

Lecture 1

Definition 0.1. A metric space is a set X equipped with a function $d : X \times X \rightarrow \mathbb{R}$, which satisfies the following axioms:

1. For any $x, y \in X$, $d(x, y) = d(y, x)$
2. For any $x, y, z \in X$, we have $d(x, y) \leq d(x, z) + d(z, y)$. This is called the “triangle inequality”
3. For any $x, y \in X$, $d(x, y) = 0$ exactly when $x = y$

Example 0.1. For $x, y \in \mathbb{R}^n$,

$$d(x, y) := \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

This is called the Euclidean distance. 2 can be replaced with any real $r \geq 1$, and it will still be a metric.

Example 0.2. In this example, $C[0, 1]$ is the set of all continuous functions $f : [0, 1] \rightarrow \mathbb{R}$. Here,

$$d(f, g) := \sup_{x \in [0, 1]} |f(x) - g(x)|$$

Example 0.3. Let $X = \mathbb{N}$, the natural numbers, including 0. Let p be a fixed prime number. The p -adic metric on \mathbb{N} is defined by

$$d_p(a, b) := \frac{1}{p^\alpha}$$

Where p^α is the largest power of p which divides $|a - b|$. So two naturals are “close” if their difference is divisible by a high power of p .

Claim. *This is a metric*

Proof. The 1st and 3rd axioms are clear. So we must prove the triangle inequality. We will consider the three quantities $d_p(a, b)$, $d_p(a, t)$, and $d_p(b, t)$, where $a, b, t \in \mathbb{N}$. Suppose p^β divides both $a - t$ and $t - b$. Then p^β divides $(a - t) + (t - b) = a - b$. Therefore,

$$d_p(a, b) \leq \frac{1}{p^\beta} \leq \max(d_p(a, t), d_p(t, b)) \leq d_p(a, t) + d_p(t, b)$$

■

Definition 0.2. Let $(X, d_X), (Y, d_Y)$ be two metric spaces. For a function $f : X \rightarrow Y$, we say that f is continuous at $x_0 \in X$ if, for all $\epsilon > 0$, there is a $\delta > 0$ such that

$$0 < |d_X(x_0, x)| < \delta \implies |d_Y(f(x_0), f(x))| < \epsilon$$

A function $f : X \rightarrow Y$ is said to be continuous if it is continuous at x for all $x \in X$.

Example 0.4. Consider a map $(\mathbb{N}, d_5) \rightarrow (\mathbb{N}, d_5)$ defined by

$$x \mapsto x^2$$

Is this continuous?

At 0, to be continuous, then for any x , if we want to get within a small distance of 0, then x has to be divisible by large powers of 5.

What about at 11?

This is continuous.

Example 0.5. What about $(\mathbb{N}, d_5) \rightarrow (\mathbb{N}, d_{17})$.

Lecture 2

Theorem 0.1. If $f : (X, d_X) \rightarrow (Y, d_Y)$ and $g : (Y, d_Y) \rightarrow (Z, d_Z)$ are both continuous, then $g \circ f : (X, d_X) \rightarrow (Z, d_Z)$

Proof. Fix $x \in X$ and $\varepsilon > 0$. Choose $\delta_1 > 0$ so that if $d_Y(f(x), y) < \delta_1$, then $d_Z(gf(x), g(y)) < \varepsilon$.

By continuity of f , we may then choose a $\delta_0 > 0$ such that if $d_X(x, x') < \delta_0$, then $d_Y(f(x), f(x')) < \delta_1$. ■

Definition 0.3. For a metric space (X, d_X) , and a real $r > 0$, the open r -ball around a point x is defined as

$$B_r(x) = \{x' \in X \mid d(x, x') < r\}$$

Exercise: State and prove some theorem about the existence of a function from $X \times X' \rightarrow Y \times Y'$, given a function $f : X \rightarrow Y$ and $g : X' \rightarrow Y'$.

Example 0.6. Balls

1. In \mathbb{R}^2 , consider

$$d_r \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) = \left(\sum_{i=1}^2 (x_i - y_i)^r \right)^{\frac{1}{r}}$$

For $r = 2$, this is the usual euclidean distance. For $r = 1$, the balls look like diamonds. In the limit, as $r \rightarrow \infty$, the metric will approach what is known as

the “box metric,” in which the distance between any point and 0 is its largest coordinate.

2. On $C[0, 1]$, the set of continuous functions from $[0, 1]$ to \mathbb{R} , we have the sup metric:

$$d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|$$

3. We also have

$$d_1(f, g) = \int_0^1 |f(x) - g(x)| dx$$

Definition 0.4. For a metric space (X, d_X) , suppose that $U \subseteq X$ is said to be “open” if, for any $x \in U$, there is a $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

Lemma 1. $B_\varepsilon(x)$ is always open.

Proof. Let $y \in B_\varepsilon(x)$. Let $t = d(x, y)$. By construction, $t < \varepsilon$. Let $\delta = \varepsilon - t$. Consider $B_\delta(y)$. For any $z \in B_\delta(y)$, we have by the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z) < \varepsilon - t + t = \varepsilon$$

and so $z \in B_\varepsilon(x)$. z was arbitrary, so we are done. ■

Lecture 3

Definition 0.5. Let (X, d) be a metric space. A set $U \subseteq X$ is said to be open if for all $x \in U$, there is an $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$.

Theorem 0.2. Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of open sets. Then

1. $\cup_{\alpha \in A} U_\alpha$ is open.
2. Let U_1, \dots, U_n be a finite subcollection. Then $\cap_{i=1}^\infty U_i$ is open.

Proof. **Lecture 4**

Definition 0.6. Two metrics d_1, d_2 are said to be equivalent on a space X if any set which is open under the d_1 -induced topology is also open under the d_2 -induced topology, and vice versa.

Example 0.7. In \mathbb{R}^2 ,

$$\begin{aligned}d_2(0, (x, y)) &= (x^2 + y^2)^{\frac{1}{2}} \\d_1(0, (x, y)) &= |x| + |y| \\d_\infty(0, (x, y)) &= \max\{|x|, |y|\}\end{aligned}$$

How do these metric's unit balls compare? In fact, d_1 's is within d_2 's, which is within d_∞ 's.

But all of these balls contain a ball of radius $\frac{1}{2}$ in any of the three metric. Thus, these are equivalent.

Definition 0.7. Two metrics d_1, d_2 are called Lipschitz equivalent if there exists some $k \in \mathbb{R}$ such that, for all $x, y \in X$, we have

$$\frac{1}{k}d_2(x, y) < d_1(x, y) < kd_2(x, y)$$

Example 0.8. This is a non-example. The 5-adic and the 17-adics are not equivalent.

Example 0.9. Let

$$\begin{aligned}d_1(f, g) &= \int_0^1 |f(x) - g(x)| dx \\d_\infty(f, g) &= \sup_{x \in [0, 1]} |f(x) - g(x)|\end{aligned}$$

One controls for area, one controls for the maximum value of f . We have

$$B_\varepsilon^{d_\infty}(0) \subset B_\varepsilon^{d_1}(0)$$

This is because if we control for the maximum size of f , we can surely control for the area under it. However, no matter how much we limit the area under f , there is some f which has that much area which has a sup greater than some ε which is fixed.

Theorem 0.3. Let d_1, d_2 be equivalent metrics on X . Then the following are equivalent

1. $f : X \rightarrow Y$ is d_1, d_Y continuous if and only if it is d_2, d_Y continuous.
2. $g : Z \rightarrow X$ is d_Z, d_1 continuous if and only if it is d_Z, d_2 continuous.

Proof. 1. Let $U \subset Y$ be open. We know that the preimage $f^{-1}(U)$ is open under the d_1 metric. But by the equivalence of d_1, d_2 , $f^{-1}(U)$ must also be open under the d_2 metric. The reverse argument also holds.

2. Let $U \subset X$ be an open set under the d_1 metric. By continuity of g , we know $g^{-1}(U)$ is open. However, U must also be open under the d_2 metric, meaning that g must be continuous with respect to both metrics. ■

Recall: If $\{U_\alpha\}_{\alpha \in A}$ is a collection of open sets, then $\cup_{\alpha \in A} U_\alpha$ is open.

We can associate to any set $R \subset X$ an open set, called the interior of R .

Definition 0.8. For any $R \subset X$, we define its interior by

$$\text{int}(R) = \bigcup_{U \text{ open}, U \subseteq R} U$$

We can say several things about $\text{int}(R)$.

1. $\text{int}(R)$ is open.
2. If U is open, then $\text{int}(U) = U$, and vice versa.
3. Suppose $A \subseteq B$. Then $\text{int}(A) \subseteq \text{int}(B)$.

Recall: If $\{C_\alpha\}_{\alpha \in A}$ are all closed, then $\cap_{\alpha \in A} C_\alpha$ is closed.

Definition 0.9. If $R \subseteq X$, then the closure of R , denoted by \overline{R} , or sometimes $\text{cl}(R)$, is defined as

$$\overline{R} := \bigcap_{C \text{ closed}, R \subseteq C} C$$

Analogously,

1. \overline{R} is closed for any R .
2. R is closed if and only if $R = \overline{R}$.
3. If $A \subset B$, then $\text{cl}(A) \subset \text{cl}(B)$.

Proposition 1. Let $x \in \text{cl}(R)$. Then, for all $\varepsilon > 0$, $B_\varepsilon(x) \cap R \neq \emptyset$, and vice-versa. We will prove this next lecture.

Lecture 4

Proof. First, suppose that $x \notin \overline{A}$. The complement of \overline{A} is open, so there exists a $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq (\overline{A})^c$, and so $B_\varepsilon(x) \cap \overline{A} = \emptyset$.

Now, suppose that there exists some $\varepsilon > 0$ such that $B_\varepsilon(x) \cap A = \emptyset$. Then $(B_\varepsilon(x))^c$ is a closed set containing A which does not contain x , thus \overline{A} cannot contain x . ■

Now, it is time for the main event.

Topological Spaces

Let X be a set. Let $\mathcal{P}(X)$ denote the power set of X , which is the set of all subsets of X .

Definition 0.10. Let $\mathcal{T} \subset \mathcal{P}(X)$. \mathcal{T} is a topology on X if it has the following properties:

1. $\emptyset, X \in \mathcal{T}$.
2. If $\{U_\alpha\}_{\alpha \in A}$ with each $U_\alpha \in \mathcal{T}$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$.
3. If $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$. Of course, we can “strengthen” this to the equivalent statement: if $U_1, \dots, U_k \in \mathcal{T}$, then $\bigcap_{i=1}^k U_i \in \mathcal{T}$.

Elements of \mathcal{T} are called “open sets.”

Example 0.10. Here are some simple examples.

1. Open sets in (X, d) form a topology (hence the definition)
2. $\mathcal{T} = \mathcal{P}(X)$ forms the discrete topology.
3. $\mathcal{T} = \{\emptyset, X\}$ forms the indiscrete topology.
4. If $X = \{0, 1\}$, then $\mathcal{T} = \{\emptyset, X, \{0\}\}$ forms a topology.
5. We can form the Zariski topology on \mathbb{R} by specifying the closed sets, which satisfy a similar but slightly different set of axioms. We define the closed sets to be \emptyset, \mathbb{R} , and any finite collection of points.

More generally, the Zariski topology on some ring R is specified by its closed sets, which are the solution locii of some set of polynomials in R .

6. Let $X = \mathbb{R}$, $\mathcal{T} = \{(r, \infty) \mid r \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$.

We have now defined a set of objects (topological spaces). Now, we want to define morphisms, the maps between spaces.

Definition 0.11. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces. We say a function $f : X \rightarrow Y$ is continuous if, for all $V \in \mathcal{T}_Y$, $f^{-1}(V) \in \mathcal{T}_X$. In other words, the preimage of any open subset of Y is an open subset of X .

Example 0.11. Some baby examples

1. Any function $f : (X, \text{discrete}) \rightarrow (Y, \mathcal{T}_Y)$ is continuous as long as X has the discrete metric, as the preimage of any open set will be a subset of X , all of which are open under the discrete topology.

2. Any function $f : (X, \mathcal{T}_X) \rightarrow (Y, \text{discrete})$ is continuous for a similar reason.
3. $\text{Id}_X : (X, \mathcal{T}_X) \rightarrow (X, \mathcal{T}_X)$ is continuous.

Theorem 0.4. *Let $f : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$ and $g : (Y, \mathcal{T}_Y) \rightarrow (Z, \mathcal{T}_Z)$ be continuous functions. Then $g \circ f : (X, \mathcal{T}_X) \rightarrow (Z, \mathcal{T}_Z)$ is continuous.*

In other words, the composition of continuous functions is continuous.

Proof. Pick $W \in \mathcal{T}_Z$. Then $g^{-1}(W) \in \mathcal{T}_Y$ since g is continuous. So, because f is continuous,

$$(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) \in \mathcal{T}_X$$

Hence, $g \circ f$ is continuous.

Definition 0.12. For \mathcal{T} a topology on X , a basis for \mathcal{T} is a subset $\mathcal{B} \subseteq \mathcal{T}$ such that every set in \mathcal{T} is the union of sets in \mathcal{B} .

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Proposition 2. *Suppose $f : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$ is a function. Then f is continuous if and only if $f^{-1}(B) \in \mathcal{T}_X$ for all $B \in \mathcal{B}$, with \mathcal{B} a basis for \mathcal{T}_Y .*

Proof. The forward direction is trivial, as a basis consists of sets which are all open. Now, suppose that $f^{-1}(B) \in \mathcal{T}_X$ for all $B \in \mathcal{B}$. Let $V \in \mathcal{T}_Y$. Write it as $\cup_{\alpha \in A} B_\alpha = V$ for $B_\alpha \in \mathcal{B}$.

Then $f^{-1}(V) = f^{-1}(\cup_{\alpha \in A} B_\alpha) = \cup_{\alpha \in A} f^{-1}(B_\alpha)$. This is the union of open subsets of X , so $f^{-1}(V)$ is open. ■

Example 0.12. Consider $B_q(\frac{r}{s})$, with $p, q, r, s \in \mathbb{Z}$, $q, s \neq 0$. In other words, the balls of rational radius and rational center. This collection of sets forms a basis for \mathbb{R} using the standard (or, as Darren calls it, the “Mother’s Knee”) topology.

Exercise: Write down the definition of interiors and closures, and check the following lemma is true:

Lemma 2. $x \in \text{cl}(A)$ if and only if, for all open $U \ni x$, $U \cap A \neq \emptyset$.

Example 0.13. Consider $(\mathbb{N}, \text{Zariski})$. What is the closure of the collection of prime numbers under this topology?

Recall that the Zariski topology on \mathbb{N} defines closed sets to be either empty, \mathbb{N} , or finite. So, for example, the closure of the integers from 1 to 10 is itself.

But the only closed set that can contain an infinite set is \mathbb{N} , so the closure of the primes is \mathbb{N} . More generally, for any infinite subset of \mathbb{N} , the closure is \mathbb{N} .

Subobjects and product objects

We have now defined our objects and morphisms, so let's talk about subobjects and product objects.

Let (X, \mathcal{T}_X) be a topological space, and $A \subseteq X$ a nonempty subset.

Denote the inclusion map $\iota : A \rightarrow X$. We want to topologize A so that ι is continuous, and is as small as possible, in the sense that any topology for which ι is continuous includes this topology on A .

For any open $U \subseteq X$, we want $\iota^{-1}(U)$ to be open. But $\iota^{-1}(U) = U \cap A$.

Definition 0.13. For (X, \mathcal{T}_X) a topological space and $A \subset X$ a subset, then the subspace topology on A , \mathcal{T}_A , consists of

$$\mathcal{T}_A := \{U \cap A \mid U \in \mathcal{T}_X\}$$

Proposition 3. Suppose we have a commutative diagram of the form

$$\begin{array}{ccc} (Z, \mathcal{T}_Z) & \xrightarrow{g} & A \\ & \searrow \iota \circ g & \downarrow \iota \\ & & X \end{array}$$

Then ι is continuous if and only if $\iota \circ g$ is continuous.

Proof. One direction is trivial, as we know the composition of continuous functions is continuous. Now, suppose that $\iota \circ g$ is continuous.

Let $W \in \mathcal{T}_A$. We know $W = W^* \cap A$ for some $W^* \in \mathcal{T}_X$. $\iota \circ g$ is continuous, so $(\iota \circ g)^{-1}W^* \in \mathcal{T}_Z$.

So, $\mathcal{T}_Z \ni (\iota \circ g)^{-1}W^* = g^{-1}(\iota^{-1}(W^*)) = g^{-1}(W)$. ■

So we can see that if we want the above to hold, we are forced into our definition of \mathcal{T}_A .

Proposition 4. \mathcal{T}_A is the only topology so that the previous proposition is true for all spaces (Z, \mathcal{T}_Z) and functions g .

Proof. Suppose that \mathcal{T}_μ (μ for “mystery” topology) such that the previous proposition holds for all choices of (Z, \mathcal{T}_Z) and $g : (Z, \mathcal{T}_Z) \rightarrow A$.

Consider

$$\begin{array}{ccc} (A, \mathcal{T}_A) & \xrightarrow{\text{Id}} & (A, \mathcal{T}_\mu) \\ & \searrow \iota \circ \text{Id} & \downarrow \iota \\ & & (X, \mathcal{T}_X) \end{array}$$