Nouvelle Revue d'Optique

A unified formalism for polarization optics by using group theory

To cite this article: Hiroshi Takenaka 1973 *Nouvelle Revue d'Optique* **4** 37

Nouv. Rev. Optique, 1973, t. *4*, n° 1, pp. 37-41

RESEARCH LABORATORY, NIPPON KOGAKU CO., Ltd.
Shinagawaku, Tokyo, Japan

A UNIFIED FORMALISM FOR POLARIZATION OPTICS BY USING GROUP THEORY

by

Hiroshi TAKENAKA

MOTS CLÉS :
Lumière Polarisée

KEY WORDS :
Polarized light

SUMMARY

A unified treatment of three polarization calculus methods — Jones matrix, Mueller matrix and Poincaré sphere — is accomplished by using the concept of the Lie group. In the case of the totally transparent system, it is related to the compact Lie group, and in the case of the partially transparent system, it is related to the homogeneous Lorentz group. The physical meaning of the Pauli's spin matrices is given and the generator is introduced and used extensively.

RÉSUMÉ

Formalisme unifié pour la polarisation, à partir de la théorie des groupes.

Le traitement unifié de trois méthodes de calcul pour la polarisation — matrices de Jones, Mueller, et la sphère de Poincaré — est fait en utilisant la notion du groupe Lie. Dans le cas du système complètement transparent, le traitement est relié au groupe compact Lie, tandis qu'il est relié au groupe homogène Lorentz en cas du système partiellement transparent. La signification optique des matrices de spin Pauli est donnée et le générateur est introduit pour l'analyse.

INTRODUCTION

Various methods are used when we analyze or synthesize complicated optical systems related to polarized light. The most well-known methods are the Jones 2 × 2 matrix, the Mueller 4 × 4 matrix, and the Poincaré sphere [1]. The relation between these is treated in many papers [2], [3]. But most papers use only algebraic calculation and do not refer to group theory and group theoretical aspects of polarization calculus. The aim of this paper is to show that the relation between these methods can be understood clearly by using group theory in polarization calculus [4] and to introduce the generator representation.

TOTALLY TRANSPARENT SYSTEM

1. — Lie group

The distinguishing characteristic of a Lie group is that the parameters of a product element are analytic functions of the parameters of the factors. This is true of O_3^+, SU(2) and the Lorentz group which will be considered later.

The concept of the generator is developed by this differentiability and it allows us to reduce the study of the whole group to a study of the group elements in the neighbourhood of the identity element. If the parameter varies in a closed interval, it is said to be compact and its important nature is that all representations of the compact groups are equivalent to the unitary repre sentation. O_3^+ and SU(2) are compact. On the other hand, the Lorentz group is not compact so that its representation is not unitary [5].

2. — Orthogonal group O_3^+

A 3 × 3 real orthogonal matrix forms a group and its independent parameters are 3. The matrix whose determinant is + 1 is called O_3^+. The rotations around the orthogonal axes are

$$\mathbf{R}_X(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{pmatrix}$$

$$(1) \qquad \mathbf{R}_Y(\psi) = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}$$

$$\mathbf{R}_Z(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The general member of O_3^+ is represented by

(2) $\mathbf{R}(\alpha, \beta, \gamma) = \mathbf{R}_Z(\gamma)\,\mathbf{R}_X(\beta)\,\mathbf{R}_Z(\alpha)$.

This is the Euler angles in classical mechanics [6].

3. — Special unitary group SU(2)

The Unitary matrix with determinant 1 also forms a group. This is called the special unitary group SU(2). The 2×2 unitary, unit determinant matrix has 3 parameters just as O_3^+ does. The general member of SU(2) is written by

(3) $\mathbf{U} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a\bar{a} + b\bar{b} = 1$

where a and b are complex numbers and – means complex conjugate.

4. — SU(2) — O_3^+ homomorphism

O_3^+ represents the rotation in the ordinary three dimensional space, and it has the characteristic of leaving $X^2 + Y^2 + Z^2$ invariant.

The operation of SU(2) on a matrix is given by a unitary transformation

(4) $\mathbf{M}' = \mathbf{UMU}^+$

where + means ajoint.

Taking $\mathbf{M}$ to be a 2×2 matrix, it can be represented by the linear combination of the unit matrix $\mathbf{1}$ and the Pauli's spin matrices $\boldsymbol{\sigma}_k$ ($k = 1, 2, 3$)

(5) $$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\mathbf{M}$ be a zero trace matrix, then

(6) $$\mathbf{M} = X\boldsymbol{\sigma}_3 + Y\boldsymbol{\sigma}_1 + Z\boldsymbol{\sigma}_2 = \begin{pmatrix} X & Y - \mathrm{i}Z \\ Y + \mathrm{i}Z & -X \end{pmatrix}.$$

Since the trace is invariant under the unitary transformation, the form of $\mathbf{M}'$ must be

(7) $$\mathbf{M}' = X'\boldsymbol{\sigma}_3 + Y'\boldsymbol{\sigma}_1 + Z'\boldsymbol{\sigma}_2 = \begin{pmatrix} X' & Y' - \mathrm{i}Z' \\ Y' + \mathrm{i}Z' & -X' \end{pmatrix}.$$

As the determinant also remains invariant under the unitary transformation, therefore

$$-(X^2 + Y^2 + Z^2) = -(X'^2 + Y'^2 + Z'^2).$$

In other words $X^2 + Y^2 + Z^2$ is invariant under the operation of SU(2). This is the same as the case of O_3^+ and so SU(2) represents the rotation.

Here let us consider a special case and see how SU(2) describes the rotation. If we put $a = \mathrm{e}^{\mathrm{i}\psi/2}$ $b = 0$ in equation (3), then

$$\mathbf{U}_X\left(\frac{\psi}{2}\right) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\psi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\psi/2} \end{pmatrix}.$$

Calculating $\mathbf{U}_X \mathbf{M} \mathbf{U}_X^+$ and using equations (4), (6), and (7) we get

$$\begin{aligned} X' &= X \\ Y' &= Y\cos\psi + Z\sin\psi \\ Z' &= -Y\sin\psi + Z\cos\psi . \end{aligned}$$

We see that the 2×2 unitary transformation of the form $\mathbf{U}_X(\psi/2)$ is equivalent to $\mathbf{R}_X(\psi)$ of equation (1)

$$\mathbf{U}_X\left(\frac{\psi}{2}\right) \leftrightarrow \mathbf{R}_X(\psi). \qquad (8)$$

Similarly

$$\mathbf{U}_Y\left(\frac{\psi}{2}\right) = \begin{pmatrix} \cos\frac{\psi}{2} & \mathrm{i}\sin\frac{\psi}{2} \\ \mathrm{i}\sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{pmatrix} \leftrightarrow \mathbf{R}_Y(\psi)$$

(8)

$$\mathbf{U}_Z\left(\frac{\psi}{2}\right) = \begin{pmatrix} \cos\frac{\psi}{2} & \sin\frac{\psi}{2} \\ -\sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{pmatrix} \leftrightarrow \mathbf{R}_Z(\psi).$$

The correspondence is two to one and SU(2) and O_3^+ are homomorphic. From this the representation of SU(2) give O_3^+ automatically.

Corresponding to equation (2) the general member of SU(2) is

(9) $$\mathbf{U}(\alpha, \beta, \gamma) = \mathbf{U}_Z\left(\frac{\gamma}{2}\right)\mathbf{U}_X\left(\frac{\beta}{2}\right)\mathbf{U}_Z\left(\frac{\alpha}{2}\right).$$

5. — The combination of polarization matrix and group theory

Considering the previous content, the optical meaning is clear. The Jones matrix of birefringence or optical activity represents the operation in SU(2) and the Mueller matrix of birefringence or optical activity represents the operation in O_3^+. The Poincaré sphere is the spatial representation of the Mueller matrix. The correspondence between the Jones matrix and the Mueller matrix is two to one.

6. — Pauli's spin matrices and generators for SU(2)

We consider the meaning of Pauli's spin matrix (5). Let $\delta\psi$ be a small angle, then $\mathbf{U}_X(\delta\psi/2)$ is developed as

$$\mathbf{U}_X\left(\frac{\delta\psi}{2}\right) = \begin{pmatrix} 1 + \mathrm{i}\frac{\delta\psi}{2} & 0 \\ 0 & 1 - \mathrm{i}\frac{\delta\psi}{2} \end{pmatrix} = \mathbf{1} + \mathrm{i}\frac{\delta\psi}{2}\boldsymbol{\sigma}_3 .$$

Similarly

$$\mathbf{U}_Y\left(\frac{\delta\psi}{2}\right) = \mathbf{1} + \mathrm{i}\frac{\delta\psi}{2}\boldsymbol{\sigma}_1 ,$$

$$\mathbf{U}_Z\left(\frac{\delta\psi}{2}\right) = \mathbf{1} + \mathrm{i}\frac{\delta\psi}{2}\boldsymbol{\sigma}_2 .$$

From the above result it is seen that $\boldsymbol{\sigma}_3$, $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ correspond to the infinitesimal rotations around the X, Y, and Z axes, respectively. Next we show that a finite rotation can be obtained by successive infinitesimal rotation by picking up $\mathbf{U}_X(\psi/2)$ as an example. Let $\delta\psi_1$, $\delta\psi_2$ be small angles, then

$$\mathbf{U}_X\left(\frac{\delta\psi_2}{2} + \frac{\delta\psi_1}{2}\right) = \left(\mathbf{1} + \mathrm{i}\frac{\delta\psi_2}{2}\boldsymbol{\sigma}_3\right)\left(\mathbf{1} + \mathrm{i}\frac{\delta\psi_1}{2}\boldsymbol{\sigma}_3\right).$$

If we put $\delta\psi = \psi/N$ and if $N \to \infty$, then

$$\text{(10)} \quad \mathbf{U}_X\left(\frac{\psi}{2}\right) = \lim_{N\to\infty}\left[\mathbf{1} + \mathrm{i}\frac{\delta\psi}{2}\boldsymbol{\sigma}_3\right]^N = \exp\left(\mathrm{i}\frac{\psi}{2}\boldsymbol{\sigma}_3\right).$$

Similarly

$$\text{(10)} \quad \mathbf{U}_Y\left(\frac{\psi}{2}\right) = \exp\left(\mathrm{i}\frac{\psi}{2}\boldsymbol{\sigma}_1\right), \quad \mathbf{U}_Z\left(\frac{\psi}{2}\right) = \exp\left(\mathrm{i}\frac{\psi}{2}\boldsymbol{\sigma}_2\right).$$

The spin matrices can be o btained by the differentiation of equations (8). If the differentiation of a matrix means a matrix of the differentiation, then

$$\text{(11)} \quad \left(\frac{\mathrm{d}\mathbf{U}_X}{\mathrm{d}\psi}\right)_{\psi=0} = \mathrm{i}\boldsymbol{\sigma}_3 , \quad \left(\frac{\mathrm{d}\mathbf{U}_Y}{\mathrm{d}\psi}\right)_{\psi=0} = \mathrm{i}\boldsymbol{\sigma}_1 ,$$

$$\left(\frac{\mathrm{d}\mathbf{U}_Z}{\mathrm{d}\psi}\right)_{\psi=0} = \mathrm{i}\boldsymbol{\sigma}_2 .$$

The validity of these equations depends on the differentiability of the Lie group.

The generators (10) can be developed by MacLaurin development and we get finally :

$$\mathbf{U}_X\left(\frac{\psi}{2}\right) = \exp\left(\frac{1}{2}\mathrm{i}\psi\boldsymbol{\sigma}_3\right) = \mathbf{1}\cos\frac{\psi}{2} + \mathrm{i}\boldsymbol{\sigma}_3\sin\frac{\psi}{2} ,$$

$$\text{(12)} \quad \mathbf{U}_Y\left(\frac{\psi}{2}\right) = \exp\left(\frac{1}{2}\mathrm{i}\psi\boldsymbol{\sigma}_1\right) = \mathbf{1}\cos\frac{\psi}{2} + \mathrm{i}\boldsymbol{\sigma}_1\sin\frac{\psi}{2} ,$$

$$\mathbf{U}_Z\left(\frac{\psi}{2}\right) = \exp\left(\frac{1}{2}\mathrm{i}\psi\boldsymbol{\sigma}_2\right) = \mathbf{1}\cos\frac{\psi}{2} + \mathrm{i}\boldsymbol{\sigma}_2\sin\frac{\psi}{2} .$$

And we get the generator representation corresponding to equation (9)

$$\text{(13)} \quad \mathbf{U}(\alpha, \beta, \gamma) = \exp(\tfrac{1}{2}\mathrm{i}\gamma\boldsymbol{\sigma}_2)\exp(\tfrac{1}{2}\mathrm{i}\beta\boldsymbol{\sigma}_3)\exp(\tfrac{1}{2}\mathrm{i}\alpha\boldsymbol{\sigma}_2) .$$

7. — Generators for O_3^+

A matrix which describes a finite rotation $\mathbf{R}_Z(\psi)$ around the Z axis comes from equation (8)

$$\text{(14)} \quad \mathbf{R}_Z(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\delta\psi$ be an infinitesimal angle, then

$$\mathbf{R}_Z(\delta\psi) = \begin{pmatrix} 1 & \delta\psi & 0 \\ -\delta\psi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{1} + \mathrm{i}\,\delta\psi\mathbf{M}_Z$$

where $\mathbf{1}$ and $\mathbf{M}_Z$ are

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{M}_Z = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\mathbf{M}_Z$ corresponds to the infinitesimal rotation around the Z axis. $\mathbf{M}_Z$ can be obtained also by the differentiation equation (14),

$$\left(\frac{\mathrm{d}\mathbf{R}_Z}{\mathrm{d}\psi}\right)_{\psi=0} = \mathrm{i}\mathbf{M}_Z .$$

It is the same as in the case of SU(2) that a finite rotation ψ can be obtained by successive infinitesimal rotation $\delta\psi$ and

$$\text{(15)} \quad \mathbf{R}_Z(\psi) = \exp(\mathrm{i}\psi\mathbf{M}_Z), \ \mathbf{M}_Z = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This is the generator of $\mathbf{R}_Z(\psi)$. Similarly

$$\left(\frac{\mathrm{d}\mathbf{R}_X}{\mathrm{d}\psi}\right)_{\psi=0} = \mathrm{i}\mathbf{M}_X , \quad \left(\frac{\mathrm{d}\mathbf{R}_Y}{\mathrm{d}\psi}\right)_{\psi=0} = \mathrm{i}\mathbf{M}_Y ,$$

$$\mathbf{R}_X(\psi) = \exp(\mathrm{i}\psi\mathbf{M}_X) , \qquad \mathbf{R}_Y(\psi) = \exp(\mathrm{i}\psi\mathbf{M}_Y) ,$$

$$\mathbf{M}_X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad \mathbf{M}_Y = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}.$$

By using these generator representations, we get the expression corresponding to equation (2)

$$\text{(16)} \quad \mathbf{R}(\alpha, \beta, \gamma) = \exp(\mathrm{i}\gamma\mathbf{M}_Z)\exp(\mathrm{i}\beta\mathbf{M}_X)\exp(\mathrm{i}\alpha\mathbf{M}_Z) .$$

There is the correspondence between the Pauli's spin matrices and the generators of O_3^+

$$\mathrm{SU(2)} \quad \begin{matrix} \boldsymbol{\sigma}_3 \leftrightarrow \mathbf{M}_X \\ \boldsymbol{\sigma}_1 \leftrightarrow \mathbf{M}_Y \\ \boldsymbol{\sigma}_2 \leftrightarrow \mathbf{M}_Z \end{matrix} \quad O_3^+ .$$

It is not difficult to show that the commutation relation of the generators of O_3^+ is

$$[\mathbf{M}_i, \mathbf{M}_j] = \mathrm{i}\varepsilon_{ijk}\mathbf{M}_k$$

and that of the Pauli's spin matrices is

$$[\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j] = 2\,\mathrm{i}\varepsilon_{ijk}\boldsymbol{\sigma}_k$$

where ε_{ijk} is the Levi-Civita symbol.

If we take into consideration $\mathbf{M}_k^3 = \mathbf{M}_k$ $(k = X, Y, Z)$, it is not difficult to develop equations (15) and we get finally, corresponding to equations (12)

$$\mathbf{R}_X(\psi) = \mathbf{1} + \mathrm{i}\mathbf{M}_X\sin\psi + \mathbf{M}_X^2(\cos\psi - 1) ,$$

$$\text{(17)} \quad \mathbf{R}_Y(\psi) = \mathbf{1} + \mathrm{i}\mathbf{M}_Y\sin\psi + \mathbf{M}_Y^2(\cos\psi - 1) ,$$

$$\mathbf{R}_Z(\psi) = \mathbf{1} + \mathrm{i}\mathbf{M}_Z\sin\psi + \mathbf{M}_Z^2(\cos\psi - 1) .$$

PARTIALLY TRANSPARENT SYSTEM

In the case of the totally transparent system, from the fact that the intensity is invariant, the analysis of the four dimensional space reduces to that of the three dimensional space. In the case of the partially transparent system, we must treat the four dimensional space itself. But as the four dimensional space can be obtained by extending the three dimensional space, we can discuss this case in the same way as the three dimensional case. So we discuss briefly in this case.

1. — Unimodular group and Lorentz group

The 2×2 matrix with determinant $+1$ forms a group. This group is called the unimodular group. On the other hand, the 4×4 orthogonal matrix which represents the rotation in the four dimensional space (X_0, X_1, X_2, X_3), that is the Lorentz transformation, also forms a group. This group is called the Lorentz group.

The unimodular group and the Lorentz group are homomorphic. The correspondence is two to one.

The optical interpretation is straightforward. The Jones matrix of dichroism or circular dichroism represents the operation in the unimodular group and the Mueller matrix of dichroism or circular dichroism represents the operation in the Lorentz group. The correspondence between the Jones and Mueller matrices is two to one.

These are expressed as :

$$\mathbf{U}_1\left(\frac{\delta}{2}\right) = \begin{pmatrix} e^{\delta/2} & 0 \\ 0 & e^{-\delta/2} \end{pmatrix} \leftrightarrow$$

$$\leftrightarrow \mathbf{L}_1(\delta) = \begin{pmatrix} \cosh\delta & \sinh\delta & 0 & 0 \\ \sinh\delta & \cosh\delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(18)\quad \mathbf{U}_2\left(\frac{\delta}{2}\right) = \begin{pmatrix} \cosh\frac{\delta}{2} & \sinh\frac{\delta}{2} \\ \sinh\frac{\delta}{2} & \cosh\frac{\delta}{2} \end{pmatrix} \leftrightarrow$$

$$\leftrightarrow \mathbf{L}_2(\delta) = \begin{pmatrix} \cosh\delta & 0 & \sinh\delta & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\delta & 0 & \cosh\delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{U}_3\left(\frac{\delta}{2}\right) = \begin{pmatrix} \cosh\frac{\delta}{2} & -\mathrm{i}\sinh\frac{\delta}{2} \\ \mathrm{i}\sinh\frac{\delta}{2} & \cosh\frac{\delta}{2} \end{pmatrix} \leftrightarrow$$

$$\leftrightarrow \mathbf{L}_3(\delta) = \begin{pmatrix} \cosh\delta & 0 & 0 & \sinh\delta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\delta & 0 & 0 & \cosh\delta \end{pmatrix}.$$

Here let us consider the meaning of the matrix $\mathbf{L}_k(\delta)$, $(k = 1, 2, 3)$. As an example we take $\mathbf{L}_1(\delta)$. If we replace X_0 by $\mathrm{i}X_0$, then we get

$$\begin{pmatrix} \mathrm{i}X'_0 \\ X'_1 \\ X'_2 \\ X'_3 \end{pmatrix} = \begin{pmatrix} \cosh\delta & \mathrm{i}\sinh\delta & 0 & 0 \\ -\mathrm{i}\sinh\delta & \cosh\delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{i}X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix}.$$

Taking into consideration

$$\cosh\delta = \cos(\mathrm{i}\delta)\,, \quad \mathrm{i}\sinh\delta = \sin(\mathrm{i}\delta)\,,$$

then the above equation becomes

$$\begin{pmatrix} \mathrm{i}X'_0 \\ X'_1 \\ X'_2 \\ X'_3 \end{pmatrix} = \begin{pmatrix} \cos(\mathrm{i}\delta) & \sin(\mathrm{i}\delta) & 0 & 0 \\ -\sin(\mathrm{i}\delta) & \cos(\mathrm{i}\delta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{i}X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

and the Stokes vectors

$$(X'_0, X'_1, X'_2, X'_3) \quad \text{and} \quad (X_0, X_1, X_2, X_3)$$

satisfy the equation

$$(\mathrm{i}X'_0)^2 + X'^2_1 + X'^2_2 + X'^2_3 = (\mathrm{i}X_0)^2 + X_1^2 + X_2^2 + X_3^2\,.$$

Therefore if we consider the $\mathrm{i}X_0$ axis as a real axis, the modified matrix of $\mathbf{L}_1(\delta)$ represents the rotation of a complex angle $\mathrm{i}\delta$ in the $\mathrm{i}X_0 - X_1$ plane.

Similarly $\mathbf{L}_2(\delta)$ and $\mathbf{L}_3(\delta)$ represent the rotations of complex angles $\mathrm{i}\delta$ in the $\mathrm{i}X_0 - X_2$ plane and $\mathrm{i}X_0 - X_3$ plane, respectively. This is the extension of the Poincaré sphere into the four dimensional space.

2. — Generators for the unimodular group and the Lorentz group

It is not difficult to show that

$$\mathbf{U}_1\left(\frac{\delta}{2}\right) = \exp\left(\frac{1}{2}\delta\boldsymbol{\sigma}_3\right) = \mathbf{1}\cosh\frac{\delta}{2} + \boldsymbol{\sigma}_3\sinh\frac{\delta}{2}\,,$$

$$(19)\quad \mathbf{U}_2\left(\frac{\delta}{2}\right) = \exp\left(\frac{1}{2}\delta\boldsymbol{\sigma}_1\right) = \mathbf{1}\cosh\frac{\delta}{2} + \boldsymbol{\sigma}_1\sinh\frac{\delta}{2}\,,$$

$$\mathbf{U}_3\left(\frac{\delta}{2}\right) = \exp\left(\frac{1}{2}\delta\boldsymbol{\sigma}_2\right) = \mathbf{1}\cosh\frac{\delta}{2} + \boldsymbol{\sigma}_2\sinh\frac{\delta}{2}\,,$$

and

$$\mathbf{L}_1(\delta) = \exp(\delta\mathbf{N}_1) = \mathbf{1} + \mathbf{N}_1\sinh\delta + \mathbf{N}_1^2(\cosh\delta - 1)\,,$$

$$\mathbf{L}_2(\delta) = \exp(\delta\mathbf{N}_2) = \mathbf{1} + \mathbf{N}_2\sinh\delta + \mathbf{N}_2^2(\cosh\delta - 1)\,,$$

$$\mathbf{L}_3(\delta) = \exp(\delta\mathbf{N}_3) = \mathbf{1} + \mathbf{N}_3\sinh\delta + \mathbf{N}_3^2(\cosh\delta - 1)$$

(20)

where $\mathbf{N}_k$ ($k = 1, 2, 3$) and $\mathbf{1}$ are

$$\mathbf{N}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{N}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{N}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

These are the generator representations, and this corresponds to the Galilean transformation. When we compare equation (12) and (19), equations (17) and (20), we recognize the very close similarities.

CONCLUSION

We have explained the structure and relationship of three polarization calculus methods using group theory.

— The generators for the Jones matrix and the Mueller matrix are introduced and considered their meanings and used extensively.

— The geometrical meaning of the matrix of dichroism or circular dichroism is shown and we see that it is related to the Lorentz transformation.

— The similarity between the matrix of birefringence and that of dichroism is pointed out.

— The relationship between the Jones matrix and the Mueller matrix is examined.

ACKNOWLEDGEMENTS. — The author wishes to express his thanks to Dr. T. Yamamoto, Nikon Research Laboratory, who read this paper and gave fruitful discussions, and Professor M. Goto, the Mathematics Department of the University of Pennsylvania, who gave useful suggestions.

REFERENCES

[1] SCHRCLIFF (W. A.). — Polarized light, *Harvard Univ. Press*, Cambridge, 1962.
[2] SCHMIEDER (R. W.). — *J. Opt. Soc. Amer.*, 1969, **59**, 297.
[3] MARATHY (A. A.). — *J. Opt. Soc. Amer.*, 1965, **55**, 969.
[4] TAKENAKA (H.). — *Japan J. appl. Phys.* (to be published).
[5] HIGMAN (B.). — Applied group-theoretic and matrix methods, *Dover*, New York 1964.
[6] GOLDSTEIN (H.). — Classical mechanics, *Addison-Wesley*. Reading, Mass. 1950.

(Manuscrit reçu le 20 décembre 1972.)