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Generalized Lorentz transformation in polarization optics

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Starting with the analogy between the Stokes quadrivector of optical polarization states and the Minkovskian quadrivector of relativistic events, the equivalent in polarization theory of the vectorial form of Lorentz boost equations is established. From these equations, the composition law of Poincaré gyrovectors and the gain equation in the interaction of dichroic devices with partially polarized light are deduced. It is shown that the equation of the gain (general Malus' law) is, up to a (slight but essential) generalization, the analog in polarization theory of the time equation of Lorentz boost. This generalization can be extended to other fields and problems of physics where the Lorentzian character of the underlying mathematics was recognized.

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1. INTRODUCTION

It was only in 1963 that what we know today to be the deep mathematical connection between the special theory of relativity (STR) and the polarization theory (PT) (more precisely the theory of action of deterministic polarization devices on the polarized light) [1–4] was noticed. Barakat [5] noted that one of the invariants of the coherency (polarization) matrix, namely, its determinant, expressed in terms of Stokes parameters: "has the form of a Lorentz line element. This fact allows us to apply group-theoretic methods employing the Lorentz group to discuss the coherency matrix. It seems surprising that no one called attention to this point." In other words, the Stokes vector of PT is a Minkowskian quadrivector, i.e., the metric of the space of optical polarization states (SOP) is the same as that of the space of events. After a decade, in 1973, Takenaka [6] treated the interaction of light with deterministic polarization devices as a Lorentz transformation, in the frame of group theory. Since then it has become firmly established that PT (precisely the field specified above) has the same mathematical underpinning as STR, and a lot of papers [7–18] have exploited this kinship, in the benefit of PT, which, from this viewpoint, was less elaborated than STR.

Another prerequisite for this paper is that, after an effort of abstraction lasting for two decades, starting with the composition law of relativistically allowed velocities [19],

$$\mathbf{w} = \mathbf{u} \oplus \mathbf{v} = \frac{\mathbf{u}+\mathbf{v}}{1+\mathbf{u}\cdot\mathbf{v}} + \frac{\gamma_u}{\gamma_u+1}\frac{\mathbf{u}\times(\mathbf{u}\times\mathbf{v})}{1+\mathbf{u}\cdot\mathbf{v}}, \tag{1}$$

Ungar [20] brought to light the world of some new mathematical entities, which he called "gyrovectors." The gyrovectors are 3D vectors which have a group-like structure, in the sense that they verify the closure condition with respect to the composition ("addition") law Eq. (1), but this addition is neither commutative nor associative; they form a "gyrogroup"; e.g., the relativistically permitted velocities form a gyrogroup due to the second postulate of STR. Ungar has not identified other gyrovectors in physics. As we shall see, there are many physical gyrovectors and the first of them consist of the Poincaré vectors of PT.

The aim of this paper is to establish the PT equivalent—in fact a generalization—of the vectorial form of Lorentz boost equations and, on this basis, the equation of the state of polarization and that of the gain resulting in the action of a dichroic device on partially polarized light. The equation of the SOP of the emergent light proves to be the PT equivalent of the law of composition of relativistic permitted velocities and that of the gain a generalized equivalent of the time equation of the Lorentz boost.

In what concerns the general problem of whether the Lorentz transformations are good enough in describing the interaction of partially coherent light with various polarization devices/media, this problem was analyzed first by Başkal and Kim [21] in the frame of group theory. They concluded that, quite generally, the decoherence processes cannot be discussed in the framework of the Lorentz group, and one way for approaching such processes is by passing to a larger symmetry group, namely, to the de Sitter group. Viewed from this perspective, the results of the present paper show that, for a complete description (SOP and gain) of even one of the simplest, but basic, deterministic (nondepolarizing) devices, the diattenuator, Lorentz transformation cannot accommodate. This aspect is hidden if we are interested only in the output SOP but comes to light when we want to determine the gain. I have introduced a coefficient of correction with respect to Lorentz transformation, imposed by the physical characteristics of the device.

2. OPERATOR OF A DICHROIC DEVICE

The general Hermitian operator, which describes the action of a dichroic device on (generally partially) polarized light in the most compact form, namely, that of the Pauli-algebraic formalism [22], is

$$H = e^{\rho} e^{\frac{\eta}{2}\mathbf{n}\cdot\boldsymbol{\sigma}} = e^{\rho}\left(\sigma_0 \cosh\frac{\eta}{2} + \mathbf{n}\cdot\boldsymbol{\sigma}\sinh\frac{\eta}{2}\right), \quad (2)$$

where

$$e^{\rho} = e^{\frac{\eta_1+\eta_2}{2}} \quad (3)$$

and

$$e^{\eta} = e^{\eta_1-\eta_2} \quad (4)$$

are the isotropic and the relative amplitude transmittances of the dichroic device, with e^{η_1} and e^{η_2} its principal (eigen-)transmittances, major and minor, respectively. The coefficients η_1 and η_2 may be both negative (for diattenuators), both positive (for diamplifiers [10,11]) or one of them positive and the other negative (amplifier in one channel, attenuator in the other). It is straightforward to write the well-known 2×2 complex matrix representation of the operator H of the dichroic device by introducing in Eq. (2) the matrix form of the components $(\sigma_1, \sigma_2, \sigma_3)$ of Pauli operator $\boldsymbol{\sigma}$ and the Cartesian components of the Poincaré unit vector (the Poincaré axis) of the dichroic device $\mathbf{n}(\cos\theta\cos\psi, \cos\theta\sin\psi, \cos\psi)$, with θ and ψ being the usual spherical coordinates of the Poincaré sphere.

The operator H is not unimodular; its squeeze part, $e^{\frac{\eta}{2}\mathbf{n}\cdot\boldsymbol{\sigma}}$, has the determinant 1, so that the determinant of H is $e^{2\rho}\neq 1$, excepting the trivial case when the device would have no isotropic transmittance. That means that the operator of a dichroic device belongs to the group GL(2, c); only its squeeze factor belongs to SL(2, c), i.e., constitutes a Lorentz transformation (more precisely a squeeze Lorentz transformation, i.e., a boost in STR language).

Usually in the literature [6–13], the isotropic factor of the operator of dichroic device [Eq. (2)] is neglected, and the transformation of SOPs produced by a dichroic device (e.g., diattenuator) is treated as a pure Lorentz transformation: "The exponential factor $e^{(\eta_1+\eta_2)/2}$ reduces both components at the same rate and *does not affect state of polarization*. The effect of polarization is determined solely by the squeeze matrix." [12,13]. The same is the situation in other problems of optics whose underlying algebra was recognized as being that of the Lorentz group. The assertion is correct as long as we are interested only in the state of polarization of the emerging light; indeed, in such problems the extra factor, e^{ρ}, can be ignored; the SOP is determined only by the squeeze (i.e., Lorentzian factor) of the operator of the device.

But if we want to obtain the full information on the modification of the state of the light that passes through the dichroic device, namely, to get also the gain of the device, the factor e^{ρ} becomes essential.

3. GENERALIZED LORENTZ TRANSFORMATION IN POLARIZATION THEORY

The most compact and elegant vectorial form of the Lorentz boost transformation (LBT) is [23,24]

$$\mathbf{r} = \gamma_u(\mathbf{r}' + \mathbf{u}t') + (\gamma_u - 1)\mathbf{n}_u \times (\mathbf{n}_u \times \mathbf{r}'), \quad (5)$$

$$t = \gamma_u(t' + \mathbf{u}\cdot\mathbf{r}'), \quad (6)$$

where $\mathbf{r}'$ and t' are the 3D position vector, and the moment of an event in an inertial reference system (IRS) K'; $\mathbf{r}$ and t the coordinates of the same event in another IRS, K; $\mathbf{u}$ the relative velocity of K' with respect to K; $\mathbf{n}_u$ the unit vector of this velocity and

$$\gamma_u = \frac{1}{\sqrt{1-u^2}} \quad (7)$$

the Lorentz factor of the LBT. In Eqs. (5) and (6), the velocities are scaled at 1 (the limit velocity, i.e., the light velocity in vacuum is taken 1) by conveniently choosing the length or time unit (the so-called "naturalized" set of units) [25].

Now, the analogy, the parallelism, between the Stokes quadrivector of the polarized light state (SOP) and the quadrivector of the event is, simply,

$$t \leftrightarrow S^0, \quad (8)$$

$$\mathbf{r} \leftrightarrow \mathbf{S}, \quad (9)$$

where S^0 is the first Stokes parameter, the light intensity, $S^0 = I$, and $\mathbf{S} = (S^1, S^2, S^3)$. At the level of basic invariants of these Lorentz transformations, the parallelism is

$$t^2 - |\mathbf{r}|^2 \leftrightarrow (S^0)^2 - |\mathbf{S}|^2 \quad (10)$$

(the first aspect of the parallelism, noticed by Barakat in 1963).

Let us further consider the normalized Stokes parameters of the incident and emergent light to and from the dichroic device:

$$\mathbf{s}_i = \mathbf{S}_i/S_i^0 = \mathbf{S}_i/I_i, \qquad \mathbf{s}_o = \mathbf{S}_o/S_o^0 = \mathbf{S}_o/I_o, \quad (11)$$

(with indices i and o for incident and output light). This way, we take advantage of the fact that these normalized vectors have a clear intuitive geometrical interpretation. They are the position vectors of the states in the Poincaré ball Σ_3^1; let us call them, firmly, the Poincaré vectors of the SOPs (sporadically, they are called so in PT):

$$\mathbf{s}_i = p_i\mathbf{n}_i, \qquad \mathbf{s}_o = p_o\mathbf{n}_o, \quad (12)$$

with p degrees of polarizations and $\mathbf{n}$ unit vectors of the SOPs in the Poincaré sphere.

Let us note already that the (3D) Poincaré vectors $\mathbf{s} = p\mathbf{n}$ are the PT equivalents of the STR 3D velocities. Indeed, from the equivalence equation [Eqs. (8) and (9)] and Eq. (11),

$$\frac{d\mathbf{r}}{dt} \leftrightarrow \frac{d\mathbf{S}}{dS^0} = \frac{d\mathbf{S}}{dI} = \mathbf{s}, \tag{13}$$

and, if we want, we may interpret them as velocities in PT, too: the rate of variation of the 3D vectorial components of the Stokes quadrivectors with respect to their scalar parts S^0.

The parameters of the transformation in Eqs. (5) and (6) are the vector $\mathbf{u}$ (the velocity of K with respect to K') and the unit vector $\mathbf{n}_u = \mathbf{u}/u$ (the boost axis). The correspondents of these vectors in PT are the Poincaré vector of the dichroic device, let us denominate it by $\mathbf{s}_d$, and its Poincaré axis $\mathbf{n}_d$:

$$\mathbf{s}_d = p_d \mathbf{n}_d \leftrightarrow \mathbf{u}, \tag{14}$$

where p_d is the degree of dichroism of the device [26].

With these correspondences, Eqs. (8), (9), and (14), we can transpose the vectorial form of the LBT, i.e., Eqs. (5) and (6) from STR in PT:

$$\mathbf{S}_o = \gamma_d(\mathbf{S}_i + \mathbf{s}_d S_i^0) + (\gamma_d - 1)\mathbf{n}_d \times (\mathbf{n}_d \times \mathbf{S}_i), \tag{15}$$

$$S_o^0 = \gamma_d(S_i^0 + \mathbf{s}_d \cdot \mathbf{S}_i), \tag{16}$$

and

$$\gamma_d = \frac{1}{\sqrt{1 - |\mathbf{s}_d|^2}} = \frac{1}{\sqrt{1 - p_d^2}}, \tag{17}$$

the Lorentz factor of the LBT induced by the dichroic device.

At this stage, we have to take into consideration that the action of a deterministic polarization device on polarized light is, in fact, a generalized Lorentz transformation, i.e., its operator (or matrix), as we have seen, pertains to the groups GO(3, 1) and GL(2, c), not to the subgroups SO(3, 1) and SL(2,c).

This fact implies a minimal but essential modification of Eqs. (15) and (16):

$$\mathbf{S}_o = G\gamma_d(\mathbf{S}_i + \mathbf{s}_d S_i^0) + G(\gamma_d - 1)\mathbf{n}_d \times (\mathbf{n}_d \times \mathbf{S}_i), \tag{18}$$

$$S_o^0 = G\gamma_d(S_i^0 + \mathbf{s}_d \cdot \mathbf{S}_i), \tag{19}$$

where G is a scalar constant, which depends on the isotropic factor of the polarization device.

Further, with Eqs. (11) and (14),

$$G\mathbf{s}_o S_o^0 = G\gamma_d S_i^0(\mathbf{s}_i + \mathbf{s}_d) + G\frac{\gamma_d - 1}{p_d^2} S_i^0 \mathbf{s}_d \times (\mathbf{s}_d \times \mathbf{s}_i), \tag{20}$$

$$S_o^0 = G\gamma_d S_i^0(1 + \mathbf{s}_d \cdot \mathbf{s}_i), \tag{21}$$

and taking into account that

$$\frac{\gamma_d - 1}{p_d^2} = \frac{\gamma_d^2}{\gamma_d + 1}, \tag{22}$$

one gets

$$GI_o\mathbf{s}_o = G\gamma_d I_i(\mathbf{s}_i + \mathbf{s}_d) + \frac{G\gamma_d^2}{\gamma_d^2 + 1} I_i \mathbf{s}_d \times (\mathbf{s}_d \times \mathbf{s}_i), \tag{23}$$

$$I_o = G\gamma_d I_i(1 + \mathbf{s}_d \cdot \mathbf{s}_i), \tag{24}$$

where I_i and I_o are the intensities of the SOPs incident on and emergent from the device, and $\mathbf{s}_i$, $\mathbf{s}_o$ are the corresponding Poincaré vectors.

The content of Eqs. (18) and (19), as well as of Eqs. (15) and (16), is that the scalar and the vectorial parts of the Stokes parameters of a state are mixed by the action of a dichroic device as the moment in time and the position vector of an event by a boost. It is worth noting that, even if in intuitive physical terms, this mixing of Eqs. (18) and (19) is not still an expressive, an intuitive one. We will process them to such forms.

Let us go further in exploiting the parallelism STR-PT. In relativistic kinematics after establishing the relationship between the viewpoint of two IRS at the level of events $(t, \mathbf{r})$, the next step is to establish the relationship between the corresponding velocities, i.e., between the velocities they report for a same movement of a material point; we have already mentioned this relationship: Eq. (1). This equation is obtained by differentiating both Eqs. (5) and (6) and dividing the results. Let us follow the same algorithm with Eqs. (18), (19) or (20), (21) or (23), (24):

$$\begin{aligned} d\mathbf{S}_o &= G\gamma_d(d\mathbf{S}_i + \mathbf{s}_d dS_i^0) + \frac{G\gamma_d^2}{\gamma_d + 1}\mathbf{s}_d(\mathbf{s}_d \times d\mathbf{S}_i) \\ &= \left[G\gamma_d\left(\frac{d\mathbf{S}_i}{dS_i^0} + \mathbf{s}_d\right) + \frac{G\gamma_d^2}{\gamma_d + 1}\mathbf{s}_d \times (\mathbf{s}_d \times \mathbf{s}_i)\right] dS_i^0, \\ &= \left[G\gamma_d(\mathbf{s}_d + \mathbf{s}_i) + \frac{G\gamma_d^2}{\gamma_d + 1}\mathbf{s}_d \times (\mathbf{s}_d \times \mathbf{s}_i)\right] dS_i^0, \end{aligned} \tag{25}$$

$$dS_o^0 = G\gamma_d(dS_i^0 + \mathbf{s}_d \cdot d\mathbf{S}_i) = G\gamma_d(1 + \mathbf{s}_d \cdot \mathbf{s}_i)dS_i^0, \tag{26}$$

where we have made use of Eqs. (13), and we have taken $\mathbf{s}_d$ and γ_d as constants because the transformation is done by a given dichroic device.

4. SOP AND GAIN EQUATIONS

Dividing Eq. (25) by Eq. (26), one gets

$$\mathbf{s}_o = \frac{\mathbf{s}_i + \mathbf{s}_d}{1 + \mathbf{s}_d \cdot \mathbf{s}_i} + \frac{\gamma_d}{\gamma_d + 1}\frac{\mathbf{s}_d \times (\mathbf{s}_d \times \mathbf{s}_i)}{1 + \mathbf{s}_d \cdot \mathbf{s}_i}, \tag{27}$$

which is the analog in PT of the composition law of velocities in STR, Eq. (1), or better the gyrovector composition law applied in PT. The Poincaré vectors of the SOPs and of the dichroic devices add according to the gyrovectors' composition law, not according to vectors' composition law (triangle rule); they are gyrovectors.

A second result is obtained directly from Eq. (26), namely, the gain given by the dichroic device:

$$g = \frac{dS_o^0}{dS_i^0} = G\gamma_d(1 + \mathbf{s}_d \cdot \mathbf{s}_i), \tag{28}$$

which easily can be handled to a completely expressive physical form, as follows.

Returning to the GL(2, c) expression of the operator of a dichroic device, Eq. (2), and having in mind that $e^{\frac{\eta}{2}\mathbf{n}\cdot\boldsymbol{\sigma}}$ is its SL(2, c), i.e., its Lorentzian part, it follows that the generic factor G is identifiable for a dichroic device with $e^{2\rho}$:

$$G = e^{2\rho} = e^{\eta_1 + \eta_2}. \tag{29}$$

Then, the equation of the gain, Eq. (28), takes the form

$$g = e^{2\rho}\gamma_d(1 + \mathbf{s}_i \cdot \mathbf{s}_d). \tag{30}$$

The factor 2 at the exponent comes physically from the fact that e^{ρ} in Eq. (2) refers to the (isotropic) amplitude transmittance, while in Eq. (30) it is about an intensity transmittance, mathematically from the fact that the determinant of operator H is $e^{2\rho}$.

This compact equation of the gain can be processed to other, physically more expressive, forms.

In STR, the Lorentz factor, γ_u, is often expressed on the basis of the hyperbolic parameter of the boost, η_u, as follows:

$$\beta = u = \tanh \eta_u \tag{31}$$

(in our case the rapidity β is equal to the velocity of the boost because we have chosen to work in the "naturalized system of units," in which $c = 1$). With Eq. (7), one obtains

$$\gamma_u = \cosh \eta_u. \tag{32}$$

Similarly, in PT the degree of dichroism as function of the coefficient of relative transmittance is [26]

$$p_d = \tanh \eta_d, \tag{33}$$

and, with Eq. (17), we obtain

$$\gamma_d = \cosh \eta_d. \tag{34}$$

Thus, Eq. (30) may be written

$$g = e^{2\rho} \cosh \eta_d (1 + \mathbf{s}_i \cdot \mathbf{s}_d), \tag{35}$$

and, with Eqs. (3) and (4), the gain, Eq. (35), takes the form

$$\begin{aligned} g &= e^{\eta_1+\eta_2} \frac{1}{2}(e^{\eta_1-\eta_2} + e^{\eta_2-\eta_1})(1 + \mathbf{s}_i \cdot \mathbf{s}_d) \\ &= \frac{1}{2}(e^{2\eta_1} + e^{2\eta_2})(1 + \mathbf{s}_i \cdot \mathbf{s}_d). \end{aligned} \tag{36}$$

Here, $e^{2\eta_1} = \tau_M$ and $e^{2\eta_2} = \tau_m$ are the principal major and minor intensity (eigen-) transmittances of the device. If we denote their mean value by

$$\frac{1}{2}(e^{2\eta_1} + e^{2\eta_2}) = \bar{\tau}, \tag{37}$$

then the gain gets the most compact physical insightful form:

$$g = \bar{\tau}(1 + \mathbf{s}_i \cdot \mathbf{s}_d). \tag{38}$$

From Eq. (38), $\bar{\tau}$ (evidently a characteristic solely of the device) is the gain given by the dichroic device for incident unpolarized light ($\mathbf{s}_i = 0$).

Also from Eq. (38), it follows that, for a given dichroic device ($\bar{\tau}$ given), the gain reaches a maximum value when the SOP of the incident light overlaps on the SOP structure of the device ($\mathbf{s}_i \uparrow\uparrow \mathbf{s}_d$):

$$g_M = \bar{\tau}(1 + p_i p_d) \tag{39}$$

and a minimum when the incident SOP is orthogonal to the polarization state of the device (i.e., their Poincaré vectors are antiparallel ($\mathbf{s}_i \uparrow\downarrow \mathbf{s}_i$):

$$g_m = \bar{\tau}(1 - p_i p_d). \tag{40}$$

Other physically expressive forms of the gain can be obtained. An exhaustive analysis of various equivalent forms is given in [26], in the frame of a Pauli algebraic approach to the problem. It would be a harmful digression from the goal of this paper to insist on them. I would transcript here only that written in the "total polarized–unpolarized dichotomy," which has plenty of physical insight [26],

$$g = (1 - p_i)\bar{\tau} + p_i\left(\tau_M \cos^2 \frac{\alpha}{2} + \tau_m \sin^2 \frac{\alpha}{2}\right), \tag{41}$$

where α is the angle between the Poincaré vectors of the incident SOP and of the dichroic device: $\alpha = \measuredangle(\mathbf{s}_i, \mathbf{s}_d)$. This is probably the most physically expressive form of the general Malus' law. For $p_i = 1$ (totally polarized incident light), $\tau_m = 0$, and $\tau_M = 1$ (ideal polarizer), it reduces to the elementary Malus' law:

$$g = \cos^2 \frac{\alpha}{2}. \tag{42}$$

Thus Eqs. (30), (36), and (38) are also various forms of the generalized Malus' law. One of the important conclusions of this paper is that the gain equation, or, the same thing, the generalized Malus' law, Eqs. (30), (36), (38), and (41) are the PT equivalent of the STR time equation in Eq. (6).

5. CONCLUSIONS

Equations (18) and (19) constitute the PT generalized Lorentz transformation equivalent to STR Lorentz transformation, namely, Eqs. (5) and (6). They provide the entire information on the state of light emergent from a dichroic device: the SOP, Eq. (27), and the gain, e.g., Eq. (38).

In regards to the SOP of the emergent light, it is obtained by dividing Eqs. (18) and (19), or Eqs. (20) and (21) or Eqs. (23) and (24) after differentiating them. Because both contain the factor G, this is eliminated in the final result of Eq. (27). The output SOP does not contain the factor G, so that the SOP would be the same if calculated on the basis of restricted Lorentz transformation, namely, Eqs. (15) and (16). This is the reason for which the widespread assertion, "The exponential factor $e^{(\eta_1+\eta_2)/2}$ reduces both components at the same rate and *does not affect state of polarization*. The effect of polarization is determined solely by the squeeze matrix" [12,13], is correct, even if the action of a dichroic device on polarized light is given, in fact, not by a Lorentz transformation but by a generalized Lorentz transformation. This is the reason for which the action of deterministic devices on partially polarized light is successfully analyzed all over in the literature, e.g., [6–18], in the frame and in terms of the Lorentz group: all these studies refer only to the modification of the light SOP, as the above quotation says explicitly.

But the Lorentz transformation, Eqs. (15) and (16), by a slight generalization, namely, Eqs. (18) and (19), covers, as we have seen (in a spectacular manner, I could say), the problem of the gain, too. The gain follows from Eq. (19), the time-like equation of the generalized Lorentz transformations. Because it comes out from this equation alone, the factor G is preserved in the gain, and, for an exhaustive approach to the interaction of deterministic devices with polarized light, we are forced to overpass the groups SO(3, 1) and SL(2, c) and appeal to GO(3, 1) and GO(2, c).

It is worth mentioning another striking aspect which points out the fact that the gain cannot be obtained in the frame of simple Lorentz transformation. Should we have extracted the ratio dS_o^0/dS_i^0 from Eq. (16), we would have obtained

$$\frac{dS_o^0}{dS_i^0} = \gamma(1 + \mathbf{s}_d \cdot \mathbf{s}_i) = \frac{1 + \mathbf{s}_d \cdot \mathbf{s}_i}{\sqrt{1 - p_d^2}}. \quad \textbf{(43)}$$

Here, the ratio on the left side of the equation means the gain, but its expression is not correct. In accordance with Eq. (43), the gain of a diattenuator would tend to infinity when the diattenuator would become an ideal polarizer, which is a completely erroneous result.

The isotropic factor $e^{2\rho}$ in the expression of the operator of any dichroic device is a *sine qua non* one, and recognizing this fact we have to overpass, slightly but essentially, the SO(3, 1) group, in fully discussing quasi-Lorentzian PT problems such as that of the interaction between deterministic devices and polarized light.

The same generalization can be extended in other fields of physics where the Lorentzian (in fact, also only the quasi-Lorentzian) character of the mathematical underpinning of various problems is recognized (multilayers [27,28], interferometry and ray optics [29,30], laser cavities [31], squeezed states of light [32], a.s.o.).

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