Jones-matrix formalism as a representation of the Lorentz group

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It is shown that the two-by-two Jones-matrix formalism for polarization optics is a six-parameter two-by-two representation of the Lorentz group. The attenuation and phase-shift filters are represented, respectively, by the three-parameter rotation subgroup and the three-parameter Lorentz group for two spatial dimensions and one time dimension. The Lorentz group has another three-parameter subgroup, which is like the two-dimensional Euclidean group. Optical filters that may have this Euclidean symmetry are discussed in detail. It is shown that the Jones-matrix formalism can be extended to some of the nonorthogonal polarization coordinate systems within the framework of the Lorentz-group representation. © 1997 Optical Society of America [S0740-3232(97)01409-9]

1. INTRODUCTION

The Lorentz group was introduced to physics as the language of space–time symmetries of relativistic particles.[1,2] However, this group serves useful purposes in many other branches of physics, including optical sciences. In recent years the Lorentz group served as the underlying language for squeezed states of light. It was Dirac who first observed that the Lorentz boost is a squeeze transformation.[3] In 1963 Dirac[4] constructed representations of the Lorentz group by using coupled harmonic oscillators. Indeed, Dirac's oscillator representation forms the theoretical foundations of squeezed states of light.[5–7] This aspect of the Lorentz group is by now well known in the optics community, and the Lorentz group is one of the theoretical tools in optics.

The squeezed state is not the only branch of optics that requires the Lorentz group. In 1981 Bacry and Cadihilac considered application of the Lorentz group for Fourier optics.[8] In 1983 Sudarshan *et al.* constructed a representation of the Lorentz group for para-axial wave optics.[9] Wavelets are known to be representations of the Lorentz group.[10] It was noted recently that the Lorentz group serves as the underlying language for reflections and refractions.[11] In addition, in their recent paper[12] Gutierrez *et al.* pointed out the relevance of the Lorentz geometry to three-dimensional concentrators.

More recently, the relevance of Lorentz transformations to polarization optics was discussed by several authors.[11,13–16] Indeed, in their recent paper[16] Opatrny and Perina showed that rotation and boost matrices from the two-by-two representation of the Lorentz group describe energy-conserving and -nonconserving optical filters, respectively. In our recent paper we pointed out that polarization optics can be formulated in terms of the six-parameter Lorentz group.[17]

The purpose of this paper is simple: The standard language for optical polarizations is the Jones-matrix formalism, and it has a long history.[18–20] Here we show that this formalism is a representation of the six-parameter Lorentz group. The Lorentz group also has a long history.

Among the many representations of this group, it was shown in one of our earlier papers[17] that the bilinear conformal representation[2] is most convenient for polarization optics. Unfortunately, however, we were not then able to connect the representation with the Jones-matrix formalism, and we complete the task in the present paper.

The six-parameter Lorentz group has three major subgroups. They are the three-parameter rotation group, the three-parameter $O(2, 1)$ group, and the three-parameter $E(2)$-like group. The mathematics of the rotation group is well known. It is shown in Ref. 17 that the rotation group is the underlying language for optical filters that cause phase shifts between the two transverse components. The $O(2, 1)$-like groups were extensively discussed recently in connection with squeezed states of light. It is known that the $O(2, 1)$-like group is the proper language for attenuation filters.[15,17]

The $E(2)$-like group is useful in helping us to understand the internal space–time symmetries of massless particles[21] but is relatively new in optics.[22] We discuss optical filters that may possess this $E(2)$-like sym-

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metry. Since this discussion requires a detailed knowledge of the six-parameter Lorentz group, and since this group is relatively new in optics, we include a systematic introduction to the Lorentz group that is applicable to three spatial dimensions and one timelike dimension.

The rotation and squeeze transformations that we discuss are directly applicable to coordinate transformations. Thus they are applicable to the case when the polarization occurs along a skewed or squeezed coordinate system. It is known that one of the $E(2)$-like transformations leads to a shear transformation. The formalism therefore is applicable also to the case in which the polarization coordinate is sheared. The polarization plane is not always perpendicular to the propagation direction of the light wave. The formalism can be extended to accommodate this case.

In Section 2 we prove that the Jones-matrix formalism is indeed a representation of the Lorentz group. In Section 3 the mathematics of Section 2 is translated into transformation matrices that correspond to optical filters. It is shown that the combined effect of attenuation and phase-shift filters leads to a six-parameter two-by-two matrix. It is noted in Section 4 that the bilinear conformal representation of the Lorentz group is the natural scientific language for polarization optics. It shown that the attenuation and phase-shift filters have their own subrepresentations and that one of those subrepresentations leads to a new kind of optical filter that has the $E(2)$-like symmetry. In Section 5 we discuss a possible new class of optical filters that have the symmetry of the two-dimensional Euclidean group, and applications of these filters are discussed in Section 6. In Section 7 we show that the formalism developed in this paper can accommodate the cases in which the polarization coordinates are squeezed or sheared. The formalism can be extended also to the case in which the polarization plane is not perpendicular to the direction of propagation.

Since the Lorentz group is relatively new in optical sciences, we present the four-by-four and two-by-two representations of this group in Appendixes A and B, respectively. Since $E(2)$ group plays the central role in this paper, we give an introduction to this group in Appendix C.

2. FORMULATION OF THE PROBLEM

In studying polarized light propagating along the z direction, the traditional approach is to consider the x and y components of the electric fields. Their amplitude ratio and the phase difference determine the state of polarization. Thus we can change the polarization by adjusting the amplitudes, by changing the relative phases, or both. For convenience, we call the optical device that changes amplitudes an attenuator and the device that changes the relative phase a phase shifter.

Let us write the electric field vector as

$$E_x = A\cos(kz - \omega t + \phi_1),$$
$$E_y = B\cos(kz - \omega t + \phi_2), \tag{2.1}$$

where A and B are the amplitudes that are real and positive numbers and ϕ_1 and ϕ_2 are the phases of the x and the y components, respectively. This form not only is useful in classical optics but also is applicable to coherent and squeezed states of light.[7,23]

The traditional language for this two-component light is the Jones-matrix formalism that is discussed in standard optics textbooks.[20] In this formalism these two components are combined into one column matrix with the exponential form for the sinusoidal function:

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{bmatrix} A\exp[i(kz - \omega t + \phi_1)] \\ B\exp[i(kz - \omega t + \phi_2)] \end{bmatrix}. \tag{2.2}$$

This column matrix is called the Jones vector. The content of polarization is determined by the ratio

$$\frac{E_y}{E_x} = \left(\frac{B}{A}\right)\exp[i(\phi_2 - \phi_1)], \tag{2.3}$$

which can be written as one complex number:

$$w = r\exp(i\phi), \tag{2.4}$$

with

$$r = B/A, \qquad \phi = \phi_2 - \phi_1 .$$

The degree of polarization is measured by these two real numbers, which are the amplitude ratio and the phase difference, respectively. The purpose of this paper is to discuss the transformation properties of this complex number w. The transformation takes place when the light beam goes through an optical filter whose transmission properties are not isotropic.

The textbook version of the Jones-matrix formalism[20] starts with the projection operator

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.5}$$

which is applicable to the Jones vector of Eq. (2.2). This operator keeps the x component and completely eliminates the y component of the electric field. This is an oversimplification of the real-world situation in which the attenuation factor in the y direction is greater than that along the x direction. We shall later replace this projection operator with an attenuation matrix that is closer to that found in the real world.

Another element in the traditional formalism is

$$P(0, \delta) = \begin{bmatrix} \exp(-i\delta/2) & 0 \\ 0 & \exp(i\delta/2) \end{bmatrix}, \tag{2.6}$$

which leads to a phase difference of δ between the x and the y components. The polarization axes are not always the x and the y axes. For this reason we need the rotation matrix

$$R(\theta) = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}. \tag{2.7}$$

The traditional Jones-matrix formalism consists of systematic combinations of the three components given in Eqs. (2.6) and (2.7) and matrix (2.5)

We now replace the projection operator of matrix (2.5) with a squeeze matrix. There are two transverse directions that are perpendicular to each other. The absorption coefficient in one transverse direction could be different from the coefficient along the other direction. Thus there is the polarization coordinate in which the absorption can be described by

$$\begin{bmatrix} \exp(-\eta_1) & 0 \\ 0 & \exp(-\eta_2) \end{bmatrix} = \exp[-(\eta_1 + \eta_2)/2] \times \begin{bmatrix} \exp(\eta/2) & 0 \\ 0 & \exp(-\eta/2) \end{bmatrix}, \tag{2.8}$$

with $\eta = \eta_2 - \eta_1$. Let us look at the projection operator of matrix (2.5). Physically, it means that the absorption coefficient along the y direction is much larger than along the x direction. The absorption matrix in Eq. (2.8) becomes the projection matrix if η_1 is close to zero and η_2 becomes infinitely large. The projection operator of matrix (2.5) is therefore a special case of the matrix given in Eq. (2.8).

The attenuation matrix of Eq. (2.8) tells us that the electric fields are attenuated at two different rates. The exponential factor $\exp[-(\eta_1 + \eta_2)/2]$ reduces both components at the same rate and does not affect the state of polarization. The effect of polarization is determined solely by the squeeze matrix

$$S(0, \eta) = \begin{bmatrix} \exp(\eta/2) & 0 \\ 0 & \exp(-\eta/2) \end{bmatrix}. \tag{2.9}$$

This type of mathematical operation is quite familiar to us from studies of squeezed states of light if not from Lorentz boosts of spinors. For convenience we call the above matrix [Eq. (2.9)] an attenuator. Thus we are expanding the Jones-matrix formalism by replacing the projection operator of matrix (2.5) by the squeeze operator in Eq. (2.9).

The phase-shifter of Eq. (2.6) can be written as

$$P(0, \delta) = \exp(-i\delta J_1), \tag{2.10}$$

with

$$J_1 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.11}$$

Our notation for the Pauli sigma matrices is different from those that appear in the conventional literature and is explained in detail in Appendix B.

The rotation operator of Eq. (2.7) takes the form

$$R(\theta) = \exp(-i\theta J_3), \tag{2.12}$$

with

$$J_3 = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

The squeeze operator of Eq. (2.9) can also be written in the exponential form

$$S(0, \eta) = \exp(-i\eta K_1), \tag{2.13}$$

with

$$K_1 = \frac{i}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It is now possible to construct a closed set of commutation relations with the above generators, J_1, J_3, and K_1. Then the result is a set of six generators given in Appendix B. They are the generators for the group $SL(2, c)$, which is locally isomorphic to the six-parameter Lorentz group. This is the mathematical content of this paper. Next we exploit the physical contents of this mathematical formalism.

3. COMBINED EFFECTS

If the polarization coordinate is the same as the xy coordinate where the electric field components take the form of Eqs. (2.1), the attenuator described in Section 2 is directly applicable to the column matrix of Eq. (2.2). If the polarization coordinate is rotated by an angle $\theta/2$, or by the matrix

$$R(\theta) = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}, \tag{3.1}$$

then the squeeze matrix becomes

$$\begin{aligned} S(\theta, \eta) &= R(\theta)S(0, \eta)R(-\theta) \\ &= \begin{bmatrix} \exp(\eta/2)\cos^2(\theta/2) + \exp(-\eta/2)\sin^2(\theta/2) & [\exp(\eta/2) - \exp(-\eta/2)]\cos(\theta/2)\sin(\theta/2) \\ [\exp(\eta/2) - \exp(-\eta/2)]\cos(\theta/2)\sin(\theta/2) & \exp(-\eta/2)\cos^2(\theta/2) + \exp(\eta/2)\sin^2(\theta/2) \end{bmatrix}. \end{aligned} \tag{3.2}$$

We can obtain the inverse of this transformation by rotating the filter or Eq. (3.2) about the z axis by 90 deg.

If we apply two squeeze matrices, the net result becomes

$$S(\theta_2, \eta_2)S(\theta_1, \eta_1) = S(\theta_3, \eta_3)R(\psi), \tag{3.3}$$

where $R(\psi)$ is a rotation about the z axis by ψ. This means that the multiplication of two squeeze matrices leads not to another squeeze matrix but to a squeeze matrix preceded by a rotation matrix. This aspect of the squeeze operation is well known from the squeezed state of light and has been discussed extensively in the literature.[7,14,15] As was noted in Section 2, the generators K_1 and J_3 are needed for the squeeze matrix of Eq. (3.2). The repeated application leads to the commutation relation for these two generators:

$$[J_3, K_1] = iK_2. \tag{3.4}$$

Indeed, J_3, K_1, and K_2 form a closed set of commutation relations for the $Sp(2)$- or $O(2, 1)$-like subgroup of $SL(2, c)$.[15] This three-parameter subgroup has been extensively discussed in connection with squeezed states of light.[5,7]

Another basic element of the filter system is the optical filter with two different values of the index of refraction along the two orthogonal directions. The effect on this filter can be written as

$$\begin{bmatrix} \exp(i\delta_1) & 0 \\ 0 & \exp(i\delta_2) \end{bmatrix} = \exp[-i(\delta_1 + \delta_2)/2] \times \begin{bmatrix} \exp(-i\delta/2) & 0 \\ 0 & \exp(i\delta/2) \end{bmatrix}, \tag{3.5}$$

with $\delta = \delta_2 - \delta_1$. In measurement processes the overall phase factor $\exp[-i(\delta_1 + \delta_2)/2]$ cannot be detected and can therefore be deleted. The polarization effect of the filter is determined solely by the matrix

$$P(0, \delta) = \begin{bmatrix} \exp(-i\delta/2) & 0 \\ 0 & \exp(i\delta/2) \end{bmatrix}. \tag{3.6}$$

In Section 2 we noted that this form is one of the basic components of the Jones-matrix formalism. This phase-shifter matrix appears as a rotation matrix around the z axis in the theory of rotation groups, but it plays a different role in this paper. Hereafter we shall call this matrix a phase shifter.

Here also, if the polarization coordinate makes an angle $\theta/2$ with the xy coordinate system, the phase shifter takes the form

$$\begin{aligned} P(\theta, \delta) &= R(\theta)P(0, \delta)R(-\theta) \\ &= \begin{bmatrix} \exp(-i\delta/2)\cos^2(\theta/2) + \exp(i\delta/2)\sin^2(\theta/2) & [\exp(-i\delta/2) - \exp(i\delta/2)]\cos(\theta/2)\sin(\theta/2) \\ [\exp(-i\delta/2) - \exp(i\delta/2)]\cos(\theta/2)\sin(\theta/2) & \exp(i\delta/2)\cos^2(\theta/2) + \exp(-i\delta/2)\sin^2(\theta/2) \end{bmatrix}. \end{aligned} \tag{3.7}$$

Here again, we can obtain the inverse of this transformation by rotating the filter about the z axis by 90 deg.

If we consider only the phase shifters, the mathematics is basically repeated applications of J_1 and J_2, resulting in applications also of J_3, where their explicit two-by-two matrix forms are given in Appendix B. Thus the phase-shift filters form an $SU(2)$- or an $O(3)$-like subgroup of the group $SL(2, c)$.

If we use both the attenuators and phase shifters, the result is the full $SL(2, c)$ group with six parameters. The transformation matrix is usually written as

$$L = \begin{bmatrix} \alpha & \beta \\ \gamma & \rho \end{bmatrix}, \tag{3.8}$$

with the condition that its determinant be 1: $\alpha\rho - \gamma\beta = 1$. The repeated application of two matrices of this kind results in

$$\begin{bmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \rho_2 \end{bmatrix}\begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \rho_1 \end{bmatrix} = \begin{bmatrix} \alpha_2\alpha_1 + \beta_2\gamma_1 & \alpha_2\beta_1 + \beta_2\rho_1 \\ \gamma_2\alpha_1 + \rho_2\gamma_1 & \gamma_2\beta_1 + \rho_2\rho_1 \end{bmatrix}. \tag{3.9}$$

The most general form of the polarization transformation is the application of this algebra to the column matrix of Eq. (2.2).

The generators of the rotation or phase shifters are Hermitian. Thus the unitary subset of the L matrices of Eq. (3.8) represents the three-parameter rotationlike subgroup or the phase shifters and their repeated applications. The generators J_3, K_1, and K_2 are all imaginary. Thus the real subset of the L matrices represents the attenuation filters and their repeated applications.

4. BILINEAR CONFORMAL REPRESENTATION OF THE LORENTZ GROUP

In the present formulation of polarization optics we are interested in calculating one complex variable defined in Eq. (2.4). As was noted in Ref. 17, we can obtain the same algebraic result by using the bilinear transformation

$$w' = \frac{\rho w + \gamma}{\beta w + \alpha}. \tag{4.1}$$

The repeated applications of these two transformations can be achieved from

$$w_1 = \frac{\rho_1 w + \gamma_1}{\beta_1 w + \alpha_1}, \qquad w_2 = \frac{\rho_2 w_1 + \gamma_2}{\beta_2 w_1 + \alpha_2}. \tag{4.2}$$

Then it is possible to write w_2 as a function of w, and the result is

$$w_2 = \frac{(\gamma_2\beta_1 + \rho_2\rho_1)w + (\gamma_2\alpha_1 + \rho_2\gamma_1)}{(\alpha_2\beta_1 + \beta_2\rho_1)w + (\alpha_2\alpha_1 + \beta_2\gamma_1)}. \tag{4.3}$$

Equation (4.3) is a reproduction of the algebra given in the matrix multiplication of Eq. (3.9). The form given in Eq. (4.1) is the bilinear representation of the Lorentz group.[2]

Let us consider the physical interpretation of this result. If we apply the matrix L of Eq. (3.8) to the column vector of Eq. (2.2), then

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \rho \end{bmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \alpha E_x + \beta E_y \\ \gamma E_x + \rho E_y \end{pmatrix}, \tag{4.4}$$

which gives

$$\frac{E_y'}{E_x'} = \frac{\gamma E_x + \rho E_y}{\alpha E_x + \beta E_y}. \tag{4.5}$$

In terms of the physical quantity w defined in Eq. (2.4), Eq. (4.5) becomes

$$w' = \frac{\gamma + \rho w}{\alpha + \beta w}. \tag{4.6}$$

Equation (4.6) is identical to the bilinear form given in Eq. (4.1), and the ratio w can now be identified with the w variable, defined as the parameter of the bilinear representation of the Lorentz group in the same equation. Indeed, the bilinear representation is clearly the natural language for polarization optics.

Let us next consider subgroups. For the phase shifters the subgroup is represented by the unitary matrix

$$\begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix}. \tag{4.7}$$

This form is preserved in the bilinear representation. As for the attenuation subgroup, all the components are real, and this aspect is also preserved in the bilinear representation.

Let us next consider another form of bilinear transformation:

$$w' = \frac{[\exp(-i\delta/2)]w}{\exp(i\delta/2) + \beta w}. \tag{4.8}$$

This is also a form-preserving transformation. This form can be transformed into matrix form:

$$\begin{bmatrix} \exp(-i\delta/2) & \beta \\ 0 & \exp(i\delta/2) \end{bmatrix}. \tag{4.9}$$

We study this subrepresentation in detail in Section 5.

5. NEW FILTERS

We should note at this point that the Lorentz group has another set of three-parameter subgroups. They are like the two-dimensional Euclidean group. Let us consider one of them, which is generated by the matrices J_1, N_2, and N_3, with

$$N_2 = J_2 + K_3, \qquad N_3 = J_3 - K_2, \tag{5.1}$$

where

$$N_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad N_3 = \begin{bmatrix} 0 & -i \\ 0 & 0 \end{bmatrix}. \tag{5.2}$$

These matrices satisfy the commutation relations

$$[J_1, N_2] = iN_3, \qquad [J_1, N_3] = -iN_2,$$
$$[N_2, N_3] = 0. \tag{5.3}$$

They indeed form a closed set of commutation relations. As is shown in Appendix C, these commutation relations are like those for the two-dimensional Euclidean group consisting of two translations and one rotation around the origin. This group has been studied extensively in connection with the space–time symmetries of massless particles, where J_1 and the two N generators correspond to the helicity and gauge degrees of freedom, respectively.[21]

The physics of J_1 is well known through the phase shifter given in Eq. (2.6). If the angle δ is $\pi/2$, the phase shifter becomes a quarter-wave shifter, which we write as

$$Q = P(0, \pi/2) = \begin{bmatrix} \exp(-i\pi/4) & 0 \\ 0 & \exp(i\pi/4) \end{bmatrix}. \tag{5.4}$$

Then J_2 and K_3 are the quarter-wave conjugates of J_3 and K_2, respectively:

$$J_2 = QJ_3Q^{-1}, \qquad K_3 = -QK_2Q^{-1}. \tag{5.5}$$

Consequently,

$$N_2 = QN_3Q^{-1}. \tag{5.6}$$

The N generators lead to the following transformation matrices:

$$T_2(\tau) = \exp(-i\tau N_2) = \begin{bmatrix} 1 & i\tau \\ 0 & 1 \end{bmatrix},$$

$$T_3(\tau) = \exp(-i\tau N_3) = \begin{bmatrix} 1 & -\tau \\ 0 & 1 \end{bmatrix}. \tag{5.7}$$

It is clear that T_2 is the quarter-wave conjugate of T_3. We can now concentrate on the transformation matrix T_3.

If T_3 is applied to the column matrix of Eq. (2.2),

$$\begin{bmatrix} 1 & -\tau \\ 0 & 1 \end{bmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_x - \tau E_y \\ E_y \end{pmatrix}. \tag{5.8}$$

This new filter superposes the y component of the electric field to the x component with an appropriate constant, but it leaves the y component invariant.

Let us examine how this superposition is achieved. The generator N_3 consists of J_3, which generates rotations around the z axis, and K_2, which generates a squeeze along the 45° axis. Physically, J_3 generates optical activities. Thus the new filter consists of a suitable combination of these two operations. In both cases we have to take into account the overall attenuation factor, which can be measured by the attenuation of the y component and is not affected by the symmetry operation of Eq. (5.8).

Is it possible to produce optical filters of this kind? Starting from an optically active material, we can introduce an asymmetry in absorption into it by either mechanical or electrical means. Another approach would be to pile up alternately J_3-type and K_2-type layers. In either case it is interesting to note that the combination of these two effects produces a special effect predicted from the Lorentz group.

The $E(2)$-like symmetry includes transformations generated by J_1, which is given in Eq. (2.11). The transformation matrix is

$$P(0, \delta) = \begin{bmatrix} \exp(-i\delta/2) & 0 \\ 0 & \exp(i\delta/2) \end{bmatrix}. \tag{5.9}$$

This matrix is given in Eq. (2.6), and its physics are well understood. Let us apply this matrix to T_3 from the left and from the right. Then

$$P(0, \delta)T_3(\tau) = \begin{bmatrix} 1 & -\exp(-i\delta/2)\tau \\ 0 & 1 \end{bmatrix},$$

$$T_3(\tau)P(0, \delta) = \begin{bmatrix} 1 & -\exp(i\delta/2)\tau \\ 0 & 1 \end{bmatrix}, \tag{5.10}$$

which lead to

$$P(0, -\delta)T_3(\tau)P(0, \delta) = T_3[\exp(i\delta)\tau] = \begin{bmatrix} 1 & -\exp(i\delta)\tau \\ 0 & 1 \end{bmatrix}. \tag{5.11}$$

We can of course obtain the $P(0, -\delta)$ filter by rotating $P(0, \delta)$ around the z axis by 90°. It is thus possible to add a phase factor to the τ variable by using phase shifters of the type $P(0, \delta)$.

6. POSSIBLE APPLICATIONS OF THE NEW FILTER

The three-dimensional rotation group occupies an important place in many branches of physics. The group $O(2, 1)$ also is useful in a number of fields, including optics.[7,24–26] Thus, traditional attenuation and phase-shift filters may be useful in construction of analog computers to perform the symmetry operations of these groups.

The group $E(2)$ is somewhat new in optics.[22] However, as noted in Appendix C, it deals with translations and rotations on a flat surface. This filter may therefore be useful as a computational device for recording and reading two-dimensional maps. Furthermore, the T_3 matrix of Eqs. (5.7) has an interesting algebraic property:

$$\begin{bmatrix} 1 & \beta_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \beta_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \beta_1 + \beta_2 \\ 0 & 1 \end{bmatrix}. \tag{6.1}$$

The matrix can therefore be used, like the logarithmic function, for converting multiplication into addition. This is one of the most basic operations in computational machines.

For a more immediate application, let us consider lens optics. It is a trivial laboratory operation to rotate a given filter about the z axis by 90°. If we rotate matrix (6.3) below, the result is a matrix of the form

$$\begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix}. \tag{6.2}$$

This form and the original form of matrix (6.3) serve as lens and translation matrices, respectively, in para-axial optics. Indeed, a system of polarization filters can serve as an analog computer for a multilens system.

Furthermore, a matrix of the form

$$\begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} \tag{6.3}$$

represents a shear transformation. This is one of the basic deformations in engineering applications.

The Lorentz group was introduced to physics as the basic language for space–time symmetries of elementary particles,[1] but it is becoming increasingly prominent in many branches of physics and engineering, including classical and quantum optics. Optical filters may provide excellent calculational tools for the Lorentz group. Thus these filters may be useful as components of future computers.

7. NONORTHOGONAL COORDINATE SYSTEMS

Since the polarization of light is caused by anisotropic crystals, the polarization coordinate is not always orthogonal. Let us consider first the case in which the polarization of the light is along a pair of skew or squeezed axes.

We have noted that the $O(2, 1)$-like subgroup describes attenuation filters. A transformation matrix of this subgroup can also transform the orthogonal coordinate system into a squeezed coordinate system. The matrix takes the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} \cosh(\eta/2) & \sinh(\eta/2) \\ \sinh(\eta/2) & \cosh(\eta/2) \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{7.1}$$

The idea is to transform the squeezed coordinate system into the orthogonal system by use of the matrix or its inverse given in Eq. (7.1). Next, we can perform the polarization algebra developed here for the orthogonal coordinate system. We then can transform the result obtained in the orthogonal system back into the original squeezed coordinate system.

It is interesting to note that the transformation matrix given in Eq. (7.1) is one of the transformation matrices within the framework of the Lorentz-group representation developed here and there is no need to make the existing mathematics more complicated. The story is the same for the shear transformation, which takes the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{7.2}$$

This transformation is also well within the framework discussed in this paper.

The polarization plane is not always perpendicular to the direction of the propagation. We can solve this problem by extending our two-by-two formalism of Eq. (3.8) into the three-by-three form

$$\begin{bmatrix} \alpha & \beta & 0 \\ \gamma & \rho & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{7.3}$$

which is applicable to the (x, y, z) coordinate system. The polarization plane can be rotated about the x axis by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\xi & -\sin\xi \\ 0 & \sin\xi & \cos\xi \end{bmatrix}. \tag{7.4}$$

Using this matrix or its inverse, we can bring the problem to the orthogonal coordinate system. After doing the standard polarization algebra, we can go back to the original coordinate system.

8. CONCLUDING REMARKS

We have replaced the projection operator of matrix (2.5) by the squeeze matrix given in Eq. (2.9), thus bringing the Jones-matrix formalism closer to the real world. This also enables us to formulate the problem within the framework of the Lorentz group.

There are now many powerful mathematical tools derivable from the Lorentz group.[7,26] We can use them to study various aspects of optics more systematically. The present paper is the first step toward a systematic exposition of polarization optics. There are several interesting problems for future solution. First, it is now possible to study the relation between the Jones vectors and the Stokes parameters by using the mathematical theorems of the Lorentz group connecting Minkowskian four-vectors and $SL(2, c)$ spinors. Furthermore, the natural language of the Lorentz group is based on circularly polarized states of photons. It would therefore be interesting to work out a Jones-matrix formalism that would be applicable to circularly polarized lights.

It is widely understood that the Lorentz group started gaining its prominence in optics through its association with squeezed states of light.[5] This is not true. This group was discussed much earlier in connection with polarization optics.[27] However, we learned the connection between squeeze transformations and Lorentz boosts while studying squeezed states of light.[7] The Lorentz group is now a powerful mathematical device because there are many squeeze transformations in physics, including the Jones-matrix formalism discussed in the present paper.

It is gratifying to note that the symmetry of the Lorentz group is useful for various engineering applications of light waves, including their polarization, reflection, and propagation media.[28] In physics it is relatively new to study Maxwell's equations in terms of the Lorentz group.[26] It will thus be a challenging problem to combine the symmetries of electromagnetic waves developed in engineering with those developed in physics.

APPENDIX A: LORENTZ TRANSFORMATIONS

Let us consider the space–time coordinates (x, y, z, t). Then the rotation about the z axis is performed by the four-by-four matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{A1}$$

This transformation is generated by

$$J_3 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{A2}$$

Likewise, we can write down the generators of rotations J_1 and J_2 around the x and y axes, respectively. These three generators satisfy the closed set of commutations relations

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{A3}$$

This set of commutation relations is for the three-dimensional rotation group.

The Lorentz boost along the z axis takes the form

$$\begin{bmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{bmatrix}, \tag{A4}$$

which is generated by

$$K_3 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}. \tag{A5}$$

Likewise, we can write generators of boosts K_1 and K_2 along the x and y axes, respectively, and they take the form

$$K_1 = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{A6}$$

These boost generators satisfy the commutation relations

$$[J_i, K_j] = i\epsilon_{ijk}K_k, \qquad [K_i, K_j] = -i\epsilon_{ijk}J_k. \tag{A7}$$

Indeed, the three rotation generators and three boost generators satisfy the closed set of commutation relations given in Eqs. (A3) and (A7). These three commutation relations form the starting point of the Lorentz group. The generators given in this appendix are four-by-four matrices, but they are not the only set that satisfy the commutation relations. We can also construct six two-by-two matrices that satisfy the same set of commutation relations. The group of transformations constructed from these two-by-matrices is often called $SL(2, c)$ or the two-dimensional representation of the Lorentz group. Throughout this paper we have used the two-by-two transformation matrices constructed from the generators of the $SL(2, c)$ group.

APPENDIX B: SPINORS AND FOUR-VECTORS IN THE LORENTZ GROUP

In Appendix A we noted that there are four-by-four and two-by-two representations of the Lorentz group. The four-by-four representation is applicable to covariant four-vectors, and the two-by-two transformation matrices are applicable to two-component spinors, which in the present case are Jones vectors. The question then is whether we can construct the four-vector from the spinors. In the language of polarization optics, the question is whether it is possible to construct the coherency matrix[29,30] from the Jones vector.

With this point in mind, let us start from the following form of the Pauli spin matrices:

$$\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{B1}$$

These matrices are written in a different convention. Here σ_3 is imaginary and σ_2 is imaginary in the traditional notation. Also in this convention, we can construct three rotation generators:

$$J_i = \frac{1}{2} \sigma_i , \tag{B2}$$

that satisfy the closed set of commutation relations

$$[J_i , J_j] = i \epsilon_{ijk} J_k . \tag{B3}$$

We can also construct three boost generators:

$$K_i = \frac{i}{2} \sigma_i , \tag{B4}$$

that satisfy the commutation relations

$$[K_i , K_j] = -i \epsilon_{ijk} J_k . \tag{B5}$$

The K_i matrices alone do not form a closed set of commutation relations, and the rotation generators J_i are needed to form a closed set:

$$[J_i , K_j] = i \epsilon_{ijk} K_k . \tag{B6}$$

The six matrices J_i and K_i form a closed set of commutation relations, and they are, like the generators of the Lorentz group, applicable to the $(3 + 1)$-dimensional Minkowski space. The group generated by the six matrices constructed above is called $SL(2, c)$ and consists of all two-by-two complex matrices with unit determinants.

To construct four-vectors we need two different spinor representations of the Lorentz group. Let us go to the commutation relations for the generators given in Eqs. (B3), (B5), and (B6). These commutators are not invariant under the sign change of the rotation generators J_i but are invariant under the sign change of the squeeze operators K_i. Thus for each spinor representation, there is another representation with the squeeze generators with opposite sign, which allows us to construct another representation with the generators

$$\dot{J}_i = \frac{1}{2} \sigma_i , \qquad \dot{K}_i = \frac{-i}{2} \sigma_i . \tag{B7}$$

We call this representation the dotted representation. If we write the transformation matrix L of Eq. (3.8) in terms of the generators as

$$L = \exp\left[-\frac{i}{2} \sum_{i=1}^{3} (\theta_i \sigma_i + i \eta_i \sigma_i) \right], \tag{B8}$$

then the transformation matrix in the dotted representation becomes

$$\dot{L} = \exp\left[-\frac{i}{2} \sum_{i=1}^{3} (\theta_i \sigma_i - i \eta_i \sigma_i) \right]. \tag{B9}$$

In both of the matrices in Eqs. (B8) and (B9), Hermitian conjugation changes the direction of rotation. However, it does not change the direction of boosts. We can achieve that only by interchanging L and $\dot{L}$, and we call this the dot conjugation.

Likewise, there are two different set of spinors. Let us use u and v for the up and the down spinors, respectively, for undotted representation. Then $\dot{u}$ and $\dot{v}$ are for the dotted representation. The four-vectors are then constructed as[31]

$$u\dot{u} = -(x - iy), \qquad v\dot{v} = (x + iy),$$
$$u\dot{v} = (t + z), \qquad v\dot{u} = -(t - z), \tag{B10}$$

leading to the matrix

$$C = \begin{bmatrix} u\dot{v} & -u\dot{u} \\ v\dot{v} & -v\dot{u} \end{bmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} (\dot{v} \quad -\dot{u}), \tag{B11}$$

where u and $\dot{u}$ are 1 if the spin is up and are 0 if the spin is down, while v and $\dot{v}$ are 0 and 1 for the spin-up and spin-down cases. The transformation matrix applicable to the column vector in Eq. (B11) is the two-by-two matrix given in Eq. (3.8). What, then, is the transformation matrix applicable to the row vector $(\dot{v}, -\dot{u})$ from the right-hand side of Eq. (B11)? It is the transpose of the matrix applicable to the column vector $(\dot{v}, -\dot{u})$. We can obtain this column vector from

$$\begin{pmatrix} \dot{v} \\ -\dot{u} \end{pmatrix} \tag{B12}$$

by applying to it the matrix

$$g = -i\sigma_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{B13}$$

This matrix also has the property that

$$g \sigma_i g^{-1} = -(\sigma_i)^T, \tag{B14}$$

where the superscript T means the transpose of the matrix. The transformation matrix applicable to the column vector of matrix (B12) is $\dot{L}$ of Eq. (B9). Thus the matrix applicable to the row vector $(\dot{v}, -\dot{u})$ in Eq. (B11) is

$$\{g^{-1} \dot{L} g\}^T = g^{-1} \dot{L}^T g. \tag{B15}$$

This is precisely the Hermitian conjugate of L.

In optics, this two-by-two matrix form appears as the coherency matrix, and it takes the form

$$C = \begin{bmatrix} \langle E_x{}^* E_x \rangle & \langle E_y{}^* E_x \rangle \\ \langle E_x{}^* E_y \rangle & \langle E_y{}^* E_y \rangle \end{bmatrix}, \tag{B16}$$

where $\langle E_i{}^* E_j \rangle$ is the time average of $E_i{}^* E_j$. This matrix is convenient when we deal with light waves whose two transverse components are only partially coherent. In terms of the complex parameter w the coherency matrix is proportional to

$$C = \begin{bmatrix} 1 & r \exp(i\delta) \\ r \exp(-i\delta) & r^2 \end{bmatrix} \tag{B17}$$

if the x and y components are perfectly coherent with the phase difference of δ. If they are totally incoherent, the off-diagonal elements vanish in the above matrix.

Let us now consider the transformation properties of the matrix. As was noted by Opatrny and Perina,[16] the matrix of Eq. (B16) is like

$$C = \begin{bmatrix} t + z & x - iy \\ x + iy & t - z \end{bmatrix}, \tag{B18}$$

where the set of variables (x, y, z, t) is transformed as a four-vector under Lorentz transformations. Furthermore, it is known that the Lorentz transformation of this four-vector is achieved through the formula

$$C' = LCL^{\dagger}, \tag{B19}$$

where the transformation matrix L is that of Eq. (3.8). The construction of four-vectors from the two-component spinors is not a trivial task.[31,32] The two-by-two representation of Eq. (B18) requires one more step.

APPENDIX C: TWO-DIMENSIONAL EUCLIDEAN GROUP

Let us consider a two-dimensional plane and use the xy coordinate system. Then L_z, defined as

$$L_z = -i\left\{x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right\}, \tag{C1}$$

will generate rotations about the origin. The translation generators are

$$P_x = -i \frac{\partial}{\partial x}, \qquad P_y = -i \frac{\partial}{\partial y}. \tag{C2}$$

These generators satisfy the commutation relations

$$[L_z, P_x] = iP_y, \qquad [L_z, P_y] = -iP_x, \qquad [P_x, P_y] = 0. \tag{C3}$$

These commutation relations are like those given in Eqs. (5.3). They become identical if L_z, P_x, and P_y are replaced by J_1, N_2, and N_3, respectively.

This group is not discussed often in physics but is intimately related to our daily life. When we drive on the streets, we make translations and rotations and thus make transformations of this $E(2)$ group. In addition, this group reproduces the internal space–time symmetry of massless particles.[1] This aspect of the $E(2)$ group has been discussed extensively in the literature.[26,31,33]

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