Nouvelle Revue d'Optique

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To cite this article: Hiroshi Takenaka 1973 *Nouvelle Revue d'Optique* **4** 105

Nouv. Rev. Optique, t. *4*, 1973, n° 2, pp. 105-107

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GENERAL BIREFRINGENCE AND GENERAL DICHROISM

by

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MOTS CLÉS :
Lumière polarisée
KEY WORDS :
Polarized light

SUMMARY

General birefringence and general dichroism are examined by applying group theory to polarization calculus.

RÉSUMÉ

La biréfringence générale et dichroïsme général

La biréfringence générale et le dichroïsme général sont étudiés en appliquant la théorie des groupes dans le calcul de polarisation.

INTRODUCTION

We have shown in the preceeding paper [1] that a unified formalism for polarization optics is possible through the application of group theory. In this paper we will use the former results to analyze general birefringence and general dichroism. General birefringence is birefringence which contains both linear and circular birefringence (optical activity). General dichroism is dichroism which contains both linear and circular dichroism. The application of group theory to these specific problems provides a more clear understanding of the polarization calculus.

GENERAL BIREFRINGENCE

Let $\mathbf{n}$ be the unit vector

$$\mathbf{n} = (\alpha, \beta, \gamma)\,, \quad \alpha^2 + \beta^2 + \gamma^2 = 1$$

and let $\boldsymbol{\sigma}$ and $\mathbf{1}$ be the vector

$$\boldsymbol{\sigma} = (\boldsymbol{\sigma}_3, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2)$$

and 2×2 unit matrix respectively, where

$$\boldsymbol{\sigma}_k \, (k = 1, 2, 3)$$

is the Pauli's spin matrix :

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A 2×2 matrix representation which corresponds to the rotation around the axis whose direction cosines are (α, β, γ) with reference to Cartesian coordinate (X, Y, Z) is given by

$$(1) \qquad \exp\left(\mathrm{i}(\mathbf{n}.\boldsymbol{\sigma})\frac{\psi}{2}\right) = \mathbf{1}\cos\frac{\psi}{2} + \mathrm{i}(\mathbf{n}.\boldsymbol{\sigma})\sin\frac{\psi}{2}$$

$$= \mathbf{1}\cos\frac{\psi}{2} + \mathrm{i}(\alpha\boldsymbol{\sigma}_3 + \beta\boldsymbol{\sigma}_1 + \gamma\boldsymbol{\sigma}_2)\sin\frac{\psi}{2}$$

$$= \begin{pmatrix} \cos\frac{\psi}{2} + \mathrm{i}\alpha\sin\frac{\psi}{2} & \mathrm{i}(\beta - \mathrm{i}\gamma)\sin\frac{\psi}{2} \\ \mathrm{i}(\beta + \mathrm{i}\gamma)\sin\frac{\psi}{2} & \cos\frac{\psi}{2} - \mathrm{i}\alpha\sin\frac{\psi}{2} \end{pmatrix}.$$

The optical meaning of this Jones matrix can be understood clearly when we consider equation (1).

$\mathrm{i}\boldsymbol{\sigma}_3$ and $\mathrm{i}\boldsymbol{\sigma}_1$ represent linear birefringence whose azimuth angle of the fast axis is 0° and 45° respectively. $\mathrm{i}\boldsymbol{\sigma}_2$ represents circular birefringence (optical activity). So the direction cosine (α, β, γ) represents the degree of a mixing (linear birefringence 0°, linear birefringence 45°, circular birefringence). Next let us consider the corresponding Mueller matrix.

Let $\mathbf{M}$ and $\mathbf{1}$ be the vector

$$\mathbf{M} = (\mathbf{M}_X, \mathbf{M}_Y, \mathbf{M}_Z)$$

and the 3×3 unit matrix respectively, where $\mathbf{M}_k$ $(k = X, Y, Z)$ are

(2) $$\mathbf{M}_X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix},$$

$$\mathbf{M}_Y = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \quad \mathbf{M}_Z = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Corresponding to equation (1) the rotation around the axis whose direction cosine is (α, β, γ) with respect to an orthogonal coordinate is given by

(3) $$\exp(\mathrm{i}\mathbf{n}.\mathbf{M}\psi) = \exp(\mathrm{i}\mathbf{R}\psi).$$

Expressing $\mathbf{R}$ explicitely

(4) $$\mathbf{R} = \mathbf{n}.\mathbf{M} = \alpha\mathbf{M}_X + \beta\mathbf{M}_Y + \gamma\mathbf{M}_Z = \begin{pmatrix} 0 & -\mathrm{i}\gamma & \mathrm{i}\beta \\ \mathrm{i}\gamma & 0 & -\mathrm{i}\alpha \\ -\mathrm{i}\beta & \mathrm{i}\alpha & 0 \end{pmatrix}$$

and taking into consideration the relation

$$\mathbf{R}^3 = \mathbf{R},$$

the equation (3) can be expressed as

(5) $$\exp(\mathrm{i}\mathbf{R}\psi) = \mathbf{1} + \mathrm{i}\mathbf{R}\sin\psi + \mathbf{R}^2(\cos\psi - 1).$$

Substituting the matrix (4) into equation (5), equation (5) becomes

(6) $$\begin{pmatrix} \cos\psi + 2\alpha^2\sin^2\frac{\psi}{2} & \gamma\sin\psi + 2\alpha\beta\sin^2\frac{\psi}{2} & -\beta\sin\psi + 2\alpha\gamma\sin^2\frac{\psi}{2} \\ -\gamma\sin\psi + 2\alpha\beta\sin^2\frac{\psi}{2} & \cos\psi + 2\beta^2\sin^2\frac{\psi}{2} & \alpha\sin\psi + 2\beta\gamma\sin^2\frac{\psi}{2} \\ \beta\sin\psi + 2\alpha\gamma\sin^2\frac{\psi}{2} & -\alpha\sin\psi + 2\beta\gamma\sin^2\frac{\psi}{2} & \cos\psi + 2\gamma^2\sin^2\frac{\psi}{2} \end{pmatrix}.$$

Again the optical meaning of this Mueller matrix can be understood clearly when we consider equation (4). It is the same as the case of the Jones matrix. When we consider a special case of equation (1) or (3), (5) or (6), we get a representation of an elementary optical system which was considered in the previous paper [1].

GENERAL DICHROISM

Let us consider the 2×2 matrix representation representing a rotation $\mathrm{i}\delta$ in the plane which contains the $\mathrm{i}X_0$ axis and an axis whose direction cosine is (α, β, γ) with respect to the (X_1, X_2, X_3) axes, in the four dimensional space.

It is represented by

(7) $$\exp\left((\mathbf{n}.\boldsymbol{\sigma})\frac{\delta}{2}\right) = \mathbf{1}\cosh\frac{\delta}{2} + (\mathbf{n}.\boldsymbol{\sigma})\sinh\frac{\delta}{2} =$$

$$= \mathbf{1}\cosh\frac{\delta}{2} + (\alpha\boldsymbol{\sigma}_3 + \beta\boldsymbol{\sigma}_1 + \gamma\boldsymbol{\sigma}_2)\sinh\frac{\delta}{2}$$

$$= \begin{pmatrix} \cosh\frac{\delta}{2} + \alpha\sinh\frac{\delta}{2} & (\beta - \mathrm{i}\gamma)\sinh\frac{\delta}{2} \\ (\beta + \mathrm{i}\gamma)\sinh\frac{\delta}{2} & \cosh\frac{\delta}{2} - \alpha\sinh\frac{\delta}{2} \end{pmatrix}.$$

The optical meaning of this Jones matrix can be obtained by considering equation (7). In this case $\boldsymbol{\sigma}_3$ and $\boldsymbol{\sigma}_1$ represent linear dichroism whose azimuth angle of the axis is 0° and 45° respectively. $\boldsymbol{\sigma}_2$ represents circular dichroism. So the direction cosine (α, β, γ) represents the degree of mixing (linear dichroism 0°, linear dichroism 45°, circular dichroism). Now let us consider the corresponding Mueller representation. Let $\mathbf{N}$ and $\mathbf{1}$ be the vector

$$\mathbf{N} = (\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3)$$

and the 4×4 unit matrix respectively, where $\mathbf{N}_k$ $(k = 1, 2, 3)$ are

(8) $$\mathbf{N}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{N}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{N}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Corresponding to equations (7), the rotation $\mathrm{i}\delta$ in the plane which contains the $\mathrm{i}X_0$ axis and an axis whose direction cosine is (α, β, γ) with respect to the (X_1, X_2, X_3) axes in the four dimensional space is given by

(9) $$\exp(\mathbf{n}.\mathbf{N}\delta) = \exp(\mathbf{L}\delta).$$

Expressing $\mathbf{L}$ explicitely

(10) $$\mathbf{L} = \mathbf{n}.\mathbf{N} = \alpha\mathbf{N}_1 + \beta\mathbf{N}_2 + \gamma\mathbf{N}_3 = \begin{pmatrix} 0 & \alpha & \beta & \gamma \\ \alpha & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 \\ \gamma & 0 & 0 & 0 \end{pmatrix}$$

and taking into consideration the relation

$$\mathbf{L}^3 = \mathbf{L},$$

then equation (9) can be expressed as

(11) $$\exp(\mathbf{L}\delta) = \mathbf{1} + \mathbf{L}\sinh\delta + \mathbf{L}^2(\cosh\delta - 1).$$

Substituting the matrix (10) into equation (11), we get

$$
\begin{pmatrix}
\cosh \delta & \alpha \sinh \delta & \beta \sinh \delta & \gamma \sinh \delta \\
\alpha \sinh \delta & 1 + \alpha^2(\cosh \delta - 1) & \alpha\beta(\cosh \delta - 1) & \alpha\gamma(\cosh \delta - 1) \\
\beta \sinh \delta & \alpha\beta(\cosh \delta - 1) & 1 + \beta^2(\cosh \delta - 1) & \beta\gamma(\cosh \delta - 1) \\
\gamma \sinh \delta & \alpha\gamma(\cosh \delta - 1) & \beta\gamma(\cosh \delta - 1) & 1 + \gamma^2(\cosh \delta - 1)
\end{pmatrix} . \tag{12}
$$

Again the optical meaning of this Mueller matrix can be understood clearly when we consider equation (10). It is the same as that of the Jones matrix. When we consider a special case of equation (7) or (9), (11) and (12), we get a representation of a familiar elementary optical system which was considered in the former paper. Now comparing equation (1) with equation (7) and equation (5) with equation (11), the similarity is very clear. This is due to the extension of the unimodular unitary group to the unimodular group, in other words, the rotation group to the Lorentz group.

CONCLUSION

The Jones and Mueller matrices of general birefringence and general dichroism are derived, and the significance is discussed. The similarities between these are also shown.

ACKNOWLEDGMENT. — The author wishes to thank Dr. T. Yamamoto, NIKON Research Laboratory, for his helpful and useful suggestions and for reading this paper.

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[1] TAKENAKA (H.). — *The previous paper* : A unified formalism for polarization optics by using group theory, *Nouv. Rev. Optique*, 1973, *4*, 37-41.

(Manuscrit reçu le 20 décembre 1972.)