Lecture 1

We begin by trying to gain a deeper understanding of the Cauchy-Riemann equations.
Let $f : X \to \mathbb{C}$, where $X \subset \mathbb{C}^n$. For now, let's say $X \subset \mathbb{C}$. In real analysis, we have a notion of differentiability for $f : \mathbb{R}^n \to \mathbb{R}^k$. We can say that f is differentiable at a point $p \in X$ when

$$f(p+h) = f(p) + (df_p)h + \rho(h)$$

Where $(df)_p : \mathbb{R}^n \to \mathbb{R}^k$ is a linear map $\in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^k)$, and $\frac{|\rho(h)|}{|h|} \to 0$ as $h \to 0$.
So we can think of the "real differential" as a linear map in $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C})$.

Definition 0.1. Let $X \subset \mathbb{C}$, and $f : X \to \mathbb{C}$. Differentiability refers to the existence of a $(df)_p \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$.
So, f is complex differentiable at $p \in X$ means that

$$f(p+h) = f(p) + f'(p)h + \rho(h)$$

Where $f'(p)$ is a complex number and $\frac{|\rho(h)|}{|h|} \to 0$.
If $A \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$, $A(z) = \alpha z, \alpha \in \mathbb{C}$.
So $(df)_p \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$.
$\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$ is a $\mathbb{C}$-vector space of dimension 1.
$\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R}^2)$ is a $\mathbb{C}$-vector space of dimension 2.
So where did the extra dimension go? What happened?
Consider an element of $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$ given by $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x - iy$.
We also have $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x + iy$.
From the real analysis point of view, these two functions are equal to their differentials. The first is called $d\overline{z}$, and the second is called dz.
$dz = dx + idy$ and $d\overline{z} = dx - idy$

$$dx \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = h_1$$
$$dy \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = h_2$$

On a complex vector space, suppose $\phi \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{C}^n, \mathbb{C})$, we have $(\overline{\phi})(v) = \overline{\phi(v)}$. So $\overline{dz} = d\overline{z}$.
Now, $\mathbb{C}$-valued real differentiable functions are just pairs of $\mathbb{R}$-valued real differentiable functions.

Example 0.1. If $k, m \in \mathbb{N}$, then $z^k\overline{z}^m : \mathbb{C} \to \mathbb{C}$ is a real smooth function (when viewed as an element of $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R}^2)$), with

$$d(z^k\overline{z}^m) = kz^{k-1}\overline{z}^m + m\overline{z}^{m-1}z^k d\overline{z}$$

We will study the differences between $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C})$ versus $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$, with complex dimensions 2 and 1, respectively.

Definition 0.2. Let V be a real vector space.
A complex structure on V is a $J \in \mathrm{End}_{\mathbb{R}}(V)$ which satisfies $J^2 = -\mathrm{Id}_V$

Proposition 1. *Define $V_J = V$ as a set and group, with a $\mathbb{C}$-action $\mathbb{C} \times V_J \to V_J$ defined by $((\alpha + i\beta), x) \mapsto \alpha x + \beta Jx$.*

Proof. Check $z(wx) = (zw)x$ for all $z, w \in \mathbb{C}$ and $x \in V_J$.

Proposition 2. *If a vector space V admits a complex structure J, then* $\dim_{\mathbb{R}} V = 2n$. *Further,* $\dim_{\mathbb{R}} V = 2\dim_{\mathbb{C}} V_J$.

Proof. First, $\det(J^2) = \det(-\mathrm{Id}_V) = (-1)^{\dim_{\mathbb{R}} V}$, so the dimension must be even. Alternatively, if $e_1, \dots, e_n$ is a basis of V_J, then check $e_1, \dots, e_n, Je_1, \dots, Je_n$ is a basis of V over R.

Example 0.2. For $\mathbb{R}^2$, let $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We see that $J_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$. This is like $i(x + iy) = ix - y$.
So $A : (\mathbb{R}^2)_{J_0} \to \mathbb{C}$ is an isomorphism of $\mathbb{C}$-vector spaces.
Let W be a vector space over $\mathbb{C}$. Consider $W_{\mathbb{R}}$, a real vector space. We see $\dim_{\mathbb{R}} W_{\mathbb{R}} = 2\dim_{\mathbb{C}} W$. Consider $J : W_{\mathbb{R}} \to W\mathbb{R}$ given by $x \mapsto ix$. Then $J^2 = -\mathrm{Id}_{W_{\mathbb{R}}}$.
Let V be a real vector space with complex structure J. Consider $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} V^*$.
$J^t : \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \to \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$, we can express $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \ni \phi = \phi_1 + \phi_2$, and by definition,

$$J^t\phi = \phi \circ J = \phi_1 \circ J + i\phi_2 \circ J$$

So $(J^t)^2 = -1$.
$J^t \in \mathrm{End}_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}))$.
Main observation: $\phi \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C})$ is $\mathbb{C}$-linear in V_j, meaning $\phi(ix) = i\phi(x)$, which is equivalent to $\phi(Jx) = i\phi(x)$.
In other words, such a ϕ is only $\mathbb{C}$-linear if ϕ is an eigenfunction of J^t with eigenvalue i.

Definition 0.3. ϕ is $\mathbb{C}$-antilinear on V_J means

$$\phi((\alpha + i\beta)x) = \overline{(\alpha + i\beta)}\phi(x)$$

for all $x \in V$.

We denote the space of $\mathbb{C}$-antilinear functionals by $\overline{\text{Hom}}_{\mathbb{C}}(V_J, \mathbb{C})$.
In fact, there is an isomorphism between $\text{Hom}_{\mathbb{C}}(V_J, \mathbb{C})$ and $\overline{\text{Hom}}_{\mathbb{C}}(V_J, \mathbb{C})$ as real vector spaces.

Theorem 0.1. *Let V be a real vector space with complex structure J. Then*

1. $\text{Hom}_{\mathbb{R}}(V, \mathbb{C}) = \text{Hom}_{\mathbb{C}}(V_J, \mathbb{C}) \oplus \overline{\text{Hom}}_{\mathbb{C}}(V_J, \mathbb{C})$.

2. *If* $\text{Hom}_{\mathbb{C}}(V_J, \mathbb{C}) := V^{1,0}$, *and* $\overline{\text{Hom}}_{\mathbb{C}}(V_J, \mathbb{C}) := V^{0,1}$, *then* $\text{Hom}_{\mathbb{R}}(V, \mathbb{C}) = V^{1,0} \oplus_{\mathbb{C}} V^{0,1}$.

3. $\dim_{\mathbb{C}} V^{1,0} = \dim_{\mathbb{C}} V^{0,1} = \frac{\dim_{\mathbb{C}}(\text{Hom}_{\mathbb{R}}(V,\mathbb{C}))}{2}$

Proof. Observe that $\phi \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$ can be written as

$$\phi = \frac{\phi - i\phi \circ J}{2} + \frac{\phi + i\phi \circ J}{2} = \frac{\phi(Jx) + i\phi(x)}{2} = i\frac{\phi - i\phi \circ J}{2}(x) = \phi$$

Further, $V^{1,0} \cap V^{0,1} = 0$ by the definitions, so we are done. ■

Thus, any differential can be split into a $\mathbb{C}$-linear and a $\mathbb{C}$-antilinear part.

Definition 0.4. $\pi^{1,0}$ is projection on the first factor, $\pi^{0,1}$ is projection onto the second. We have

$$\phi = \pi^{1,0}\phi + \pi^{0,1}\phi$$

Corollary 0.2. *If $\phi \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$, then ϕ being $\mathbb{C}$-linear (i.e $\phi \in V^{1,0}$) if and only if $\pi^{0,1}\phi = 0$.*

Definition 0.5. Applying to $(df)_p \in \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$, then

$$(df)_p = \pi^{1,0} df_p + \pi^{0,1} df_p$$

Say $\pi^{1,0} df_p = \underbrace{\partial f_p}_{\text{complex linear}}$ and $\pi^{0,1} df_p = \underbrace{\overline{\partial} df_p}_{\text{complex antilinear}}$

Theorem 0.3. *A function $f : X \to \mathbb{C}$ is $\mathbb{C}$-differentiable at $p \in X$ if and only if f is $\mathbb{R}$-differentiable at p and $df_p = \partial f_p$, which happens if and only if $\overline{\partial} f_p = 0$.*

Proof. We have $\mathbb{C} \cong \mathbb{R}^2_{J_0}$, which has standard basis $\mathbb{R}^2 = \langle e_1, e_2 \rangle_{\mathbb{R}^2}$. This has a dual basis in $\text{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R})$ given by dx and dy. That is, $\text{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{C}) = \langle dx, dy \rangle_{\mathbb{C}}$.
$J_0 e_1 = e_2$ and $J_0 e_2 = -e_1$, so $dx \circ J_0 = -dy$ and $dy \circ J_0 = dx$.

$$\begin{aligned}\pi^{0,1} dx &= \frac{1}{2}(dx - idx \circ J_0) \\ &= \frac{1}{2}(dx + idy) \overset{\text{def}}{=} dz\end{aligned}$$

Further,

$$\pi^{0,1}dx = \frac{1}{2}d\overline{z}$$
$$\pi^{1,0}dy = \frac{1}{2}dz$$

So

$$df = f_x dx + f_y dy = \frac{f_x - if_y}{2}dz + \frac{f_x + if_y}{d}\overline{z} = \partial f + \overline{\partial} f$$

Definition 0.6.

$$\frac{\partial}{\partial z} = \frac{\partial_x - i\partial_y}{2}$$
$$\frac{\partial}{\partial \overline{z}} = \frac{\partial_x + i\partial_y}{2}$$

So

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}$$

So analyticity is equivalent to $df = \partial f$, meaning $\overline{\partial} f = 0$, which is equivalent to $\frac{\partial f}{\partial \overline{z}} = 0$, which means

$$\frac{\partial(u + iv)}{\partial \overline{z}} = 0$$

So $(\partial_x + i\partial_y)(u + iv) = 0$. Multiplying out, we get

$$u_x = v_y$$
$$u_y = -v_x$$

Lecture 2

The focus for the first bit of this course will be the so-called (by Dennis) $\overline{\partial}$-calculus. Suppose $f : X \to \mathbb{C}$ is differentiable for some $X \subseteq \mathbb{R}^{2n}$. It has a differential $df_p \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{C})$.
f is holomorphic if and only if $df_p \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$. Last time, we talked about how the second sits in the first, and how they interact.
Question: How to make $\mathbb{C}^n$ out of $\mathbb{R}^{2n}$. The abstract algebra way to do it is with a complex structure J. If V is a vector space over $\mathbb{R}$, then $\dim_{\mathbb{R}} V = 2n$ $J \in \mathrm{End}_{\mathbb{R}}(V)$ with $J^2 = -\mathrm{Id}_V$.

For all $x \in V_J$, we define $ix = Jx$, so V_J is a vector space over $\mathbb{C}$, and $\dim_{\mathbb{C}} V_j = \frac{\dim_{\mathbb{R}} V}{2}$

Example 0.3. Let $V = \mathbb{R}^2$, $J = J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. $\mathbb{R}^2_{J_0} \cong \mathbb{C}$

Last time, we showed that for any $\phi \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = H$, then

$$\phi = \underbrace{\frac{\phi - i\phi \circ J}{2}}_{\in \mathrm{Hom}_{\mathbb{C}}(V_J, \mathbb{C})} + \underbrace{\frac{\phi + i\phi \circ J}{2}}_{\in \overline{\mathrm{Hom}_{\mathbb{C}}}(V_J, \mathbb{C})}$$

So, $H = \mathrm{Hom}_{\mathbb{C}}(V_J, \mathbb{C}) \oplus \overline{\mathrm{Hom}_{\mathbb{C}}(V_J, \mathbb{C})} = V^{1,0} \oplus V^{0,1}$
Where $V^{1,0}$ and $V^{0,1}$ are what we call $\mathrm{Hom}_{\mathbb{C}}(V_J, \mathbb{C})$ and $\overline{\mathrm{Hom}}_{\mathbb{C}}(V_J, \mathbb{C})$ by tradition.
Let $V = \mathbb{R}^2$, $J = J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
Let $\phi = df_p = \frac{\partial f(p)}{\partial x} dx_p + \frac{\partial f(p)}{\partial y} dy_p$
After doing some computations, we get

$$d_p f = \pi^{1,0} df + \pi^{0,1} df = A dz + B d\overline{z}$$

Where $dz = dx + idy$ and $d\overline{z} = dx - idy$. You can check that the former is in $V^{1,0}$ and the latter in $V^{0,1}$.
The coefficient A is denoted by tradition as $\frac{\partial f}{\partial z}(p)$, and B as $\frac{\partial f}{\partial \overline{z}}(p)$.
Here, the presence of ∂ does not imply any limit taking or anything, they are just notation.
Note $A = \frac{1}{2}(\partial_x f - i\partial_y f)|_p$ and $B = \frac{1}{2}(\partial_x f + i\partial_y f)|_p$.

Definition 0.7. We define

$$\partial f_p \overset{\text{def}}{=} f_z(p) dz_p$$
$$\overline{\partial} f_p \overset{\text{def}}{=} f_{\overline{z}}(p) d\overline{z}_p$$

The former is $\mathbb{C}$-linear, and the second $\mathbb{C}$-antilinear.

Claim. *f is $\mathbb{C}$-differentiable at p if and only if f is $\mathbb{R}$-differentiable at p and* $\overline{\partial} f_p = 0$

Proof. If $f = u + iv$, we get

$$\overline{\partial} f_p = \frac{\partial f}{\partial \overline{z}}(p) = 0$$

Which gives you the Cauchy-Riemann equations.
So f is analytic if $d_p f = f_z dz$.

Example 0.4. What are the following?

1. $\frac{\partial |z|}{\partial z}$

2. $\frac{\partial |z|}{\partial \overline{z}}$

How do we manage these problems?

Claim. *If $m, k \in \mathbb{Z} \backslash \{0\}$, we will consider $d(z^m \overline{z}^k)$. We have $f(z+h) = (z+h)^m(\overline{z} + \overline{h})^m = f(z) + (mz^{m-1}\overline{z}^k)h + (k\overline{z}^{k-1}z^m)\overline{h} + \mathcal{O}(h^2)$*
Where $\frac{|LHS-RHS|}{|z-2|} \to 0$ as $z \to 2$
Then $\overline{\partial}(z^m\overline{z}^k) = (k\overline{z}^{k-1}z^m)d\overline{z}$

Proof. Let's do some examples. Consider $\frac{\overline{z}-1}{z+1}$. We have

$$\begin{aligned}
\frac{\overline{z}-1}{z+1} &= \frac{(\overline{z}-2)+2-1}{(z-2)+2+1} \\
&= \frac{(\overline{z}-2)+1}{3}\frac{1}{1+\frac{z-2}{3}} \\
&= \frac{\overline{z}-1}{3}\left(1 - \frac{z-2}{3} + \mathcal{O}(H^2)\right) \\
&= c + Az + B\overline{z} + \rho(z)
\end{aligned}$$

There are two building blocks for doing problems:

1. First, remember you are really doing real analysis.

2. Use the formula $d_pf = \pi^{1,0}\cdots$

Example 0.5. We will calculate $d|z| = d\sqrt{z\overline{z}}$.

$$\begin{aligned}
d|z| &= d\sqrt{z\overline{z}} \\
&= \frac{1}{2\sqrt{z\overline{z}}}d(z\overline{z}) \\
&= \frac{1}{2|z|}d(z\overline{z}) \\
&= \frac{1}{2|z|}zd\overline{z} + \overline{z}dz
\end{aligned}$$

So the answer would be $\frac{z}{2|z|}$.
So, just express your function as a function of $z\overline{z}$ and proceed to do real analysis.
We know $\partial_z f, \partial_{\overline{z}} f$, and want to find $\partial_z(\overline{f}), \partial_{\overline{z}}(\overline{f}) =?$
We have

$$df = \partial f + \overline{\partial} f$$
$$d(\overline{f}) = \overline{df} = \overline{(\partial f)} + \overline{(\overline{\partial} f)}$$

Now, $\overline{\partial}(\overline{f}) = \overline{\partial f}$. The bottom is equal to $\frac{\partial \overline{f}}{\partial \overline{z}} d\overline{z}$
So

$$\frac{\partial \overline{f}}{\partial \overline{z}} = \overline{\frac{\partial f}{\partial z}}$$

The conjugate of something which is complex anti-linear is complex linear.
What is $\frac{\partial \overline{f}}{\partial z}$? This is $\overline{\frac{\partial f}{\partial \overline{z}}}$.
So, the general procedure is to decompose your function as something linear + something antilinear, and use the sentence I just wrote.
How to compute $\frac{\partial f \circ g}{\partial \overline{z}}$?
Well $\underbrace{d(f \circ g) = df \circ df}_{\text{The chain rule expresses the functoriality of the derivative}}$, which is equal to
$(\partial f + \overline{\partial} f) \circ (\partial g + \overline{\partial} g) = \partial f \circ \partial g + \overline{\partial f} \circ dg + \cdots$.
Now, $\overline{\partial} f \circ \partial g = f_{\overline{z}} d\overline{z} \circ (g_z dz) = f_{\overline{z}} \overline{g_z}$.

Definition 0.8. Suppose $\frac{\partial f}{\partial \overline{z}}(p) = 0$. Then $df_p = \frac{\partial f}{\partial z}(p) dz_p$, and we write this as $f'(p)dz$.

Homework problem: We know f holomorphic, compute $\frac{\partial}{\partial} \overline{z} F(|f|)$, where $F : \mathbb{R} \to \mathbb{R}$ is smooth.
So

$$dF(|f|) = F'(|f|) d|f| = F'(|f|) d\sqrt{f\overline{f}}$$

And $d\sqrt{u} = \frac{1}{2u} du$, so the above is equal to

$$F'(|f|) \frac{1}{2|f|} d(f\overline{f})^{-1} = \frac{F'(|f|)}{2|f|}(\overline{f} df + f d\overline{z}) = \frac{F'(|f|)}{2|f|}(\overline{f} f_z dz + f \overline{f_z} d\overline{z})$$

Our answer is thus whatever we get in front of $d\overline{z}$, so in this case the solution is

$$\frac{\partial}{\partial \overline{z}} F(|f|) = \frac{F'(|f|)}{2|f|} f \overline{f'}$$

The $|z| = \sqrt{z\overline{z}}$ is a very useful trick.
Complex analysis is kind of a local study of $f : X \to \mathbb{C}$ for some $X \subseteq \mathbb{C}$, where f is differentiable, $\overline{\partial} f = 0$, or equivalently $\frac{\partial f}{\partial \overline{z}} = 0$ in X. Really, we are studying solutions to a certain PDE (Cauchy-Riemann equation).

Suppose you want $u_{xx} - u_{yy} = 0$. If this is the case (and it turns out exactly when this is the case), we can express $u = \phi(x-y) + \psi(x+y)$ for ϕ, ψ arbitrary of 1 variable. What if we want to study $u_y = u_{xx}$ in, say, $y > -\varepsilon$? We actually have a formula:

$$u(x,y) = \frac{1}{2\pi y}\int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4y}} u(s,0)ds, y > 0$$

Suppose we want to study $u_{yy} + u_{xx} = 0$ in $y > -\varepsilon$, or maybe an open ball around the origin. We have a formula

$$u(x,y) = \int_{-\infty}^{\infty} P_H(x, y-s)u(s,0)ds$$

Where P_H is the Poisson Kernel.

MAIN LOCAL THM

Let $f : B_{R+\varepsilon} \to \mathbb{C}$. Then the following statements are all equivalent.

1. For any $p \in B_R$, f is differentiable and $df_p = \partial f_p$, so $\overline{\partial} f_p = 0$.

2. $f(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{z-w}dw$

3. Like 8 other things

What exactly is complex integration?
Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth map. We integrate functions over maps. If ϕ is a continuous function, the formula is

$$\int_\gamma \phi(z)dz = \int_a^b \phi(\gamma(t))\dot{\gamma}(t)dt$$

where $\dot{\gamma}$ is the time-derivative of γ, so will be a complex number.
Let $t \in [0, 2\pi]$, $C_R(t) = Re^{it}$, a circle going counterclockwise. Let $\gamma(t) = Re^{-it}$. Then $\gamma^{-1} : [a,b] \to \mathbb{C}$ is the same but in the opposite direction. We have

$$\int_{\gamma^{-1}} \phi\, dz = \int_\gamma \phi\, dz$$

Lecture 3

We continue with the Local Theorem for analytic functions. Let $f : B_{R_0+\varepsilon} \to \mathbb{C}$. Then the following statements are all equivalent.

(i) $\forall z \exists$ a finite $\lim_{\Delta z \to \infty} \frac{f(z+\Delta z)-f(z)}{\Delta z} \overset{\text{def}}{=} f'(z)$.

(ii) $\forall z$, f is $\mathbb{R}$-differentiable at z and $df_z = \partial f_z$, which is equivalent to $\overline{\partial} f_z \equiv 0$, which gives the Cauchy-Riemann equations.

(iii) f is continuous, and for all $R < R_0$, for all z such that $|z-a| < R$, then

$$f(z) = \frac{1}{2\pi i}\int_{C_R(a)} \frac{f(w)}{w-z}dw$$

(iv) $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ in $B_R(a)$ with $a_n = \frac{1}{2\pi i}\int_{C_R(a)} \frac{f(w)}{(w-a)^{n+1}}dw$ for all $R < R_0$.

(v) For some coefficient c_n, $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ for all $|z-a| < R_0$ for some c_n.

(vi) For all z, $\exists f'(z), f''(z), \ldots, f^{(n)}(z), \ldots$. Moreover, $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ for all $|z-a| < R_0$ with

$$c_n = \frac{f^{(n)}(a)}{n!}$$

and

$$f'(z) = \sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$$

for all $|z-a| < R_0$.

Proof. We already proved (i) and (ii) equivalent. $(ii) \implies (iii)$ will come later, and the steps $(iv) \implies (v) \implies (vi)$ are proven by undergrad power series techniques. Then of course $(vi) \implies (i)$ is a triviality.

Claim. $(iii) \implies (iv)$.

Proof. Recall the setup: (iii) holds in a disk of radius R, which is δ less than R_0. We have $f(z) = \frac{1}{2\pi i}\int_{C_R(a)} \frac{f(w)}{z-w}dw$. Now,

$$\frac{1}{w-z} = \frac{1}{(w-z)-(z-a)} = \frac{1}{w-a}\cdot\frac{1}{1-\frac{z-a}{w-a}} = \sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}$$

This works because $|z-a| < |w-a|$. So

$$f(z) = \frac{1}{2\pi i}\int_{C_R(a)}\left(\sum_{n=0}^{\infty}(z-a)^n\frac{f(w)}{(w-a)^{n+1}}\right)dw$$

We want to swap the integral and the summation, and then we will be done. We can swap an integral and a sum provided the sum converges uniformly on $[a,b]$. We will show that this sum converges uniformly using the M test.

Let $M_n = \sup_{[a,b]} |f_n|$. Then if $\sum_{n\in\mathbb{N}} M_n$ converges, then $\sum_{n\in\mathbb{N}} f_n$ converges uniformly .

We have to calculate $\sup_{|w-a|=R} |(z-a)^n \frac{f(w)}{(w-a)^{n+1}}| = M_n$. We find

$$M_n = \frac{(R-\delta)^n}{R^{n+1}} \left(\frac{\sup_{\overline{B_R(a)}} |f|}{R} \right)$$

so $\sum_{n\in\mathbb{N}} M_n < \infty$

■

We make use of the trick where $\frac{1}{A+B} = \frac{1}{A} \cdots \frac{1}{1+\frac{B}{A}}$, and if $\frac{B}{A}$ is small we can do a series expansion.

Now, we will prove the $\triangle$ inequality for complex integrals.

Let $\gamma : [a,b] \to \mathbb{C}$ be C^1 or piecewise smooth. , and let $\phi : \mathbb{C} \to \mathbb{C}$ be continuous. By definition, we have

$$\begin{aligned} \left| \int_\gamma f(z)\, dz \right| &= \int_a^b \phi(\gamma(t))\dot{\gamma}(t)\, dt \\ &\leq \int_a^b |\phi(\gamma(t))||\dot{\gamma}(t)|\, dt \\ &=: \int_\gamma |\phi(z)||dz| \end{aligned}$$

We have used the fact that for continuous real functions $f : \mathbb{R} \to \mathbb{R}$ we have

$$\left| \int_a^b f(t)\, dt \right| \leq \int_a^b |f(t)|\, dt$$

We are moving towards proving $(ii) \implies (iii)$. The next result we need Goursat's Lemma.

Definition 0.9. Let $X \subset \mathbb{C}$ be open. If $f'(z)$ exists for all $z \in X$, we say $f \in A(X)$ or $f \in \mathscr{H}(X)$.

Lemma 1. *(Goursat's Lemma)*
Let a solid $\triangle \subset \Omega \subset \mathbb{C}$, with Ω open ($\triangle$ is a 2-simplex).
Denote the boundary of $\triangle$ by T, with the counter-clockwise orientation. Then for any $f \in \mathscr{H}(\Omega)$, we have

$$\int_T f\, dz = 0$$

Proof. Recall the Cauchy-Riemann system

$$u_x = v_y$$
$$u_y = -v_x$$

We have from Calc 3 the classic Green's theorem:

$$\int_{\partial\Omega} P dx + Q dy = \int\int_{\Omega} (\text{something}) dx dy$$

Using these together will give the proof.

Step 1

First, we will seek a contradiction. Suppose that there is some $\varepsilon_0 > 0$ such that

$$\left| \int_T f\, dz \right| \geq \varepsilon_0$$

We can perform barycentric subdivision to $\triangle$ (pictures forthcoming until I figure out how to insert them in TeXwriter), and we can express this integral as the sum of the integrals of four sub-simplices, $T_1, \ldots, T_4$. We must have $|\int_{T_i} f\, dz| \geq \frac{\varepsilon_0}{4}$. Denote this sub-simplex by T_1, and it's interior by $\triangle_1$.
So we have

$$\left| \int_{T_1} f\, dz \right| \geq \frac{\varepsilon_0}{4}$$

Now, $\text{length}(T_1) = \frac{\text{length}(T)}{2} = \frac{c_0}{2}$, and $\text{diam}(\triangle_1) = \frac{\text{diam}(\triangle)^2}{2} = \frac{c_1}{2}$.
We can keep proceeding by doing barycentric subdivision to T_1.
So we have a sequence $\triangle_1 \supset \triangle_2 \supset \ldots$, with $\partial\triangle_i = T_i$, with $|\int_{T_i} f\, dz| \geq \frac{\varepsilon_0}{4^i}$. Further, the length of T_j is $\frac{c_0}{2^j}$, and the diameter of $\triangle_j$ is $\frac{c_1}{2^j}$.
We can see that $\cap_{j=1}^{\infty} \triangle_j = \{p\}$, and without loss of generality we can assume $p = 0 \in \Omega$.

Step 2

We have $f(z) = f(0) + f'(0)z + \rho(z)$, where $\frac{|\rho(z)|}{|z|} \to 0$. So

$$\int_\gamma 1\, dz = \int_a^b 1 \cdot \dot{\gamma}(t) dt = \gamma(b) - \gamma(a)$$

So
$$\int_\gamma z\,dz = \int_a^b \gamma(t)\dot\gamma(t)\,dt = \frac{1}{2}\int_a^b ((\gamma(\dot{t})^2))\,dt = \frac{\gamma(b)^2-\gamma(a)^2}{2}$$
So
$$\int_{T_j} f\,dz = \int_{T_j} f(0)\,dz + \int_{T_j} f'(0)\,dz + \int_{T_j}\rho(z)\,dz = 0+0+\left|\int_{T_j}\rho(z)\,dz\right|$$
So
$$\begin{aligned}\left|\int_{T_j} f\,dz\right| &\le \int_{T_j}|\rho(z)||dz| = \int_{T_j}\frac{|\rho(z)|}{|z|}|z||dz|\\ &\le \sup_{\triangle_j}\frac{|\rho(z)|}{|z|}\operatorname{diam}(\triangle_j)\operatorname{length}(T_j)\end{aligned}$$

But $\frac{\varepsilon_0}{c_0c_14^i} \le \sup_{bigt_j}\frac{|\rho(z)|}{|z|}\frac{c_0c_1}{4^j}$, which for large d contradicts our assumption that $\frac{|\rho(z)|}{|z|}\to 0$, proving Gousat's Lemma.
Now that we have Goursat's Lemma, we can prove $(ii) \implies (iii)$.

Proposition 3. *$(ii) \implies (iii)$.*

Proof. **Step 1**

By Goursat, for $F\in\mathscr{H}(B_{R+\varepsilon}(a))$, $\int_{C_R(a)} F\,dw = \int_{C_\varepsilon(z)} F\,dw$ when $z\in B_R(a)$. We will show this in step 2.
Now, consider the map $w\mapsto\frac{f(w)}{w-z}$. We have

$$\int_{C_R(a)}\frac{f(w)}{w-z}\,dz = \int_{C_\varepsilon(z)}\frac{f(w)}{w-z}\,dz = \int_0^{2\pi}\frac{f(z+\varepsilon e^{it})}{\varepsilon e^{it}}\,dz = i\int_0^{2\pi} f(z+\varepsilon e^{it})\,dt = if(z)2\pi i$$

Lecture 4

Suppose we have a function F, which is holomorphic in $B_a(R+\varepsilon)\backslash\{z\}$. $F(w)=\frac{f(w)}{w-z}$, with $f\in\mathscr{H}(B_{R+\varepsilon}(a)$. We have

$$\int_{C_R(a)} F(w)\,dw =_* \int_{C_\varepsilon(z)} F(w)\,dw$$
Once we have established this, we can take the limit as $\varepsilon\to 0$ to get the Cauchy formula $f(z)=\cdots$.
We need some facts to prove $*$

Fact 1

Suppose a circle has N points distributed on it's circumference. Any two adjacent points forms an angle with the center. Let φ_N be the max of these angles. Note $\varphi_N \to 0$ as $N \to \infty$. Let L_n be the polygonal curve formed by the secants between adjacent points. Then

$$\int_{L_n} \phi(w)\, dw \to \int_{C_r(a)} \phi(w)\, dw$$

for ϕ continuous. The proof is easy, just break up the integrals and use uniform convergence.

Fact 2

Let A_1, A_2, A_3, A_4 be points in a a ball, and consider the loop going from $A_1 \to A_2$, then $A_2 \to A_3$, then $A_3 \to A_4$, and finally $A_4 \to A_1$.
Suppose that z is not in the convex hull of $\{A_1, A_2, A_3, A_4\}$.
Then $\int_{A_1 \to \cdots \to A_1} F(w)\, dw = 0$. We calculate this by breaking this loop up into the sum of two simplices, and the integral over the simplices will be zero by Goursat's lemma.
We can then approximate the region in between $B_a(R+\varepsilon)$ and $B_z(\varepsilon)$ by a collection of loops of the above form.
This completes the proof

$\blacksquare$

Theorem 0.4. *(Inverse Function Theorem)*
Let $f \in \mathscr{H}(\Omega)$, with $\Omega \subseteq \mathbb{C}$ open. Let $z_0 \in \Omega$.
Suppose that $f'(z_0) \neq 0$. Then there exists a $\varepsilon > 0$ such that $f|_{B_\varepsilon(z_0)}$ is a biholomorphism onto its image, which is open.
Moreover, there is a $g : f(B_\varepsilon(z_0)) \to B_\varepsilon(z_0))$ which is holomorphic, such that $g \circ f = \mathrm{Id}$.
In other words, g is a "local inverse" to f (think two branches of square root, etc.)

Proof. Let $z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$
Just looking at f purely as a smooth function from $\mathbb{R}^2 \to \mathbb{R}^2$, we have

$$\det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = u_x v_y - u_y v_x$$

By Cauchy-Riemann, we have $f' = u_x + iv_x$, so $u_x^2 + v_x^2 = |f'|^2|_{(x_0,y_0)} \neq 0$. This immediately gives bijectivity of the map f, and the fact that its image is open.
Now $f^{-1} : f(B_\varepsilon(z_0)) \to B_\varepsilon(z_0)$ is a real C^∞ function. We just need to prove it is analytic.

We know $\mathrm{Id}_{B_\varepsilon(z_0)} = g \circ f$, so

$$\begin{aligned}\mathrm{Id} &= dg \circ df \\ &= (\partial g + \overline{\partial} g) \circ (\partial f + \overline{\partial} f) \\ &= \partial g \circ \partial f + \overline{\partial} g \circ \partial f\end{aligned}$$

We must conclude that $\overline{\partial} g \circ \partial f$ is both $\mathbb{C}$-linear and $\mathbb{C}$-antilinear, meaning it is identically zero. Now, ∂f is linear and bijective, thus, $\overline{\partial} g = 0$, so $dg = \partial g$, so g is holomorphic.

■

If $\det df = 0$ for a real smooth function, then we can't really say anything about f. However, suppose $f'(z) = 0$, but $f'(z) \not\equiv 0$. Then we know a lot about the structure of f.

A homework problem asks to show the following: let $K \subseteq \Omega \subseteq \mathbb{C}$, with $\Omega \neq K$, K compact, Ω open.

By compactness, $d(K, \mathbb{C} \backslash \Omega) = \delta > 0$. So

$$f(z) = \frac{1}{2\pi i} \int_{C_{\frac{\delta}{2}}(a)} \frac{f(w)}{w - z} \, dz$$

Then

$$f'(a) = \frac{1}{2\pi i} \int_{C_{\frac{\delta}{2}}(a)} \frac{f(w)}{(w - a)^2} \, dw$$

And now you can do some bounding shenanigans to get $\leq \frac{\sup|f|}{(\frac{\delta}{2})^2}$

By controlling the derivative, we may control the derivative. The reverse is not true: no matter how we restrict the variance of f, the derivative can do strange things.

But for solutions to PDEs, the control goes the other way. That is, if you control $\sup|f|$, this controls the derivative, a fact which is not true in general. And every holomorphic function is a solution to a PDE.

So if a sequence of functions converges uniformly, so does the sequence of their derivatives, and so on.

Theorem 0.5. *(Liouville)*

Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function (meaning holomorphic on all of $\mathbb{C}$), such that $\sup_{\mathbb{C}} |f| < \infty$. Then $f(z)$ is constant.

Remark. This gives the easiest proof of the fundamental theorem of algebra:

Suppose we have a nice non-constant polynomial p without a root. Then $\frac{1}{p}$ is entire and bounded, so $\frac{1}{p}$ must be constant, so p is constant, a contradiction (unless p is constant in the first place).

Thus p must have a root.

Proof. Write

$$f(z) = \frac{1}{2\pi i} \int_{C_R(z_0)} \frac{f(w)}{w - z_0} \, dw$$

Then

$$f' = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w - z_0)^2} \, dw$$

The complex triangle inequality tells us

$$|f'| \leq c \left| \int_{C_r(z_0)} \frac{f(w)}{(w - z_0)^2} \right| |dw| \leq c \frac{\sup_{\mathbb{C}} |f|}{R^2} 2\pi R = c \frac{\sup_{\mathbb{C}} |f|}{R}$$

Taking a limit as $R \to \infty$, we see that $|f'(z_0)| = 0$. But z_0 was arbitrary, so f is constant.

Theorem 0.6. *(Factorization)*
Let $f : B_\varepsilon(a) \to \mathbb{C}$ be holomorphic, and suppose $f(a) = 0$ and $f' \not\equiv 0$.
Then there exists an $N \in \mathbb{N}$ such that $f(z) = (z - a)^N \phi(z)$, with ϕ holomorphic in $B_\varepsilon(a)$, and $\phi(a) \neq 0$.

Proof. Write
Find the first N such that the Nth derivative of f is nonzero.

$$f(z) = \sum_{n=1}^{\infty} c_n (z - a)^n = \sum_{n=N}^{\infty} c_n (z - a)^n = (z - a)^n (c_N + c_{N+1}(z - a) + \cdots)$$

Corollary 0.7. *The zeroes of a holomorphic function are isolated, unless the function is identically zero.*

Corollary 0.8. *(analytic continuation principle)*
Let $f, g : B_\varepsilon(a) \to \mathbb{C}$ be holomorphic, and let $f(z_j) = g(z_j)$ for some sequence $z_j \to a$, $z_j \neq a$ for all j.
Then $f = g$ in $B_\varepsilon(a)$.
This follows directly from the zeroes of non-zero holomorphic functions being isolated.

■

Remark. Let Ω_1, Ω_2 be open, $\Omega_1 \subset \Omega_2 \subset \mathbb{C}$, with $f, g : \Omega_2 \to \mathbb{C}$ holomorphic. Suppose that $f|_{\Omega_1} = g|_{\Omega_1}$.
Then $f = g$ in Ω_z.

Proof. Let $a \in \Omega_1$, let $\gamma : I \to \Omega_2$ be a path from a to $p \in \Omega_2$. Let $G = \{t \in I \mid |f(\gamma(t)) - g(\gamma(t))| = 0$. This set is closed, and moreover $[0, \varepsilon) \subset G$ for any $0 < \varepsilon < 1$. We can then prove that G is open.
But for G to be a clopen subset of I means that $G = I$ (also because G is nonempty).

Lecture 4

Remark. Dennis wants to comment on a question on the homework relating to expansions in an annulus.
Consider $\frac{z^2}{z-3i}$. Expand in powers of $(z-1)$ in an annulus containing i. You don't need anything about Laurent series. The only thing you need to know is

$$\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n$$

when $|q| < 1$.
What if $|z| > 1$?
Note that

$$\frac{1}{1-z} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}$$

Let's do the problem.
At $3i$ there is a singularity, and we want an annulus centered at 1.
Take

$$\begin{aligned}\frac{z^2}{z-3i} &= ((z-1)+1))^2\frac{1}{(z-1)+(1-3i)} \\ &= \frac{(1+(z-1))^2}{1-3i}\frac{1}{1-\frac{z-1}{3i-1}} \\ &= \frac{(1+w^2)}{1-3i}\sum_{n=0}^{\infty}\frac{1}{(3i-1)^n}w^n, w = z-1, |w| < |3i-1| = \sqrt{10}\end{aligned}$$

What if we want to do it in an annulus containing $4i$?
The same calculations hold, up to the second line:

$$\begin{aligned}
\frac{z^2}{z-3i} &= ((z-1)+1))^2 \frac{1}{(z-1)+(1-3i)} \\
&= \frac{(1+(z-1))^2}{1-3i} \frac{1}{1-\frac{z-1}{3i-1}} \\
&= \frac{1+(z-1)^2}{(1-3i)\frac{z-1}{3i-1}} \left(\frac{1}{-1+\frac{3i-1}{z-1}} \right) \\
&= \frac{1+(z-1)^2}{(z-1)} \frac{1}{1-\frac{3i-1}{z-1}} \\
&= \frac{1+\cdots}{\cdots} \sum_{n=0}^{\infty} (3i-1)^n \frac{1}{(z+1)^n}
\end{aligned}$$

For the last inequality, $|z-1| > \sqrt{10}$.
What if we want an expansion in the maximal annulus containing $4i$?
Now, to elaborate on something in the main local theorem.

Theorem 0.9. *(Morera's)*
Let $f : B_{R_0}(a) \to \mathbb{C}$. The following are equivalent:

- $f \in \mathscr{H}(B_{R_0}(a))$
- *f is continuous in $B_{R_0}(a)$ and for all solid 2-simplices $\triangle$ with boundary T, we have*
$$\int_T f(z)\,dz = 0$$
for some appropriate parameterization of T.

Definition 0.10. Let ϕ be a function. The support of ϕ, $\operatorname{supp}\phi$, is defined as $S = \overline{\{z \mid \phi(z) \neq 0\}}$.
Remark. The support is a closed set, and $\phi|_{S^c} \equiv 0$.

Definition 0.11. Let X be an open subset of C.
Then

$$C_0^\infty(X, \mathbb{C}) \overset{\text{def}}{=} C_0^\infty(x) = \{f : X \to \mathbb{C} \mid \text{ all} \partial_x^\alpha, \partial_y^\beta f \text{ exist in } X, \operatorname{supp} f \text{ is compact}\}$$

This is often called $\mathscr{D}(X)$
Any element of C_0^∞ is not holomorphic, as it is identically zero on an open set.

An example:

$$f(x) = \begin{cases} e^{-\frac{1}{|z|^2-1}} & |z| < 1 \\ 0 & |z| \geq 1 \end{cases}$$

Then $f(x) \in \mathscr{D}(\mathbb{C})$.

Theorem 0.10. *The following are equivalent:*

(a) $f \in \mathscr{H}(B_{R_0}(a))$

(b) $\forall R < R_0$, $f \in L^1(B_{R_0}(a))$ *(in the Lebesgue sense) and* $\frac{\partial f}{\partial \bar{z}} = 0$. *This means that for any* $\phi \in \mathscr{D}(B_{R_0}(a))$, $\int_{B_{R_0}(a)} f \frac{\partial \phi}{\partial \bar{z}}\, d\lambda^2 = 0$.

Proof. of Morera's theorem
Showing $(i) \implies (ii)$ is Goursat's lemma. So we will now prove $(ii) \implies (i)$.

Step 1

Define $F : B_{R_0}(a) \to \mathbb{C}$ defined by $z \mapsto \int_{S(z)} f(w)\, dw$, where $S(z) : I \to B_{R_0}(a)$ is given by $t \mapsto (1-t)a + tz$.

Goal

Our goal is to show that F is holomorphic in $B_{R_0}(a)$, and $F'(z) = f(z)$ for all z.

Step 2

By definition,

$$F' = \lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = \frac{1}{h}\left(\int_{I_2} + \int_{I_1}\right) = \frac{1}{h} \int_{I_3} f\, dz$$

Where I_1 is a line from z to a, I_2 is from a to a third point, and I_3 is from that third point to z. The last term goes to 0 as h goes to 0.

Theorem 0.11. *(Removable singularity)*
Let $f : B_R \setminus \{0\} \to \mathbb{C}$ *satisfy*

(a) $f \in \mathscr{H}(B_R \setminus \{0\})$

(b) $\frac{|f(z)|}{|z|} \to 0$ *as* $z \to 0$.

Then we can define $f(0)$ *such that* $f \in \mathscr{H}(B_R)$.

Proof. Define $f(z)z = F(z)$ for $z \neq 0$.
Define $F(0) = 0$.
Then F is continuous in $B_R(0)$: we will prove $F \in \mathscr{H}(B_R(0))$
So

$$\begin{aligned} f(z)z = F(z) &= \sum_{n=1}^{\infty} a_n z^{n-1} \\ &= \sum_{n=0}^{\infty} b_n z^n \end{aligned}$$

And we are done.
By Goursat, we have to prove that for all $\triangle$, $\int_T F\,dz = 0$.
There are some cases:

1. In case 1, $\int_T F\,dz = 0$, so we're done by Goursat.

2. In case 2, where T has 0 as a vertex, we can chop off a tiny piece of T, and then the integral is the integral over the new trapezoid we've created, plus the integral over a smaller triangle. The magnitude of this second integral is bounded above by the max of F on the triangle times the length of the triangle. This is bounded above by a constant times ε. So $\left|\int_T F\,dz\right| = 0$.

3. In case 3, T has zero on the boundary, but not a vertex. In this case, we do more subdivision shenanigans, and it just works by case 2.

4. In case 4, 0 is an interior point of $\triangle$. We can do more subdivision shenanigans, and use case 3 and it just works.

We are now done

■

We also have:
Suppose $\Delta u = 0$ in $B_R \setminus \{0\}$, it suffices to have $\frac{|u(z)|}{\ln|z|} \to 0$. Then $u \in C^2(B_R(a))$ and $\Delta u = 0$.

Theorem 0.12. $(i) \implies (ii)$
Let $f \in \mathscr{H}$. Then for all k, m, $\partial_x^k \partial_y^m f$ exists and is continuous, and $\frac{\partial f}{\partial \bar{z}} = \frac{1}{z}(\partial_x + i\partial_y)f = 0$.
Then $\frac{1}{2}(\partial_x + i\partial_y)f\phi\,dxdy = 0$ for all $\phi \in \mathscr{D}(B_R(a))$.

Proof. By Fubini,

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{C}}(\partial_x f)\phi dxdy + \cdots &= \frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\partial_x f(x,y)\phi(x,y)dx\right)dy \\
&= -\frac{1}{2}\int_{\mathbb{R}}\left(\int_{a_y}^{b_y} f\partial_x\phi\, dx\right)dy + 0 \\
&= -\frac{1}{2}\int_{\mathbb{C}} f\frac{\partial\phi}{\partial\overline{z}}\, d\lambda^2
\end{aligned}$$

■

We actually proved the following statement:
Let $x \in \Omega \subseteq \mathbb{R}^N$, u smooth, f smooth. Suppose we have a constant coefficient differential operator $f = \sum_{|\alpha|\leq N} C_\alpha \partial_x^\alpha(u)$
Then $u \in L^1(\Omega)$ and

$$\int u\left(\sum_{|\alpha|\leq N}(-1)^{|\alpha|}C_\alpha\partial_x^\alpha\phi\right)d\lambda^2 = \int_\Omega f\phi\, d\lambda^N$$

for all $\phi \in \mathscr{D}(\Omega)$.
So for a differential operator with constant coefficients, we can define a distribution of solutions to a PDE, such as $\triangle u = f$.

Lecture 5

Let $\alpha, \beta : I \to X$ be two loops in an open, connected subset of $\mathbb{C}$. When may we "continuously deform" these loops into each other?
What even is a "continuous deformation"?

Definition 0.12. If α, β are two loops in X, we say that α, β are freely homotopic, or $\alpha \overset{\circ}{\simeq} \beta$, if there exists a continuous $F : I \times I \to X$, such that

$$\begin{aligned}
F(t,0) &= \alpha(t), t \in I \\
F(t,1) &= \beta(t), t \in I \\
F(0,s) &= F(1,s), s \in I
\end{aligned}$$

Theorem 0.13. *(Homotopy Cauchy Formula)*

Let $X \subseteq \mathbb{C}$ be connected and open, and $f : X \to \mathbb{C}$ is holomorphic. Let $\gamma_0, \gamma_1 : I \to X$ be two piecewise smooth loops. If γ_0, γ_1 are freely homotopic in X, then

$$\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$$

Proof. **Step 1**

Claim. By assumption, there is a homotopy $H \in C(I \times I, X)$ between γ_0, γ_1. This function can be approximated for any $\varepsilon > 0$ by $H_\varepsilon : I \times I \to X$ in the following sense:

1. *H_ε is continuous.*

2. *We can chop $I \times I$ into squares. Every $[t_j, t_{j+1}] \times [s_k, s_{k+1}]$ is mapped into some $B_\rho(a) \subseteq X$, where a, ρ depend on j, k, where, t_i, s_i are elements of two partitions of I.*

3. *For all s, $H_\varepsilon(-, s_j)$ is linear on $[t_k, t_{k+1}]$, and for all t, $H_\varepsilon(t_j, -)$ is linear on $[s_k, s_{k+1}]$*

4. *$H_\varepsilon(0, s) = H_\varepsilon(1, s)$ for all $s \in I$.*

5.

$$\int_{\gamma_0} f\,dz - \int_{H_\varepsilon(-,s)} f\,dt \le \varepsilon$$
$$\int_{\gamma_1} f\,dz - \int_{H_\varepsilon(t,-)} f\,ds \le \varepsilon$$

Proof. of claim.
The proof of the existence of H_ε follows the same idea as the proof of:

Claim. Let $\phi : I \to \mathbb{R}$ be continuous. Then, for every $\varepsilon > 0$, there is an N such that there exists a piecewise linear ϕ_ε, with

$$\sup_I |\phi - \phi_\varepsilon| \le \varepsilon$$

And, if ϕ' exists and is continuous, then ϕ'_ε exists and is continuous, and further

$$\sup_I |\phi' - \phi'_\varepsilon| \le \varepsilon$$

Proof. On homework.

Step 2

Let $\tilde{\gamma_0} = H_\varepsilon(-, 0)$, and $\tilde{\gamma_1} = H_\varepsilon(-, 1)$ be piecewise linear loops.
We want to show that $\int_{\tilde{\gamma_0}} f\,dz = \int_{\tilde{\gamma_1}} f\,dz$, and then by part 5 of step 1, we would be finished. Let's do that now.
Dennis has drawn a picture of the unit square, with the bottom $\frac{1}{N}$ of it noted. This box, of length 1 and height $\frac{1}{N}$, he integrates over the boundary.
Goal:

$$\int_{\tilde{\gamma_0}} f\,dz = \int_{H_\varepsilon(-,\frac{1}{N})} f\,dz$$

The integrals of the left and right side of the box are equal, but have opposite orientation, so they cancel. If we can show the integral over the entire boundary of the small box is 0, we would be done.
We can chop up $[0, 1] \times [0, \frac{1}{N}]$ into a bunch of boxes. The integral over each of the small boxes is zero, because each small box is mapped into $B_\rho(a)$. The sum of these integrals, which is the integral over the box, is zero.

■

Index of a curve

Definition 0.13. Let $\gamma : I \to \mathbb{C} \setminus \{0\}$ be a piecewise smooth loop. We define the index as follows.

$$\text{ind}(\gamma, \rho) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z}$$

If you expand this out, things will cancel and this will always be an integer.
By the theorem we just proved, we can show that the index of two freely homotopic loops is the same.
Ahlfor's computation:
Let $h(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s)}\,ds$. This function is piecewise smooth by the fundamental theorem of calculus. $h(0) = 0$, $h(1) = \int_\gamma \frac{dz}{z}$. Ahlfors observes that if we consider $g(t) = e^{-h(t)}\gamma(t)$ is a nice, continuous, piecewise smooth function, so it has derivative

$$\begin{aligned} g' &= e^{-h}(-h')\gamma + e^{-h}\gamma' \\ &= e^{-h}\frac{\gamma'}{\gamma}\gamma + -e^{-h}\gamma' = 0 \end{aligned}$$

The derivative being zero tells us $g(0) = g(1)$. So $\gamma(0) = e^{-h(1)}\gamma(1)$, so $h(1) = 2i\pi N$ for some integer N.

Theorem 0.14. *Let λ be a piecewise smooth loop in $\mathbb{C}$, such that $p \notin \lambda(I)$. Then*

1. *The index* $\text{ind}(\gamma, p)$ *is an integer (we already proved this)*

2. $\text{ind}(\gamma, -) : \mathbb{C} \setminus \lambda(I) \to \mathbb{Z}$ *is continuous. That is, it is locally constant (constant on every connected component).*

3. *Suppose* $\lambda \overset{\circ}{\simeq} \nu$ *in* $\mathbb{C} \setminus \{p\}$*. Then* $\text{ind}(\lambda, p) = \text{ind}(\nu, p)$*.*

4. *If* $\gamma(I) \subseteq B_\rho(a)$*, with* $p \notin B_\rho(a)$*, then* $\text{ind}(\lambda, p) = 0$*.*

Let $X = \mathbb{C} \setminus \{0\}$*, also called* $\mathbb{C}^\times$*. Let* γ *be a loop around 0, and* β *be a loop not around zero. How do we show* $\alpha \overset{\circ}{\not\simeq} \beta$ *?*
First, you show that $\mathbb{C} \setminus \{0\} \cong S^1$ *is a homotopy equivalence.*
Then you life α, β *into the cover of* S^1*, etc.*
Alternatively, if $\alpha \overset{\circ}{\simeq} \beta$*, then* $\text{ind}(\alpha, 0) = \text{ind}(\beta, 0)$*. We can thus clearly see that* $\alpha \overset{\circ}{\not\simeq} \beta$*.*

Lecture 6

Theorem 0.15. *Suppose* $f \in \mathscr{H}(\{z \mid r_0 - \varepsilon < |a - z| < r + \varepsilon\}$
Then

(i)

$$f(z) = \frac{1}{2\pi i} \int_{C_{r_1}(a)} \frac{f(w)}{w - z} \, dw - \frac{1}{2\pi i} \int_{C_{r_0}(a)} \frac{f(w)}{w - z} \, dw$$

(ii) There exists $g_1(z) = \sum_{n=0}^\infty (z-a)^n c_n$ *which converges in* $|z-a| < r_1$*, and* $g_2(z) = \sum_{n=1}^\infty \frac{d_n}{(z-a)^n}$ *which converges in* $|z - a| > r_0$*, such that* $f(z) \equiv g_1(z) + g_2(z)$ *for* $r_0 < |z - a| < r_1$*.*

(iii) c_n, d_n *are unique.*

(iv) For all piecewise smooth loops $\lambda : S^1 \to \{r_0 \leq |z - a| \leq r_1\}$*,*

$$\begin{aligned} \text{ind}(\lambda, a) c_n &= \frac{1}{2\pi i} \int_\lambda \frac{f(w)}{(w - a)^{n+1}} \, dw, \; n \geq 0 \\ \text{ind}(\lambda, a) d_n &= \frac{1}{2\pi i} \int_\lambda (w - a)^{n-1} f(w) \, dw, \; n \geq 1 \end{aligned}$$

Corollary 0.16. *Suppose* $f : B_R(a) \setminus \{a\} \to \mathbb{C}$ *is holomorphic, and* $\lambda : S^1 \to B_R(a) \setminus \{a\}$ *is piecewise smooth. Then*

$$\int_\lambda f(w) \, dw = \text{ind}(\lambda, a) d_1$$

$f(z) = \frac{d_1}{z-a} + \sum_{n\neq -1} d_n(z-a)^n$.

Definition 0.14. The coefficient d_1 is called the residue of f at a, denoted $\text{res}_a f$.

Proof. (i) Fix z with $r_0 < |z-a| < r_1$, and consider the function

$$F(w) = \frac{f(w) - f(z)}{w - z}$$

This is holomorphic in the annulus $\{r_0 \leq |w-a| \leq r_1\} \setminus \{z\}$. Notice that $\lim_{w\to z} F(w) = f'(z)$, so F is bounded in a neighborhood of the singularity at z, so this singularity is removable.

Now, we have $r_0 e^{it} \overset{\circ}{\simeq} r_1 e^{it}$, with $H(s,t) = (1-s)e^{ir_0 t} + se^{ir_1 t}$. So

$$\int_{C_{r_0}(a)} F(w)\, dw = \int_{C_{r_1}(a)} F(w)\, dw$$

So

$$-f(z) \int_{C_{r_1}(a)} \int \frac{dw}{w-z} + \int_{C_{r_0}(a)} \frac{f(w)}{w-z}\, dw$$

Which is equal to

$$-f(z) \int_{C_{r_1}(a)} \frac{dw}{w-z} + \int_{C_{r_1}(a)} \frac{f(w)}{w-z}\, dw$$

Details of the rest of the proof posted by Denis on The Gauchu.

Remark. In the proof of (iv) you use the fact that if f is holomorphic and $\gamma : I \to \mathbb{C}$ smooth, then $\frac{d}{dt}(f \circ \gamma(t)) = f'(\gamma(t))\dot{\gamma}(t)$.
$d(f\circ\gamma) = df \circ d\gamma = \partial f \circ (\dot{\gamma}dt) = f'dz \circ (\dot{\gamma}dt) = (f'\dot{\gamma})dt$
We now study isolated singularities.

Definition 0.15. Let $f : B_\delta(a) \setminus \{a\} \to \mathbb{C}$ be holomorphic. The order of f at a, denoted $\omega(f,a)$, is the minimal power in Laurent's expansion of f in $0 < |z-a| < \delta$.

$$\omega(f,a) = N <=> f(z) = \sum_{n=N}^{\infty} b_n(z-a)^n, b_N \neq 0$$

$$\omega(f,a) = -\infty <=> f(z) = \sum_{n=-\infty}^{\infty} b_n(z-a)^n, b_n \neq 0 \text{ for infinitely many } n$$

Theorem 0.17. *Let $f : B_\delta(a) \setminus \{a\} \to \mathbb{C}$ be holomorphic.*

(i) $\omega(f,a) = -N$ for a finite $N > 0$ if and only if $f(z) = \frac{\phi(z)}{(z-a)^N}$, $N > 0$ finite, with ϕ holomorphic in $B_\delta(a)$, $\phi(a) \neq 0$, which happens if and only if $|f(z)| \to \infty$ as $z \to a$.

(ii) If $\omega(f,a) = -\infty$, then for all $\varepsilon > 0$, $\overline{f(B_\varepsilon(a) \setminus \{a\})} = \mathbb{C}$.

Proof. of (ii) Assume for the sake of contradiction that for some $\varepsilon > 0$, $\overline{f(\cdots)} \neq \mathbb{C}$. That means there exists a point $w_0 \in \mathbb{C}$ and a real $\rho > 0$, such that $\overline{B_\rho(w_0)} \cap f(\cdots) = \varnothing$.
So for all $0 < |z-a| < \delta$, $|f(z) - w_0| \geq \rho$.
Now, consider $F(z) = \frac{1}{f(z)-w_0}$. This is holomorphic in $B_\varepsilon(a) \setminus \{a\}$, and $|F(z)| \leq \frac{1}{\rho} < \infty$. So we can remove the singularity at a to make F holomorphic in $B_\varepsilon(a)$. We can factorize it as

$$F(z) = (z-a)^N \phi(z),\ \phi \in \mathscr{H}(B_\varepsilon(a)),\ \phi(a) \neq 0$$

Then

$$F = \frac{1}{f - w_0}$$

which is equivalent to

$$f(z) = w_0 + \frac{1}{F} = w_0 + \frac{\frac{1}{\phi(z)}}{(z-a)^N} = w_0 + \frac{1}{(z-a)^N}\psi(z) = \frac{1}{(z-a)^N}\sum_{n=0}^{\infty} c_n (z-a)^n$$

But then $\omega(f,a) \neq -\infty$, a contradiction.

Proof. of (i)
(i) is a string of 3 equivalent statements. You show $1 \implies 2 \implies 3 \implies 1$. This is easy, and for the final implication, you use part (ii).
Let $\hat{\mathbb{C}}$ denote the Riemann sphere, which is topologically the one point compactification of $\mathbb{C}$.

Definition 0.16. Let $a \in X \subset \hat{\mathbb{C}}$, X open in $\mathscr{T}_\infty$, and consider $f : X \to \hat{\mathbb{C}}$.
f is holomorphic in a neighborhood of a if

1. $a \in \mathbb{C}$, $f(a) \in \mathbb{C}$, with f holomorphic in the old sense.

2. $a \in \mathbb{C}$, $f(a) = \infty$ if and only if $F(z) = \begin{cases} \frac{1}{f(z)} & z \neq a \\ 0 & z = a \end{cases}$ is holomorphic in a neighborhood of 0.

3. $a = \infty$, $f(a) \in \mathbb{C}$ is holomorphic if and only if $F(z) = \begin{cases} f(\frac{1}{z}) & z \neq 0 \\ f(a) & z = 0 \end{cases}$ is holomorphic

4. $a = \infty, f(a) = \infty$ is holomorphic if and only if $F(z) = \begin{cases} \frac{1}{f(\frac{1}{z})} & z \neq 0 \\ 0 & z = 0 \end{cases}$ is holomorphic near 0 in the old sense.

In case 2, a is a pole of f, meaning $\omega(f,a) = N < 0$ is finite.

Theorem 0.18. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be holomorphic in a neighborhood of ∞. Then*

(i) $f(\infty) = 0$ *if and only if* $f(z) = \frac{1}{z^N}\phi(z)$ *for* $|z| > R$, $0 < N < \infty$, ϕ *holomorphic on* $|z| > R$, *and* $\phi \to A \neq 0, \neq \infty$ *as* $|z| \to \infty$.

(ii) $f(\infty) = \infty$ *if and only if* $f(z) = z^N\phi(z), |z| > R, 0 < N < \infty, \phi \to A \neq 0, \neq \infty$ *as* $|z| \to \infty$.

Exercise:
Suppose $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is holomorphic on $\hat{\mathbb{C}}$. Show that this is equivalent to f being a rational function.

Definition 0.17. Let $X \subset \mathbb{C}$ be open, connected. A map $f : X \to \hat{\mathbb{C}}$ is meromorphic if it is holomorphic in a neighborhood of every point of X in the extended sense.
The set of meromorphic functions is called $\mathscr{M}$. This is equivalent to a function $f : \mathbb{C} \to \mathbb{C}$ being holomorphic everywhere except an at most countable family of isolated poles.

Proposition 4. *Let $X \subset \mathbb{C}$ be open.*

(i) $\mathscr{M}(X)$ *is a field*

(ii) $f \in \mathscr{M}(X)$ *if and only if* $\frac{1}{f} \in \mathscr{M}(X)$.

(iii) $S_{\frac{1}{f}} = Z_f Z_{\frac{1}{f}} = S_f$

All of the above can be proven using the factorization theorem.

Lecture 7

Lecture 8

Let $X, \tilde{X}, \cdots \subset \mathbb{C}$ be domains. We say a map $p : X \to \tilde{X}$ is a cover if p is

1. $p(\tilde{X}) \supseteq X$ *(p is onto)*

2. *For all $a \in X$, there is an open $a \in V \subset X$, such that $p^{-1}(V) = \coprod_\alpha \mathcal{O}_\alpha$, where the $\mathcal{O}_\alpha$ are open, disjoint, connected sets such that $p|_{\mathcal{O}_\alpha} : \mathcal{O}_\alpha \to V$ is a homeomorphism for all α.*

Example 0.6. The basic example is the exponential map $\mathbb{C} \to \mathbb{C}^\times$. We defined $V_a = \mathbb{C} \setminus \{ta \mid t \leq 0\}$. The preimage of V_a under the exponential map will be disjoint strips in $\mathbb{C}$, with width depending on the argument of a.

A lift of the exponential function would be an F making this diagram commute

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "exp"] \\ X \arrow[ur, "F"] \arrow[r, "f"] & \mathbb{C}^\times \end{tikzcd}$$

Theorem 0.19. *(Uniqueness of lift)*
Let $\tilde{X} \xrightarrow{p} X$ *be a cover. Let* B *be a domain, and let* $\phi : B \to X$ *be continuous. Lt* $f_1, f_2 : B \to X$ *satisfy* $\phi = p \circ f_i$. *If there exists a* $b_0 \in B$ *such that* $f_1(b_0) = f_2(b_0)$, *then* $f_1 \equiv f_2$ *on* B.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ B \arrow[ur, "f_i"] \arrow[r, "\phi"] & X \end{tikzcd}$$

Theorem 0.20. *(Existence of lift)*
Let $\tilde{X} \xrightarrow{p} X$ *be a cover. Let* B *be a domain, and* $\phi : B \to X$ *continuous. For any* $b_0 \in B$, *fix* $\tilde{x}_0 \in p^{-1}(\{\phi(B_0)\})$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ B \arrow[r, "\phi"] & X \end{tikzcd}$$

$$\begin{tikzcd} & \tilde{x}_0 \arrow[d, "p"] \\ b_0 \arrow[r, "\phi"] & \phi(b_0) \end{tikzcd}$$

Then there exists continuous $f : B \to \tilde{X}$ *such that* $f(b_0) = x_0, p \circ f = \phi$ *on* B *if and only if* $\phi_*(\pi_1(B, b_0)) = (p \circ f)_*(\pi_1(B, b_0)) = p_* \circ f_*(\pi_1(B, b_0) \lhd p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ B \arrow[ur, "f"] \arrow[r, "\phi"] & X \end{tikzcd}$$

Corollary 0.21. *For*

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ X \arrow[r, "f"] & C^\times \end{tikzcd}$$

with f holomorphic, fix any $z_0 \in X$, fix any $\tilde{w}_0 \in \{\ln f(z_0)\}$ (so $e^{\tilde{w}_0} = f(z_0)$). Then

(i) There is a unique holomorphic F such that $f = e^F$ in X, $F(z_0) = \tilde{w}_0$ IF AND ONLY IF $f_*(\pi_1(X, z_0)) = \{1\}$.

(ii) If X is simply connected, then F always exists.

(iii) If $f(X)$ is a simply connected subset of a domain, then F always exists.

Theorem 0.22. *Let $f : X \to \mathbb{C}^\times$ be holomorphic (i.e. $f(z) \neq 0$ for all $z \in X$)*
Fix any $z_0 \in X$ and any w_0 such that $e^{w_0} = f(z_0)$ (i.e $w_0 \in \{\ln f(z_0)\}$).
Then

(i) If X is simply connected, then there is a unique $F \in \mathscr{H}(X)$ such that $e^F = f$ in X, and $F(z_0) = w_0$. F is given by

$$F(z) = w_0 + \int_{z_0}^{z} \frac{f'(w)}{f(w)}\, dw$$

The notation above makes sense because the integral doesn't depend on the curve between z and z_0.

(ii) If $f(X) \subset \mathcal{D} \subset \mathbb{C}^\times$, with $\mathcal{D}$ a simply connected domain, then there is a unique $F \in \mathscr{H}(X)$ such that $e^F = f$ in $X, F(z_0) = w_0$ given by

$$F(z) = w_0 + \int_{f(z_0)}^{f(z)} \frac{dw}{w}$$

Again, this makes sense because $\mathcal{D}$ is a simply connected domain on which $\frac{1}{z}$ is holomorphic, so the integral doesn't depend on the curve between z_0 and z.

Proof. (a) We have

$$F(z_0) = w_0 + \int_{z_0}^{z_0} dw = w_0$$

Here, we have used the fact that we are on a simply connected set to say $\int_{z_0}^{z_0} dw = 0$. We want $\frac{e^F}{f}|_{z_0} = \frac{e^{w_0}}{f(z_0)} = 1$. Let's compute the derivative. We have

$$\left(\frac{e^F}{f}\right)' = \frac{e^F F' f - e^F f'}{f^2}$$

$F' = \frac{f'}{f}$, so the above is identically zero.

(b) You can prove yourself .

■

We will consider the situation

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\exp"] \\ X \arrow[r, "\iota"] & \mathbb{C}^\times \end{tikzcd}$$

where ι is an inclusion. We want to think about branches of ln.
The standard branch is the one given by $X = \mathbb{C} \setminus \{t \mid t \in \mathbb{R}, t \leq 0\}$. Let $z_0 = 1$. Then $0 = w_0 \in \exp^{-1}(\{1\})$
So $\underbrace{\ln z}_{\text{in standard branch}} = \underbrace{\int_1^z \frac{dw}{w}}_{\ln \text{ in } \mathbb{R}} = \underbrace{\ln |z|}_{\text{in real analysis sense}} + i\phi$, with $-\pi < \phi < \pi$.
What about $\{\sqrt{z}\}$. Unfortunately, for the function $z \mapsto z^n$, this is a surjection from $\mathbb{C}^\times \to \mathbb{C}^\times$, so we don't have simply connectedness. That is $p_*(\pi_1(\mathbb{C}^\times)) \ncong \{1\}$. Luckily, the definition of roots can be expressed using logs:

Theorem 0.23. *Let X be simply connected, $f : X \to \mathbb{C}^\times$ holomorphic, $z_0 \in X$, w_0 such that $w_0^n = f(z_0)$. Then there exists a unique $g \in \mathscr{H}(X)$ such that $g^n = f$ in X, $g(z_0) = w_0$, given by*

$$g(z_0) = w_0 e^{\frac{1}{n}\int_{z_0}^z \frac{f'(w)}{f(w)}\, dw}$$

Proof. $\cdot \smile \cdot$

■

Denis would like to solve some problems now.
Suppose we have a domain consisting of the complement of of the boundary of a semicircle of radius 1 at 1, and the line $\{x + iy \mid x = 2, y \geq 0\}$. That is, we want $F \in \mathscr{H}(X)$ such that $e^{F(z)} = z$ in X. $F(1) = 0$.

$$F(3) = \text{“}\ln_* 3\text{”} = 0 + \underbrace{\int_1^3 \frac{dz}{z}}_{\text{needs to be in } X}$$

A line from 1 to 3 is not entirely in X. We take the integral over a semicircle from 1 to -1, and get

$$F(3) = i\pi + \int_0^1 \frac{-2dt}{-1-2t} = 2\underbrace{\int_0^1}_{=\ln(3)} + i\pi = \ln 3 + 2i\pi$$

Of course in the homework we have to compute more complicated things.
Let f be holomorphic in an open $X \ni a$, with $f'(a) \neq 0$. Then there is a small neighbourhood of a which is mapped biholomorphically onto its image by the inverse funtion theorem. What if $f'(a) = 0$?

Theorem 0.24. Let f be holomorphic in an open $X \ni a$. Let f' have a zero of order m at a. We know then that $f'(z) = (z-a)^m \phi(z)$, with $\phi(a) = 0$. Then

$$f(z) - f(a) = (\phi(z))^{m+1}$$

for some holomorphic ϕ, with $\phi(a) = 0, \phi'(a) \neq 0$.

Proof. ∎

Lecture 9

Last time, we showed that if $\Omega \subseteq \mathbb{C}$ is a simply connected domain, then for any f, there is a g, defined by

$$g(z) = w_0 + \int_{z_0}^{z} \frac{f'(w)}{f(w)}\, dw$$

So that $e^g = f$ in Ω. In the above, w_0 is such that $e^{w_0} = z_0$.
Now let w_0 be such that $w_0^n = f(z_0)$. Let

$$g(z) = w_0 e^{\frac{1}{n}\int_{z_0}^{z} \frac{f'(w)}{f(w)}\, dw}$$

Then $g^n(z) = f(z)$.
If we are on a domain $f(\Omega)$ which lies in a simply connected domain, then we can define

$$g(z) = w_0 + \int_{f(z_0)}^{f(z)} \frac{dw}{w}$$

Theorem 0.25. Let f be holomorphic, with $f(0) = 0$. Let $f'(0) = \cdots = f^{(N-1)}(0) = 0$, $f^{(N)}(0) \neq 0$.
Then there exists an open $V \subseteq \mathbb{C}$ containing 0 such that

1. *There exists a biholomorphism g between V and $B_\varepsilon(0)$ for some $\varepsilon > 0$.*

2. *$f(z) = (g(z))^N$ for $z \in V$. That is, the following diagram commutes*

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "g"] & B_{\varepsilon^N} \\
B_\varepsilon \arrow[ur, "z \mapsto z^N"'] &
\end{tikzcd}$$

3. *For all $w \in B_{\varepsilon^N}$, $w \neq 0$, there exist distinct $z_1, \ldots, z_n \in V$ such that $f(z_j) = w$ for all j.*

Corollary 0.26. *(Open Mapping Theorem)*
Let $X \subseteq \mathbb{C}$, be open $f \in \mathscr{H}(X)$. Then, unless f is constant, $f(X)$ will be open.

Proof. Let $w_0 \in f(X)$. Then there is a point $z_0 \in X$ with $f(z_0) = w_0$. By the theorem, $f(V) \supseteq B_\varepsilon$ for some ε.

■

Corollary 0.27. *(Maximum Principal)*
Let $X \subseteq \mathbb{C}$ be open, and let $f \in \mathscr{H}(X)$. Then for all $p \in X$, $|f(p)| < \sup_{p \in X} |f(p)|$. That is, f does not achieve its sup on X. If f can be continuously extended to the boundary of X, then $|f|$ achieves its maximum on this boundary.

Proof. Let $p \in X$. Then there exists an open neighborhood V of p such that the image under f is an open ball contained in $f(X)$, so there is a point in this ball which achieves a larger absolute value than p.

■

We will now prove the theorem given at the beginning of this lecture.

Proof. By our factorization theorem, $f(z) = z^N \phi(z)$, with ϕ holomorphic in a neighborhood of 0, with $\phi'(0) \neq 0$, and $\phi \neq 0$ in some neighborhood of 0.
We can find a ψ such that $\psi(z) =$ "$\sqrt[N]{\phi(z)}$", meaning $\psi(z)^N = \phi(z)$ in $B_\rho(0)$ for some $\rho > 0$.
So then $f(z) = (z\psi)^N = z^N \psi^N = g(z)^N$. Taking the derivative,

$$g(z)' = (z\psi(z))' = \psi(z) + z\psi'(z)$$

So $g(z)'|_{z=0} = \phi(0) \neq 0$, allowing us to apply a previous theorem.

■

Here's an example of a typical problem:

Example 0.7. Find a regular branch of $\ln(iz+\sqrt{1-z^2})$ in $\Omega = \mathbb{C} \backslash ((-\infty, -1] \cup [1, +\infty))$.
We want a $g(z)$ such that $e^g = \ln(iz + \sqrt{1-z^2})$ in this domain, which satisfies $g(0) = 0$. In the homework, all the branches you can find will be unique, although we do not need to prove this.
Let $f : \Omega \to \mathbb{C}^\times$ be given by $z \mapsto 1-z^2$. We want $g_1 \in \mathscr{H}(\Omega)$ such that $g_1(z)^2 = 1-z^2$ in Ω, and $g_1(0) = 1$. By a theorem, we have

$$g_1(z) = ie^{\frac{1}{2}\int_0^z \frac{f'}{f} dw} = e^{\frac{1}{2}\int_0^z -\frac{w}{1-w^2} dw}$$

Let g_2 be defined on Ω by $z \mapsto iz + g_1(z)$. We can see $g_2(0) = 0 \cdot 0 + 1 = 1$. If $g_2(z) = 0$, then $iz = g_1(z)$, so $-z^2 = g_1(z)^2$, which implies $-z^2 = 1 - z^2$, and so $0 = 1$. So $g_2(z) \neq 0$ for all $z \in \Omega$, so $g_2 : \Omega \to \mathbb{C}^\times$.

Now, $0 \in \{\ln(1)\}$, so by a theorem we can define

$$g(z) = 0 + \int_0^z \frac{g_2'(w)}{g_2(w)}\,dw = \frac{i + g_1'(z)}{iz + g_1(z)}, z \in \Omega$$

Let's solve a problem like this from the homework.
In $\Omega = \mathbb{C} \setminus [0, +\infty)$, find a regular branch $g(z)$ of $\ln(1+\sqrt{z})$ such that $g(-1) \in \{\ln(1 \pm i)\} = \ln(\sqrt{2}) \pm i\frac{\pi}{4} + i2\pi\mathbb{Z}$. Let's pick $\ln(\sqrt{2}) + \frac{i\pi}{4}$ and compute the limit

$$\lim_{\varepsilon \to 0^+} g(4 - i\varepsilon) = ?$$

Let's denote by f a branch of $\{1 + \sqrt{z}\}$ which is holomorphic in Ω such that $f(-1) = 1 + i$. We have $f(-1) = 1 + i$, $i \in \{\sqrt{-1}\}$, so by a previous theorem, we can define

$$f = 1 + ie^{\frac{1}{2}\int_{-1}^z \frac{1}{w}\,dw}$$

So $f(z) \in \{1 + \sqrt{\mathbb{Z}}\}$ for all $z \in \Omega$. Because $\ln(\sqrt{2}) + \frac{i\pi}{4} \in \{\ln(1+i)\}$, the same theorem then allows us to define

$$g(z) = \ln(\sqrt{2}) + \frac{i\pi}{4} + \int_{-1}^z \frac{f'(w)}{f(w)}\,dw$$

This is a valid branch, but it makes solving for $\lim_{\varepsilon\to 0^+} g(4 - 1\varepsilon)$ a pain in the neck. So let's use the statement from earlier about $f(\Omega)$. What is the range of f?
For $0 < \varphi_z < 2\pi$, $z = |z|e^{i\varphi_z}$. Consider the curve $t \mapsto e^{i[(1-t)\pi + t\varphi_\zeta]}$. If $|\zeta| = 1$, then

$$ie^{\frac{1}{2}\int_{-1}^\zeta \frac{dw}{w}} = e^{\frac{i\pi}{2}}e^{i\frac{\varphi_\zeta - \pi}{2}} = e^{i\frac{\varphi_\zeta}{2}}$$

Anyways, the punchline is $f : \Omega \to \{\operatorname{Im} z > 0\}$.
So we use the formula $g =$ "$\ln f$" $= \ln(\sqrt{2}) + \frac{i\pi}{2} + \int_{f(-1)}^{f(z)} \frac{dw}{w}$. We calculate $f(-1) = i + i$, so we integrate over a curve starting at $1 + i$ and going to $f(z)$.
We have $f(4 - i0^+) = -1 + i0^+$. There are several curves we could integrate over, but Denis says the final answer will be the result of plugging in $\ln(\sqrt{2}) + \frac{i\pi}{2} - \ln(\sqrt{2}) + i\left(\frac{\pi}{4} + \frac{\pi}{2}\right) = i\pi$.

Lecture 9

Let $f \in \mathscr{H}(B_r(a) \setminus \{a\})$. Then $f(z) = \sum_{n=0}^\infty c_n(z-a)^n + \sum_{n=1}^\infty \frac{dn}{(z-a)^n}$, and we call $d_1 = \operatorname{res}_a f$ the residue of f at a.
$\gamma \overset{\circ}{\simeq} C_\rho(a)$ in $B_r(a) \setminus \{a\}$ implies $\int_\gamma f\,dz = \int_{C_\rho(a)} dz = 2\pi i d_1$.

Theorem 0.28. *(Residue Theorem)*
Let $\Omega \subseteq \mathbb{C}$ be simply connected. Let (a_n) be a sequence of isolated points in Ω. Let $S = \cup_n \{a_n\}$, and let $f \in \mathscr{H}(\Omega \setminus S)$. Then for any $\gamma : S^1 \to \Omega \setminus S$ which is piecewise smooth, we have

$$\int_\gamma f\, dz = 2\pi i \sum_n \operatorname{ind}(\gamma, a_n) \operatorname{res}_{a_n} f$$

Proof. A proof goes something like this: if $\gamma \overset{\circ}{\simeq} p$ for some $p \in \Omega$, then there is a homotopy $\phi : S^1 \times I \to \Omega$. $\phi(S^1 \times I) = M$ which is compact. Let $P = \underline{\{z \in \Omega \mid \operatorname{ind}(\gamma, z) \neq 0\}}$. This is a compact subset of Ω. Take $(M \cup P) \cap S$. This is a finite set $\{a_1, \dots, a_n\}$.
So, $f \to \tilde{f} = f - \sum_{n=1}^\infty \frac{dn}{(z-a_1)^n} - \sum_{n=1}^\infty \frac{d_{2n}}{(z-a_2)^n} - \cdots - \sum_{n=1}^\infty \frac{d_{Nn}}{(z-a)^n}$. We've killed off the singularities, in the words of Denis, so the integral of $\tilde{f}$ is zero. This gives the result vaguely.
A careful proof can be found in the notes.

Theorem 0.29. *(Rouche's Theorem)*
Let $\Omega \subseteq \mathbb{C}$ be simply connected, and suppose $f \in \mathscr{H}(\Omega)$, $r : S^1 \to \Omega \setminus z_f$ a piecewise smooth loop, where z_f is the set of points where f is zero. Let $\Delta f \in \mathscr{H}(\Omega)$ be a perturbation, $|\Delta f| < |f|$ on $\operatorname{im} \gamma$.
Then $f + \Delta f \neq 0$ on $\operatorname{im} \gamma$ and

$$\sum_{a \in z_f} \operatorname{ind}(\gamma, a)\omega(a, f) = \sum_{b \in z_{f+\Delta f}} \operatorname{ind}(\gamma, b)\omega(b, f + \Delta f)$$

Example 0.8. For all real $\lambda > 1$, $z = \lambda - e^{-z}$ has exactly one solution in $\{\operatorname{Re} z \geq 0\}$ which is real
We can rewrite $\underbrace{(z-\lambda)}_{f} + \underbrace{e^{-z}}_{\Delta f} = 0$. $f(z)$ is only zero at $z = \lambda$ if $\operatorname{Re} z \geq 0$. If we draw a box around λ and make it larger, then something goes to zero.
$|\Delta f| \leq |f|$ on γ, and $|\Delta f| = |e^{iy}| = 1$. So

$$\begin{aligned} f(z) &= |iy - \lambda| \\ &= \sqrt{\lambda^2 + y^2} \\ &> 1 \end{aligned}$$

Now

$$\begin{aligned} |\Delta f(z)| &= |e^{-N-iy}| \\ &= e^{-N} \\ &< 1 \end{aligned}$$

Detailed work in Denis' notes.

Theorem 0.30. *(Argument Principle)*
Suppose $\Omega \subseteq \mathbb{C}$ is a simply connected domain, and suppose $f \in \mathscr{M}(\Omega)$ (f is meromorphic on Ω). Let $\gamma : S^1 \to \Omega \setminus (S_f \cup Z_f)$ (the singular parts and zeroes) be piecewise smooth.
Then $f \circ \gamma$ is a piecewise smooth loop. Then

$$\operatorname{ind}(f \circ \gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'}{f} \, dw = \sum_{a \in S_f \cup Z_f} \operatorname{ind}(\gamma, a) \omega(a, f)$$

Remark. Z_f - zero set, S_f - set of poles are discrete subsets of Ω.

Proof. The above is equal to $\frac{1}{2\pi i} \int_{f \circ \gamma} \frac{dw}{w} = f' |_{\gamma(t)}^{\dot{\gamma}(t)}$. Idk I'm very lost and sleepy lol.
Denis writes $\frac{f'}{f} \in \mathscr{H}(\Omega \setminus (S_f \cup Z_f))$.
By the residue theorem,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f} \, dz = \sum_{a \in S_f \cup Z_f} \operatorname{ind}(\gamma, a) \operatorname{res}_a \frac{f'}{f}$$

If $a \in Z_f$, then $f(z) = (z-a)^N \phi(z)$, $\phi'(a) \neq 0$. So $f'(z) = N(z-a)^{N-1}\phi + (z-a)^N \phi'(z)$.
So

$$\frac{f'}{f} = \frac{N(z-a)^{N-1}\phi + (z-a)^N \phi'(z)}{(z-a)^N \phi} = \frac{\omega(f, a)}{z-a} + \underbrace{\frac{\phi'}{\phi}}_{\text{holomorphic}}$$

So $\operatorname{res}_a \frac{f'}{f} = \omega(f, a)$.
Now we can prove Rouche's theorem.

Proof. of Rouche's Theorem
We have $f \in \mathscr{H}(\Omega)$, $\gamma : S^1 \to \Omega \setminus Z_f$, and $f + \Delta(f) \in \mathscr{H}(\Omega)$ a perturbation. $\gamma : S^1 \to (\Omega \setminus Z_{f+\Delta f})$. We want to compare $\sum_{z \in Z_f} \operatorname{ind}(\gamma, a)\omega(a, f)$ with the same expression but with f replaced by $f + \Delta f$.
It will be enough to show

$$\int_\gamma \frac{f'}{f} \, dz = \int_\gamma \frac{f' + \Delta f'}{f + \Delta f} \, dz$$

Let $h = \frac{f + \Delta f}{f}$. $S_h \subset Z_f$, $Z_h \subset Z_{f+\Delta f}$. h is meromorphic, so h is holomorphic in a neighborhood of $\operatorname{im} \gamma$.
Now

$$\frac{f' + \Delta f'}{f + \Delta f} = \frac{f'}{f} + \frac{h'}{h}$$

Lecture 10

Recap

- Let $X \subset \mathbb{C}$ be a simply connected domain, $f \in \mathscr{H}(X \setminus S)$, with $S = \cup_n \{a_n\}$ a discrete set and $\gamma : S^1 \to X \setminus S$ piecewise smooth loop, then the residue theorem states
$$\int_\gamma f\,dz = \sum_{a \in S} \operatorname{ind}(\gamma, a_n) \operatorname{res}_{a_n} f$$
- Let $f \in \mathscr{M}(X)$, and $\gamma : S^1 \to X \setminus (S_f \cup Z_f)$. Then
$$\operatorname{ind}(f \circ \gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'}{f}\,dz = \sum_{a \in S_f \cup Z_f} \operatorname{ind}(\gamma, a) \omega(a, f)$$
- Rouche: Let X be simply connected, $f \in \mathscr{H}(X)$, $\gamma : S^1 \to X \setminus Z_f$ piecewise smooth, and $\Delta f \in \mathscr{H}(X)$ such that $|\Delta f| < |f|$ on $\gamma(S^1)$. Then
$$\sum_{a \in Z_f} \operatorname{ind}(\gamma, a)\omega(a, f) = \sum_{b \in Z_{f+\Delta f}} \operatorname{ind}(\gamma, b)\omega(b, f)$$

Proof. (Of Roche's Theorem (continued))
$\gamma : S^1 \to X \setminus Z_{f+\Delta f}$. We need to show
$$\int_\gamma \frac{f'}{f}\,dz = \int_\gamma \frac{f' + \Delta f'}{f + \Delta f}\,dz$$
Last time, we got to
$$\frac{f' + \Delta f'}{f + \Delta f} = \frac{f'}{f} + \frac{h'}{h},\ h = \frac{f + \Delta f}{f}$$
Comparing these two integrals, the only thing to verify is that indeed
$$\int_\gamma \frac{h'}{h}\,dz = 0$$
By definition, $Z_h = Z_{f+\Delta f}, S_h = Z_f$. $\gamma : S^1 \to X \setminus (S_h \cup Z_f)$. We must compute $\operatorname{ind}(h \circ \gamma, 0)$. This can be seen to be zero using certain geometric considerations, which can be found on the Gaucho.

Claim. $\operatorname{im}(h \circ \gamma)$ *is either in the upper or lower half plane, but not both.*

Proof. This follows from stuff on Gaucho I guess sorry.
I guess this finishes the proof...

∎

Conformal Maps

Let $X \subset \mathbb{C}$ be a domain.

Definition 0.18. We define the automorphism group of X to be

$$\mathrm{Aut}(X) \stackrel{\text{def}}{=} \{f \mid f : X \to X \text{ bijective, } f \in \mathscr{H}(X)\}$$

It turns out that if $f \in \mathrm{Aut}(X)$, then $f^{-1} \in \mathscr{H}(X)$ as well, so $f^{-1} \in \mathrm{Aut}(X)$. This can be done using the inverse function theorem, and arguing by cases, as well as remembering that if $f'(a) = 0$ then $f(z) - f(a) = (\phi(z))^N$ for some reason? for some ϕ which is a bijection and $\phi'(a) \neq 0$, and some $N > 1$. But then f is not injective near a.

Theorem 0.31. *(Riemann Mapping Theorem)*
Let $X \subset \mathbb{C}$ be a simply connected domain. Then there exists a biholomorphic $f : X \to B_1(0)$ which is bijective. Moreover, for all f_1, f_2 biholomorphic bijections between X and $B_1(0)$ such that $f_1(p) = f_2(p) = 0$ for some fixed p, then $f_1 = e^{it_0} f_2$ for some $t_0 \in \mathbb{R}$.

Proof. We will prove this later.

■

Theorem 0.32. $\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b \mid a, b \in \mathbb{C}, a \neq 0\}$.

Proof. Take any $f \in \mathrm{Aut}(\mathbb{C})$. Then $f = \sum_{n=0}^{\infty} c_n z^n$. If we can show that $c_n \equiv 0$ for all n above a certain point, then f will be a polynomial, so we must conclude it is linear by the fundamental theorem of algebra (due to f being bijective).
Suppose $c_n \neq 0$ for infinitely many n. Then

$$g(z) = f(\frac{1}{z}) = \sum_{n=0}^{\infty} \frac{c_n}{z^n}$$

has the property that $g(B_\varepsilon)$ is dense in $\mathbb{C}$ for all $\varepsilon > 0$. So $f(\mathbb{C} \setminus B_R) \cap f(B_R) \neq \varnothing$, which contradicts that f is injective (remember $g(B_\varepsilon) = f(\mathbb{C} \setminus B_{\frac{1}{\varepsilon}})$.
So we must conclude that $f(z) = c_n(z-a)^n$. If $n > 1$, then f is not injective, hence $c_n = c_1 \neq 0$, and $n = 1$.

■

Theorem 0.33. $\mathrm{Aut}(\hat{\mathbb{C}}) = Mob$*, where*

$$Mob \stackrel{\text{def}}{=} \{z \mapsto \frac{az+b}{cz+d} \mid ad - bc \neq 0\}$$

It is a good exercise to verify that $Mob \cong \mathrm{GL}_2(\mathbb{R})$.

Proof. Let $f \in \mathrm{Aut}(\hat{\mathbb{C}})$.
If $f(\infty) = \infty$, then $f \in \mathrm{Aut}(\mathbb{C})$, and we're done.
Without loss of generality, suppose $f(a) = \infty$. Let $g(z) = a + \frac{1}{z}$. Then $f \circ g \in \mathrm{Aut}(\hat{\mathbb{C}})$.
But then $(f \circ g)(\infty) = \infty$, so we're done.

■

Let $\triangle = B_1(0)$, the open unit disk.

Lemma 2. (Schwartz Lemma)
Let $f \in \mathscr{H}(\triangle), f(\triangle) \subseteq \triangle$.

1. *If $f(0) = 0$, then $|f(z)| \leq |z|$ for all z. Moreover, if $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then f is a rotation.*

2. *To be covered next time.*

Proof. (Sketch)
Consider $\phi(z) = \frac{f(z)}{z}$. If $f(0) = 0$ and f is holomorphic, then $\frac{1}{z}\sum_{n=0}^{\infty} c_n z^n = \sum_{n=1}^{\infty} c_n z^{n-1} \in \mathscr{H}(\triangle)$. What is the sup of $|\phi|$ over $\triangle$? This is $\lim_{r\to 1^-} \max_{\partial B_r} |\phi|$.
This is equal to $\lim_{r\to 1^-} \max_{|z|=r} \frac{|f(z)|}{|z|} = \lim_{r\to 1^-}(\frac{1}{r}\max_{|z|=r}|f(z)|) \leq \lim_{r\to 1^-}(\frac{1}{r}1) = 1$.

Lecture 11

We will now actually do the proof instead of just a sketch.

Proof. The key idea will be using $f(z) + \sum_{n=1}^{\infty} c_n z^n$, and

$$\begin{cases} \phi(z) = \frac{f(z)}{z} = \sum_{n=0}^{\infty} c_{n+1} z^n \in \mathscr{H}(\triangle) \\ \phi(0) = f'(0) \end{cases}$$

We will proceed with the proof.
We know that ϕ is holomorphic. So by the maximum principle,

$$\begin{aligned} \sup_{\triangle} |\phi| &= \lim_{r\to 1^-}\left(\max_{|z|=r} |\phi(z)|\right) \\ &= \lim_{r\to 1^-}\left(\frac{1}{r}\max_{|z|=r}|f(z)|\right) \\ &\leq \lim_{r\to 1^-}(\frac{1}{r}1) \\ &= 1 \end{aligned}$$

So for all $z \in \triangle$, $|f(z)| \leq |z|$. By the first part, $|\phi(0)| \leq 1 = \sup_{\triangle}|\phi|$. But $\phi(0) = f'(0)$, so $|f'(0)| \leq 1$, and $|f'(0)| = 1 \implies |\phi(0)| = 1$. So $\phi(z)$ is constant by the maximum principal, because it achieves it's max on an interior point. So $\phi(z) = e^{it_0}$ for some t_0, meaning $f(z) = ze^{it_0}$.

$\blacksquare$

Now, our goal is to figure out what $\mathrm{Aut}(\triangle)$ looks like.

Remark. For any simply connected $X \subseteq \mathbb{C}$, with $X \neq \mathbb{C}$, then there is a biholomorphism between X and $\triangle$. This is the Riemann mapping theorem.
However, there is NO biholomorphism between $\triangle$ and $\mathbb{C}$. To see this, suppose there is such a biholomorphism. But this is a bounded entire function, and so is constant. There is a homeomorphism, and even a diffeomorphism between $\mathbb{C}$ and $\triangle$. But a biholomorphism is, morally, too much to ask for: holomorphic functions satisfy a system of PDEs, and any smooth function satisfying these systems must be highly exceptional. To ask for this map to also be bijective, it is too much.

Automorphisms of $\triangle$

"And now exposition will get a little bit violent" - Denis. For any $|a| < 1$, consider

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}, \ |z| \le 1$$

We can see that

- For all $|a| < 1$, $\varphi_a \in \mathscr{H}(B_{1+\varepsilon_a})$, for some $\varepsilon_a > 0$.
- $\varphi_0(z) = z, \varphi_a(a) = 0, \varphi_a(0) = -a$.
- Moreover, we can quickly compute the derivative to get $\varphi_a'(0) = 1 - |a|^2$, $\varphi_a'(a) = \frac{1}{1-|a|^2}$.

Our goal will be to develop some sort of calculus or formalisms for this family of maps parameterized by a. We can see that

- $(\varphi_{-a} \circ \varphi_a)(z) \equiv z$ for $|z| \le 1$.
- $|\varphi_a(e^{it})| = \cdots = e^{-it}\frac{e^{it}-a}{\overline{(e^{it}-a)}}$, so $\varphi_a(S^1) \subseteq S^1$.
- The above implies $\varphi_a : \triangle \to \triangle$ is bijective. In other words, $\varphi_a \in \mathrm{Aut}(\triangle)$ for all $|a| < 1$.

What does $\mathrm{Aut}(\triangle)$ look like without these φ_a? It has to at least include rotations: each rotation fixes the origin, and only one φ_a fixes the origin (namely φ_0). As it turns out, this is all of them.

Theorem 0.34. *Let $f \in \mathrm{Aut}(\triangle)$, let $a \in \triangle$ satisfy $f(a) = 0$ (such a point exists because f must be a biholomorphism, so in particular a bijection). Then*

$$f = e^{i\theta}\varphi_a$$

for some $\theta \in \mathbb{R}$.

Proof. First, note $(f \circ \varphi_{-a})(0) = f(\varphi_{-a}(0)) = f(a) = 0$. So $f \circ \varphi_{-a}$ fixes the origin, so we may apply the Schwartz lemma to get $\left|(f \circ \varphi_{-a})'\right| \leq 1$.
Further,

$$\begin{aligned}(f \circ \varphi_{-a})'(0) &= f'(a)\varphi_{-a}'(0) \\ &= f'(a)(1 - |a|^2)\end{aligned}$$

So $\left|f'(a)\right|(1 - |a|^2) \leq 1$. Consider $\varphi_a \circ f^{-1}$. This fixes the origin once again, so we may once again apply the Schwartz lemma. By a similar computation,

$$\left|(\varphi_a \circ f^{-1})'(0)\right| \leq 1$$

So $(\varphi_a \circ f^{-1})'|_0 = \varphi_a'|_a (f^{-1})'|_0 = \frac{1}{1-|a|^2}\frac{1}{f'(a)}$, and so $\left|(f \circ \varphi_{-a})'(0)\right| = 1$. But by the Schwartz lemma, this is just a rotation:

$$f(\varphi_{-a}(z)) \equiv e^{i\theta}z$$

for some θ. So $f(z) \equiv e^{i\theta}\varphi_a(z)$

■