Lecture 1

Why measure theory?

We want to answer questions like the following: what is the "total length" of an arbitrary $E \subseteq \mathbb{R}$? What about the "total area" of an arbitrary $E \subseteq \mathbb{R}^2$?
In other words, can we define a function $\mu : 2^{\mathbb{R}^d} \to [0, +\infty)$ so that $\mu(E)$ is sufficiently "nice?"
What properties would we like a function μ (called a measure) to have? Let's stick to $\mathbb{R}$ for now.

1. For $E = [a, b]$ (or (a, b)), we'd like $\mu(E) = b - a$.

2. For a sequence of disjoint intervals $I_i \subseteq \mathbb{R}$,

$$\mu\left(\bigcup_{i=1}^{n} I_i\right) = \sum_{i=1}^{n} \mu(I_i)$$

What about $\mathbb{Q} \cap [0, 1]$? What about the area under a curve?
What if E is any arbitrary set????

Pre measure theory

In the mid 1800s, Riemann first defined the Riemann integral in terms of upper and lower sums.
Fortunately, it's good enough for most "ordinary" functions a student might encounter when doing calculus for the first time.
Unfortunately, it's not good for taking limits.
For example, given $f_1, f_2, f_3, \cdots : [a, b] \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) =: f(x)$ exists for all x, when can we conclude that

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{n\to\infty} f_n(x)]\, dx = \int_a^b f(x)\, dx$$

We learn in undergraduate real analysis that we may only conclude the above if the f_n converge uniformly to f.

Measure theory

Measure theory allows us to define a much more powerful theory of integration, giving us

- More integrable functions
- An integral which behaves better with limits
- An integral ideally suited for probability theory.

Our first goal will be to define a function $\mu : 2^{\mathbb{R}} \to [0, \infty)$ satisfying the following:

1. If $E_1, E_2, \ldots$ is a countable sequence of disjoint sets, then

$$\mu\left(\bigcup E_i\right) = \sum \mu(E_i)$$

If μ satisfies this, we say it is "countably additive."

2. $\mu([a, b]) = b - a$ for all such intervals.

3. μ is translation invariant, i.e. for any $t \in \mathbb{R}$,

$$\mu(E + t) = \mu(E)$$

Where "$E + t$" $\stackrel{\text{def}}{=} \{x + t \mid x \in E\}$

Theorem 0.1. *(Vitali) There is no such μ.*

Proof. Suppose that such a μ exists.

Claim. *If $A \subseteq B$, then $\mu(A) \leq \mu(B)$.*

Proof. Note $B = A \coprod (B \backslash A)$, so

$$\mu(B) = \mu(A) + \mu(B \backslash A)$$

And because μ is always nonnegative, we may conclude that $\mu(B) \geq \mu(A)$. ∎

Now, define an equivalence relation on $\mathbb{R}$ as follows:

$$x \sim y <=> x - y \in \mathbb{Q}$$
$$[x] \stackrel{\text{def}}{=} \{y \in \mathbb{R} \mid x \sim y\}$$

Claim. *Every equivalence class contains a point in $[0, 1]$.*

Proof. Homework exercise. ∎

Now, for each equivalence class, choose an element in $[0, 1]$ belonging to that class. For this step, we are using choice and, it turns out, there is no way not to in this proof.

Call the resulting set A. So $A \subseteq [0,1]$, and for any x, $[x] \cap A$ is a singleton.
Let $B = \bigcup_{q \in \mathbb{Q} \cap [-1,1]} A + q$
Note that this is a disjoint union: indeed, if $A + q$ intersects nontrivially with $A + q'$ for $q \neq q'$, then there are x, x' in A such that $x = x' + q$, and so $x \sim x'$, which by construction is impossible.

Claim. $[0,1] \subseteq B \subseteq [-1,2]$.

Proof. First, if $x \in [0,1]$, then $x = a + q$ for some $a \in A \subseteq [0,1], q = x - a \in [-1,1]$. Thus, $x \in B$.
Next, if $b \in B$, then $b = a + q$, for $q \in A = [0,1]$ and $q \in [-1,1]$, so $b \in [-1,2]$.
So we must conclude by the lemma that

$$1 = \mu([1,0]) \leq \mu(B) \leq \mu([-1,2]) = 3$$

But by the properties of μ, we also have

$$\mu(B) = \sum_{q \in \mathbb{Q} \cap [0,1]} \mu(A + q) = \sum_{q \in \mathbb{Q} \cap [0,1]} \mu(A)$$

The sum on the right hand side is either 0 or ∞. But we just showed that it is between 1 and 3, a contradiction. Therefore, $\mu(A)$ cannot be defined.

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So, if this is impossible, which criterion should we weaken to make it possible?
If we weaken the first to get finite additivity, we run into problems for $d \geq 3$, for example the Banach-Tarski paradox.
If we weaken the other two, then μ is no longer compactible with the usual notion of "length."

Two good choices

- Given a measure on a family of sets, it extends to an outer measure on all sets.
- Similarly, given an outer measure, you can single out "nice sets" on which it is a measure.

What kind of family of subsets should we restrict to?
Let X be a set.

Definition 0.1. $\mathcal{A}$ is an <u>algebra of subsets of X</u> if $\mathcal{A} \neq \varnothing$, and

- $E_1, \ldots, E_n \in \mathcal{U} \implies \bigcup_{i=1}^n E_i \in \mathcal{U}$. In other words, it is "closed under finite unions."
- $E \in \mathcal{U} \implies X \backslash E \in \mathcal{U}$. In other words it is "closed under compliments."

- $\varnothing, X \in \mathcal{A}$.

Lemma 1. *If $\mathcal{A}$ is an algebra of subsets, then $\mathcal{A}$ is closed under finite intersections.*

Proof. Homework 2

Example 0.1. (i) $\mathcal{A} = 2^X$

(ii) $\mathcal{A} = \{\varnothing, X\}$

(iii) $\mathcal{A}$ = all finite OR cofinite subsets of X (cofinite means the complement is finite).

Definition 0.2. A σ-algebra $\mathcal{A}$ is an algebra that is closed under countable unions.

Remark. σ-algebras are closed under countable intersections.

Example 0.2. Above, (i) and (ii) are σ-algebras, but (iii) is not.

Proposition 1. *Given any family $\mathcal{E}$ of subsets of X, there is a smallest σ-algebra $\mu(\mathcal{E})$ containing $\mathcal{E}$, meaning that if $\mathcal{F}$ is a σ-algebra containing $\mathcal{E}$, then $\mu(\mathcal{E}) \subseteq \mathcal{F}$.*

Lecture 2

Definition 0.3. Given a nonempty set X and $\mathcal{M}$ a σ-algebra of subsets of X, we call $(X, \mathcal{M})$ a measurable space.
Recall:

Proposition 2. *Given any family $\mathcal{E}$ of subsets of X, there is a smallest σ-algebra $\mathcal{M}(\mathcal{E})$ containing $\mathcal{E}$, meaning that if $\mathcal{F}$ is a σ-algebra containing $\mathcal{E}$, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{F}$.*

Proof. We begin with a claim.

Claim. Given any nonempty collection $\mathcal{C}$ of σ-algebras on X, then

$$\cap \mathcal{C} \stackrel{\text{def}}{=} \{E \subseteq X \mid E \in \mathcal{A} \forall \mathcal{A} \in \mathcal{C}\}$$

is a σ-algebra.

Proof. Homework 2

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Let $\mathcal{C} = \{\mathcal{A} \mid \mathcal{A}$ is a σ-algebra on X and $\mathcal{E} \subseteq \mathcal{A}\}$. $\mathcal{C}$ is nonempty, because $2^X \in \mathcal{C}$. By the claim, $\cap \mathcal{C}$ is a σ-algebra. By the definition of $\mathcal{C}$, $\mathcal{E} \subseteq \mathcal{C}$ and for any σ-algebra $\mathcal{A}$ such that $\mathcal{E} \subseteq \mathcal{A}$, $\cap \mathcal{C} \subseteq \mathcal{A}$.
Thus $\mathcal{M}(\mathcal{E}) = \cap \mathcal{C}$ is the smallest σ-algebra containing $\mathcal{E}$.

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Remark. Intuitively, $\mathcal{M}(\mathcal{E})$ is a σ-algebra containing the sets in $\mathcal{E}$ by "going from the outside in," starting with σ-algebras that are "too big" and taking intersections.
Recall: a topology τ is a collection of subsets of a set X (called open sets), which is closed under arbitrary unions and finite intersections, and $X, \varnothing \in \tau$
Let (X, τ) be a topoological space.

Definition 0.4. The Borel σ-algebra of X, denoted $\mathcal{B}X$, is the σ-algebra generated by the open subsets of X. Its members are known as Borel sets.
What do the Borel sets look like? Let's go from the "inside out."
Let $\mathcal{F}$ =open sets in X, $\mathcal{F}^\sigma$ all countable unions of sets in $\mathcal{F}$, $\mathcal{F}^\delta$ all countable intersections, and $\overline{\mathcal{F}}$ complements of sets in $\mathcal{F}$.
To build Borel sets:

$$\mathcal{F} \to \mathcal{F}^\delta \cup \overline{\mathcal{F}^\delta} \to \cdots \to \mathcal{B}X$$

To learn more, look up the "Borel hierarchy."

Proposition 3. *The Borel σ-algebra on $\mathbb{R}$, which we denote $\mathcal{B}_\mathbb{R}$, is generated by each of the following.*

(i) Open intervals $\mathcal{E}_1 = \{(a, b) \mid a < b, a, b \in \mathbb{R}\}$

(ii) Closed intervals $\mathcal{E}_2 = \{[a, b] \mid a \leq b, a, b, \in \mathbb{R}\}$

(iii) Half-open intervals $\mathcal{E}_3 = \{[a, b) \mid a < b, a, b, \in \mathbb{R}\}$

(iv) Open rays $\mathcal{E}_4 = \{(a, \infty) \mid a \in \mathbb{R}\}$

(v) Closed rays $\mathcal{E}_5 = \{[a, \infty) \mid a \in \mathbb{R}\}$

That is, $\mathcal{M}(\mathcal{E}_i) = \mathcal{B}_\mathbb{R}$ *for any* $i \in \{1, \ldots, 5\}$.

Proof. Homework 2 ∎

Let

- $\{(X_1, \mathcal{M}_i)\}_{i=1}^\infty$ be a collection of measurable spaces.

$$X \stackrel{\text{def}}{=} \prod_{i=1}^\infty X_i$$

- π_i be the projection $X \to X_i$

Example 0.3. If $(X_i, \mathcal{M}_i) = (\mathbb{R}, \mathcal{B}_\mathbb{R})$, for $i \in \{1, \ldots, n\}$. Then $X = \mathbb{R}^n$.

Definition 0.5. The product σ-algebra

$$\bigotimes_{i\in\mathbb{N}} \mathcal{M}_i \stackrel{\text{def}}{=} \mathcal{M}\left(\{\prod_{i\in\mathbb{N}} E_i \mid E_i \in \mathcal{M}_i\}\right)$$

Our goal is to show that $\mathcal{B}_{\mathbb{R}^n} = \otimes_{i=1}^n \mathcal{B}_{\mathbb{R}}$.

Proposition 4. *Given $\mathcal{E}_i \subseteq 2^{X_i}$ such that $X_i \in \mathcal{E}_i$, let $\mathcal{M}_i = \mathcal{M}(\mathcal{E}_i)$. Then*

$$\bigotimes_{i\in\mathbb{N}} \mathcal{M}_i = \mathcal{M}\left(\{\prod_{i\in\mathbb{N}} E_i \mid E_i \in \mathcal{E}_i\}\right)$$

Note: If $\mathcal{E} \subseteq \mathcal{F}$, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$. If $\mathcal{E} \subseteq \mathcal{M}(\mathcal{F})$, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}\mathcal{F}$.
Recall: Given a function $f : X \to Y$ between arbitrary nonempty sets, then

(i) $f^{-1}\left(\cup_{i\in\mathbb{N}} E_i\right) = \cup_{i\in\mathbb{N}} f^{-1}(E_i)$ for all $E_i \subseteq X$.

(ii) $f^{-1}(E^c) = (f^{-1}(C))^c$, for all $E \subseteq X$.

Proof. By the first statement of the preceding note,

$$\mathcal{M}\left(\{\prod_{i\in\mathbb{N}} E_i \mid E_i \in \mathcal{E}_i\}\right) \subseteq \mathcal{M}\left(\{\prod_{i\in\mathbb{N}} E_i \mid E_i \in \mathcal{M}_i\}\right) = \bigotimes_{i\in\mathbb{N}} \mathcal{M}_i$$

For equality, it suffices to show

$$\mathcal{M}\left(\{\prod_{i\in\mathbb{N}} E_i \mid E_i \in \mathcal{M}_i\}\right) \subseteq \mathcal{M}\left(\{\prod_{i\in\mathbb{N}} E_i \mid E_i \in \mathcal{E}_i\}\right)$$

Let $\mathcal{M}\left(\{\prod_{i\in\mathbb{N}} E_i \mid E_i \in \mathcal{E}_i\}\right) = \mathcal{A}$.
Note that

$$\begin{aligned}\prod_{i\in\mathbb{N}} E_i &= \{x \in X \mid \pi_i(x) \in E_i \forall i\} \\ &= \bigcap_{i\in\mathbb{N}} \{x \in X : \pi_i(x) \in E_i\} \\ &= \bigcap_{i\in\mathbb{N}} \pi_i^{-1}(E_i)\end{aligned}$$

Because $\mathcal{A}$ is a σ-algebra, it suffices to show that $\pi_i^{-1}(E_i) \in \mathcal{A}$ for all $E_i \in \mathcal{M}_i$.

Claim. *Let* $\mathcal{F}_i \stackrel{\text{def}}{=} \{E_i \subseteq X_i \mid \pi_i^{-1}(E_i) \in \mathcal{A}\}$.
This is a σ*-algebra.*

Proof.

$$\bigcup_{i=1}^{\infty} \pi_i^{-1}(E_i) = \pi_i^{-1}(\bigcup_{i=1}^{\infty} E_i)$$

So $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{F}_i$.
Similarly,

$$\pi_i^{-1}(E^c) = (\pi_i^{-1}(E))^c$$

so $E \in \mathcal{F}_i$ implies $E^c \in \mathcal{F}_i$.

Lecture 3

Because $X_i \in \mathcal{E}_i$and $\pi_i^{-1}(E_i) = X_1 \times X_2 \times \cdots \times E_i \times \cdots$, we know $\pi_i^{-1}(E_i) \in \mathcal{A}$ for all $E_i \in \mathcal{E}_i$.
In other words, $\mathcal{E}_i \subseteq \mathcal{F}_i$. Since $\mathcal{F}_i$ is a σ-algebra, $\mathcal{M}(\mathcal{E}_i) = \mathcal{M}_i \subseteq \mathcal{F}_i$.
Thus, $\pi_i^{-1}(\mathcal{E}_i) \in \mathcal{A}$ for all $E_i \in \mathcal{M}_i$ and all i.

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In order to characterize the Borel product σ-algebra, it will be convenient to assume our undderlying spaces have a metric that induces the topology.
Let $(X_i, d_i), i = 1, \ldots, n$ be metric spaces. Let

$$X = \prod_{i=1}^{n} X_i$$

Endow the product space with the metric

$$d_{\max}((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \max_{i=1,\ldots,n} (d_1(x_1, y_1), d_2(x_2, y_2), \ldots, d_n(x_n, y_n))$$

Theorem 0.2. *Given metric spaces* $X_1, X_2, \ldots, X_d$ *and their product*

$$X = \prod_{i=1}^{d} X_i$$

endowed with the metric $d_{\max}$*, then* $\otimes_{i=1}^{n} \mathcal{B}_{X_i} \subseteq \mathcal{B}_X$
If the X_i *are all seperable, then* $\otimes_{i=1}^{n} \mathcal{B}_{X_i} = \mathcal{B}_X$

Remark. Since the definition of $\mathcal{B}_X$ only depends on the topology of X, then this statement holds even if $d_{\max}$ is replaced by an equivalent metric, where "equivalent" means "generates the same topology."

Remark. $d_{\max}$ is convenient because:

$$\begin{aligned} B_r(x_1, \ldots, x_m) &= \{(y_1, \ldots, y_n) | d_{\max}(\vec{x}, \vec{y}) < r\} \\ &= \{(y_2, \ldots, y_n) \mid d_i(x_i, y_i) < r \forall i\} \\ &= \prod_{i=1}^{n} B_r(X_i) \end{aligned}$$

Recall:
Fact 1: If $X_1, \ldots, X_m$ are seperable, so is $\prod_{i=1}^{m} X_i$.
Fact 2: In a seperable metric space, every open set can be written as a countable union of balls, $\mathcal{U} = \cup_{i=1}^{\infty} B_i$
Fact 3: $\{\prod_{i=1}^{n} E_i \mid E_i \subseteq X_i, \text{open}\} \subseteq \{\text{open subsets of } X\}$

Proof. : By the previous proposition, $\otimes_{i=1}^{n} \mathcal{B}_{X_i}$ is generated by

$$\{\prod_{i=1}^{n} E_i \mid E_i \subseteq X_{\text{open}}\} \subseteq \{\text{open subsets of } X\}$$

Thus $\otimes_{i=1}^{n} \mathcal{B}_{X_i} \subseteq \mathcal{B}_X$.
Now, suppose $X_1, \ldots, X_n$ are seperable. By facts 1 and 2, every open subset of X can be written as a countable union of balls.
To prove $\otimes_{i=1}^{n} \mathcal{B}_{X_i} = \mathcal{B}_X$, it suffices to show that

$$\{\text{open subsets of } X\} \subseteq \otimes_{i=1}^{n} \mathcal{B}_{X_i}$$

The left hand side is equal to $\{\cup_{j=1}^{\infty} B_j \mid B_j \subseteq X_{\text{open}} \text{ball}\}$, and the right hand side is equal to $\mathcal{M}\left(\{\prod_{i=1}^{n} E_i \mid E_i \text{ open}\}\right)$
This will hold, as long as we can show $B_j \in \mathcal{M}\left(\{\prod_{i=1}^{n} E_i \mid E_i \text{ open}\}\right)$
Since X is endowed with $d_{\max}$, we know that any open ball in X can be expressed as $B = \prod_{i=1}^{n} B_i$, where $B_i \subseteq X_i$ is a ball. This gives the result. ∎

Now, it is finally time to talk about measures.

Measures

Call $(X, \mathcal{M})$ a measurable space when X is a set and $\mathcal{M}$ is a σ-algebra on X.

Definition 0.6. A measure on a measurable space $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, +\infty]$ such that

(i) $\mu(\varnothing) = 0$

(ii) If $\{E_i\}$ is a countable disjoint collection of sets, then

$$\mu(\bigcup E_i) = \sum \mu(E_i)$$

This is called "countable (disjoint) additivity"

Example 0.4. (Dirac mass/Dirac measure)
Let $(X, \mathcal{M}) = (X, 2^X)$.
Fix $x_0 \in X$ and define

$$\mu(A) = \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{otherwise} \end{cases}$$

Example 0.5. (Counting measure)
Let $(X, \mathcal{M}) = (X, 2^X)$. Define

$$\mu(A) = |A| = \text{ the number of elements in } A$$

Given a measurable space $(X, \mathcal{M})$ and a measure μ, we call $(X, \mathcal{M}, \mu)$ a measure space and $E \in \mathcal{M}$ a measurable set

Theorem 0.3. *For any measure space $(X, \mathcal{M}, \mu)$ and measurable sets $A, B, A_1, A_2, \dots \in \mathcal{M}$,*

(i) $A \subseteq B \implies \mu(A) \leq \mu(B)$. This is called "monotonicity"

(ii) $\mu(\bigcup_i A_i) \leq \sum_i \mu(A_i)$. This is called "(countable) sub additivity)."

(iii) If $A_i \subseteq A_{i+1}$, then $\mu(\bigcup_i A_i) = \lim_{n\to\infty} \mu(A_i)$. This is called "continuity from below."

(iv) If $A_{i+1} \subseteq A_i$ for all i, and $\mu(A_i) < \infty$, then $\mu(\bigcap A_i) = \lim_{i\to\infty} \mu(A_i)$. This is called "continuity from above."

Remark. For (iv), why do we need the additional hypothesis $\mu(A_1) < \infty$?.
Consider the counting measure on $(\mathbb{N}, 2^{\mathbb{N}})$, and $A_i = \{n \in \mathbb{N} \mid n \geq i\}$, which satisfies $A_{i+1} \subseteq A_i$, but it fails $\mu(A_1) < \infty$:

$$0 = \mu(\varnothing) = \mu(\cap_{i=1}^{\infty} A_i) \neq \lim_{i\to\infty} \mu(A_i) = +\infty$$

Proof. (i) Since $A \subseteq B$, $B = A \cup (B \backslash A)$, so $\mu(B) = \mu(A) + \mu(B \backslash A)$ by countable additivity. $\mu(B \backslash A) \geq 0$, so (i) follows.

(ii) Define $B_1 = A_1$, $B_2 = A_2 \backslash A_1$, $B_3 = A_3 \backslash (A_1 \cup A_2), \dots, B_n = A_n \backslash (\cup_{i=1}^{n-1} A_i)$. Then $\cup_i A_i = \cup_i B_i$, so by countable disjoint additivity,

$$\mu(\cup_i A_i) = \mu(\cup_i B_i) = \sum_i \mu(B_i) \leq \sum_i \mu(A_i)$$

(iii) Define $B_1 = A_1$, and $B_i = A_i \backslash A_{i-1}$. Then $A_n = \cup_{i=1}^n B_i$, so $\cup_{i=1}^\infty A_i = \cup_{i=1}^\infty B_i$. Thus $\mu(A_n) = \mu(\cup_{i=1}^n B_i) = \sum_{i=1}^n \mu(B_i)$. Consequently,

$$\mu(\cup_{n=1}^\infty A_n) = \mu(\cup_{i=1}^\infty B_i) = \sum_{i=1}^\infty \mu(B_i) = \lim_{n\to\infty} \sum_{i=1}^n \mu(B_i) = \lim_{n\to\infty} \mu(A_n)$$

(iv) Next time!

Lecture 4

Recall
Let

- (X_i, d_i), $i = 1, \dots, n$ metric spaces
- $\{(X_i, \mathscr{M}_i)\}_{i=1}^n$ a collection of measurable spaces.
- $\mathcal{X} = \prod_{i=1}^n X_i$ product space.
- $d_{\max}((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max\{d_i(x_i, y_i)\}$.

Definition 0.7.

$$\bigotimes_{\alpha \in A} \mathscr{M}_\alpha = \mathcal{M}\left(\{\prod_{\alpha \in A} E_\alpha \mid E_\alpha \in \mathscr{M}_\alpha\}\right)$$

We have the following theorem

Theorem 0.4.

$$\mathcal{B}_{\mathcal{X}} = \otimes_{i=1}^n \mathcal{B}_{X_i}$$

That is, the Borel σ-algebra generated by the products of the X_i is equal to the products of the Borel σ-algebras generated by the X_i.
Now, back to measure spaces.

Remark. The definition of Borel sets only depends on the notion of open sets, do $d_{\max}$ could be replaced with any equivalent metric.
We will now prove that a measure satisfies continuity from above.

Proof. Let $\{A_i\}_{i\in\mathbb{N}}$ be a descending sequence of measurable sets.
Define $B_i = A_1 \backslash A_i$
We have $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$
Note $\mu(A_1) = \mu(B_i \cup A_i) = \mu(B_i) + \mu(A_i)$ by disjoint additivity.

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty}(A_1\backslash A_i) = \bigcup_{i=1}^{\infty} A_1 \cap A_i^c = A_1 \backslash \left(\bigcap_{i=1}^{\infty} A_i\right)$$

So

$$\begin{aligned}
\mu(A_1) &= \mu\left(\left(A_1 \backslash \left(\bigcap_{i=1}^{\infty} A_i\right)\right) \cup \bigcap_{i=1}^{\infty} A_i\right) \\
&= \mu\left(A_1 \backslash \left(\bigcap_{i=1}^{\infty} A_i\right)\right) + \mu\left(\bigcap_{i=1}^{\infty} A_i\right) \\
&= \mu\left(\bigcap_{i=1}^{\infty} B_i\right) + \mu\left(\bigcap_{i=1}^{\infty} A_i\right) \\
&= \lim_{i\to\infty} \mu(B_i) + \mu\left(\bigcap_{i=1}^{\infty} A_i\right)
\end{aligned}$$

Since $\mu(A_1) < \infty$, by monotonicity, $\mu(B_i), \mu(A_i)$ are also finite, and, recalling from before,

$$\mu(B_i) = \mu(A_1) - \mu(A_i)$$

So,

$$\mu(A_1) = \lim_{i\to\infty}(\mu(A_1) - \mu(A_i)) + \mu(\cap_{i=1}^{\infty} A_i)$$

So $\lim_{i\to\infty} \mu(A_i) = \mu(\cap_{i=1}^{\infty} A_i)$.

Measure Terminology

- μ is a finite measure if $\mu(\mathcal{X}) < +\infty$.
- μ is a σ-finite measure if there exists $\{E_i\}_{i=1}^{\infty} \in \mathcal{M}^{\mathbb{N}}$ such that $\cup_{i=1}^{\infty} E_i = \mathcal{X}$ and $\mu(E_i) < +\infty$. In other words, we can chop $\mathcal{X}$ into countably many measurable pieces of finite size.

- E is a null set of μ if $E \in \mathcal{M}$ and $\mu(E) = 0$.
- We say that a property holds for μ-almost every $x \in \mathcal{X}$ if the set of points where it doesn't hold is a null set.

Recall our ultimate goal: a measure μ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ where $\mu((a,b)) = b - a$, and it is translation invariant.

Outer Measures

Definition 0.8. An outer measure on a set $\mathcal{X}$ is a function $\mu^* : 2^{\mathcal{X}} \to [0, +\infty]$ satisfying

(i) $\mu^*(\varnothing) = 0$

(ii) $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$

(iii) $\mu^*(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$

Remark. *(ii)* + *(iii)* is equivalent to the statement that if $E \subseteq \cup_{i=1}^{\infty} A_i$, then $\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$.

Example 0.6. Let $\mathcal{X} = \mathbb{R}$. The Lebesgue Outer Measure is defined by

$$\mu^*(A) = \inf\{\sum_{i=1}^{\infty} |b_i - a_i| : A \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i]\}$$

We will prove that μ^* is an outer measure. We will also show $\mu^*((a,b]) = b - a$, and μ^* is translation-invariant.
Is μ^* countably additive? No, by Vitali's theorem.
While we will be able to show that μ^* is an outer measure, it is not a measure on $2^{\mathbb{R}}$.

Definition 0.9. Let $\mathcal{X}$ be a nonempty set, and μ^* an outer measure on $\mathcal{X}$. We say $A \subseteq \mathcal{X}$ is μ^*-measurable if, for all $E \subseteq \mathcal{X}$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Remark. We know that if, in the above expression, "=" is replaced by "≤", it holds for any $E \subseteq \mathcal{X}$ by countable subadditivity

Proposition 5. *If $\mu^*(B) = 0$ for $B \subseteq \mathcal{X}$, then B is μ^*-measurable.*

Proof. Fix an arbitrary $E \subseteq \mathcal{X}$. Then, by monotonicity, $\mu^*(E) \geq \mu^*(E \cap B^c) = \mu^*(B) + \mu^*(E \cap B^c)$, so $\mu^*(E) = E \cap B) + \mu^*(E \cap B^c)$.

Theorem 0.5. *(Caratheodory): Given an outer measure μ^* on $\mathcal{X}$, let*

$$\mathscr{M} \stackrel{\text{def}}{=} \{A \subseteq X : A \text{ is } \mu^* - \text{measurable}\}$$

Then

(i) $\mathscr{M}$ is a σ-algebra

(ii) μ^* is a measure on $\mathscr{M}$.

Question: Is this the "largest" σ-algebra on which μ^* can be defined as a measure? In general, the answer is no - see hw3.

Proof. $\mathscr{M}$ is nonempty, because by the proposition, $\varnothing$ is μ^*-measurable.
Now we want to see that $\mathscr{M}$ is closed under complements. This clearly holds by the definition of μ^*.
We will now show $\mathscr{M}$ is closed under finite unions. It will suffice to show that if $A, B \in \mathscr{M}$, then $A \cup B \in \mathscr{M}$.
Fix an arbitrary $E \subseteq \mathcal{X}$. We have

$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\
&= \mu^*(E \cap A) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\
&\geq \mu^*((E \cap A) \cup (E \cap A^c \cap B^c)) + \mu^*(E \cap A^c \cap B^c) \\
&= \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)
\end{aligned}$$

So $A \cup B$ is μ^*-measurable.
Remark. "$\leq$" always holds by countable subadditivity.
Now, we will show that $\mu^*|_{\mathscr{M}}$ is finitely additive.

Claim. given $\{B_i\}_{i=1}^n \subseteq \mathscr{M}$ disjoint, then for all $A \subseteq \mathcal{X}$,

$$\mu^*(E \cap (\cup_{i=1}^n)) = \sum_{i=1}^n \mu^*(E \cap B_i)$$

Proof. We will proceed by induction. The base case is obvious.
Now, assume the result holds for $n-1$. We will show it holds for n. We have

$$\begin{aligned}
\mu^*(E \cap (\cup_{i=1}^n B_i)) &= \mu^*(E \cap (\cup_{i=1}^n) \cap B_n) + \mu^*(E \cap (\cup_{i=1}^n B_i) \cap B_n^c \\
&= \mu^*(E \cap B_n) + \mu^*(E \cap (\cup_{i=1}^{n-1} B_i)) \\
&= \mu^*(E \cap B_n) + \sum_{i=1}^{n-1} \mu^*(E \cap B_i)
\end{aligned}$$

We will finish next time!

Lecture 5

Now for the exciting conclusion.
Taking $E = \mathcal{X}$ in the above claim, we see $\mu^*|_{\mathscr{M}}$ is finitely additive.

Claim. Given $\{B_i\}_{i=1}^\infty \subseteq \mathscr{M}$ *disjoint, for all* $E \subseteq X$,

$$\mu^*(E) = \sum_{i=1}^\infty (\mu^*(E \cap B_i)) + \mu^*(E \cap (\cup_{i=1}^\infty B_i)^c)$$

Proof. The left hand side is immediately seen to be less than the right hand side due to the countable subadditivity of μ^*, since

$$\begin{aligned} E &= (E \cap (\cup_{i=1}^\infty B_i)) \cup (E \cap (\cup_{i=1}^\infty B_i)^c) \\ &= (\cup_{i=1}^\infty (E \cap B_i)) \cup (E \cap (\cup_{i=1}^\infty B_i)^c) \end{aligned}$$

It remains to show that the left hand side is greater than or equal to the right hand side.
Since $\mathscr{M}$ is closed under finite unions, $\cup_{i=1}^n B_i \in \mathscr{M}$, so by the definition of μ^*-measurable,

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap (\cup_{i=1}^n B_i)) + \mu^*(E \cap (\cup_{i=1}^n B_i)^c) \\ &= \sum_{i=1}^n \mu^*(E \cap B_i) + \mu^*(E \cap (\cup_{i=1}^\infty B_i)^c) \end{aligned}$$

Taking the limit as $n \to \infty$ gives the result.
Now, to show $\mathscr{M}$ is closed under countable unions. Fix $\{C_i\}_{i=1}^\infty \subseteq \mathscr{M}$. We want to show $\cup_{i=1}^\infty C_i \in \mathscr{M}$.
Define $B_1 = C_1$, and in general $B_n = C_n \backslash (\cup_{i=1}^{n-1} C_i)$
Then $B_n \in \mathscr{M}$ for each n, and $\cup_{i=1}^\infty B_i = \cup_{i=1}^\infty C_i$.
Fix $E \subseteq \mathcal{X}$. Then, by a previous claim, we know

$$\begin{aligned} \mu^*(E) &= \sum_{i=1}^n \mu^*(E \cap B_i) + \mu^*(E \cap (\cup_{i=1}^\infty B_i)^c) \\ &\geq \mu^*(E \cap (\cup_{i=1}^\infty B_i)) + \mu^*(E \cap (\cup_{i=1}^\infty B_i)^c) \end{aligned}$$

Since we already have the inequality going in the other direction, we have shown $\cup_{i=1}^\infty B_i = \cup_{i=1}^\infty C_i \in \mathscr{M}$.

Taking $E = \cup_{i=1}^{\infty} B_i$, for B_i disjoint, nonempty, then

$$\begin{aligned}\mu^*(\cup_{i=1}^{\infty} B_i) &= \mu^*(E) \\ &= \sum_{i=1}^{\infty} \mu^*(E \cap B_i) + \mu^*(E \cap (\cup_{i=1}^{\infty} B_i)^c) \\ &= \sum_{i=1}^{\infty} \mu^*(B_i) + \mu^*(\varnothing)\end{aligned}$$

Thus $\mu^*(\cup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} B_i$.

■

Back to Lebesgue outer measure. Let $\mathcal{X} = \mathbb{R}$. Recall we define

$$\mu^*(A) = \inf\{\sum_{i=1}^{\infty} |b_i - a_i| : A \subseteq \cup_{i=1}^{\infty}(a_i, b_i]\}$$

We want to show that

1. μ^* is an outer measure

2. It gives the correct lengths to $(c, d]$,

3. It is translation invariant

4. $\mathscr{B}_{\mathbb{R}}$ is contained in the collection of μ^*-measurable sets.

In fact, we will study a generalization of Lebesgue outer measure that will give rise to Lebesgue-Stieljes measures.
Recall: $F : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is right continuous if for all $x \in \mathbb{R}$,

$$\lim_{y \to x^+} F(y) = F(x)$$

Definition 0.10. Given $F : \mathbb{R} \to \mathbb{R}$ non-decreasing and right-continuous, define

$$\mu_F^*(A) = \inf\{\sum_{i=1}^{\infty}(F(b_i) - F(a_i)) : A \subseteq \cup_{i=1}^{\infty}(a_i, b_i]\}$$

Note: Katy HATES the term "non-decreasing," and will use it interchangeably with the term "increasing." To denote something which is not constant anywhere, she will say "strictly" increasing.
Why do we require F to be nondecreasing?

Spoiler: We will show any finite measure μ on $\mathscr{B}_\mathbb{R}$ satisfies $\mu = \mu_F^*|_{\mathscr{B}_\mathbb{R}}$, for

$$F(x) = \mu((-\infty, x])$$

We call F the Cumulative Distribution Function, or CDF.
Note that if μ is a finite measure on $\mathscr{B}_\mathbb{R}$ and $F(x)$ is it's CDF, then F is

- Nondecreasing: If $x \leq y$, then $(-\infty, x] \subseteq (-\infty, y]$, which implies $F(x) \leq F(y)$.
- Right-continuous: For any decreasing sequence x_n whose limit is x, $\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \mu((-\infty, x_n]) = \mu((-\infty, x]) = F(x)$. The penultimate equality is due to μ being continuous from above, as it is a finite measure.

Theorem 0.6. For any nondecreasing right-continuous F, μ_F^* is an outer measure.

Proof.

- First, $\mu_F^*(\varnothing) = \inf\{\sum_{i=1}^\infty F(b_i) - F(a_i) : \varnothing \subseteq \cup_{i=1}^\infty (a_i, b_i]\} = 0$, as every interval contains $\varnothing$ as a subset so we may set $a_i = b_i \equiv 1$ for all i. Since $\mu_F^* \geq 0$ by definition, $\mu_F^*(\varnothing) = 0$
- Now, we want to show that if $A \subseteq \cup_{i=1}^\infty B_i$, then $\mu_F^*(A) \leq \sum_{i=1}^\infty \mu_F^*(B_i)$.

  If $\mu_F^*(B_i) = \infty$ for some i, we are done. Without loss of generality, suppose $\mu_F^*(B_i) < \infty$ for each i.

  By the definition of inf, for all $\varepsilon > 0$ and each i there exists $\{I_i^\varepsilon\}_{i=1}^\infty$ of intervals depending on ε such that

$$B_j \subseteq \cup_{i=1}^\infty I_i^{j,\varepsilon}$$
$$\mu_F^*(B_j) \leq \sum_{i=1}^\infty |I_i^{j,\varepsilon}| \leq \mu_F^*(B_j) + \frac{\varepsilon}{2j}$$

  Thus

$$\begin{aligned} A &\subseteq \cup_{j=1}^\infty \cup_{i=1}^\infty I_i^{j,\varepsilon} \\ \mu_F^*(A) &\leq \sum_{i,j=1}^\infty |I|_F^{j,\varepsilon} \\ &\leq \sum_{j=1}^\infty \mu_F^*(B_j) + \frac{\varepsilon}{2j} \\ &= \sum_{j=1}^\infty \mu_F^*(B_j) + \varepsilon \end{aligned}$$

Sending $\varepsilon \to 0$ completes the proof.

■

Theorem 0.7. *For all* $a, b \in \mathbb{R}$, $a \leq b$,

$$\mu^*((a,b]) = F(b) - F(a)$$

Proof. $\leq$ follows quickly, since we know $(a,b] \subseteq (a,b] \cup \varnothing \cup \varnothing \cup \cdots$, so the definition of μ_F^* ensures

$$\mu_F^*((a,b]) \leq \sum_{i=1}^{\infty} F(b_i) - F(a_i) = F(b) - F(a)$$

Now we turn to $\geq$. Note that if $a = b$, we already showed that $\mu_F^*((a,b]) = \mu_F^*(\varnothing) = 0 = F(b) - F(a)$, so without loss of generality $a < b$.
It suffices to show that if $(a,b] \subseteq \cup_{i=1}^{\infty}(a_i, b_i]$, then $F(b) - F(a) \leq \sum_{i=1}^{\infty} F(b_i) - F(a_i)$
Since F is right continuous, for all $\varepsilon > 0$ we can find $\delta_i > 0$ such that $F(b_i + \delta_i) < F(b_i) + \frac{\varepsilon}{2^i}$.
Note that

$$[a+\varepsilon, b] \subseteq (a,b] \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i] \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i + \delta_i)$$

Since $[a+\varepsilon, b]$ is compact and $\{(a_i, b_i + \delta_i)\}_{i=1}^{\infty}$ is an open cover, there exists a finite subcover

$$[s + \varepsilon] \subseteq \bigcup_{i=1}^{N}(a_i, b_i + \delta_i)$$

Without loss of generality we may throw away any unnecessary elements of the cover. The "first" element of the cover must overlap with exactly one other element of the cover, the "second" interval. Thus we may assume that

$$b_i + \delta_i \in (a_{i+1}, b_{i+1} + \delta_{i+1}) \forall i = 1, \ldots, N-1$$

Tune in next time for the continuation!

Lecture 6

And now, for the exciting conclusion...

Since F is nondecreasing,

$$\begin{aligned} F(b) - F(a+\varepsilon) &\leq f(b_N + \delta_N) - F(a_1) \\ &= F(b_N + \delta_N) - F(a_N) + \sum_{i=1}^{N-1} (F(a_{i+1}) - F(a_i)) \\ &= F(b_N + \delta_N) - F(a_N) + \sum_{i=1}^{N-1} (F(b_i + \delta_i) - F(a_i)) \\ &= \sum_{i=1}^{N} (F(b_i + \delta_i) - F(a_i)) \\ &\leq \sum_{i=1}^{N} (F(b_i) - F(a_i) + \frac{\varepsilon}{2^i}) \\ &\leq \left| \sum_{i=1}^{\infty} F(b_i) - F(a_i) \right| + \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, and F is right-continuous, sending $\varepsilon \to 0$ gives the result.

■

Definition 0.11. By Carathéodory's theorem, we know μ_F^* is a measure when restricted to $\mathscr{M}_{\mu_F^*}$, the collection of μ_F^*-measurable sets. We will denote this measure by μ_F, and call it the Legesgue-Stieljes measure associated to F.
How does this help our goals?
Is μ_F a Borel measure (that is, a measure when restricted to the Borel σ-algebra)?
Yes

Theorem 0.8. $\mathscr{B}_{\mathbb{R}} \subseteq \mathscr{M}_{\mu_F^*}$

Proof. It suffices to show that, for all $b \in \mathbb{R}$, $(-\infty, b] \in \mathscr{M}_{\mu_F^*}$.
That is, we must show for all $E \subseteq \mathbb{R}$

$$\mu_F^*(E) \geq \mu_F^*(E \cap (-\infty, b]) + \mu_F^*(E \cap (-\infty, b]^c)$$

We already have $\leq$ by countable additivity.
Fix a $\varepsilon > 0$. By definition of μ_F^*, there exists a cover $\{(a_i, b_i]\}_{i=1}^{\infty}$ such that $E \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i]$ and

$$\sum_{i=1}^{\infty} (F(b_i) - F(a_i)) \leq \mu_F^*(E) + \varepsilon$$

Note that

$$(a_i, b_i] \cap (-\infty, b] \subseteq (a_i, b]$$
$$(a_i, b_i] \cap [b, \infty) \subseteq (b, b_i]$$

so

$$E \cap (-\infty, b] \subseteq \bigcup_{i=1}^{\infty} (a_i, b]$$
$$E \cap [b, +\infty) \subseteq \bigcup_{i=1}^{\infty} (b, b_i]$$

$$\begin{aligned}\mu_F^*(E \cap (-\infty, b]) + \mu_F^*(E \cap [b, +\infty)) &\leq \sum_{i=1}^{\infty} (F(b) - F(a_i)) + \sum_{j=1}^{\infty} (F(b_j) - F(b)) \\ &= \sum_{i=1}^{\infty} (F(b_i) - F(a_i)) \\ &\leq \mu_F^*(E) + \varepsilon\end{aligned}$$

Sending $\varepsilon \to 0$ gives us the result. ■

Definition 0.12. When $F(x) = x$, we write $\lambda^* \overset{\text{def}}{=} \mu_F^*$,
and we call it the Lebesgue outer measure. Similarly, we write $\lambda \overset{\text{def}}{=} \mu_F$, and call it the Lebesgue measure. Finally, $\mathscr{M}_{\lambda^*} \overset{\text{def}}{=} \mathscr{M}_{\mu_F^*}$, and we call this collection the Lebesgue measurable sets.
Thus, we know all Borel sets are Lebesgue measurable.
In this way, we have found a Borel measure that gives the "right" length to intervals $(a, b]$.
The last "intuitive" property of λ that we seek to show is translation invariance.

Theorem 0.9. *λ^* is translation invariant on $2^{\mathbb{R}}$, and λ is translation invariant on $\mathscr{M}_{\lambda^*}$.*

Proof. For any $a \in \mathbb{R}$, $A \subseteq \mathbb{R}$, $A \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i]$ is equivalent to
$A + a \subseteq \bigcup_{i=1}^{\infty} (a_i + a, b_i + a]$.
Therefore $\lambda^*(A) = \lambda^*(A + a)$. The only thing left to show is $\mathscr{M}_\lambda$ is translation invariant.

Claim. Let $A \in \mathscr{M}_{\lambda^*}$. Then $A + a \in \mathscr{M}_{\lambda^*}$.

Proof. Fix $E \subseteq \mathbb{R}$. We want to show

$$\lambda^*(E) = \lambda^*(E \cap (A+a)) + \lambda^*(E \cap (A+a)^c)$$

We know λ^* is translation invariant, so

$$\lambda^*(E) = \lambda^*((E-a) \cap A) + \lambda^*((E-a) \cap A^c)$$

For any $S \subseteq \mathbb{R}$,

$$\begin{aligned}(E-a) \cap S &= \{x - a \mid x \in E, x - a \in S\} \\ &= \{x \mid x \in E, x \in S + a\} \\ &= E \cap (S+a) - a\end{aligned}$$

So

$$\begin{aligned}(S+a)^c &= \{y \mid y \notin S + a\} \\ &= \{y \mid y - a \notin S\} \\ &= \{y - a \mid y - a \notin S\} \\ &= S^c + a\end{aligned}$$

Therefore

$$\begin{aligned}\lambda^*(E) &= \lambda^*(E \cap (A+a)) + a) + \lambda^*((E \cap (A^c + a)) - a) \\ &= \lambda^*((E \cap (A+a)) - a) + \lambda^*((E \cap (A+a)^c) - a) \\ &= \lambda^*(E \cap (A+a)) + \lambda^*(E \cap (A+a)^c)\end{aligned}$$

Thus, for any $A \in \mathscr{M}_{\lambda^*}$, we have $\lambda(A) \overset{\text{def}}{=} \lambda^*(A) = \lambda^*(A+a) \overset{\text{def}}{=} \lambda(A+a)$ ■

In fact, <u>all</u> finite Borel measures are of this form.

Theorem 0.10. *Suppose μ is a finite Borel measure. Then $\mu = \mu_F$, where F is the cumulative distribution function, $F(x) = \mu((-\infty, x))$.*

Proof. Recall, we already showed that for any finite measure μ on $\mathscr{B}_{\mathbb{R}}$, $F(x) = \mu((-\infty, x])$ is nondecreasing and right-continuous.

We seek to show $\mu(E) = \mu_F(E)$ for all $E \in \mathscr{B}_{\mathbb{R}}$.
First, consider the half-open interval $(a, b]$, $a \leq b$. μ is a measure by hypothesis, so in particular is finitely additive:

$$\mu((a,b]) + \underbrace{\mu((-\infty,a])}_{=F(a)} = \underbrace{\mu((-\infty,b])}_{=F(b)}$$

So, $\mu((a,b]) = F(b) - F(a) = \mu_F((a,b])$.
Now, fix $E \in \mathscr{B}_{\mathbb{R}}$. Consider $\{(a_i, b_i]\}_{i=1}^{\infty}$ such that $E \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i]$.
By countable subadditivity,

$$\mu(E) \leq \sum_{i=1}^{\infty} \mu((a_i,b_i]) = \sum_{i=1}^{\infty}(F(b_i) - F(a_i))$$

Taking the infimum over all such covers, $\mu(E) \leq \mu_F(E)$. It remains to show the opposite inequality.
Since $E \in \mathscr{B}_{\mathbb{R}}$ was arbitrary,

$$\mu(E^c) \leq \mu_F(E^c)$$

Thus $\mu(E) + \mu(E^c) = \mu(\mathbb{R})$. Then $\mu(E) = \mu(\mathbb{R}) - \mu(E^c)$. So

$$\begin{aligned} \mu(E) &\geq \mu(\mathbb{R}) - \mu_F(E^c) \\ &= \mu_F(R) - \mu_F(E^c)) \\ &= \mu_F(E) \end{aligned}$$

But the above has a gap: in particular, we don't know that $\mu(\mathbb{R}) = \mu_F(\mathbb{R})$. If we prove that, we are done.

Claim. $\mu(\mathbb{R}) = \mu_F(\mathbb{R})$.

Proof.

$$\begin{aligned} \mu(\mathbb{R}) = \mu\left(\bigcup_{i=1}^{\infty}[-i,i]\right) &\overbrace{=}^{\text{by upper continuity}} \lim_{i\to\infty} \mu((-i,i]) \\ = \lim_{i\to\infty} \mu_F((-i,i]) &\underbrace{=}_{\text{by lower continuity}} \mu_F\left(\bigcup_{i=1}^{\infty}[-i,i]\right) = \mu_F(\mathbb{R}) \end{aligned}$$

■

We conclude our study of Borel measures on the real line with some regularity properties of Lebesgue-Stieljes measures.

Lemma 2. Given $F : \mathbb{R} \to \mathbb{R}$ nondecreasing, right-continuous, for all $E \in \mathscr{M}_{\mu_F^*}$,

$$\mu_F(E) = \inf\{\sum_{i=1}^{\infty} \mu_F((a_i, b_i)) \mid E \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i), a_i \leq b_i\}$$

Proof. By HW3Q2,

$$\mu_F(E) = \inf\{\sum_{i=1}^{\infty} \mu_F(A_i) \mid E \subseteq \bigcup_{i=1}^{\infty} A_i, \{A_i\}_{i=1}^{\infty} \in \mathscr{M}_{\mu_F^*}^{\mathbb{N}}\}$$

Thus, "$\leq$" must hold. It remains to show the opposite inequality.
By definition, for all $E \in \mathscr{M}_{\mu_F^*}$,

$$\mu_F^*(E) = \inf\{\sum_{i=1}^{\infty}(F(b_i) - F(a_i)) \mid E \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i], a_i \leq b_i\}$$

Fix $\varepsilon > 0$. Then there exists a sequence of intervals $\{(a_i, b_i]\}_{i=1}^{\infty}$ such that $E \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i]$ and

$$\mu_F(E) + E \geq \sum_{i=1}^{\infty}(F(b_i) - F(a_i))$$

Furthermore, for any $(a_i, b_i]$, we may define $B_n \stackrel{\text{def}}{=} (a_i, b_i + \frac{1}{n})$, and since $\mu_F(B_1) < +\infty$, continuity from above ensures

$$\lim_{n\to\infty} \mu(B_n) = \mu_F\left(\bigcap_{i=1}^{\infty} B_n\right) = \mu_F((a_i, b_i])$$

Thus, for all i, there exists a $\delta_i > 0$ such that

$$\mu_F((a_i, b_i + \delta_i]) \leq \mu_F((a_i, b_i]) + \frac{\varepsilon}{2^i}$$

Thus,

$$\begin{aligned}
\mu_F(E) &\leq \sum_{i=1}^{\infty} \mu_F((a_i, b_i + \delta_i]) \\
&\leq \sum_{i=1}^{\infty} \mu_F((a_i, b_i]) + \frac{\varepsilon}{2^i} \\
&\leq \mu_F(E) + 2\varepsilon
\end{aligned}$$

Letting $\varepsilon \to 0$, this shows the other direction.

■

Lecture 7

Recall: Given $F : \mathbb{R} \to \mathbb{R}$ nondecreasing and right continuous,

$$\mu_F^*(A) \stackrel{\text{def}}{=} \inf\{\sum_i ((F(b_i) - F(a_i)) \mid A \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i], a_i \leq b_i\}$$

Theorem 0.11.

$$\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}_{\mu_F^*}$$

We will show that in general, this is a strict containment.

Theorem 0.12. *For any $E \in \mathcal{M}_{\mu_F^*}$,*

$$\begin{aligned} \mu_F(E) &= \inf\{\mu_F(U) \mid E \subseteq U, U\,open\} \\ &= \sup\{\mu_F(K) \mid K \subseteq E, K\,compact\} \end{aligned}$$

Proof. Fix $E \in \mathcal{M}_{\mu_F^*}$.

Step 1

Fix $\varepsilon > 0$. The lemma proven in the previous lecture ensures that there exists $\{(a_i, b_i)\}_{i=1}^{\infty}$ such that $E \subseteq \cup_{i=1}^{\infty}(a_i, b_i)$ and

$$\mu_F \underbrace{\left(\bigcup_{i=1}^{\infty}(a_i, b_i)\right)}_{\stackrel{\text{def}}{=} U} \leq \sum_{i=1}^{\infty} \mu_F((a_i, b_i)) \leq \mu_F(E) + \varepsilon$$

The first inequality is by countable additivity of μ, and the second by construction.

Step 2

There are a few cases:

1. In the first case, assume that E is bounded. If E is closed, then by Heine-Borel E is compact, and taking $K = E$ gives the result. Next, suppose that E is not closed. Fix $\varepsilon > 0$. By step 1, there exists an open $U \supseteq \overline{E} \setminus E$ such that

$$\mu_F(U) \leq \mu_F(\overline{E} \setminus E) + \varepsilon$$

Define $K = \overline{E} \setminus U$. Then K is compact. By definition,

$$\begin{aligned} K &= \overline{E} \cap U^c \subseteq \overline{E} \cap (\overline{E} \cap E^c)^c \\ &= \overline{E} \cap (\overline{E}^c \cup E) = E \end{aligned}$$

So

$$\mu_F(E \cap U) + \mu_F(K) \geq \mu_F(E \cap U) + \mu_F(E \setminus U) = \mu_F(E)$$

Since E is bounded, $\mu_F(E \cap U) < +\infty$, $\mu_F(U) < +\infty$, so

$$\begin{aligned}
\mu_F(K) &\geq \mu_F(E) - \mu_F(E \cap U) \\
&= \mu_F(E) - (\mu_F(U) - \mu_F(E \setminus U)) \\
&\geq \mu_F(E) - \mu_F(U) + \mu_F(\overline{E} \setminus E) \\
&\geq \mu_F(E) - \varepsilon
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this gives the result.

2. In the second case, assume that E is unbounded. Define $E_j = E \cap (j, j+1], j \in \mathbb{Z}$. E_j is clearly bounded, so by case 1, we know that for all $\varepsilon > 0$, there exists a compact $K_j \subseteq E_j$, such that

$$\mu_F(K_j) \geq \mu_F(E_j) - \frac{\varepsilon}{2^{|j|}}$$

Then $H_n = \cup_{j=-n}^{n} E_j$ is compact, with $H_n \subseteq E$. By additivity,

$$\begin{aligned}
\mu_F(H_n) = \sum_{j=-n}^{n} \mu_F(K_j) &\geq \sum_{j=-n}^{n} \mu_F(E_j) - 2\varepsilon \\
&\geq \mu_F\left(\bigcup_{j=-n}^{n} E_j\right) - 2\varepsilon
\end{aligned}$$

By continuity from below, we may pick $N \in \mathbb{N}$ sufficiently large so that

$$\mu_F\left(\bigcup_{j=-N}^{N} E_j\right) \geq \mu_F(E) - \varepsilon$$

Thus,

$$\mu_F(H_N) \geq \mu_F(E) - \varepsilon$$

■

Time for an important example.

Example 0.7. The Cantor Set
Warmup:
$\lambda(\{a\}) = 0$, $\lambda(\mathbb{Q}) = \lambda(\cup_{i=1}^{\infty}\{r_i\})$, where r_i is some enumeration of $\mathbb{Q}$. Then the above is equal to $\sum_{i=1}^{\infty} \lambda(\{r_i\}) = 0$.

On the other hand, fix $\varepsilon > 0$ and define

$$U = (0,1) \cap \left(\bigcup_{j=1}^{\infty} (r_j - \frac{\varepsilon}{2^{j+1}}, r_j + \frac{\varepsilon}{2^{j+1}}) \right)$$

Then U is open and dense in $(0,1)$. From a topological perspective, this means that U is "large" (comeagre).
However, in a measurable sense, U is "small:"

$$\lambda(U) \leq \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon$$

Note: U depends on ε, so we have not shown that $\mu(U) = 0$.
Now we construct the Cantor set.
Start with $U_1 = (0,1)$. Let U_2 be U_1 with the middle third removed, so is two disjoint intervals. Inductively, let U_i be U_{i-1}, with the middle thirds of all intervals removed. Then the Cantor set is the intersection of all U_i.
Alternatively, the Cantor set is every real in $(0,1)$ whose base 3 expansion does not contain a 2.

Theorem 0.13. *Let C be the Cantor set. Then*

(i) C is compact, nowhere dense, and totally disconnected (meaning the only connected subsets are singletons). Further, C has no isolated points.

(ii) $\lambda(C) = 0$.

(iii) C has cardinality of the continuum.

Measurable Functions

Definition 0.13. Given $f : X \to Y$, $f^{-1}(E) = \{x \in X \mid f(x) \in E\}$ is called the preimage of f
These are basic set theory facts from HW 1:

$$f^{-1}(\cup_\alpha E_\alpha) = \cup_\alpha f^{-1}(E_\alpha)$$
$$f^{-1}(E^c) = (f^{-1}(E))^c$$
$$f^{-1}(\cap_\alpha E_\alpha) = \cap_\alpha f^{-1}(E\alpha)$$

Definition 0.14. Suppose $(X,\mathcal{M}),(Y,\mathcal{N})$ are measurable spaces, and $f : X \to Y$. Then $\{f^{-1}(E) \mid E \in \mathcal{N}\}$ is the pullback of $\mathcal{N}$, and $\{E \mid f^{-1}(E) \in \mathcal{M}\}$ is the pushforward of $\mathcal{M}$.

Definition 0.15. A function $f : X \to Y$ is $(\mathcal{M},\mathcal{N})$-measurable if for all $E \in \mathcal{N}$, $f^{-1}(E) \in \mathcal{M}$.
Equivalently, the pullback of $\mathcal{M}$ is a subset of N. Equivalently, $\mathcal{N}$ is a subset of the pushforward of $\mathcal{M}$.
Informally, "the inverse image of every measurable set is measurable" (Katy says the reason this isn't formal is because "measurable set" already means something specific in the context of an outer measure).
If $f : X \to \mathbb{R}(\overline{\mathbb{R}})$, we will suppose that the range is endowed with $\mathscr{B}_{\mathbb{R}}(\mathscr{B}_{\overline{\mathbb{R}}})$.

Definition 0.16. (a) $f : \mathbb{R} \to \overline{\mathbb{R}}$ is Lebesgue Measurable if it is $(\mathcal{M}_{\lambda^*},\mathscr{B}_{\overline{\mathbb{R}}})$-measurable.

(b) Given topological spaces X, Y, $f : X \to Y$ is Borel Measurable if it is $(\mathscr{B}_X,\mathscr{B}_Y)$-measurable.

Remark. Given $f : \mathbb{R} \to \overline{\mathbb{R}}$, which is a stronger criteria: being Borel measurable, or being Lebesgue measurable? Borel measurable implies Lebesgue measurable since $\mathscr{B}_{\mathbb{R}} \subseteq \mathcal{M}_{\lambda^*}$.

Proposition 6. *Given measureable spaces $(X,\mathcal{M}),(Y,\mathcal{N})$, where $\mathcal{N}$ is generated by $\mathcal{E}$. Then $f : X \to Y$ is $(\mathcal{M},\mathcal{N})$-measurable is equivalent to $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.*

Proof. One direction is immediate. For the other direction, since $\{E : f^{-1}(E) \in \mathcal{M}\}$ (the pushforward of $\mathcal{M}$) is a σ-algebra containing $\mathcal{E}$. By assumption, $\mathcal{N}$ is generated by $\mathcal{E}$, meaning $\mathcal{N} \subseteq \{E \mid f^{-1}(E) \in \mathcal{M}\}$.

Corollary 0.14. *If X and Y are topological spaces, then every continuous function $f : X \to Y$ is Borel measurable.*

Proof. Since the open subsets of Y generate the σ-algebra $\mathcal{N} = \mathcal{B}_Y$, the previous proposition ensures that it suffices to check $f^{-1}(U) \in \mathcal{B}_X$ for all U open, and this is true, since $f^{-1}(U)$ is open.

Corollary 0.15. *If $(X,\mathcal{M})$ is a measurable space and $f : X \to \mathbb{R}$ the following are equivalent:*

(i) f is $(\mathcal{M},\mathscr{B}_{\mathbb{R}})$-measurable.

(ii) $f^{-1}((a,+\infty)) \in \mathcal{M}$ for all $a \in \mathbb{R}$.

(iii) $f^{-1}((-\infty,a)) \in \mathcal{M}$ for all $a \in \mathbb{R}$

If $(X,\mathcal{M})$ is a measurable space and $f : X \to \overline{\mathbb{R}}$ the following are equivalent:

(i) f is $(\mathcal{M},\mathscr{B}_{\mathbb{R}})$-measurable.

(ii) $f^{-1}((a,+\infty]) \in \mathcal{M}$ *for all* $a \in \mathbb{R}$.

(iii) $f^{-1}([-\infty,a)) \in \mathcal{M}$ *for all* $a \in \mathbb{R}$