Lecture 1

We begin by trying to gain a deeper understanding of the Cauchy-Riemann equations.
Let $f : X \to \mathbb{C}$, where $X \subset \mathbb{C}^n$. For now, let's say $X \subset \mathbb{C}$. In real analysis, we have a notion of differentiability for $f : \mathbb{R}^n \to \mathbb{R}^k$. We can say that f is differentiable at a point $p \in X$ when

$$f(p+h) = f(p) + (df_p)h + \rho(h)$$

Where $(df)_p : \mathbb{R}^n \to \mathbb{R}^k$ is a linear map $\in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^k)$, and $\frac{|\rho(h)|}{|h|} \to 0$ as $h \to 0$.
So we can think of the "real differential" as a linea map in $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C})$.

Definition 0.1. Let $X \subset \mathbb{C}$, and $f : X \to \mathbb{C}$. Differentiability refers to the existence of a $(df)_p \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$.
So, f is complex differentiable at $p \in X$ means that

$$f(p+h) = f(p) + f'(p)h + \rho(h)$$

Where $f'(p)$ is a complex number and $\frac{|\rho(h)|}{|h|} \to 0$.
If $A \in \mathrm{Hom}_{\mathbb{C}})(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$, $A(z) = \alpha z, \alpha \in \mathbb{C}$.
So $(df)_p \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$.
$\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$ is a $\mathbb{C}$-vector space of dimension 1.
$\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R}^2)$ is a $\mathbb{C}$-vector space of dimension 2.
So where did the extra dimension go? What happened?
Consider an element of $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$ given by $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x - iy$.
We also have $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x + iy$.
From the real analysis point of view, these two functions are equal to their differentials. The first is called $d\overline{z}$, and the second is called dz.
$dz = dx + idy$ and $d\overline{z} = dx - idy$

$$dx \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = h_1$$
$$dy \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = h_2$$

On a complex vector space, suppose $\phi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{C}^n, \mathbb{C})$, we have $(\overline{\phi})(v) = \overline{\phi(v)}$. So $\overline{dz} = d\overline{z}$.
Now, $\mathbb{C}$-valued real differentiable functions are just pairs of $\mathbb{R}$-valued real differentiable functions.

Example 0.1. If $k, m \in \mathbb{N}$, then $z^k \overline{z}^m : \mathbb{C} \to \mathbb{C}$ is a real smooth function (when viewed as an element of $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R}^2)$), with

$$d(z^k \overline{z}^m) = kz^{k-1}\overline{z}^m + m\overline{z}^{m-1}z^k d\overline{z}$$

We will study the differences between $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C})$ versus $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$, with complex dimensions 2 and 1, respectively.

Definition 0.2. Let V be a real vector space.
A complex structure on V is a $J \in \mathrm{End}_{\mathbb{R}}(V)$ which satisfies $J^2 = -\mathrm{Id}_V$

Proposition 1. *Define $V_J = V$ as a set and group, with a $\mathbb{C}$-action $\mathbb{C} \times V_J \to V_J$ defined by $((\alpha + i\beta), x) \mapsto \alpha x + \beta J x$.*

Proof. Check $z(wx) = (zw)x$ for all $z, w \in \mathbb{C}$ and $x \in V_J$.

Proposition 2. *If a vector space V admits a complex structure J, then $\dim_{\mathbb{R}} V = 2n$. Further, $\dim_{\mathbb{R}} V = 2\dim_{\mathbb{C}} V_J$.*

Proof. First, $\det(J^2) = \det(-id_V) = (-1)^{\dim_{\mathbb{R}} V}$, so the dimension must be even. Alternatively, if $e_1, \ldots, e_n$ is a basis of V_J, then check $e_1, \ldots, e_n, Je_1, \ldots, Je_n$ is a basis of V over R.

Example 0.2. For $\mathbb{R}^2$, let $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We see that $J_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$. This is like $i(x + iy) = ix - y$.
So $A : (\mathbb{R}^2)_{J_0} \to \mathbb{C}$ is an isomorphism of $\mathbb{C}$-vector spaces.
Let W be a vector space over $\mathbb{C}$. Consider $W_{\mathbb{R}}$, a real vector space. We see $\dim_{\mathbb{R}} W_{\mathbb{R}} = 2\dim_{\mathbb{C}} W$. Consider $J : W_{\mathbb{R}} \to W\mathbb{R}$ given by $x \mapsto ix$. Then $J^2 = -\mathrm{Id}_{W_{\mathbb{R}}}$.
Let V be a real vector space with complex structure J. Consider $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} V^*$.
$J^t : \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \to \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$, we can express $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \ni \phi = \phi_1 + \phi_2$, and by definition,

$$J^t \phi = \phi \circ J = \phi_1 \circ J + i\phi_2 \circ J$$

So $(J^t)^2 = -1$.
$J^t \in \mathrm{End}_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}))$.
Main observation: $\phi \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ is $\mathbb{C}$-linear in V_j, meaning $\phi(ix) = i\phi(x)$, which is equivalent to $\phi(Jx) = i\phi(x)$.
In other words, such a ϕ is only $\mathbb{C}$-linear if ϕ is an eigenfunction of J^t with eigenvalue i.

Definition 0.3. ϕ is $\mathbb{C}$-antilinear on V_J means

$$\phi((\alpha + i\beta)x) = \overline{(\alpha + i\beta)}\phi(x)$$

for all $x \in V$.

We denote the space of $\mathbb{C}$-antilinear functionals by $\overline{\text{Hom}}_{\mathbb{C}}(V_J, \mathbb{C})$.
In fact, there is an isomorphism between $\text{Hom}_{\mathbb{C}}(V_J, \mathbb{C})$ and $\overline{\text{Hom}}_{\mathbb{C}}(V_J, \mathbb{C})$ as <u>real</u> vector spaces.

Theorem 0.1. *Let V be a real vector space with complex structure J. Then*

1. $\text{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \text{Hom}_{\mathbb{C}}(V_J, \mathbb{C}) \oplus \overline{\text{Hom}}_{\mathbb{C}}(V_J, \mathbb{C})$.

2. *If* $\text{Hom}_{\mathbb{C}}(V_J, \mathbb{C}) := V^{1,0}$, *and* $\overline{\text{Hom}}_{\mathbb{C}}(V_J, \mathbb{C}) := V^{0,1}$, *then* $\text{Hom}_{\mathbb{R}}(V, \mathbb{C}) = V^{1,0} \oplus_{\mathbb{C}} V^{0,1}$.

3. $\dim_{\mathbb{C}} V^{1,0} = \dim_{\mathbb{C}} V^{0,1} = \frac{\dim_{\mathbb{C}}(\text{Hom}_{\mathbb{R}}(V,\mathbb{C}))}{2}$

Proof. Observe that $\phi \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$ can be written as

$$\phi = \frac{\phi - i\phi \circ J}{2} + \frac{\phi + i\phi \circ J}{2} = \frac{\phi(Jx) + i\phi(x)}{2} = i\frac{\phi - i\phi \circ J}{2}(x) = \phi$$

Further, $V^{1,0} \cap V^{0,1} = 0$ by the definitions, so we are done. ■

Thus, any differential can be split into a $\mathbb{C}$-linear and a $\mathbb{C}$-antilinear part.

Definition 0.4. $\pi^{1,0}$ is projection on the first factor, $\pi^{0,1}$ is projection onto the second. We have

$$\phi = \pi^{1,0}\phi + \pi^{0,1}\phi$$

Corollary 0.2. *If $\phi \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$, then ϕ being $\mathbb{C}$-linear (i.e $\phi \in V^{1,0}$) if and only if $\pi^{0,1}\phi = 0$.*

Definition 0.5. Applying to $(df)_p \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$, then

$$(df)_p = \pi^{1,0}df_p + \pi^{0,1}df_p$$

Say $\pi^{1,0}df_p = \underbrace{\partial f_p}_{\text{complex linear}}$ and $\pi^{0,1}df_p = \underbrace{\overline{\partial} df_p}_{\text{complex antilinear}}$

Theorem 0.3. *A function $f : X \to \mathbb{C}$ is $\mathbb{C}$-differentiable at $p \in X$ if and only if f is $\mathbb{R}$-differentiable at p and $df_p = \partial f_p$, which happens if and only if $\overline{\partial} f_p = 0$.*

Proof. We have $\mathbb{C} \cong \mathbb{R}^2_{J_0}$, which has standard basis $\mathbb{R}^2 = \langle e_1, e_2 \rangle_{\mathbb{R}^2}$. This has a dual basis in $\text{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R})$ given by dx and dy. That is, $\text{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{C}) = \langle dx, dy \rangle_{\mathbb{C}}$.
$J_0 e_1 = e_2$ and $J_0 e_2 = -e_1$, so $dx \circ J_0 = -dy$ and $dy \circ J_0 = dx$.

$$\begin{aligned} \pi^{0,1}dx &= \frac{1}{2}(dx - idx \circ J_0) \\ &= \frac{1}{2}(dx + idy) && := dz \end{aligned}$$

Further,

$$\pi^{0,1} dx = \frac{1}{2} d\overline{z}$$
$$\pi^{1,0} dy = \frac{1}{2} dz$$

So

$$df = f_x dx + f_y dy = \frac{f_x - if_y}{2} dz + \frac{f_x + if_y}{d} \overline{z} = \partial f + \overline{\partial} f$$

Definition 0.6.

$$\frac{\partial}{\partial z} = \frac{\partial_x - i\partial_y}{2}$$
$$\frac{\partial}{\partial \overline{z}} = \frac{\partial_x + i\partial_y}{2}$$

So

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \overline{z}} d\overline{z}$$

So analyticity is equivalent to $df = \partial f$, meaning $\overline{\partial} f = 0$, which is equivalent to $\frac{\partial f}{\partial \overline{z}} = 0$, which means

$$\frac{\partial(u + iv)}{\partial \overline{z}} = 0$$

So $(\partial_x + i\partial_y)(u + iv) = 0$. Multiplying out, we get

$$u_x = v_y$$
$$u_y = -v_x$$