1 Lecture 1

Definition 1.1 (1)**.** Metric Space
A metric space is a set X equipped with a function $d : X \times X \to \mathbb{R}$, which satisfies the following axioms:

1. For any $x, y \in X$, $d(x, y) = d(y, x)$

2. For any $x, y, z \in X$, we have $d(x, y) \leq d(x, z) + d(z, y)$. This is called the triangle inequality

3. For any $x, y \in X$, $d(x, y) = 0$ exactly when $x = y$.

Example 1.1. For $x, y \in \mathbb{R}^n$,

$$d(x, y) := \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

This is called the Euclidean distance. 2 can be replaced with any real $r \geq 1$, and it will still be a metric.

Example 1.2. In this example, $C[0,1]$ is the set of all continuous functions $f : [0,1] \to \mathbb{R}$. Here,

$$d(f, g) := \sup_{x \in [0,1]} |f(x) - g(x)|$$

Example 1.3. Let $X = \mathbb{N}$, the natural numbers, including 0. Let p be a fixed prime number. The p-adic metric on $\mathbb{N}$ is defined by

$$d_p(a, b) := \frac{1}{p^\alpha}$$

Where p^α is the largest power of p which divides $|a - b|$. So two naturals are "close" if their difference is divisible by a high power of p.

Claim (1)**.** *This is a metric*

Proof. The 1st and 3rd axioms are clear. So we must prove the triangle inequality. We will consider the three quantities $d_p(a, b), d_p(a, t)$, and $d_p(b, t)$, where $a, b, t \in \mathbb{N}$. Suppose p^β divides both $a - t$ and $t - b$. Then p^β divides $(a - t) + (t - b) = a - b$. Therefore,

$$d_p(a, b) \leq \frac{1}{p^\beta} \leq \max(d_p(a, t), d_p(t, b)) \leq d_p(a, t) + d_p(t, b)$$

■

Definition 1.2 (2)**.** Continuity
Let $(X, d_X), (Y, d_Y)$ be two metric spaces. For a function $f : X \to Y$, we say that f is continuous at $x_0 \in X$ if, for all $\epsilon > 0$, there is a $\delta > 0$ such that

$$0 < |d_X(x_0, x)| < \delta \implies |d_Y(f(x_0), f(x))| < \epsilon$$

A function $f : X \to Y$ is said to be continuous if it is continuous at x for all $x \in X$.

Example 1.4. Consider a map $(\mathbb{N}, d_5) \to (\mathbb{N}, d_5)$ defined by

$$x \mapsto x^2$$

Is this continuous?
At 0, to be continuous, then for any x, if we want to get within a small distance of 0, then x has to be divisible by large powers of 5.
What about at 11?
This is continuous.

Example 1.5. What about $(\mathbb{N}, d_5) \to (\mathbb{N}, d_{17})$.

2 Lecture 2

Theorem 2.1. *If $f : (X, d_X) \to (Y, d_Y)$ and $g : (Y, d_Y) \to (Z, d_Z)$ are both continuous, then $g \circ f : (X, d_X) \to (Z, d_Z)$*

Proof. Fix $x \in X$ and $\varepsilon > 0$. Choose $\delta_1 > 0$ so that if $d_Y(f(x), y) < \delta_1$, then $d_Z(gf(x), g(y)) < \varepsilon$.
By continuity of f, we may then choose a $\delta_0 > 0$ such that if $d_X(x, x') < \delta_2$, then $d_Y(f(x), f(x')) < \delta_1$.

■

Definition 2.1. For a metric space (X, d_X), and a real $r > 0$, the open r-ball around a point x is defined as

$$B_r(x) = \{x' \in X \mid d(x, x') < r\}$$

Exercise: State and prove some theorem about the existence of a function from $X \times X' \to Y \times Y'$, given a function $f : X \to Y$ and $g : X' \to Y'$.

Example 2.1. Balls

1. In $\mathbb{R}^2$, consider

$$d_r\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = \left(\sum_{i=1}^{2} (x_i - y_i)^r\right)^{\frac{1}{r}}$$

For $r = 2$, this is the usual euclidean distance. For $r = 1$, the balls look like diamonds. In the limit, as $r \to \infty$, the metric will approach what is known as the "box metric," in which the distance between any point and 0 is it's largest coordinate.

2. On $C[0,1]$, the set of continuous functions from $[0,1]$ to $\mathbb{R}$, we have the sup metric:

$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

3. We also have

$$d_1(f,g) = \int_0^1 d|f(x) - g(x)|dx$$

Definition 2.2. For a metric space (X, d_X), suppose that $U \subseteq X$ is said to be "open" if, for any $x \in U$, there is a $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

Lemma 1. *$B_\varepsilon(x)$ is always open.*

Proof. Let $y \in B_\varepsilon(x)$. Let $t = d(x,y)$. By construction, $t < \varepsilon$. Let $\delta = \varepsilon - t$. Consider $B_\delta(y)$. For any $z \in B_\delta(y)$, we have by the triangle inequality

$$d(x,z) \leq d(x,y) + d(y,z) < \varepsilon - t + t = \varepsilon$$

and so $z \in B_\varepsilon(x)$. z was arbitrary, so we are done.

∎