Lecture 1

Definition 0.1. A metric space is a set X equipped with a function $d : X \times X \to \mathbb{R}$, which satisfies the following axioms:

1. For any $x, y \in X$, $d(x, y) = d(y, x)$

2. For any $x, y, z \in X$, we have $d(x, y) \leq d(x, z) + d(z, y)$. This is called the "triangle inequality"

3. For any $x, y \in X$, $d(x, y) = 0$ exactly when $x = y$

Example 0.1. For $x, y \in \mathbb{R}^n$,

$$d(x, y) \stackrel{\text{def}}{=} \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

This is called the Euclidean distance. 2 can be replaced with any real $r \geq 1$, and it will still be a metric.

Example 0.2. In this example, $C[0, 1]$ is the set of all continuous functions $f : [0, 1] \to \mathbb{R}$. Here,

$$d(f, g) \stackrel{\text{def}}{=} \sup_{x \in [0,1]} |f(x) - g(x)|$$

Example 0.3. Let $X = \mathbb{N}$, the natural numbers, including 0. Let p be a fixed prime number. The p-adic metric on $\mathbb{N}$ is defined by

$$d_p(a, b) \stackrel{\text{def}}{=} \frac{1}{p^\alpha}$$

Where p^α is the largest power of p which divides $|a - b|$. So two naturals are "close" if their difference is divisible by a high power of p.

Claim. *This is a metric*

Proof. The 1st and 3rd axioms are clear. So we must prove the triangle inequality. We will consider the three quantities $d_p(a, b), d_p(a, t)$, and $d_p(b, t)$, where $a, b, t \in \mathbb{N}$. Suppose p^β divides both $a - t$ and $t - b$. Then p^β divides $(a - t) + (t - b) = a - b$. Therefore,

$$d_p(a, b) \leq \frac{1}{p^\beta} \leq \max(d_p(a, t), d_p(t, b)) \leq d_p(a, t) + d_p(t, b)$$

■

Definition 0.2. Let $(X, d_X), (Y, d_Y)$ be two metric spaces. For a function $f : X \to Y$, we say that f is continuous at $x_0 \in X$ if, for all $\epsilon > 0$, there is a $\delta > 0$ such that

$$0 < |d_X(x_0, x)| < \delta \implies |d_Y(f(x_0), f(x))| < \epsilon$$

A function $f : X \to Y$ is said to be continuous if it is continuous at x for all $x \in X$.

Example 0.4. Consider a map $(\mathbb{N}, d_5) \to (\mathbb{N}, d_5)$ defined by

$$x \mapsto x^2$$

Is this continuous?
At 0, to be continuous, then for any x, if we want to get within a small distance of 0, then x has to be divisible by large powers of 5.
What about at 11?
This is continuous.

Example 0.5. What about $(\mathbb{N}, d_5) \to (\mathbb{N}, d_{17})$.

Lecture 2

Theorem 0.1. *If $f : (X, d_X) \to (Y, d_Y)$ and $g : (Y, d_Y) \to (Z, d_Z)$ are both continuous, then $g \circ f : (X, d_X) \to (Z, d_Z)$*

Proof. Fix $x \in X$ and $\varepsilon > 0$. Choose $\delta_1 > 0$ so that if $d_Y(f(x), y) < \delta_1$, then $d_Z(gf(x), g(y)) < \varepsilon$.
By continuity of f, we may then choose a $\delta_0 > 0$ such that if $d_X(x, x') < \delta_2$, then $d_Y(f(x), f(x')) < \delta_1$.

■

Definition 0.3. For a metric space (X, d_X), and a real $r > 0$, the open r-ball around a point x is defined as

$$B_r(x) = \{x' \in X \mid d(x, x') < r\}$$

Exercise: State and prove some theorem about the existence of a function from $X \times X' \to Y \times Y'$, given a function $f : X \to Y$ and $g : X' \to Y'$.

Example 0.6. Balls

1. In $\mathbb{R}^2$, consider

$$d_r\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = \left(\sum_{i=1}^{2} (x_i - y_i)^r\right)^{\frac{1}{r}}$$

For $r = 2$, this is the usual euclidean distance. For $r = 1$, the balls look like diamonds. In the limit, as $r \to \infty$, the metric will approach what is known as the "box metric," in which the distance between any point and 0 is it's largest coordinate.

2. On $C[0,1]$, the set of continuous functions from $[0,1]$ to $\mathbb{R}$, we have the sup metric:

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$$

3. We also have

$$d_1(f,g) = \int_0^1 d|f(x) - g(x)|dx$$

Definition 0.4. For a metric space (X, d_X), suppose that $U \subseteq X$ is said to be "open" if, for any $x \in U$, there is a $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

Lemma 1. *$B_\varepsilon(x)$ is always open.*

Proof. Let $y \in B_\varepsilon(x)$. Let $t = d(x,y)$. By construction, $t < \varepsilon$. Let $\delta = \varepsilon - t$. Consider $B_\delta(y)$. For any $z \in B_\delta(y)$, we have by the triangle inequality

$$d(x,z) \leq d(x,y) + d(y,z) < \varepsilon - t + t = \varepsilon$$

and so $z \in B_\varepsilon(x)$. z was arbitrary, so we are done.

■

Lecture 3

Definition 0.5. Let (X,d) be a metric space. A set $U \subseteq X$ is said to be open if for all $x \in U$, there is an $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq U$.

Theorem 0.2. *Let $\{U_\alpha\}_{\alpha\in A}$ be a collection of open sets. Then*

1. *$\cup_{\alpha\in A} U_\alpha$ is open.*

2. *Let $U_1, \ldots, U_n$ be a finite subcollection. Then $\cap_{i=1}^\infty U_i$ is open.*

Proof. **Lecture 4**

Definition 0.6. Two metrics d_1, d_2 are said to be equivalent on a space X if any set which is open under the d_1-induced topology is also open under the d_2-induced topology, and vice versa.

Example 0.7. In $\mathbb{R}^2$,

$$\begin{aligned} d_2(0,(x,y)) &= (x^2+y^2)^{\frac{1}{2}} \\ d_1(0,(x,y)) &= |x|+|y| \\ d_\infty(0,(x,y)) &= \max\{|x|,|y|\} \end{aligned}$$

How do these metric's unit balls compare? In fact, d_1's is within d_2's, which is within d_∞'s.
But all of these balls contain a ball of radius $\frac{1}{2}$ in any of the three metric. Thus, these are equivalent.

Definition 0.7. Two metrics d_1, d_2 are called Lipschitz equivalent if there exists some $k \in \mathbb{R}$ such that, for all $x, y \in X$, we have

$$\frac{1}{k} d_2(x,y) < d_1(x,y) < k d_2(x,y)$$

Example 0.8. This is a non-example. The 5-adic and the 17-adics are not equivalent.

Example 0.9. Let

$$\begin{aligned} d_1(f,g) &= \int_0^1 |f(x)-g(x)|\,dx \\ d_\infty(f,g) &= \sup_{x\in[0,1]} |f(x)-g(x)| \end{aligned}$$

One controls for area, one controls for the maximum value of f. We have

$$B_\varepsilon^{d_\infty}(0) \subset B_\varepsilon^{d_1}(0)$$

This is because if we control for the maximum size of f, we can surely control for the area under it. However, no matter how much we limit the area under f, there is some f which has that much area which has a sup greater than some ε which is fixed.

Theorem 0.3. *Let d_1, d_2 be equivalent metrics on X. Then the following are equivalent*

1. *$f : X \to Y$ is d_1, d_Y continuous if and only if it is d_2, d_Y continuous.*

2. *$g : Z \to X$ is d_Z, d_1 continuous if and only if if is d_Z, d_2 continuous.*

Proof. **1.** Let $U \subset Y$ be open. We know that the preimage $f^{-1}(U)$ is open under the d_1 metric. But by the equivalence of d_1, d_2, $f^{-1}(U)$ must also be open under the d_2 metric. The reverse argument also holds.

2. Let $U \subset X$ be an open set under the d_1 metric. By continuity of g, we know $g^{-1}(U)$ is open. However, U must also be open under the d_2 metric, meaning that g must be continuous with respect to both metrics.

■

Recall: If $\{U_\alpha\}_{\alpha \in A}$ is a collection of open sets, then $\cup_{\alpha \in A} U_\alpha$ is open.
We can associate to any set $R \subset X$ an open set, called the interior of R.

Definition 0.8. For any $\mathbb{R} \subset X$, we define it's <u>interior</u> by

$$\text{int}(R) = \bigcup_{U \text{ open}, U \subseteq R} U$$

We can say several things about $\text{int}(R)$.

1. $\text{int}(R)$ is open.

2. If U is open, then $\text{int}(U) = U$, and vice versa.

3. Suppose $A \subseteq B$. Then $\text{int}(A) \subseteq \text{int}(B)$.

Recall: If $\{C_\alpha\}_{\alpha \in A}$ are all closed, then $\cap_{\alpha \in A} C_\alpha$ is closed.

Definition 0.9. If $R \subseteq X$, then the <u>closure</u> of R, denoted by $\overline{R}$, or sometimes $\text{cl}(R)$, is defined as

$$\overline{R} \overset{\text{def}}{=} \bigcap_{C \text{ closed}, C \supseteq R} C$$

Analagously,

1. $\overline{R}$ is closed for any R.

2. R is closed if and only if $R = \overline{R}$.

3. If $A \subset B$, then $\text{cl}(A) \subset \text{cl}(B)$.

Proposition 1. *Let $x \in \text{cl}(R)$. Then, for all $\varepsilon > 0$, $B_\varepsilon(x) \cap R \neq \varnothing$, and vice-versa. We will prove this next lecture.*

Lecture 4

Proof. First, suppose that $x \notin \overline{A}$. The complement of $\overline{A}$ is open, so there exists a $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq (\overline{A})^c$, and so $B_\varepsilon(x) \cap \overline{A} = \varnothing$.
Now, suppose that there exists some $\varepsilon > 0$ such that $B_\varepsilon(x) \cap A = \varnothing$. Then $(B_\varepsilon(x))^c$ is a closed set containing A which does not contain x, thus $\overline{A}$ cannot contain x.

■

Now, it is time for the main event.

Topological Spaces

Let X be a set. Let $\mathscr{P}(X)$ denote the power set of X, which is the set of all subsets of X.

Definition 0.10. Let $\mathscr{T} \subset \mathscr{P}(X)$. $\mathscr{T}$ is a topology on X if it has the following properties:

1. $\varnothing, X \in \mathscr{T}$.

2. If $\{U_\alpha\}_{\alpha \in A}$ with each $U_\alpha \in \mathscr{T}$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathscr{T}$.

3. If $A, B \in \mathscr{T}$, then $A \cap B \in \mathscr{T}$. Of course, we can "strengthen" this to the equivalent statement: if $U_1, \ldots, U_k \in \mathscr{T}$, then $\cap_{i=1}^k U_i \in \mathscr{T}$

Elements of $\mathscr{T}$ are called "open sets."

Example 0.10. Here are some simple examples.

1. Open sets in (X, d) form a topology (hence the definition)

2. $\mathscr{T} = \mathscr{P}(X)$ forms the discrete topology.

3. $\mathscr{T} = \{\varnothing, X\}$ forms the indescrete topolgy.

4. If $X = \{0, 1\}$, then $\mathscr{T} = \{\varnothing, X, \{0\}\}$ forms a topology.

5. We can form the Zariski topology on $\mathbb{R}$ by specifying the closed sets, which satisfy a similar but slightly different set of axioms. We define the closed sets to be $\varnothing, \mathbb{R}$, and any finite collection of points.

   More generally, the Zariski topology on some ring R is specified by its closed sets, which are the solution locii of some set of polynomials in R.

6. Let $X = \mathbb{R}$, $\mathscr{T} = \{(r, \infty) \mid r \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$.

We have now defined a set of objects (topological spaces). Now, we want to define morphisms, the maps between spaces.

Definition 0.11. Let $(X, \mathscr{T}_X), (Y, \mathscr{T}_Y)$ be topological spaces. We say a function $f : X \to Y$ is continuous if, for all $V \in \mathscr{T}_Y$, $f^{-1}(V) \in \mathscr{T}_X$. In other words, the preimage of any open subset of Y is an open subset of X.

Example 0.11. Some baby examples

1. Any function $f : (X, \text{discrete}) \to (Y, \mathscr{T}_Y)$ is continuous as long as X has the discrete metric, as the preimage of any open set will be a subset of X, all of which are open under the discrete topology.

2. Any function $f : (X, \mathscr{T}_X) \to (Y,\ \text{discrete})$ is continous for a similar reason.

3. $\text{Id}_X : (X, \mathscr{T}_X) \to (X, \mathscr{T}_X)$ is continuous.

Theorem 0.4. *Let $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ and $g : (Y, \mathscr{T}_Y) \to (Z, \mathscr{T}_Z)$ be continuous functions. Then $g \circ f : (X, \mathscr{T}_X) \to (Z, \mathscr{T}_Z)$ is continuous.*
In other words, the composition of continuous functions is continuous.

Proof. Pick $W \in \mathscr{T}_Z$. Then $g^{-1}(W) \in \mathscr{T}_Y$ since g is continuous. So, because f is continuous,

$$(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) \in \mathscr{T}_X$$

Hence, $g \circ f$ is continuous.

Definition 0.12. For $\mathscr{T}$ a topology on X, a <u>basis</u> for $\mathscr{T}$ is a subset $\mathscr{B} \subseteq \mathscr{T}$ such that every set in $\mathscr{T}$ is the union of sets in $\mathscr{B}$.

Lecture 5

Proposition 2. *Suppose $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is a function. Then f is continuous if and only if $f^{-1}(B) \in \mathscr{T}_X$ for all $B \in \mathscr{B}$, with $\mathscr{B}$ a basis for $\mathscr{T}_Y$.*

Proof. The forward direction is trivial, as a basis consists of sets which are all open.
Now, suppose that $f^{-1}(B) \in \mathscr{T}_X$ for all $B \in \mathscr{B}$. Let $V \in \mathscr{T}_Y$. Write it as $\cup_{\alpha \in A} B_\alpha = V$ for $B_\alpha \in \mathscr{B}$.
Then $f^{-1}(V) = f^{-1}(\cup_{\alpha \in A} B_\alpha) = \cup_{\alpha \in A} f^{-1}(B_\alpha)$. This is the union of open subsets of X, so $f^{-1}(V)$ is open.

■

Example 0.12. Consider $B_{\frac{p}{q}}(\frac{r}{s})$, with $p, q, r, s \in \mathbb{Z}$, $q, s \neq 0$. In other words, the balls of rational radius and rational center. This collection of sets forms a basis for $\mathbb{R}$ using the standard (or, as Darren calls it, the "Mother's Knee") topology.
Exercise: Write down the definition of interiors and closures, and check the following lemma is true:

Lemma 2. *$x \in \text{cl}(A)$ if and only if, for all open $U \ni x$, $U \cap A \neq \varnothing$.*

Example 0.13. Consider ($\mathbb{N}$, Zariski). What is the closure of the collection of prime numbers under this topology?
Recall that the Zariski topology on $\mathbb{N}$ defines closed sets to be either empty, $\mathbb{N}$, or finite. So, for example, the closure of the integers from 1 to 10 is itself.
But the only closed set that can contain an infinite set is $\mathbb{N}$, so the closure of the primes is $\mathbb{N}$. More generally, for any infinite subset of $\mathbb{N}$, the closure is $\mathbb{N}$.

Subobjects and product objects

Subobjects

We have now defined our objects and morphisms, so let's talk about subobjects and product objects.
Let $(X, \mathscr{T}_X)$ be a topological space, and $A \subseteq X$ a nonempty subset.
Denote the inclusion map $\iota : A \to X$. We want to topologize A so that ι is continuous, and is as small as possible, in the sense that any topology for which ι is continuous includes this topology on A.
For any open $U \subseteq X$, we want $\iota^{-1}(U)$ to be open. But $\iota^{-1}(U) = U \cap A$.

Definition 0.13. For $(X, \mathscr{T}_X)$ a topological space and $A \subset X$ a subset, then the subspace topology on A, $\mathscr{T}_A$, consists of

$$\mathscr{T}_A \overset{\text{def}}{=} \{U \cap A \mid U \in \mathscr{T}_X\}$$

Proposition 3. *Suppose we have a commutative diagram of the form*

$$\begin{tikzcd}
(Z, \mathscr{T}_Z) \arrow[r, "g"] \arrow[dr, "\iota\circ g"'] & A \arrow[d, "\iota"] \\
& X
\end{tikzcd}$$

Then ι is continuous if and only if $\iota \circ g$ is continuous.

Proof. One direction is trivial, as we know the composition of continuous functions is continuous. Now, suppose that $\iota \circ g$ is continuous.
Let $W \in \mathscr{T}_A$. We know $W = W^* \cap A$ for some $W^* \in \mathscr{T}_X$. $\iota \circ g$ is continuous, so $(\iota \circ g)^{-1}W^* \in \mathscr{T}_Z$.
So, $\mathscr{T}_Z \ni (\iota \circ g)^{-1} = g^{-1}(\iota^{-1}(W^*)) = g^{-1}(W)$.

■

So we can see that if we want the above to hold, we are forced into our definition of $\mathscr{T}_A$.

Proposition 4. *$\mathscr{T}_A$ is the only topology so that the previous proposition is true for all spaces $(Z, \mathscr{T}_Z)$ and functions g.*

Proof. Suppose that $\mathscr{T}_\mu$ (μ for "mystery" topology) such that the previous proposition holds for all choices of $(Z, \mathscr{T}_Z)$ and $g : (Z, \mathscr{T}_Z) \to A$.

Consider

$$\begin{tikzcd}
(A, \mathscr{T}_A) \arrow[r, "\mathrm{Id}"] \arrow[dr, "\iota\circ\mathrm{Id}"'] & (A, \mathscr{T}_\mu) \arrow[d, "\iota"] \\
& (X, \mathscr{T}_X)
\end{tikzcd}$$

Lecture 6

We are trying to prove that $\mathscr{T}_A$ is the unique topology on A such that

$$\begin{tikzcd}
(A, \mathscr{T}_A) \arrow[r, "\mathrm{Id}"] \arrow[dr, "\iota\circ\mathrm{Id}"'] & (A, \mathscr{T}_\mu) \arrow[d, "\iota"] \\
& (X, \mathscr{T}_X)
\end{tikzcd}$$

Proof. Suppose $\mathscr{T}_\mu$ is such a topology on A. Then

$$\begin{tikzcd}
(A, \mathscr{T}_A) \arrow[r, "\mathrm{Id}"] \arrow[dr, "\iota\circ\mathrm{Id}"'] & (A, \mathscr{T}_\mu) \arrow[d, "\iota"] \\
& (X, \mathscr{T}_X)
\end{tikzcd}$$

We have $\iota \circ \mathrm{Id}$ is continuous, so $(A, \mathscr{T}_A) \xrightarrow{\mathrm{Id}} (A, \mathscr{T}_\mu)$ is continuous. So if $W \in \mathscr{T}_\mu$, then $W \in \mathscr{T}_A$. Therefore $\mathscr{T}_\mu \subseteq \mathscr{T}_A$.
We know that $\iota \circ \mathrm{Id}$ is continuous. Let $W \in \mathscr{T}_X$. By continuity, $(\iota \circ \mathrm{Id})^{-1} W \in \mathscr{T}_\mu \implies (\mathrm{Id})^{-1} \circ \iota^{-1} W \in \mathscr{T}_\mu$.
So then $(\mathrm{Id})^{-1}(W \cap A) \in \mathscr{T}_\mu$, and $W \cap A \in \mathscr{T}_\mu$. Therefore for any $W \in \mathscr{T}_\mu$, $W \cap A \in \mathscr{T}_\mu$, so $\mathscr{T}_A \subset \mathscr{T}_\mu$.

Product Objects

Let $(X, \mathscr{T}_X), (Y, \mathscr{T}_Y)$ be topological spaces. We want to topologize $X \times Y$.
Well, if that holds a topology, then the projections damn well better be continuous.

In other words, we want to arrange such that

$$\begin{tikzcd} X \times Y \arrow[r, "\rho_X"] \arrow[d, "\rho_Y"'] & X \\ Y & \end{tikzcd}$$

both ρ_X, ρ_Y are continuous.
Let $U \in \mathscr{T}_X$. Then $\rho_X^{-1} U = U \times Y$. Similarly, for $V \in \mathscr{T}_Y$, $\rho_Y^{-1} V = X \times V$.
Thus, for the projections to be continuous, we need that the intersection of $U \times Y$ and $X \times V$ are in $\mathscr{T}_{X \times Y}$ for all $U \in \mathscr{T}_X, V \in \mathscr{T}_Y$. The sets of this form, $X \times V \cap U \times Y$, form a basis for a topology.
In other words, the product topology has basis $\{U \times V \mid U \in \mathscr{T}_X, V \in \mathscr{T}_Y\}$.

Theorem 0.5. *Consider the commutative diagram*

$$\begin{tikzcd} & Z \arrow[dl, "\rho_X \circ g"'] \arrow[d, "g"] \arrow[dr, "\rho_Y \circ g"] & \\ X & X \times Y \arrow[l, "\pi_j"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

Then g is continuous if and only if $\rho_X \circ g$ and $\rho_Y \circ g$ are continuous.

Proof. Next time

Lecture 7

Relevant digression

Let $(X_i, \mathscr{T}_i)_{i \geq 1}$ be a family of topological spaces. A basis for a topology on $\prod_{i \geq 1} X_i$ is $\{\prod_{i \geq 1} u_i \mid u_i \in \mathscr{T}_i\}$
We can ask that the projection $\prod_{i \geq 1} X_i \to X_j$ is continuous for each j, but then

$$p_j^{-1}(V_j) = X_1 \times \cdots \times X_{j-1} \times V_j \times X_{j+1} \times \cdots$$

Definition 0.14. Let $(X_i, \mathscr{T}_i)_{i \geq 1}$ be a family of topological spaces. Then the Tychonoff's product topology has a basis consisting of sets of the form $\prod_{i \geq 1} V_i$, where $V_j \subseteq X_j$, and $V_j = X_j$ for all but finitely many j.

Example 0.14. Let p be prime. Then $\mathbb{Z}/p^k$ will denote $\mathbb{Z}/p^k\mathbb{Z}$. We may equip it with the discrete topology. Consider $\prod_{k\geq 1}\mathbb{Z}/p^k$. Let's put a metric on this. We will define

$$d(x,y)=\sum_{k\geq 1}\frac{1}{2^k}d_k(x_k,y_k)$$

It is easy to convince yourself this is a metric.
Note that there exists a map $\mathbb{Z}\to\prod_{k\geq 1}\mathbb{Z}/p^k$, given by $x\mapsto(x \pmod p), x \pmod{p^2}), \dots)$
An integer m is close to zero in this metric if m is divisible by large powers of p.
This is (basically) the p-adic metric.
Exercise: What is the closure of $\mathbb{Z}$ in this metric space?

Hausdorff Spaces

Definition 0.15. Here is what Darren calls a "reasonable definition" of convergence of a sequence in a general topological space.
Let x_n be a seequence in $(X,\mathscr{T})$. We say that x_n converges to x if, for any $x\in U\in\mathscr{T}$, there exists an N such that $x_k\in U$ for all $k\geq N$.

Example 0.15. Let $\mathcal{P}$ be the set of prime numbers, and let $\mathbb{N}$ have the Zariski topology. What does $\mathcal{P}$ converge to?
Consider 193. An open neighborhood U of this would be a set which contains 193, and all but finitely many primes. So, for any such neighborhood, there will be an N such that $p_k\in U$ for all $k\geq N$. So, the sequence $x_k=p_k$ converges to 193. But 193 was arbitrary, so the sequence converges to any natural number.
The problem here is that the open sets are too big. The Hausdorff condition will get us around this.

Definition 0.16. A topological space $(X,\mathscr{T})$, is said to be Hausdorff if, for any $x,y\in X$, when $x\neq y$, there exists $U_x,U_y\in\mathscr{T}$, such that $x\in U_x$, $y\in U_y$, and $U_x\cap U_y=\varnothing$.

Lemma 3. *(a) Metric spaces are Hausdorff*

(b) If (x_n) has a limit in a Hausdorff space $(X,\mathscr{T})$, then it's unique

Proof. (a) Pick $x\neq y$ in (X,d) with $d(x,y)=\varepsilon>0$. Consider $B_{\frac{\varepsilon}{3}}(x)$ and $B_{\frac{\varepsilon}{3}}(y)$. These obviously are disjoint by the triangle inequality.

(b) Let x_n converge to x, and let $y\neq x$ be some other point besides x. Then, there is some neighborhood of x which is disjoint from a neighborhood of y. Eventually, every x_k is in this neighborhood, meaning none are in the neighborhood of y. Thus, x_n cannot converge to y.

Definition 0.17. Let $(X, \mathscr{T}_X), (Y, \mathscr{T}_Y)$ be topological spaces. X and Y are said to be homeomorphic if there exist continuous maps $f : X \to Y$, $g : Y \to X$, such that $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$.

Lecture 8

Theorem 0.6. *(a) Suppose X, Y are Hausdorff spaces. Then $X \times Y$ is also Hausdorff.*

(b) If $A \subseteq X$, and X is Hausdorff, then A is also Hausdorff under the subspace topology.

(c) If $(X, \mathscr{T}_X)$ is Hausdorff, and X is homeomorphic to $(Y, \mathscr{T}_Y)$, then Y is also Hausdorff.

Proof. (a) Pick distinct points in $X \times Y$, (x_1, y_1) and (x_2, y_2). Either $x_1 \neq x_2$, or $y_1 \neq y_2$, or both. Without loss of generality, suppose $x_1 \neq x_2$. Because X is Hausdorff, there are open sets $U_1, U_2 \subseteq X$, such that $x_1 \in U_1, x_2 \in U_2$, and $U_1 \cap U_2 = \varnothing$. Let $O_i = U_i \cap Y$. $(x_i, y_i) \in O_i$, and $O_1 \cap O_2 = \varnothing$, as no x-coordinate in O_1 is the same as any x-coordinate in O_2.

(b) Exercise

(c) Pick $y_1 \neq y_2 \in Y$. Then there is a unique x_1, x_2 such that $f(x_i) = y_i$. Because X is Hausdorff, there are disjoint open neighborhoods $x_1 \in U_1, x_2 \in U_2$. $f(U_1)$ and $f(U_2)$ are both open, as f is a homeomorphic; each $f(U_i)$ contains y_i; finally, $f(U_1) \cap f(U_2) = \varnothing$, as f was bijective. Thus, Y is Hausdorff.

■

Compact Spaces

Compactness is a very good property that a space can have. For metric spaces, we can think of compactness as a generalization of being a finite collection of points.

Example 0.16. (Exercise)

(a) Let (X, d) be a metric space, $|X| < \infty$. Given $X \xrightarrow{f} X$ such that $d(f(x), f(y)) \geq d(x, y)$ for all $x, y \in X$. Then f is an isometry. Give yourself one line to prove this.

(b) Do the same thing for (X, d) a compact metric space.

Definition 0.18. A topological space $(X, \mathscr{T})$ is called compact if, for any open cover $X = \cup_{\alpha \in A} U_\alpha$, $U_\alpha \in \mathscr{T}$, there exists $\alpha_1, \dots, \alpha_k \in \alpha$ such that $\cup_{i \le j \le k} U_{\alpha_j} = X$. In other words, any open cover admits a finite subcover.

Example 0.17. 1. Let X be any set, and consider $\mathscr{T} = \{\varnothing, X\}$. This is trivially compact.

2. ($\mathbb{R}$,Zariski) is compact easily. Given an open covering, pick a set, and this covers every point of $\mathbb{R}$ except finitely many points. For each of those points, it is covered by another element of the covering. Thus, we can produce a finite subcover.

3. $(\mathbb{R}, \mathscr{T}_{MK})$. This is not compact, as $\mathbb{R} = \cup_{N \ge 0}(-n, n)$. This is an open cover which clearly does not have a finite subcover.

Definition 0.19. A subset $C \subseteq (X, d)$ is bounded if for any $\xi \in C$,, there is a K_ξ such that $B_{K_\xi}(\xi) \supseteq C$.

Lemma 4. *If $C \subseteq (X, d)$ is compact, then C is bounded.*

Proof. Clearly, $C = \cup_{i=1}^{\infty} B_i(x)$ for any particular $x \in C$. This must have a finite subcover, so in particular there is a largest n such that $C \subseteq B_n(x)$.

■

Lemma 5. *Suppose $A \subseteq (X, \mathscr{T})$ is compact in the subspace topology with X Hausdorff. Then A is closed.*

Proof. Take $\xi \notin A$. We want to construct an open neighborhood of ξ which is disjoint from A.
For any point $a \in A$, there are open neighborhoods U_a, V_a, with $a \in U_a, \xi \in V_a$, with $U_a \cap V_a = \varnothing$. Do this for every $a \in A$. Then $\{U_a \cap A \mid a \in A\}$ is an open covering of A in $\mathscr{T}_A$. A is compact, so there exists a finite subcover $(U_{a_1} \cap A) \cup \cdots \cup (U_{a_k} \cap A) \supseteq A$. Each U_{a_i} has a corrseponding V_{a_i}. Let $V^* = \cap_{1 \le i \le k} V_{a_i}$. this is an open set, because the intersection of finitely many sets is open, and is disjoint from A, as it is disjoint from U_{a_i} for each i.
So, ξ is in the interior of A^c. However, ξ was arbitrary, so A^c is open, so A is closed.

■

Lecture 9

Theorem 0.7. *(Heine-Borel)*
$[a, b]$ is compact.

Lecture 10

Theorem 0.8. *Suppose $A \subseteq (X, \mathscr{T})$ is a closed subset of a compact space. Then A is compact.*

Proof. Let $\{U_\alpha \cap A\}_{\alpha \in A}$ be an open (in the subspace topology) cover of A. Of course, $U_\alpha \in \mathscr{T}_X$ for all α.
Then $\{X \setminus A, U_\alpha\}_{\alpha \in A}$ is an open covering of X. By hypothesis, X is compact, so this admits a finite subcover $\{X \setminus A, U_{\alpha_1} \cup \cdots \cup U_{\alpha_k}$ of X.
Then $\{U_{\alpha_i} \cap A\}_{i=1}^k$ is an open cover of A. So, $\{U_\alpha \cap A\}_{\alpha \in A}$ admits a finite subcover. But this cover was arbitrary, so we may conclude that any such open cover of A admits a finite subcover.
Thus, A is compact. ■

Corollary 0.9. *Let $C \subseteq (\mathbb{R}, \mathscr{T}_{MK})$. Then C is compact if and only if it is closed and bounded.*

Proof. Compact implies closed and bounded is easy. For the other direction, if C is bounded, then it is contained in some interval $[a, b]$. By Heine-Borel, such an interval is compact. So then C is a closed subset of $[a, b]$, which is compact, so C is a closed subset of a compact space, hence compact.

Remark. Darren has several remarks:

1. Closed and bounded does not equate to compactness in a general metric space.

2. This is a digression, but the above tells us that the Cantor set is compact.

Definition 0.20. Let (X, d) be a metric space. We say that X is Sequentially Compact if, for any sequence $(x_n) \in X^{\mathbb{N}}$, there is a convergent subsequence (x_{n_k}) of (x_n).

Theorem 0.10. *Let (X, d) be metric space. Then X is compact if and only if it is sequentially compact.*

Proof. of easy direction
Let $(x_n) \in X^{\mathbb{N}}$ be a sequence in X. If (x_n) takes on some value $x \in X$ infinitely many times, we're done.
So we can assume that doesn't happen.
Suppose for contradiction that there is no convergent subsequence. For each $x \in X$, there is an $\varepsilon = \varepsilon(x) > 0$ such that $B_{\varepsilon(x)}(x)$ contains only finitely many points of (x_n). Then $\bigcup_{x \in X} B_{\varepsilon(x)}(x)$ is an open cover, so there exists a finite subcover $B_{\varepsilon(x_1)}(x_1) \cup \cdots \cup B_{\varepsilon(x_n)}(x_n)$. But this implies there are only finitely many elements of our sequence, a contradiction. ■

Definition 0.21. Given $\varepsilon > 0$, an ε-net is a set of points $x_1, \dots, x_k \in X$ so that $B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_k) = X$.

Theorem 0.11. *Suppose a metric space (X, d) is sequentially compact. Then X has a (finite) ε-net for every $\varepsilon > 0$.*

Proof. We proceed by contraposition. Suppose that there exists a bad $\varepsilon > 0$, such that there is no finite ε-net.
Pick a point x_1. By hypothesis, $B_\varepsilon(x_1)$ is <u>not</u> all of X. So we can pick a point $x_2 \in B_\varepsilon(x_1)^c$. Then, by hypothesis, $B_\varepsilon(x_1) \cup B_\varepsilon(x_2)$ is <u>not</u> all of X. We can keep doing this indefinitely, to get a sequence $(x_n) \in X^{\mathbb{N}}$, such that $d(x_i, x_j) \geq \varepsilon > 0$ for all $i \neq j$.
Thus, $(x_n) \in X^{\mathbb{N}}$ is a sequence which admits no cauchy subsequence, meaning it admits no convergent subsequence, meaning that X cannot be compact. ∎

Lecture 11

Lemma 6. *Suppose (X, d) is sequentially compact. Then, for all $\varepsilon > 0$, (X, d) has a finite ε-net.*
We proved this last time.

Definition 0.22. Let (X, d) be a metric space, and $\mathscr{O} = \{U_\alpha\}_{\alpha \in A}$ an open covering. Then a <u>Lebesgue Number</u> for $\mathscr{O}$ is a $\delta > 0$ such that for all $x \in X$, $B_{\delta(X)} \subset U_\beta$ for at least one $\beta \in A, U_\beta \in \mathscr{O}$.

Lemma 7. *If (X, d) is sequentially compact, then every open covering has a Lebesgue Number.*

Proof. Let $\mathscr{O} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of X. Suppose $\mathscr{O}$ has no Lebesgue number. $\frac{1}{2}$ is not a Lebesgue number, so there exists a point x_2 such that $B_{\frac{1}{2}}(x_2)$ is not inside any single U_α. Next, $\frac{1}{3}$ is not a Lebesgue number, so $B_{\frac{1}{3}}(x_3)$ is not in any individual U_α. We can continue, and generate a sequence $\{x_2, x_3, \dots\}$, such that $B_{\frac{1}{i}}(x_i)$ is not contained in any U_α.
By hypothesis, this sequence has a convergent subsequence x_{n_k}, which converges to $x \in X$. For any $\varepsilon > 0$, $B_\varepsilon(x)$ contains infinitely many x_i. In particular, for a U_α containing x, there is a $\varepsilon > 0$ such that $B_\varepsilon(x)$ contains infinitely many of the x_i. But then eventually the ball of radius $\frac{1}{i}$ around x_i will be entirely in this ball, contradicting the hypothesis.
Explicitly, pick $n_k >> 0$ such that $\frac{1}{n_k} < \frac{\varepsilon}{10}$, then $B_{\frac{1}{n_k}}(x_{n_k}) \subset B_\varepsilon(x) \subset U_\beta$. ∎

Now, we prove the converse of the theorem from the end of last lecture.

Proof. that sequential compactness implies compactness.
Given $\mathscr{O} = \{U_\alpha\}_{\alpha \in A}$ an open covering. By the previous lemma, there is a Lebesgue number $\varepsilon^* > 0$ for $\mathscr{O}$. By another previous lemma, there is a finite ε^*-net for $\mathscr{O}$. That is, there are points $\{x_1, \dots, x_k\}$, such that $B_{\varepsilon^*}(x_1) \cup \dots \cup B_{\varepsilon^*}(x_k)$. But each of these lies inside one of the U_α, so the corresponding U_α yields a finite subset of $\mathscr{O}$. ■

Example 0.18. Consider $(C[0,1], \sup)$. Let f_0 be the zero function, and consider $\overline{B_1(f_0)}$. This is closed, and bounded. However, it is not compact.
Consider a function which is zero, except for a spike at some point. Each such function is distance 1 from f_0, but is also distance 1 from each other such function. A sequence of such functions can't have a convergent subsequence.
So, this ball is not sequentially compact, and thus not compact.

Example 0.19. Consider the infinite product

$$X = \prod_{k \geq 1} \mathbb{Z}/p^k$$

where p is a prime fixed beforehand. Earlier, we defined a metric by

$$d(x,y) = \sum_{k \geq 1} \frac{1}{2^k} d_k(x_k, y_k)$$

This is sequentially compact. Can you see why?
<u>Exercise</u>: Write this down.
<u>Exercise</u>: Suppose $f : (X, d_X) \to (Y, d_Y)$ is continuous, and X is compact. Prove f is uniformly continuous. i.e. given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon$$

for <u>any</u> $x, x' \in X$.

Lecture 12

Theorem 0.12. *Let $(X, \mathscr{T}_X)$, $(Y, \mathscr{T}_Y)$ be compact. Then $(X \times Y, \mathscr{T}_{X \times Y})$ is compact.*

Remark. Suppose $(X, d_X), (Y, d_Y)$ are metric spaces, both compact. Then $(X \times Y, d_X + d_Y)$ is compact. This is easy, as $X \times Y$ in this case is sequentially compact.

Proof. Suppose $\{U_\alpha\}_{\alpha \in A}$ is an open covering of $X \times Y$.
Fix $y \in Y$. Then $X \times \{y\}$ (with the subspace topology that $X \times Y$ induces) is homeomorphic to X. X is compact, so $X \times \{y\}$ is compact.
For each $x \in X$, $(x, y) \in U_\beta$ for some $\beta \in A$. $U_\beta \in \mathscr{T}_{X \times Y}$, so $U_\beta = \bigcup$(open in X)$\times$(open in Y) .

So there exists $A_{(x,y)} \times B_{(x,y)} \subset U_\beta$, $A_{(x,y)} \in \mathscr{T}_X, B_{(x,y)} \in \mathscr{T}_Y$.
$\{A_{(x,y)}\}_{x\in X}$ is an open covering of X, and thus admits a finite subcover, $A_{(x_1,y)} \cup \cdots \cup A_{(x_k,y)} = X$. For each $A_{(x_i,y)}$, there is a $B_{(x_1,y)}$.
Set $B^*_y = \bigcap_{1\le i\le k} B_{(x_i,y)}$. Then the entire strip $X \times B^*_y$ is covered by a finite number of U_β.
$\{B^*_y\}_{y\in Y}$ is an open covering of Y, so by compactness of Y admits a finite subcover $B^*_{y_1} \cup \cdots \cup B^*_{y_\ell}$. Each $B^*_{y_i}$ has the property that $X \times B^*_{y_i}$ is covered by finitely many U_β.
So $X \times Y$ is covered by a finite number of U_β's.
The cover $\{U_\alpha\}_{\alpha\in A}$ was arbitrary, so we are done.

∎

Before we started this proof, Daren said "First I'm gonna take a sip of the real stuff," and took a drink from his can of diet Coke. What did he mean by this?

Connected Spaces

Definition 0.23. A topological space $(X, \mathscr{T})$ is <u>not connected</u> if there exist nonempty, disjoint, open sets A, B such that $X = A \coprod B$.

Definition 0.24. A topological space $(X, \mathscr{T})$ is <u>not connected</u> if there is a continuous surjection $f : (X, \mathscr{T}) \to \{0, 1\}$.

Proposition 5. *The above two definitions are equivalent.*

Proof. First, suppose that there is a continuous surjection $f : (X, \mathscr{T}) \to \{0, 1\}$. Then $f^{-1}(0), f^{-1}(1)$ are nonempty disjoint open subsets of X, whose union is X.
Now, suppose that we can express X as the union of two disjoint nonempty open sets A, B. Then $f(x) = \begin{cases} 1 & x \in A \\ 0 & x \in B \end{cases}$ is a continuous surjection from X to $\{0, 1\}$.

Lemma 8. *Suppose X is connected, and $f : X \to Y$ is continuous. Then $f(X)$ is connected.*

Proof. If there exists $\pi : f(X) \to \{0, 1\}$ which is continuous and surjective, then $\pi \circ f : X \to \{0, 1\}$ is continuous and surjective.

∎

Corollary 0.13. *Connectedness is a topological property.*

Theorem 0.14. *$X \times Y$ is a connected topological space if and only if X and Y are connected.*

Proof. If $X \times Y$ is connected, then $p_X : X \times Y \to X$ is continuous, so by the above lemma, X is connected. A similar argument shows Y is connected.
Now, suppose X, Y are connected. Let $f : X \times Y \to \{0, 1\}$ be continuous. $X \times \{y\}$ is connected for any $y \in Y$, $f(X \times \{y\}) = \{0\}$ without loss of generality.

Lecture 13

Lemma 9. *$S \subseteq (\mathbb{R}, \mathscr{T}_{MK})$ is an interval if and only if it satisfies the inbetweenness (sp?) property: if $x, y \in S$, and $x < u < y$, then $u \in S$.*

Proof. One direction is easy.
Now, suppose that S has the inbetweenness property. We will abuse notation somewhat by using square brackets for any interval, even if one of the bounds is $\pm\infty$.
Let $\lambda = \inf S$, and $\nu = \sup S$.

Claim. $[\lambda, \nu] \subseteq S \subseteq [\lambda, \nu]$.

Proof. This is immediate. S is certainly contained inside $[\lambda, \nu]$. For containment the other way, pick $u \in [\lambda, \nu]$. Then there exists an $x \in S$ such that $x < u$. Similarly, there is a $y \in S$ with $u < y$. So $u \in S$.
To correctly interpret this proof, simply replace the brackets with either (or [. Make sure to pick the ones which make the statements correct.

■

Theorem 0.15. *The connected subsets of $\mathbb{R}$ are precisely the intervals.*

Proof. Let $S \subset \mathbb{R}$. If S is not an interval, then there is some $x, y, z \in \mathbb{R}$, with $x < y < z$, with $x, z \in S$, $y \notin S$. Then $(-\infty, y) \coprod (y, \infty)$ is a disconnection of S. So if S is connected, it is an interval.
Now, suppose that S is an interval that is disconnected. That is, $S = A \coprod B$, A, B open, nonempty, disjoint, etc. There is some $x \in A$ and $y \in B$, as by assumption they are nonempty.
It will suffice to reduce to the case of $[x, y]$. So $(A \cap [x, y]) \coprod (B \cap [x, y])$ is a disconnection of $[x, y]$. Consider $d : (A \cap [x, y]) \times (B \cap [x, y]) \to \mathbb{R}$ which gives the distance between a point in $A \cap [x, y]$ and a point in $B \cap [x, y]$. This is the product of compact spaces, and thus compact. Therefore, the infimum is achieved, and must be greater than 0.
So there exist $\alpha \in A \cap [x, y], \beta \in B \cap [x, y]$, such that $d(\alpha, \beta) \leq d(a, b)$ for any $a \in A \cap [x, y], b \in B \cap [x, y]$. But $d(l, r) = |l - r|$ for any l, r.
Take $\xi = \frac{1}{2}(\alpha + \beta)$. Then

$$|\beta - \frac{1}{2}(\alpha + \beta)| = \frac{1}{2}|\beta - \alpha| < |\alpha - \beta|$$

So $\frac{1}{2}(\alpha + \beta) \notin B$. Similarly, $\frac{1}{2}(\alpha + \beta) \notin A$, a contradiction.

■

Theorem 0.16. *Suppose A is connected, with $A \subseteq K \subseteq \overline{A}$. Then K is connected.*

Proof. Suppose K is not connected. Then there is a continuous, surjective function $f : K \to \{0, 1\}$. Because A is connected, $f|_A : A \to \{0, 1\}$ is not surjective. Suppose there is a $k \in K$ such that $f(k) = 1$. Then $f^{-1}(1)$ is an open neighborhood of k. But, by the definition of the closure, k $f^{-1}(1) \cap A \neq \varnothing$. But f is identically 0 on A, a contradiction. Thus, there is no such f.

■

Example 0.20. Time for a classic example. Consider $g : (0, \infty) \to \mathbb{R}^2$ given by $t \mapsto (t, \sin(1/t))$. This is the so-called "topologists sine curve". This is continuous, so its image is connected as a subset of $\mathbb{R}^2$. Then, by the theorem, $\overline{\operatorname{im} g}$ is connected. Look at a picture.

Lecture 14

Definition 0.25. Let $(X, \mathscr{T})$ be a topological space. Define the equivalence relation $x \sim y$ if, for all $f : X \to \{0, 1\}$ continuous and onto, $f(x) = f(y)$.
The equivalence classes are called connected components of X.

Example 0.21. **1.** $[0, 1] \cup [2, 3]$ has two connected components.

2. In $\mathbb{Q}$ the only connected components are singletons. A space with this property is said to be totally disconnected.

3. By the lemma from last time (closure of connected set is connected), components are closed.

Proposition 6. *Let $\mathscr{C}(X, \mathscr{T})$ denote the components of f. Suppose that $f : X \to Y$ is a homeomorphism. Then there is a bijection $f_* : \mathscr{C}(X) \to \mathscr{C}(Y)$.*

Proof. For a point $x \in X$, define $C_x \in \mathscr{C}(X)$ as the component containing x. Then we define $f_*(C_x) = C_{f(x)}$.
We need to show this is well defined. Suppose $x' \in C_x$. Then $f(x') \in f(C_x)$, which is a connected set that contains $f(x)$ and $f(x')$. So $f_*(C_x) = f_*(C_{x'})$. So this is well defined.

$$\begin{tikzcd}
\mathscr{C}(X) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathscr{C}(Y) \arrow[d, "g_*"] \\
 & \mathscr{C}(Z)
\end{tikzcd}$$

I claim this commutes.

$$g_* f_* C_x = g_* C_{f(x)} = C_{g(f(x))} = (gf)_*(C_x)$$

Further, I_* is the identity, so we are done.

Definition 0.26. Let $(X, \mathscr{T})$ be a topological space. We say it is path connected if, for any $x, y \in X$, there exists a continuous $\sigma : [0,1] \to X$ such that $\sigma(0) = x, \sigma(1) = y$.

Lemma 10. *Suppose X is a path connected topological space, and $f : X \to Y$ is a continuous function. Then $f(X) \subseteq Y$ is path connected.*

Proof. Let $\xi, \eta \in f(X)$. Then there exist $x_\xi, x_\eta \in X$ such that $f(x_\xi) = \xi, f(x_\eta) = \eta$. By hypothesis, there is a continuous $\sigma : [0,1] \to X$ such that $\sigma(0) = x_\xi, \sigma(1) = x_\eta$. Then $f \circ \sigma : [0,1] \to f(X)$ is continuous, and $(f \circ \sigma)(0) = \xi, (f \circ \sigma)(1) = \eta$. ■

Corollary 0.17. *Path connectedness is invariant under homeomorphism.*

Proposition 7. *Suppose X is path connected. Then X is connected.*

Proof. If X is not connected, then $X = A \coprod B$, for $A, B \subseteq X$ open, disjoint, nonempty. Let $\alpha \in A, \beta \in B$.
If there exists a path $\sigma : [0,1] \to X$ with $\sigma(0) = a, \sigma(1) = \beta$, then $\sigma^{-1}(A) \coprod \sigma^{-1}(B)$ is a partition of $[0,1]$. But we know this space is connected. ■

The converse of the above proposition is false.
Let $g : (0, \infty) \to \mathbb{R}^2$ be defined by

$$t \mapsto (t, \sin(\frac{1}{t}))$$

Claim. $\overline{\text{im}(g)}$ *is not path connected.*

Proof. Let A be a point on the graph, and $(0,0)$ the origin (which lies not in $\text{im}(g)$, but in its closure).
Suppose there is a continuous $\sigma[0,1] \to \overline{\text{im}(g)}$ with $\sigma(0) = A, \sigma(1) = (0,0)$, and consider the projection onto the y-axis, ρ_y. $\rho_y \circ \sigma$ is uniformly continuous.
Given $\delta = \frac{1}{3}$, there is a $\varepsilon = \frac{1}{n}$ such that $|x - x'| < \frac{1}{n}$, and $|\rho_y \sigma(x) - \rho_y \sigma(x')|$.
To traverse between to zeroes of this function, you need to travel $\frac{6}{n}$.
Alternatively, consider the sequence $x_n = \frac{1}{\frac{\pi}{2} + 2\pi n}$. We have

$$\lim_{n \to \infty} \sigma(x_n) = (1,1) \neq (0,0) = \sigma(0) = \sigma(\lim_{n \to \infty} x_n)$$

Theorem 0.18. *Let $U \subseteq (\mathbb{R}^n, \mathscr{T}_{mk})$ be open and connected. Then U is path connected.*

Proof. Pick $\xi \in U$. Consider $A = \{\alpha \in U \mid \alpha \text{ can be connected to } \beta \text{ by a path}\}$, and let $B = A^c$.

Claim. *A, B are both open.*

Proof. Suppose $\alpha \in A$. Then there is an open ball around α lying entirely inside U, and these balls are path connected. So we must conclude that one of A, B is empty. A is not, so $A = U$.

Lecture 15

Completeness

Let (X, d) be a metric space. Suppose $x_n \to x$. This means that given $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that, for all $k \geq N$, $d(x_k, x) < \varepsilon$.

Definition 0.27. A sequence $(x_n)_{n=1}^{\infty}$ in a metric space is said to be Cauchy if, given $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for $k, \ell \geq N$, $d(x_k, x_\ell) < \varepsilon$.
It is an easy consequence of definitions that if (x_n) converges, then it is Cauchy. The converse is not true in general.

Example 0.22. In $(\mathbb{Q}, |\cdot|_{MK})$, we can find a sequence of rationals which is Cauchy, and converges to $\sqrt{2}$, which is not rational.

Definition 0.28. A metric space (X, d) is said to be complete if every Cauchy sequence (x_n) converges to some $x \in X$.

Theorem 0.19. *$(\mathbb{R}, |\cdot|_{MK})$ is complete.*

Proof. Let (x_n) be a Cauchy sequence in $\mathbb{R}$.
Then $(x_n) \subset \mathbb{R}$ is bounded. So, there exists an $A \in \mathbb{N}$ such that $[-A, A] \supseteq \{x_n\}$. But $[-A, A]$ is compact, and therefore sequentially compact. So, (x_n) has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$.
Because (x_n) is Cauchy, it can be shown (as an exercise!) that if a Cauchy sequence has a convergent subsequence, then the whole sequence converges.

■

Remark. Completeness is not a topological invariant. For example, $(\mathbb{R}, |\cdot|_{MK})$ and $((0,1), |\cdot|_{MK})$ are homeomorphic, but one is complete and the other is not.

Proposition 8. *Let $(X, d_X), (Y, d_Y)$ be complete metric spaces. Then $(X \times Y, d_{X \times Y} = \max(d_X, d_Y)$. That is, $d_{X \times Y}((x_1, y_1), (x_2, y_2)) = \max(d_X(x_1, x_2), d_Y(y_1, y_2))$.*

Proof. Think

■

Corollary 0.20. *$(\mathbb{R}, |\cdot|_{MK})$ is complete.*

Theorem 0.21. *Let (X, d) be complete. Then if $Y \subseteq X$ is closed, then (Y, d) is complete.*

Proof. Let (y_n) be a Cauchy sequence in Y. By hypothesis, this converges to some $\lambda \in X$. So, for any open $U \ni \lambda$, U contains points in Y. Thus, $\lambda \in \overline{Y}$. However, Y is closed, so $Y = \overline{Y}$, so $\lambda \in Y$.

■

Theorem 0.22. *Let (X, d) be a metric space, and Y is a complete subset of X. Then Y is closed.*

Proof. Pick $y^\star \in \overline{Y}$. We will show $y^\star \in Y$. For any open ball containing $y^\star$, this ball intersects Y nontrivially by the definition of $\overline{Y}$. So for each $n \in \mathbb{N}$, $B_{\frac{1}{n}}(y^\star)$ contains a point in Y. Let $y_n \in Y \cap B_{\frac{1}{n}}(y^\star)$.
The sequence $y_n \to y^\star$, so it is Cauchy. But Y is complete, so $\lim_{n\to\infty} y_n = y^* \in Y$. $y^\star$ was arbitrary, so $Y = \overline{Y}$, so Y is closed.

■

Theorem 0.23. *Let (X, d) be compact. Then (X, d) is complete.*

Proof. Let (x_n) be a Cauchy sequence in X. In metric spaces, compactness is equivalent to sequential compactness, so this has a convergent subsequence. But the limit of this subsequence must be the limit of (x_n), because (x_n) is Cauchy. Thus, (x_n) converges. (x_n) was an arbitrary Cauchy sequence, so every Cauchy sequence in X converges. Thus, X is complete.

■

Definition 0.29. (Old definition)
A metric space (X, d) is totally bounded if given $\varepsilon > 0$, there exist $x_1, \dot{;} x_k \in X$ such that

$$X = \cup_{1 \leq i \leq k} B_\varepsilon(x_i)$$

Recall that in a metric space, compactness implies total boundedness.

Theorem 0.24. *If (X, d) is complete and totally bounded, then it is compact.*
Proof will be next time.

Lecture 16

Proof. We will show that X is sequentially compact. Let (x_n) be a sequence in X. Fix $\varepsilon = \frac{1}{2}$. By hypothesis, there exists a finite $\frac{1}{2}$-net $B_{\frac{1}{2}}(c_1) \cup \cdots \cup B_{\frac{1}{2}}(c_k) = X$.
Wew can extract a subsequence $x_1^{(1)}, x_2^{(1)}, \ldots$, such that one of these $\frac{1}{2}$-balls contains $x_i^{(1)}$ for all i.
Now, let $\varepsilon = \frac{1}{4}$. By hypothesis, there exists a subsequence of $(x_n^{(1)})$, $x_1^{(2)}, x_2^{(2)}, \cdots$, with $x_i^{(2)}$ contained in the same $\frac{1}{4}$-ball.
We continue this construction inductively, with $x_j^{(i)}$ being the jth element of a subsequence of x lying entirely in the intersection of balls.

Then $x_i^{(i)}$ is a Cauchy subsequence: $x_i^{(i)}, x_j^{(j)}$ lie in the same ball of radius $\frac{1}{2^{\max(i,j)}}$. That is, $d(x_i^{(i)}, x_j^{(j)}) \leq \frac{1}{2^{\min(i,j)}}$. So this sequence is a Cauchy subsequence of (x_n). But (x_n) was arbitrary, so we are done with the proof.

■

Theorem 0.25. *Any metric space* (X, d) *has a canonical completion* $(\tilde{X}, \tilde{d})$. *In other words, there exists an isometry* $i : X \to \tilde{X}$ *such that* $\overline{i(X)} = \tilde{X}$ *(i.e.* $i(X)$ *is* <u>*dense*</u> *in* $\tilde{X}$*). We also stipulate that* $\tilde{X}$ *is complete.*
That is, if $Y \supset X$ *is a complete metric space, then there is an injective* $f : \tilde{X} \to Y$ *which makes the following diagram commutative:*

$$\begin{tikzcd} & Y \\ X \arrow[ur, "\iota"] \arrow[r, "\iota_X"'] & \tilde{X} \arrow[u, "f"'] \end{tikzcd}$$

Proof.

Proposition 9. *Let* $\mathscr{B}(X, \mathbb{R}) \stackrel{\text{def}}{=} \{f : X \to \mathbb{R} \mid f \text{ is bounded}\}$. *This is a metric space, with metric* $d(f, g) = \sup_{x \in X} |f(x) - g(x)|$.
Then $\mathscr{B}(X, \mathbb{R})$ *is complete.*

Remark. In this definition, we can replace $\mathbb{R}$ with any metric space, and $\sup_{x \in X} |f(x) - g(x)|$ with $\sup_{x \in X} d(f(x), g(x))$, and this remains true.

Proof. of proposition.
Let (f_k) be a Cauchy sequence of functions in $\mathscr{B}(X, \mathbb{R})$. Fix $x \in X$. Then $\{f_k(x)\}$ is a Cauchy sequence in $\mathbb{R}$, and so converges to a point we will call $f(x)$. We need to show that $f(x)$ is a bounded function, and we need to show that $f_k \to f$ (we know this happens pointwise).
To show that f is bounded, take $\varepsilon = 1$. Then there is an $N \in \mathbb{N}$ such that $d(f_k, f_\ell) < 1$ for $k, \ell \geq N$. Therefore, $|f_k(x) - f_\ell(x)| < 1$ for all $x \in X$ by definition.
In particular, $\sup_{x \in X} |f_N(x) - f_\ell(x)| < 1$ for all ℓ. For a fixed $x \in X$, we can see $|f_N(x), f_\ell(x)| \leq 1$. Letting $\ell \to \infty$, we have $|f(x)| \leq 1+$the bound for $f_N(x)$. So, f is bounded.
Now for the second part. We will show $f_n \to f$ in the sup metric. Darren will leave this as an exercise, but it is easy if you use the fact $d(f, g) = 0$ if and only if $f = g$ everywhere.

■

Now we have proven the proposition, we can prove the theorem.
We will isometrically embed $(X, d) \to (\mathscr{B}(X, \mathbb{R}), \text{sup metric})$. Our first guess is to map $x \mapsto f_x$, where $f_x(\xi) = d(x, \xi)$. The problem is that this is not necessarily a bounded function.

Fix a base point $x^\star \in X$. Then we map $x \mapsto f_x$, where $f_x(\xi) = d(x,\xi) - d(\xi, x^\star)$. This is a bounded function (by triangle inequality, this is bounded above in magnitude by $d(x, x^\star)$).
We will now show that $x \mapsto f_x$ is isometric. Let $x, y \in X$. What is $d(f_x, f_y)$ under the sup metric? By definition, it is given by

$$\sup_{\xi \in X} |f_x(\xi) - f_y(\xi)| = \sup_{\xi \in X} |d(x,\xi) - d(y,\xi)|$$

This last term is bounded above by $d(x,y)$. This supremum can be achieved by letting $\xi = x$ or $\xi = y$. So, $d(f_x, f_y) = d(x,y)$. So $x \mapsto f_x$ is an isometry, so we are done.

■

Lecture 17

Theorem 0.26. *(Banach Fixed Point Theorem)*
Let (X,d) be a complete metric space, and let $f : X \to X$ be complete, with $d(f(x), f(y)) \leq kd(x,y)$ for all x, y, where $0 \leq k < 1$ (such an f is called a k-contraction). Then f has a unique fixed point, i.e. a point $p \in X$ such that $f(p) = p$.

Proof. Let $x_0 \in X$, and let $x_n = f^{(n)}(x_0)$.

Claim. *$\{x_n\}$ is Cauchy*

Proof. Indeed, for $n \leq m$,

$$\begin{aligned} d(x_n, x_m) &= d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq kd(x_{n-1}, x_n) + \cdots \\ &\leq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + \cdots + k^{m-1} d(x_0, x) \\ &= k^n d(x_0, x_1)(1 + k + \cdots + k^{m-1-n}) < \frac{k^n}{1-k} d(x_0, x_1) \end{aligned}$$

$k < 1$, so as $n \to \infty$, this gets arbitrarily small, so $\{x_n\}$ is Cauchy, proving the claim.
Because (X,d) is complete, $x_n \to x$. Because of $d(f(x), f(y)) \leq kd(x,y)$ for all x, y, f is uniformly continuous. So, $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x)$. So, $f(x) = x$.
The uniqueness comes from the fact that if x, y are both fixed points, then the distance between them is multiplied by k. This can only happen if $k = 0$, or $d(x,y) = 0$, implying $x = y$.

■

Corollary 0.27. *If $f : X \to X$ is a continuous function such that $f^{(t)}$ is a k-contraction for some t, then f has a unique fixed point.*

Proof. By Banach, $f^{(t)}$ has a fixed point $x^\star$. But then $f^{(t)} \circ f(x^\star) = f^{(t+1)}(x^\star) = f \circ f^{(t)}(x^\star) = f(x^\star)$. So $f(x^\star)$ is a fixed point of $f^{(t)}$. But the above theorem gives us that the fixed point of $f^{(t)}$ is unique, so $x^\star = f(x^\star)$, so $x^\star$ is a fixed point of f.

Theorem 0.28. *(Cantor)*
Let (X, d) be a metric space. Then X is complete if for every sequence of nonempty closed sets $V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$, such that $\operatorname{diam}(V_j) \to 0$, there exists a point $x^\star \in \cap_{k\geq 1} V_k$.

Remark. The point $x^\star$ (if it exists) is unique, which is immediate from the condition that the diameter of V_j goes to 0.

Proof. <u><=</u>
Suppose (X, d) is not complete. Let (x_n) be a Cauchy sequence which does not converge. Because it does not converge, it has no limit points, and hence the image of the sequence is closed. So, $V_k \overset{\text{def}}{=} \{x_k, x_{k+1}, \dots\}$ is closed for any k.
We can see that $V_k \supseteq V_{k+1}$ for all k easily, and the diameter goes to zero because it is Cauchy.
However, the intersection of all these sets is empty. So, X does not satisfy the given condition.
<u>=></u>
Suppose (X, d) is complete. Pick $x_k \in V_k$. Then (x_k) is a Cauchy sequence, because the diameter of V_j goes to zero. X is complete, so x_k converges to some x.

Claim. $x^\star \in \cap_{k\geq 1} V_k$.

Proof. Each V_k is closed, so, because (x_k) eventually lies completely in any V_j, we must conclude that each V_k contains $x^\star$.
This completes the proof.

■

Definition 0.30. Let (X, d) be a metric space (although this definition doesn't require the metric). A subset $H \subseteq X$ is called <u>nowhere dense</u> if $\operatorname{int}(\operatorname{cl}(H)) = \varnothing$.

Theorem 0.29. *(Baire)*
Let (X, d) be a complete metric space, and suppose $\{H_n\}_{n\geq 1}$ is a countable collection of nowhere dense sets. Then $X \setminus (\cup_{k\geq 1} V_k)$ is dense.
We will prove this next time.

Lecture 18

Lemma 11. *Let (X, d) be a metric space. Then a subset H is nowhere dense if and only if for all nonempty open $U \subset X$, U contains a ball disjoint from H.*

Proof. <u>$<=$</u>

We want to show that $\text{int}(\text{cl}(H)) = \varnothing$.
Let $y \in U$, $\delta > 0$. By hypothesis, there is some $z \in B_\delta(y)$, and $\varepsilon > 0$, such that $B_\varepsilon(z) \subset B_\delta(y)$, and $B_\varepsilon(z) \cap H = \varnothing$. Thus, $z \notin \text{cl}(H)$. But z was chosen to be an interior point of $B_\varepsilon(y)$, so the interior of $\text{cl}(H)$ does not contain any open set.

<u>$=>$</u>

Pick $x \in U$ and $\varepsilon > 0$ so that $B_\varepsilon(x) \subset U$. By hypothesis, $\text{int}(\text{cl}(H)) = \varnothing$. So $B_\varepsilon(x)$ is not contained in $\text{cl}(H)$, and so $z \in \underbrace{(X \setminus \text{cl}(H))}_{\text{open}} \cap \overbrace{B_\varepsilon(x)}^{\text{open}}$. So this set is open, and so z admits a ball which is entirely contained in that set.

■

We are now ready to prove Baire's Theorem.

Proof. of Baire's Theorem
We will make use of Cantor's Theorem.
Pick a nonempty open $U \subseteq X$. By the lemma, there exists a ball $B_{r_1}(x_1) \subseteq U$, such that $B_{r_1}(x) \cap H_1 = \varnothing$.
Let $V_1 \overset{\text{def}}{=} \{x \mid d(x, x_1) \leq \frac{r_1}{2}\}$. This is a closed subset of $B_{r_1}(x)$. Now, $B_{\frac{r_1}{3}}(x_1)$ is open. So, by the lemma, there exists a $B_{r_2}(x_2) \subset B_{\frac{r_1}{3}}(x_1)$ which is disjoint from H_2. Let $V_2 = \{x \mid d(x, x_2) \leq \frac{r_2}{2^2}\}$.
Continuing in this manner, we have a decreasing collection $V_1 \supset V_2 \supset \cdots$, with $\text{diam}\, V_i \to 0$ as $n \to \infty$, so their intersection contains a point. But by construction $V_k \cap H_k = \varnothing$ for all k, so this point does not intersect any of the H_k. This means the point is in $X \setminus \cup_{k\geq 1} H_k$. U was arbitrary, so we have shown that any open set intersects $X \setminus \cup_{k\geq 1} H_k$, and so this set is dense.

■

Lecture 19

<u>Fundamental Group</u>

Suppose X is a topological space. We want a way to associate with X some algebraic object, $G(X)$. We want to do this so that if $f : X \to Y$ is continuous, it "descends" to some homomorphism $f_\star : G(X) \to G(Y)$. Further, in analogy with the fact that the composition of continuous maps is continuous, we want the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

to become

$$\begin{tikzcd} G(X) \arrow[r, "f_\star"] \arrow[dr, "(g\circ f)_\star"'] & G(Y) \arrow[d, "g_\star"] \\ & G(Z) \end{tikzcd}$$

We also want the identity homeomorphism on X to be sent to the identity isomorphism on $G(X)$.

Theorem 0.30. *Suppose we have some way of doing the above. Then if X is homeomorphic to Y, then $G(X)$ is isomorphic to $G(Y)$.*

Proof. Suppose $f : X \to Y$, $g : Y \to X$ are inverse homeomorphisms. Then the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\mathrm{Id}"'] & Y \arrow[d, "g"] \\ & X \end{tikzcd}$$

becomes

$$\begin{tikzcd} G(X) \arrow[r, "f_\star"] \arrow[dr, "\mathrm{Id}"'] & G(Y) \arrow[d, "g_\star"] \\ & G(X) \end{tikzcd}$$

But this means $g_\star f_\star = \mathrm{Id}_{G(X)}$. A similar argument gives $f_\star g_\star = \mathrm{Id}_{G(Y)}$. ∎

Why might we want to do this? The above theorem shows that we can use such a group as a way of distinguishing spaces. If we can show $G(X) \not\cong G(Y)$, then we may assume $X \not\cong Y$.

We will begin this construction by defining a weaker equivalence relation than homeomorphism, which will preserve the fundamental group (which we will later construct).

For the rest of time, X, Y will be topological spaces, and f, g continuous.

Definition 0.31. Consider $f, g : X \to Y$. We say that $\underline{f \text{ is homotopic to } g}$, which we write as $f \simeq g$, if there exists a continuous $F : X \times \underbrace{[0,1]}_{\text{“}I\text{”}} \to Y$, such that for all $x \in X$

$$F(x,t) = \begin{cases} f(x) & t = 0 \\ g(x) & t = 1 \end{cases}$$

We sometimes write $F(\cdot, t)$ as f_t or F_t, depending on convience.

Lemma 12. *Homotopy, as defined above, is an equivalence relation between elements of* $\mathrm{Hom}(X, Y)$.

Proof. It is trivial to verify that $f \simeq f$.
Next, suppose $f \simeq g$. If $F : X \times [0,1] \to Y$ is a homotopy between f, g, then $G : X \times [0,1] \to Y$ given by $G(x,t) = F(x, 1-t)$ is a homotopy which shows $g \simeq f$.
Finally, suppose $f \simeq g$ and $g \simeq h$. Then there are homotopies F_t, G_t. We can define H_t by

$$H_t(x) = \begin{cases} F_{2t}(x) & t \in [0, \frac{1}{2}] \\ G_{2t-1}(x) & t \in [\frac{1}{2}, 1] \end{cases}$$

We have to check this is continuous at $t = \frac{1}{2}$. Indeed, $F_{2\frac{1}{2}}(x) = G_{2\frac{1}{2}-1}(x)$, so H is continuous.

■

Example 0.23. **1.** $f, g : X \to Y \subseteq \mathbb{R}^n$ with Y convex, then $F(x,t) = (1-t)f(x) + tg(x)$ is a homotopy between f, g, so $f \simeq g$. We are using the convexity hypothesis when we assert that $(1-t)f(x) + tg(x) \in Y$.

2. $f, g : X \to S^{n-1} \subset \mathbb{R}^n$. If $f(x) \neq -g(x)$ for all $x, \in x$, then $f \simeq g$. How can we see this? We will prove it in the special case of $S^2 \subset \mathbb{R}^3$. Consider $f(x), g(x) \in S^2$. There is a line going between these two points, and this line misses the origin. We can normalize every point on this line, and we get a path between $f(x)$ and $g(x)$. Explicitly,

$$F(x,t) = \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}$$

Note this is a sufficient but not necessary condition.