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A Unified Formalism for Polarization Optics by Using Group Theory II (Generator Representation)

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Generator representations of Jones and Mueller matrices are derived and their physical implications are discussed. The coexistence of linear birefringence with linear dichroism and that of circular birefringence with circular dichroism are given as concrete applications of these generator representations.

§1. Introduction

In the previous paper[1)] we have shown that the unified formalism for polarization optics is possible by utilizing group theory. As an extension of that paper, we will derive the most general generator representations for both the Jones matrix and Mueller matrix. It is shown that all generator representations already derived[2)] can be deduced from the general generator representations.

§2. Most General Jones and Mueller Matrices

The most general Mueller matrix corresponding to the most general Jones matrix expressed by

$$U=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1}$$

is given in a form,[1,3,4)]

$$L=\begin{pmatrix} \frac{1}{2}(a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d}) & \frac{1}{2}(a\bar{a}-b\bar{b}+c\bar{c}-d\bar{d}) & \mathrm{Re}(\bar{a}b+\bar{c}d) & -\mathrm{Im}(\bar{a}b+\bar{c}d) \\ \frac{1}{2}(a\bar{a}+b\bar{b}-c\bar{c}-d\bar{d}) & \frac{1}{2}(a\bar{a}-b\bar{b}-c\bar{c}+d\bar{d}) & \mathrm{Re}(\bar{a}b-\bar{c}d) & -\mathrm{Im}(\bar{a}b-\bar{c}d) \\ \mathrm{Re}(\bar{a}c+\bar{b}d) & \mathrm{Re}(\bar{a}c-\bar{b}d) & \mathrm{Re}(\bar{a}d+\bar{b}c) & -\mathrm{Im}(\bar{a}d-\bar{b}c) \\ \mathrm{Im}(\bar{a}c+\bar{b}d) & \mathrm{Im}(\bar{a}c-\bar{b}d) & \mathrm{Im}(\bar{a}d+\bar{b}c) & \mathrm{Re}(\bar{a}d-\bar{b}c) \end{pmatrix} \tag{2}$$

The derivation of matrix (2) demonstrates no constraints among four variables of a, b, c and d. When, however, we impose the condition

$$ad-bc=1, \tag{3}$$

matrices (1) and (2) form the unimodular group and the Lorentz group, respectively. Accordingly, generator representations of these matrices can be derived.[5)]

§3. Generators for the Jones and Mueller Matrices

3.1 *Generators for the Jones matrix*

Under the constraint (3), matrix (1) will contain three complex parameters, *i.e.* six real parameters. When these six real paramaters are introduced into matrix (1), it can be written as

$$U=\begin{pmatrix} 1+\alpha_1+i\alpha_2 & \alpha_3+i\alpha_4 \\ \alpha_5+i\alpha_6 & d(\alpha_s) \end{pmatrix} \tag{4}$$

where

$$d(\alpha_s)=\frac{1+(\alpha_3+i\alpha_4)(\alpha_5+i\alpha_6)}{1+\alpha_1+i\alpha_2}.$$

Matrix (4) has six infinitesimal transformations $I_k(k=1, \ldots\ldots, 6)$ which can be easily constructed. For example, I_1 can be obtained from eq. (4) in which all the α_s except α_1 are directly set equal to zero and α_1 is set equal to zero after differentiation of eq. (4) with respect to α_1. In a similar way, other five infinitesimal transformations $I_k(k=2, \ldots\ldots, 6)$ can be obtained. Thus, we finally have the following six infinitesimal transformations,

$$I_1=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad I_2=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\quad I_3=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$I_4=\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix},\quad I_5=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\quad I_6=\begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}. \tag{5}$$

These matrices satisfy 15 commutation relationships, some of which, for example, are

$$I_1I_3 - I_3I_1 = 2I_3,$$
$$I_1I_2 - I_2I_1 = 0.$$

Any unimodular matrix U can be expressed in the following form,

$$U = \exp\left(\sum_{s=1}^{6} \lambda_s I_s\right) \tag{6}$$

where λ_s is a real number.

We will now consider an implication of matrices (5). Infinitesimal transformations of a totally transparent system were previously derived[2,6] as

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{7}$$

while those of the partially transparent system were derived[2,6] as

$$i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{8}$$

Using these transformations we obtain the relationships

$$\begin{aligned} &I_1 = \sigma_3, && I_4 = \tfrac{1}{2}(i\sigma_1 - \sigma_2), \\ &I_2 = i\sigma_3, && I_5 = \tfrac{1}{2}(-i\sigma_2 + \sigma_1), \\ &I_3 = \tfrac{1}{2}(i\sigma_2 + \sigma_1), && I_6 = \tfrac{1}{2}(i\sigma_1 + \sigma_2). \end{aligned} \tag{9}$$

Thus, matrices (5) and matrices (7) and (8) have the same implication, and the most general generator representation of the Jones matrix for birefringence and dichroism is eq. (6).

3.2 *Generators for the Mueller matrix*

Following the method with the Jones matrix, we now derive infinitesimal transformations of the Mueller matrix. To do this, we let

$$a = 1 + \alpha_1 + i\alpha_2, \quad b = \alpha_3 + i\alpha_4,$$
$$c = \alpha_5 + i\alpha_6, \quad d = \frac{1 + bc}{a},$$

in matrix (2).

By the same calculation used in 3.1, we then find that

$$I'_1 = 2\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad I'_2 = 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$I'_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad I'_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix},$$

$$I'_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad I'_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}. \tag{10}$$

These matrices satisfy the same 15 commutation relationships as matrices (5). For example

$$I'_1I'_3 - I'_3I'_1 = 2I'_3,$$
$$I'_1I'_2 - I'_2I'_1 = 0.$$

Matrix (2) can be expressed in the form,

$$L = \exp\left(\sum_{s=1}^{6} \lambda_s I'_s\right) \tag{11}$$

where λ_s is a real number.

Let us now consider an implication of matrices (10). When 3×3 matrices are changed into 4×4 matrices as was previously shown,[2] infinitesimal transformations of the birefringent system become

$$M_X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad M_Y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$M_Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{12}$$

and those of the dichroic system are

$$N_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$N_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

By means of matrices (12) and (13), matrices (10) can be expressed as

$$\begin{aligned} &I'_1 = 2N_1, && I'_4 = iM_Y - N_3 \\ &I'_2 = 2iM_X, && I'_5 = -iM_Z + N_2 \\ &I'_3 = iM_Z + N_2, && I'_6 = iM_Y + N_3. \end{aligned} \tag{14}$$

Thus, matrices (10) have the same meaning as matrices (12) and (13), and the most general generator representation for the Mueller matrix of birefringence and dichroism is eq. (11).

Due to the following relations,

$$\boldsymbol{\sigma}_3 \leftrightarrow \boldsymbol{M}_X, \quad \boldsymbol{\sigma}_1 \leftrightarrow \boldsymbol{M}_Y, \quad \boldsymbol{\sigma}_2 \leftrightarrow \boldsymbol{M}_Z,$$
$$i\boldsymbol{\sigma}_3 \leftrightarrow \boldsymbol{N}_1, \quad i\boldsymbol{\sigma}_1 \leftrightarrow \boldsymbol{N}_2, \quad i\boldsymbol{\sigma}_2 \leftrightarrow \boldsymbol{N}_3,$$

we can show correspondences between $\boldsymbol{I}_s$ (Jones matrix) and $\boldsymbol{I}'_s$ (Mueller matrix) which is given by the homomorphism of the unimodular and Lorentz groups.

§4. Applications

The representations derived above are the most general generator representations. As examples of application we will now show the coexistence of linear birefringence with linear dichroism and that of circular birefringence with circular dichroism.

4.1 *Coexistence of linear birefringence with linear dichroism*

Let δ and δ' be a phase difference and an absorption difference. If we let

$$\lambda_1 = \frac{\delta'}{2}, \quad \lambda_2 = \frac{\delta}{2}, \quad \lambda_s = 0 (s = 3, \ldots, 6)$$

in eq. (6), then it becomes

$$\boldsymbol{U} = \exp\left(\frac{1}{2}(\delta' \boldsymbol{I}_1 + \delta \boldsymbol{I}_2)\right) = \begin{pmatrix} e^{\frac{1}{2}(i\delta + \delta')} & 0 \\ 0 & e^{-\frac{1}{2}(i\delta + \delta')} \end{pmatrix}, \qquad (15)$$

which is the Jones matrix of this case. If we let

$$\lambda_1 = \frac{\delta'}{2}, \quad \lambda_2 = \frac{\delta}{2}, \quad \lambda_s = 0 (s = 3, \ldots, 6)$$

in eq. (11), we obtain

$$\boldsymbol{L} = \exp\left(\frac{1}{2}(\delta' \boldsymbol{I}'_1 + \delta \boldsymbol{I}'_2)\right) = \begin{bmatrix} \cosh\delta' & \sinh\delta' & 0 & 0 \\ \sinh\delta' & \cosh\delta' & 0 & 0 \\ 0 & 0 & \cos\delta & \sin\delta \\ 0 & 0 & -\sin\delta & \cos\delta \end{bmatrix} \qquad (16)$$

which is the Mueller matrix corresponding to eq. (15).[7]

4.2 *Coexistence of circular birefringence with circular dichroism*

Let θ and θ' be a circular phase difference and a circular absorption difference. If we let

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = \frac{\theta}{2}, \quad \lambda_4 = -\frac{\theta'}{2},$$
$$\lambda_5 = -\frac{\theta}{2}, \quad \lambda_6 = \frac{\theta'}{2},$$

in eq. (6), we obtain

$$\boldsymbol{U} = \exp\left(\frac{1}{2}\theta(\boldsymbol{I}_3 - \boldsymbol{I}_5) + \frac{1}{2}\theta'(-\boldsymbol{I}_4 + \boldsymbol{I}_6)\right) = \begin{pmatrix} \cos\frac{1}{2}(\theta' + i\theta) & \sin\frac{1}{2}(\theta' + i\theta) \\ -\sin\frac{1}{2}(\theta' + i\theta) & \cos\frac{1}{2}(\theta' + i\theta) \end{pmatrix} \qquad (17)$$

which is the Jones matrix. If we let

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = \frac{\theta}{2}, \quad \lambda_4 = -\frac{\theta'}{2},$$
$$\lambda_5 = -\frac{\theta}{2}, \quad \lambda_6 = \frac{\theta'}{2},$$

in eq. (11), then we find that

$$\boldsymbol{L} = \exp\left(\frac{1}{2}\theta(\boldsymbol{I}'_3 - \boldsymbol{I}'_5) + \frac{1}{2}\theta'(-\boldsymbol{I}'_4 + \boldsymbol{I}'_6)\right) = \begin{bmatrix} \cosh\theta' & 0 & 0 & \sinh\theta' \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ \sinh\theta' & 0 & 0 & \cosh\theta' \end{bmatrix} \qquad (18)$$

which is the Mueller matrix corresponding to the Jones matrix (17).

§5. Conclusion

The most general generators have been derived and their physical implications have been considered. The author is grateful to Dr. T. Yamamoto who gave fruiteful suggestions.

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