$f(z) = \sum_{n=1}^{\infty} c_n(z-a)^n = \sum_{n=N}^{\infty} c_n(z-a)^n = (z-a)^n(c_N + c_{N+1}(z-a) + \cdots)$$

Corollary 0.7. *The zeroes of a holomorphic function are isolated, unless the function is identically zero.*

Corollary 0.8. *(analytic continuation principle)*
Let $f, g : B_\varepsilon(a) \to \mathbb{C}$ be holomorphic, and let $f(z_j) = g(z_j)$ for some sequence $z_j \to a$, $z_j \neq a$ for all j.
Then $f = g$ in $B_\varepsilon(a)$.
This follows directly from the zeroes of non-zero holomorphic functions being isolated.

■

Remark. Let Ω_1, Ω_2 be open, $\Omega_1 \subset \Omega_2 \subset \mathbb{C}$, with $f, g : \Omega_2 \to \mathbb{C}$ holomorphic. Suppose that $f|_{\Omega_1} = g|_{\Omega_1}$.
Then $f = g$ in Ω_z.

Proof. Let $a \in \Omega_1$, let $\gamma : I \to \Omega_2$ be a path from a to $p \in \Omega_2$. Let $G = \{t \in I \mid |f(\gamma(t)) - g(\gamma(t))| = 0$. This set is closed, and moreover $[0, \varepsilon) \subset G$ for any $0 < \varepsilon < 1$. We can then prove that G is open.
But for G to be a clopen subset of I means that $G = I$ (also because G is nonempty).