Lecture 1

We begin by trying to gain a deeper understanding of the Cauchy-Riemann equations.
Let $f : X \to \mathbb{C}$, where $X \subset \mathbb{C}^n$. For now, let's say $X \subset \mathbb{C}$. In real analysis, we have a notion of differentiability for $f : \mathbb{R}^n \to \mathbb{R}^k$. We can say that f is differentiable at a point $p \in X$ when

$$f(p+h) = f(p) + (df_p)h + \rho(h)$$

Where $(df)_p : \mathbb{R}^n \to \mathbb{R}^k$ is a linear map $\in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^k)$, and $\frac{|\rho(h)|}{|h|} \to 0$ as $h \to 0$.
So we can think of the "real differential" as a linear map in $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C})$.

Definition 0.1. Let $X \subset \mathbb{C}$, and $f : X \to \mathbb{C}$. Differentiability refers to the existence of a $(df)_p \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$.
So, f is complex differentiable at $p \in X$ means that

$$f(p+h) = f(p) + f'(p)h + \rho(h)$$

Where $f'(p)$ is a complex number and $\frac{|\rho(h)|}{|h|} \to 0$.
If $A \in \mathrm{Hom}_{\mathbb{C}})(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$, $A(z) = \alpha z, \alpha \in \mathbb{C}$.
So $(df)_p \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$.
$\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$ is a $\mathbb{C}$-vector space of dimension 1.
$\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R}^2)$ is a $\mathbb{C}$-vector space of dimension 2.
So where did the extra dimension go? What happened?
Consider an element of $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$ given by $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x - iy$.
We also have $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x + iy$.
From the real analysis point of view, these two functions are equal to their differentials. The first is called $d\bar{z}$, and the second is called dz.
$dz = dx + idy$ and $d\bar{z} = dx - idy$

$$dx \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = h_1$$
$$dy \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = h_2$$

On a complex vector space, suppose $\phi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{C}^n, \mathbb{C})$, we have $(\overline{\phi})(v) = \overline{\phi(v)}$. So $\overline{dz} = d\bar{z}$.
Now, $\mathbb{C}$-valued real differentiable functions are just pairs of $\mathbb{R}$-valued real differentiable functions.

Example 0.1. If $k, m \in \mathbb{N}$, then $z^k\overline{z}^m : \mathbb{C} \to \mathbb{C}$ is a real smooth function (when viewed as an element of $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R}^2)$), with

$$d(z^k\overline{z}^m) = kz^{k-1}\overline{z}^m + m\overline{z}^{m-1}z^k d\overline{z}$$

We will study the differences between $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C})$ versus $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$, with complex dimensions 2 and 1, respectively.

Definition 0.2. Let V be a real vector space.
A complex structure on V is a $J \in \mathrm{End}_{\mathbb{R}}(V)$ which satisfies $J^2 = -\mathrm{Id}_V$

Proposition 1. *Define $V_J = V$ as a set and group, with a $\mathbb{C}$-action $\mathbb{C} \times V_J \to V_J$ defined by $((\alpha + i\beta), x) \mapsto \alpha x + \beta Jx$.*

Proof. Check $z(wx) = (zw)x$ for all $z, w \in \mathbb{C}$ and $x \in V_J$.

Proposition 2. *If a vector space V admits a complex structure J, then $\dim_{\mathbb{R}} V = 2n$. Further, $\dim_{\mathbb{R}} V = 2\dim_{\mathbb{C}} V_J$.*

Proof. First, $\det(J^2) = \det(-id_V) = (-1)^{\dim_{\mathbb{R}} V}$, so the dimension must be even. Alternatively, if $e_1, \ldots, e_n$ is a basis of V_J, then check $e_1, \ldots, e_n, Je_1, \ldots, Je_n$ is a basis of V over R.

Example 0.2. For $\mathbb{R}^2$, let $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We see that $J_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$. This is like $i(x + iy) = ix - y$.
So $A : (\mathbb{R}^2)_{J_0} \to \mathbb{C}$ is an isomorphism of $\mathbb{C}$-vector spaces.
Let W be a vector space over $\mathbb{C}$. Consider $W_{\mathbb{R}}$, a real vector space. We see $\dim_{\mathbb{R}} W_{\mathbb{R}} = 2\dim_{\mathbb{C}} W$. Consider $J : W_{\mathbb{R}} \to W\mathbb{R}$ given by $x \mapsto ix$. Then $J^2 = -\mathrm{Id}_{W_{\mathbb{R}}}$.
Let V be a real vector space with complex structure J. Consider $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} V^*$.
$J^t : \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \to \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$, we can express $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \ni \phi = \phi_1 + \phi_2$, and by definition,

$$J^t\phi = \phi \circ J = \phi_1 \circ J + i\phi_2 \circ J$$

So $(J^t)^2 = -1$.
$J^t \in \mathrm{End}_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}))$.
Main observation: $\phi \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ is $\mathbb{C}$-linear in V_j, meaning $\phi(ix) = i\phi(x)$, which is equivalent to $\phi(Jx) = i\phi(x)$.
In other words, such a ϕ is only $\mathbb{C}$-linear if ϕ is an eigenfunction of J^t with eigenvalue i.

Definition 0.3. ϕ is $\mathbb{C}$-antilinear on V_J means

$$\phi((\alpha + i\beta)x) = \overline{(\alpha + i\beta)}\phi(x)$$

for all $x \in V$.

We denote the space of $\mathbb{C}$-antilinear functionals by $\overline{\text{Hom}}_{\mathbb{C}}(V_J, \mathbb{C})$.
In fact, there is an isomorphism between $\text{Hom}_{\mathbb{C}}(V_J, \mathbb{C})$ and $\overline{\text{Hom}}_{\mathbb{C}}(V_J, \mathbb{C})$ as real vector spaces.

Theorem 0.1. *Let V be a real vector space with complex structure J. Then*

1. $\text{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \text{Hom}_{\mathbb{C}}(V_J, \mathbb{C}) \oplus \overline{\text{Hom}}_{\mathbb{C}}(V_J, \mathbb{C})$.

2. *If* $\text{Hom}_{\mathbb{C}}(V_J, \mathbb{C}) := V^{1,0}$, *and* $\overline{\text{Hom}}_{\mathbb{C}}(V_J, \mathbb{C}) := V^{0,1}$, *then* $\text{Hom}_{\mathbb{R}}(V, \mathbb{C}) = V^{1,0} \oplus_{\mathbb{C}} V^{0,1}$.

3. $\dim_{\mathbb{C}} V^{1,0} = \dim_{\mathbb{C}} V^{0,1} = \frac{\dim_{\mathbb{C}}(\text{Hom}_{\mathbb{R}}(V,\mathbb{C}))}{2}$

Proof. Observe that $\phi \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$ can be written as

$$\phi = \frac{\phi - i\phi \circ J}{2} + \frac{\phi + i\phi \circ J}{2} = \frac{\phi(Jx) + i\phi(x)}{2} = i\frac{\phi - i\phi \circ J}{2}(x) = \phi$$

Further, $V^{1,0} \cap V^{0,1} = 0$ by the definitions, so we are done. ∎

Thus, any differential can be split into a $\mathbb{C}$-linear and a $\mathbb{C}$-antilinear part.

Definition 0.4. $\pi^{1,0}$ is projection on the first factor, $\pi^{0,1}$ is projection onto the second. We have

$$\phi = \pi^{1,0}\phi + \pi^{0,1}\phi$$

Corollary 0.2. *If $\phi \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$, then ϕ being $\mathbb{C}$-linear (i.e $\phi \in V^{1,0}$) if and only if $\pi^{0,1}\phi = 0$.*

Definition 0.5. Applying to $(df)_p \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$, then

$$(df)_p = \pi^{1,0}df_p + \pi^{0,1}df_p$$

Say $\pi^{1,0}df_p = \underbrace{\partial f_p}_{\text{complex linear}}$ and $\pi^{0,1}df_p = \underbrace{\overline{\partial} df_p}_{\text{complex antilinear}}$

Theorem 0.3. *A function $f : X \to \mathbb{C}$ is $\mathbb{C}$-differentiable at $p \in X$ if and only if f is $\mathbb{R}$-differentiable at p and $df_p = \partial f_p$, which happens if and only if $\overline{\partial} f_p = 0$.*

Proof. We have $\mathbb{C} \cong \mathbb{R}^2_{J_0}$, which has standard basis $\mathbb{R}^2 = \langle e_1, e_2 \rangle_{\mathbb{R}^2}$. This has a dual basis in $\text{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R})$ given by dx and dy. That is, $\text{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{C}) = \langle dx, dy \rangle_{\mathbb{C}}$.
$J_0 e_1 = e_2$ and $J_0 e_2 = -e_1$, so $dx \circ J_0 = -dy$ and $dy \circ J_0 = dx$.

$$\begin{aligned} \pi^{0,1}dx &= \frac{1}{2}(dx - idx \circ J_0) \\ &= \frac{1}{2}(dx + idy) && := dz \end{aligned}$$

Further,

$$\pi^{0,1}dx = \frac{1}{2}d\overline{z}$$
$$\pi^{1,0}dy = \frac{1}{2}dz$$

So

$$df = f_x dx + f_y dy = \frac{f_x - if_y}{2}dz + \frac{f_x + if_y}{d}\overline{z} = \partial f + \overline{\partial} f$$

Definition 0.6.

$$\frac{\partial}{\partial z} = \frac{\partial_x - i\partial_y}{2}$$
$$\frac{\partial}{\partial \overline{z}} = \frac{\partial_x + i\partial_y}{2}$$

So

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}$$

So analyticity is equivalent to $df = \partial f$, meaning $\overline{\partial} f = 0$, which is equivalent to $\frac{\partial f}{\partial \overline{z}} = 0$, which means

$$\frac{\partial(u+iv)}{\partial \overline{z}} = 0$$

So $(\partial_x + i\partial_y)(u+iv) = 0$. Multiplying out, we get

$$u_x = v_y$$
$$u_y = -v_x$$

Lecture 2

The focus for the first bit of this course will be the so-called (by Dennis) $\overline{\partial}$-calculus. Suppose $f : X \to \mathbb{C}$ is differentiable for some $X \subseteq \mathbb{R}^{2n}$. It has a differential $df_p \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{C})$.

f is holomorphic if and only if $df_p \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$. Last time, we talked about how the second sits in the first, and how they interact.

Question: How to make $\mathbb{C}^n$ out of $\mathbb{R}^{2n}$. The abstract algebra way to do it is with a complex structure J. If V is a vector space over $\mathbb{R}$, then $\dim_{\mathbb{R}} V = 2n$ $J \in \mathrm{End}_{\mathbb{R}}(V)$ with $J^2 = -\mathrm{Id}_V$.

For all $x \in V_J$, we define $ix = Jx$, so V_J is a vector space over $\mathbb{C}$, and $\dim_{\mathbb{C}} V_j = \frac{\dim_{\mathbb{R}} V}{2}$

Example 0.3. Let $V = \mathbb{R}^2$, $J = J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. $\mathbb{R}^2_{J_0} \cong \mathbb{C}$
Last time, we showed that for any $\phi \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = H$, then

$$\phi = \underbrace{\frac{\phi - i\phi \circ J}{2}}_{\in \mathrm{Hom}_{\mathbb{C}}(V_J, \mathbb{C})} + \underbrace{\frac{\phi + i\phi \circ J}{2}}_{\in \overline{\mathrm{Hom}_{\mathbb{C}}}(V_J, \mathbb{C})}$$

So, $H = \mathrm{Hom}_{\mathbb{C}}(V_J, \mathbb{C}) \oplus \overline{\mathrm{Hom}_{\mathbb{C}}(V_J, \mathbb{C})} = V^{1,0} \oplus V^{0,1}$
Where $V^{1,0}$ and $V^{0,1}$ are what we call $\mathrm{Hom}_{\mathbb{C}}(V_J, \mathbb{C})$ and $\overline{\mathrm{Hom}}_{\mathbb{C}}(V_J, \mathbb{C})$ by tradition.
Let $V = \mathbb{R}^2$, $J = J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
Let $\phi = df_p = \frac{\partial f(p)}{\partial x} dx_p + \frac{\partial f(p)}{\partial y} dy_p$
After doing some computations, we get

$$d_p f = \pi^{1,0} df + \pi^{0,1} df = A dz + B d\overline{z}$$

Where $dz = dx + idy$ and $d\overline{z} = dx - idy$. You can check that the former is in $V^{1,0}$ and the latter in $V^{0,1}$.
The coefficient A is denoted by tradition as $\frac{\partial f}{\partial z}(p)$, and B as $\frac{\partial f}{\partial \overline{z}}(p)$. Here, the presence of ∂ does not imply any limit taking or anything, they are just notation.
Note $A = \frac{1}{2}(\partial_x f - i\partial_y f)|_p$ and $B = \frac{1}{2}(\partial_x f + i\partial_y f)|_p$.

Definition 0.7. We define

$$\partial f_p = f_z(p) dz_p$$
$$\overline{\partial} f_p = f_{\overline{z}}(p) d\overline{z}_p$$

The former is $\mathbb{C}$-linear, and the second $\mathbb{C}$-antilinear.

Claim. *f is $\mathbb{C}$-differentiable at p if and only if f is $\mathbb{R}$-differentiable at p and $\overline{\partial} f_p = 0$*

Proof. If $f = u + iv$, we get

$$\overline{\partial} f_p = \frac{\partial f}{\partial \overline{z}}(p) = 0$$

Which gives you the Cauchy-Riemann equations.
So f is analytic if $d_p f = f_z dz$.

Example 0.4. What are the following?

1. $\frac{\partial |z|}{\partial z}$

2. $\frac{\partial |z|}{\partial \overline{z}}$

How do we manage these problems?

Claim. *If $m, k \in \mathbb{Z}\backslash\{0\}$, we will consider $d(z^m\overline{z}^k)$. We have $f(z+h) = (z+h)^m(\overline{z} + \overline{h})^m = f(z) + (mz^{m-1}\overline{z}^k)h + (k\overline{z}^{k-1}z^m)\overline{h} + \mathcal{O}(h^2)$*
Where $\frac{|LHS-RHS|}{|z-2|} \to 0$ ad $z \to 2$
Then $\overline{\partial}(z^m\overline{z}^k = (k\overline{z}^{k-1}z^m)d\overline{z}$
Let's do some examples. Consider $\frac{\overline{z}-1}{z+1}$. We have

$$\frac{\overline{z}-1}{z+1} = \frac{(\overline{z}-2)-2-1}{(z-2)+2-1} = \frac{(\overline{z}-z)+1}{3}\frac{1}{1+\frac{z-2}{3}} = \frac{\overline{z}-1}{3}(1-\frac{z-2}{3}+\mathcal{O}(H^2)) = cAz+B\overline{z}+\rho(z)$$

There are two building blocks for doing problems:

1. *First, remember you are really doing real analysis.*

2. *Use the formula $d_p f = \pi^{1,0}\cdots$*

Example 0.5. We will calculate $d|z| = d\sqrt{z\overline{z}}$.

$$d|z| = d\sqrt{z\overline{z}} = \frac{1}{2\sqrt{z\overline{z}}}d(z\overline{z}) = \frac{1}{2|z|}d(z\overline{z}) = \frac{1}{2|z|}zd\overline{z} + \overline{z}dz$$

So the answer would be $\frac{z}{2|z|}$.
So, just express your function as a function of $z\overline{z}$ and proceed to do real analysis.
We know $\partial_z f, \partial_{\overline{z}} f$, and want to find $\partial_z(\overline{f}), \partial_{\overline{z}}(\overline{f}) =?$
We have

$$\begin{aligned} df &= \partial f + \overline{\partial} f \\ d(\overline{f}) &= \overline{df} = \overline{(\partial f)} + \overline{(\overline{\partial} f)} \end{aligned}$$

Now, $\overline{\partial}(\overline{f}) = \overline{\partial f}$. The bottom is equal to $\frac{\partial \overline{f}}{\partial \overline{z}} d\overline{z}$
So

$$\frac{\partial \overline{f}}{\partial \overline{z}} = \overline{\frac{\partial f}{\partial z}}$$

The conjugate of something which is complex anti-linear is complex linear.
What is $\frac{\partial \overline{f}}{\partial z}$? This is $\overline{\frac{\partial f}{\partial \overline{z}}}$.
So, the general procedure is to decompose your function as something linear + something antilinear, and use the sentence I just wrote.

How to compute $\frac{\partial f \circ g}{\partial \overline{z}}$?
Well $\underbrace{d(f \circ g) = df \circ df}_{\text{The chain rule expresses the functoriality of the derivative}}$, which is equal to
$(\partial f + \overline{\partial} f) \circ (\partial g + \overline{\partial} g) = \partial f \circ \partial g + \overline{\partial f} \circ dg + \cdots$.
Now, $\overline{\partial} f \circ \partial g = f_{\overline{z}} d\overline{z} \circ (g_z dz) = f_{\overline{z}} \overline{g_z}$.

Definition 0.8. Suppose $\frac{\partial f}{\partial \overline{z}}(p) = 0$. Then $df_p = \frac{\partial f}{\partial z}(p) dz_p$, and we write this as $f'(p)dz$.
Homework problem: We know f holomorphic, compute $\frac{\partial}{\partial}\overline{z} F(|f|)$, where $F : \mathbb{R} \to \mathbb{R}$ is smooth.
So

$$dF(|f|) = F'(|f|)d|f| = F'(|f|)d\sqrt{f\overline{f}}$$

And $d\sqrt{u} = \frac{1}{2u} du$, so the above is equal to

$$F'(|f|)\frac{1}{2|f|} d(f\overline{f})^{-1} = \frac{F'(|f|)}{2|f|}(\overline{f} df + f d\overline{z}) = \frac{F'(|f|}{2|f|}(\overline{f} f_z dz + f \overline{f_z} d\overline{z})$$

Our answer is thus whatever we get in front of $d\overline{z}$, so in this case the solution is

$$\frac{\partial}{\partial \overline{z}} F(|f|) = \frac{F'(|f|)}{2|f|} f\overline{f'}$$

The $|z| = \sqrt{z\overline{z}}$ is a very useful trick.
Complex analysis is kind of a local study of $f : X \to \mathbb{C}$ for some $X \subseteq \mathbb{C}$, where f is differentiable, $\overline{\partial} f = 0$, or equivalently $\frac{\partial f}{\partial \overline{z}} = 0$ in X. Really, we are studying solutions to a certain PDE (Cauchy-Riemann equation).
Suppose you want $u_{xx} - u_{yy} = 0$. If this is the case (and it turns out exactly when this is the case), we can express $u = \phi(x-y) + \psi(x+y)$ for ϕ, ψ arbitrary of 1 variable.
What if we want to study $u_y = u_{xx}$ in, say, $y > -\varepsilon$? We actually have a formula:

$$u(x,y) = frac12\pi y \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4y}} u(s,0)ds, y > 0$$

Suppose we want to study $u_{yy} + u_{xx} = 0$ in $y > -\varepsilon$, or maybe an open ball around the origin. We have a formula

$$u(x,y) = \int_{-\infty}^{\infty} P_H(x, y-s)u(s,0)ds$$

Where P_H is the Poisson Kernel.

MAIN LOCAL THM

Let $f : B_{R+\varepsilon} \to \mathbb{C}$. Then the following statements are all equivalent.

1. For any $p \in B_R$, f is differentiable and $df_p = \partial f_p$, so $\overline{\partial} f_p = 0$.

2. $f(z) = \int_{C_R} \frac{f(w)}{z-w} dw$

3. Like 8 other things

What exactly is complex integration?
Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth map. We integrate functions over maps. If ϕ is a continuous function, the formula is

$$\int_\gamma \phi(z)dz = \int_a^b \phi(\gamma(t))\dot{\gamma}(t)dt$$

where $\dot{\gamma}$ is the time-derivative of γ, so will be a complex number.
Let $t \in [0, 2\pi]$, $C_R(t) = Re^{it}$, a circle going counterclockwise. Let $\gamma(t) = Re^{-it}$. Then $\gamma^{-1} : [a, b] \to \mathbb{C}$ is the same but in the opposite direction. We have

$$\int_{\gamma^{-1}} \phi \, dz = \int_\gamma \phi \, dz$$