Lecture 1

Definition 0.1. A metric space is a set X equipped with a function $d : X \times X \to \mathbb{R}$, which satisfies the following axioms:

1. For any $x, y \in X$, $d(x, y) = d(y, x)$

2. For any $x, y, z \in X$, we have $d(x, y) \leq d(x, z) + d(z, y)$. This is called the "triangle inequality"

3. For any $x, y \in X$, $d(x, y) = 0$ exactly when $x = y$

Example 0.1. For $x, y \in \mathbb{R}^n$,

$$d(x, y) := \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

This is called the Euclidean distance. 2 can be replaced with any real $r \geq 1$, and it will still be a metric.

Example 0.2. In this example, $C[0, 1]$ is the set of all continuous functions $f : [0, 1] \to \mathbb{R}$. Here,

$$d(f, g) := \sup_{x \in [0,1]} |f(x) - g(x)|$$

Example 0.3. Let $X = \mathbb{N}$, the natural numbers, including 0. Let p be a fixed prime number. The p-adic metric on $\mathbb{N}$ is defined by

$$d_p(a, b) := \frac{1}{p^\alpha}$$

Where p^α is the largest power of p which divides $|a - b|$. So two naturals are "close" if their difference is divisible by a high power of p.

Claim. *This is a metric*

Proof. The 1st and 3rd axioms are clear. So we must prove the triangle inequality. We will consider the three quantities $d_p(a, b), d_p(a, t)$, and $d_p(b, t)$, where $a, b, t \in \mathbb{N}$. Suppose p^β divides both $a - t$ and $t - b$. Then p^β divides $(a - t) + (t - b) = a - b$. Therefore,

$$d_p(a, b) \leq \frac{1}{p^\beta} \leq \max(d_p(a, t), d_p(t, b)) \leq d_p(a, t) + d_p(t, b)$$

■

Definition 0.2. Let $(X, d_X), (Y, d_Y)$ be two metric spaces. For a function $f : X \to Y$, we say that f is <u>continuous</u> at $x_0 \in X$ if, for all $\epsilon > 0$, there is a $\delta > 0$ such that

$$0 < |d_X(x_0, x)| < \delta \implies |d_Y(f(x_0), f(x))| < \epsilon$$

A function $f : X \to Y$ is said to be continuous if it is continuous at x for all $x \in X$.

Example 0.4. Consider a map $(\mathbb{N}, d_5) \to (\mathbb{N}, d_5)$ defined by

$$x \mapsto x^2$$

Is this continuous?
At 0, to be continuous, then for any x, if we want to get within a small distance of 0, then x has to be divisible by large powers of 5.
What about at 11?
This is continuous.

Example 0.5. What about $(\mathbb{N}, d_5) \to (\mathbb{N}, d_{17})$.

Lecture 2

Theorem 0.1. *If $f : (X, d_X) \to (Y, d_Y)$ and $g : (Y, d_Y) \to (Z, d_Z)$ are both continuous, then $g \circ f : (X, d_X) \to (Z, d_Z)$*

Proof. Fix $x \in X$ and $\varepsilon > 0$. Choose $\delta_1 > 0$ so that if $d_Y(f(x), y) < \delta_1$, then $d_Z(gf(x), g(y)) < \varepsilon$.
By continuity of f, we may then choose a $\delta_0 > 0$ such that if $d_X(x, x') < \delta_2$, then $d_Y(f(x), f(x')) < \delta_1$.

■

Definition 0.3. For a metric space (X, d_X), and a real $r > 0$, the open r-ball around a point x is defined as

$$B_r(x) = \{x' \in X \mid d(x, x') < r\}$$

Exercise: State and prove some theorem about the existence of a function from $X \times X' \to Y \times Y'$, given a function $f : X \to Y$ and $g : X' \to Y'$.

Example 0.6. Balls

1. In $\mathbb{R}^2$, consider

$$d_r\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = \left(\sum_{i=1}^{2} (x_i - y_i)^r\right)^{\frac{1}{r}}$$

For $r = 2$, this is the usual euclidean distance. For $r = 1$, the balls look like diamonds. In the limit, as $r \to \infty$, the metric will approach what is known as the "box metric," in which the distance between any point and 0 is it's largest coordinate.

2. On $C[0,1]$, the set of continuous functions from $[0,1]$ to $\mathbb{R}$, we have the sup metric:

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$$

3. We also have

$$d_1(f,g) = \int_0^1 d|f(x) - g(x)|dx$$

Definition 0.4. For a metric space (X, d_X), suppose that $U \subseteq X$ is said to be "open" if, for any $x \in U$, there is a $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

Lemma 1. *$B_\varepsilon(x)$ is always open.*

Proof. Let $y \in B_\varepsilon(x)$. Let $t = d(x,y)$. By construction, $t < \varepsilon$. Let $\delta = \varepsilon - t$. Consider $B_\delta(y)$. For any $z \in B_\delta(y)$, we have by the triangle inequality

$$d(x,z) \leq d(x,y) + d(y,z) < \varepsilon - t + t = \varepsilon$$

and so $z \in B_\varepsilon(x)$. z was arbitrary, so we are done. ∎

Lecture 3

Definition 0.5. Let (X,d) be a metric space. A set $U \subseteq X$ is said to be open if for all $x \in U$, there is an $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq U$.

Theorem 0.2. *Let $\{U_\alpha\}_{\alpha\in A}$ be a collection of open sets. Then*

1. *$\cup_{\alpha\in A} U_\alpha$ is open.*

2. *Let $U_1, \ldots, U_n$ be a finite subcollection. Then $\cap_{i=1}^{\infty} U_i$ is open.*

Proof.

Lecture 4

Definition 0.6. Two metrics d_1, d_2 are said to be equivalent on a space X if any set which is open under the d_1-induced topology is also open under the d_2-induced topology, and vice versa.

Example 0.7. In $\mathbb{R}^2$,

$$\begin{aligned} d_2(0,(x,y)) &= (x^2+y^2)^{\frac{1}{2}} \\ d_1(0,(x,y)) &= |x|+|y| \\ d_\infty(0,(x,y)) &= \max\{|x|,|y|\} \end{aligned}$$

How do these metric's unit balls compare? In fact, d_1's is within d_2's, which is within d_∞'s.
But all of these balls contain a ball of radius $\frac{1}{2}$ in any of the three metric. Thus, these are equivalent.

Definition 0.7. Two metrics d_1, d_2 are called Lipschitz equivalent if there exists some $k \in \mathbb{R}$ such that, for all $x, y \in X$, we have

$$\frac{1}{k} d_2(x,y) < d_1(x,y) < k d_2(x,y)$$

Example 0.8. This is a non-example. The 5-adic and the 17-adics are not equivalent.

Example 0.9. Let

$$\begin{aligned} d_1(f,g) &= \int_0^1 |f(x)-g(x)|\, dx \\ d_\infty(f,g) &= \sup_{x\in[0,1]} |f(x)-g(x)| \end{aligned}$$

One controls for area, one controls for the maximum value of f. We have

$$B_\varepsilon^{d_\infty}(0) \subset B_\varepsilon^{d_1}(0)$$

This is because if we control for the maximum size of f, we can surely control for the area under it. However, no matter how much we limit the area under f, there is some f which has that much area which has a sup greater than some ε which is fixed.

Theorem 0.3. *Let d_1, d_2 be equivalent metrics on X. Then the following are equivalent*

1. *$f : X \to Y$ is d_1, d_Y continuous if and only if it is d_2, d_Y continuous.*

2. *$g : Z \to X$ is d_Z, d_1 continuous if and only if if is d_Z, d_2 continuous.*

Proof. **1.** Let $U \subset Y$ be open. We know that the preimage $f^{-1}(U)$ is open under the d_1 metric. But by the equivalence of d_1, d_2, $f^{-1}(U)$ must also be open under the d_2 metric. The reverse argument also holds.

2. Let $U \subset X$ be an open set under the d_1 metric. By continuity of g, we know $g^{-1}(U)$ is open. However, U must also be open under the d_2 metric, meaning that g must be continuous with respect to both metrics.

■

Recall: If $\{U_\alpha\}_{\alpha \in A}$ is a collection of open sets, then $\cup_{\alpha \in A} U_\alpha$ is open.
We can associate to any set $R \subset X$ an open set, called the interior of R.

Definition 0.8. For any $\mathbb{R} \subset X$, we define it's interior by

$$\text{int}(R) = \bigcup_{U \text{ open}, U \subseteq R} U$$

We can say several things about $\text{int}(R)$.

1. $\text{int}(R)$ is open.

2. If U is open, then $\text{int}(U) = U$, and vice versa.

3. Suppose $A \subseteq B$. Then $\text{int}(A) \subseteq \text{int}(B)$.

Recall: If $\{C_\alpha\}_{\alpha \in A}$ are all closed, then $\cap_{\alpha \in A} C_\alpha$ is closed.

Definition 0.9. If $R \subseteq X$, then the closure of R, denoted by $\overline{R}$, or sometimes $\text{cl}(R)$, is defined as

$$\overline{R} := \bigcap_{C \text{ closed}, C \supseteq R} C$$

Analagously,

1. $\overline{R}$ is closed for any R.

2. R is closed if and only if $R = \overline{R}$.

3. If $A \subset B$, then $\text{cl}(A) \subset \text{cl}(B)$.

Proposition 1. *Let $x \in \text{cl}(R)$. Then, for all $\varepsilon > 0$, $B_\varepsilon(x) \cap R \neq \varnothing$, and vice-versa. We will prove this next lecture.*

Lecture 4

Proof. First, suppose that $x \notin \overline{A}$. The complement of $\overline{A}$ is open, so there exists a $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq (\overline{A})^c$, and so $B_\varepsilon(x) \cap \overline{A} = \varnothing$.
Now, suppose that there exists some $\varepsilon > 0$ such that $B_\varepsilon(x) \cap A = \varnothing$. Then $(B_\varepsilon(x))^c$ is a closed set containing A which does not contain x, thus $\overline{A}$ cannot contain x.

■

Now, it is time for the main event.

TOPOLOGICAL SPACES

Let X be a set. Let $\mathscr{P}(X)$ denote the power set of X, which is the set of all subsets of X.

Definition 0.10. Let $\mathscr{T} \subset \mathscr{P}(X)$. $\mathscr{T}$ is a topology on X if it has the following properties:

1. $\varnothing, X \in \mathscr{T}$.

2. If $\{U_\alpha\}_{\alpha \in A}$ with each $U_\alpha \in \mathscr{T}$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathscr{T}$.

3. If $A, B \in \mathscr{T}$, then $A \cap B \in \mathscr{T}$. Of course, we can "strengthen" this to the equivalent statement: if $U_1, \ldots, U_k \in \mathscr{T}$, then $\cap_{i=1}^k U_i \in \mathscr{T}$

Elements of $\mathscr{T}$ are called "open sets."

Example 0.10. Here are some simple examples.

1. Open sets in (X, d) form a topology (hence the definition)

2. $\mathscr{T} = \mathscr{P}(X)$ forms the discrete topology.

3. $\mathscr{T} = \{\varnothing, X\}$ forms the indescrete topolgy.

4. If $X = \{0, 1\}$, then $\mathscr{T} = \{\varnothing, X, \{0\}\}$ forms a topology.

5. We can form the Zariski topology on $\mathbb{R}$ by specifying the closed sets, which satisfy a similar but slightly different set of axioms. We define the closed sets to be $\varnothing, \mathbb{R}$, and any finite collection of points.

More generally, the Zariski topology on some ring R is specified by its closed sets, which are the solution locii of some set of polynomials in R.

6. Let $X = \mathbb{R}$, $\mathscr{T} = \{(r, \infty) \mid r \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$.

We have now defined a set of objects (topological spaces). Now, we want to define morphisms, the maps between spaces.

Definition 0.11. Let $(X, \mathscr{T}_X), (Y, \mathscr{T}_Y)$ be topological spaces. We say a function $f : X \to Y$ is continuous if, for all $V \in \mathscr{T}_Y$, $f^{-1}(V) \in \mathscr{T}_X$. In other words, the preimage of any open subset of Y is an open subset of X.

Example 0.11. Some baby examples

1. Any function $f : (X, \text{discrete}) \to (Y, \mathscr{T}_Y)$ is continuous as long as X has the discrete metric, as the preimage of any open set will be a subset of X, all of which are open under the discrete topology.

2. Any function $f : (X, \mathscr{T}_X) \to (Y,\ \text{discrete})$ is continous for a similar reason.

3. $\mathrm{Id}_X : (X, \mathscr{T}_X) \to (X, \mathscr{T}_X)$ is continuous.

Theorem 0.4. *Let $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ and $g : (Y, \mathscr{T}_Y) \to (Z, \mathscr{T}_Z)$ be continuous functions. Then $g \circ f : (X, \mathscr{T}_X) \to (Z, \mathscr{T}_Z)$ is continuous.*
In other words, the composition of continuous functions is continuous.

Proof. Pick $W \in \mathscr{T}_Z$. Then $g^{-1}(W) \in \mathscr{T}_Y$ since g is continuous. So, because f is continuous,

$$(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) \in \mathscr{T}_X$$

Hence, $g \circ f$ is continuous.

Definition 0.12. For $\mathscr{T}$ a topology on X, a <u>basis</u> for $\mathscr{T}$ is a subset $\mathscr{B} \subseteq \mathscr{T}$ such that every set in $\mathscr{T}$ is the union of sets in $\mathscr{B}$.