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A Unified Formalism for Polarization Optics by Using Group Theory I (Theory)

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Group theory provides a method for a unified treatment of the Poincaré sphere, the Jones matrix and the Mueller matrix. We first introduce the Poincaré sphere by a stereographic projection of the plane representation of elliptically polarized light. Next we consider the unitary group and the rotation group by interpreting the stereographic projection through group theory. By extending these two groups, we finally consider the unimodular group and the Lorentz group.

The unitary group and the rotation group are related to the Jones and Mueller matrices of totally transparent systems, while the unimodular group and the Lorentz group are related to the Jones and Mueller matrices of most general systems including partially transparent systems. By this approach we can grasp the unified treatment of these three methods and understand clearly their relationship and structure.

§1. Introduction

Various methods are used when we analyze or synthesize complicated optical systems related to polarized light. The most well-known methods are the Jones 2×2 matrix, the Mueller 4×4 matrix, and the Poincaré sphere.[1] The relation among these methods is treated in many papers.[2] But most papers use only algebraic calculation and do not refer to group theory and group theoretical aspects of polarization calculus. The aim of this paper is to show that the relation among these methods can be understood clearly by using group theory in polarization calculus.

1.1 *Elliptically polarized light*[3]

With reference to a Cartesian coordinate system (X, Y, Z), let the monochromatic, elliptically polarized light travel along the direction of Z axis. The fast and slow components of the light vector along the axes X and Y respectively are then represented by the equations

$$E_X=A_X\exp(i\omega t), \tag{1.a}$$

$$E_Y=B_Y\exp(i\omega t), \tag{1.b}$$

where ω is an angular velocity.

If A and B are the amplitudes of A_X and B_Y and Δ is the phase lag of the component of Y behind that of X, then

$$\frac{E_Y}{E_X}=\frac{B}{A}\cos\Delta+i\frac{B}{A}\sin\Delta=X+iY. \tag{2}$$

Eliminating t from eqs. (1a) and (1b), we have

$$\frac{X^2}{A^2}+\frac{Y^2}{B^2}-\frac{2XY}{AB}\cos\Delta=\sin^2\Delta. \tag{3}$$

This equation shows an ellipse inscribed in a rectangle of sides $2A$ and $2B$ of Fig. 1.
The angles ε and ν shown in Fig. 1 are:

$$\tan\varepsilon=\frac{b}{a}, \tag{4}$$

$$\tan\nu=\frac{B}{A}. \tag{5}$$

There are the following relations between these quantities:

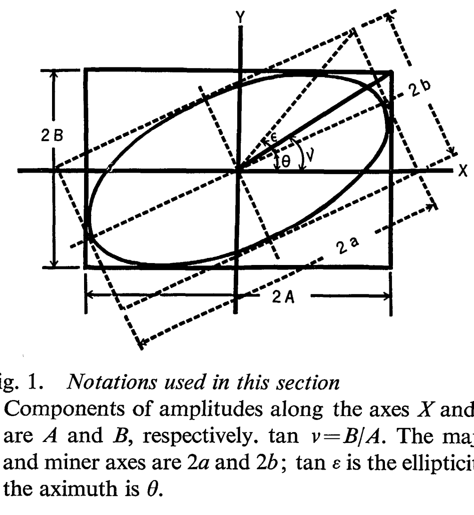

Fig. 1. *Notations used in this section*
Components of amplitudes along the axes X and Y are A and B, respectively. $\tan\nu=B/A$. The major and miner axes are $2a$ and $2b$; $\tan\varepsilon$ is the ellipticity; the aximuth is θ.

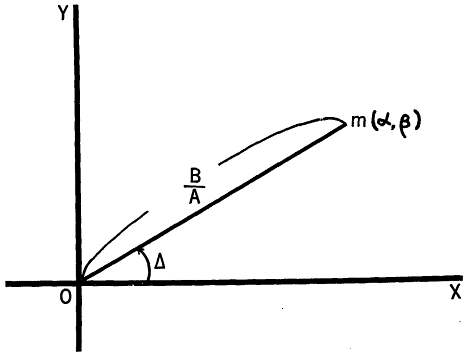

Fig. 2. *The plane representation of elliptically polarized light E_Y/E_X.*
The ratio of the components of amplitudes is B/A.
The relative phase difference is Δ.

$$\pm\sin 2\varepsilon = \sin 2\nu \cdot \sin\Delta, \tag{6}$$

$$\tan 2\theta = \tan 2\nu \cdot \cos\Delta, \tag{7}$$

$$\cos 2\nu = \cos 2\varepsilon \cdot \cos 2\theta. \tag{8}$$

In eq. (6), the upper and lower signs refer to the left and right rotation, respectively.

In eq. (2), it is understood that the ratio E_Y/E_X determines the shape and rotation direction of elliptical vibration given by eq. (3). This vibration can be represented by a point m on the plane (refer to Fig. 2). The distance $\overline{om}$ and the angle of $\overline{om}$ against the axis X represent the ratio B/A and phase difference Δ.

From eq. (2), the representative point $m(\alpha, \beta)$ is given by

$$(\alpha, \beta) = \left(\frac{B}{A}\cos\Delta, \frac{B}{A}\sin\Delta\right). \tag{9}$$

1.2 *Stereographic projection*

We consider a stereographic projection which establishes a correspondence between the points of a plane and those of a sphere.[4)]

Let C be a unit radius sphere, which has its center at the origin of a Cartesian coordinate system (X, Y, Z). Let S be a point with coordinate $(-1, 0, 0)$ and let M be a variable point on the unit sphere (refer to Fig. 3). If P is a point where the line SM and the $Y-Z$ plane intersect, then a correspondence between a plane and a sphere is established by this line SM. This is a stereographic projection of the sphere onto the plane.

Next, let us consider the equation describing the stereographic projection. Let MN be the perpendicular dropped from the point to the axis X. Since $\overline{SO}=1$ (—means length), we get from similar triangles

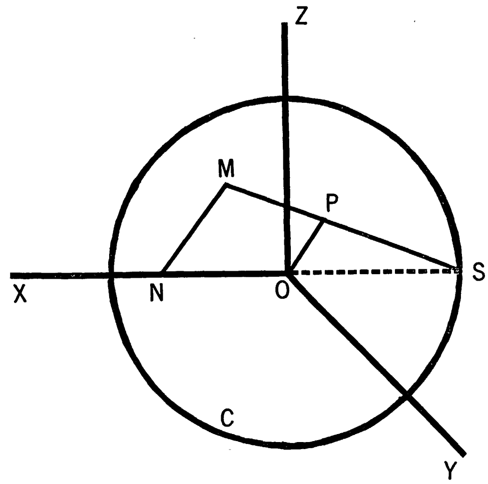

Fig. 3. *Stereographic projection*
The unit sphere is C. The point on C is M. The point in the plane $Y-Z$ and on the line is P.

$$\overline{MN} = (1+\overline{ON})\overline{OP}. \tag{10}$$

If the coordinates of the point M and those of the point P are (X, Y, Z) and (α, β), respectively, we have the relations

$$\overline{MN} = (1+X)\overline{OP}$$

or

$$Y = (1+X)\alpha, \quad Z = (1+X)\beta. \tag{11}$$

Substituting these into $X^2+Y^2+Z^2=1$ produces

$$(\alpha^2+\beta^2)(1+X)^2+X^2=1.$$

From this equation and eq. (11), we have

$$X = \frac{1-\alpha^2-\beta^2}{1+\alpha^2+\beta^2}, \quad Y = \frac{2\alpha}{1+\alpha^2+\beta^2},$$

$$Z = \frac{2\beta}{1+\alpha^2+\beta^2}. \tag{12}$$

(Normalized Stokes rector)

Now if we consider the $Y-Z$ plane of Fig. 3 as the $X-Y$ plane of Fig. 2, the point $P(\alpha, \beta)$ becomes the point $m(\alpha, \beta)$. Then, substituting eq. (9) into eq. (12) and using the relations (6), (7) and (8) yield

$$X = \cos 2\varepsilon \cdot \cos 2\theta,$$

$$Y = \cos 2\varepsilon \cdot \sin 2\theta, \tag{13}$$

$$Z = \sin 2\varepsilon,$$

which is the Poincaré sphere representation.

Next, instead of the two real numbers α and β in the plane, we can introduce a single complex number, $\zeta=\alpha+i\beta$. Let $\bar{\zeta}$ be a conjugate complex number, then we get from eq. (12),

$$X=\frac{1-\zeta\bar{\zeta}}{1+\zeta\bar{\zeta}},\quad Y+iZ=\frac{2\zeta}{1+\zeta\bar{\zeta}},$$
$$Y-iZ=\frac{2\bar{\zeta}}{1+\zeta\bar{\zeta}}. \tag{14}$$

Now when we consider eq. (2) and eq. (9), we find

$$\zeta=\frac{E_Y}{E_X}. \tag{15}$$

E_X, E_Y are called homogeneous complex coordinates in the plane. By using eq. (15), eq. (14) becomes

$$X=\frac{E_X\bar{E}_X-E_Y\bar{E}_Y}{E_X\bar{E}_X+E_Y\bar{E}_Y},\quad Y=\frac{\bar{E}_XE_Y+E_X\bar{E}_Y}{E_X\bar{E}_X+E_Y\bar{E}_Y},$$
$$Z=\frac{1}{i}\cdot\frac{\bar{E}_XE_Y-E_X\bar{E}_Y}{E_X\bar{E}_X+E_Y\bar{E}_Y}. \tag{16}$$

§2. The Unitary Group and the Rotation Group

Let us consider the unitary transformation of variables whose determinant is $+1$. All such unitary transformation is represented by

$$E'_X=aE_X+bE_Y,\quad E'_Y=-\bar{b}E_X+\bar{a}E_Y, \tag{17}$$

where a and b are complex numbers which satisfy the following condition

$$a\bar{a}+b\bar{b}=1. \tag{18}$$

Since this transformation is unitary,

$$E'_X\bar{E}'_X+E'_Y\bar{E}'_Y=E_X\bar{E}_X+E_Y\bar{E}_Y. \tag{19}$$

These new variables E_X, E_Y give a new point on the unit sphere by the relation:

$$X'=\frac{E'_X\bar{E}'_X-E'_Y\bar{E}'_Y}{E'_X\bar{E}'_X+E'_Y\bar{E}'_Y},\quad Y'=\frac{\bar{E}'_XE'_Y+E'_X\bar{E}'_Y}{E'_X\bar{E}'_X+E'_Y\bar{E}'_Y},$$
$$Z'=\frac{1}{i}\cdot\frac{\bar{E}'_XE'_Y-E'_X\bar{E}'_Y}{E'_X\bar{E}'_X+E'_Y\bar{E}'_Y}. \tag{20}$$

Substituting eq. (17) into E'_X, E'_Y of eq. (20) and using the condition (19) and the transformation (16), we get the representation for X', Y', Z' as the linear combination of X, Y, Z. The result is:

$$\begin{aligned}X'&=(a\bar{a}-b\bar{b})X+(\bar{a}b+a\bar{b})Y+i(\bar{a}b-a\bar{b})Z,\\ Y'&=-(ab+\bar{a}\bar{b})X+\frac{1}{2}(a^2+\bar{a}^2-b^2-\bar{b}^2)Y\\ &\quad+\frac{i}{2}(\bar{a}^2+\bar{b}^2-a^2-b^2)Z,\\ Z'&=i(\bar{a}\bar{b}-ab)X+\frac{i}{2}(a^2+\bar{b}^2-\bar{a}^2-b^2)Y\\ &\quad+\frac{1}{2}(a^2+\bar{a}^2+b^2+\bar{b}^2)Z.\end{aligned} \tag{21}$$

All unitary transformation (17) transforms the $Y-Z$ plane onto itself and then the stereographic projection transforms the unit sphere onto itself accordingly. Thus, the transformation (21) is a three real variables' orthogonal transformation satisfying the condition:

$$X'^2+Y'^2+Z'^2=X^2+Y+Z^2.$$

Further, since it is shown that the determinant of eq. (21) is $+1$, this transformation (21) represents a spatial transformation around the origin.

It is shown that this three dimensional rotation group is homomorphic to the unitary transformation with determinant $+1$.

It is not difficult to interpret these results optically. The Jones matrix of a totally transparent system is:

$$\begin{pmatrix} a & b\\ -\bar{b} & \bar{a}\end{pmatrix},\quad a\bar{a}+b\bar{b}=1. \tag{17'}$$

Corresponding to the Jones matrix the Mueller matrix is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & a\bar{a}-b\bar{b} & a\bar{b}+\bar{a}b & i(\bar{a}b-a\bar{b})\\ 0 & -(ab+\bar{a}\bar{b}) & \frac{1}{2}(a^2+\bar{a}^2-b^2-\bar{b}^2) & \frac{i}{2}(\bar{a}^2+\bar{b}^2-a^2-b^2)\\ 0 & i(\bar{a}\bar{b}-ab) & \frac{i}{2}(a^2+\bar{b}^2-\bar{a}^2-b^2) & \frac{1}{2}(a^2+\bar{a}^2+b^2+\bar{b}^2)\end{pmatrix} \tag{21'}$$

The space representation of this three dimensional rotation group is the Poincaré sphere.

Here we notice that in eq. (21) two complex parameters, a and b, are included which must satisfy eq. (18). Each complex parameter includes two real parameters, that is,

$$a=q_0+iq_1, \quad b=q_2+iq_3,$$

and the condition (18) is equivalent to

$$q_0^2+q_1^2+q_2^2+q_3^2=1.$$

This fact shows that the totally transparent system can be treated by using quaternions.

§3. The Unimodular Group and Lorentz Group

We have established the close relation between the three dimensional rotation group and the two dimensional unitary transformation with determinant $+1$. Similarly we can establish a close relationship between the Lorentz group and the general two dimensional linear transformation with determinant $+1$, the so called unimodular group. The unitary transformation whose determinant is 1, is a subgroup of the unimodular group.

We introduce four new variables: X_k $(k=0, 1, 2, 3)$. If we put

$$X=\frac{X_1}{X_0}, \quad Y=\frac{X_2}{X_0}, \quad Z=\frac{X_3}{X_0}, \tag{22}$$

in eq. (16) which defines the stereographic projection, then we get

$$\frac{X_1}{X_0}=\frac{E_X\bar{E}_X-E_Y\bar{E}_Y}{E_X\bar{E}_X+E_Y\bar{E}_Y}, \quad \frac{X_2}{X_0}=\frac{\bar{E}_XE_Y+E_X\bar{E}_Y}{E_X\bar{E}_X+E_Y\bar{E}_Y},$$
$$\frac{X_3}{X_0}=\frac{1}{i}\cdot\frac{\bar{E}_XE_Y-E_X\bar{E}_Y}{E_X\bar{E}_X+E_Y\bar{E}_Y}.$$

Apart from the common factor, these equations can be written:

$$X_0=E_X\bar{E}_X+E_Y\bar{E}_Y, \quad X_2=\bar{E}_XE_Y+E_X\bar{E}_Y, \tag{23}$$
$$X_1=E_X\bar{E}_X-E_Y\bar{E}_Y, \quad X_3=\frac{1}{i}(\bar{E}_XE_Y-E_X\bar{E}_Y).$$

As the old variables X, Y, Z satisfy the equation

$$X^2+Y^2+Z^2=1,$$

the new variables defined by eq. (23) satisfy the equation

$$X_1^2+X_2^2+X_3^2-X_0^2=0 \tag{24}$$

for arbitrary complex variables E_X, E_Y.

The representation $E_X\bar{E}_X+E_Y\bar{E}_Y$, which is the variable given by eq. (23), remains invariant under the unitary transformation of E_X, E_Y. In this case, therefore, the three dimensional relation is obtained. Now let us weeken the condition that the transformation is unitary. And consider the general group of the linear transformation whose forms are

$$E'_X=aE_X+bE_Y, \quad E'_Y=cE_X+dE_Y. \tag{25}$$

At first we make

$$X_0+X_1=2E_X\bar{E}_X, \quad X_2+iX_3=2\bar{E}_XE_Y, \tag{26}$$
$$X_0-X_1=2E_Y\bar{E}_Y, \quad X_2-iX_3=2E_X\bar{E}_Y.$$

For new variables E'_X and E'_Y, X'_k gains new values

$$X'_0+X'_1=2E'_X\bar{E}'_X, \quad X'_2+iX'_3=2\bar{E}'_XE'_Y, \tag{27}$$
$$X'_0-X'_1=2E'_Y\bar{E}'_Y, \quad X'_2-iX'_3=2E'_X\bar{E}'_Y.$$

Substituting eq. (25) into eq. (27) produces

$$\begin{aligned}
X'_0+X'_1&=a\bar{a}(X_0+X_1)+b\bar{b}(X_0-X_1)+\bar{a}b(X_2+iX_3)+a\bar{b}(X_2-iX_3),\\
X'_0-X'_1&=c\bar{c}(X_0+X_1)+d\bar{d}(X_0-X_1)+\bar{c}d(X_2+iX_3)+c\bar{d}(X_2-iX_3),\\
X'_2+iX'_3&=\bar{a}c(X_0+X_1)+\bar{b}d(X_0-X_1)+\bar{a}d(X_2+iX_3)+\bar{b}c(X_2-iX_3),\\
X'_2-iX'_3&=a\bar{c}(X_0+X_1)+b\bar{d}(X_0-X_1)+b\bar{c}(X_2+iX_3)+a\bar{d}(X_2-iX_3).
\end{aligned} \tag{28}$$

Solving the equations for X'_k, we get

$$\begin{aligned}
X'_0&=\frac{1}{2}(a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d})X_0+\frac{1}{2}(a\bar{a}-b\bar{b}+c\bar{c}-d\bar{d})X_1+\mathrm{Re}(\bar{a}b+\bar{c}d)X_2-\mathrm{Im}(\bar{a}b+\bar{c}d)X_3,\\
X'_1&=\frac{1}{2}(a\bar{a}+b\bar{b}-c\bar{c}-d\bar{d})X_0+\frac{1}{2}(a\bar{a}-b\bar{b}-c\bar{c}+d\bar{d})X_1+\mathrm{Re}(\bar{a}b-\bar{c}d)X_2-\mathrm{Im}(\bar{a}b-\bar{c}d)X_3,\\
X'_2&=\mathrm{Re}(\bar{a}c+\bar{b}d)X_0+\mathrm{Re}(\bar{a}c-\bar{b}d)X_1+\mathrm{Re}(\bar{a}d+\bar{b}c)X_2-\mathrm{Im}(\bar{a}d-\bar{b}c)X_3,\\
X'_3&=\mathrm{Im}(\bar{a}c+\bar{b}d)X_0+\mathrm{Im}(\bar{a}c-\bar{b}d)X_1+\mathrm{Im}(\bar{a}d+\bar{b}c)X_2+\mathrm{Re}(\bar{a}d-\bar{b}c)X_3.
\end{aligned} \tag{29}$$

This optical interpretation is given by the following. The Jones matrix of the general optical system, including the partially transparent system, is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{25'}$$

corresponding to eq. (25).

On the other hand, the Mueller matrix has the following form

$$\begin{pmatrix} \frac{1}{2}(a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d}) & \frac{1}{2}(a\bar{a}-b\bar{b}+c\bar{c}-d\bar{d}) & \mathrm{Re}(\bar{a}b+\bar{c}d) & -\mathrm{Im}(\bar{a}b+\bar{c}d) \\ \frac{1}{2}(a\bar{a}+b\bar{b}-c\bar{c}-d\bar{d}) & \frac{1}{2}(a\bar{a}-b\bar{b}-c\bar{c}+d\bar{d}) & \mathrm{Re}(\bar{a}b-\bar{c}d) & -\mathrm{Im}(\bar{a}b-\bar{c}d) \\ \mathrm{Re}(\bar{a}c+\bar{b}d) & \mathrm{Re}(\bar{a}c-\bar{b}d) & \mathrm{Re}(\bar{a}d+\bar{b}c) & -\mathrm{Im}(\bar{a}d-\bar{b}c) \\ \mathrm{Im}(\bar{a}c+\bar{b}d) & \mathrm{Im}(\bar{a}c-\bar{b}d) & \mathrm{Im}(\bar{a}d+\bar{b}c) & \mathrm{Re}(\bar{a}d-\bar{b}c) \end{pmatrix} \tag{29'}$$

where we used the relation

$$\mathrm{Re}(X)=\frac{X+\bar{X}}{2}, \quad \mathrm{Im}(X)=\frac{X-\bar{X}}{2i}.$$

Here we consider the four dimensional transformation. The new variables X_k satisfy the equation

$$X_1'^2+X_2'^2+X_3'^2-X_0'^2=0, \tag{30}$$

just the same as the old variables X_k. But the left-side terms of eq. (24) and eq. (30) may be different by the common factor, that is:

$$X_1'^2+X_2'^2+X_3'^2-X_0'^2=K(X_1^2+X_2^2+X_3^2-X_0^2).$$

Here K is a constant. By making use of eq. (28) this is equal to

$$K=|ad-bc|^2.$$

If eq. (29) shows the Lorentz transformation, K must be 1 (this means that the determinant of the transformation (25′) is 1) and the transformation is the unimodular transformation

$$ad-bc=1. \tag{31}$$

The linear transformation (29) which is subject to eq. (31) is the proper orthochronous Lorentz transformation. The proper orthochronous Lorentz transformation is homomorphic to the unimodular transformation.

§4. Conclusion

The relationship among the Jones matrix, the Mueller matrix and the Poincaré sphere and the relationship of these representations to the group theory are summarized in the following chart:

Jones Matrix ⋮ Mueller Matrix
⋮
Unimodular Group — Homomorphism — Lorentz Group ···· Most General Optical System
↑ ↑
Subgroup Subgroup
| |
Unitary Group — Homomorphism — Rotation Group ···· Transparent Optical System
↑ ↑
The Plane Representation of Elliptically Polarized Light — Stereographic Projection → Poincaré Sphere

In the stereographic projection we put the $X-Y$ plane onto the $Y-Z$ plane of the Cartesian coordinate system inside the sphere. This is because of the characteristic traditional order of the Stokes parameters. If we put the $X-Y$ plane onto the $X-Y$ plane (refer to Fig. 4), then the resultant form of the Lorentz group representation becomes that of Schmieder.[2)]

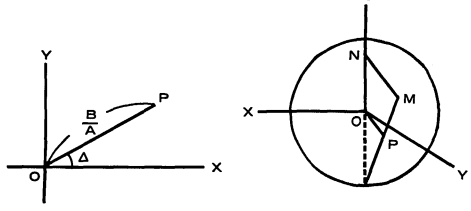

Fig. 4. *Natural order of Schmieder*
The $X-Y$ plane of the left figure is overlapped on the $X-Y$ plane of the right figure. Notations are the same with those of Figs. 2 and 3.

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