CS 177 Midterm Exam 1

Applications of Probability in Computer Science, Fall 2024

Wednesday, October 30, 2024 from 3:00-3:50pm

Your Name: **Row/Seat Number:**

Your ID #(e.g., 123456789) **UCINetID (e.g., ucinetid@uci.edu)**

- Total time is **50 minutes**. Please look through all questions and organize your time wisely.
- Please put your name and ID **on every page**.
- Please write your **row and seat number** on the cover page.
- **Electronic devices are not allowed.**
- For full credit, be sure to **write clearly** and **show all your work**.
- When computing probabilities or expected values, **enter the numerical value of your answer in the provided box**, and also include equations showing your calculations. You may express numerical values as either fractions or decimals.
- You may use **one** (two-sided) 8.5x11-inch sheet of (your own) handwritten notes.
  You may keep these notes, they do not need to be turned in.
- The final page is a sheet containing potentially useful formulas.
  Please separate the formula sheet from the other exam pages, and do not turn it in.
- Have a question? **Please raise your hand for help.**

Problems

Total: *(50 points)*

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Name: $\boxed{}$ ID#: $\boxed{}$

Problem 1 The Personal Identification Number. *(20 points)*

An account has a personal identification number (PIN) that is exactly three digits long, where each digit can be any of $\{0, 1, 2, \ldots, 9\}$. Unfortunately you recently changed your PIN, and now you can't remember it! But, you do know that your PIN is defined by non-repeating digits in increasing or decreasing order. For example, it might be *358* or *853*, but not *355* and not *385*.

a) *You try to access your account by entering random PINs. If you randomly guess one PIN with non-repeating digits in increasing or decreasing order, what is the probability you are correct?*

$$P(\textit{Guess PIN}) = \boxed{}$$

b) *You decide to repeatedly enter random PINs to access your account. Each entry is an independent sample as in part (a), so it is possible that the same code will be tried more than once. If it takes 5 seconds to try one PIN, what is the expected time needed to access the account?*

$$E[\textit{Time to Access}] = \boxed{}$$

c) *Suppose that the first 70 PINs you randomly guess are incorrect. You continue guessing random PINs, possibly with repetition, until the account is accessed. Given that the first 70 PINs failed, what is the expected total time (including the 70 failures) needed to access the account?*

$$E[\textit{Time to Access}] = \boxed{}$$

d) *Suppose the account has a security feature where if the incorrect PIN is entered 15 times, your account becomes locked and can no longer be accessed. What is the probability that your account becomes locked before you correctly guess the PIN?* ***Your answer may involve an exponent, and does not need to be evaluated numerically.***

$$P(\textit{Account Lock}) = \boxed{}$$

Name: ID#:

Problem 2 The CAPTCHA. *(22 points)*

You are developing a web security system that asks users to recognize partially-hidden objects in images, to verify that they are people and not AI agents. You show each user 3 images, and they label them independently. Let $Y = 1$ if the user is a person, $Y = 0$ if the user is an AI agent, and $X \in \{0, 1, 2, 3\}$ be the number of correctly recognized objects. Assume that $P(Y = 1) = \frac{1}{3}$.

If the user is a person ($Y = 1$), they have probability 3/4 of recognizing each object correctly. If the user is an AI agent ($Y = 0$), they have probability 1/2 of recognizing each object correctly.

a) *What is $P(Y = 1 \mid X = 3)$, the probability that the user is a person given that they recognize all three objects correctly?*

$$P(Y = 1 \mid X = 3) = \boxed{}$$

b) *What is $P(Y = 1 \mid X = 1)$, the probability that the user is a person given that they recognize only one object correctly?*

$$P(Y = 1 \mid X = 1) = \boxed{}$$

c) *What is $E[X]$, the expected number of objects that are correctly recognized?*

$$E[X] = \boxed{}$$

Name: ______ ID#: ______

Problem 3 Truth and Falsehood. *(8 points)*

For each statement below, please **completely fill in the bubble** next to either **True** or **False**. Only select **True** if a statement is *always* true, for any distribution of the referenced discrete random variables. You do *not* need to explain or justify your answers.

a) $\sum_y p_{X|Y}(x \mid y) = 1$.

○ **True** ○ **False**

b) $p_{X|Y}(x \mid y) \geq p_{XY}(x, y)$.

○ **True** ○ **False**

c) If X and Y are conditionally independent given $Z = z$, then they are also independent.

○ **True** ○ **False**

d) The expected value of a geometric random variable is equal to its median.

○ **True** ○ **False**

e) $g(E[x]) = E[g(x)]$ for any continuous and differentiable function g.

○ **True** ○ **False**

f) If X is a random variable and $Y = aX + b$, then $E[Y^2] = a^2 E[X^2] + b^2$.

○ **True** ○ **False**

g) For the coupon collectors' problem, if the number of coupon types N is large, then the expected number of trials needed to collect all N distinct coupon types is approximately N^2.

○ **True** ○ **False**

h) Consider a hash table with $N = 2^{32} \approx 10^9$ bins. If only 10,000 items have been stored in the table, then the probability of a collision (two items being hashed to the same bin) is small.

○ **True** ○ **False**

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Common Discrete Random Variables

Geometric For a geometric variable X with success probability $0 < q \le 1$,

$$p_X(x) = (1-q)^{x-1}q, \qquad x \in \{1,2,3,\ldots\}.$$
$$F_X(x) = P(X \le x) = 1-(1-q)^x, \qquad x \in \{1,2,3,\ldots\}.$$
$$E[X] = \sum_{x=1}^{\infty} x p_X(x) = \frac{1}{q}.$$
$$\mathrm{Var}[X] = E[X^2] - E[X]^2 = \frac{1-q}{q^2}.$$

Binomial For a binomial variable X with n trials and success probability $0 \le q \le 1$,

$$p_X(x) = \binom{n}{x} q^x (1-q)^{n-x} = \frac{n!}{(n-x)!x!} q^x (1-q)^{n-x}, \qquad x \in \{0,1,\ldots,n\}.$$
$$E[X] = \sum_{x=0}^{n} x p_X(x) = nq.$$
$$\mathrm{Var}[X] = E[X^2] - E[X]^2 = nq(1-q).$$

Uniform For a random variable X with a discrete uniform distribution between a and b,

$$p_X(x) = \begin{cases} 1/(b-a+1), & \text{if } x \in \{a, a+1, \ldots, b-1, b\}. \\ 0, & \text{otherwise.} \end{cases}$$
$$E[X] = \sum_{x=-\infty}^{\infty} x p_X(x) = \frac{a+b}{2}.$$
$$\mathrm{Var}[X] = E[X^2] - E[X]^2 = \frac{(b-a+1)^2 - 1}{12}.$$

Summations and Series

$$\sum_{i=0}^{\infty} q^i = \frac{1}{1-q}, \qquad \text{if } 0 < q < 1.$$
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$